Classical solutions of the divergence equation with Dini continuous data

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Abstract

We consider the boundary value problem associated with the divergence operator on a bounded regular subset of $\mathbb{R}^n$, with homogeneous Dirichlet boundary condition. We prove the existence of a classical solution under slight assumptions on the datum.

1 Introduction

In this paper we deal with the existence of classical solutions for the first order boundary value problem

$$\begin{cases} \operatorname{div} u = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We look for solutions $u : \Omega \to \mathbb{R}^n$, belonging at least to $C^1(\Omega) \cap C^0(\overline{\Omega})$ but, actually, we will prove a sharper result of regularity at the boundary (see Theorem 1). Here, Ω is a smooth, bounded, open subset of $\mathbb{R}^n$, $n \geq 2$, while F is a given continuous function (as expected, F will be required to fulfill a condition slightly stronger than the bare continuity), satisfying the compatibility condition $\int_\Omega F(x)\,dx = 0$. This is a classical problem in mathematical fluid mechanics, strictly connected with the Helmholtz decomposition and the div-curl lemma (see Kozono and Yanagisawa [20]). We recall that, if the boundary condition is dropped, a solution of the divergence equation can be readily obtained by taking the gradient of the Newtonian potential of F, provided it is in $C^2(\Omega)$. These aspects are extensively covered in Galdi [15, Ch. III], with special attention to the work of Bogovskiĭ [7], where the problem (1) is solved in the setting of the Sobolev spaces $H_0^{1,p}(\Omega)$. Further developments may also be found in Borchers and Sohr [8]. For different approaches and results, the reader should consider the books by Ladyzhenskaya [21] and Tartar [26], which especially cover the Hilbert case, while Amrouche and Girault [1] devised an approach based on the negative norm theory developed in Nečas [23].

Our approach follows closely the Bogovskiĭ's one, where the representation formula (2) below, in analogy with the Sobolev's "cubature" formulae, provides explicitly a special solution of the problem (1). We recall that, *per se*, problem (1) has infinitely many solutions. The representation formula (2) turns out to be extremely flexible in the applications to many different

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settings as, for instance, in the recent results for weighted and $L^{p(x)}$-spaces (see Huber [17]). Classical results in Hölder spaces have been shown in Kapitanskiĭ and Piletskas [18], as corollaries of a more general result, which seems to be obtained in a way different from ours. We point out that our methods, which can be considered as classical, can be also easily modified to obtain the corresponding results in Hölder spaces, for which we also mention the recent review in Csató, Dacorogna, and Kneuss [11]. In addition, we also note that the non-uniqueness feature of the first order system (1) allows some existence results with more regularity than expected from the usual Sobolev machinery, as the striking results of Bourgain and Brezis [9], which come from a non-linear selection principle (see also the extensions to the Dirichlet problem and Triebel-Lizorkin spaces setting in Bousquet, Mironescu, and Russ [10]).

Our interest in the problem is twofold: on one side, we want to investigate the results close to the limiting case $F \in L^\infty(\Omega) \cap C^0(\Omega)$, where counterexamples to the existence of a solution are known (see Bourgain and Brezis [9], Dacorogna, Fusco, and Tartar [12]; Maremonti [22], from the point of view of Hydrodynamics); on the other side, we are interested in relaxing as much as possible the assumptions needed to prove the existence of classical solutions, with the aim of finding weaker assumptions allowing to construct classical solutions to fluid mechanics problems.

Since the bare continuity of F is not enough to that purpose, we went back to the pioneering results by Dini [14] and Petrini [24] about the Poisson equation, and consider the problem with the additional hypothesis that F is Dini continuous, which we denote in the following by $F \in C_D(\Omega)$ (see Section 2.2 for a formal definition). The main tool will be to combine the formula (2) with a modification of an argument used by Korn to obtain a similar regularity result for the second order derivatives of the Newtonian potential (see Gilbarg and Trudinger [16, Ch. 4]). In fact, for the Poisson equation, these tools allow to exploit the property of the Dini continuity to "mitigate" the singularity of the integrand in the formula representing the second order derivatives of the potential, without the need to apply the Calderòn-Zygmund theory for singular integral operators.

About the homogeneous boundary condition present in (1), our proof is based on some new insight on the formula (2), in the sense that we made some simple observations on the Bogovskiĭ formula that we cannot find stated explicitly elsewhere in literature (see Theorem 2). These observations allow us to consider the case when the datum F cannot be approximated by compactly supported smooth functions. Such an approximation seems to play a fundamental role for the previous results in Sobolev or Orlicz spaces, and hence the argument used in [7, 8, 15] does not apply immediately to our setting, unless $F_{|\partial\Omega} = 0$, which is an unnecessary assumption (see also Remark 5).

For the sake of completeness and to put the present work into a wider perspective, we also wish to mention that the link between Dini continuity and existence of classical solutions in fluid mechanics started with the work of Shapiro [25] in the steady-state case and found a very interesting application in the paper of Beirão da Veiga [2], where the 2D Euler equations for incompressible fluids are solved in the "critical" space of vorticity $C^0(0, T; C_D(\Omega))$. This is also very close (if one thinks about scaling) to the Besov spaces used by Vishik [27]. More recently, the same results with Dini continuous vorticity have also been employed in Koch [19] and in [5] to study the fine properties of the long-time behavior of the 2D Euler equations. In addition, the interest for classical solutions of the Stokes system has been revived in the recent papers of Beirão da Veiga [3, 4], and provided a further motivation to our analysis of the divergence and curl operator, since they are among the basic building blocks of the theory. Finally, the "inversion" of the divergence operator in the continuous setting is also one of the tools giving rise to the celebrated series of results of De Lellis and Székelyhidi (see, for instance, [13, Sec. 4]) on the Onsager conjecture.

The main result of this paper is the following theorem of classical regularity up to the boundary (see Section 2.1 and Section 2.2 below for the definition of C^2-boundary and $C_D(\Omega)$ respectively, and Remark 23 for the dependencies of the constant c).

Theorem 1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a C^2-boundary. Then, there exists a constant c such that, for any $F \in C_D(\Omega)$ satisfying $\int_\Omega F(x)\,dx = 0$, there exists a solution $u \in C^1(\overline{\Omega})$ of the problem* (1) *verifying*

$$\|u\|_{C^1(\overline{\Omega})} \leq c\,\|F\|_{C_D(\Omega)}\,.$$

We also want to point out that, since the system (1) is not elliptic, the well-known results for elliptic equations (and systems) do not apply directly. While the interior regularity (see Theorem 16) can be obtained by adapting standard results (see Section 2 and Section 3), the regularity up to the boundary requires an *ad hoc* treatment (see Section 4).

To conclude the introduction, we also mention that the problem of the existence of a $C^1(\Omega)\cap C^0(\overline{\Omega})$ solution for the curl equation (with homogeneous Dirichlet boundary condition) is treated elsewhere (see [6]), following the same method, by using the similar (but more complicated) representation formula valid in that case.

2 Basic concepts, notations, and preliminary results

In this section we recall the main definitions we will use, as well as some basic facts about the representation formula developed by Bogovskiĭ. Most of the results of this section are well-known. However, some of them about the behaviour at the boundary (that will be crucial in order to fulfill the homogeneous boundary conditions) are, as far as we know, not explicitly available in the literature.
In the following we denote by $B(x,R) = \{y \in \mathbb{R}^n : |y-x| < R\}$, the open ball of radius R centered at x, by $S^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$ the unit sphere of $\mathbb{R}^n$, and by $|S^{n-1}|$ its $(n-1)$-dimensional measure.

2.1 Definition of a $C^{k,\lambda}$-boundary

By following Nečas [23], we say that $\partial\Omega$ is of class $C^{k,\lambda}$ (and write $\partial\Omega \in C^{k,\lambda}$) if, for any $P \in \partial\Omega$, there exist a rotation A_P, positive numbers δ_P and Δ_P, and $h_P : [-\delta_P, \delta_P]^{n-1} \to (-\Delta_P, \Delta_P)$ verifying $h_P \in C^{k,\lambda}([-\delta_P, \delta_P]^{n-1})$ such that:

- $(x', x_n) \in A_P(\Omega - P) \Longleftrightarrow x_n > h_P(x')$,
- $(x', x_n) \in A_P(\partial\Omega - P) \Longleftrightarrow x_n = h_P(x')$,

for any $x' \in [-\delta_P, \delta_P]^{n-1}$ and any $x_n \in (-\Delta_P, \Delta_P)$.
Here $\Omega - P := \{A - P : A \in \Omega\}$ and $\partial\Omega - P := \{A - P : A \in \partial\Omega\}$.

The hypothesis that $\partial\Omega \in C^{0,1}$ will be used in Section 3, to obtain a more general inner-regularity result. For the oncoming results of Section 4.2, let us remark explicitly that, if $\partial\Omega$ is of class C^2 as in Theorem 1 (but it is enough that h_P is at least differentiable in P for any $P \in \partial\Omega$), in the previous definition it is possible to choose A_P so that it maps the normal to $\partial\Omega$ at P onto the x_n axis, $\Delta_P \leq \delta_P/2$ and to have, for some $0 < R_P \leq \delta_P$, that the following properties hold true

- $\Big\{P + A_P^{-1}\big((-R_P, R_P)^n\big), \quad P \in \partial\Omega\Big\}$ is an open covering of $\partial\Omega$,
- $\Omega_P := P + A_P^{-1}\big(\big\{x' \in (-R_P, R_P)^{n-1},\, h(x') < x_n < h(x') + R_P\big\}\big) \subseteq \Omega$.

If, in addition, Ω is bounded, it follows immediately that there exist a finite number of boundary points $P_1, \ldots, P_k$ such that $\partial\Omega \subseteq \cup_{i=1}^{k} \overline{\Omega_{P_i}}$, and then $\Omega \setminus \cup_{i=1}^{k} \Omega_{P_i} \subset\subset \Omega$, and it may be covered by a finite number of open balls contained in Ω. This feature, together with a localization argument, will allow to treat the problem of the regularity at the boundary (see Section 4).

2.2 The Dini continuous functions

We denote by $C_D(\Omega)$ the space of the Dini continuous functions F, *i.e.* the functions $F \in C^0(\overline{\Omega})$ such that, if one introduces the (uniform) modulus of continuity

$$\omega(F, \rho) := \sup_{\substack{x,y \in \Omega \\ |x-y| < \rho}} |F(x) - F(y)|,$$

then the function $\omega(F, \rho)/\rho$ is integrable around 0^+. The space so defined may be equipped with the following norm

$$\|F\|_{C_D(\Omega)} := \max_{x \in \overline{\Omega}} |F(x)| + \int_0^{\mathrm{diam}(\Omega)} \frac{\omega(F, \rho)}{\rho}\, d\rho,$$

and turns out to be a Banach space. In literature, the space $C_D(\Omega)$ is often referred to as the space of uniformly Dini continuous functions. We remark that, by uniform continuity, any function in $C_D(\Omega)$ may be extended up to the boundary of Ω with the same modulus of continuity. We also observe that $C^{0,\alpha}(\overline{\Omega}) \subset C_D(\Omega)$ for any $\alpha \in]0, 1]$, and recall that the relevance of Dini continuity in partial differential equations theory comes from the result stating that, if $f \in C_D(\Omega)$, then the solution of the Poisson equation

$$\Delta u = f,$$

with zero Dirichlet conditions satisfies $D^2 u \in C^0(\Omega)$ (see, e.g, Gilbarg and Trudinger [16, Pb. 4.2]; see also Dini [14] and Petrini [24]).

As usual, we denote by $C^1(\overline{\Omega})$ the space of the functions in $C^0(\overline{\Omega})$ whose first order derivatives are uniformly continuous in Ω, and so they may be continuously extended to the closure $\overline{\Omega}$. We do not distinguish between scalar and vector valued functions, since the meaning is clear from the context.

2.3 Bogovskiĭ's formula and its variants

Unless differently specified (namely, in the last two sections), the following notation and hypotheses are tacitly assumed throughout all the paper:

- The symbol B denotes the open unit ball of $\mathbb{R}^n$, $n \geq 2$, centered at the origin;
- The symbol ψ denotes a non-identically vanishing scalar function verifying $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\mathrm{supp}\, \psi \subseteq B$;
- By $\partial_j \psi$ we denote the partial derivative of ψ with respect to its j-th argument;
- The domain Ω is a bounded open subset of $\mathbb{R}^n$, star-shaped with respect to any point of $\overline{B}$ (This strong geometric restriction on Ω will be relaxed in the last two sections).

The aim of this section is to recall the representation formula for a solution of the divergence problem, due to Bogovskiĭ [7], as well as several useful variants and consequences. We start with a theorem, which recalls the integral formula and gives a first uniform estimate.

Theorem 2. *Let $q > n$ and let $F \in L^q(\Omega)$. Then:*

i) The Bogovskiĭ's formula

$$v(x) := \int_\Omega F(y) \left[\frac{x-y}{|x-y|^n} \int_{|x-y|}^{+\infty} \psi \left(y + \xi \frac{x-y}{|x-y|} \right) \xi^{n-1} d\xi \right] dy, \tag{2}$$

defines for any $x \in \mathbb{R}^n$ (and not only almost everywhere) a vector-valued function $v : \mathbb{R}^n \to \mathbb{R}^n$;

ii) The vector field verifies $v(x) = 0$ for all $x \in \mathbb{R}^n \backslash \Omega$;

iii) For any $q > n$

$$|v(x)| \leq c \, \|F\|_{L^q(\Omega)} \qquad \forall \, x \in \mathbb{R}^n,$$

where the constant c depends only on n, ψ, $\operatorname{diam} \Omega$, and q;

Formula (2) can be also rewritten in the following three equivalent ways

iv) $v(x) = \int_\Omega F(y) \left[(x-y) \int_1^\infty \psi \left(y + \alpha(x-y) \right) \alpha^{n-1} d\alpha \right] dy$;

v) $v(x) = \int_\Omega F(y) \left[\frac{x-y}{|x-y|^n} \int_0^\infty \psi \left(x + r \frac{x-y}{|x-y|} \right) (|x-y| + r)^{n-1} dr \right] dy$;

vi) $v(x) = \int_{x-\Omega} F(x-z) \frac{z}{|z|^n} \int_0^\infty \psi \left(x + r \frac{z}{|z|} \right) (|z| + r)^{n-1} \, dr \, dz$,
where $x - \Omega = \{ z \in \mathbb{R}^n : \exists \, y \in \Omega$ such that $z = x - y \}$.

Definition 3 (Bogovskiĭ's kernel)**.** *For $x, y \in \mathbb{R}^n$ with $x \neq y$, we define the Bogovskiĭ's kernel (associated to ψ) by*

$$N(x,y) := \frac{x-y}{|x-y|^n} \int_{|x-y|}^{+\infty} \psi \left(y + \xi \frac{x-y}{|x-y|} \right) \xi^{n-1} d\xi,$$

and remark that we can rewrite the Bogovskiĭ's formula as follows

$$v(x) = \int_\Omega N(x,y) \, F(y) \, dy.$$

Occasionally, we refer to the vector field v as the Bogovskiĭ's potential, in analogy with the classical theory of Newtonian potentials (see [16]). Equivalent expressions for the Bogovskiĭ's potential can be written by using the formulae iv),v), and vi) from Theorem 2.

The proof of Theorem 2 requires some preliminary results, we will use extensively later on.

Lemma 4. *The following properties of the Bogovskiĭ's kernel hold true:*

i) The kernel $N(x,y)$ verifies

$$N(x,y) \equiv 0 \qquad \forall \, x \notin \Omega \text{ and } \forall \, y \in \Omega;$$

ii) There exists a constant $c > 0$, depending only on n, ψ, and $\operatorname{diam} \Omega$, such that

$$|N(x,y)| \leq c \, |x-y|^{1-n} \quad \forall \, x, y \in \mathbb{R}^n : \; x \neq y.$$

The proof of Theorem 2 follows immediately from the direct inspection of the kernel and by observing that if $x \notin \Omega$ and $\psi\left(y + \xi \frac{x-y}{|x-y|}\right) \neq 0$ holds true for some $\xi > |x-y|$, then $y \notin \Omega$. By using the above lemma, Theorem 2 follows by some straightforward arguments, left to the reader. We just point out that since Ω is bounded, it turns out that $N(x, \cdot) \in L^{q'}(\Omega)$ for all $x \in \Omega$ and for all $q' \in [1, \frac{n}{n-1}[$.

Remark 5. *It is useful to point out explicitly that the Bogovskiĭ's potential $v(x)$ is well-defined and vanishes at the boundary $\partial\Omega$ for any $F \in L^q(\Omega)$, with $q > n$, without any other assumption but those made on Ω and ψ in Theorem 2. Next, the property $v_{|\partial\Omega} = 0$ does not come by approximating F by $C_0^\infty(\Omega)$ functions and by taking limits, but it descends directly from the formula (2) for a large class of data. This will be crucial in the rest of the paper, since a function in $C_D(\Omega)$ cannot be approximated uniformly by regular functions with compact support, unless it vanishes at the boundary.*

Very relevant consequences of the properties of $N(x, y)$ and of Theorem 2 are the two interior and boundary regularity results for the potential v of a smooth, compactly supported F.

Theorem 6. *Under the same hypotheses of Theorem 2, if in addition $F \in C_0^\infty(\Omega)$ then $v \in C_0^\infty(\Omega)$.*

Theorem 7. *Under the same assumptions of Theorem 2, it follows that $v \in C^0(\mathbb{R}^n)$ and, by restriction, $v \in C^0(\overline{\Omega})$.*

Theorem 6 is classical and the proof of may be found, *e.g.*, in Galdi [15, Lemma III.3.1], while the short proof Theorem 7, which is based on Theorem 2, is original and given below.

Proof of Theorem 7. Let $\{F_k\} \subset C_0^\infty(\Omega)$ be such that $F_k \to F$ in $L^q(\Omega)$ for some $q > n$, and assume that F_k is extended by zero outside Ω. Let v_k and v be the corresponding Bogovskiĭ's potentials. By Theorem 2 *ii)* and *iii)*, it follows that v_k converge uniformly to v in $\mathbb{R}^n$. Since, by Theorem 6, $\{v_k\} \subset C_0^\infty(\mathbb{R}^n)$, the theorem follows immediately. $\square$

We also recall some further properties of the Bogovskiĭ's kernel, which will turn out to be useful in the following, see again Galdi [15, Ch. III.3]. First, we have an identity about the derivatives of the kernel in Theorem 2, *iv)*, which can be obtained observing that differentiating under the sign of integral is completely justified also in the sense of Riemann integrals (Note that this is the expression of $N(x, y)$ fow which derivatives are better handled). Next, we have the fundamental estimates for $\partial_{x_j} N_i(x, y)$ (which allowed to exploit the Calderòn-Zygmund theory to obtain the original Bogovskiĭ's results about the $H_0^{1,p}(\Omega)$ regularity of v), which will be essential for our results in the setting of Dini continuous functions.

Lemma 8. *For any fixed $x, y \in \Omega$, such that $x \neq y$ let, for $i = 1, \dots, n$*

$$N_i(x, y) = (x_i - y_i) \int_1^\infty \psi(y + \alpha(x - y))\, \alpha^{n-1}\, d\alpha.$$

Then, it follows that for all $i, j = 1, \dots, n$

$$\partial_{x_j} N_i(x, y) = (x_i - y_i) \int_1^\infty \partial_j \psi(y + \alpha(x - y))\, \alpha^{n-1}\, d\alpha - \partial_{y_j} N_i(x, y).$$

Moreover, $\partial_{x_j} N_i(x, y) = K_{ij}(x, x - y) + G_{ij}(x, y)$, where $K_{ij}(x, \cdot)$ is a Calderòn-Zygmund singular kernel and G_{ij} is a weakly singular kernel, hence it holds

$$|\partial_{x_j} N_i(x, y)| \leq M\, |x - y|^{-n} \qquad \forall\, x, y \in \mathbb{R}^n : \; x \neq y,$$

for some constant $M = M(\psi, n, \text{diam } \Omega)$.

2.4 The representation formula for derivatives of the Bogovskiĭ's potential

The main tool we take advantage of in this paper is an old aged argument exploited by Korn (see, e.g., Gilbarg and Trudinger [16, Ch. 4]) in the study of the existence of classical solutions of the Poisson equation, based on a suitable smoothing of the singularity present in the representation formula for the second order derivatives of the Newtonian potential. A similar argument provides us a way to approximate the Bogovskiĭ's potential by regular functions. We give full details since the technique is different from the one with truncation, used to get results in the Lebesgue space setting and also because we will need to use the theorems about uniform convergence of sequences of continuous functions. To this purpose, we fix a function $\eta \in C^\infty(\mathbb{R}^+)$ such that $\eta(t) \equiv 0$ on $[0,1]$, $\eta(t) \equiv 1$ if $t \geq 2$, and $|\eta'(t)| \leq 2$ for all $t \in \mathbb{R}^+$.

Definition 9. *For any $F \in L^p(\Omega)$, with $p \geq 1$, and $\epsilon > 0$ let us set*

$$\begin{aligned} v^\epsilon(x) &:= \int_\Omega F(y)\,(x-y)\,\left[\int_1^\infty \psi(y+\alpha(x-y))\,\alpha^{n-1}d\alpha\right]\,\eta\left(\frac{|x-y|}{\epsilon}\right)\,dy \\ &= \int_\Omega F(y)\,N(x,y)\,\eta\left(\frac{|x-y|}{\epsilon}\right)\,dy. \end{aligned}$$

We remark that

$$N(x,y)\,\eta\left(\frac{|x-y|}{\epsilon}\right) \equiv 0 \qquad \text{for } |x-y| < \epsilon,$$

and therefore the above truncated kernel belongs to $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and is bounded by Lemma 4. On the other hand, if $|x-y| \geq \epsilon$ the set of α such that $|y+\alpha(x-y)| \leq 1$, where ψ could be non vanishing, is bounded as well. It follows immediately that v^ϵ is well-defined for all $x \in \mathbb{R}^n$ and it belongs to $C^\infty(\mathbb{R}^n)$. Moreover, under the same hypotheses of Theorem 6 and following the same argument, one proves that it actually belongs to $C_0^\infty(\Omega)$. The following theorem is the cornerstone of the resolution of the divergence equation.

Theorem 10. *Assume the same hypotheses of Theorem 2, and let η and v^ϵ be as above.*

i) If $F \in L^\infty(\Omega)$, then

$$\lim_{\epsilon \to 0^+} v^\epsilon(x) = v(x) \quad \textit{uniformly in } \mathbb{R}^n.$$

ii) If, in addition, $\int_{\mathbb{R}^n} \psi(x)\,dx = 1$ and $F \in C^0(\overline{\Omega})$, then

$$\lim_{\epsilon \to 0^+} \operatorname{div} v^\epsilon(x) = F(x) - \psi(x) \int_\Omega F(y)\,dy \qquad \forall\, x \in \Omega,$$

and consequently, if $\int_\Omega F(x)\,dx = 0$, then

$$\lim_{\epsilon \to 0^+} \operatorname{div} v^\epsilon(x) = F(x) \qquad \forall\, x \in \Omega.$$

Proof. The fact that the Bogovskiĭ's potential is a right inverse of the divergence is a classical result. Nevertheless we report the proof here since it is slightly different from that in Lebesgue spaces (cf. [15, Lemma III.3.1]), where the truncation is not smooth and a surface integral (not present here) has to be studied. To prove *i)*, let us fix any $x \in \Omega$. We remark that, by Lemma 4, for any $\epsilon < \operatorname{dist}(x, \partial\Omega)/2$

$$\begin{aligned} &|v_i^\epsilon(x) - v_i(x)| \\ &\quad \leq \int_\Omega |\,F(y) N_i(x,y)|\,\left|\eta\left(\frac{|x-y|}{\epsilon}\right) - 1\right|\,dy \\ &\quad \leq c\|F\|_{L^\infty(\Omega)} \int_{|x-y|<2\epsilon} \frac{1}{|x-y|^{n-1}}\,dy \leq c\|F\|_{L^\infty(\Omega)} \int_{|z|<2\epsilon} |z|^{1-n}\,dz, \end{aligned}$$

and by the absolute continuity of the Lebesgue integral, it follows that the last integral tends to zero, independently of $x \in \Omega$. To complete the proof of *i)* it is enough to remark that, again by Lemma 4, $v^\epsilon(x) = v(x) = 0$ for all $x \notin \Omega$.

To prove *ii)*, by differentiating v_i^ϵ at any $x \in \Omega$ it follows that

$$\begin{aligned}
&\partial_{x_j} v_i^\epsilon(x)\\
&= \int_\Omega F(y)\,\delta_{ij}\left[\int_1^\infty \psi(y+\alpha(x-y))\,\alpha^{n-1} d\alpha\right]\eta\left(\frac{|x-y|}{\epsilon}\right)\,dy\\
&+ \int_\Omega F(y)(x_i-y_i)\left[\int_1^\infty (\partial_j\psi(y+\alpha(x-y))\,\alpha^{n} d\alpha\right]\eta\left(\frac{|x-y|}{\epsilon}\right)\,dy\\
&+ \int_\Omega F(y)(x_i-y_i)\left[\int_1^\infty \psi(y+\alpha(x-y))\,\alpha^{n-1} d\alpha\right]\eta'\left(\frac{|x-y|}{\epsilon}\right)\frac{x_j-y_j}{|x-y|}\frac{dy}{\epsilon}\\
&= \int_\Omega F(y)\,\delta_{ij}\left[\int_1^\infty \psi(y+\alpha(x-y))\,\alpha^{n-1} d\alpha\right]\eta\left(\frac{|x-y|}{\epsilon}\right)\,dy\\
&+ \int_\Omega F(y)(x_i-y_i)\left[\int_1^\infty (\partial_j\psi(y+\alpha(x-y))\,\alpha^{n} d\alpha\right]\eta\left(\frac{|x-y|}{\epsilon}\right)\,dy\\
&+ \int_\Omega F(y)\frac{(x_i-y_i)(x_j-y_j)}{|x-y|}\left[\int_1^\infty \psi(y+\alpha(x-y))\,\alpha^{n-1} d\alpha\right]\eta'\Big(\frac{|x-y|}{\epsilon}\Big)\frac{dy}{\epsilon},
\end{aligned}$$

and therefore we have

$$\begin{aligned}
&\operatorname{div} v^\epsilon(x)\\
&= n\int_\Omega F(y)\left[\int_1^\infty \psi(y+\alpha(x-y))\,\alpha^{n-1} d\alpha\right]\eta\left(\frac{|x-y|}{\epsilon}\right)\,dy\\
&+ \sum_{i=1}^n\int_\Omega F(y)(x_i-y_i)\left[\int_1^\infty (\partial_i\psi(y+\alpha(x-y))\,\alpha^{n} d\alpha\right]\eta\left(\frac{|x-y|}{\epsilon}\right)\,dy\\
&+ \int_\Omega F(y)|x-y|\left[\int_1^\infty \psi(y+\alpha(x-y))\,\alpha^{n-1} d\alpha\right]\eta'\left(\frac{|x-y|}{\epsilon}\right)\frac{dy}{\epsilon}\\
&=: I_1+I_2+I_3.
\end{aligned}$$

Now,

$$\begin{aligned}
&I_1+I_2\\
&= \int_\Omega F(y)\,\eta\left(\frac{|x-y|}{\epsilon}\right)\quad\times\\
&\times\int_1^\infty\Big[\psi(y+\alpha(x-y))\,n\,\alpha^{n-1}+\alpha^n\sum_{i=1}^n\partial_i\psi(y+\alpha(x-y))(x_i-y_i)\Big]\,d\alpha dy\\
&= \int_\Omega F(y)\,\eta\left(\frac{|x-y|}{\epsilon}\right)\int_1^\infty\frac{d}{d\alpha}\Big[\psi(y+\alpha(x-y))\alpha^n\Big]\,d\alpha\,dy\\
&= -\psi(x)\int_\Omega F(y)\,\eta\left(\frac{|x-y|}{\epsilon}\right)\,dy\quad\longrightarrow\quad -\psi(x)\int_\Omega F(y)\,dy,
\end{aligned}$$

as ϵ tends to zero.

Moreover, introducing in I_3 first the change of variables $\alpha := \xi/|x-y|$, next $\xi := r+|x-y|$, and finally $z := \epsilon^{-1}(x-y)$ one obtains

$$
\begin{aligned}
I_3 &= \int_\Omega \frac{F(y)}{|x-y|^{n-1}}\,\eta'\left(\frac{|x-y|}{\epsilon}\right)\frac{1}{\epsilon}\,\left[\int_{|x-y|}^\infty \psi\left(y+\xi\frac{x-y}{|x-y|}\right)\xi^{n-1}\,d\xi\right]\,dy \\
&= \int_{\epsilon<|x-y|<2\epsilon} \frac{F(y)}{|x-y|^{n-1}}\,\eta'\left(\frac{|x-y|}{\epsilon}\right)\frac{1}{\epsilon}\times \\
&\qquad\times\left[\int_0^\infty \psi\left(x+r\frac{x-y}{|x-y|}\right)(r+|x-y|)^{n-1}\,dr\right]\,dy \\
&= \int_{1<|z|<2} \frac{F(x-\epsilon z)}{|z|^{n-1}}\,\eta'(|z|)\,\left[\int_0^\infty \psi\left(x+r\frac{z}{|z|}\right)(r+\epsilon|z|)^{n-1}\,dr\right]\,dz.
\end{aligned}
$$

We now claim that, as ϵ goes to 0, the latter converges to

$$
F(x)\int_{1<|z|<2}\frac{1}{|z|^{n-1}}\eta'(|z|)\,\left[\int_0^\infty \psi\left(x+r\frac{z}{|z|}\right)r^{n-1}\,dr\right]\,dz. \tag{3}
$$

In fact, since $\psi\left(x+r\frac{z}{|z|}\right)$ vanishes when $r>1+\mathrm{diam}\ \Omega$, then

$$
\left|\int_0^\infty \psi\left(x+r\frac{z}{|z|}\right)(r+\epsilon|z|)^{n-1}\,dr\right| \le \max|\psi|\,(1+2\,\mathrm{diam}\ \Omega)^n =: M.
$$

Hence, we get

$$
\begin{aligned}
&\left|\int_{1<|z|<2}\frac{F(x-\epsilon\,z)-F(x)}{|z|^{n-1}}\,\eta'(|z|)\,\left[\int_0^\infty \psi\left(x+r\frac{z}{|z|}\right)(r+\epsilon|z|)^{n-1}\,dr\right]\,dz\right| \\
&\le 2M\int_{1<|z|<2}\frac{|F(x-\epsilon\,z)-F(x)|}{|z|^{n-1}}\,dz \le 2M\int_{1<|z|<2}\frac{\omega(F,\epsilon\,|z|)}{|z|^{n-1}}\,dz,
\end{aligned}
$$

where we recall that $\omega(F,\,\cdot\,)$ is the modulus of continuity of F. By the uniform continuity of F on Ω and the Lebesgue theorem on dominated convergence, the last integral vanishes as ϵ goes to zero and therefore the claim is proved.

Finally, by introducing in (3) the radial and angular coordinates $\rho := |z|$ and $u := z/|z|$, one gets

$$
\int_1^2 \eta'(\rho)\,d\rho\int_{S^{n-1}}\int_0^\infty \psi(x+r\,u)\,r^{n-1}\,dr\,du = (\eta(2)-\eta(1))\int_{\mathbb{R}^n}\psi(w)\,dw = 1.
$$

Therefore, $I_3 \to F(x)$ as ϵ goes to 0, and the lemma follows. □

The next lemma will be useful in proving the representation formula in Theorem 12.

Lemma 11.

$$
\begin{aligned}
\partial_{x_j}\left[N_i(x,y)\,\eta\left(\frac{|x-y|}{\epsilon}\right)\right] &= -\partial_{y_j}\left[N_i(x,y)\,\eta\left(\frac{|x-y|}{\epsilon}\right)\right] + \\
&\quad + \eta\left(\frac{|x-y|}{\epsilon}\right)(x_i-y_i)\int_1^\infty \partial_j\psi(y+\alpha(x-y))\,\alpha^{n-1}d\alpha.
\end{aligned}
$$

Proof. It is an immediate consequence of Lemma 8 and the opposite sign in the derivatives of $\eta(|x-y|/\epsilon)$. □

As usual in potential theory, getting a representation formula for the derivatives of the function v_i is a crucial goal. We will obtain it by taking the limit of the derivatives of its "regular approximation" v_i^ϵ. Thus, let us start by differentiating its components

$$v_i^\epsilon(x) = \int_\Omega F(y) N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right) dy = \int_{B_\Lambda} F(y) N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right) dy,$$

where $F \in L^\infty(\Omega)$ is extended by zero outside Ω and $B_\Lambda := B(0, 2\,\mathrm{diam}\,\Omega)$ is a ball of radius large enough to have $\Omega \subset\subset B_\Lambda$. By the previous lemma and Bogovskiĭ's formula in Theorem 2, *iv)* it follows that, for any $x \in \Omega$,

$$\begin{aligned}
&\partial_{x_j} v_i^\epsilon(x) \\
&= \int_{B_\Lambda} F(y)\, \partial_{x_j}\left[N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy \\
&= \int_{B_\Lambda} [F(y) - F(x)]\, \partial_{x_j}\left[N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy \\
&\quad + F(x) \int_{B_\Lambda} \partial_{x_j}\left[N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy \\
&= \int_{B_\Lambda} [F(y) - F(x)]\, \partial_{x_j}\left[N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy \\
&\quad + F(x) \int_{B_\Lambda} \eta\left(\frac{|x-y|}{\epsilon}\right) (x_i - y_i) \int_1^\infty \partial_j \psi(y + \alpha(x-y))\, \alpha^{n-1} d\alpha\, dy \\
&\quad - F(x) \int_{B_\Lambda} \partial_{y_j}\left[N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy.
\end{aligned}$$

Since $\eta\left(\frac{|x-y|}{\epsilon}\right) \equiv 1$ for any $x \in \Omega$, any $y \in \partial B_\Lambda$ and $\epsilon < \mathrm{diam}\,\Omega/2$, by the Gauss-Green formula the last integral is equal to $\int_{\partial B_\Lambda} N_i(x,y)\, \nu_j(y)\, d\sigma_y$, where $\nu_j(y)$ is the j-th component of the outward unit normal vector at the point $y \in \partial B_\Lambda$.

The previous computations suggest to put forward a conjecture about the limit as ϵ goes to zero: it will be proved in the next theorem, which is the main original result of this section.

Theorem 12 (Representation formula and estimate for the derivatives of the potential)**.** *Assume all the hypotheses of Theorem 2, and let η, v^ϵ, and B_Λ be as above. Furthermore, let $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$ and let $F \in C_D(\Omega)$. For all $i, j = 1, \dots, n$ define*

$$\begin{aligned}
V_i^j(x) := &\int_{B_\Lambda} [F(y) - F(x)]\, \partial_{x_j} N_i(x,y)\, dy \\
&+ F(x) \int_{B_\Lambda} (x_i - y_i) \int_1^\infty \partial_j \psi(y + \alpha(x-y))\, \alpha^{n-1}\, d\alpha\, dy \\
&- F(x) \int_{\partial B_\Lambda} N_i(x,y)\, \nu_j(y)\, d\sigma_y.
\end{aligned}$$

Then, for all $i, j = 1, \dots, n$, we have:

i) The function $V_i^j(x)$ is well-defined for all $x \in \Omega$;

ii) As $\epsilon \to 0$, the partial derivative $\partial_{x_j} v_i^\epsilon$ converges uniformly to V_i^j on any $K \subset\subset \Omega$;

iii) It holds $\partial_{x_j} v_i(x) = V_i^j(x)$ for all $x \in \Omega$;

iv) The potential $v \in C^1(\Omega)$*;*

v) For any $K \subset\subset \Omega$ *there exists a constant c, depending only on* $n, \psi,$ diam Ω*, and* $d(K, \partial\Omega)$*, such that*

$$\|v\|_{C^1(K)} \leq c\|F\|_{C_D(\Omega)}.$$

Proof. To prove *i)*, fix any $x \in \Omega$. Remark that, after its extension by zero outside Ω, $F \in L^\infty(\mathbb{R}^n)$. Fix any ζ, with $\mathrm{dist}(x, \partial\,\Omega)/2 < \zeta < \mathrm{dist}(x, \partial\,\Omega)$. One has

$$\begin{aligned}
&\int_{B_\Lambda} |F(y) - F(x)| \left|\partial_{x_j} N_i(x,y)\right| \, dy \\
&= \int_{B(x,\zeta)} |F(y) - F(x)| \left|\partial_{x_j} N_i(x,y)\right| \, dy \\
&\quad + \int_{\{|x-y| \geq \zeta\} \cap B_\Lambda} |F(y) - F(x)| \left|\partial_{x_j} N_i(x,y)\right| \, dy \\
&=: I_4 + I_5.
\end{aligned}$$

Since $B(x, \zeta) \subset \Omega$, by Lemma 8 it follows that

$$\begin{aligned}
I_4 &\leq \int_{B(x,\zeta)} \frac{|F(y) - F(x)|}{|y - x|} |y - x| \, |\partial_{x_j} N_i(x,y)| \, dy \\
&\leq \int_{B(x,\zeta)} \frac{\omega(F, |y-x|)}{|y - x|} \, \frac{M}{|y - x|^{n-1}} \, dy,
\end{aligned}$$

where $\omega(F, \cdot)$ is the modulus of continuity of F in Ω. By introducing the radial and angular coordinates it follows that

$$I_4 \leq M\, |S^{n-1}| \int_0^\zeta \frac{\omega(F, \rho)}{\rho} \, d\rho,$$

and, therefore, it is bounded by $M\,|S^{n-1}|\, \|F\|_{C_D(\Omega)}$.

Furthermore, since both F and $\partial_{x_j} N_i(x,y)$ are bounded on $\{|x - y| \geq \zeta\}$, the term I_5 is bounded as well by $M'\,\|F\|_\infty$, where M' depends only on $n, \psi, \mathrm{diam}\ \Omega$, and $\mathrm{dist}(x, \partial\Omega)$.

Finally, since $\partial_j \psi \in C_0^\infty(\mathbb{R}^n)$ and $\mathrm{supp}\, \partial_j \psi \subset B$, it follows that

$$\int_\Omega (x_i - y_i) \int_1^\infty \partial_j \psi(y + \alpha(x - y))\, \alpha^{n-1} d\alpha \, dy,$$

is the Bogovskiĭ's potential (2) associated to $\partial_j \psi$ instead of ψ, and corresponding to the smooth and bounded function $F \equiv 1$. By Theorem 2 *iii)*, it is globally bounded by some constant M'', depending only on n, ψ, and $\mathrm{diam}\,\Omega$. Since $x \in \Omega$ and $|y| = 2\,\mathrm{diam}\,\Omega$, then $N_i(x,y)$ is bounded on ∂B_Λ and therefore the surface integral is finite as well. Thus *i)* follows.

Moreover, we remark explicitly that the previous computations imply immediately that, for any $x \in \Omega$,

$$|V_i^j(x)| \leq M|S^{n-1}|\, \|F\|_{C_D(\Omega)} + (M' + M'')\|F\|_\infty \leq c\|F\|_{C_D(\Omega)} \tag{4}$$

where c depends only on $n, \psi, \mathrm{diam}\ \Omega$ and $\mathrm{dist}(x, \partial\Omega)$.

To prove *ii)*, fix any $K \subset\subset \Omega$. Thus, for any $x \in K$ and $\epsilon > 0$ such that $\epsilon < \mathrm{dist}(K, \partial\Omega)/2$, it follows that

$$
\begin{aligned}
&|\partial_{x_j} v_i^\epsilon(x) - V_i^j(x)| \leq \\
&\leq \left| \int_{B_\Lambda} [F(y) - F(x)]\, \partial_{x_j} \left\{ N_i(x,y) \left[\eta\left(\frac{|x-y|}{\epsilon}\right) - 1 \right] \right\} dy \right| \\
&+ \left| F(x) \int_{B_\Lambda} (x_i - y_i) \int_1^\infty \partial_{x_j} \psi(y + \alpha(x-y)) \left[\eta\left(\frac{|x-y|}{\epsilon}\right) - 1 \right] \alpha^{n-1} d\alpha dy \right| \\
&\leq \int_{B(x,2\epsilon)} |F(x) - F(y)|\, |\partial_{x_j} N_i(x,y)|\, dy + \\
&+ \int_{B(x,2\epsilon)} |F(y) - F(x)|\, |N_i(x,y)| \left| \eta'\left(\frac{|x-y|}{\epsilon}\right) \frac{x_j - y_j}{|x-y|} \frac{1}{\epsilon} \right| dy \\
&+ \int_{B(x,2\epsilon)} |F(x)|\, |x_i - y_i| \int_1^\infty |\partial_{x_j} \psi(y + \alpha(x-y))\,| \alpha^{n-1} d\alpha\, dy \\
&=: I_6 + I_7 + I_8.
\end{aligned}
$$

As above, by Lemma 8 it follows that

$$
I_6 \leq M \int_{B(x,2\epsilon)} \frac{|F(x) - F(y)|}{|y-x|^n} dy \leq M\, |S^{n-1}| \int_{\rho < 2\epsilon} \frac{\omega(F,\rho)}{\rho} d\rho.
$$

By the Dini continuity of F and the consequent absolute continuity of the integral, the last term vanishes as ϵ goes to zero, independently of $x \in K$.

In order to estimate the second term I_7 remark that, by Theorem 2 *v)* and the hypothesis on η'

$$
\begin{aligned}
I_7 \leq & \int_{\epsilon \leq |x-y| \leq 2\epsilon} |F(x) - F(y)| \left| \eta'\left(\frac{|x-y|}{\epsilon}\right) \right| \frac{|x_j - y_j|}{|x-y|} \frac{1}{\epsilon} \times \\
& \times \frac{|x_i - y_i|}{|x-y|^n} \int_0^\infty \left| \psi\left(x + r\frac{x-y}{|x-y|}\right) \right| (|x-y| + r)^{n-1} dr\, dy \\
\leq & \, 4 \int_{\epsilon \leq |x-y| \leq 2\epsilon} |F(x) - F(y)| \frac{1}{|x-y|^n} \times \\
& \times \int_0^\infty \left| \psi\left(x + r\frac{x-y}{|x-y|}\right) \right| (|x-y| + r)^{n-1} dr\, dy.
\end{aligned}
$$

By introducing the variable $y := x + \rho\, u$, since

$$
\begin{aligned}
\int_0^\infty |\psi(x + ru)| (\rho + r)^{n-1}\, dr &= \int_0^{1+|x|} |\psi(x + ru)|\, (\rho + r)^{n-1}\, dr \\
&\leq \max_{\mathbb{R}^n} |\psi|\, (1 + \text{diam}\ \Omega + 2\epsilon)^{n-1},
\end{aligned}
$$

it follows as above that the last term is bounded by a multiple of $\int_\epsilon^{2\epsilon} \frac{\omega(F,\rho)}{\rho} d\rho$ and, again by the absolute continuity of the Lebesgue integral, the term I_7 vanishes as ϵ goes to zero, independently of $x \in K$.

Finally, by using $\partial_j \psi$ instead of ψ, as in the proof of the previous *i)* from Theorem 2 *iii)*, it follows that for any fixed $q > n$ and suitable constants c', c''

$$
\begin{aligned}
|I_8| &\leq c' \max_{\overline{\Omega}} |F(x)| \left\| \eta\left(\frac{|x-y|}{\epsilon}\right) - 1 \right\|_{L^q(\Omega)} \\
&\leq c'' \left\| \eta\left(\frac{|x-y|}{\epsilon}\right) - 1 \right\|_{L^q(B(x,\, \text{diam}\ \Omega))}.
\end{aligned}
$$

Since the last norm vanishes as ϵ goes to zero, for any $q > n$ and independently of $x \in \Omega$, *ii)* follows.
From *ii)*, by the classical theorem on a converging sequences of functions whose derivatives converge uniformly, it follows *iii)*, while *iv)* follows immediately from *ii)*, *iii)*, due to the fact that $v^\epsilon \in C^\infty(\mathbb{R}^n)$.

Finally, the estimate *v)* follows immediately from *iii)*, the above bound for V_i^j in (4), and the bound for the potential in Theorem 2 *iii)*. $\square$

By using a well-known argument based on translation and rescaling, as in Galdi [15, Lemma III.3.1] with $x \mapsto \frac{x-x_0}{R}$, the previous results lead to the following theorem.

Theorem 13 (Interior regularity for bounded star-shaped domains)**.** *Let $B(x_0, R)$ be an open ball in $\mathbb{R}^n, n \geq 2$, and let Ω be a bounded open subset of $\mathbb{R}^n$, star-shaped with respect to every point of $\overline{B(x_0, R)}$. Then, for any $F \in C_D(\Omega)$ verifying $\int_\Omega F(x)\,dx = 0$, there exists a solution $u \in C^1(\Omega) \cap C^0(\mathbb{R}^n)$ of the problem*

$$\begin{cases} \operatorname{div} u(x) = F(x) & \text{in } \Omega, \\ u \equiv 0 & \text{on } \complement\Omega, \end{cases}$$

verifying, for any $K \subset\subset \Omega$,

$$\|v\|_{C^1(K)} \leq c\|F\|_{C_D(\Omega)},$$

where the constant c depends only on $n, \psi, \operatorname{diam}\Omega, R,$ and $dist(K, \partial\Omega)$.

3 Classical solutions for the divergence problem in the interior of Lipschitz domains.

The aim of this brief section is to relax the very strong geometric restrictions on the domain Ω requested in the previous results, although at the price to renounce the simplicity of the solution in the form of a single Bogovskiĭ's potential.

The next lemma (which follows strictly the one proved in Galdi [15, Lemma III.3.4]) provides the localization apparatus we will use to prove the existence of a classical solution in a wider class of bounded domains including, for instance, those with a smooth boundary. We start with a suitable "partition of unity" lemma.

Lemma 14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\mathcal{G} = \{G_1, \ldots, G_{m+p}\}$ be an open covering of $\overline{\Omega}$. Assume that, if $\Omega_i := \Omega \cap G_i$, then:*

- *a)* $\partial\Omega \subset \cup_{i=1}^m G_i$;
- *b)* $\overline{G_i} \subset \Omega$, *for any* $i = m+1, \ldots, m+p$;
- *c)* $\Omega = \cup_{i=1}^{m+p} \Omega_i$.

Next, for each $i = 1, \ldots, m+p$, there exist $\zeta_i \in C_0^\infty(G_i)$, $m_i \in \mathbb{N}$ and, for $k = 1, \ldots, m_i$, $\theta_k \in C_0^\infty(\Omega_i)$ and $\phi_k \in C_0^\infty(\overline{\Omega})$ such that, if one sets

$$F_i(x) := \zeta_i(x)F(x) + \sum_{k=1}^{m_i} \theta_k(x) \int_\Omega \phi_k(y)F(y)\,dy,$$

for any $F \in C_D(\Omega)$ with $\int_\Omega F(x)\,dx = 0$, then *i)* $F_i \equiv 0$ *in* $\overline{\Omega}\backslash\overline{\Omega_i}$*, for all* $i = 1, \dots, m+p$*;*

ii) $\|F_i\|_{C_D(\Omega_i)} \leq c\,\|F\|_{C_D(\Omega)}$*, where* c *is a constant depending only on* Ω*;*

iii) $\int_\Omega F_i(x)\,dx = 0$*, for all* $i = 1, \dots, m+p$*;*

iv) $F(x) = \sum_{i=1}^{m+p} F_i(x)$*, for all* $x \in \overline{\Omega}$*.*

Proof. The proof of this result may be obtained by following that in [15, Lemma III.3.4], simply by replacing C_0^∞ with C_D in any occurrence involving f, f_i or g_i (by assuming Ω as their domain of definition) and by extending ψ_i and χ_i by zero outside their supports. □

Remark 15. *In order to apply the regularity result in Theorem 13 to the divergence problem "localized" in* Ω_i*, we remark explicitly that:*

a) The compatibility condition for the "localized" datum F_i*, i.e.*

$$\int_{\Omega_i} F_i(x)\,dx = 0,$$

follows immediately from i) and iii) of Lemma 14.

b) If $\partial\Omega \in C^{0,1}$ *then, by following the proof of [15, Lemma III.3.4], it can be shown that any* Ω_i*,* $i = 1, \dots, m+p$*, in the previous lemma may be chosen as star-shaped with respect to any point of some closed ball contained in it.*

c) If $\partial\Omega \in C^2$*, then the open sets* Ω_i*,* $i = 1, \dots, m$*, may be chosen as the sets* Ω_{P_i} *in Section 2.1, while the remaining ones are open balls.*

The next result, which extends Theorem 13 to a considerably wider class of domains, will now be obtained by a localization argument.

Theorem 16 (Interior regularity for Lipschitz domains)**.** *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$*, with* $\partial\Omega \in C^{0,1}$*. Then, for any* $F \in C_D(\Omega)$ *with* $\int_\Omega F(x)\,dx = 0$*, there exists a solution* $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ *of the problem*

$$\begin{cases} \operatorname{div} u(x) = F(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

verifying for any $K \subset\subset \Omega$*,*

$$\|v\|_{C^1(K)} \leq c\|F\|_{C_D(\Omega)},$$

where the constant c *depends only on* n, ψ, Ω *and* dist$(K, \partial\Omega)$*.*

Proof. Let Ω_i and F_i be defined as in Lemma 14 and Remark 15 b), and let u_i be the solution in $C^1(\Omega_i) \cap C^0(\mathbb{R}^n)$ of the problem

$$\begin{cases} \operatorname{div} u_i(x) = F_i(x) & \text{in } \Omega_i, \\ u_i \equiv 0 & \text{on } \complement\Omega_i, \end{cases}$$

whose existence is ensured by Theorem 13, Lemma 14, and Remark 15 a)-b). Thus, by setting

$$u(x) = \sum_{i=1}^{m+p} u_i(x),$$

one obtains $u \in C^1(\Omega) \cap C^0(\mathbb{R}^n)$. Moreover, $\operatorname{div} u(x) = F(x)$ and, since u_i vanishes on $\complement\Omega_i$, then $u = 0$ on $\partial\Omega$. The norm estimate follows immediately from the one in Theorem 13, applied to each u_i. □

Remark 17. *The relevant point is the fact that Ω can be decomposed as a finite union of Ω_i, each one star-shaped with respect to some closed ball contained in it. The Lipschitz regularity of $\partial\Omega$ is only a sufficient condition for it, without any direct relationship with the interior regularity.*

Remark 18. *The constant in the norm estimate from Theorem 16 does not depend only on* $\operatorname{diam}\Omega$*, but also on its geometry. In fact, both the number of the star-shaped subsets of its decomposition and the radii of the closed balls with respect to whose points they are star-shaped are to be taken into account in the expression of the constant.*

4 Classical solutions regular up to the boundary

In this section we prove the main result of the paper, that is the existence of a solution of the divergence equation regular up to the boundary and vanishing on it, as previously outlined in Theorem 1. Our approach follows, as far as possible, the classical one for the Poisson equation, which is essentially based on the following steps: first, a suitable localization, together with a change of variables to reduce the domain of the problem to a special one, namely a half-ball or a half-cube, while the portion of boundary under exam is mapped onto a subset of the hyperplane $\{x \in \mathbb{R}^n : \ x_n = 0\}$; next, a separate treatment of the "tangential" derivatives, as opposed to the "normal" ones in the x_n-direction; finally, a suitable "reflection" of the solution across the hyperplane $\{x \in \mathbb{R}^n : \ x_n = 0\}$, in order to put the portion of the boundary of interest in the interior and then to be able to take advantage of the already proved result of regularity at the interior. A favourable feature of the Laplace operator is that the regularity of any selected second order derivative (namely the "normal" one) may be deduced from those of the other derivatives and of the datum, simply by using pointwise the equation. On the contrary the particular structure of the divergence operator, which is non-elliptic, allows to employ a similar argument only for the "normal" derivative of the "normal" component u_n of the unknown vector field, requiring an *ad hoc* treatment for all the other partial derivatives in the normal direction.

In the former of the following sections it will be established a result of regularity at the boundary for a half-cube; in the latter, the localization in Lemma 14 and Remark 15, and a standard "flattening" change of variables are exploited to extend the previous result to the general domain with a C^2-boundary.

4.1 The case of a half-cube

Let us define, for $a > 0$, the cube $Q_a := (-a, a)^n$ and the upper and lower half-cubes $Q_a^+ := (-a, a)^{n-1} \times (0, a)$ and $Q_a^- := (-a, a)^{n-1} \times (-a, 0)$, respectively. Furthermore, we set $x' := (x_1, \dots, x_{n-1})$.

The object of this section is to prove the following theorem.

Theorem 19. *For any $F \in C_D(Q_a^+)$ verifying $\int_{Q_a^+} F(x)\,dx = 0$, and such that* $\operatorname{supp} F \subset \overline{Q}^+_{a/2}$*, there exists a solution $u \in C^1(\overline{Q}_a^+) \cap C^0(\mathbb{R}^n)$ of the problem*

$$\begin{cases} \operatorname{div} u(x) = F(x) & \text{in } Q_a^+, \\ u \equiv 0 & \text{on } \complement Q_a^+, \end{cases}$$

verifying

$$\|u\|_{C^1(\overline{Q}_a^+)} \le c\|F\|_{C_D(Q_a^+)},$$

where the constant c depends only on n, ψ, a.

The proof will be postponed until the end of the section, and requires several considerations and lemmas. We immediately observe that, by (possibly) extending F by zero outside its support and rescaling the variables as in Theorem 13, we can reduce ourselves to consider only the case where $a = 1$, and $\operatorname{supp} F \subseteq \overline{Q}^+_{1/2}$.

To this aim, let us define

$$F^*(x) := \begin{cases} F(x) & \text{in } \overline{Q}^+_1, \\ F(x^*) & \text{in } \overline{Q}^-_1, \end{cases}$$

where, as usual, the starred variable $x^* := (x', x_n)^* = (x', -x_n)$ denotes that one obtained by reflection across the $\{x_n = 0\}$ hyperplane. We observe that

$$\int_{Q_1^-} F^*(x)\,dx = \int_{Q_1^+} F(x)\,dx,$$

and, obviously, if $F \in C_D(Q_1^+)$ then $F^* \in C_D(Q_1)$, with the same norm.

By the previous Theorem 13, there exists $W \in C^0(\overline{Q}_1) \cap C^1(Q_1)$, such that

$$\begin{cases} \operatorname{div} W = F^* & \text{in } Q_1, \\ W = 0 & \text{on } \partial Q_1. \end{cases}$$

Thus, we define a vector field $w : Q_1 \to \mathbb{R}^n$ by setting

$$\begin{aligned} w(x) &:= \big(w_1(x), \dots, w_{n-1}(x), w_n(x)\big) \\ &= \frac{1}{2}\big(W_1(x) + W_1(x^*), \dots, W_{n-1}(x) + W_{n-1}(x^*), W_n(x) - W_n(x^*)\big), \end{aligned} \tag{5}$$

and we observe that it satisfies

$$\begin{cases} \operatorname{div} w = F^* & \text{in } Q_1, \\ w = 0 & \text{on } \partial Q_1, \\ w_n = 0 & \text{on } \big((-1,1)^{n-1} \times \{0\}\big), \end{cases}$$

and hence, by restriction to the upper cube Q_1^+,

$$\begin{cases} \operatorname{div} w = F & \text{in } Q_1^+, \\ w_\alpha = 0 & \text{on } \partial Q_1^+ \backslash \big((-1,1)^{n-1} \times \{0\}\big), \text{ for } \alpha = 1, \dots, n-1, \\ w_n = 0 & \text{on } \partial Q_1^+. \end{cases}$$

Therefore, in order to show that there exists a $C^1(\overline{Q}^+_1)$ solution of (1), it is enough to subtract from w any divergence-free vector field $\phi \in C^1(\overline{Q}^+_1)$ such that $\phi_{|\partial Q_1^+} = w_{|\partial Q_1^+}$. In particular, it is enough to find a function ϕ vanishing on $\partial Q_1^+ \backslash \big((-1,1)^{n-1} \times \{0\}\big)$ and verifying, for $\alpha = 1, \dots, n-1$,

$$\phi_\alpha(x', 0) = w_\alpha(x, 0) \in C^1((-1,1)^{n-1}) \qquad \text{and} \qquad \phi_n(x', 0) = 0.$$

Observe that, since $\operatorname{supp} w_\alpha(x', 0) \subset\subset (-1,1)^{n-1}$, then $w_\alpha(x', 0)$ can be extended by zero and considered as it belongs to $C^1_0(\mathbb{R}^{n-1})$.

One may be tempted to set

$$\widetilde{\Phi}_\alpha(x) := x_n(1-x_n)^2 w_\alpha(x',0) \qquad \text{for } \alpha = 1,\dots,n-1,$$

$$\widetilde{\phi}(x) := \Big(\frac{\partial\widetilde{\Phi}_1}{\partial x_n},\dots,\frac{\partial\widetilde{\Phi}_{n-1}}{\partial x_n},-\sum_{\beta=1}^{n-1}\frac{\partial\widetilde{\Phi}_\beta}{\partial x_\beta}\Big).$$

Observe that $\widetilde{\Phi}_\alpha$ vanishes on ∂Q_1^+, for any $\alpha = 1,\dots,n-1$. A direct computation shows that $\widetilde{\phi}$ assumes the requested value on ∂Q_1^+. Moreover, by the already proved inner regularity of w in Q_1, it follows that $\partial_{x_\alpha}\partial_{x_n}\widetilde{\Phi}_\alpha$ is continuous and thus, by the Schwarz theorem on mixed derivatives, $\widetilde{\phi}$ turns out to be divergence-free. Nevertheless, it is not evident as to whether $\widetilde{\phi} \in C^1(\overline{Q_1^+})$, because the function $\widetilde{\Phi}_\alpha(x)$ can be differentiated, in principle, only one time with respect to x'. To overcome this difficulty, we regularized each component $\widetilde{\Phi}_\alpha$, for $\alpha = 1,\dots,n-1$, in such a way that the trace of its normal derivative on $\{x_n = 0\}$ is equal to $w_\alpha(x',0)$, by adapting some classical mollification tools as those used in Nečas [23] for the extension of traces; see also [9, 10] for related results about Sobolev spaces.

As above, by applying the rescaling already used in Theorem 13 after a possible extension of $w_\alpha(x',0)$ by zero to the whole subspace $\{x_n = 0\}$, we can reduce ourselves to the case where supp $w_\alpha(x',0) \subseteq (-1/2,1/2)^{n-1}$.

Definition 20 (A compactly supported regular extension to the upper half-plane)**.** *Let $\rho \in C_0^\infty(\mathbb{R}^{n-1})$ be such that $\int_{\mathbb{R}^{n-1}}\rho(x')\,dx' = 1$ and $\operatorname{supp}\rho \subset B(0,1)$. Moreover, let $\theta \in C^2(\mathbb{R})$ be such that $\theta(0) = 1$ and $\theta(t) = 0$ for $t \geq \frac{1}{2}$. We define $\Phi_\alpha : \mathbb{R}^{n-1}\times[0,+\infty) \to \mathbb{R}$ by setting*

$$\Phi_\alpha(x',x_n) := \begin{cases} \dfrac{\theta(x_n)}{x_n^{n-2}}\displaystyle\int_{\mathbb{R}^{n-1}} w_\alpha(y',0)\,\rho\Big(\frac{x'-y'}{x_n}\Big)\,dy' & \text{for } x_n > 0,\\[2ex] 0 & \text{for } x_n = 0.\end{cases}$$

We observe that

$$\Phi_\alpha(x) = x_n\,\theta(x_n)\,(w_\alpha(\cdot,0) * \rho_{x_n}(\cdot))(x') \qquad \text{for } x_n > 0,$$

where the symbol "$*$" denotes the convolution operator in $\mathbb{R}^{n-1}$ and, for any function g defined on $\mathbb{R}^{n-1}$ and any $\epsilon > 0$, we use the standard notation $g_\epsilon(x') := \frac{1}{\epsilon^{n-1}}g\big(\frac{x'}{\epsilon}\big)$. We recall the following elementary result on convolutions, which we will use extensively.

Lemma 21. *Let $f \in C_0^0(\mathbb{R}^{n-1})$ and $g \in C_0^\infty(\mathbb{R}^{n-1})$. Then:*

$$\begin{aligned}
&\operatorname{supp}(f*g) \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g},\\
&\lim_{\epsilon\to 0^+}(f*g_\epsilon)(x') = f(x')\int_{\mathbb{R}^{n-1}} g(y')\,dy' \qquad \textit{uniformly on } \mathbb{R}^{n-1},\\
&|(f*g_\epsilon)(x')| \leq c\,\|f\|_\infty \quad \textit{and} \quad |\lim_{\epsilon\to 0^+}(f*g_\epsilon)(x')| \leq c\,\|f\|_\infty \quad \forall\, x' \in \mathbb{R}^{n-1},
\end{aligned}$$

where $c = \|g\|_1$.

The next lemma provides the required regular extension of the boundary data and a bound for its norm.

Lemma 22. *For $\alpha = 1, \dots, n-1$ it holds $\Phi_\alpha \in C^2(\mathbb{R}^{n-1} \times [0, +\infty))$, supp $\Phi_\alpha \subset \overline{Q_1^+}$, and the vector field*

$$\phi := \Big(\frac{\partial \Phi_1}{\partial x_n}, \dots, \frac{\partial \Phi_{n-1}}{\partial x_n}, -\sum_{\beta=1}^{n-1} \frac{\partial \Phi_\beta}{\partial x_\beta}\Big),$$

verifies

$$\phi \in C^1(\overline{Q_1^+}), \tag{6}$$
$$\operatorname{div} \phi = 0 \ \ in \ Q_1^+, \tag{7}$$
$$\phi_\alpha(x', 0) = w_\alpha(x', 0) \qquad for \ \alpha = 1, \dots, n-1, \tag{8}$$
$$\phi_n(x', 0) = 0, \tag{9}$$
$$\|\phi\|_{C^1\left(\overline{Q_1^+}\right)} \leq c\, \|w\|_{C^1(Q_1^+ \cap \{x_n = 0\})}, \tag{10}$$

where w is defined as in Theorem 19, and c is a constant depending only on n, θ, and ρ.

Proof. We observe that, by the standard properties of the mollifiers, it follows that $\Phi_\alpha \in C^\infty(\mathbb{R}^{n-1} \times (0, +\infty))$, but the relevant fact is how it behaves as $x_n \to 0^+$.

From Lemma 21 it follows immediately that $\Phi_\alpha \in C^0(\mathbb{R}^{n-1} \times [0, +\infty))$, and that supp $\Phi_\alpha \subset \overline{Q_1^+}$, while the fact that div ϕ vanishes in the interior follows (formally) from direct computation, but it will be justified only after it will be proved that $\Phi_\alpha \in C^2(Q_1^+)$, $\quad \alpha = 1, \dots, n-1$.

First, we consider the tangential derivatives and we have, for $x_n > 0$,

$$\begin{aligned} \partial_{x_\beta} \Phi_\alpha(x', x_n) &= \frac{\theta(x_n)}{x_n^{n-1}} \int_{\mathbb{R}^{n-1}} w_\alpha(y', 0)\, \partial_{x_\beta} \rho\Big(\frac{x' - y'}{x_n}\Big)\, dy' \\ &= \theta(x_n)\, (w_\alpha(\cdot, 0)\, * \eta^\beta_{x_n}(\cdot))(x'), \end{aligned} \tag{11}$$

with $\eta^\beta(x') := \partial_{x_\beta} \rho(x')$. Since $\int_{\mathbb{R}^{n-1}} \eta^\beta(x')\, dx' = 0$, by Lemma 21, it follows that

$$\lim_{x_n \to 0^+} \partial_{x_\beta} \Phi_\alpha(x', x_n) = 0 = \partial_{x_\beta} \Phi_\alpha(x', 0),$$

and therefore $\partial_{x_\beta} \Phi_\alpha(x', x_n) \in C^0(\mathbb{R}^{n-1} \times [0, +\infty))$, for $\alpha, \beta = 1, \dots, n-1$, and moreover (9) is satisfied.

For $x_n > 0$, we have by direct computation that

$$\begin{aligned} \partial_{x_n} \Phi_\alpha(x', x_n) = &\frac{\theta'(x_n)}{x_n^{n-2}} \int_{\mathbb{R}^{n-1}} w_\alpha(y', 0)\, \rho\Big(\frac{x' - y'}{x_n}\Big)\, dy' \\ &+ (2 - n) \frac{\theta(x_n)}{x_n^{n-1}} \int_{\mathbb{R}^{n-1}} w_\alpha(y', 0)\ \rho\Big(\frac{x' - y'}{x_n}\Big)\, dy' \\ &- \frac{\theta(x_n)}{x_n^{n-1}} \int_{\mathbb{R}^{n-1}} w_\alpha(y', 0) \sum_{\beta=1}^{n-1} \Big(\frac{x_\beta - y_\beta}{x_n}\Big)\, \partial_{x_\beta} \rho\Big(\frac{x' - y'}{x_n}\Big)\ dy'. \end{aligned}$$

By defining

$$\Psi(x') := \sum_{\beta=1}^{n-1} x_\beta \partial_{x_\beta} \rho(x') = \sum_{\beta=1}^{n-1} x_\beta\, \eta^\beta(x'),$$

we can rewrite the normal derivative as follows

$$\begin{aligned} \partial_{x_n} \Phi_\alpha(x', x_n) = &\Big[x_n \theta'(x_n) + (2 - n)\theta(x_n)\Big]\, (w_\alpha(\cdot, 0)\, * \rho_{x_n}(\cdot))(x') \\ &- \theta(x_n)\, (w_\alpha(\cdot, 0)\, * \Psi_{x_n}(\cdot))(x'). \end{aligned} \tag{12}$$

Since, integrating by parts, one gets

$$\int_{\mathbb{R}^{n-1}} \Psi(x')\,dx' = -(n-1)\int_{\mathbb{R}^{n-1}} \rho(x')\,dx' = -(n-1),$$

we obtain by Lemma 21 that

$$\lim_{x_n\to 0^+} \partial_{x_n}\Phi_\alpha(x',x_n) = \Big[(2-n)\theta(0)-(1-n)\theta(0)\Big]w_\alpha(x',0) = w_\alpha(x',0),$$

and then $\Phi \in C^1(\mathbb{R}^{n-1}\times[0,+\infty))$ and (8) holds true.

Next, we need to prove the continuity of the second order derivatives of Φ to show (6). First, we consider the second order derivatives different from $\frac{\partial^2}{\partial x_n^2}$ and, for $x_n > 0$, by using the commutativity of the convolution operator, we have for all $\alpha, \beta, \gamma = 1, \dots, n-1$

$$\begin{aligned}\partial_{x_\gamma}\partial_{x_\beta}\Phi_\alpha(x',x_n) &= \frac{\theta(x_n)}{x_n^{n-1}}\int_{\mathbb{R}^{n-1}} \partial_{x_\gamma} w_\alpha(x'-y',0)\,\partial_{x_\beta}\rho\Big(\frac{y'}{x_n}\Big)\,dy' \\ &= \theta(x_n)\,(\partial_{x_\gamma}w_\alpha(\cdot,0) \,*\, \eta^\beta_{x_n}(\cdot))(x'),\end{aligned} \tag{13}$$

$$\begin{aligned}&\partial_{x_\gamma}\partial_{x_n}\Phi_\alpha(x',x_n) \\ &\quad= \frac{x_n\theta'(x_n)+(2-n)\theta(x_n)}{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_\gamma} w_\alpha(x'-y',0)\rho\Big(\frac{y'}{x_n}\Big)\,dy' \\ &\qquad - \frac{\theta(x_n)}{x_n^{n-1}}\int_{\mathbb{R}^{n-1}} \partial_{x_\gamma} w_\alpha(x'-y',0)\,\Psi\Big(\frac{y'}{x_n}\Big)\,dy' \\ &\quad= \Big[x_n\theta'(x_n)+(2-n)\theta(x_n)\Big]\,(\partial_{x_\gamma}w_\alpha(\cdot,0)\,*\,\rho_{x_n}(\cdot))(x') \\ &\qquad - \theta(x_n)\,(\partial_{x_\gamma}w_\alpha(\cdot,0)\,*\,\Psi_{x_n}(\cdot))(x'),\end{aligned} \tag{14}$$

and they are all continuous up to $\{x_n = 0\}$, by the C^1-regularity of $w_\alpha(x',0)$. The analogous result about $\partial_{x_n}\partial_{x_\gamma}\Phi_\alpha(x',x_n)$ follows immediately from the Schwarz theorem on mixed derivatives.

Next, by introducing the change of variables $y' = x_n z'$, we get

$$(w_\alpha(\cdot,0) * \Psi_{x_n}(\cdot))(x') = \int_{\mathbb{R}^{n-1}} w_\alpha(x'-x_n\,z',0)\,\Psi(z')\,dz',$$

and

$$\begin{aligned}&\partial_{x_n}\partial_{x_n}\Phi_\alpha(x',x_n) \\ &\quad= \Big[\theta'(x_n)+x_n\theta''(x_n)+(2-n)\theta'(x_n)\Big]\,(w_\alpha(\cdot,0)\,*\,\rho_{x_n}(\cdot))(x') \\ &\qquad - \theta'(x_n)\,(w_\alpha(\cdot,0)\,*\,\Psi_{x_n}(\cdot))(x') \\ &\qquad + \Big[x_n\theta'(x_n)+(2-n)\theta(x_n)\Big]\,(w_\alpha(\cdot,0)\,*\,\Psi_{x_n}(\cdot))(x') \\ &\qquad + \theta(x_n)\int_{\mathbb{R}^{n-1}}\sum_{\beta=1}^{n-1} z_\beta\,\partial_{x_\beta}w_\alpha(x'-x_n z',0)\,\Psi_{x_n}(z')\,dz'.\end{aligned} \tag{15}$$

Therefore $\Phi_\alpha(x',x_n) \in C^2(\mathbb{R}^{n-1}\times[0,+\infty))$ for $\alpha = 1,\dots,n-1$, and thus (6) follows. By the Schwarz theorem, also (7) follows.
Finally, the estimate (10) follows directly by applying the bounds from Lemma 21 to the expressions for the first and second order derivatives of Φ in (11), (12), (13), (14), and (15). □

Now, we can get the result of regularity at the boundary for a half-cube as outlined at the beginning of this section.

Proof of Theorem 19. By setting

$$u(x) = w(x) - \phi(x),$$

where w is defined as in (5) and ϕ as in Lemma 22, it follows that u is the aimed classical (of class C^1) solution for the divergence equation vanishing at the boundary and regular up to it. The estimate follows immediately from those of Theorem 13 and Lemma 22. □

We observe that the solution provided by the last theorem is not simply a possibly scaled Bogovskiĭ's potential for some B and ψ: in fact, it is the difference between a superposition of such a potential and its "reflected" one, as in a sort of image-charge method in Electrostatics, and a suitable regular extension of the boundary datum.

4.2 The proof of the main theorem in the general case

In order to exploit the results in the previous sections and to prove the main theorem, we first introduce the hypothesis $\partial\Omega \in C^2$ and apply Lemma 14 and Remark 15 a) and c) to get an open covering of Ω whose members are either balls contained in Ω or sets $\Omega_P = \Omega_{P_i}$, defined in Section 2.1, verifying $\cup_{i=1}^m \overline{\Omega}_{P_i} \supseteq \partial\Omega$. Hence, to build up a global solution from the "localized" ones (in the same way we have obtained it in Section 3), it will be sufficient to apply to each of these sets Ω_P a well-known regular change of variables, boundary- and divergence-preserving. It transforms each Ω_P into a half-cube where, by Theorem 19, we have a regular solution, and then we transform it back to obtain the aimed "localized" regular solution. The argument is well-known but some details in the classical setting are worth to be emphasized, especially to deduce the requested regularity on the boundary of the domain.

To this purpose, fix $\Xi := A_P(\Omega_P - P)$ as in Section 2.1. By the same divergence-preserving rescaling used in Theorem 13 we may assume $R_P = 1$. Now, let us define the smooth transformation $T : \Xi \to Q_1^+$ by setting

$$y = T(x) := \big(x', x_n - h(x')\big),$$

where we have set $h := h_P$. The map T is invertible, its inverse being

$$x = T^{-1}(y) = \big(y', y_n + h(y')\big).$$

Observe that both T, T^{-1} have Jacobian determinant equal to 1 and therefore, for any given $f : \Xi \to \mathbb{R}$, if one defines $\widetilde{f} : \ Q_1^+ \to \mathbb{R}$ by setting

$$\widetilde{f}(y) := f(T^{-1}(y)),$$

it follows immediately that

$$\int_{Q_1^+} \widetilde{f}(y)\, dy = \int_{\Xi} f(x)\, dx.$$

Thus, given $F \in C_D(\Xi)$ such that $\int_{\Xi} F(x)\, dx = 0$, it follows immediately

$$\int_{Q_1^+} \widetilde{F}(y)\, dy = 0,$$

while, to show that $\widetilde{F} \in C_D(Q_1^+)$, it would be enough to prove that the mapping $T^{-1} : Q_1^+ \to \Xi$ is α-Hölder continuous for some $\alpha > 0$ (see, for instance, [2, Lemma 4.1]). Since we are actually assuming that $\partial\Omega \in C^2$, we have immediately that $|\nabla h(x)|$ is bounded and therefore

$$\omega(\widetilde{F}, \rho) \le \omega(F, \|\nabla h\|_\infty \, \rho),$$

which proves that $\widetilde{F} \in C_D(Q_1^+)$.

Hence, it is possible to apply Theorem 19, which provides a solution $\widetilde{u} \in C^1(\overline{Q_1^+}) \cap C^0(\mathbb{R}^n)$ of the problem

$$\begin{cases} \operatorname{div}_y \widetilde{u}(y) = \widetilde{F}(y) & \text{in } Q_1^+, \\ \widetilde{u} \equiv 0 & \text{on } \complement Q_1^+, \end{cases}$$

where we denote by div_y the divergence operator with respect to the variables $y = (y', y_n)$. Observe also that $\operatorname{supp} \widetilde{u} \subset \overline{Q_1^+}$.

Now, we need to transform back the vector field $\widetilde{u}$ to find a solution in Ξ. Vector fields $u : \Xi \to \mathbb{R}^n$ are transformed in the "covariant" way into $\widetilde{u} : \ Q_1^+ \to \mathbb{R}^n$ (where $x = T^{-1}(y)$) as follows:

$$\begin{cases} \widetilde{u}_\alpha(y) := u_\alpha(x) & \text{for } \alpha = 1, \ldots, n-1, \\ \widetilde{u}_n(y) := u_n(x) - \displaystyle\sum_{\beta=1}^{n-1} u_\beta(x) \, \partial_{x_\beta} h(x'). \end{cases}$$

Analogously, the inverse transformation (where $y = T(x)$) is given by

$$\begin{cases} u_\alpha(x) = \widetilde{u}_\alpha(y) & \text{for } \alpha = 1, \ldots, n-1, \\ u_n(x) = \widetilde{u}_n(y) + \displaystyle\sum_{\beta=1}^{n-1} \widetilde{u}_\beta(y) \, \partial_{x_\beta} h(y'). \end{cases} \tag{16}$$

We observe that

$$\begin{aligned} \operatorname{div}_x u(x) &= \sum_{j=1}^{n} \partial_{x_j} u_j(x) \\ &= \sum_{\alpha=1}^{n-1} \partial_{x_\alpha} \widetilde{u}_\alpha(T(x)) + \partial_{x_n} \Big(\widetilde{u}_n(T(x)) + \sum_{\beta=1}^{n-1} \widetilde{u}_\beta(T(x)) \, \partial_{x_\beta} h(x') \Big) \\ &= \sum_{\alpha=1}^{n-1} \partial_\alpha \widetilde{u}_\alpha(y) - \sum_{\alpha=1}^{n-1} \partial_\alpha h(y') \, \partial_n \widetilde{u}_\alpha(y) \\ &\qquad + \partial_n \widetilde{u}_n(y) + \sum_{\beta=1}^{n-1} \partial_\beta h(y') \, \partial_n \widetilde{u}_\beta(y) = \sum_{j=1}^{n} \partial_{y_j} \widetilde{u}_j(y) \\ &= \operatorname{div}_y \widetilde{u}(y), \end{aligned}$$

where $h(y') = h(x')$, since the transformation T is the identity on the first $n-1$ variables.

Furthermore, the "lower boundary" of Ξ, that is the set

$$\Big\{ x = (x', x_n) \in \mathbb{R}^n : \ x' \in (-1,1)^{n-1} \text{ and } x_n = h(x') \Big\},$$

is mapped on $(-1,1)^{n-1} \times \{0\}$, the lower face of the cube Q_1^+, and since

$$\widetilde{u}(y', 0) = 0 \quad \text{implies} \quad u(x', x_n - h(x')) = 0,$$

then the vector u satisfies $u_{|\partial\Omega} = 0$.

Thus, being u defined by (16), in order to get the regularity up to the boundary of u it will be sufficient to recall that $h \in C^2\big((-1,1)^{n-1}\big)$.

Now, in order to pass from the local coordinates in each set Ω_{P_i}, $i = 1, \dots, m$, to the global ones in the original domain Ω, one can apply to each of them the inverse of the rotation and the translation introduced in Section 2.1, which preserve regularity, divergence, and boundary values. Finally, we observe that, by (16), the C^1-norm of u on $\overline{\Omega}_{P_i}$ is bounded by the C^1-norm of $\widetilde{u}$ on $[-R_{P_i}, R_{P_i}]^{n-1}$, multiplied by a constant depending on the C^2-norm of h_{P_i}. Since, by (5), the C^1-norm of $\widetilde{u}$ is bounded in turn by the C_D-norm of F_i it follows that

$$\|u\|_{C^1(\overline{\Omega}_{P_i})} \leq c\,\|F_i\|_{C_D(\Omega)},$$

and Theorem 1 follows by localization.

Remark 23. *The estimate in Theorem 1 is a consequence of those in all the previous regularity results. It follows that the constant appearing there depends on the dimension n, the diameter and the geometry of Ω and its boundary, and the functions ψ used to define the local Bogovskiĭ's potentials occurring in the construction of the provided global solution.*

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