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On the analysis of a geometrically selective turbulence model

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Abstract: In this paper we propose some new non-uniformly-elliptic/damping regularizations of the Navier-Stokes equations, with particular emphasis on the behavior of the vorticity. We consider regularized systems which are inspired by the Baldwin-Lomax and by the selective Smagorinsky model based on vorticity angles, and which can be interpreted as Large Scale methods for turbulent flows. We consider damping terms which are active at the level of the vorticity. We prove the main *a priori* estimates and compactness results which are needed to show existence of weak and/or strong solutions, both in velocity/pressure and velocity/vorticity formulation for various systems. We start with variants of the known ones, going later on to analyze the new proposed models.

Keywords: Navier-Stokes equations, vorticity, turbulence, Large Eddy Simulation models

MSC: Primary 35Q30; Secondary 76F65,76D03

1 Introduction

In this paper we consider families of Large Eddy Simulation models which are variants of the classical Smagorinsky model [1]. We follow an approach similar to the modeling done by Cottet, Jiroveanu, and Michaux [2], proposing a selective model based on the local behavior of the angle of the vorticity direction, at neighbouring points. We recall that “regularizations” of the Navier-Stokes equations

$$\begin{aligned}\partial_t v + (v \cdot \nabla) v - \nu \Delta v + \nabla q &= f, \\ \operatorname{div} v &= 0,\end{aligned}\tag{1.1}$$

with perturbation obtained by a monotone nonlinear operator A , produce widely studied models as the one below

$$\begin{aligned}\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla \pi + A(u) &= f, \\ \operatorname{div} u &= 0.\end{aligned}\tag{1.2}$$

We observe that we denoted by v the unknown velocity field for a Newtonian fluid described by the Navier-Stokes equations (1.1), while by u we denote the field which is solution of (1.2). We recall that in the turbulence modeling the latter system is generally associated with the modeling or description of the large scales (eddies) of the solution. With the usual terminology if v is split into the mean flow \bar{v} and the (turbulent) fluctuations $v' = v - \bar{v}$, then u is a computable approximation/model for \bar{v} .

From the point of view of the mathematical theory, early studies of (1.2) date back to O.A. Ladyzhenskaya [3] and J.L. Lions [4] in the spirit of looking for modifications of the Navier-Stokes equations which

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allow to prove theorems of global existence and uniqueness for regular enough distributional solutions. In particular, a wide class of models is associated with the following nonlinear monotone operator

$$A(u) = -\operatorname{div}(v_0 + v_1|Du|)^{p-2}Du \quad p > 2, v_0 = 0, \text{ and } v_1 > 0,$$

where $Du = 2^{-1}(\nabla u + \nabla u^T)$ is the deformation tensor, that is the symmetric part of the gradient matrix. (In fact J.-L. Lions considered a model involving the full gradient instead of Du , and this turns out to have better mathematical properties, but produces equations which are not invariant by change of reference system).

The use of this models with $p = 3$ was already implemented by Smagorinsky [1] in some geophysical model; the use of stabilization obtained by adding this term dates back to von Neumann and Richtmyer [5] in the analysis of shocks in compressible flows.

It turns out that models as those obtained by (1.2) play a fundamental role in the modeling of turbulent flows for $p > 2$ and also in the study of families of non-Newtonian fluids, when $p < 2$. Especially when $p > 2$ the model is associated to an *eddy viscosity* assumption, and $v_t = v_1|Du|^{p-2}$ is the so-called turbulent viscosity, see for instance the discussion in [6, 7]. On the other hand, completely different motivations lead to the study of the model associated with smaller values of p , see for instance [8]. The mathematical theory for the above model, depending also on the values of the constants v_0, v_1, p is particularly complex and wide, since the equations can be degenerate or singular and special features appear, depending on the regime of the various parameters.

Here, we consider a variant of the Smagorinsky model, which can be derived in the wake of the classical Baldwin-Lomax model [9]. This is a one-equation-model, which fits in the class of eddy viscosity models with eddy viscosity $v_t = v_t(\bar{u})$ and which can be used to derive a whole family of simplified rotational models, see Rong, Layton, and Zhao [10]. Several models have been proposed and the analysis of the so-called “*relaxed rotational backscatter*” and “*relaxed rotational backscatter Baldwin Lomax*” models will be done in a forthcoming paper [11], together with the comparison in the derivation of the different systems.

By using the Kolmogorov-Prandtl relation for the turbulent viscosity v_t

$$v_t = c \ell \sqrt{k'},$$

(here c is a constant, ℓ the mixing length, and k' the kinetic energy associated to fluctuations u') one can derive the following model, where $\omega := \operatorname{curl} u$

$$\partial_t u + \beta^2 \operatorname{curl}(\ell^2(x)\partial_t \omega) + (u \cdot \nabla)u - \nu \Delta u + \nabla \pi + C_{BL} \operatorname{curl}(\ell^2(x)|\omega|\omega) = f. \tag{1.3}$$

This is called the “*backscatter Baldwin-Lomax*” model with β and C_{BL} non-negative parameters, while $\ell(x)$ is a smooth function which can be related to the mixing length. Apart the classical sources, a modern discussion with the derivation of the Baldwin-Lomax model can be found in [10–12].

An interesting link with near-wall-models arises when tuning the function $\ell(x)$ to be the distance from the boundary $\partial\Omega$ of the physical domain, hence reducing the dissipative effect, as the point x approaches the boundary layer. A simple application to channel flows is the one implemented with the van Driest damping, which is an early approach to obtain reliable and accurate turbulence simulations near a solid flat wall, see Pope [13] and also [7, Sec 3.3.1].

The analysis of the model (1.3) reveals that the first term $\beta^2 \operatorname{curl}(\ell^2(x)\partial_t \omega)$ is a sort of “dispersive” operator, hence the equations (1.3) are of pseudo-parabolic type. In fact, if $\ell^2(x) = \ell_0^2 \in \mathbb{R}^+$, for all $x \in \Omega$ then by using the fundamental identity of vector calculus

$$\operatorname{curl}(\operatorname{curl} f) = -\Delta f + \nabla(\operatorname{div} f), \tag{1.4}$$

it holds $\beta^2 \operatorname{curl}(\ell_0^2 \partial_t \omega) = -\beta^2 \ell_0^2 \partial_t \Delta u$, producing the dispersive term which characterizes the Voigt model. In this paper we focus on models dealing with the second term in (1.3), which is one of the most common in literature. Many of its variants have been already implemented and studied, hence in the sequel we will always assume that

$$\beta = 0 \quad \text{and} \quad C_{BL} > 0,$$

hence we consider the so called *Baldwin-Lomax* model. The second additional term in (1.3), namely $C_{BL} \operatorname{curl}(\ell^2(x)|\omega|\omega)$ is very similar to the dissipative term appearing in the Smagorinsky model, and we begin by considering the equations with extra dissipation given by $C_{BL} \operatorname{curl}(\ell^2(x)|\omega|^{p-2}\omega)$, with $2 < p \leq 3$. We start with the simplest case of $\ell(x) = \ell_0$. In this case, the properties of the solutions are rather similar to those known for the classical generalized Ladyžhenskaya-Lions-Smagorinsky model. There is a critical dependency of the dispersive and smoothing operators, when the function $\ell = \ell(x)$ is assigned as a multiple of the (regularized) distance from the boundary $d(x, \partial\Omega)$ of the point $x \in \Omega$. The study of models with smoothing operators which are degenerate near to the boundary will be the object of a forthcoming work [12]. In Section 4 we will consider a modified (in terms of differential operators) method in which ℓ is a function itself of the solution, as used in certain turbulence models.

Plan of the paper. In section 2 we summarize the notation and the functional setting. Next, in Section 3 we are first proving some fine existence and regularity properties for the model with $\beta = 0$ and a constant $\ell_0 > 0$, since this allows for interesting generalizations. In Section 4 we start considering a new model which introduces the smoothing in a less strong way, working directly on the vorticity equation, hence considering a model which is more related with damping than with smoothing. Finally, taking inspiration from the work of Cottet *et al.* [2] we propose a new version of the selective Baldwin-Lomax-type model taking into account the possible variations of the direction of the vorticity and linking the choice of the function $\ell(x)$ to the relative alignment of the vorticity. The approach we follow makes a stricter connections with the celebrated geometric criterion of Constantin and Fefferman [14] for the regularity of the solution and some of its variants/improvements as developed in [15, 16].

2 Functional setting and comparison with previous results

In this section we first introduce the notation and the precise definitions of functions spaces we will need to deal with. We will use the customary Sobolev spaces $(W^{s,p}(\Omega), \|\cdot\|_{W^{s,p}})$ and we denote the L^p -norm simply by $\|\cdot\|_p$ and since the Hilbert case plays a special role we denote the $L^2(\Omega)$ -norm simply by $\|\cdot\|$. In most cases $\Omega \subset \mathbb{R}^3$ will be an open bounded set with smooth boundary $\partial\Omega$. Due to well-known technical problems arising when dealing with the vorticity equation, in some cases we will restrict to the Cauchy problem in \mathbb{R}^3 or with the problem in the space periodic setting. By χ_A we denote the indicator function of the measurable set A .

As usual by p' we denote the conjugate exponent. For $(X, \|\cdot\|)$ a Banach space we will also denote the usual Bochner spaces of functions defined on $[0, T]$ and with values in X by $(L^p(0, T; X), \|\cdot\|_{L^p(X)})$.

For the variational formulation of the Navier-Stokes equations (1.1), and more generally of all systems of partial differential equations with the constraint of incompressibility we shall consider, we introduce the space \mathcal{V} of smooth and divergence-free vectors fields, with compact support in Ω . We then denote the completion of \mathcal{V} in $(L^2(\Omega))^3$ by $L^2_\sigma(\Omega)$ and the completion in $(H^1_0(\Omega))^3$ by $H^1_{0,\sigma}(\Omega)$, see also [17] for further details. Since some results are set in the Hilbert space $L^2_\sigma(\Omega)$ is endowed with the natural L^2 -norm $\|\cdot\|_2$ and inner product (\cdot, \cdot) , while $H^1_{0,\sigma}(\Omega)$ with the norm $\|\nabla v\|_2$ and inner product $((u, v)) := (\nabla u, \nabla v)$. As usual, we do not distinguish between scalar and vector valued function spaces. The dual pairing between V and V' is denoted by $\langle \cdot, \cdot \rangle$, and the dual norm by $\|\cdot\|_*$.

Since we need also to consider the space-periodic setting, in this case we consider scalar and vector defined on the torus $\mathbb{T}^3 = (\mathbb{R}/(2\pi\mathbb{Z}))^3$, which are with zero mean value, that is

$$\int_{\mathbb{T}^3} f(x) \, dx := \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} f(x) \, dx = 0.$$

In that case the linear space \mathcal{V} is made of divergence-free vector fields, 2π -periodic in each of the coordinate directions and with zero mean value. Its closure in $(L^2(\mathbb{T}^3))^3$ and $(H^1(\mathbb{T}^3))^3$ is then denoted by $L^2_\sigma(\mathbb{T}^3)$ and $H^1_\sigma(\mathbb{T}^3)$, respectively and the norms are the same as those employed in the case of a bounded domain.

In the case of a bounded domain Ω it holds the Poincaré inequality

$$\|v\|_p \leq C(p, \Omega) \|\nabla v\|_p,$$

valid for all smooth enough vector fields such that at least $v \cdot n = 0$ on the boundary $\partial\Omega$ (or in the space periodic case with zero mean value).

Since we have mainly estimates on the vorticity, we also need to estimate the full gradient, by means of the vorticity, whenever it is possible. We have the following inequality (cf. Bourguignon and Brezis [18]): There exists a constant $C = C(s, p, \Omega)$ such that, for all $s \geq 1$

$$\|\varphi\|_{s,p} \leq C \left[\|\nabla \cdot \varphi\|_{s-1,p} + \|\operatorname{curl} \varphi\|_{s-1,p} + \|\varphi \cdot n\|_{s-1/p,p,\Gamma} + \|\varphi\|_{s-1,p} \right], \tag{2.1}$$

for all vector fields $\varphi \in (W^{s,p}(\Omega))^3$, where n denotes the outward unit normal vector on $\partial\Omega$. This same result has been later improved by von Wahl [19] obtaining, under geometric conditions on the domain, an estimate without lower order terms: Let Ω be such that $b_1(\Omega) = b_2(\Omega) = 0$, where $b_i(\Omega)$ denotes the i -th Betti number, that is the dimension of the i -th homology group $H^i(\Omega, \mathbb{Z})$. Then, there exists C depending only on p and Ω such that

$$\|\nabla \varphi\|_p \leq C (\|\nabla \cdot \varphi\|_p + \|\operatorname{curl} \varphi\|_p), \tag{2.2}$$

for all $\varphi \in (W^{1,p}(\Omega))^3$ satisfying either $(\varphi \cdot n)|_\Gamma = 0$ or $(\varphi \times n)|_\Gamma = 0$.

In addition we will use the Sobolev inequality

$$\|v\|_6 \leq C \|\nabla v\|_2 \quad \text{for all } v \in H_0^1(\Omega).$$

The basic compactness results for space-time functions we use is the Aubin-Lions lemma. If X_0, X_1 are separable and reflexive Banach spaces; if $X_0 \hookrightarrow X$ (with compact embedding) and $X \hookrightarrow X_1$ (with continuous embedding), then for $1 < \alpha, \beta < \infty$ it holds that

$$\left\{ v \in L^\alpha(0, T; X_0); \partial_t v \in L^\beta(0, T; X_1) \right\} \hookrightarrow L^\alpha(0, T; X). \tag{2.3}$$

3 On large scale models based on the vorticity

We start describing the models we want to study. In the simplest form we have the following equations with $\ell = \ell_0 \in \mathbb{R}^+$

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla \pi + (C_{BL} \ell_0^2) \operatorname{curl}(|\omega|^{p-2} \omega) = f, \tag{3.1}$$

and setting $\bar{\nu} := C_{BL} \ell_0^2 > 0$ we consider the following initial boundary value problem

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla \pi + \bar{\nu} \operatorname{curl}(|\omega|^{p-2} \omega) &= f && \text{in } (0, T) \times \Omega, \\ \operatorname{div} u &= 0 && \text{in } (0, T) \times \Omega, \\ \omega &= \operatorname{curl} u && \text{in } (0, T) \times \Omega, \\ u &= 0 && \text{on } (0, T) \times \partial\Omega, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{3.2}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth and bounded open set. (In this case we can allow Dirichlet boundary conditions and obviously very similar computations are also valid in the space periodic setting).

For the above Baldwin-Lomax type initial boundary value problem (3.2) we can prove the following result:

Theorem 1. *Let be given $2 < p \leq 3$, $u_0 \in L_\sigma^2(\Omega)$, and $f \in L^2((0, T) \times \Omega)$. Then, there exists at least a weak solution to (3.2). Furthermore, if $p > \frac{5}{2}$, then the weak solution is unique.*

The proof of Theorem 1 can be obtained by an adaption of the one valid for general shear dependent fluids. We will give a sketch of the proof in order to precise the functional setting, but many results are close to those

reported in [20]. The result is valid also for smaller values of the exponent p , but the proof requires some technical adjustments. Here, since we consider turbulence type problems we restrict to the range $2 < p < 3$, but the reader can easily combine the results with those of the cited references to study the problems also a) in the shear thickening case $p < 2$; b) for values of p larger than 2 (for technical reasons related with the embedding $W^{2,2}(\Omega) \hookrightarrow L^p(\Omega)$), at least for $p \leq 6$.

Proof of Theorem 1. One main starting point is the *a priori* energy estimate which is obtained by using as test function u itself and integrating over Ω . Observe that, due to the Gauss-Green formulas, we get

$$\int_{\Omega} \operatorname{curl}(|\omega|^{p-2}\omega) \cdot u \, dx = \int_{\Omega} |\omega|^{p-2}\omega \cdot \omega \, dx + \int_{\partial\Omega} |\omega|^{p-2}(\omega \times n) \cdot u \, dS, \tag{3.3}$$

and the boundary integral vanishes, since $u = 0$ at the boundary. This shows that, for smooth enough solutions, then

$$\frac{1}{2} \|u(T)\|^2 + \nu \int_0^T \|\nabla u(s)\|^2 \, ds + \frac{\bar{\nu}}{2} \int_0^T \|\omega(s)\|_p^p \, ds \leq \frac{1}{2} \|u_0\|^2 + C(p, \bar{\nu}) \|f\|^2, \tag{3.4}$$

hence that the following *a priori* estimate holds true (and it is valid for Galerkin approximate functions)

$$u \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_{0,\sigma}(\Omega)) \cap L^p(0, T; (W^{1,p}(\Omega))^3).$$

We observe that the L^p -bound for the vorticity implies a bound on the whole gradient thanks to (2.2). The estimates are justified by using a Galerkin approximation and in the following we will need to make use of a basis made by eigenfunctions of the Stokes operator (Spectral basis).

Once the *a priori* estimate is established, a fair standard application of the “monotonicity argument” (cf. [4, 21] for the restricted range $\frac{5}{2} \leq p \leq 3$) or of Vitali’s lemma ensures the existence of a weak solution. This can be done as in [20, Ch. 5], for $2 < p \leq 3$

As in the classical results by Lions and Ladyžhenskaya, the solution is unique if $p > 5/2$. This follows by observing that in such range of exponents the solution u is smooth enough to be used as test function. The following differential inequality holds true for the difference $U = u_1 - u_2$ of two solutions with the same initial datum $u_0 \in L^2_\sigma(\Omega)$ and external force $f \in L^2((0, T) \times \Omega)$

$$\frac{d}{dt} \|U\| \leq C \|\nabla u_1\|_p^{\frac{2p}{2p-3}} \|U\|^2.$$

Since $\frac{2p}{2p-3} \leq p$ holds true if $p \geq \frac{5}{2}$, then the coefficient $\|\nabla u_1\|_p^{\frac{2p}{2p-3}}$ of the term $\|U\|^2$ from right-hand side belongs to $L^1(0, T)$ and the Gronwall lemma implies that $U \equiv 0$ if $U(0) = 0$. Details can be found in [20, Ch. 5.4]. □

Remark 3.1. *The same computations as in (3.3) show that the boundary integral vanishes also when using the curl-based Navier-type boundary conditions*

$$\begin{cases} u \cdot n = 0 & \text{on } (0, T) \times \partial\Omega, \\ \omega \times n = 0 & \text{on } (0, T) \times \partial\Omega. \end{cases} \tag{3.5}$$

These boundary conditions are particularly relevant in free-boundary problems and in turbulence modeling, see [22, 23]. Hence, the same argument used in the proof of Theorem 1 applies also to the problem in the setting of Navier-type conditions. The proof can be obtained by adapting the approach outlined in [24], as in Conca [25] and Temam and Ziane [26] and does not present particular additional technical difficulties, hence we do not reproduce it here.

The next step in the analysis of the model regards the existence of strong solutions and the vorticity behavior, especially when $p < 5/2$. The balance equation for the curl is obtained by taking the curl of (3.1) hence getting the following equation

$$\partial_t \omega - \nu \Delta \omega + \bar{\nu} \operatorname{curl}(\operatorname{curl}(|\omega|^{p-2}\omega)) + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u + \operatorname{curl} f.$$

As usual, dealing with the equation satisfied by ω , requires to deal with the problem without boundaries (Cauchy or periodic) or the curl-type Navier conditions (3.5). Here, in order to avoid inessential details, let us assume that $\text{curl} f = 0$ and observe that the system satisfied by the vorticity ω is the following initial value problem (we used (1.4))

$$\begin{aligned} \partial_t \omega - \nu \Delta \omega - \bar{\nu} \Delta(|\omega|^{p-2} \omega) + \bar{\nu} \nabla \operatorname{div}(|\omega|^{p-2} \omega) \\ + (u \cdot \nabla) \omega &= (\omega \cdot \nabla) u && \text{in } (0, T) \times \mathbb{T}^3, \\ \operatorname{div} u = \operatorname{div} \omega &= 0 && \text{in } (0, T) \times \mathbb{T}^3, \\ \omega &= \operatorname{curl} u && \text{in } (0, T) \times \mathbb{T}^3, \\ \omega(0) &= \omega_0 && \text{in } \mathbb{T}^3. \end{aligned} \tag{3.6}$$

Let us analyze the space periodic case and then we will explain changes needed to handle the other cases. In the space periodic case we can prove in a slightly simpler and alternate way the results stating that the critical exponent to have global existence of strong solution is $p = 11/5$. Earlier results in this direction can be found in [20, 27].

Theorem 2. *Let $p \geq 11/5$ and let be given $u_0 \in H^1_\sigma(\mathbb{T}^3)$. Then, there exists a unique solution ω of (3.6) such that $\omega \in L^\infty(0, T; L^2_\sigma(\mathbb{T}^3)) \cap L^2(0, T; H^1_\sigma(\mathbb{T}^3))$. Hence, by using (2.1) it follows $u \in L^\infty(0, T; H^1_\sigma(\mathbb{T}^3)) \cap L^2(0, T; H^2_\sigma(\mathbb{T}^3))$.*

Proof. The proof is mainly based on estimate obtained multiplying (3.6) by ω and integrating by parts. This can be transferred to Galerkin-Fourier approximate equations, with the basis of complex exponential, to prove existence in a standard way. With this technique one obtains for the extra-nonlinear term the following equality

$$\begin{aligned} &\int_{\mathbb{T}^3} \operatorname{curl}(\operatorname{curl}(|\omega|^{p-2} \omega)) \cdot \omega \, dx \\ &= - \int_{\mathbb{T}^3} \Delta(|\omega| \omega) \cdot \omega \, dx + \int_{\mathbb{T}^3} \nabla(\operatorname{div} |\omega|^{p-2} \omega) \cdot \omega \, dx \\ &= - \int_{\mathbb{T}^3} |\omega|^{p-2} \omega \cdot \Delta \omega \, dx \\ &= 4 \frac{p-2}{p^2} \int_{\mathbb{T}^3} |\nabla |\omega|^{p/2}|^2 \, dx + \frac{1}{2} \int_{\mathbb{T}^3} |\omega|^{p-2} |\nabla \omega|^2 \, dx, \end{aligned}$$

where we used that $\int_{\mathbb{T}^3} \nabla q \cdot \omega \, dx = 0$ and classical integration by parts as developed for –at least for the velocity field– the NSE in [28], but applied to the vorticity. In this way, we can apply the Sobolev embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$ to obtain that

$$\exists C_{Sob} > 0 : \quad \|\omega\|_{3p}^p \leq C_{Sob} \|\nabla |\omega|^{p/2}\|^2.$$

(All calculations are also valid in the whole space case \mathbb{R}^3 , by assuming sufficient decay at infinity of the integrated fields.) Next, we recall that by integration by parts $\int_{\mathbb{T}^3} (u \cdot \nabla) \omega \cdot \omega \, dx = 0$, while, by using Hölder inequality and explicit integral representation formulas we get

$$\left| \int_{\mathbb{T}^3} (\omega \cdot \nabla) u \cdot \omega \, dx \right| \leq \|\omega\|_3^2 \|\nabla u\|_3 \leq C_{CZ} \|\omega\|_3^3,$$

since the gradient can be written as a proper Calderón-Zygmund singular integral in terms of the vorticity, by using (1.4) and differentiating the Biot-Savart formula. The constant C_{CZ} from the right-hand-side is the one coming from the estimate for singular integrals $\|\nabla u\|_3 \leq C_{CZ} \|\omega\|_3$, and C_{CZ} depends (being $p = 3$ fixed) only on the fact that we are working in three space dimensions.

In this way we arrive to the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\omega\|^2 + \nu \|\nabla \omega\|^2 + \frac{4(p-2)\bar{\nu}}{p C_{Sob}} \|\omega\|_{3p}^p \leq C_{CZ} \|\omega\|_3^3,$$

We now use the following two convex-interpolation inequalities

$$\|\omega\|_3 \leq \|\omega\|_2^{\frac{2(p+1)}{3p-2}} \|\omega\|_6^{\frac{p}{3p-2}} \quad \text{and} \quad \|\omega\|_3 \leq \|\omega\|_p^{\frac{p-1}{2}} \|\omega\|_{\frac{3}{2}}^{\frac{3-p}{2}}.$$

We split the term from the right-hand side of the differential inequality as $\|\omega\|_3^{3(1-\alpha)} \|\omega\|_3^{3\alpha}$, with $\alpha = \frac{p(3p-5)}{6(p-1)}$. With this splitting and the two above inequalities we get

$$\|\omega\|_3^3 \leq (\|\omega\|_2^2)^{\frac{3-p}{2}} \|\omega\|_p^{\frac{p(3p-5)}{4}} \|\omega\|_{\frac{3}{2}}^{\frac{3p(3-p)}{4}},$$

and by Young inequality with $\delta = \frac{4}{9-3p}$ and $\delta' = \frac{4}{3p-5}$ (all calculations are justified for $p > 5/3$) we finally show that for all $\epsilon > 0$ exists $C_\epsilon > 0$ such that

$$\|\omega\|_3^3 \leq C_\epsilon (\|\omega\|_2^2)^{\frac{2(p-3)}{3p-5}} \|\omega\|_p^p + \epsilon \|\omega\|_{\frac{3}{2}}^p.$$

Observe that for $p > 5/3$, then $\frac{2(p-3)}{3p-5} < 1$ if and only if $p > 11/5$, with equality in the limit case. Consequently, for $p \geq 11/5$, we have proved the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\omega\|^2 + \nu \|\nabla \omega\|^2 + \frac{2(p-2)\bar{\nu}}{p C_{Sob}} \|\omega\|_{\frac{3}{2}}^p \leq c \|\omega\|_p^p (\|\omega\|_2^2)^\beta \quad \text{with } \beta \leq 1.$$

Being $\int_0^T \|\omega(s)\|_p^p ds < \infty$ from the a-priori estimate valid for weak solutions, the basic theory of differential inequalities shows that in this case,

$$\|\omega(T)\|^2 + \int_0^T \left(\nu \|\nabla \omega(s)\|^2 + \frac{2(p-2)\bar{\nu}}{p C_{Sob}} \|\omega(s)\|_{\frac{3}{2}}^p \right) ds \leq C,$$

where the constant C depends only on the data of the problem.

The above *a priori* estimates can be used in a spectral Galerkin method to prove existence of strong solutions. The uniqueness of strong solutions, follows in the same manner as before in the larger range $p \geq 11/5$, but now for solutions starting with the smoother initial datum in $H_\sigma^1(\mathbb{T}^3)$. \square

Remark 3.2. *In the proof of the a priori estimate needed to show existence of strong solutions, we used the fact that also the basic energy estimate (3.4) valid for weak solutions is still true. This point should be recalled later on in Section 4.1 in the analysis of the new models, since the techniques are rather different in that case, where the formulation in velocity/pressure is not so easy to be handled.*

Remark 3.3. *The case with Navier-type boundary conditions (3.5) can be handled in a similar way, by using the following two identities (see [23, Lemmas 1-2]). The first one*

$$\begin{aligned} - \int_{\Omega} \Delta f \cdot f |f|^{p-2} dx &= \frac{1}{2} \int_{\Omega} |f|^{p-2} |\nabla f|^2 dx + 4 \frac{p-2}{p^2} \int_{\Omega} |\nabla |f|^{\frac{p}{2}}|^2 dx \\ &\quad - \int_{\Gamma} |f|^{p-2} (n \cdot \nabla) f \cdot f dS, \end{aligned}$$

is valid for all smooth enough vector fields, while the following one is related with the vorticity field

$$- \frac{\partial \omega}{\partial n} \cdot \omega = (\epsilon_{1jk} \epsilon_{1\beta\gamma} + \epsilon_{2jk} \epsilon_{2\beta\gamma} + \epsilon_{3jk} \epsilon_{3\beta\gamma}) \omega_j \omega_\beta \partial_k n_\gamma \quad \text{on } \Gamma,$$

with summation over repeated indices. Since $\omega \times n = 0$, it holds $\omega = (\omega \cdot n) n$ at the boundary and we obtain that $(\omega \cdot \nabla) \omega = (\omega \cdot n) \frac{\partial \omega}{\partial n}$. This shows the following two inequalities are valid

$$\begin{aligned} - \int_{\Omega} \Delta \omega \cdot \omega |\omega|^{p-2} dx &\geq 4 \frac{p-2}{p^2} \int_{\Omega} |\nabla |\omega|^{\frac{p}{2}}|^2 dx - c \int_{\Gamma} |\omega|^p dS, \\ - \int_{\Omega} \Delta \omega \cdot \omega dx &\geq \int_{\Omega} |\nabla \omega|^2 dx - c \int_{\Gamma} |\omega|^2 dS. \end{aligned}$$

By using the trace theorem and the embedding $H^{1/2}(\Gamma) \subset L^4(\Gamma)$, applied to ω and to $|\omega|^{p/2}$, we can easily see that the two surface integrals can be bounded with the terms from the left hand side, obtaining the following differential inequality (cf. [24, Eq. (23)] and [23, Eq. (2.3)])

$$\frac{1}{2} \frac{d}{dt} \|\omega\|^2 + \nu \|\nabla \omega\|^2 + \frac{4(p-2)\bar{\nu}}{p C_{Sob}} \|\omega\|_{3p}^p \leq C(\|\omega\|_3^3 + \|\omega\|_p^p + \|\omega\|_2^2),$$

for a constant C depending only on p and Ω . At this point the same reasoning as before can be applied to prove results of existence and uniqueness, by using a functional setting with the Hilbert space $H_\sigma^1(\Omega) := (H^1(\Omega))^3 \cap L_\sigma^2(\Omega)$, which replaces the space $H_{0,\sigma}^1(\Omega)$, due to the fact that the usual Sobolev machinery works also in this setting, as explained in the cited references.

4 A couple of new turbulence models

>From the results of the previous section it turns out that the relevant case $p = 3$ produces a unique strong solution but it is also well-known that the dissipation/stabilization introduced by the Smagorinsky like term is generally too strong and, in particular, there is an extremely artificial stabilizing effect on the boundary layer. The Baldwin-Lomax model associated with a function of the distance from the boundary implies –by results on weighted spaces– a control of a less strong dissipation norm. This observation comes from the use of Poincaré type estimate

$$\left\| u(x) - \int_{\Omega} u(y) dy \right\|_q \leq C \|d^\delta(x) \nabla u(x)\|_p,$$

where $q \leq \frac{3p}{3-p(1-\delta)}$ and $d(x) = \text{dist}(x, \partial\Omega)$, see Hurri-Syrjänen [29]. This result, coupled with the machinery developed as in Kufner [30] and with other several technical tools, can be used to produce similar results, at least for a model with a nonlinear perturbation given by

$$A(u) = -C_S \text{div}(d^\delta(x) |\nabla u|) \nabla u,$$

for an appropriate exponent $\delta > 0$. Considering the operator depending only on the deformation tensor or the curl version as in Baldwin-Lomax, requires also to use appropriate variants also of other tools as the Korn inequality and the way to pass to the limit in the approximate system. This will be addressed in a forthcoming work, which is out of the scopes of the present one. Here, we consider a different approach, based not on the knowledge and enough smoothness of the function $\ell(x)$. We study a weakly-dissipative model, based on an anisotropic and selective choice of the dissipation term, producing a non uniformly elliptic regularization, but with an approach more oriented towards computations.

Since the Smagorinsky term is too strong and the model associated does not well-reproduce smaller scales a single universal constant C_S for different turbulent fields in rotating or sheared flows, near solid walls, or in transitional regimes is not likely to be determined; the approach of Lilly (and its variants in [31]) seem to work only in the homogeneous and isotropic case of fully developed turbulence; several methods has been designed to overcome this fact. Apart Obukhov approach and the already cited Van Driest damping in the channel flow setting, an early method is the dynamic one introduced in Germano, Piomelli, Moin, and Cabot [32]. Several modifications, leading also the multiscale variational methods have been proposed, see [7] but here we focus on a method which seems to be the closest to the analysis coming from the Partial Differential Equations framework. It has also strong connections with the use of a *geometric* approach in the study of the regularity of the weak solutions to the Navier-Stokes equations [33].

One important modeling idea in many variations of the eddy viscosity models is that the

eddy viscosity terms need to be active only in regions where the solution is not regular, or where there is a strong generation of small scales.

In the “dynamic model” introduced in [32], the Smagorinsky model’s “parameter” C_S is not anymore a constant but it is chosen locally in space and time, so to make the Smagorinsky model to agree (in a least squares sense) as closely as possible with the Bardina scale similarity model.

In the selective model of Cottet, Jiroveanu, and Michaux [2] the parameter C_S is not obtained by a solution of a local minimization problem, but by detecting the regions where *vorticity is active* as measured by the non-alignment of the direction of ω . This can be achieved in a computational way by considering a function filter $\Psi(t, x)$ which is the indicator function of the points where the vorticity direction is badly behaved, that is in regions where the angle of vorticity is not a Lipschitz or even Hölder function of the space position. Hence, the model studied in the cited reference has the eddy viscosity term

$$\nu_t(t, x, u) = C_S^2 \ell_0^2 \Psi(t, x) |Du(t, x)|,$$

leading to the LES system (1.2), which is considered also in a bounded domain for numerical tests.

Both these methods need to be calibrated and their implementations are well documented, but in any case they suffer from some limitations in the mathematical formulation. In particular, the selective term acts at the level of the velocity field, but the localization regards the vorticity activity, and the two facts are not treated with full mathematical rigour in the aforementioned papers, while the computational aspects and the results of implementation are instead well-documented. Here, we mainly propose a model which is more amenable to a precise treatment, which do not produce strong dissipation at the level of the velocity. Instead our approach produces a model acting directly on the vorticity balance equation and which gives a precise bound on the growth of the vorticity magnitude.

4.1 A new model, with an eddy viscosity damping the vorticity

Here, we consider the NSE in the velocity/vorticity formulation and we analyze the enstrophy behavior for the following model

$$\begin{aligned} \partial_t \omega - \nu \Delta \omega + C_{S,\omega} |\omega|^{p-2} \omega + (u \cdot \nabla) \omega &= (\omega \cdot \nabla) u, \\ \operatorname{div} u &= \operatorname{div} \omega = 0, \\ \operatorname{curl} u &= \omega, \end{aligned} \tag{4.1}$$

which is much weaker in terms of dissipation than the Baldwin-Lomax type previously studied. In the Cauchy setting, or in the space periodic, or even with curl-Navier condition we consider the above system where $C_{S,\omega} > 0$ is a given constant and u is formally obtained from the vorticity ω after solving $u = \operatorname{curl}^{-1} \omega$. More precisely u is obtained by ω through the Biot-Savart law.

We deal primary with the space-periodic case. The interpretation of this model is that we are adding a zeroth order damping term directly at the vorticity level, which is needed to balance in a sharp form the vortex-stretching term

$$\int_{\mathbb{T}^3} (\omega \cdot \nabla) u \cdot \omega \, dx,$$

which behaves, roughly speaking, as the integral of $|\omega|^3$. In fact, by multiplying by ω and integrating over the physical domain, one gets

$$\frac{1}{2} \frac{d}{dt} \|\omega\|^2 + \nu \|\nabla \omega\|^2 + C_{S,\omega} \|\omega\|_p^p \leq C_{CZ} \|\omega\|_3^3.$$

The constant C_{CZ} from the right-hand-side is the one coming from the standard estimate with singular integrals. Hence, if $p = 3$ and if $C_{S,\omega} \geq C_{CZ}$, then we get the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\omega\|^2 + \nu \|\nabla \omega\|^2 + (C_{S,\omega} - C_{CZ}) \|\omega\|_3^3 \leq 0. \tag{4.2}$$

The choice of the parameter $p = 3$ seems obliged in this setting, but the above estimate allows us to show global existence of a solution, once the constant $C_{S,\omega}$ is chosen large enough, but in an universal (not depending on the solution) way. The life-span of the solution turns out to be independent of the size of the

initial datum. We recall that without the damping term (that is when $C_{S,\omega} = 0$) one can prove results which are local or that are global only for unrealistic extremely small initial data. In this case there are not other *a priori* estimates immediately available even at the level of the velocity, this explains the fact that smaller values of p seem not treatable. Hence, also results as those of Zhou [34] with smaller critical exponents p , but with the damping $|u|^{p-2}u$ at the level of velocity, are not available in this case.

Remark 4.1. *The above model (4.1) in the velocity/vorticity formulation can be considered also in the velocity/pressure formulation. To write this we need a proper right-inverse of the curl operator, to write “formally” system (4.1) as the curl of the following one*

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla \pi + \bar{\nu} \operatorname{curl}^{-1}(|\omega|\omega) = 0.$$

The linear operator “ curl^{-1} ” is defined in (4.3) by using the following argument: From the vector calculus identity (1.4), valid for smooth vector fields f , we obtain that if \mathcal{F} is such that $-\Delta \mathcal{F} = f$, (or equivalently $\mathcal{F} = (-\Delta)^{-1}f$ in the whole space or in the torus with periodic boundary conditions) then

$$\operatorname{curl}(\operatorname{curl} \mathcal{F}) = -\Delta \left((-\Delta)^{-1}f \right) + \nabla(\operatorname{div} \mathcal{F}) = f + \nabla q,$$

for some q ; hence, the vector field $\mathcal{V} := \operatorname{curl} \mathcal{F} = \operatorname{curl} \left((-\Delta)^{-1}f \right)$ satisfies $\operatorname{curl} \mathcal{V} = f + \nabla q$.

Consequently, in absence of boundaries, we use as right inverse of the curl operator the linear operator defined in the following way

$$\operatorname{curl}^{-1} := \operatorname{curl}(-\Delta)^{-1}, \tag{4.3}$$

and it is a right inverse of the curl modulo a gradient term, which is nevertheless inessential in the dynamic equation for the enstrophy. Observe that the differential operators we are using are invariant by rotations, hence the equations we obtain are invariant by change of reference frame and are physically meaningful.

We need to apply curl^{-1} to $f = |\omega|\omega$ and in the case of the torus, in order to invert the Laplace operator, the zero mean value is needed. Observe that clearly $\int_{\mathbb{T}^3} \omega \, dx = 0$, but in general $\int_{\mathbb{T}^3} |\omega|\omega \, dx \neq 0$.

To conclude, the model with constant damping (4.1) should be written in the velocity/pressure formulation as follows

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla \pi + \bar{\nu} \operatorname{curl}(-\Delta)^{-1} \left(|\omega|\omega - \int_{\mathbb{T}^3} |\omega|\omega \, dy \right) = 0. \tag{4.4}$$

In terms of vorticity we obtain, in fact, taking the curl

$$\begin{aligned} \partial_t \omega - \nu \Delta \omega + \bar{\nu}(-\Delta)(-\Delta)^{-1} \left(|\omega|\omega - \int_{\mathbb{T}^3} |\omega|\omega \, dx \right) \\ + \bar{\nu} \nabla(-\Delta)^{-1} \left(|\omega|\omega - \int_{\mathbb{T}^3} |\omega|\omega \, dx \right) + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u, \end{aligned}$$

or, with obvious simplifications,

$$\partial_t \omega - \nu \Delta \omega + \bar{\nu} \left(|\omega|\omega - \int_{\mathbb{T}^3} |\omega|\omega \, dx \right) + \bar{\nu} \nabla q + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u.$$

The introduction of the velocity/pressure formulation does not change the enstrophy balance. In fact, with this system we have the following identity, obtained by integration by parts,

$$\begin{aligned} \int_{\mathbb{T}^3} \operatorname{curl} \operatorname{curl}(-\Delta)^{-1} \left(|\omega|\omega - \int_{\mathbb{T}^3} |\omega|\omega \, dx \right) \cdot \omega \, dx &= \\ &= \int_{\mathbb{T}^3} \left(|\omega|\omega - \int_{\mathbb{T}^3} |\omega|\omega \, dx \right) \cdot \omega \, dx = \\ &= \int_{\mathbb{T}^3} |\omega|\omega \cdot \omega \, dx = \|\omega\|_3^3, \end{aligned}$$

where we used again (1.4) and the fact that gradients and divergence-free vector fields are orthogonal, together with the fact that ω (as well as ∇u and Du) have zero mean value.

Remark 4.2. Concerning the velocity/pressure formulation, we observe that results on the Fractional Laplacian in Córdoba and Córdoba [35] show the following “positivity lemma”

$$\int |\theta|^{p-2} \theta (-\Delta)^\alpha \theta \, dx \geq 0 \quad \forall \alpha \in [0, 1], \tag{4.5}$$

for all values of $p \geq 1$.

In our case the energy inequality for weak solutions of the model (4.4) will hold if one would be able to prove the following inequality

$$\int |\theta|^{p-2} \theta (-\Delta)^{-1} \theta \, dx \geq 0 \tag{4.6}$$

which correspond to the negative value $\alpha = -1$ in (4.5). From the latter estimate in fact one could show

$$\begin{aligned} & \int_{\mathbb{T}^3} \operatorname{curl}(-\Delta)^{-1} \left(|\omega|\omega - \int_{\mathbb{T}^3} |\omega|\omega \, dx \right) \cdot u \, dx \\ &= \int_{\mathbb{T}^3} (-\Delta)^{-1} \left(|\omega|\omega - \int_{\mathbb{T}^3} |\omega|\omega \, dx \right) \cdot \omega \, dx \\ &= \int_{\mathbb{T}^3} \left(|\omega|\omega - \int_{\mathbb{T}^3} |\omega|\omega \, dx \right) \cdot (-\Delta)^{-1} \omega \, dx \\ &= \int_{\mathbb{T}^3} |\omega|\omega \cdot (-\Delta)^{-1} \omega \, dx \geq 0, \end{aligned}$$

which derives immediately from integration by parts.

At present, the validity of (4.6) represents an open problem, and we are forced to restrict to work only in the setting of velocity/vorticity formulation. We are then considering a system for which the energy inequality does not follow directly: Some of the basic machinery and estimates available for the 3D Navier-Stokes equations cannot be applied directly, and a slightly different treatment for system (4.1) is needed.

The main result of this section is the following

Theorem 3. Let be given $\omega_0 \in L^2_\sigma(\mathbb{T}^3)$ and let $C_{S,\omega}$ large enough, but independent of the data of the problem. Then, there exists a unique weak¹ solution to the following initial value problem

$$\begin{aligned} \partial_t \omega - \nu \Delta \omega + C_{S,\omega} \left(|\omega|\omega - \int_{\mathbb{T}^3} |\omega|\omega \, dx \right) \\ + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u, & \quad \text{in } (0, T) \times \mathbb{T}^3, \\ \operatorname{div} u = \operatorname{div} \omega = 0 & \quad \text{in } (0, T) \times \mathbb{T}^3, \\ \operatorname{curl} u = \omega & \quad \text{in } (0, T) \times \mathbb{T}^3, \\ \omega(0) = \omega_0 & \quad \text{in } \mathbb{T}^3, \end{aligned} \tag{4.7}$$

Proof. We start by observing that natural a-priori estimate for system (4.7) is (4.2), and if $C_{S,\omega} \geq C_{CZ}$, then it follows that

$$\omega \in L^\infty(0, T; L^2_\sigma(\mathbb{T}^3)) \cap L^2(0, T; H^1_\sigma(\mathbb{T}^3)) \cap L^3(0, T; (L^3(\mathbb{T}^3))^3),$$

¹ Here by weak solution we mean a distributional solution in $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$. Clearly this is a strong solution if read in terms of u which belongs to $L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$, and strong is in the usual sense for the Navier-Stokes equations [17].

which is enough to ensure the existence through a Fourier-Galerkin approximation. In fact, the bound on ω proves by comparison also that

$$\partial_t \omega \in L^2(0, T; (H^1_\sigma)')$$

This is obtained by multiplying by a periodic and with zero mean value $\varphi \in \mathcal{V}$ it follows that

$$\begin{aligned} |\langle \partial_t \omega, \varphi \rangle| &= \nu \langle \nabla \omega, \nabla \varphi \rangle - C_{S,\omega} \langle |\omega| \omega, \varphi \rangle + \langle u \otimes \omega, \nabla \varphi \rangle - \langle u \otimes \omega, (\nabla \varphi)^T \rangle \\ &\leq \nu \|\nabla \omega\|_2 \|\nabla \varphi\|_2 + C_{S,\omega} \|\omega\|_{12/5}^2 \|\varphi\|_6 + 2\|u\|_6 \|\omega\|_3 \|\nabla \varphi\|_2 \\ &\leq c(\|\nabla \omega\|_2 \|\nabla \varphi\|_2 + \|\omega\|_2 \|\omega\|_3 \|\nabla \varphi\|_2 + \|\nabla u\|_2 \|\omega\|_3 \|\nabla \varphi\|_2) \\ &\leq c(\|\nabla \omega\|_2 + \|\omega\|_2 \|\omega\|_3) \|\nabla \varphi\|_2 \\ &\leq c(\|\nabla \omega\|_2 + \|\omega\|_2^{3/2} \|\nabla \omega\|_2^{1/2}) \|\nabla \varphi\|_2, \end{aligned}$$

showing the requested property by using the already obtained bound.

This implies, by using the classical Aubin-Lions lemma, that from the approximating sequences we can extract a sub-sequence converging strongly in $L^2(0, T; (L^2(\mathbb{T}^3))^3)$, cf. (2.3). Next by interpolation and by using the *a priori* bounds this shows also strong convergence in $L^3(0, T; (L^3(\mathbb{T}^3))^3)$. This is enough to pass to the limit in all terms and to show that the limit is a solution of the problem. We do not give details for this well-known step, which can be found in [17], simply translating the results on the velocity to those for the vorticity.

We show now that the obtained regularity is enough to show uniqueness. In fact the duality between $\partial_t \omega$ and ω is well-defined. We take two solutions ω_1, ω_2 corresponding to the same initial datum $\omega_0(x)$ and, multiplying by the difference $\Omega = \omega_1 - \omega_2$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Omega\|^2 + \nu \|\nabla \Omega\|^2 + C_{S,\omega} \|\Omega\|_3^3 + \int_{\mathbb{T}^3} (u_1 \cdot \nabla) \Omega \cdot \Omega \\ + \int_{\mathbb{T}^3} ((u_1 - u_2) \cdot \nabla) \omega_2 \cdot \Omega = \int_{\mathbb{T}^3} (\Omega \cdot \nabla) u_1 \cdot \Omega + \int_{\mathbb{T}^3} (\omega_2 \cdot \nabla) (u_1 - u_2) \cdot \Omega, \end{aligned}$$

and we observe that by integrating by parts and $\int_{\mathbb{T}^3} (u_1 \cdot \nabla) \Omega \cdot \Omega \, dx = 0$, while by usual Hölder and Sobolev estimates

$$\begin{aligned} \int_{\mathbb{T}^3} ((u_1 - u_2) \cdot \nabla) \omega_2 \cdot \Omega &\leq \|u_1 - u_2\|_6 \|\nabla \omega_2\|_2 \|\Omega\|_3 \leq \|\Omega\|_2^{3/2} \|\nabla \omega_2\|_2 \|\nabla \Omega\|_2^{1/2}, \\ \int_{\mathbb{T}^3} (\Omega \cdot \nabla) u_1 \cdot \Omega &\leq \|\Omega\|_3^2 \|\nabla u_1\|_3 \leq \|\omega_1\|_3 \|\Omega\|_2 \|\nabla \Omega\|_2, \\ \int_{\mathbb{T}^3} (\omega_2 \cdot \nabla) (u_1 - u_2) \cdot \Omega &\leq \|\omega_2\|_3 \|\nabla (u_1 - u_2)\|_3 \|\Omega\|_3 \leq \|\omega_2\|_3 \|\Omega\|_2 \|\nabla \Omega\|_2. \end{aligned}$$

Hence, with Young inequality we get

$$\frac{d}{dt} \|\Omega\|^2 + \nu \|\nabla \Omega\|^2 \leq c(\|\nabla \omega_2\|^{4/3} + \|\omega_1\|_2^2 + \|\omega_2\|_2^2) \|\Omega\|_2^2,$$

and due to the available information on the two weak solutions it follows that $(\|\nabla \omega_2\|^{4/3} + \|\omega_1\|_2^2 + \|\omega_2\|_2^2) \in L^1(0, T)$, hence we get $\Omega \equiv 0$.

The exponent $p = 3$ represents the natural one in order to control the growth of the nonlinear term. \square

Remark 4.3. We observe that in the case of the Navier-Stokes equations having a the vorticity (or the gradient) in $L^3(0, T; L^3(\Omega))$ implies the smooth continuation of the solution, by scaling invariant results, see Beirão da Veiga [36]. Here the situation is different: For the model (4.1) we do not have the basic energy estimate valid for the Navier-Stokes equations (3.4), hence some of the results typical of the Navier-Stokes do not follow directly.

Remark 4.4. Next, we detect, if any, the critical exponent p to ensure existence of strong solutions for the system

$$\partial_t \omega - \nu \Delta \omega + C_{S,\omega} \left(|\omega|^{p-2} \omega - \int_{\mathbb{T}^3} |\omega|^{p-2} \omega \, dx \right) + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u.$$

We use the two convex interpolation inequalities

$$\|\omega\|_3 \leq \|\omega\|_2^{\frac{1}{2}} \|\omega\|_6^{\frac{1}{2}} \quad \text{and} \quad \|\omega\|_3 \leq \|\omega\|_p^{\frac{p}{6-p}} \|\omega\|_6^{\frac{6-2p}{6-p}}$$

and we split the right-hand side as $\|\omega\|_3^3 = \|\omega\|_3^{3\alpha} \|\omega\|_3^{3(1-\alpha)}$, with α chosen as $\alpha = 2 \frac{2p-3}{3p}$, to get

$$\|\omega\|_3^3 \leq \|\omega\|_2^{\frac{2p-3}{p}} \|\omega\|_p \|\omega\|_6^{\frac{3}{p}} \leq C_S \|\omega\|_2^{\frac{2p-3}{p}} \|\omega\|_p \|\nabla \omega\|_2^{\frac{3}{p}}.$$

Hence, by using Young inequality with exponents $x = \frac{2p}{3}$ and $x' = \frac{2p}{2p-3}$ we show

$$\|\omega\|_3^3 \leq \frac{\nu}{2} \|\nabla \omega\|_2^2 + \frac{C}{\nu^{\frac{2}{2p-3}}} \|\omega\|_2^2 \|\omega\|_p^{\frac{2p}{2p-3}}.$$

In particular we have $\frac{2p}{2p-3} \leq p$ if $p \geq \frac{5}{2}$. Hence, in the limiting case $p = 5/2$ we have the differential inequality.

$$\frac{1}{2} \frac{d}{dt} \|\omega\|^2 + \nu \|\nabla \omega\|^2 + \left(C_{S,\omega} - C \nu^{-\frac{2}{2p-3}} \|\omega\|_2^2 \right) \|\omega\|_p^p \leq 0,$$

and this shows that, in the case of $p = \frac{5}{2}$ (which is smaller than the critical exponent $p = 3$) one can show global existence (on the whole half-line \mathbb{R}^+) provided that the coefficient $C_{S,\omega}$ is larger than a constant depending on the size of the initial datum and on the viscosity.

4.2 A new selective model, based on the vorticity direction

As we have seen before the control of the $L^3((0, T) \times \Omega)$ of ω can be used to infer existence and uniqueness of the solutions. The results of the previous section are related with a control of global (integral) quantities, being based on the classical Sobolev machinery. Even if the results of the previous section are original, they are based on a fair application of well-known techniques. Observe that having $\omega \in L^3_{t,x}$ would be enough as additional assumption if applied to the NSE. In our setting due to the fact that we have a nonlinear additional term and lack of the energy inequality, this is not enough to ensure a direct result of global existence, unless not some restrictions on the size of the damping coefficient $C_{S,\omega}$ is added, cf. Remark 4.3.

In this section we use a rather different approach, which is based on a selective damping, where selection is based on the criterion of adding damping of the vorticity only in regions where there is a large vortex stretching. We now propose an alternate (with respect to the ones already present in literature, cf. especially [2, 37]) selective method with a non-constant turbulent coefficient, in such a way to ensure damping only at the level of the vorticity, and for which the mathematical theory can be formulated in a rigorous way.

To detect the regions where the vorticity is highly active we formulate a criterion based not on the relative size of the vorticity, but by considering a turbulent viscosity functions which is a multiple of the indicator of the region with intense vortex activity; this is detected by considering the local behavior of the direction of the vorticity

$$\xi(t, x) := \frac{\omega(t, x)}{|\omega(t, x)|},$$

itself in the neighborhood of the point $(t, x) \in (0, T) \times \Omega$.

We observe that a similar model has been already considered by Cottet, Jiroveanu, and Michaux [2], but in their case they worked directly with the equation for the velocity, proposing the following LES model

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla \pi - \operatorname{div}(C_S^2 \ell_0^2 \Psi |Du| Du) &= f & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u &= 0 & \text{in } (0, T) \times \Omega, \end{aligned} \tag{4.8}$$

with Dirichlet conditions, where

$$\Psi(t, x) = \begin{cases} 1 & \text{if } \beta_0 < \beta_\lambda \leq \pi - \beta_0, \\ 0 & \text{otherwise.} \end{cases}$$

The constant β_0 is a threshold angle, while

$$\beta_\lambda(t, x) := \arcsin \frac{\omega(t, x) \times \omega_\lambda(t, x)}{|\omega(t, x)| |\omega_\lambda(t, x)|}, \tag{4.9}$$

and ω_λ is the average over a the surface of ball or radius $\lambda > 0$

$$\omega_\lambda(t, x) = \frac{1}{4\pi\lambda^2} \int_{|r|=\lambda} \omega(x+r) dS,$$

where the length λ is related to the mesh size and the integral can be numerically estimated by averaging over the six closest neighboring points (in a cubic uniform mesh).

The interpretation of the rationale behind the criterion (see also Guermond and Prudhomme [38]) is based on the fact that if the angle between the vorticity is well behaved, then weak solutions of the Navier-Stokes equations are smooth, cf. [14, 15]. Hence, the lack of (anti) alignment of the vorticity is a measure of the potential singular behavior of the solutions. The main result is the theorem stating that if

$$\sin(\theta(t, x, y)) \leq C|x-y|^{1/2} \quad \text{a.e. } x, y \in \mathbb{R}^3, \text{ a.e. } t \in [0, T], \tag{4.10}$$

where $\theta(t, x, y) := \angle(\xi(t, x), \xi(t, y))$, then weak solutions of the Navier-Stokes equations are smooth, see [14–16, 39].

Here, we consider and rigorously analyze a variation of the model (4.8), in such a way that the *a priori* estimate can be put in a precise quantitative relation with the growth of the enstrophy and the behavior of the vorticity direction. We are in fact introducing a new model with the aim of being able to perform some quantitative estimates, missing in the cited references [2, 37]) which were more focused on the implementation and on the effective numerical results).

The new model we propose and for which we are able to prove quantitative *a priori* estimates is the following: in the velocity/vorticity formulation we consider

$$\begin{aligned} \partial_t \omega - \nu \Delta \omega + C_{S,\omega} \Psi |\omega| \omega + (u \cdot \nabla) \omega &= (\omega \cdot \nabla) u, & \text{in } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} u &= \operatorname{div} \omega = 0 & \text{in } (0, T) \times \mathbb{R}^3, \\ \omega &= \operatorname{curl} u & \text{in } (0, T) \times \mathbb{R}^3, \\ \omega(0) &= \omega_0 & \text{in } \mathbb{R}^3, \end{aligned} \tag{4.11}$$

with the scalar function Ψ defined as follows:

$$\Psi(t, x) = \begin{cases} 1 & \text{where “jumps” of the vorticity direction are large,} \\ 0 & \text{elsewhere.} \end{cases}$$

The precise quantitative notion of “large jumps” will be specified later in the formulation of Theorem 4. We remark that the expected smoothness of Ψ is very low (nothing more than L^∞ can be inferred) and not enough to establish probably good regularity properties, nevertheless it can be at least used to prove a priori bounds.

The fact that the operator is with non-constant coefficients makes the treatment more complex than in [20] and in addition the damping is not uniform, producing additional L^3_{loc} estimates which are not uniform in the whole spatial domain.

The main original point is to work directly with the vorticity balance equation, as in the model of the previous Section 4.1 and to establish or identify a “good set” $\Omega_\lambda \subset (0, T) \times \mathbb{R}^3$ such that if $\Psi(t, x) = \chi_{\Omega_\lambda}(t, x)$, then it follows that the degenerate damping term

$$C_{S,\omega} \Psi(t, x) |\omega(t, x)| \omega(t, x),$$

in the equations for the vorticity (4.11) is enough to control the growth of the enstrophy. The results in [14] and further developments in [15, 24] are devoted to the analysis of the weaker possible alignment of ξ able to ensure the regularity of weak solution. By studying a special setting with “type I singularities,” Giga and

Miura [39] have been able to study also a case of mere continuity, without any other further requirement (Lip-schitz or Hölder) as in the above references. Nevertheless, the modulus of continuity is not easily computable in numerical tests, since the uniform control requires rather extensive computations.

The LES model, as introduced with the determination of the threshold angle (4.9), is based on this rationale from condition (4.10). On the other hand, we consider here a criterion built up on the results from Ref. [16] which states regularity for the Navier-Stokes equations if the possible jumps of angles between the vorticity at neighboring points is small enough. This is more fit in the framework of the determination of the criterion, since averaging is not needed.

Theorem 4. *Let us assume ² that $|\omega(t, x)| \geq M$ for all $(t, x) \in \mathbb{R}^3 \times (0, T)$, for some $M > 0$. Let us fix the two constant $0 < \epsilon_0 \ll 1$ and $\lambda > 0$ and define the following set*

$$\Omega_\lambda(t) = \left\{ (t, x) \in (0, T) \times \mathbb{R}^3 : \exists y \in B(0, \lambda) : |\xi(t, x + y) \times \xi(t, x)| \geq \epsilon_0 \right\}.$$

Then, if $\omega_0 \in L^2_\sigma(\mathbb{R}^3)$, if ϵ_0 is small enough, and if λ is large enough, then there exists a family of uniformly bounded and global in time approximate solutions to model (4.11).

Proof. As usual the main part of the proof is to properly estimate the vortex stretching term. We work in the whole space where explicit formulas are neat, but a similar treatment can be easily done also in the space-periodic setting. We split the term responsible of the vortex stretching into two parts as follows

$$\begin{aligned} \int_{\mathbb{R}^3} (\omega \cdot \nabla) u \cdot \omega \, dx &= \int_{\mathbb{R}^3} S[\omega](x) \omega(x) \cdot \omega(x) \, dx \\ &= \int_{\Omega_\lambda} S[\omega](x) \omega(x) \cdot \omega(x) \, dx + \int_{\mathbb{R}^3 \setminus \Omega_\lambda} S[\omega](x) \omega(x) \cdot \omega(x) \, dx, \end{aligned}$$

where $S[\omega](x) = Du$ is the deformation matrix which can be represented as a singular operator integral, while

$$\begin{aligned} S[\omega](x) \omega(x) \cdot \omega(x) &= |\omega(x)|^2 P.V. \int_{\mathbb{R}^3} \frac{\widehat{y}_i \widehat{y}_k - 3\delta_{ik}}{|y|^3} \epsilon_{ijk} \xi_l(x + y) \xi_i(x) \xi_j(x) |\omega(x + y)| \, dy, \end{aligned}$$

where $[a \times b]_j = \epsilon_{ijk} a_k b_l$ is the exterior product (with the totally anti-symmetric Ricci tensor ϵ_{ijk}) and $\widehat{y} = y/|y|$.

The integral in dy is split into two parts: the inner where $|y| < \lambda$ and the outer one, by means of a smooth cut-off non-increasing function $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which equals 1 for $0 < s < \lambda/2$ and zero for $s \geq 3\lambda/2$. In particular, we set

$$F(x, y) := \frac{\widehat{y}_i \widehat{y}_k - 3\delta_{ik}}{|y|^3} \epsilon_{ijk} \xi_l(x + y) \xi_i(x) \xi_j(x) |\omega(x + y)| \, dy,$$

and we have that

$$\begin{aligned} \int_{\mathbb{R}^3} S[\omega](x) \omega(x) \omega(x) \, dx &= \int_{\mathbb{R}^3} |\omega(x)|^2 P.V. \int_{\mathbb{R}^3} (1 - \rho(|y|)) F(x, y) \, dx dy \\ &+ \int_{\mathbb{R}^3} |\omega(x)|^2 P.V. \int_{\mathbb{R}^3} \rho(|y|) F(x, y) \, dx dy. \end{aligned}$$

² This assumption is needed to be sure that the vorticity direction is well-defined for all points. Nevertheless the regions where the vorticity is bounded are not regions of high turbulent activity, hence the restriction is not relevant. Anyway, by a cut-off of the vorticity into $\omega = \omega_1 + \omega_2$, with $\omega_1 = \omega(t, x)\chi_{\{|\omega| < M\}}$ and $\omega_2 = \omega - \omega_1$, as in [15], one can easily deal also with the general case.

In the first term we have that the inner part of the integral in dy vanishes, hence we can estimate the double integral as follows

$$\begin{aligned} \int_{\mathbb{C}\Omega_\lambda} |\omega(x)|^2 P.V. \frac{3}{4\pi} \int_{\mathbb{R}^3} (1 - \rho(|y|)) F(x, y) dy dx \\ \leq \frac{3}{4\pi} \sqrt{\frac{2}{\lambda}} \int_{\mathbb{R}^3} |\omega(x)|^2 \int_{\mathbb{R}^3} \frac{|\omega(x+y)|}{|y|^{5/2}} dy dx. \end{aligned}$$

We use now the Hardy-Littlewood-Sobolev inequality to infer

$$\left\| \int_{\mathbb{R}^3} |\omega(x+y)| \frac{dy}{|y|^{5/2}} \right\|_{L^3} \leq c_{HLS} \|\omega\|_{L^2}, \quad \text{with } c_{HLS} = 2^{7/3} \pi^{7/6}.$$

By the above formula and the Hölder inequality in $\mathbb{C}\Omega_\lambda$ with exponents $3/2$ and 3 , with the usual interpolation and Young’s inequality, we get

$$\begin{aligned} \int_{\mathbb{C}\Omega_\lambda} |\omega(x)|^2 P.V. \frac{3}{4\pi} \int_{\mathbb{R}^3} (1 - \rho(|y|)) F(dx, dy) dx dy &\leq \frac{C}{\sqrt{\lambda}} \|\omega\|_{3,\Omega}^2 \|\omega\|_2 \\ &\leq \frac{C}{\lambda} \|\omega\|_2 \|\nabla \omega\|_2 + \|\omega\|^2. \end{aligned}$$

The other term is the one such that the singular integral is not truncated by the cut-off function but we can use on it the smallness condition on the angle to show

$$\begin{aligned} \int_{\mathbb{C}\Omega_\lambda} |\omega(x)|^2 P.V. \frac{3}{4\pi} \int_{\mathbb{R}^3} \rho(|y|) F(dx, dy) dx dy &\leq \epsilon_0 \|\omega\|_{3,\Omega_\lambda}^2 \|\omega\|_2 \\ &\leq \epsilon_0^2 C \|\omega\|_2 \|\nabla \omega\|_2 + \|\omega\|^2. \end{aligned}$$

Next, we consider the integral over the set Ω_λ , where do not have control on the direction of the vorticity. First we use the Hölder inequality and the norm of $S[\omega](x)$ is a proper singular integral on the whole space, this proves the following inequality

$$\left| \int_{\Omega_\lambda} S[\omega](x) \omega(x) \cdot \omega(x) dx \right| \leq C \|\omega\|_{3,\Omega_\lambda}^2 \|\omega\|_{3,\mathbb{R}^3},$$

and the constant C is independent of the data of the problem (depends only on the space dimension). Next, we use the usual Sobolev and convex interpolation inequalities (with constants independent of the solution) in the whole space and the Hölder inequality with exponents $3/2$, 12 , and 4 to prove

$$\begin{aligned} \left| \int_{\Omega_\lambda} S[\omega](x) \omega(x) \cdot \omega(x) dx \right| &\leq C \|\omega\|_{3,\Omega_\lambda}^2 \|\omega\|_{2,\mathbb{R}^3}^{1/3} \|\omega\|_{2,\mathbb{R}^3}^{1/6} \|\nabla \omega\|_{2,\mathbb{R}^3}^{1/2} \\ &\leq C v^{-\frac{3}{8}} \|\omega\|_{3,\Omega_\lambda}^3 \|\omega\|_{2,\mathbb{R}^3}^{1/2} + \|\omega\|_{2,\mathbb{R}^3}^2 + \frac{v}{2} \|\nabla \omega\|_{2,\mathbb{R}^3}^2. \end{aligned}$$

Hence, by collecting all the inequalities, we get

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \left[\frac{v}{2} - (\epsilon_0^2 + \frac{1}{\lambda}) \right] \|\omega\|_2^{4/3} \|\nabla \omega\|_2^2 + (C_{S,\omega} - C v^{-\frac{3}{8}} \|\omega\|_2^{1/2}) \|\omega\|_{3,\Omega_\lambda}^3 \leq \|\omega\|_2^2.$$

This shows that by an appropriate choice of the constant $C_{S,\omega}$ large enough, then the following *a priori* estimate holds true

$$\omega \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)),$$

provided ϵ_0 is chosen small enough and λ large enough. □

Remark 4.5. *In the definition of the set Ω_λ there are two parameters, one related with the relative non-alignment of the vorticity and the other with the size of the set where the condition has to be checked. In some sense both depend on the viscosity, since the a priori estimate requires the smallness of ϵ_0 and the largeness of λ in order to have non negative terms. In this way we quantitatively link the amount of damping on the vorticity with the size of the set where this has to be fulfilled. In practice, controlling the vorticity direction in a small set imposes as counterpart to have larger constants for the dissipative term.*

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