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Luigi C. Berselli[1], Argus Adrian Dunca[2], Roger Lewandowski[3], and Dinh Duong Nguyen[4]

[1] Università di Pisa, Dipartimento di Matematica, Via Buonarroti 1/c, I-56127 Pisa, Italy, E-mail: luigi.carlo.berselli@unipi.it
[2]Department of Mathematics and Informatics, University Politehnica of Bucharest, Romania, E-mail: argus_dunca@yahoo.com
[3,4]IRMAR, UMR CNRS 6625, University of Rennes 1 and FLUMINANCE Team, INRIA Rennes, France, E-mail: Roger.Lewandowski@univ-rennes1.fr, dinh-duong.nguyen@univ-rennes1.fr

Abstract

This paper is devoted to study the rate of convergence of the weak solutions $\mathbf{u}_\alpha$ of α-regularization models, namely the Leray-α, Navier-Stokes-α, modified Leray-α and simplified Bardina models, to the weak solution $\mathbf{u}$ of the Navier-Stokes equations in the two-dimensional case with subjecting to the periodic boundary conditions as the regularization parameter α goes to zero.

Key words : Rate of convergence, α-turbulence models, Navier-Stokes Equations.
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1 Introduction

In this work we study the rate of convergence of weak solutions of several α-models of turbulence to the weak solution of the Navier-Stokes equations (NSE for short) in the 2D periodic context. The motivation to study reduced order turbulence models is that according to the well-known K41 turbulence theory of Kolmogorov, at high Reynolds number the NSE cannot be resolved numerically because at high Reynolds number Re they require a huge number of DOF per unit volume, $\mathcal{O}(Re^{d^2/4})$ where $d = 2, 3$, [20, page 2], [29, page 4], which leads to a huge computational cost. The α-models under study herein are the Leray-α, Navier-Stokes-α, modified Leray-α and simplified Bardina models, which have been introduced and analyzed in [5, 6, 7, 12, 13, 17], [8] (Leray-α), (Navier-Stokes-α), [18] (modified Leray-α) and [3, 22, 23] (simplified Bardina).

In the sequel $\mathbf{u}(t,\mathbf{x})$ and $p(t,\mathbf{x})$ for $t > 0$, $\mathbf{x} \in \Omega$ denote the velocity and the pressure of the fluid, respectively, which satisfy the NSE, i.e.,

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \tag{1.1}$$
$$\nabla\cdot\mathbf{u} = 0, \tag{1.2}$$
$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \tag{1.3}$$

where the constant $\nu > 0$ denotes the kinematic viscosity, $\mathbf{u}_0$ and $\mathbf{f}$ are given as the initial velocity and the external forces. The four α-models of turbulence can be obtained by replacing the nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ by $N(\mathbf{u}_\alpha)$ in (1.1), i.e.,

$$\partial_t \mathbf{u}_\alpha + N(\mathbf{u}_\alpha) - \nu\Delta\mathbf{u}_\alpha + \nabla p_\alpha = \mathbf{f}, \tag{1.4}$$

$$\nabla\cdot\mathbf{u}_\alpha = 0, \tag{1.5}$$

$$\mathbf{u}_\alpha|_{t=0} = \mathbf{u}_0, \tag{1.6}$$

where both the initial velocity $\mathbf{u}_0$ and the body forces $\mathbf{f}$ have been considered as in the NSE (1.1)-(1.3). For instance, in the Leray-α model the operator N is given by

$$N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha \quad \text{where} \quad \mathbf{u}_\alpha = \overline{\mathbf{u}}_\alpha - \alpha^2\Delta\overline{\mathbf{u}}_\alpha$$

for some $\alpha \geq 0$. The nonlinear operator N for each α-model will be defined in the next section.

The convergence of weak solutions of the above α-models as α goes to zero with assuming various regularity of the problem data has been investigated in several papers, see [1, 2, 4, 9, 11, 13]. The most recent results available in the literature will be summarized bellow.

In the above setting Cao and Titi proved in [2, Theorems 4.4-4.7] that in the 2D periodic case for all α-models there holds

$$\sup_{t\in[0,T]} ||(\mathbf{u}-\overline{\mathbf{u}}_\alpha)(t)||^2 \leq C\alpha^2\left(CT\left(1+\log\left(\frac{L}{2\pi\alpha}\right)\right)+C\right) \quad \forall\alpha\leq L/2\pi, \tag{1.7}$$

where C is a constant and $\overline{\mathbf{u}}_\alpha$ is the unique solution of (2.5) subject to periodic boundary conditions. Here T denotes the final time. The logarithmic factor appears in (1.7) following an application of the Brezis-Gallouet inequality. Their result is obtained under the regularity condition $\mathbf{u}_0 \in \mathcal{D}(A)$ and $\mathbf{f}\in\mathbf{L}^2(0,T;\mathbf{H})$, see the notation in the next section. We emphasize here a significant difference between their anaylsis and ours in that their α-models of turbulence assumes the initial condition as

$$\overline{\mathbf{u}}_\alpha(0,\cdot) = \mathbf{u}_0 \quad \text{and not} \quad \mathbf{u}_\alpha(0,\cdot) = \mathbf{u}_0,$$

as is in our analysis below or in [9].

Another result concerning the rate of convergence under study has been obtained in the 3D case [4] where the authors show that the error $\mathbf{e} = \mathbf{u}-\mathbf{u}_\alpha$ is bounded by, see [4, Theorem 5.1],

$$\int_0^T ||\mathbf{e}||\,dt \leq C(T)\alpha. \tag{1.8}$$

Their analysis is carried out in the 3D periodic setting and assumes a small data condition (in which the existence and uniqueness of weak solutions $\mathbf{u}$ of the 3D NSE is ensured). Here $\mathbf{u}$ and $\mathbf{u}_\alpha$ are the weak solutions of the NSE and Navier-Stokes-α, respectively, with periodic boundary conditions. The norm $||\cdot||$ always be denoted as the $\mathbf{L}^2(\Omega)$-norm (or $\mathbf{H}$-norm) throughout this paper, see the definitions in the next section.

Another result concerning the convergence rate of α-models of turbulence has been obtained in [9, Theorems 3.1 and 3.6] (both for 2D and 3D) where it's proved that

$$\sup_{t\in[0,T]} ||\mathbf{e}(t)||^2 + \int_0^T ||\nabla\mathbf{e}||^2\,dt \leq C(T)\alpha^2. \tag{1.9}$$

The result is obtained with $\mathbf{u}_\alpha(0,\cdot) = \mathbf{u}_0 \in \mathbf{V}, \mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{H})$ and under an extra assumption that the weak solution of the 3D NSE $\mathbf{u} \in \mathbf{L}^4(0,T;\mathbf{H}^1(\Omega))$. The latter condition ensures that the existence and uniqueness of weak solutions are established, see Section 6 for more details. The logarithmic term in (1.7) is removed in his results for both 2D and 3D periodic cases.

One common feature of the above three papers is that the convergence rate is determined on a finite time interval and the constant $C(T)$ that appears in the final estimates (1.7, 1.8, 1.9) depends on the final time T. In this report we are interested in obtaining estimates like (1.7, 1.8, 1.9) but uniform in $t \in \mathbb{R}_+$. Our analysis requires the usual regularity of the data $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$. Then there holds for all α-models herein,

$$||\mathbf{e}(s)||^2 + \nu \int_0^s ||\nabla \mathbf{e}||^2 \, dt \leq C\alpha^3 \quad \forall s \geq 0, \tag{1.10}$$

where C is a time-independent constant, see Theorem 4.1. There holds the higher order estimates for all $s \geq 0$

$$||\nabla \mathbf{e}(s)||^2 + \nu \int_0^s ||\Delta \mathbf{e}||^2 \, dt \leq \begin{cases} C\alpha^2, & (a) \\ C\alpha^2 \left(C \log\left(\frac{L}{2\pi\alpha}\right) + C \right). & (b) \end{cases} \tag{1.11}$$

The constants C in (1.10)-(1.11) do not depend on time. The inequalities (1.11)-(a) and (1.11)-(b) hold, respectively, for Leray-α, Navier-Stokes-α and modified Leray-α, simplified Bardina models, see Theorem 4.2. These are the main results in the present work. Somehow, we improve the mentioned results above.

Thanks to (1.10)-(1.11) we are able to study the rates of convergence of the pressure which is represented by Theorem 5.1 in Section 5. It is proved that for $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$:

$$\int_0^s ||\nabla q||^2 \, dt \leq \begin{cases} C\alpha^{5/2} & \text{for L-}\alpha \text{ and NS-}\alpha \text{ models,} \\ C\alpha^2 \left(C \log\left(\frac{L}{2\pi\alpha}\right) + C \right) & \text{for ML-}\alpha \text{ and SB models,} \end{cases} \tag{1.12}$$

where C do not depend on times and q denotes the difference between p and p_α the corresponding pressures of the NSE and all α-models, respectively.

The story is totally different in the 3D case which will be mentioned in Section 6. In this case it is well-known that the uniqueness of Leray-Hopf weak solutions is unknown so far as well as the existence of global strong solution. The singularity might occur in finite times.

Plan of the paper. The paper is organized as follows: In Section 2 we recall the mathematical context and notations which are used throughout the paper. Then in Section 3 uniform in time energy estimates are established for the weak solutions of the NSE and for all α-models as well. This is the main step before investigating the rate of convergence in Section 4, where it is provided that the error is uniformly bounded in some suitable norms in terms of the parameter α. The rate of convergence corresponding to the pressure is also studied in Section 5. In Section 6, we say some words about the 3D case. The paper is ended by the conclusions in Section 7 and the Appendix in Section 8.

2 Mathematical context

For a real $L > 0$, $\Omega = [0, L]^2$ will denote the 2D periodic domain. In our analysis, for $1 \le p \le \infty$, $m \in \mathbb{N}$, $\mathbf{L}^p(\Omega)$ and $\mathbf{H}^m(\Omega)$ will be the standard Lebesgue and Sobolev spaces on Ω, respectively. Moreover, the $\mathbf{L}^p(\Omega)$-norm is denoted by $||\cdot||_p$ for all $1 \le p \le \infty$, expect for the case $p = 2$ where $||\cdot|| \equiv ||\cdot||_2$. The bold symbols are used for vectors, matrices, or space of vectors. We also denote Π, the set of all trigonometric polynomials of two variables periodic on Ω with spatial zero mean, i.e.,

$$\int_\Omega \phi(\mathbf{x})\, d\mathbf{x} = 0, \quad \forall \phi \in \Pi.$$

Let us define

$$\mathbf{\Lambda} := \left\{ \boldsymbol{\varphi} \in \Pi^2 \, : \, \nabla \cdot \boldsymbol{\varphi} = 0 \right\}.$$

As usual when studying the NSE we define the following standard Hilbert functional spaces

$$\mathbf{H} := \text{the closure of } \mathbf{\Lambda} \text{ in } \mathbf{L}^2(\Omega),$$
$$\mathbf{V} := \text{the closure of } \mathbf{\Lambda} \text{ in } \mathbf{H}^1(\Omega).$$

Let $(\cdot,\cdot)$ and $||\cdot||$ be the standard inner product and norm on $\mathbf{H}$, that are

$$(\mathbf{u}, \mathbf{v}) := \int_\Omega \mathbf{u} \cdot \mathbf{v}\, d\mathbf{x} \quad \text{and} \quad ||\mathbf{u}||^2 := \int_\Omega |\mathbf{u}|^2\, d\mathbf{x}.$$

The inner product $(\mathbf{u}, \mathbf{v})_{\mathbf{V}}$ and the corresponding norm $||\mathbf{u}||_{\mathbf{V}}$ on $\mathbf{V}$ are defined as follow

$$(\mathbf{u}, \mathbf{v})_{\mathbf{V}} := (\nabla \mathbf{u}, \nabla \mathbf{v}) \quad \text{and} \quad ||\mathbf{u}||_{\mathbf{V}} := ||\nabla \mathbf{u}||.$$

In the sequel, we use the notation P_σ for denoting the Helmholtz-Leray orthogonal projection operator of $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$. We next consider an orthonormal basis $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \dots, \boldsymbol{\varphi}_n, \dots$ of $\mathbf{H}$ consisting of eigenfunctions of the the Laplace operator

$$-\Delta \, : \, \mathbf{H}^2(\Omega) \cap \mathbf{V} \longrightarrow \mathbf{H}$$

and for $m \ge 1$, $\mathbf{H}_m = \text{span}\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \dots, \boldsymbol{\varphi}_m\}$ denotes the finite dimensional space.

It is well-known that in the periodic boundary conditions $A = -P_\sigma \Delta$ the Stokes operator with its domain $\mathcal{D}(A) := \mathbf{H}^2(\Omega) \cap \mathbf{V}$ satisfies [2, 13]:

$$A\mathbf{u} = -P_\sigma \Delta \mathbf{u} = -\Delta \mathbf{u} \quad \forall \mathbf{u} \in \mathcal{D}(A). \tag{2.1}$$

Let $\lambda_1 > 0$ be the first eigenvalue of A, i.e., $A\boldsymbol{\varphi}_1 = \lambda_1 \boldsymbol{\varphi}_1$, and the above setting leads to $\lambda_1 = (2\pi/L)^2$. By the virtue of the Poincaré inequality we have

$$\lambda_1 ||\mathbf{u}||^2 \le ||\nabla \mathbf{u}||^2 \quad \forall \mathbf{u} \in V, \tag{2.2}$$

$$\lambda_1 ||\nabla \mathbf{u}||^2 \le ||A\mathbf{u}||^2 = ||\Delta \mathbf{u}||^2 \quad \forall \mathbf{u} \in \mathcal{D}(A). \tag{2.3}$$

Then it follows by (2.2)-(2.3) that there exist positive dimensionless constants c_1, c_2 such that

$$c_1 ||A\mathbf{u}|| \le ||\mathbf{u}||_{\mathbf{H}^2(\Omega)} \le c_2 ||A\mathbf{u}|| \quad \forall \mathbf{u} \in \mathcal{D}(A). \tag{2.4}$$

The filter operator used to construct the turbulence models is the Helmholtz filter, see Germano [15], or [1, 10, 24]. Given a parameter $\alpha > 0$ (which will be called the filter radius), for each $\mathbf{u} \in \mathbf{H}$, its mean (or filter) $\overline{\mathbf{u}} \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$ is the unique solution of the following Helmholtz equation:

$$\overline{\mathbf{u}} - \alpha^2 \Delta \overline{\mathbf{u}} = \mathbf{u} \quad \text{in } \Omega \tag{2.5}$$

with periodic boundary conditions. The filter (2.5) implies that

$$||\mathbf{u} - \overline{\mathbf{u}}|| = \alpha^2 ||\Delta \overline{\mathbf{u}}|| \quad \forall \mathbf{u} \in \mathbf{H}.$$

One can easily check that in the periodic context the filter satisfies the inequality, see [9, formula 2.5]:

$$||\overline{\mathbf{u}}|| + \alpha ||\nabla \overline{\mathbf{u}}|| + \alpha^2 ||\Delta \overline{\mathbf{u}}|| \leq C ||\mathbf{u}|| \quad \forall \mathbf{u} \in \mathbf{H}, \tag{2.6}$$

where C is a Sobolev constant. It follows that

$$||\nabla \mathbf{u} - \nabla \overline{\mathbf{u}}|| = \alpha^2 ||\nabla \Delta \overline{\mathbf{u}}|| \leq C\alpha ||\Delta \mathbf{u}|| \quad \forall \mathbf{u} \in \mathcal{D}(A). \tag{2.7}$$

As usual a pair $(\mathbf{u}, p)$ is considered as a solution of the NSE (1.1)-(1.3). Moreover, it is not difficult to recover the pressure p from the velocity $\mathbf{u}$. Then it is enough to seek for the velocity $\mathbf{u}$. In order to focus on the velocity, we must find a way which eliminates the pressure. There are several ways to do that and an usual one to get rid of the pressure term is to apply the Helmholtz-Leray orthogonal projection to both the NSE and α-models. Given $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in \mathbf{H}$, the NSE (1.1)-(1.3) is equivalent to the functional differential equation

$$\begin{aligned} \frac{d\mathbf{u}}{dt} + P_\sigma[(\mathbf{u} \cdot \nabla)\mathbf{u}] - \nu \Delta \mathbf{u} &= \mathbf{f}, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0, \end{aligned} \tag{2.8}$$

and the α-models (1.4)-(1.6) investigated herein are equivalent to

$$\begin{aligned} \frac{d\mathbf{u}_\alpha}{dt} + P_\sigma[N(\mathbf{u}_\alpha)] - \nu \Delta \mathbf{u}_\alpha &= \mathbf{f}, \\ \mathbf{u}_\alpha|_{t=0} &= \mathbf{u}_0, \end{aligned} \tag{2.9}$$

where one has used the facts that $P_\sigma \mathbf{f} \equiv \mathbf{f}$ since $\mathbf{f} \in \mathbf{H}$, $P_\sigma \Delta \mathbf{u} = \Delta \mathbf{u}$ and $P_\sigma(\nabla p) = P_\sigma(\nabla p_\alpha) = 0$.

Remark 2.1. *Thanks to the Leray-Helmholtz decomposition and for simplicity we assume that* $\mathbf{f}$ *is divergence free. Otherwise, the gradient part of* $\mathbf{f}$ *can be added to the modified pressure and* $P_\sigma \mathbf{f}$ *is replaced by* $\mathbf{f}$.

The nonlinear operator N is defined for each α-model as follows:

$$N(\mathbf{u}_\alpha) = \begin{cases} (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha & \text{in the Leray-}\alpha \text{ model (L-}\alpha\text{)}, \\ (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha & \text{in the modified Leray-}\alpha \text{ model (ML-}\alpha\text{)}, \\ (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha & \text{in the simplified Bardina model (SB)}, \\ -\overline{\mathbf{u}}_\alpha \times (\nabla \times \mathbf{u}_\alpha) & \text{in the Navier-Stokes-}\alpha \text{ model (NS-}\alpha\text{)}. \end{cases}$$

Remark 2.2. *A common property of all* α*-models which are considered in the present paper is that these models reduce to the NSE when* $\alpha = 0$*. It can be seen directly from the equality (2.5).*

The well-known 2D-Ladyzhenskaya inequality [21, Lemma 1 page 8] reads:

$$||\mathbf{u}||_4 \leq C||\mathbf{u}||^{1/2}||\nabla \mathbf{u}||^{1/2}, \tag{2.10}$$

where C is a non-negative dimensionless constant. The existence and uniqueness of solutions of the NSE and all α-models herein are summarized by the following remark:

Remark 2.3 (**Summary**). *We will give a quick review about the existence and uniqueness of solutions for the NSE and all α-models herein. In the 2D case, the solution of the NSE is known to be smooth for all time, see Temam [33, Theorem 3.2], [34]. In short, the proof for the existence and uniqueness of solution of four α-models in this report, with the periodic boundary conditions, which can be established by using the standard Galerkin method. It can be followed by the proof of the NSE, see Temam [32, 34]. The NS-α model, also known as the viscous Camassa-Holm or Lagrangian averaged NS-α model, which was introduced and studied in a series of papers, see [5, 6, 7, 12, 13, 17]. It is also the first one in the family of α-models, see Cao-Titi [2]. Later the L-α model was introduced and implemented computationally by Cheskidov-Holm-Olson-Titi [8]. It is known that this model was inspired by the celebrated Leray-1934's paper, see [26, page 206], where another filter is applied instead of (2.5) and also by the NS-α model. Followed by the two first α-models, Ilyin-Lunasin-Titi introduced and studied the ML-α model in the 3D periodic case, see [18]. It is also tested numerically in [16]. However, the global existence and uniqueness for 2D can be proved in the similar way. For the last α-model in this report, the Bardina closure model of turbulence was firstly introduced by Bardina-Ferziger-Reynolds in [19]. Then it is simplified, i.e., the SB model, which was firstly introduced and studied by Layton-Lewandowski in [22, 23] and then by Cao-Lunasin-Titi in [3]. Layton and Lewandowski provided the well-posedness for the 3D case subject to periodic boundary conditions and then the results were improved by Cao and his colleagues where the condition for the initial data has been relaxed.*

3 A priori estimates

Before going to estimate the error between the α-models and the NSE, we need some bounds on the solutions which are given by following lemmas. Although these are standard we give all the proofs for completeness. Notice that all estimates require the initial data $\mathbf{u}_0 \in \mathbf{V}$ instead of $\mathbf{u}_0 \in \mathcal{D}(A)$ as in Cao-Titi [2]. For simplicity, we denote $\mathcal{F} := ||\mathbf{f}||^2_{\mathbf{L}^2(\mathbb{R}_+;\mathbf{H})}$ and C is stand for a non-negative dimensionless constant.

Remark 3.1 (**Existence and uniqueness**). *As mentioned in Remark 2.3 the NSE and all α-models herein admit a unique weak solution in the classical class $\mathbf{L}^\infty(\mathbb{R}_+;\mathbf{H}) \cap \mathbf{L}^2(\mathbb{R}_+;\mathbf{V})$ in the case $\mathbf{u}_0 \in \mathbf{H}$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$.*

Lemma 3.1 (**NSE**). *Let $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$. Then the unique weak solution $\mathbf{u}$ of the NSE satisfies*

$$||\mathbf{u}(s)||^2 + \nu \int_0^s ||\nabla \mathbf{u}||^2\, dt \leq ||\mathbf{u}_0||^2 + \frac{\mathcal{F}}{\nu\lambda_1} =: C_{NSE1} \quad \forall s \geq 0, \tag{3.1}$$

and

$$||\nabla \mathbf{u}(s)||^2 + \nu \int_0^s ||\Delta \mathbf{u}||^2\, dt \leq ||\nabla \mathbf{u}_0||^2 + \frac{\mathcal{F}}{\nu} =: C_{NSE2} \quad \forall s \geq 0. \tag{3.2}$$

Remark 3.2. *Estimate (3.1) in the previous theorem can be obtained more generally requiring* $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{V}')$ *where as usual* $\mathbf{V}'$ *denotes the dual space of* $\mathbf{V}$. *We use the condition* $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$ *for both estimates (3.1) and (3.2) for conciseness.*

Proof. Take the scalar product of the NSE (2.8) with $\mathbf{u}$ and use the identity $(P_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}],\mathbf{u}) = 0$, which lead to the following estimate

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}||^2 + \nu||\nabla\mathbf{u}||^2 \leq ||\mathbf{f}||||\mathbf{u}||. \tag{3.3}$$

Using Poincaré and Young inequalities on the r.h.s (right-hand side) of (3.3) yields:

$$\frac{d}{dt}||\mathbf{u}||^2 + \nu||\nabla\mathbf{u}||^2 \leq \frac{1}{\nu\lambda_1}||\mathbf{f}||^2. \tag{3.4}$$

Integrating (3.4) on $[0,s]$ for $s \geq 0$, one has

$$||\mathbf{u}(s)||^2 + \nu\int_0^s ||\nabla\mathbf{u}||^2\,dt \leq ||\mathbf{u}_0||^2 + \frac{1}{\nu\lambda_1}\int_0^s ||\mathbf{f}||^2\,dt. \tag{3.5}$$

Finally, the estimate (3.1) follows by (3.5) since s can be chosen arbitrary. In order to prove the other estimate (3.2), instead of $\mathbf{u}$, we take $-\Delta\mathbf{u}$ as a test in the NSE (2.8). In the 2D case periodic the nonlinear term vanishes, see [32, 33, Lemma 3.1], i.e.,

$$(P_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}], -\Delta\mathbf{u}) = 0. \tag{3.6}$$

By the Young inequality the term corresponding to the body forces can be estimated by

$$(\mathbf{f}, -\Delta\mathbf{u}) \leq \frac{\nu}{2}||\mathbf{f}||^2 + \frac{\nu}{2}||\Delta\mathbf{u}||^2. \tag{3.7}$$

Therefore, the rest of the proof follows as the same as what have been done above. Thus, the proof is complete. □

Lemma 3.2 (L-α). *Let* $\mathbf{u}_0 \in \mathbf{V}$ *and* $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$. *Then the unique weak solution* $\mathbf{u}_\alpha$ *of the L-α satisfies* $\forall s \geq 0$

$$||\nabla\mathbf{u}_\alpha(s)||^2 + \nu\int_0^s ||\Delta\mathbf{u}_\alpha||^2\,dt \leq \frac{CC_{L1}^2}{\nu^4}\left(||\mathbf{u}_0||^2 + \frac{\mathcal{F}}{\nu\lambda_1}\right) + \frac{2\mathcal{F}}{\nu} =: C_L, \tag{3.8}$$

where C_{L1} *is given in (3.13).*

Proof. For the L-α model, the nonlinear term is given by

$$N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha \quad \text{where} \quad \overline{\mathbf{u}}_\alpha - \alpha^2\Delta\overline{\mathbf{u}}_\alpha = \mathbf{u}_\alpha.$$

Taking $\mathbf{u}_\alpha$ as a test in the L-α model (2.9) gives

$$\frac{d}{dt}||\mathbf{u}_\alpha||^2 + \nu||\nabla\mathbf{u}_\alpha||^2 \leq \frac{1}{\nu\lambda_1}||\mathbf{f}||^2. \tag{3.9}$$

Here $(P_\sigma[(\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha],\mathbf{u}_\alpha) = 0$ has been used, see [2, formula 2.16]. It leads to for all $s \geq 0$

$$||\mathbf{u}_\alpha(s)||^2 + \nu\int_0^s ||\nabla\mathbf{u}_\alpha||^2\,dt \leq ||\mathbf{u}_0||^2 + \frac{\mathcal{F}}{\nu\lambda_1}. \tag{3.10}$$

Testing (2.9) by $-\Delta\overline{\mathbf{u}}_\alpha$ and replacing $\mathbf{u}_\alpha$ by $\overline{\mathbf{u}}_\alpha - \alpha^2\Delta\overline{\mathbf{u}}_\alpha$ yield

$$\frac{d}{dt}\left(||\nabla\overline{\mathbf{u}}_\alpha||^2 + \alpha^2||\Delta\overline{\mathbf{u}}_\alpha||^2\right) + \nu||\Delta\overline{\mathbf{u}}_\alpha||^2 + \nu\alpha^2||\nabla\Delta\overline{\mathbf{u}}_\alpha||^2 \leq \frac{||\mathbf{f}||^2}{\nu}. \tag{3.11}$$

Here the vanishing of the nonlinear term has been used, i.e.,

$$(P_\sigma[(\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha], -\Delta\overline{\mathbf{u}}_\alpha) = ((\overline{\mathbf{u}}_\alpha\cdot\nabla)(\overline{\mathbf{u}}_\alpha - \alpha^2\Delta\overline{\mathbf{u}}_\alpha), -\Delta\overline{\mathbf{u}}_\alpha) = 0.$$

Therefore, by (3.11) for all $s \geq 0$

$$||\nabla\overline{\mathbf{u}}_\alpha(s)||^2 + \alpha^2||\Delta\overline{\mathbf{u}}_\alpha(s)||^2 + \nu\int_0^s \left(||\Delta\overline{\mathbf{u}}_\alpha||^2 + \alpha^2||\nabla\Delta\overline{\mathbf{u}}_\alpha||^2\right) dt \leq C_{L1}, \tag{3.12}$$

where C_{L1} is given by

$$||\nabla\overline{\mathbf{u}}_0||^2 + \alpha^2||\Delta\overline{\mathbf{u}}_0||^2 + \frac{\mathcal{F}^2}{\nu} \leq (1+\lambda_1)||\nabla\mathbf{u}_0||^2 + \frac{\mathcal{F}^2}{\nu} =: C_{L1}, \tag{3.13}$$

here the facts $||\nabla\overline{\mathbf{u}}_0|| \leq ||\nabla\mathbf{u}_0||$, $\alpha^2||\Delta\overline{\mathbf{u}}_0||^2 \leq ||\mathbf{u}_0||^2$ given by (2.6) and the Poincaré inequality have been applied. We test (2.9) again by $-\Delta\mathbf{u}_\alpha$ which leads to

$$\frac{1}{2}\frac{d}{dt}||\nabla\mathbf{u}_\alpha||^2 + \nu||\Delta\mathbf{u}_\alpha||^2 = (P_\sigma[(\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha], \Delta\mathbf{u}_\alpha) + (\mathbf{f}, -\Delta\mathbf{u}_\alpha). \tag{3.14}$$

The first term on the r.h.s of (3.14) can be estimated by:

$$\begin{aligned}(P_\sigma[(\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha], \Delta\mathbf{u}_\alpha) &\leq C||\overline{\mathbf{u}}_\alpha||_4||\nabla\mathbf{u}_\alpha||_4||\Delta\mathbf{u}_\alpha|| \\ &\leq C||\nabla\overline{\mathbf{u}}_\alpha||||\nabla\mathbf{u}_\alpha||^{1/2}||\Delta\mathbf{u}_\alpha||^{3/2} \\ &\leq \frac{C}{\nu^3}||\nabla\overline{\mathbf{u}}_\alpha||^4||\nabla\mathbf{u}_\alpha||^2 + \frac{\nu}{4}||\Delta\mathbf{u}_\alpha||^2. \end{aligned} \tag{3.15}$$

Here one has used the Hölder, 2D-Ladyzhenskaya, Sobolev and Young inequalities, respectively. From (3.14)-(3.15) one obtains

$$\frac{d}{dt}||\nabla\mathbf{u}_\alpha||^2 + \nu||\Delta\mathbf{u}_\alpha||^2 \leq \frac{2}{\nu}||\mathbf{f}||^2 + \frac{C}{\nu^3}||\nabla\overline{\mathbf{u}}_\alpha||^4||\nabla\mathbf{u}_\alpha||^2. \tag{3.16}$$

The previous estimate yields for all $s \geq 0$

$$||\nabla\mathbf{u}_\alpha(s)||^2 + \nu\int_0^s ||\Delta\mathbf{u}_\alpha||^2\, dt \leq \frac{2\mathcal{F}}{\nu} + \frac{C}{\nu^3}\int_0^s ||\nabla\overline{\mathbf{u}}_\alpha||^4||\nabla\mathbf{u}_\alpha||^2\, dt. \tag{3.17}$$

Finally, both estimates (3.10) and (3.12) are applied in (3.17) to get (3.8) and end the proof. □

Lemma 3.3 (**NS-α**). *Let $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$. Then the unique weak solution $\mathbf{u}_\alpha$ of the NS-α satisfies*

$$||\nabla\mathbf{u}_\alpha(s)||^2 + \nu\int_0^s ||\Delta\mathbf{u}_\alpha||^2\, dt \leq ||\nabla\mathbf{u}_0||^2 + \frac{\mathcal{F}}{\nu} =: C_{NS_a} \quad \forall s \geq 0. \tag{3.18}$$

Proof. The nonlinear term of this model is given by $N(\mathbf{u}_\alpha) = -\overline{\mathbf{u}}_\alpha \times (\nabla\times\mathbf{u}_\alpha)$. Taking $-\Delta\mathbf{u}_\alpha$ as a test in (2.9) yields

$$\frac{1}{2}\frac{d}{dt}||\nabla\mathbf{u}_\alpha||^2 + \nu||\Delta\mathbf{u}_\alpha||^2 = (P_\sigma[-\overline{\mathbf{u}}_\alpha\times(\nabla\times\mathbf{u}_\alpha)], \Delta\mathbf{u}_\alpha) - (\mathbf{f}, \Delta\mathbf{u}_\alpha). \tag{3.19}$$

Thanks to the equality (8.4) in Lemma 8.1, the first term on the r.h.s of (3.19) disappears, i.e.,

$$(P_\sigma[-\overline{\mathbf{u}}_\alpha \times (\nabla \times \mathbf{u}_\alpha)], \Delta \mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \times (\nabla \times \mathbf{u}_\alpha), -\Delta \mathbf{u}_\alpha) = 0.$$

Then the rest of the proof is complete as the same way as in the NSE case, see again Lemma 3.1. □

Lemma 3.4 (**SB**)**.** *Let* $\mathbf{u}_0 \in \mathbf{V}$ *and* $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$*. Then the unique weak solution* $\mathbf{u}_\alpha$ *of the SB model satisfies*

$$||\nabla \mathbf{u}_\alpha(s)||^2 + \nu \int_0^s ||\Delta \mathbf{u}_\alpha||^2 \, dt \le \frac{CC_S^2}{\nu^2 \lambda_1} + \frac{2\mathcal{F}}{\nu} =: C_{SB} \quad \forall s \ge 0, \tag{3.20}$$

where C *is a positive constant and* C_S *is given by (3.22).*

Proof. For this model, the nonlinear term is given by $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha$. Taking $-\Delta \overline{\mathbf{u}}_\alpha$ as a test in (2.9) and using the fact $\mathbf{u}_\alpha = \overline{\mathbf{u}}_\alpha - \alpha^2 \Delta \overline{\mathbf{u}}_\alpha$ give us

$$\frac{d}{dt}\left(||\nabla \overline{\mathbf{u}}_\alpha||^2 + \alpha^2 ||\Delta \overline{\mathbf{u}}_\alpha||^2\right) + \nu ||\Delta \overline{\mathbf{u}}_\alpha||^2 + \nu \alpha^2 ||\nabla \Delta \overline{\mathbf{u}}_\alpha||^2 \le \frac{1}{\nu}||\mathbf{f}||^2, \tag{3.21}$$

where the identity $(P_\sigma[(\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], -\Delta \overline{\mathbf{u}}_\alpha) = 0$ has been used. Thus, for all $s \ge 0$

$$||\nabla \overline{\mathbf{u}}_\alpha(s)||^2 + \alpha^2 ||\Delta \overline{\mathbf{u}}_\alpha(s)||^2 + \nu \int_0^s \left(||\Delta \overline{\mathbf{u}}_\alpha||^2 + \alpha^2 ||\nabla \Delta \overline{\mathbf{u}}_\alpha||^2\right) dt \le C_S, \tag{3.22}$$

where $C_S := C_{L1}$ as given in (3.13). Then one takes $-\Delta \mathbf{u}_\alpha$ as a test in (2.9) to obtain

$$\frac{1}{2}\frac{d}{dt}||\nabla \mathbf{u}_\alpha||^2 + \nu ||\Delta \mathbf{u}_\alpha||^2 = (P_\sigma[(\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \Delta \mathbf{u}_\alpha) - (\mathbf{f}, \Delta \mathbf{u}_\alpha). \tag{3.23}$$

The nonlinear term on the r.h.s of (3.23) is estimated by:

$$\begin{aligned}
(P_\sigma[(\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \Delta \mathbf{u}_\alpha) &\le C ||\overline{\mathbf{u}}_\alpha||_4 ||\nabla \overline{\mathbf{u}}_\alpha||_4 ||\Delta \mathbf{u}_\alpha|| \\
&\le C ||\overline{\mathbf{u}}_\alpha||^{1/2} ||\nabla \overline{\mathbf{u}}_\alpha|| ||\Delta \overline{\mathbf{u}}_\alpha||^{1/2} ||\Delta \mathbf{u}_\alpha|| \\
&\le \frac{C}{\nu} ||\overline{\mathbf{u}}_\alpha|| ||\nabla \overline{\mathbf{u}}_\alpha||^2 ||\Delta \overline{\mathbf{u}}_\alpha|| + \frac{\nu}{4} ||\Delta \mathbf{u}_\alpha||^2.
\end{aligned} \tag{3.24}$$

In the above inequalities the Hölder, 2D-Ladyzhenskaya and Young inequalities have been applied, respectively. The estimates (3.23)-(3.24) lead to

$$\frac{d}{dt}||\nabla \mathbf{u}_\alpha||^2 + \nu ||\Delta \mathbf{u}_\alpha||^2 \le \frac{2}{\nu}||\mathbf{f}||^2 + \frac{C}{\nu} ||\overline{\mathbf{u}}_\alpha|| ||\nabla \overline{\mathbf{u}}_\alpha||^2 ||\Delta \overline{\mathbf{u}}_\alpha||. \tag{3.25}$$

and by (3.22) for all $s \ge 0$

$$\begin{aligned}
||\nabla \mathbf{u}_\alpha(s)||^2 + \nu \int_0^s ||\Delta \mathbf{u}_\alpha||^2 \, dt &\le \frac{2\mathcal{F}}{\nu} + \frac{C}{\nu} \int_0^s ||\overline{\mathbf{u}}_\alpha|| ||\nabla \overline{\mathbf{u}}_\alpha||^2 ||\Delta \overline{\mathbf{u}}_\alpha|| \, dt \\
&\le \frac{2\mathcal{F}}{\nu} + \frac{CC_{SB}}{\nu \lambda_1} \int_0^s ||\Delta \overline{\mathbf{u}}_\alpha||^2 \, dt \\
&\le \frac{2\mathcal{F}}{\nu} + \frac{CC_{SB}^2}{\nu^2 \lambda_1}.
\end{aligned} \tag{3.26}$$

Therefore, the proof is complete. □

Lemma 3.5 (**ML-α**). *Let $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$. Then the unique weak solution $\mathbf{u}_\alpha$ of the ML-α model satisfies*

$$(3.27) \qquad ||\nabla \mathbf{u}_\alpha(t)||^2 + \nu \int_0^s ||\Delta \mathbf{u}_\alpha||^2 \, dt \le \frac{C_{ML4}}{\nu^4} + \frac{2\mathcal{F}}{\nu} =: C_{ML_a} \quad \forall s \ge 0,$$

where $C_{ML4} = C C_{ML1} C_{ML2} C_{ML3}$ with C is a positive constant and for $i = 1,2,3$, C_{MLi} are given by (3.30), (3.34) and (3.39), respectively.

Proof. The nonlinear term of this model is given by $N(\mathbf{u}_\alpha) = (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha$. Taking $\overline{\mathbf{u}}_\alpha$ as a test in (2.9) and replacing $\mathbf{u}_\alpha$ by $\overline{\mathbf{u}}_\alpha - \alpha^2 \Delta \overline{\mathbf{u}}_\alpha$ to obtain

$$(3.28) \qquad \frac{d}{dt}\left(||\overline{\mathbf{u}}_\alpha||^2 + \alpha^2 ||\nabla \overline{\mathbf{u}}_\alpha||^2\right) + \nu ||\nabla \overline{\mathbf{u}}_\alpha||^2 + \nu\alpha^2 ||\Delta \overline{\mathbf{u}}_\alpha||^2 \le \frac{1}{\nu\lambda_1}||\mathbf{f}||^2.$$

Here the fact $(P_\sigma[(\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha], \overline{\mathbf{u}}_\alpha) = 0$ and the Poincaré inequality have been used on the r.h.s. Then One gets from (3.28) for all $s \ge 0$

$$(3.29) \qquad ||\overline{\mathbf{u}}_\alpha(s)||^2 + \alpha^2 ||\nabla \overline{\mathbf{u}}_\alpha(s)||^2 + \nu \int_0^s \left(||\nabla \overline{\mathbf{u}}_\alpha||^2 + \alpha^2 ||\Delta \overline{\mathbf{u}}_\alpha||^2\right) ds \le C_{ML1},$$

where as in (3.13) above C_{ML1} is given by

$$(3.30) \qquad ||\overline{\mathbf{u}}_0||^2 + \alpha^2 ||\nabla \overline{\mathbf{u}}_0||^2 + \frac{\mathcal{F}^2}{\nu\lambda_1} \le (1+\lambda_1)||\mathbf{u}_0||^2 + \frac{\mathcal{F}^2}{\nu\lambda_1} =: C_{ML1}.$$

Taking $\mathbf{u}_\alpha$ as a test in (2.9) yields

$$(3.31) \qquad \frac{1}{2}\frac{d}{dt}||\mathbf{u}_\alpha||^2 + \nu ||\nabla \mathbf{u}_\alpha||^2 = -((\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, \mathbf{u}_\alpha) + (\mathbf{f}, \mathbf{u}_\alpha).$$

The nonlinear term on the r.h.s of (3.31) can be estimated by

$$\begin{aligned} ((\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, \mathbf{u}_\alpha) &\le C||\mathbf{u}_\alpha||_4^2 ||\nabla \overline{\mathbf{u}}_\alpha|| \\ &\le C||\mathbf{u}_\alpha|| ||\nabla \mathbf{u}_\alpha|| ||\nabla \overline{\mathbf{u}}_\alpha|| \\ &\le \frac{C}{\nu}||\mathbf{u}_\alpha||^2 ||\nabla \overline{\mathbf{u}}_\alpha||^2 + \frac{\nu}{4}||\nabla \mathbf{u}_\alpha||^2. \end{aligned}$$

Here we have used the Hölder, 2D-Ladyzhenskaya and Young inequalities, respectively. Using the Young inequality for the other term on the r.h.s of (3.31) gives

$$(3.32) \qquad \frac{d}{dt}||\mathbf{u}_\alpha||^2 + \nu ||\nabla \mathbf{u}_\alpha||^2 \le \frac{2}{\lambda_1 \nu}||\mathbf{f}||^2 + \frac{C}{\nu}||\mathbf{u}_\alpha||^2 ||\nabla \overline{\mathbf{u}}_\alpha||^2.$$

Using the estimate (3.29) for all $s \ge 0$ leads to

$$\begin{aligned} \int_0^s ||\mathbf{u}_\alpha||^2 ||\nabla \overline{\mathbf{u}}_\alpha||^2 \, dt &= \int_0^s \left(||\overline{\mathbf{u}}_\alpha||^2 + 2\alpha^2 ||\nabla \overline{\mathbf{u}}_\alpha||^2 + \alpha^4 ||\Delta \overline{\mathbf{u}}_\alpha||^2\right) ||\nabla \overline{\mathbf{u}}_\alpha||^2 \, dt \\ (3.33) \qquad &\le \frac{4C_{ML1}^2}{\nu}. \end{aligned}$$

Here one has used the following identity

$$||\mathbf{u}_\alpha||^2 = ||\overline{\mathbf{u}}_\alpha||^2 + 2\alpha^2 ||\nabla \overline{\mathbf{u}}_\alpha||^2 + \alpha^4 ||\Delta \overline{\mathbf{u}}_\alpha||^2.$$

Therefore, by (3.32)-(3.33) for all $s \geq 0$

$$||\mathbf{u}_\alpha(s)||^2 + \nu \int_0^s ||\nabla \mathbf{u}_\alpha||^2 \, dt \leq \frac{2\mathcal{F}}{\nu\lambda_1} + \frac{4CC_{ML1}^2}{\nu^2} =: C_{ML2}. \tag{3.34}$$

Next, we take $-\Delta\mathbf{u}_\alpha$ as a test in (2.9) to obtain

$$\frac{1}{2}\frac{d}{dt}||\nabla\mathbf{u}_\alpha||^2 + \nu||\Delta\mathbf{u}_\alpha||^2 = (P_\sigma[(\mathbf{u}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha], \Delta\mathbf{u}_\alpha) - (\mathbf{f}, \Delta\mathbf{u}_\alpha). \tag{3.35}$$

The nonlinear integral can be estimated by

$$\begin{aligned}(P_\sigma[(\mathbf{u}_\alpha \cdot \nabla)\bar{\mathbf{u}}_\alpha], \Delta\mathbf{u}_\alpha) &\leq C||\mathbf{u}_\alpha||_4||\nabla\bar{\mathbf{u}}_\alpha||_4||\Delta\mathbf{u}_\alpha||_2 \\ &\leq C||\mathbf{u}_\alpha||^{1/2}||\nabla\mathbf{u}_\alpha||^{1/2}||\nabla\bar{\mathbf{u}}_\alpha||^{1/2}||\Delta\mathbf{u}_\alpha||^{3/2} \\ &\leq \frac{C}{\nu^3}||\mathbf{u}_\alpha||^2||\nabla\mathbf{u}_\alpha||^2||\nabla\bar{\mathbf{u}}_\alpha||^2 + \frac{\nu}{4}||\Delta\mathbf{u}_\alpha||^2. \end{aligned} \tag{3.36}$$

Here one has used the Hölder, 2D-Ladyzhenskaya, Sobolev and Young inequalities, respectively. From (3.35)-(3.36) we obtain:

$$\frac{d}{dt}||\nabla\mathbf{u}_\alpha||^2 + \nu||\Delta\mathbf{u}_\alpha||^2 \leq \frac{C}{\nu^3}||\mathbf{u}_\alpha||^2||\nabla\bar{\mathbf{u}}_\alpha||^2||\nabla\mathbf{u}_\alpha||^2 + \frac{2}{\nu}||\mathbf{f}||^2 \tag{3.37}$$

and in particular

$$\frac{d}{dt}||\nabla\mathbf{u}_\alpha||^2 \leq \frac{CC_{ML2}}{\nu^3}||\nabla\bar{\mathbf{u}}_\alpha||^2||\nabla\mathbf{u}_\alpha||^2 + \frac{2}{\nu}||\mathbf{f}||^2. \tag{3.38}$$

Hence, by (3.38) for all $s \geq 0$

$$||\nabla\mathbf{u}_\alpha(s)||^2 \leq \left(||\nabla\mathbf{u}_0||^2 + \frac{2\mathcal{F}}{\nu}\right) \exp\left\{\frac{CC_{ML2}^2}{\nu^4}\right\} =: C_{ML3}. \tag{3.39}$$

Together (3.37) and (3.39) one obtains (3.27). Thus, the proof is complete for this model. □

4 The rate of convergence of $\mathbf{u}_\alpha$ to $\mathbf{u}$

In this section, we study the rate of convergence of the weak solutions $\mathbf{u}_\alpha$ of the four α-models to the weak solution $\mathbf{u}$ of the NSE in some suitable norms in terms of α as α tends to zero. For simplicity, throughout this section $\mathbf{e} = \mathbf{u} - \mathbf{u}_\alpha$ denotes the error between $\mathbf{u}$ and $\mathbf{u}_\alpha$ which are the weak solutions of the NSE (2.8) and α-models (2.9), respectively. The first main result in this section is given by the following theorem:

Theorem 4.1. *Let $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$. Then*

$$||\mathbf{e}(s)||^2 + \nu \int_0^s ||\nabla\mathbf{e}||^2 \, dt \leq C_r\alpha^3 \quad \forall s \geq 0, \tag{4.1}$$

where C_r is given by

$$\begin{cases} (4.9) & \text{for the L-}\alpha\text{ model,} \\ (4.10) & \text{for the ML-}\alpha\text{ model,} \\ (4.12) & \text{for the SB model,} \\ (4.20) & \text{for the NS-}\alpha\text{ model.} \end{cases}$$

Proof. We subtract (2.9) from (2.8) and by multiplying $\mathbf{e}$ and integrating the result reads

$$\frac{1}{2}\frac{d}{dt}||\mathbf{e}||^2 + \nu||\nabla\mathbf{e}||^2 = (-P_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] + P_\sigma[N(\mathbf{u}_\alpha)], \mathbf{e}). \tag{4.2}$$

We add and subtract on the r.h.s of (4.2) the term $((\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e})$ and then rewrite it in the following form:

$$\begin{aligned} RHS &= (-P_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] + P_\sigma[N(\mathbf{u}_\alpha)], \mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + N(\mathbf{u}_\alpha), P_\sigma\,\mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + N(\mathbf{u}_\alpha), \mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}) + (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + N(\mathbf{u}_\alpha), \mathbf{e}). \end{aligned} \tag{4.3}$$

We will deal with the two terms on the r.h.s of (4.3) separately. Replacing $\mathbf{u}_\alpha$ by $\mathbf{u}-\mathbf{e}$ the first term in (4.3) is rewritten as follows:

$$\begin{aligned} (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}) &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)(\mathbf{u}-\mathbf{e}), \mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}, \mathbf{e}) \\ &= ((-\mathbf{e}\cdot\nabla)\mathbf{u}, \mathbf{e}) \\ &= ((\mathbf{e}\cdot\nabla)\mathbf{e}, \mathbf{u}), \end{aligned}$$

where $(\mathbf{u}_\alpha\cdot\nabla)\mathbf{e}, \mathbf{e}) = 0$ has been used and the result is continuous estimated by

$$\begin{aligned} ((\mathbf{e}\cdot\nabla)\mathbf{e}, \mathbf{u}) &\leq C||\mathbf{e}||_4||\nabla\mathbf{e}||||\mathbf{u}||_4 \\ &\leq C||\mathbf{e}||^{1/2}||\nabla\mathbf{e}||^{3/2}||\mathbf{u}||^{1/2}||\nabla\mathbf{u}||^{1/2} \\ &\leq \frac{C}{\nu^3}||\mathbf{u}||^2||\nabla\mathbf{u}||^2||\mathbf{e}||^2 + \frac{\nu}{4}||\nabla\mathbf{e}||^2. \end{aligned} \tag{4.4}$$

The first inequality from above is due to the Hölder inequality with the pairing (1/4, 1/2, 1/4), the second one is obtained by applying the 2D-Ladyzhenskaya inequality and the last one comes from using the Young inequality with the pairing (1/4, 3/4). The residual term will be estimated for each model separately, one after the others.

Proof of the L-α model. For this model the nonlinear term is given by $N(\mathbf{u}_\alpha) = (\bar{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha$. The residual term is written as

$$R = (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\bar{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}) = -(((\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}).$$

The Hölder, 2D-Ladyzhenskaya, (2.5), (2.7), Sobolev, Poincaré and Young inequalities are going to apply to get the following estimates:

$$\begin{aligned} R &\leq C||\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha||_4||\nabla\mathbf{u}_\alpha||||\mathbf{e}||_4 \\ &\leq C||\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha||^{1/2}||\nabla\mathbf{u}_\alpha - \nabla\bar{\mathbf{u}}_\alpha||^{1/2}||\nabla\mathbf{u}_\alpha||\,||\mathbf{e}||^{1/2}||\nabla\mathbf{e}||^{1/2} \\ &\leq \frac{CC_L^{1/2}}{\lambda_1^{1/2}}\alpha^{3/2}||\Delta\bar{\mathbf{u}}_\alpha||^{1/2}\,||\Delta\mathbf{u}_\alpha||^{1/2}\,||\nabla\mathbf{e}|| \\ &\leq \frac{CC_L^{1/2}}{\lambda_1^{1/2}}\alpha^{3/2}||\Delta\mathbf{u}_\alpha||\,||\nabla\mathbf{e}|| \\ &\leq \frac{CC_L\alpha^3}{\nu\lambda_1}||\Delta\mathbf{u}_\alpha||^2 + \frac{\nu}{4}||\nabla\mathbf{e}||^2. \end{aligned} \tag{4.5}$$

Notice that $||\nabla \mathbf{u}_\alpha(t)||$ in the above estimate is uniformly bounded by $C_L^{1/2}$ where C_L given by Lemma 3.2. Putting (4.4) and (4.5) into (4.2) gives us

$$\frac{d}{dt}||\mathbf{e}||^2 + \nu||\nabla \mathbf{e}||^2 \leq \frac{CC_L\alpha^3}{\nu\lambda_1}||\Delta \mathbf{u}_\alpha||^2 + \frac{C}{\nu^3}||\mathbf{u}||^2||\nabla \mathbf{u}||^2||\mathbf{e}||^2. \tag{4.6}$$

Here we are going to apply the Gronwall's lemma for (4.6). Although the argument is standard we still provide the details for this model and for the other models the details will be shipped. Let

$$A(s) := -\frac{C}{\nu^3}\int_0^s ||\mathbf{u}||^2||\nabla \mathbf{u}||^2\, dt \quad \forall s \geq 0,$$

where C is given in (4.6). Multiplying both sides of (4.6) by $A(t)$ yields for all $s \geq 0$

$$||\mathbf{e}(s)||^2 \leq \frac{CC_L\alpha^3}{\nu\lambda_1}\exp\{-A(s)\}\int_0^s ||\Delta \mathbf{u}_\alpha||^2\, dt, \tag{4.7}$$

where one has used the fact $\mathbf{e}_0 = 0$. Thus, combine (4.7) with Lemmas 3.3 and 3.2 to obtain

$$||\mathbf{e}(s)||^2 \leq \frac{CC_L^2\alpha^3}{\nu^2\lambda_1}\exp\left\{\frac{C_{NSE1}^2}{\nu^4}\right\} =: E_L\alpha^3 \quad \forall s \geq 0, \tag{4.8}$$

where C_L and C_{NSE1} are given by Lemmas 3.2 and 3.1, respectively. Finally, we combine (4.6) and (4.8) to get (4.1) with C_r given by

$$C_{r_L} = C\left(\frac{C_L}{\nu^2} + \frac{C_{NSE1}^2 E_L}{\nu^4}\right). \tag{4.9}$$

Proof of the ML-α model. In this case the residual term is rewritten as

$$\begin{aligned} R &= (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha, \mathbf{e}) \\ &= ((\mathbf{u}_\alpha\cdot\nabla)\mathbf{e}, \mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha), \end{aligned}$$

and is handled precisely as in the L-α case. Then the proof for this case follows by that of the L-α model with C_r is given by

$$C_{r_{ML_a}} = C\left(\frac{C_{ML_a}}{\nu^2} + \frac{C_{NSE1}^2 E_{ML_a}}{\nu^4}\right), \tag{4.10}$$

where C_{ML_a} is given by Lemma 3.5 and

$$E_{ML_a} = \frac{CC_{ML_a}^2}{\nu^2\lambda_1}\exp\left\{\frac{C_{NSE1}^2}{\nu^4}\right\}.$$

Proof of the SB model. In this case the residual term is given by

$$\begin{aligned} R &= (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha, \mathbf{e}) \\ &= (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha - (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha, \mathbf{e}) \\ &= -(((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\cdot\nabla)\mathbf{u}_\alpha, \mathbf{e}) - ((\overline{\mathbf{u}}_\alpha\cdot\nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha), \mathbf{e}) \\ &= R_1 + R_2. \end{aligned} \tag{4.11}$$

The term R_1 on the r.h.s of (4.11) can be handled as (4.5) in the case of L-α model. The second term R_2 can be estimated as in the ML-α case. Therefore, the constant C_r in this case has the following form

$$C_{r_{SB}} = C_{r_L} + C_{r_{ML}}. \tag{4.12}$$

Proof of the NS-α model. In this case the nonlinear term is given by $N(\mathbf{u}_\alpha) = -\overline{\mathbf{u}}_\alpha \times (\nabla \times \mathbf{u}_\alpha)$. This model is treated in a different way from the others. Firstly, we rewrite the NSE in the rotational form as

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} - \mathbf{u} \times (\nabla \times \mathbf{u}) + \nabla\left(p + \frac{|\mathbf{u}|^2}{2}\right) = \mathbf{f}. \tag{4.13}$$

Here we have used the following formula

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\mathbf{u} \times (\nabla \times \mathbf{u}) + \nabla\left(\frac{|\mathbf{u}|^2}{2}\right).$$

Secondly, applying Helmholtz-Leray orthogonal projection P_σ on both sides of (4.13), note that $P_\sigma[\mathbf{f}] \equiv \mathbf{f}$, and then take the different between the result and the NS-α model (2.9) to obtain

$$\frac{d\mathbf{e}}{dt} - \nu\Delta\mathbf{e} = -P_\sigma[-\mathbf{u} \times (\nabla \times \mathbf{u})] + P_\sigma[-\overline{\mathbf{u}}_\alpha \times (\nabla \times \mathbf{u}_\alpha)]. \tag{4.14}$$

Taking $\mathbf{e}$ as a test in (4.14) to obtain:

$$\frac{1}{2}\frac{d}{dt}||\mathbf{e}||^2 + \nu||\nabla\mathbf{e}||^2 = (\mathbf{u} \times (\nabla \times \mathbf{u}) - \overline{\mathbf{u}}_\alpha \times (\nabla \times \mathbf{u}_\alpha), \mathbf{e}). \tag{4.15}$$

Adding and subtracting the r.h.s of (4.15) a term $(\mathbf{u}_\alpha \times (\nabla \times \mathbf{u}_\alpha), \mathbf{e})$ and then using the following formulas

$$\begin{aligned}(\mathbf{u} \times (\nabla \times \mathbf{u}) - \mathbf{u}_\alpha \times (\nabla \times \mathbf{u}_\alpha), \mathbf{e}) &= (\mathbf{u} \times (\nabla \times \mathbf{e}), \mathbf{e}),\\ (\mathbf{u}_\alpha \times (\nabla \times \mathbf{u}_\alpha) - \overline{\mathbf{u}}_\alpha \times (\nabla \times \mathbf{u}_\alpha), \mathbf{e}) &= ((\mathbf{u} - \overline{\mathbf{u}}_\alpha) \times (\nabla \times \mathbf{u}_\alpha), \mathbf{e}),\end{aligned}$$

to rewrite (4.15) as

$$\frac{1}{2}\frac{d}{dt}||\mathbf{e}||^2 + \nu||\nabla\mathbf{e}||^2 = (\mathbf{u} \times (\nabla \times \mathbf{e}), \mathbf{e}) + ((\mathbf{u} - \overline{\mathbf{u}}_\alpha) \times (\nabla \times \mathbf{u}_\alpha), \mathbf{e}). \tag{4.16}$$

By doing the same way as in (4.4)-(4.5) the two terms on the r.h.s of (4.16) are estimated respectively by:

$$\begin{aligned}(\mathbf{u} \times (\nabla \times \mathbf{e}), \mathbf{e}) &\le C||\mathbf{u}||_4||\nabla\mathbf{e}||||\mathbf{e}||_4\\ &\le C||\mathbf{u}||^{1/2}||\nabla\mathbf{u}||^{1/2}||\nabla\mathbf{e}||^{3/2}||\mathbf{e}||^{1/2}\\ &\le \frac{C}{\nu^3}||\mathbf{u}||^2||\nabla\mathbf{u}||^2||\mathbf{e}||^2 + \frac{\nu}{4}||\nabla\mathbf{e}||^2 \end{aligned} \tag{4.17}$$

and

$$((\mathbf{u}-\overline{\mathbf{u}}_\alpha)\times(\nabla\times\mathbf{u}_\alpha),\mathbf{e}) \le C||\mathbf{u}-\overline{\mathbf{u}}_\alpha||_4||\nabla\mathbf{u}_\alpha||||\mathbf{e}||_4$$
$$\le \frac{C}{\lambda_1^{1/2}}||\mathbf{u}-\overline{\mathbf{u}}_\alpha||^{1/2}||\nabla(\mathbf{u}-\overline{\mathbf{u}}_\alpha)||^{1/2}||\nabla\mathbf{u}_\alpha||||\nabla\mathbf{e}||$$
$$\le \frac{C}{\lambda_1^{1/2}}\alpha^{3/2}||\Delta\overline{\mathbf{u}}_\alpha||^{1/2}||\Delta\mathbf{u}_\alpha||^{1/2}||\nabla\mathbf{u}_\alpha||||\nabla\mathbf{e}||$$
$$\le \frac{C}{\nu\lambda_1}\alpha^3||\Delta\mathbf{u}_\alpha||^2||\nabla\mathbf{u}_\alpha||^2+\frac{\nu}{4}||\nabla\mathbf{e}||^2$$
$$\le \frac{CC_{NS_a}}{\nu\lambda_1}\alpha^3||\Delta\mathbf{u}_\alpha||^2+\frac{\nu}{4}||\nabla\mathbf{e}||^2. \tag{4.18}$$

Here C_{NS_a} is given by Lemma 3.3. Combining (4.16)-(4.18) yields

$$\frac{d}{dt}||\mathbf{e}||^2+\nu||\nabla\mathbf{e}||^2 \le \frac{C}{\nu^3}||\mathbf{u}||^2||\nabla\mathbf{u}||^2||\mathbf{e}||^2+\frac{CC_{NS_a}}{\nu\lambda_1}\alpha^3||\Delta\mathbf{u}_\alpha||^2. \tag{4.19}$$

It can be seen that (4.19) shares the similar structure with (4.6). Therefore, the constant C_r in this case will be given by the following form

$$C_{r_{NS_a}} = C\left(\frac{C_{NS_a}}{\nu^2}+\frac{C_{NSE1}^2E_{NS_a}}{\nu^4}\right), \tag{4.20}$$

where

$$E_{NS_a} = \frac{CC_{NS_a}^2}{\nu^2\lambda_1}\exp\left\{\frac{C_{NSE1}^2}{\nu^4}\right\}.$$

□

From Theorem 4.1 we have immediately the following results:

Corollary 4.1. *Let $\mathbf{u}_0\in\mathbf{V}$ and $\mathbf{f}\in\mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$. Then for all $s\ge 0$*

$$||\overline{\mathbf{e}}(s)||^2+\nu\int_0^s||\nabla\overline{\mathbf{e}}||^2\,dt+\nu\int_0^s||\nabla\times\overline{\mathbf{e}}||^2\,dt+\nu\int_0^s||\nabla\times\mathbf{e}||^2 \le 3C_r\alpha^3, \tag{4.21}$$

where $\overline{\mathbf{e}}=\overline{\mathbf{u}}-\overline{\mathbf{u}}_\alpha$ and C_r is given by Theorem 4.1 for each α-model.

Proof. The result follows by Theorem 4.1, (2.6) and $||\nabla\times\mathbf{e}||=||\nabla\mathbf{e}||$, $||\nabla\times\overline{\mathbf{e}}||=||\nabla\overline{\mathbf{e}}||$ by using

$$-\Delta\mathbf{e}=\nabla\times(\nabla\times\mathbf{e})-\nabla(\nabla\cdot\mathbf{e})=\nabla\times(\nabla\times\mathbf{e}).$$

□

Corollary 4.2. *Let $\mathbf{u}_0\in\mathbf{V}$ and $\mathbf{f}\in\mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$. Then for all $s\ge 0$*

$$||(\mathbf{u}-\overline{\mathbf{u}}_\alpha)(s)||^2+\nu\int_0^s||\nabla(\mathbf{u}-\overline{\mathbf{u}}_\alpha)||^2\,dt+\nu\int_0^s||\nabla\times(\mathbf{u}-\overline{\mathbf{u}}_\alpha)||^2\,dt \le C_{cor}(\alpha^2+\alpha^3), \tag{4.22}$$

where C_{cor} is given by (4.25).

Proof. The triangle inequality, Theorem 4.1, Lemma 3.1, relation (2.6) and Poincaré inequality yield for all $s \geq 0$

$$\begin{aligned}
||\mathbf{u} - \bar{\mathbf{u}}_\alpha(s)||^2 &\leq 2\left(||(\mathbf{u} - \mathbf{u}_\alpha)(s)||^2 + ||(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)(s)||^2\right) \\
&\leq 2C_r\alpha^3 + 2\alpha^4||\Delta\bar{\mathbf{u}}_\alpha(s)||^2 \\
&\leq 2C_r\alpha^3 + 2C\alpha^2||\mathbf{u}_\alpha(s)||^2 \\
&\leq 2C_r\alpha^3 + 2C\frac{C_E}{\lambda_1}\alpha^2. \qquad (4.23)
\end{aligned}$$

Here for each α-model C_E is given by C_L, C_{NS_a}, C_{SB} or C_{ML_a} in Lemmas 3.2, 3.3, 3.4 and 3.5, respectively. Moreover, C_r is given by Theorem 4.1. Similarity, we have for all $s \geq 0$

$$\begin{aligned}
\nu\int_0^s ||\nabla(\mathbf{u} - \bar{\mathbf{u}}_\alpha)||^2\,dt &\leq 2\nu\left(\int_0^s ||\nabla(\mathbf{u} - \mathbf{u}_\alpha)||^2\,dt + \int_0^s ||\nabla(\mathbf{u}_\alpha - \bar{\mathbf{u}}_\alpha)||^2\,dt\right) \\
&\leq 2C_r\alpha^3 + 2C\alpha^2\nu\int_0^s ||\Delta\mathbf{u}_\alpha||^2\,dt \\
&\leq 2C_r\alpha^3 + 2CC_E\alpha^2. \qquad (4.24)
\end{aligned}$$

Moreover, we have $||\nabla(\mathbf{u} - \bar{\mathbf{u}}_\alpha)|| = ||\nabla \times (\mathbf{u} - \bar{\mathbf{u}}_\alpha)||$. Thus, (4.22) follows by (4.23) and (4.24) with the constant C given by

$$C_{cor} = 2\max\{C_r, CC_E, CC_E/\lambda_1\}. \qquad (4.25)$$

$\square$

Remark 4.1 (**Compare results**). *Corollary 4.2 improves the results which are given by Cao-Titi [2]. Their results require* $\alpha \leq L/2\pi$ *and only for the* $L^\infty(L^2)$ *norm with* $\forall s \in [0, T]$

$$||(\mathbf{u} - \bar{\mathbf{u}}_\alpha)(s)||^2 \leq C\alpha^2\left(TC\left(1 + \log\left(\frac{L}{2\pi\alpha}\right)\right) + C\right), \qquad (4.26)$$

where C *is a constant (unnecessary be the same) and* T *is the final time. Notice that their estimates depend on* T*. That is not of our case. Moreover, it is required* $\mathbf{u}_0 \in \mathcal{D}(A)$ *instead of* $\mathbf{u}_0 \in \mathbf{V}$*. Of course if* $\alpha > L/2\pi$ *then the r.h.s of (4.26) has the same order with that of (4.22).*

The next main result in this section is given as the following theorem:

Theorem 4.2. *Let* $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+; \mathbf{H})$ *and*

$$D(s) := ||\nabla\mathbf{e}(s)||^2 + \nu\int_0^s ||\Delta\mathbf{e}||^2\,dt \quad \forall s \geq 0. \qquad (4.27)$$

Then

1. For the L-α *and NS-*α *models*

$$D(s) \leq C_R\,\alpha^2,$$

where C_R *is given by (4.39) and (4.56), respectively.*

2. For the ML-α *model*

$$D(s) \leq \begin{cases} C_R\,\alpha^2\left(K_1\log(L/2\pi\alpha) + K_2 + C_{ML_a}\right) & \text{with } C_R \text{ given by } (4.46), \\ C_R\,\alpha^2 & \text{with } C_R \text{ given by } (4.47), \end{cases}$$

in the case $\alpha < L/2\pi$ and $\alpha \geq L/2\pi$, respectively.
3. For the SB model

$$D(s) \leq \begin{cases} C_R\,\alpha^2\,(K_1 \log(L/2\pi\alpha) + K_2 + C_{SB}) & \textit{with } C_R \textit{ given by } (4.49), \\ C_R\,\alpha^2 & \textit{with } C_R \textit{ given by } (4.50), \end{cases}$$

in the case $\alpha < L/2\pi$ and $\alpha \geq L/2\pi$, respectively. Here C_{ML_α}, C_{SB} and K_1, K_2 are given by Lemmas 3.5, 3.4 and 8.2, respectively.

Remark 4.2. *It can be seen that*

$$x^2 \leq x^2 \log(1/x) \quad \forall x \in (0, 1/e] \text{ and otherwise if } \forall x \in [1/e, \infty).$$

Proof. Subtracting (2.9) from (2.8) and taking $-\Delta\mathbf{e}$ as a test yield:

$$\frac{1}{2}\frac{d}{dt}||\nabla\mathbf{e}||^2 + \nu||\Delta\mathbf{e}||^2 = (-P_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] + P_\sigma[N(\mathbf{u}_\alpha)], -\Delta\mathbf{e}). \tag{4.28}$$

Adding and subtracting the term $((\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e})$ to the r.h.s of (4.28):

$$RHS = (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) + (-(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + N(\mathbf{u}_\alpha), -\Delta\mathbf{e}). \tag{4.29}$$

By using $\mathbf{e} = \mathbf{u} - \mathbf{u}_\alpha$ the first term on the r.h.s of (4.29) can be estimated by

$$\begin{aligned} I_1 &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha, -\Delta\mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + ((\mathbf{u}-\mathbf{e})\cdot\nabla)(\mathbf{u}-\mathbf{e}), -\Delta\mathbf{e}) \\ &= (-(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{e} - (\mathbf{e}\cdot\nabla)\mathbf{u} + (\mathbf{e}\cdot\nabla)\mathbf{e}, -\Delta\mathbf{e}) \\ &= ((\mathbf{u}\cdot\nabla)\mathbf{e}, \Delta\mathbf{e}) + (\mathbf{e}\cdot\nabla)\mathbf{u}, \Delta\mathbf{e}) = I_{11} + I_{12}, \end{aligned} \tag{4.30}$$

where the vanishing of the term $((\mathbf{e}\cdot\nabla)\mathbf{e}, -\Delta\mathbf{e})$ has been used. The first term on the r.h.s of (4.30) is bounded by

$$\begin{aligned} I_{11} = ((\mathbf{u}\cdot\nabla)\mathbf{e}, \Delta\mathbf{e}) &\leq C||\mathbf{u}||_4||\nabla\mathbf{e}||_4||\Delta\mathbf{e}|| \\ &\leq C||\mathbf{u}||^{1/2}||\nabla\mathbf{u}||^{1/2}||\nabla\mathbf{e}||^{1/2}||\Delta\mathbf{e}||^{3/2} \\ &\leq \frac{C}{\nu^3}||\mathbf{u}||^2||\nabla\mathbf{u}||^2||\nabla\mathbf{e}||^2 + \frac{\nu}{6}||\Delta\mathbf{e}||^2. \end{aligned} \tag{4.31}$$

In (4.31), the Hölder, 2D-Ladyzhenskaya and Young inequalities have been applied. Similarity, the other term on the r.h.s of (4.30) can be handled by:

$$\begin{aligned} I_{12} = ((\mathbf{e}\cdot\nabla)\mathbf{u}, \Delta\mathbf{e}) &\leq C||\mathbf{e}||_4||\nabla\mathbf{u}||_4||\Delta\mathbf{e}|| \\ &\leq C||\mathbf{e}||^{1/2}||\nabla\mathbf{e}||^{1/2}||\nabla\mathbf{u}||^{1/2}||\Delta\mathbf{u}||^{1/2}||\Delta\mathbf{e}|| \\ &\leq \frac{C}{\lambda_1^{1/2}}||\nabla\mathbf{e}||||\nabla\mathbf{u}||^{1/2}||\Delta\mathbf{u}||^{1/2}||\Delta\mathbf{e}|| \\ &\leq \frac{C}{\nu\lambda_1}||\nabla\mathbf{e}||^2||\nabla\mathbf{u}||||\Delta\mathbf{u}|| + \frac{\nu}{6}||\Delta\mathbf{e}||^2. \end{aligned} \tag{4.32}$$

Using (4.31)-(4.32) the quantity I_1 in (4.30) can be bounded by

$$I_1 \leq \left(\frac{C}{\nu^3}||\mathbf{u}||^2||\nabla\mathbf{u}||^2 + \frac{C}{\nu\lambda_1}||\nabla\mathbf{u}||||\Delta\mathbf{u}||\right)||\nabla\mathbf{e}||^2 + \frac{\nu}{3}||\Delta\mathbf{e}||^2. \tag{4.33}$$

In the following parts, we will estimate the second term on the r.h.s of (4.29) for each α-model. We start with the first one as follows.

Proof of the L-α model. The nonlinear term is given by $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$. Therefore, the residual term can be estimated by

$$
\begin{aligned}
I_2 &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha, -\Delta \mathbf{e}) \\
&= ((\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, -\Delta \mathbf{e}) \\
&\le C||\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha||_4 ||\nabla \mathbf{u}_\alpha||_4 ||\Delta \mathbf{e}|| \\
&\le C||\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha||^{1/2} ||\nabla\overline{\mathbf{u}}_\alpha - \nabla\mathbf{u}_\alpha||^{1/2} ||\nabla \mathbf{u}_\alpha||^{1/2} ||\Delta \mathbf{u}_\alpha||^{1/2} ||\Delta \mathbf{e}|| \\
&\le C\alpha ||\nabla \mathbf{u}_\alpha|| ||\Delta \mathbf{u}_\alpha|| ||\Delta \mathbf{e}|| \\
&\le \frac{C}{\nu}\alpha^2 ||\nabla \mathbf{u}_\alpha||^2 ||\Delta \mathbf{u}_\alpha||^2 + \frac{\nu}{6}||\Delta \mathbf{e}||^2 \\
&\le \frac{CC_L\alpha^2}{\nu} ||\Delta \mathbf{u}_\alpha||^2 + \frac{\nu}{6}||\Delta \mathbf{e}||^2. \qquad (4.34)
\end{aligned}
$$

Here the Hölder, 2D-Ladyzhenskaya, (2.5)-(2.6) and Young inequalities have been applied. Moreover, C_L is given by Lemma 3.2. Using estimates (4.28)-(4.34) leads to

$$
\begin{aligned}
\frac{d}{dt}||\nabla \mathbf{e}||^2 + \nu||\Delta \mathbf{e}||^2 \le & \left(\frac{C}{\nu^3}||\mathbf{u}||^2||\nabla \mathbf{u}||^2 + \frac{C}{\nu\lambda_1}||\nabla \mathbf{u}|| ||\Delta \mathbf{u}|| \right) ||\nabla \mathbf{e}||^2 \\
& + \frac{CC_L\alpha^2}{\nu}||\Delta \mathbf{u}_\alpha||^2 \qquad (4.35)
\end{aligned}
$$

and in particular

$$
y'(t) - g(t)y(t) \le h(t) \quad \forall t \ge 0, \qquad (4.36)
$$

where for all $t \ge 0$

$$
\begin{cases}
y(t) &= ||\nabla \mathbf{e}(t)||^2, \\
g(t) &= \dfrac{C}{\nu^3}||\mathbf{u}(t)||^2||\nabla \mathbf{u}(t)||^2 + \dfrac{C}{\nu\lambda_1}||\nabla \mathbf{u}(t)|| ||\Delta \mathbf{u}(t)||, \\
h(t) &= \dfrac{CC_L\alpha^2}{\nu}||\Delta \mathbf{u}_\alpha(t)||^2.
\end{cases}
$$

Therefore, the Gronwall's lemma gives for all $s \ge 0$

$$
||\nabla \mathbf{e}(s)||^2 \le \frac{CC_L}{\nu^2} \exp\left\{ \frac{C^2_{NSE1}}{\nu^4} + \frac{C^{1/2}_{NSE1}C^{1/2}_{NSE2}}{\nu^2\lambda_1} \right\} \alpha^2 =: R_L\alpha^2. \qquad (4.37)
$$

Here we use the fact that $\nabla \mathbf{e}(0) = 0$. Finally, combine (4.35) and (4.37) yield

$$
||\nabla \mathbf{e}(s)||^2 + \nu \int_0^s ||\Delta \mathbf{e}||^2 \, dt \le C_R\,\alpha^2 \quad \forall s \ge 0, \qquad (4.38)
$$

where

$$
C_R = \left(\frac{C^2_{NSE1}}{\nu^4} + \frac{C^{1/2}_{NSE1}C^{1/2}_{NSE2}}{\nu^2\lambda_1} \right) R_L + \frac{CC_L^2}{\nu^2}. \qquad (4.39)
$$

Thus, the proof is complete.

Proof of the ML-α model. The nonlinear term is given by $N(\mathbf{u}_\alpha) = (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha$. Hence, the residual term can be rewritten by

$$
\begin{aligned}
I_2 &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta \mathbf{e}) \\
&= ((\mathbf{u}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta \mathbf{e}) \\
&= (((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha) \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta \mathbf{e}) + ((\overline{\mathbf{u}}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta \mathbf{e}) \\
&= I_{21} + I_{22}. \qquad (4.40)
\end{aligned}
$$

The first term on the r.h.s of (4.40) can be estimated by

$$
\begin{aligned}
I_{21} &= (((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha) \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta \mathbf{e}) \\
&\le C||\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha||_4 ||\nabla \overline{\mathbf{u}}_\alpha - \nabla \mathbf{u}_\alpha||_4 ||\Delta \mathbf{e}||_2 \\
&\le C||\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha||^{1/2} ||\nabla \overline{\mathbf{u}}_\alpha - \nabla \mathbf{u}_\alpha|| \, ||\Delta \overline{\mathbf{u}}_\alpha - \Delta \mathbf{u}_\alpha||^{1/2} ||\Delta \mathbf{e}|| \\
&\le C\alpha ||\Delta \mathbf{u}_\alpha|| ||\nabla \mathbf{u}_\alpha|| ||\Delta \mathbf{e}|| \\
&\le \frac{C}{\nu}\alpha^2 ||\nabla \mathbf{u}_\alpha||^2 ||\Delta \mathbf{u}_\alpha||^2 + \frac{\nu}{12}||\Delta \mathbf{e}||^2 \\
&\le \frac{CC_{ML_a}}{\nu}\alpha^2 ||\Delta \mathbf{u}_\alpha||^2 + \frac{\nu}{12}||\Delta \mathbf{e}||^2, \qquad (4.41)
\end{aligned}
$$

where C_{ML_a} is given by Lemma 3.5. Next, we bound the second term on the r.h.s of (4.40) as follows:

$$
\begin{aligned}
I_{22} &= ((\overline{\mathbf{u}}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta \mathbf{e}) \\
&\le C||\overline{\mathbf{u}}_\alpha||_\infty ||\nabla \overline{\mathbf{u}}_\alpha - \nabla \mathbf{u}_\alpha|| ||\Delta \mathbf{e}|| \\
&\le C\alpha ||\overline{\mathbf{u}}_\alpha||_\infty ||\Delta \mathbf{u}_\alpha|| ||\Delta \mathbf{e}|| \\
&\le \frac{C\alpha^2}{\nu} ||\overline{\mathbf{u}}_\alpha||_\infty^2 ||\Delta \mathbf{u}_\alpha||^2 + \frac{\nu}{12}||\Delta \mathbf{e}||^2 \\
&\le \frac{C\alpha^2}{\nu}\left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2\right) ||\Delta \mathbf{u}_\alpha||^2 + \frac{\nu}{12}||\Delta \mathbf{e}||^2, \qquad (4.42)
\end{aligned}
$$

in the case $\alpha < L/2\pi$. Here K_1 and K_2 are given in Lemma 8.2 in the Appendix. Otherwise,

$$
I_{22} \le \frac{CK_2\alpha^2}{\nu}||\Delta \mathbf{u}_\alpha||^2 + \frac{\nu}{12}||\Delta \mathbf{e}||^2 \quad \text{for } \alpha \ge L/2\pi. \qquad (4.43)
$$

Here in (4.41)-(4.42), we have used the inequalities Hölder, 2D-Ladyzhenshaya, Young and formula (8.13) in Lemma 8.2. Putting (4.33) and (4.41)-(4.42) into the r.h.s of (4.28), we obtain

$$
\begin{aligned}
\frac{d}{dt}||\nabla \mathbf{e}||^2 + \nu ||\Delta \mathbf{e}||^2 &\le \left(\frac{C}{\nu^3}||\mathbf{u}||^2||\nabla \mathbf{u}||^2 + \frac{C}{\nu\lambda_1}||\nabla \mathbf{u}|| ||\Delta \mathbf{u}||\right) ||\nabla \mathbf{e}||^2 \\
&\quad + \frac{C\alpha^2}{\nu}\left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_{ML_a}\right) ||\Delta \mathbf{u}_\alpha||^2, \qquad (4.44)
\end{aligned}
$$

if $\alpha < L/2\pi$ or

$$
\begin{aligned}
\frac{d}{dt}||\nabla \mathbf{e}||^2 + \nu ||\Delta \mathbf{e}||^2 &\le \left(\frac{C}{\nu^3}||\mathbf{u}||^2||\nabla \mathbf{u}||^2 + \frac{C}{\nu\lambda_1}||\nabla \mathbf{u}|| ||\Delta \mathbf{u}||\right) ||\nabla \mathbf{e}||^2 \\
&\quad + \frac{C\alpha^2}{\nu}(K_2 + C_{ML_a})||\Delta \mathbf{u}_\alpha||^2, \qquad (4.45)
\end{aligned}
$$

if $\alpha \geq L/2\pi$. Since both (4.44) and (4.45) have similar structure with (4.35) then the rest of the proof follows by that of the L-α model. The constant C_R in this case is given by

$$C_R = \left(\frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right) R_{ML_a} + \frac{C C_{ML_a}}{\nu^2} \tag{4.46}$$

if $\alpha < L/2\pi$ or

$$C_R = \left(\frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right) R_{ML_a} + \frac{C C_{ML_a}(K_2 + C_{ML_a})}{\nu^2} \tag{4.47}$$

if $\alpha \geq L/2\pi$. Here

$$R_{ML_a} := \frac{C C_{ML_a}}{\nu^2} \exp \left\{ \frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right\}.$$

Thus, the proof is complete.

Proof of the SB model. The nonlinear term is given by $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha$. Adding and subtracting the term $(\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$ give

$$\begin{aligned} I_2 &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta \mathbf{e}) \\ &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha - (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta \mathbf{e}) \\ &= I_{21} + I_{22}. \end{aligned} \tag{4.48}$$

Here, the first term on the r.h.s of (4.48) can be handled by

$$\begin{aligned} I_{21} &= (-(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha, -\Delta \mathbf{e}) \\ &= ((\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha) \cdot \nabla)\mathbf{u}_\alpha, \Delta \mathbf{e}), \end{aligned}$$

which is similar to (4.34) in the L-α model. The other term is rewritten as

$$\begin{aligned} I_{22} &= (-(\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha \cdot \nabla)\overline{\mathbf{u}}_\alpha, -\Delta \mathbf{e}) \\ &= ((\overline{\mathbf{u}}_\alpha \cdot \nabla)(\overline{\mathbf{u}}_\alpha - \mathbf{u}_\alpha), -\Delta \mathbf{e}), \end{aligned}$$

which is similar to (4.42) in the ML-α model. Therefore, the constant C_R in this case is similar as in the ML-α model and has the form

$$C_R = \left(\frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right) R_{SB} + \frac{C C_{SB}}{\nu^2} \tag{4.49}$$

if $\alpha < L/2\pi$ or

$$C_R = \left(\frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right) R_{SB} + \frac{C C_{SB}(K_2 + C_{SB})}{\nu^2} \tag{4.50}$$

if $\alpha \geq L/2\pi$. Here C_{SB} is given by Lemma 3.4 and

$$R_{SB} := \frac{C C_{SB}}{\nu^2} \exp \left\{ \frac{C_{NSE1}^2}{\nu^4} + \frac{C_{NSE1}^{1/2} C_{NSE2}^{1/2}}{\nu^2 \lambda_1} \right\}.$$

Thus, the proof is complete.

Proof of the NS-α model. The mathematical setting for this model can be done as in the previous part, see again Theorem 4.1, in the proof for the NS-α model. Taking $-\Delta\mathbf{e}$ as a test in (4.14) leads to

$$\frac{1}{2}\frac{d}{dt}||\nabla\mathbf{e}||^2 + \nu||\Delta\mathbf{e}||^2 = (\mathbf{u}\times(\nabla\times\mathbf{u}), -\Delta\mathbf{e}) + (-\overline{\mathbf{u}}_\alpha\times(\nabla\times\mathbf{u}_\alpha), -\Delta\mathbf{e}). \tag{4.51}$$

As doing in the case of L-α model, we add and subtract on the r.h.s of (4.51) by $(\mathbf{u}_\alpha\times(\nabla\times\mathbf{u}_\alpha), -\Delta\mathbf{e})$. It leads us to estimate the following quantities

$$I_1 = (\mathbf{u}\times(\nabla\times\mathbf{u}) - \mathbf{u}_\alpha\times(\nabla\times\mathbf{u}_\alpha), -\Delta\mathbf{e}), \tag{4.52}$$

$$I_2 = (\mathbf{u}_\alpha\times(\nabla\times\mathbf{u}_\alpha) - \overline{\mathbf{u}}_\alpha\times(\nabla\times\mathbf{u}_\alpha), -\Delta\mathbf{e}). \tag{4.53}$$

Firstly, focusing on I_1 in (4.52), where we can do as the same way as in the equality (4.30). In fact, I_1 can be rewritten by using (8.4) in Lemma 8.1 as

$$I_1 = (-\mathbf{u}\times(\nabla\times\mathbf{e}), -\Delta\mathbf{e}) + (-\mathbf{e}\times(\nabla\times\mathbf{u}), -\Delta\mathbf{e}). \tag{4.54}$$

It is easily seen that (4.54) has the same structure with (4.30). Secondly, for I_2 in (4.53), we have

$$I_2 = ((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)\times(\nabla\times\mathbf{u}_\alpha), -\Delta\mathbf{e}), \tag{4.55}$$

where it can be estimated in the similar way as in the inequality (4.34). From (4.54)-(4.55), the rest of the proof follows by that of the L-α model. The constant C_R has the form

$$C_R = \left(\frac{C^2_{NSE1}}{\nu^4} + \frac{C^{1/2}_{NSE1}C^{1/2}_{NSE2}}{\nu^2\lambda_1}\right)R_{NS_a} + \frac{CC^2_{NS_a}}{\nu^2}, \tag{4.56}$$

where C_{NS_a} is given by Lemma 3.3 and

$$R_{NS_a} := \frac{CC_{NS_a}}{\nu^2}\exp\left\{\frac{C^2_{NSE1}}{\nu^4} + \frac{C^{1/2}_{NSE1}C^{1/2}_{NSE2}}{\nu^2\lambda_1}\right\}.$$

Thus, the proof is complete. □

The following results are immediately follow by Theorem 4.2:

Corollary 4.3. *Let* $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{f}\in\mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$ *and*

$$\overline{D}(s) := ||\nabla\overline{\mathbf{e}}(s)||^2 + \nu\int_0^s ||\Delta\overline{\mathbf{e}}||^2\,dt \quad \forall s\geq 0.$$

Then

$$\overline{D}(s) \leq D(s) \leq C_R\,h(\alpha) \quad \forall s\geq 0,$$

where C_R *and* $h(\alpha)$ *are given by Theorem 4.2 for each* α*-model. In particular,*

$$||\nabla\times\mathbf{e}(s)||^2 + \nu\int_0^s ||\nabla\times(\nabla\times\mathbf{e})||^2\,dt \leq C_R\,h(\alpha) \quad \forall s\geq 0,$$

and

$$||\nabla\times\overline{\mathbf{e}}(s)||^2 + \nu\int_0^s ||\nabla\times(\nabla\times\overline{\mathbf{e}})||^2\,dt \leq C_R\,h(\alpha) \quad \forall s\geq 0.$$

Proof. The proof follows by applying Theorem 4.2 and (2.6). ☐

Corollary 4.4. *Let* $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$ *and*

$$E(s) = ||\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)(s)||^2 + \nu \int_0^s ||\Delta(\mathbf{u} - \overline{\mathbf{u}}_\alpha)||^2\, dt \quad \forall s \geq 0. \tag{4.57}$$

Then

$$E(s) \leq 2C_R\, h(\alpha) + 2CC_E\alpha^2 \quad \forall s \geq 0. \tag{4.58}$$

Moreover, for all $s \geq 0$

$$||\nabla \times (\mathbf{u} - \overline{\mathbf{u}}_\alpha)(s)||^2 + \nu \int_0^s ||\nabla \times (\nabla \times (\mathbf{u} - \overline{\mathbf{u}}_\alpha))||^2\, dt \leq C(C_R\, h(\alpha) + C_E\alpha^2).$$

Proof. The proof shares the same idea with Corollary 4.2. We start with

$$\begin{aligned} ||\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)(s)||^2 &\leq 2(||\nabla(\mathbf{u} - \mathbf{u}_\alpha)(s)||^2 + ||\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)(s)||^2) \\ &\leq 2C_R\, h(\alpha) + 2\alpha^4||\nabla\Delta\overline{\mathbf{u}}_\alpha(s)||^2 \\ &\leq 2C_R\, h(\alpha) + 2C\alpha^2||\nabla \mathbf{u}_\alpha(s)||^2 \\ &\leq 2C_R\, h(\alpha) + 2CC_E\alpha^2 \quad \forall s \geq 0, \end{aligned} \tag{4.59}$$

where (2.6) has been used in the third inequality. As previous parts C_E is given as in Corollary 4.2. Similarity, for all $s \geq 0$

$$\begin{aligned} I &= \nu \int_0^s ||\Delta\mathbf{u} - \Delta\overline{\mathbf{u}}_\alpha||^2\, dt \\ &\leq 2\nu \int_0^s ||\Delta\mathbf{u} - \Delta\mathbf{u}_\alpha||^2\, dt + 2\nu \int_0^s ||\Delta\mathbf{u}_\alpha - \Delta\overline{\mathbf{u}}_\alpha||^2\, dt \\ &\leq 2C_R h(\alpha) + 2\nu\alpha^4 \int_0^s ||\Delta\Delta\overline{\mathbf{u}}_\alpha||^2\, dt \\ &\leq 2C_R h(\alpha) + 2\alpha^2\nu \int_0^s ||\Delta\mathbf{u}_\alpha||^2\, dt \\ &\leq 2C_R h(\alpha) + 2CC_E\alpha^2. \end{aligned} \tag{4.60}$$

Therefore, (4.58) follows by combining (4.59) and (4.60). ☐

5 The rate of convergence of p_α to p

In this section we focus on the error of the pressure by using the results from the previous sections. Let p and p_α are the pressures associated to the weak solutions $\mathbf{u}$ and $\mathbf{u}_\alpha$ of the NSE (1.1)-(1.3) and all α-models (1.4)-(1.6), respectively. It will be shown that the difference $q = p - p_\alpha$ is bounded in terms of the parameter α uniformly in time in a suitable norm.

Theorem 5.1. *Let* $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+;\mathbf{H})$ *and*

$$I(s) := \int_0^s ||\nabla q||^2\, dt \quad \forall s \geq 0. \tag{5.1}$$

Then
1. For for the L-α and NS-α models

$$I(s) \leq C\alpha^{5/2} + C\alpha^3,$$

where C given by (5.9).
2. For for the ML-α model

$$I(s) \leq \begin{cases} C\alpha^4 + C\alpha^3 + C(\alpha^{5/2} + \alpha^2)(\log(L/2\pi\alpha) + 1) & \text{if } \alpha < L/2\pi, \\ C\alpha^4 + C\alpha^3 + C\alpha^{5/2} + C\alpha^2 & \text{if } \alpha \geq L/2\pi. \end{cases}$$

where C given by (5.14) and (5.15).
3. For for the SB model

$$I(s) \leq \begin{cases} C\alpha^3 + C\alpha^{5/2}(\log(L/2\pi\alpha) + 1)^{1/2} + C\alpha^2(\log(L/2\pi\alpha) + 1) & \text{if } \alpha < L/2\pi, \\ C\alpha^3 + C\alpha^{5/2} + C\alpha^2 & \text{if } \alpha \geq L/2\pi, \end{cases}$$

where C given by (5.20) and (5.21).

Proof. It follows from the NSE (1.1)-(1.3) and α-models (1.4)-(1.6) that

$$-\Delta q = \nabla \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)] =: \nabla \cdot \mathbf{g}. \tag{5.2}$$

Assumes that p and p_α are periodic and zero averages. The vanishing of the mean values of p and p_α ensure their uniqueness determined. Multiplying (5.2) by q and integrating on Ω with using Cauchy-Schwarz inequality yield

$$||\nabla q||^2 \leq ||\mathbf{g}||^2 = \int_\Omega |(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)|^2 \, d\mathbf{x}. \tag{5.3}$$

In order to estimate the error of the pressure we are led to bound the r.h.s of (5.3). Replacing $\mathbf{e}$ by $\mathbf{u} - \mathbf{u}_\alpha$, adding and subtracting the term $(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha$ give

$$\begin{aligned} ||\mathbf{g}||^2 &= \int_\Omega |(\mathbf{u} \cdot \nabla)\mathbf{u} - N(\mathbf{u}_\alpha)|^2 \, d\mathbf{x} \\ &= \int_\Omega |(\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2 \, d\mathbf{x} \\ &= \int_\Omega | - (\mathbf{e} \cdot \nabla)\mathbf{u} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{e} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2 \, d\mathbf{x} \\ &= C \int_\Omega \left(|(\mathbf{e} \cdot \nabla)\mathbf{u}|^2 + |(\mathbf{u}_\alpha \cdot \nabla)\mathbf{e}|^2 + |(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha - N(\mathbf{u}_\alpha)|^2 \right) \, d\mathbf{x}. \end{aligned} \tag{5.4}$$

By (5.4) one has for all $t \geq 0$:

$$I(t) = \int_0^t ||\mathbf{g}||^2 \, ds \leq C(I_1 + I_2 + I_3). \tag{5.5}$$

The proof is given for each α-model separately below one after the others.

Proof of the L-α model. We have $N(\mathbf{u}_\alpha) = (\overline{\mathbf{u}}_\alpha \cdot \nabla)\mathbf{u}_\alpha$ in this case. Each term on the r.h.s of (5.5) will be estimated below. Firstly,

$$
\begin{aligned}
I_1 &= \int_0^t \int_\Omega |(\mathbf{e}\cdot\nabla)\mathbf{u}|^2\,d\mathbf{x}\,ds \\
&\le \int_0^t ||\mathbf{e}||_4^2 ||\nabla\mathbf{u}||_4^2\,ds \\
&\le \int_0^t ||\mathbf{e}||\,||\nabla\mathbf{e}||\,||\nabla\mathbf{u}||\,||\Delta\mathbf{u}||\,ds \\
&\le C_r^{1/2} C_{NSE2}^{1/2}\,\alpha^{3/2}\left(\int_0^t ||\nabla\mathbf{e}||^2\,ds\right)^{1/2}\left(\int_0^t ||\Delta\mathbf{u}||^2\,ds\right)^{1/2} \\
&\le \frac{C_r C_{NSE}}{\nu}\alpha^3 \quad \forall t\ge 0, \qquad (5.6)
\end{aligned}
$$

here we have used the Hölder and 2D-Ladyzhenskaya inequalities, Lemma 3.1, 4.1 and 4.2, respectively. Secondly,

$$
\begin{aligned}
I_2 &= \int_0^t \int_\Omega |(\mathbf{u}_\alpha\cdot\nabla)\mathbf{e}|^2\,d\mathbf{x}\,ds \\
&\le \int_0^t ||\mathbf{u}_\alpha||\,||\nabla\mathbf{u}_\alpha||\,||\nabla\mathbf{e}||\,||\Delta\mathbf{e}||\,ds \\
&\le \frac{C_L}{\lambda^{1/2}}\left(\int_0^t ||\nabla\mathbf{e}||^2\,ds\right)^{1/2}\left(\int_0^t ||\Delta\mathbf{e}||^2\,ds\right)^{1/2} \\
&\le \frac{C_L C_r^{1/2} C_R^{1/2}}{\nu}\alpha^{5/2} \quad \forall t\ge 0, \qquad (5.7)
\end{aligned}
$$

here we have used the Hölder and 2D-Ladyzhenskaya inequalities, Lemma 3.2, Theorems 4.1 and 4.2, respectively. Thirdly,

$$
\begin{aligned}
I_3 &= \int_0^t \int_\Omega |((\mathbf{u}_\alpha-\overline{\mathbf{u}}_\alpha)\cdot\nabla)\mathbf{u}_\alpha|^2\,d\mathbf{x}\,ds \\
&\le \int_0^t ||\mathbf{u}_\alpha-\overline{\mathbf{u}}_\alpha||\,||\nabla(\mathbf{u}_\alpha-\overline{\mathbf{u}}_\alpha)||\,||\nabla\mathbf{u}_\alpha||\,||\Delta\mathbf{u}_\alpha||\,ds \\
&\le 2CC_L\alpha^3\int_0^t ||\Delta\overline{\mathbf{u}}_\alpha||||\Delta\mathbf{u}_\alpha||\,ds \\
&\le \frac{2CC_L^{3/2}}{\nu}\alpha^3 \quad \forall t\ge 0, \qquad (5.8)
\end{aligned}
$$

here in additional as above we have used (2.5), (4.59) and Lemma 3.2. Thus the proof for this model follows by (5.5)-(5.8)

$$
I(s) \le \frac{C_L C_r^{1/2} C_R^{1/2}}{\nu}\alpha^{5/2} + \left(\frac{C_r C_{NSE}}{\nu} + \frac{2CC_L^{3/2}}{\nu}\right)\alpha^3. \qquad (5.9)
$$

Proof of the ML-α model. For this model I_1 is estimated as above. We start with I_2

by

$$I_2 \leq \int_0^t ||\mathbf{u}_\alpha|| \, ||\nabla \mathbf{u}_\alpha|| \, ||\nabla \mathbf{e}|| \, ||\Delta \mathbf{e}|| \, ds$$

$$\leq \frac{C_{ML_a}}{\lambda_1^{1/2}} \left(\int_0^t ||\nabla \mathbf{e}||^2 \, ds \right)^{1/2} \left(\int_0^t ||\Delta \mathbf{e}||^2 \, ds \right)^{1/2}$$

$$\leq \begin{cases} \dfrac{C_{ML_a} C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} \left(K_1 \log\left(\dfrac{L}{2\pi\alpha}\right) + K_2 + C_{ML_a} \right)^{1/2} & \text{if } \alpha < L/2\pi, \\ \dfrac{C_{ML_a} C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} & \text{if } \alpha \geq L/2\pi, \end{cases} \tag{5.10}$$

for all $t \geq 0$. One has used the results Lemma 3.5, Theorems 4.1 and 4.2. The term I_3 is bounded by

$$I_3 = \int_0^t \int_\Omega |(\mathbf{u}_\alpha \cdot \nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x} \, ds \leq 2(I_{31} + I_{32}). \tag{5.11}$$

By (4.59) and Lemma 3.5 yield

$$I_{31} = \int_0^t \int_\Omega |((\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha) \cdot \nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x} \, ds$$

$$\leq \int_0^t ||\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha||_4^2 \, ||\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)||_4^2 \, ds$$

$$\leq \int_0^t ||\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha|| \, ||\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)||^2 ||\Delta(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)|| \, ds$$

$$\leq \frac{C C_{ML_a}^2}{\nu} \alpha^4 \quad \forall t \geq 0. \tag{5.12}$$

The other term can be estimated for all $t \geq 0$ by

$$I_{32} = \int_0^t \int_\Omega |(\overline{\mathbf{u}}_\alpha \cdot \nabla)(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x} \, ds$$

$$\leq \int_0^t ||\overline{\mathbf{u}}_\alpha||_\infty^2 ||\nabla(\mathbf{u}_\alpha - \overline{\mathbf{u}}_\alpha)||^2 \, ds$$

$$\leq \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2 \int_0^t ||\Delta \mathbf{u}_\alpha||^2 \, ds$$

$$\leq \begin{cases} \dfrac{C_{ML_a}}{\nu} \left(K_1 \log\left(\dfrac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2 & \text{if } \alpha < L/2\pi, \\ \dfrac{C_{ML_a}}{\nu} K_2 \alpha^2 & \text{if } \alpha \geq L/2\pi, \end{cases} \tag{5.13}$$

here Lemma 8.2 and (2.7) have been applied. There for the proof for this model is finished by (5.10)-(5.13)

$$I(s) \leq \frac{C_r C_{NSE}}{\nu} \alpha^3 + \frac{C_{ML_a} C_r^{1/2} C_R^{1/2}}{\nu} \alpha^{5/2} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_{ML_a} \right)^{1/2}$$

$$+ \frac{C C_{ML_a}^2}{\nu} \alpha^4 + \frac{C_{ML_a}}{\nu} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 \right) \alpha^2, \tag{5.14}$$

if $\alpha < L/2\pi$ and

$$I(s) \leq \frac{C_r C_{NSE}}{\nu}\alpha^3 + \frac{C_{ML_a} C_r^{1/2} C_R^{1/2}}{\nu}\alpha^{5/2} + \frac{CC_{ML_a}^2}{\nu}\alpha^4 + \frac{C_{ML_a} K_2}{\nu}\alpha^2, \tag{5.15}$$

if $\alpha \geq L/2\pi$.

Proof of the SB model. For this model we have for all $t \geq 0$

$$\begin{aligned} I(t) &= \int_0^t \int_\Omega |(\mathbf{u}\cdot\nabla)\mathbf{u} - (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha + (\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{u}_\alpha - (\overline{\mathbf{u}}_\alpha\cdot\nabla)\overline{\mathbf{u}}_\alpha|^2\, d\mathbf{x}ds \\ &\leq 4(I_1 + I_2 + I_3). \end{aligned} \tag{5.16}$$

One has used the fact that $\mathbf{u}_\alpha = \mathbf{u} - \mathbf{e}$ in the second term inside the integral. Similarity, by Corollary 4.1 and Lemma 3.4

$$\begin{aligned} I_1 &= \int_0^t \int_\Omega |((\mathbf{u} - \overline{\mathbf{u}}_\alpha)\cdot\nabla)\mathbf{u}|^2\, d\mathbf{x}ds \\ &\leq \int_0^t ||\mathbf{u} - \overline{\mathbf{u}}_\alpha||\, ||\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)||\, ||\nabla\mathbf{u}||\, ||\Delta\mathbf{u}|| \\ &\leq \frac{C_{NSE2}^{1/2}}{\nu} C_{cor}^{1/2}(\alpha^2 + \alpha^3)^{1/2} \left(\nu\int_0^t ||\nabla(\mathbf{u} - \overline{\mathbf{u}}_\alpha)||^2\, ds\right)^{1/2} \left(\nu\int_0^t ||\Delta\mathbf{u}||^2\, ds\right)^{1/2} \\ &\leq \frac{C_{NSE2}}{\nu} C_{cor}(\alpha^2 + \alpha^3) \quad \forall t \geq 0. \end{aligned} \tag{5.17}$$

We deal with the second integral by using Lemma 3.4 and Theorems 4.1, 4.2

$$\begin{aligned} I_2 &= \int_0^t \int_\Omega |(\overline{\mathbf{u}}_\alpha\cdot\nabla)\mathbf{e}|^2\, d\mathbf{x}ds \\ &\leq \int_0^t ||\overline{\mathbf{u}}_\alpha||||\nabla\overline{\mathbf{u}}_\alpha||||\nabla\mathbf{e}||||\Delta\mathbf{e}||\, d\mathbf{x}ds \\ &\leq \frac{CC_{SB}}{\nu\lambda_1^{1/2}} \left(\nu\int_0^t ||\nabla\mathbf{e}||^2\, ds\right)^{1/2} \left(\nu\int_0^t ||\Delta\mathbf{e}||^2\, ds\right)^{1/2} \\ &\leq \begin{cases} \dfrac{CC_{SB}}{\nu\lambda_1^{1/2}} C_r^{1/2} C_R^{1/2}\alpha^{5/2}\left(K_1 \log\left(\dfrac{L}{2\pi\alpha}\right) + K_2 + C_{SB}\right)^{1/2} & \text{if } \alpha < L/2\pi, \\ \dfrac{CC_{SB}}{\nu\lambda_1^{1/2}} C_r^{1/2} C_R^{1/2}\alpha^{5/2} & \text{if } \alpha < L/2\pi. \end{cases} \end{aligned} \tag{5.18}$$

Similarity, the last term can be estimated for all $t \geq 0$ by

$$
\begin{aligned}
I_3 &= \int_0^t \int_\Omega |(\bar{\mathbf{u}}_\alpha \cdot \nabla)(\mathbf{u} - \bar{\mathbf{u}}_\alpha)|^2 \, d\mathbf{x} ds \\
&\leq \int_0^t ||\bar{\mathbf{u}}_\alpha|| ||\nabla \bar{\mathbf{u}}_\alpha|| ||\nabla(\mathbf{u} - \bar{\mathbf{u}}_\alpha)|| ||\Delta(\mathbf{u} - \bar{\mathbf{u}}_\alpha)|| \, ds \\
&\leq \frac{C_{SB}^{1/2}}{\nu \lambda_1^{1/2}} (2C_R \, h(\alpha) + 2CC_{SB}\alpha^2)^{1/2} \left(\nu \int_0^t ||\nabla \mathbf{u}_\alpha||^2 \, ds \right)^{1/2} \left(\nu \int_0^t ||\Delta(\mathbf{u} - \bar{\mathbf{u}}_\alpha)||^2 \, ds \right)^{1/2} \\
&\leq \frac{C_{SB}}{\nu \lambda_1^{1/2}} (2C_R \, h(\alpha) + 2CC_{SB}\alpha^2)
\end{aligned}
$$

(5.19)
$$
= \begin{cases} \dfrac{CC_{SB}}{\nu \lambda_1^{1/2}} \left[C_R \, \alpha^2 \left(K_1 \log\left(\dfrac{L}{2\pi\alpha}\right) + K_2 + C_{SB} \right) + C_{SB}\alpha^2 \right] & \text{if } \alpha < L/2\pi, \\ \dfrac{CC_{SB}}{\nu \lambda_1^{1/2}} (C_R + C_{SB})\alpha^2 & \text{if } \alpha < L/2\pi. \end{cases}
$$

Therefore, by (5.17)-(5.19)

$$
I(s) \leq \frac{C_{NSE2}}{\nu} C_{cor}(\alpha^2 + \alpha^3) + \frac{CC_{SB}}{\nu \lambda_1^{1/2}} C_r^{1/2} C_R^{1/2} \alpha^{5/2} \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_{SB} \right)^{1/2}
$$

$$
+ \frac{CC_{SB}}{\nu \lambda_1^{1/2}} \left[C_R \left(K_1 \log\left(\frac{L}{2\pi\alpha}\right) + K_2 + C_{SB} \right) + C_{SB} \right] \alpha^2 \tag{5.20}
$$

if $\alpha < L/2\pi$ and

$$
I(s) \leq \frac{C_{NSE2}}{\nu} C_{cor}(\alpha^2 + \alpha^3) + \frac{CC_{SB}}{\nu \lambda_1^{1/2}} C_r^{1/2} C_R^{1/2} \alpha^{5/2} + \frac{CC_{SB}}{\nu \lambda_1^{1/2}} (C_R + C_{SB})\alpha^2, \tag{5.21}
$$

if $\alpha \geq L/2\pi$. Thus the proof for this model is completed.

Proof of the NS-α model. First we rewrite the NSE (1.1)-(1.3) and α-models (1.4)-(1.6) in the rotational forms as in the proof of Theorem 4.1. Adding and subtracting $\mathbf{u}_\alpha \times (\nabla \times \mathbf{u}_\alpha)$ instead of $(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha$ in (5.4). It can be seen that the proof of this model follows by that of the L-α model with the details are skipped. The bound of $I(s)$ for $s \geq 0$ shares the similar form as in (5.9). □

6 The 3D case

In this section is devoted for the rate of convergence of weak solutions of the α-models to that of the NSE in the 3D case. If $\mathbf{u} \in \mathbf{L}^4([0,T]; \mathbf{H}^1(\Omega))$ the standard Sobolev embedding implies that $\mathbf{u} \in \mathbf{L}^4([0,T]; \mathbf{L}^6(\Omega))$ which is a special case of the well-known Leray-Serrin-Prodi (LSP) 3D uniqueness assumption, where $r = 4$ and $s = 6$, see formula (6.1) below, see Leray [25], Prodi [27] and Serrin [31]. More specifically, that is

$$
\mathbf{u} \in \mathbf{L}^r([0,T]; \mathbf{L}^s(\Omega)) \quad \text{where} \quad \frac{3}{s} + \frac{2}{r} = 1, \; s \geq 3. \tag{6.1}
$$

It is also known that, see for example Galdi [14, Definition 2.1, Theorem 4.2], weak solutions satisfy the LSP condition are unique and regular in the set of all Leray-Hopf weak solutions. Recently, under the conditions $\mathbf{f} \in \mathbf{L}^2([0,T];\mathbf{H})$, $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ and an extra condition $\mathbf{u} \in \mathbf{L}^4([0,T];\mathbf{H}^1(\Omega))$, the author of [9] showed that the rate of convergence of weak solutions $\mathbf{u}_\alpha$ of the four α-models to $\mathbf{u}$ is $\mathcal{O}(\alpha)$ for some suitable norms. More precisely, that is

$$\sup_{t\in[0,T]} ||\mathbf{e}(t)||^2 + \nu \int_0^T ||\nabla \mathbf{e}||^2 \, dt \le C(T)\alpha^2, \tag{6.2}$$

where C is the Sobolev constant and C_T is given by

$$C(T) = C_1 \exp\left\{ \frac{C}{\nu^3} \int_0^T ||\nabla \mathbf{u}||^4 \, ds \right\}. \tag{6.3}$$

here $C_1 = C_1(\mathbf{u}_0, \mathbf{f}, \nu)$. On one hand, it follows that in the case $\mathbf{u} \in \mathbf{L}^4(\mathbb{R}_+;\mathbf{H}^1(\Omega))$, which satisfies (6.1), we get the error is uniformly bounded in time, i.e.,

$$\sup_{t\ge 0} ||\mathbf{e}(t)||^2 + \nu \int_0^\infty ||\nabla \mathbf{e}||^2 \, dt \le C_\infty \alpha^2, \tag{6.4}$$

where

$$C_\infty = C_1 \exp\left\{ \frac{C}{\nu^3} \int_0^\infty ||\nabla \mathbf{u}||^4 \, ds \right\}. \tag{6.5}$$

On the other hand, if a weak solution $\mathbf{u}$ of the NSE regular up to a limit time $T_* < \infty$ or we say that $\mathbf{u}$ becomes irregular at the time T_*. Assume that T_* is the first time that $\mathbf{u}$ becomes irregular, see Definition 6.1 in Galdi [14], then it is proved that the $\mathbf{H}^1(\Omega)$-norm of $\mathbf{u}$, $||\nabla \mathbf{u}(t)||^2$ will blow-up as t closes to T_* from below, see for instance [14, Theorem 6.4], Leray [25] and Scheffer [30]. That is given in the following form: there exists $\epsilon = \epsilon_{T_*} > 0$ small enough such that

$$||\nabla \mathbf{u}(t)|| \ge \frac{C\nu^{3/4}}{(T_* - t)^{1/4}} \quad \forall t \in (T_* - \epsilon, T_*), \tag{6.6}$$

where $C > 0$ only depending on Ω. In that case, by (6.6), we consider $C(T)$ in (6.3) with $T_* - \epsilon < T < T_*$, which will also blow-up as in the following way

$$\begin{aligned} C(T) &= C_1 \exp\left\{ \frac{C}{\nu^3} \int_0^T ||\nabla \mathbf{u}||^4 \, ds \right\} \\ &\ge C_1 \exp\left\{ \frac{C}{\nu^3} \int_{T_*-\epsilon}^T ||\nabla \mathbf{u}||^4 \, ds \right\} \\ &\ge C_1 \exp\left\{ C \int_{T_*-\epsilon}^T \frac{1}{T_* - s} \, ds \right\} \\ &= C_1 \frac{\epsilon^C}{(T_* - T)^C}. \end{aligned} \tag{6.7}$$

7 Conclusions

In this report under a regularity assumption $\mathbf{u}_0 \in \mathbf{V}$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}_+, \mathbf{H})$, we provide the rate of convergence of $\mathbf{u}_\alpha$ to $\mathbf{u}$ as well as p_α to p. In addition our argument is tied up to the periodic case mostly because special properties of the Stokes operator A. The extension of the results to other boundary conditions such as the Dirichlet boundary conditions or to the Euler equations are left as future works. In the 3D case extra assumptions should be assumed for the uniqueness of solution of the NSE before studying the rate of convergence.

Remark 7.1. *It seems to be the case that all results herein can be established when the periodic domain $\Omega = [0, L]^2$ is replaced by the whole space $\mathbb{R}^2$. However, the existence and uniqueness of weak solutions of all α-models herein need to be studying carefully first. That will be investigated for the forthcoming works.*

8 Appendix

This section is devoted to give the proof of some results which have been applied before. We give all details for completeness.

It is known that the trilinear form associated to the rotational 2D NSE $\tilde{B}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \times (\nabla \times \mathbf{v}), \mathbf{w})$ satisfies, see [2, formula 2.19 page 1237],

$$\tilde{B}(\mathbf{u}, \mathbf{v}, \mathbf{u}) = 0 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}. \tag{8.1}$$

Moreover, in the 2D periodic setting of $\tilde{B}$ is given by

$$\tilde{B}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\boldsymbol{\Omega}} \left(\frac{\partial \mathbf{v_1}}{\partial \mathbf{y}} - \frac{\partial \mathbf{v_2}}{\partial \mathbf{x}} \right) (\mathbf{u_2 w_1} - \mathbf{u_1 w_2})\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \tag{8.2}$$

where $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$, $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$, $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$. In addition, $\tilde{B}$ satisfies the following property, see [2, formula 2.18 page 1237],

$$\tilde{B}(\mathbf{u}, \mathbf{u}, \Delta\mathbf{u}) = 0 \quad \forall \mathbf{u} \in \mathcal{D}(A). \tag{8.3}$$

The property (8.3) extends to the trilinear form associated to the NS-α model and that is stated in the next lemma:

Lemma 8.1. *In the 2D periodic case there holds for all $\alpha \geq 0$*

$$\tilde{B}(\overline{\mathbf{u}}, \mathbf{u}, \Delta\mathbf{u}) = (\overline{\mathbf{u}} \times (\nabla \times \mathbf{u}), \Delta\mathbf{u}) = 0 \quad \forall \mathbf{u} \in \mathcal{D}(A). \tag{8.4}$$

Remark 8.1. *Property (8.4) of the trilinear form $\tilde{B}$ is mentioned in Rebholz, [28] and its proof is most likely present somewhere in the related literature. We make no claim of novelty and we provide a proof bellow only for completeness.*

Proof. We replace successively on the r.h.s of (8.4) the first term $\overline{\mathbf{u}}$ by $\alpha^2\Delta\overline{\mathbf{u}} + \mathbf{u}$, the last term $\Delta\mathbf{u}$ by $-\alpha^2\Delta\Delta\overline{\mathbf{u}} + \Delta\overline{\mathbf{u}}$, and the middle term $\mathbf{u}$ by $-\alpha^2\Delta\overline{\mathbf{u}} + \overline{\mathbf{u}}$ and use (8.1) and (8.3) to obtain that

$$\tilde{B}(\overline{\mathbf{u}}, \mathbf{u}, \Delta\mathbf{u}) = -\alpha^4 \tilde{B}(\Delta\overline{\mathbf{u}}, \overline{\mathbf{u}}, \Delta\Delta\overline{\mathbf{u}}). \tag{8.5}$$

Using (8.2) to rewrite the above equality we have that

$$I := \tilde{B}(\Delta\overline{\mathbf{u}}, \overline{\mathbf{u}}, \Delta\Delta\overline{\mathbf{u}}) = \int_\Omega \left(\frac{\partial \overline{\mathbf{u}}_1}{\partial y} - \frac{\partial \overline{\mathbf{u}}_2}{\partial x} \right) (\Delta\overline{\mathbf{u}}_2 \Delta\Delta\overline{\mathbf{u}}_1 - \Delta\overline{\mathbf{u}}_1 \Delta\Delta\overline{\mathbf{u}}_2)\, dxdy.$$

We expand the double Laplacian:

$$\begin{aligned} I = &\int_\Omega \left(\frac{\partial \overline{\mathbf{u}}_1}{\partial y} - \frac{\partial \overline{\mathbf{u}}_2}{\partial x} \right) \Delta\overline{\mathbf{u}}_2 \left(\frac{\partial^2 \Delta\overline{\mathbf{u}}_1}{\partial x^2} + \frac{\partial^2 \Delta\overline{\mathbf{u}}_1}{\partial y^2} \right) dxdy \\ &- \int_\Omega \left(\frac{\partial \overline{\mathbf{u}}_1}{\partial y} - \frac{\partial \overline{\mathbf{u}}_2}{\partial x} \right) \Delta\overline{\mathbf{u}}_1 \left(\frac{\partial^2 \Delta\overline{\mathbf{u}}_2}{\partial x^2} + \frac{\partial^2 \Delta\overline{\mathbf{u}}_2}{\partial y^2} \right) dxdy =: I_1 - I_2. \end{aligned} \tag{8.6}$$

We integrate by parts to get rid of the second order derivatives. The first integral from above is equal to:

$$I_1 = -\int_\Omega \left[\left(\frac{\partial^2 \bar{\mathbf{u}}_1}{\partial x \partial y} - \frac{\partial^2 \bar{\mathbf{u}}_2}{\partial x^2} \right) \Delta \bar{\mathbf{u}}_2 \frac{\partial \Delta \bar{\mathbf{u}}_1}{\partial x} + \left(\frac{\partial \bar{\mathbf{u}}_1}{\partial y} - \frac{\partial \bar{\mathbf{u}}_2}{\partial x} \right) \frac{\partial \Delta \bar{\mathbf{u}}_2}{\partial x} \frac{\partial \Delta \bar{\mathbf{u}}_1}{\partial x} \right] dxdy$$
$$- \int_\Omega \left[\left(\frac{\partial^2 \bar{\mathbf{u}}_1}{\partial y^2} - \frac{\partial^2 \bar{\mathbf{u}}_2}{\partial x \partial y} \right) \Delta \bar{\mathbf{u}}_2 \frac{\partial \Delta \bar{\mathbf{u}}_1}{\partial y} + \left(\frac{\partial \bar{\mathbf{u}}_1}{\partial y} - \frac{\partial \bar{\mathbf{u}}_2}{\partial x} \right) \frac{\partial \Delta \bar{\mathbf{u}}_2}{\partial y} \frac{\partial \Delta \bar{\mathbf{u}}_1}{\partial y} \right] dxdy. \tag{8.7}$$

Similarity, the second integral in (8.6) is equal to

$$I_2 = -\int_\Omega \left[\left(\frac{\partial^2 \bar{\mathbf{u}}_1}{\partial x \partial y} - \frac{\partial^2 \bar{\mathbf{u}}_2}{\partial x^2} \right) \Delta \bar{\mathbf{u}}_1 \frac{\partial \Delta \bar{\mathbf{u}}_2}{\partial x} + \left(\frac{\partial \bar{\mathbf{u}}_1}{\partial y} - \frac{\partial \bar{\mathbf{u}}_2}{\partial x} \right) \frac{\partial \Delta \bar{\mathbf{u}}_1}{\partial x} \frac{\partial \Delta \bar{\mathbf{u}}_2}{\partial x} \right] dxdy$$
$$- \int_\Omega \left[\left(\frac{\partial^2 \bar{\mathbf{u}}_1}{\partial y^2} - \frac{\partial^2 \bar{\mathbf{u}}_2}{\partial x \partial y} \right) \Delta \bar{\mathbf{u}}_1 \frac{\partial \Delta \bar{\mathbf{u}}_2}{\partial y} + \left(\frac{\partial \bar{\mathbf{u}}_1}{\partial y} - \frac{\partial \bar{\mathbf{u}}_2}{\partial x} \right) \frac{\partial \Delta \bar{\mathbf{u}}_1}{\partial y} \frac{\partial \Delta \bar{\mathbf{u}}_2}{\partial y} \right] dxdy. \tag{8.8}$$

Subtracting (8.8) from (8.7), we obtain that all terms containing $\left(\frac{\partial \bar{\mathbf{u}}_1}{\partial y} - \frac{\partial \bar{\mathbf{u}}_2}{\partial x} \right)$ will sum up to 0 and therefore

$$I = -\int_\Omega \left[\left(\frac{\partial^2 \bar{\mathbf{u}}_1}{\partial x \partial y} - \frac{\partial^2 \bar{\mathbf{u}}_2}{\partial x^2} \right) \Delta \bar{\mathbf{u}}_2 \frac{\partial \Delta \bar{\mathbf{u}}_1}{\partial x} + \left(\frac{\partial^2 \bar{\mathbf{u}}_1}{\partial y^2} - \frac{\partial^2 \bar{\mathbf{u}}_2}{\partial x \partial y} \right) \Delta \bar{\mathbf{u}}_2 \frac{\partial \Delta \bar{\mathbf{u}}_1}{\partial y} \right] dxdy$$
$$+ \int_\Omega \left[\left(\frac{\partial^2 \bar{\mathbf{u}}_1}{\partial x \partial y} - \frac{\partial^2 \bar{\mathbf{u}}_2}{\partial x^2} \right) \Delta \bar{\mathbf{u}}_1 \frac{\partial \Delta \bar{\mathbf{u}}_2}{\partial x} + \left(\frac{\partial^2 \bar{\mathbf{u}}_1}{\partial y^2} - \frac{\partial^2 \bar{\mathbf{u}}_2}{\partial x \partial y} \right) \Delta \bar{\mathbf{u}}_1 \frac{\partial \Delta \bar{\mathbf{u}}_2}{\partial y} \right] dxdy. \tag{8.9}$$

Since $\nabla \cdot \mathbf{u} = 0$ then $\nabla \cdot \bar{\mathbf{u}} = 0$ or $\partial \bar{\mathbf{u}}_1 / \partial x = -\partial \bar{\mathbf{u}}_2 / \partial y$ and we have that

$$-\Delta \bar{\mathbf{u}}_2 = \frac{\partial^2 \bar{\mathbf{u}}_1}{\partial x \partial y} - \frac{\partial^2 \bar{\mathbf{u}}_2}{\partial x^2} \quad \text{and} \quad \Delta \bar{\mathbf{u}}_1 = \frac{\partial^2 \bar{\mathbf{u}}_1}{\partial y^2} - \frac{\partial^2 \bar{\mathbf{u}}_2}{\partial x \partial y}.$$

Replacing in (8.9) gives

$$I = -\int_\Omega \left[(-\Delta \bar{\mathbf{u}}_2) \Delta \bar{\mathbf{u}}_2 \frac{\partial \Delta \bar{\mathbf{u}}_1}{\partial x} + \Delta \bar{\mathbf{u}}_1 \Delta \bar{\mathbf{u}}_2 \frac{\partial \Delta \bar{\mathbf{u}}_1}{\partial y} \right] dxdy$$
$$+ \int_\Omega \left[(-\Delta \bar{\mathbf{u}}_2) \Delta \bar{\mathbf{u}}_1 \frac{\partial \Delta \bar{\mathbf{u}}_2}{\partial x} + \Delta \bar{\mathbf{u}}_1 \Delta \bar{\mathbf{u}}_1 \frac{\partial \Delta \bar{\mathbf{u}}_2}{\partial y} \right] dxdy. \tag{8.10}$$

We let $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ with $\mathbf{v}_1 = \Delta \bar{\mathbf{u}}_1$, $\mathbf{v}_2 = \Delta \bar{\mathbf{u}}_2$ and upon replacing in (8.10) it follows that we need to show that

$$\int_\Omega \left(\mathbf{v}_2 \mathbf{v}_2 \frac{\partial \mathbf{v}_1}{\partial x} - \mathbf{v}_1 \mathbf{v}_2 \frac{\partial \mathbf{v}_1}{\partial y} - \mathbf{v}_2 \mathbf{v}_1 \frac{\partial \mathbf{v}_2}{\partial x} + \mathbf{v}_1 \mathbf{v}_1 \frac{\partial \mathbf{v}_2}{\partial y} \right) dxdy = 0. \tag{8.11}$$

Using $\nabla \cdot \mathbf{v} = 0$ (since $\nabla \cdot \bar{\mathbf{u}} = 0$) and integration by parts we can immediately show that the first and the last terms in (8.11) will vanish, i.e.,

$$\int_\Omega \mathbf{v}_2 \mathbf{v}_2 \frac{\partial \mathbf{v}_1}{\partial x} \, dxdy = 0 \quad \text{and} \quad \int_\Omega \mathbf{v}_1 \mathbf{v}_1 \frac{\partial \mathbf{v}_2}{\partial y} \, dxdy = 0. \tag{8.12}$$

Then using (8.12) and integration by parts we obtain that the second and the third terms in (8.11) will vanish, i.e.,

$$\int_\Omega \mathbf{v}_1 \mathbf{v}_2 \frac{\partial \mathbf{v}_1}{\partial y} \, dxdy = 0, \quad \text{and} \quad \int_\Omega \mathbf{v}_2 \mathbf{v}_1 \frac{\partial \mathbf{v}_2}{\partial x} \, dxdy = 0.$$

Therefore going back to (8.10) gives

$$I = \tilde{B}(\Delta \bar{\mathbf{u}}, \bar{\mathbf{u}}, \Delta \Delta \bar{\mathbf{u}}) = 0,$$

and (8.5) then the proof of this lemma is finished. □

Thanks to the a priori estimates given in Section 3 the result [2, Proposition 4.2] follows with requiring only $\mathbf{u}_0 \in \mathbf{V}$ instead of $\mathbf{u}_0 \in \mathcal{D}(A)$. We give a proof here for completeness.

Lemma 8.2. *Let $\mathbf{u}_\alpha$ be the weak solutions of any α-model. Then there exist K_1 and K_2 such that for all $t \geq 0$*

$$||\overline{\mathbf{u}}_\alpha(t)||_\infty^2 \leq \begin{cases} K_1 \log\left(\dfrac{L}{2\pi\alpha}\right) + K_2 & \text{if } \alpha < L/2\pi, \\ K_2 & \text{if } \alpha \geq L/2\pi. \end{cases} \tag{8.13}$$

Proof. Assume that $0 \leq \alpha < \lambda_1^{-1/2} = L/2\pi$. Thanks to the Brezis-Gallouet inequality, see for example Cao-Titi [2, Proposition 2.3], there exists a constant $C = C(\Omega) > 0$ such that

$$\begin{aligned}
||\overline{\mathbf{u}}_\alpha||_\infty^2 &\leq C||\nabla\overline{\mathbf{u}}_\alpha||^2 \left[1 + \log\left(\frac{L}{2\pi}\frac{||\Delta\overline{\mathbf{u}}_\alpha||}{||\nabla\overline{\mathbf{u}}_\alpha||}\right)\right] \\
&= C\left[||\nabla\overline{\mathbf{u}}_\alpha||^2\left(1 + \log\left(\frac{L}{2\pi\alpha}\right)\right) + ||\nabla\overline{\mathbf{u}}_\alpha||^2 \log\left(\frac{\alpha||\Delta\overline{\mathbf{u}}_\alpha||}{||\nabla\overline{\mathbf{u}}_\alpha||}\right)\right] \\
&\leq C\left[C_E\left(1 + \log\left(\frac{L}{2\pi\alpha}\right)\right) + \frac{C_1^2}{\lambda_1^{1/2}}\frac{||\nabla\overline{\mathbf{u}}_\alpha||}{C_1}\log\left(\frac{C_1}{||\nabla\overline{\mathbf{u}}_\alpha||}\right)\right] \\
&\leq C\left[C_E\left(1 + \log\left(\frac{L}{2\pi\alpha}\right)\right) + \frac{C_1^2}{e\lambda_1^{1/2}}\right].
\end{aligned}$$

Here one has used the following estimate in the third inequality above

$$\alpha^2||\Delta\overline{\mathbf{u}}_\alpha||^2 \leq C||\mathbf{u}_\alpha||^2 \leq C\frac{||\nabla\mathbf{u}_\alpha||^2}{\lambda_1} \leq C\frac{C_E}{\lambda_1} =: C_1^2, \quad (C_1 > 0) \tag{8.14}$$

and

$$||\nabla\overline{\mathbf{u}}_\alpha|| \leq C||\nabla\mathbf{u}_\alpha|| \leq CC_E^{1/2} \leq \frac{CC_1}{\lambda_1^{1/2}}, \tag{8.15}$$

where the two first estimates in (8.14)-(8.15) are given by (2.5) and the constant C_E is given as in Corollary 4.1. Thus, in this case the proof follows by choosing $K_1 = CC_E$ and $K_2 = K_1 + C_1^2/\left(e\lambda_1^{1/2}\right)$. In the other case $\alpha \geq L/2\pi$ we have $\log(L/2\pi\alpha) \leq 0$. Therefore, the proof follows by the previous case. □

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