A REMARK ON THE MAYER-VIETORIS DOUBLE COMPLEX FOR SINGULAR COHOMOLOGY

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Abstract. Given an open cover of a paracompact topological space X, there are two natural ways to construct a map from the cohomology of the nerve of the cover to the cohomology of X. One of them is based on a partition of unity, and is more topological in nature, while the other one relies on the Mayer-Vietoris double complex, and has a more algebraic flavour. In this paper we prove that these two maps coincide, thus answering a question posed by N. V. Ivanov.

Let X be a paracompact space, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. We denote by $N(\mathcal{U})$ the *nerve* of $\mathcal{U}$, i.e. the simplicial set having I as set of vertices, in which a finite subset $\{i_0, \ldots, i_k\} \subseteq I$ spans a simplex if and only if $U_{i_0} \cap \ldots \cap U_{i_k} \neq \varnothing$. As usual, we endow the geometric realization $|N(\mathcal{U})|$ of $N(\mathcal{U})$ with the weak topology associated to the natural CW structure of $|N(\mathcal{U})|$.

Any partition of unity $\Phi = \{\varphi_i \colon X \to \mathbb{R}\}_{i\in I}$ subordinate to $\mathcal{U}$ induces a map

$$f_\Phi \colon X \to |N(\mathcal{U})| , \qquad f_\Phi(x) = \sum_{i\in I} \varphi_i(x) \cdot i \ .$$

Moreover, the homotopy class of f_Φ does not depend on the chosen partition of unity Φ. Indeed, if Ψ is another partition of unity, then we have a well-defined homotopy $tf_\Psi + (1-t)f_\Phi$ between f and g. Therefore, if R is any ring with unity, the map f_Φ induces a map

$$f^* = f_\Phi^* \colon H^*(|N(\mathcal{U})|, R) \to H^*(X, R) \ ,$$

which does not depend on the choice of Φ. Throughout this paper, we fix a ring with unity R, and for any topological space Y we denote by $C^*(Y) = C^*(Y, R)$ (resp. $H^*(Y) = H^*(Y, R)$) the singular cochain complex (resp. the singular cohomology algebra) of Y with coefficients in R.

There is another natural way to define a map from the (simplicial) cohomology of $N(\mathcal{U})$ to the singular cohomology of X. Let $C^{*,*}(\mathcal{U})$ be the Mayer-Vietoris double complex associated to $\mathcal{U}$, i.e. for every $(p, q) \in \mathbb{N}^2$ let

$$C^{p,q}(\mathcal{U}) = \prod_{\underline{i}\in I_p} C^q(U_{\underline{i}}) \ ,$$

where I_p denotes the set of ordered $(p+1)$-tuples $(i_0, \ldots, i_p) \in I^{p+1}$ such that $U_{\underline{i}} := U_{i_0} \cap \ldots \cap U_{i_p} \neq \varnothing$ (in particular, $I_0 = \{i \in I \,|\, U_i \neq \varnothing\}$). We refer the reader to Section 1 for the precise definition of this double complex.

To the double complex $C^{*,*}(\mathcal{U})$ there is associated the *total complex* T^*, and we have maps

$$\alpha_X\colon H^*(X) \to H^*(T^*)\ , \qquad \beta\colon H^*(N(\mathcal{U})) \to H^*(T^*)$$

from the singular cohomology of X to the cohomology of T^* and from the simplicial cohomology of $N(\mathcal{U})$ to the cohomology of T^*. Moreover, the map α turns out to be an isomorphism (see Section 1).

Let now $\nu\colon H^*(|N(\mathcal{U})|) \to H^*(N(\mathcal{U}))$ be the canonical isomorphism between the simplicial cohomology of $N(\mathcal{U})$ and the singular cohomology of its geometric realization (see Section 2). By setting $\eta = \alpha_X^{-1} \circ \beta \circ \nu$ we have thus defined a map

$$\eta\colon H^*(|N(\mathcal{U})|) \to H^*(X)\ .$$

The main result of this paper shows that the maps f^* and η coincide:

Theorem 1. *The maps*

$$f^*\colon H^*(|N(\mathcal{U})|) \to H^*(X)\,, \quad \eta\colon H^*(|N(\mathcal{U})|) \to H^*(X)$$

coincide.

Theorem 1 answers a question posed by Ivanov in [Iva87, page 1113] and in [Iva, page 71].

1. The Mayer-Vietoris double complex

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of the topological space X. We now thoroughly describe the horizontal and the vertical differentials of the double complex $C^{*,*}(\mathcal{U})$ defined in the introduction, also fixing the notation we will need later.

If $\varphi \in C^{p,q}(\mathcal{U})$ and $\underline{i} \in I_p$, then we denote by $\varphi_{\underline{i}}$ the projection of φ on $C^q(U_{\underline{i}})$. For every $(p,q) \in \mathbb{N}^2$ we denote by

$$\delta_v^{p,q}\colon C^{p,q}(\mathcal{U}) \to C^{p,q+1}(\mathcal{U})$$

the "vertical" differential which restricts to the usual differential $C^q(U_{\underline{i}}) \to C^{q+1}(U_{\underline{i}})$ for every $\underline{i} \in I_p$, and by

$$\delta_h^{p,q}\colon C^{p,q}(\mathcal{U}) \to C^{p+1,q}(\mathcal{U})$$

the "horizontal" differential such that, for every $\underline{i} = (i_0, \dots, i_{p+1}) \in I_{p+1}$ and every $\varphi \in C^{p,q}(\mathcal{U})$,

$$(\delta_h^{p,q}(\varphi))_{\underline{i}} = \sum_{k=0}^{p+1} (-1)^k \big(\varphi_{(i_0,\dots,\hat{i}_k,\dots,i_{p+1})}\big)|_{U_{\underline{i}}}\ . \tag{1}$$

We augment the double complex $C^{*,*}(\mathcal{U})$ as follows. We define $C_q^{\mathcal{U}}$ as the subcomplex of the singular chain complex $C_q(X)$ generated (over R) by those singular simplices $s\colon \Delta^q \to X$ such that $s(\Delta^q)$ is contained in U_i for some $i \in I$. We then set $C^{-1,q}(\mathcal{U}) = C^q_{\mathcal{U}} = \mathrm{Hom}(C_q^{\mathcal{U}}, R)$. The usual boundary maps of the complex $C_*^{\mathcal{U}}$ induce dual coboundary maps, which endow $C^*_{\mathcal{U}}$ with the structure of a complex. The inclusion of the complex $C_*^{\mathcal{U}}$ in the full complex of singular chains induces a map of complexes $\widetilde{\gamma} : C^*(X) \to C^*_{\mathcal{U}}$. It is known that the map γ induced in cohomology is an isomorphism (see e.g. [Hat02, Proposition 2.21]) and we will identify the singular cohomology of X with the cohomology of the complex $C^q_{\mathcal{U}}$ via γ. The augmentation maps $\delta^{-1,q} \colon C^{-1,q}(\mathcal{U}) \to C^{0,q}(\mathcal{U})$ are defined by setting, for every $i \in I_0$,

$$(\delta^{-1,q}(\varphi))_i = \varphi|_{U_i} \ .$$

In order to define the augmentation of the vertical complexes, we consider the Cech complex given by $C^{p,-1}(\mathcal{U}) = \check{C}^p(\mathcal{U}) = \prod_{\underline{i} \in I_p} R$, with boundary maps defined as in formula (1). We then define the augmentation maps $\delta^{p,-1}$: $C^{p,-1}(\mathcal{U}) \to C^{p,0}(\mathcal{U})$ by setting

$$(\delta^{p,-1}(\varphi))_{\underline{i}}(s) = \varphi_{\underline{i}} \ \in \ R$$

for every $\varphi \in C^{p,-1}(\mathcal{U})$, every $\underline{i} = (i_0, \ldots, i_p) \in I_p$ and every singular simplex $s \colon \Delta^0 \to U_{i_0} \cap \ldots \cap U_{i_p}$.

Remark 1.1. The complex $\check{C}^*(\mathcal{U})$ computes the Cech cohomology of the cover $\mathcal{U}$ with coefficients in the *constant* presheaf R. Such cohomology, which is usually denoted by $\check{H}(\mathcal{U})$, is tautologically isomorphic to the simplicial cohomology of the nerve $N(\mathcal{U})$. It is costumary to rather study the Cech cohomology of $\mathcal{U}$ with coefficients in the *locally constant* sheaf R. However this cohomology does not always coincide with the cohomology of $N(\mathcal{U})$. They coincide, for example, under the assumption that every $U_{\underline{i}}$, $i \in I_p$, $p \in \mathbb{N}$, is path connected.

In the next lemma we prove that the rows of the augmented double complex are exact.

Lemma 1.2. *For every $q \in \mathbb{N}$, the complex*

$$0 \longrightarrow C^{-1,q}(\mathcal{U}) \xrightarrow{\delta_h^{-1,q}} C^{0,q}(\mathcal{U}) \xrightarrow{\delta_h^{0,q}} \cdots \xrightarrow{\delta_h^{p-1,q}} C^{p,q}(\mathcal{U}) \xrightarrow{\delta_h^{p,q}} \cdots$$

is exact.

Proof. Let $s \colon \Delta^q \to X$ be a singular simplex such that $s(\Delta^q)$ is contained in U_i for some $i \in \ I$. We set

$$C_s^{-1,q}(\mathcal{U}) = \{\varphi \in C^{-1,q}(\mathcal{U}) \,|\, \varphi(s') = 0 \text{ for every } s' \neq s\} \ ,$$

and for every $p \geqslant 0$ and every $\underline{i} \in I_p$ we define

$$C_s^{p,q}(U_{\underline{i}}) = \{\varphi \in C^q(U_{\underline{i}}) \,|\, \varphi(s') = 0 \text{ for every } s' \neq s\} \ .$$

We also set $I(s) = \{i \in I \,|\, s(\Delta^q) \subseteq U_i\}$, $I_p(s) = (I(s))^{p+1} \subseteq I_p$, and

$$C_s^{p,q}(\mathcal{U}) = \prod_{\underline{i} \in I_p(s)} C_s^{p,q}(U_{\underline{i}})$$

(according to our definition, $C_s^{p,q}(U_{\overline{i}}) = 0$ whenever $\underline{i} \notin I_p(s)$). We observe that $C_s^{*,q}(\mathcal{U})$ is a subcomplex of $C^{*,q}(\mathcal{U})$, and that

$$C^{p,q}(\mathcal{U}) = \prod_{s \colon \Delta^q \to X} C_s^{p,q}(\mathcal{U}) \ .$$

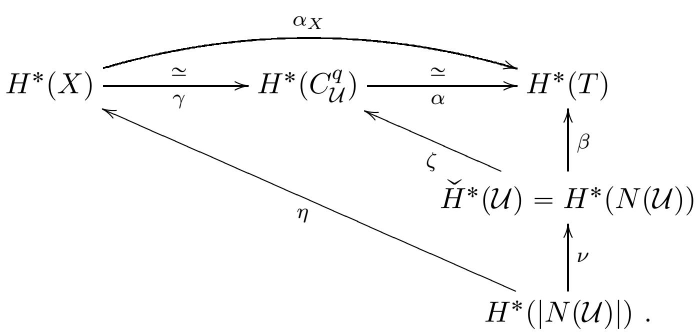
Hence, in order to conclude it is sufficient to show that each $C_s^{*,q}(\mathcal{U})$ is exact. However, the complex $C_s^{*,q}(\mathcal{U})$ is isomorphic to the simplicial cohomology complex of the full simplex with vertices $I(s)$, whence the conclusion. $\square$

As a consequence of the previous lemma the cohomology groups of the complex $C^{-1,*}$ are isomorphic to the cohomology of the *total complex* T^* associated to the double complex. Recall that T^* is defined by setting

$$T^n = \bigoplus_{\substack{(p,q)\in\mathbb{N}^2 \\ p+q=n}} C^{p,q}(\mathcal{U})$$

with differential $\delta^n \colon T^n \to T^{n+1}$ given by $\delta^n = \bigoplus_{p+q=n}(\delta_h^{p,q} + (-1)^p \delta_v^{p,q})$. The augmentation maps induce morphisms of complexes $\widetilde{\alpha}^* : C^*_{\mathcal{U}} \to T^*$ and $\widetilde{\beta}^* \colon \check{C}^* \to T^*$ and we denote by α, β the maps induced by α^*, β^* on cohomology. By Lemma 1.2 α is an isomorphism in every degree and the map $\alpha \circ \gamma \colon H^*(X) \to H^*(T^*)$ is the isomorphism α_X defined in the introduction. We define $\zeta = \alpha^{-1} \circ \beta$ and $\eta = \alpha_X^{-1} \circ \beta \circ \nu$.

The notation introduced so far is summarized in the following diagram:



When we want to stress the dependence of these constructions on the cover $\mathcal{U}$ we write $\alpha_{\mathcal{U}}$, $\beta_{\mathcal{U}}$, etc.

2. The case of a simplicial complex

In this section we analyze the Mayer-Vietoris double complex when $X = |S|$ is the geometric realization of a simplicial complex S. Let I be the vertex set of S. We consider the open cover $\mathcal{U}^* = \{U_i^*\}_{i\in I}$ of $|S|$ given by the open stars of the vertices, i.e. for every $i \in I$ we set $U_i = \{x \in |S| : x_i > 0\}$, where x_i denotes the barycentric coordinate of the point x relative to the vertex i. Observe that the simplical complexes $N(\mathcal{U}^*)$ and S on the set of vertices I are equal and we will identify them. Hence, in this case $\eta_{\mathcal{U}^*} : H^*(|S|) \to H^*(|S|)$. Notice also that in this case all intersections $U_{\underline{i}}^*$ are contractible, hence, also the columns of the augmented double complex are exact. As a consequence, β and ζ are isomorphisms. The next proposition shows that the map η is the identity in this case.

Proposition 2.1. *If S is a simplicial complex and $\mathcal{U}^*$ is the cover described above then $\eta = Id$.*

To prove this proposition we will perform a computation by describing a lift of ζ at the level of cochains. To simplify the computations we will use *alternating* cochains, whose definition is recalled below.

Construction of $\widetilde{\zeta}$. We start by describing a lift

$$\widetilde{\zeta}\colon \check{C}(\mathcal{U}) \to C^{-1,p}(\mathcal{U}) = C^p_{\mathcal{U}}$$

of the map ζ at the level of cochains. We first construct chain homotopies

$$K^{p,q}\colon C^{p,q}(\mathcal{U}) \to C^{p-1,q}(\mathcal{U})\,, \quad p \geqslant 0\,, \quad q \geqslant 0\ .$$

For each singular simplex s with image contained in some open subset U_i we fix an index $i(s)$ such that $\operatorname{Im} s \subseteq U_{i(s)}$. For all $\varphi \in C^{p,q}(\mathcal{U})$ and for all singular simplices s with image contained in $U_{\underline{i}}$ for some $\underline{i} \in I_{p-1}$, $p \geqslant 0$, we define

$$\big(K^{p,q}(\varphi)_{\underline{i}}\big)(s) = \varphi_{i(s),\underline{i}}(s)$$

(when $p = 0$ there is no index $\underline{i}$ and we just take $s \in C^{\mathcal{U}}_q$). It is easy to check that $\delta_h^{p-1,q}K^{p,q} + K^{p+1,q}\delta_h^{p,q} = \mathrm{Id}$ for every $p \geqslant 0$, $q \geqslant 0$. Hence, if we define

$$\widetilde{\zeta} = (-1)^{\frac{p(p+1)}{2}}\, K^{0,p} \circ \delta_v^{0,p-1} \circ K^{1,p-1} \circ \cdots \circ K^{p-1,1} \circ \delta_v^{p-1,0} \circ K^{p,0} \circ \delta_v^{p,-1}$$

then for every cocycle $\varphi \in \check{C}^p(\mathcal{U})$ we have $\zeta([\varphi]) = [\widetilde{\zeta}(\varphi)]$ in $H^p(C^*_{\mathcal{U}})$.

Singular and algebraic simplices. Let us now recall the construction of the isomorphism ν between the simplicial cohomology $H^*(S)$ of S and the singular cohomology $H^*(|S|)$ of its geometric realization. Let $C_*(S)$ be the chain complex of simplicial chains on S, i.e. let C_p be the free R-module with basis

$$\{(i_0,\dots,i_p) \in I^{p+1} \mid \{i_0,\dots,i_p\} \text{ is a simplex of } S\}\ ,$$

and let $C^*(S)$ be the dual chain complex of $C_*(S)$. Elements of the basis just described are usually called *algebraic* simplices.

For any algebraic simplex $\sigma = (i_0,\dots,i_p)$ of S, one defines the singular simplex $\langle\sigma\rangle\colon \Delta^p \to |S|$ by setting

$$\langle\sigma\rangle(t_0,\dots,t_p) = t_0 i_0 + \cdots + t_p i_p\ .$$

The map $\sigma \mapsto \langle\sigma\rangle$ extends to a chain map $C_*(S) \to C_*(|S|)$, whose dual map $\widetilde{\nu} : C^*(|S|) \to C^*(S)$ induces the isomorphism $\nu\colon H^*(|S|) \to H^*(S)$ (see e.g. [Hat02] Theorem 2.27). We write $\nu_S, \widetilde{\nu}_S$ when we want to stress the dependence on the simplicial complex.

Alternating cochains. To compute $\zeta \circ \nu$ it is convenient to use *alternating* cochains. Let $\mathfrak{S}_{p+1}$ be the permutation group of $\{0,\dots,p\}$. We say that a simplicial cochain $\varphi \in C^p(S)$ is alternating if $\varphi(i_{\tau(0)},\dots,i_{\tau(p)}) = \varepsilon(\tau)\varphi(i_0,\dots,i_p)$ for every $\tau \in \mathfrak{S}_{p+1}$, and $\varphi(i_0,\dots,i_p) = 0$ whenever $i_j = i_{j'}$ for some $j \neq j'$. Alternating cochains form a subcomplex of the complex of cochains which is homotopy equivalent to the full complex (see e.g. [Sta18, Chap.20, Section 23]).

Alternating cochains may be defined also in the context of singular homology as follows. For every $\tau \in \mathfrak{S}_{p+1}$ denote by $\rho_\tau\colon \Delta^p \to \Delta^p$ the affine automorphism of Δ^p defined by $\rho_\tau(t_0,\dots,t_p) = (t_{\tau(0)},\dots,t_{\tau(p)})$. If X is a topological space, we say that a singular cochain $\varphi \in C^p(X)$ is *alternating* if $\varphi(s \circ \rho_\tau) = \varepsilon(\tau)\,\varphi(s)$ for every $\tau \in \mathfrak{S}_{p+1}$ and every singular simplex $s\colon \Delta^p \to X$, and $\varphi(s) = 0$ for every singular simplex s such that $s = s \circ \rho_\tau$ for an odd permutation $\tau \in \mathfrak{S}_{p+1}$. Alternating singular cochains form a subcomplex the complex of singular cochains which is homotopy equivalent to the full complex (see e.g. [Bar95]). Moreover, the map $\widetilde{\nu}$ introduced above sends alternating singular cochain to alternating simplicial cochains, and both the homotopy maps $K^{p,q}$ and the vertical differential send alternating cochains to alternating ones.

We want to compute $\widetilde{\zeta}(\varphi)$ on singular simplices of the form $\langle\sigma\rangle$, as σ varies among the algebraic simplices of S. However, simplices of S are not contained in any U_i^*. We will then make use of the barycentric subdivision S' of S, together with a suitable simplicial approximation of the identity $S' \to S$. Let I' be the set of vertices of S'. This set is in bijective correspondence with the set of simplices of S: for $i' \in I'$ we denote by $\Delta_{i'}$ the simplex of S of which i' is the barycenter; in the opposite direction, if Δ is a simplex of S we denote by i'_Δ its barycenter. The p-simplices of S' are then the subsets $\{i'_0, \dots, i'_p\}$ where $\Delta_{i'_0} \subset \cdots \subset \Delta_{i'_p}$.

If for every simplex Δ of S we denote by $b_\Delta \in |S|$ the geometric barycenter of Δ then the map $b\colon |S'| \to |S|$ defined by $b(\sum_\Delta t_\Delta i'_\Delta) = \sum_\Delta t_\Delta b_\Delta$ is a homeomorphism, and we will identify the geometric realization of S' and S via this map. We construct a second map from $|S'|$ to $|S|$ as follows. We fix an auxiliary total ordering on I, and we define a simplicial map $g\colon S' \to S$ by setting

$$g(i') = \max \Delta_{i'}$$

for every vertex i' of S'. The geometric realization $|g|\colon |S| = |S'| \to |S|$ of g is homotopic to b via the homotopy $tb + (1-t)|g|$, $t \in [0,1]$.

We may define the map i used to construct the homotopies $K^{p,q}$ in such a way that, for every algebraic simplex $\sigma' = (i'_0, \dots, i'_p)$ of $C_*(S')$,

$$i(\langle\sigma'\rangle) = \min\{g(i'_0), \dots, g(i'_p)\} \ .$$

For simplicity, we will denote $i(\langle\sigma'\rangle)$ by $i(\sigma')$. With this choice, the singular simplex $\langle\sigma'\rangle$ is supported in $U^*_{i(\sigma')}$ as required in the definition of the map i.

Let $\alpha = (\alpha_{\underline{i}}) \in C^{h,k}(\mathcal{U}^*)$ and let $\sigma' = (i'_0, \dots, i'_{k+1}) \in C_{k+1}(U^*_{\underline{i}})$, $\underline{i} \in I_h$, be an algebraic $(k+1)$-simplex of S'. If $\partial_h \sigma' = (i'_0, \dots, \widehat{i'_h}, \dots, i'_{k+1})$ denotes the algebraic h-th face of σ', then

$$\begin{aligned} \left(\delta_v^{h-1,k} K^{h,k}(\alpha)\right)(\langle\sigma'\rangle) &= \sum_{h=0}^{k+1} (-1)^h \left(K^{h,k}(\alpha)\right)_{\underline{i}}(\langle\partial_h\sigma'\rangle) \\ &= \sum_{h=0}^{k+1} (-1)^h \alpha_{i(\partial_h\sigma'),\underline{i}}(\langle\partial_h\sigma'\rangle) \ . \end{aligned} \tag{2}$$

Lemma 2.2. *Let* φ *be an alternating cocycle in* $C^p(N(\mathcal{U}^*)) = \check{C}^p(\mathcal{U}^*)$*, and let* $\sigma' \in C_p(S')$ *be an algebraic simplex. Then*

$$\big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle) = \varphi(g_*(\sigma')) \ ,$$

where $g_*\colon C_p(S') \to C_p(S)$ *is the map induced by* $g\colon S' \to S$.

Proof. Let $\sigma' = (i'_0, \dots, i'_p)$ and set $\Delta_\ell = \Delta_{i'_\ell}$ for $\ell = 0, \dots, p$ and $i_\ell = g(i'_\ell)$. Recall that simplices of S' corresponds to comparable subsets of a simplex of S. Moreover, since φ is alternating, both $g^*(\varphi)$ and $\widetilde{\zeta}(\varphi)$ are alternating, thus in order to check that the equality of the statement holds we may assume that

$$\Delta_0 \subsetneq \Delta_1 \cdots \subsetneq \Delta_p \ .$$

By definition we have $i_\ell = \max \Delta_\ell$, hence in particular $i_0 \leqslant i_1 \leqslant \cdots \leqslant i_p$. Since φ is alternating, this implies at once that

$$\varphi(g_*(\sigma')) = \begin{cases} \varphi_{i_0, i_1, \dots, i_p} & \text{if } i_0 < \cdots < i_p \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Let us now compute $\big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle)$. For every algebraic simplex $\tau'_k \in C_k(S')$, we write $\tau'_{k-1} < \tau'_k$ if τ'_{k-1} is an algebraic face of τ'_k, i.e. if there exists $h = 0, \dots, k$ such that $\tau'_{k-1} = \partial_h \tau'_k$. By iterating (2) we get

$$\big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle) = (-1)^{\frac{p(p+1)}{2}} \sum_{\sigma'_0 < \cdots < \sigma'_p = \sigma'} \pm \varphi_{i(\sigma'_0), i(\sigma'_1), \dots, i(\sigma'_p)} \ . \tag{4}$$

Let now $\sigma'_0 < \cdots < \sigma'_p$ be a fixed descending sequence of faces of σ'. Since the map i is given by taking a minimum we have $i(\sigma'_0) \geqslant i(\sigma'_1) \geqslant \cdots \geqslant i(\sigma'_p)$ and all these elements belong to the set $\{i_0, \dots, i_p\}$. Hence if $\varphi_{i(\sigma'_0), i(\sigma'_1), \dots, i(\sigma'_p)} \neq 0$ we have $i_0 < \cdots < i_p$ and $i(\sigma'_\ell) = i_{p-\ell}$ for every ℓ. In particular $\big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle)$ agrees with $\varphi(g_*(\sigma'))$ in the second case of formula (3).

Assume now $i_0 < \cdots < i_p$. As just observed, if $\varphi_{i(\sigma'_0), i(\sigma'_1), \dots, i(\sigma'_p)} \neq 0$ then $i(\sigma'_\ell) = i_{p-\ell}$ for every ℓ, and this readily implies that the unique non-trivial addend in the right-hand sum in (4) corresponds to the sequence

$$\overline{\sigma}'_0 = (i'_p), \quad \overline{\sigma}'_1 = (i'_{p-1}, i'_p), \quad \dots \quad , \overline{\sigma}'_p = (i'_0, \dots, i'_{p-1}, i'_p) \ .$$

In particular, for every $j = 0, \dots, p-1$ we have $\overline{\sigma}'_j = (-1)^0 \partial_0 \overline{\sigma}'_{j+1}$. Hence

$$\begin{aligned} \big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle) &= (-1)^{\frac{p(p+1)}{2}} \varphi_{i(\overline{\sigma}'_0), i(\overline{\sigma}'_1), \dots, i(\overline{\sigma}'_p)} \\ &= (-1)^{\frac{p(p+1)}{2}} \varphi_{i_p, i_{p-1}, \dots, i_0} = \varphi_{i_0, i_1, \dots, i_p} \end{aligned}$$

settling also the first case in formula (3). □

Before proving the proposition we notice that the map $C_*(S) \to C_*(|S|)$ constructed above does not factor through $C_*^{\mathcal{U}^*}$ because no positive-dimensional simplex of S is contained in U_i^* for any $i \in I$. However the analogous map from $C_*(S')$ to $C_*(|S|)$ does. Hence the map $\widetilde{\nu}_{S'}\colon C^*(|S'|) \to C^*(S')$ factors as $\widetilde{\nu}_{S'} = \widetilde{\mu} \circ \widetilde{\gamma}$, where $\widetilde{\gamma}\colon C^*(|S'|) \to C^*_{\mathcal{U}^*}$ is the map defined in Section 1, and

$\widetilde{\mu}\colon C^*_{\mathcal{U}^*} \to C^*(S')$. We denote by $\mu\colon H^*(C^*_{\mathcal{U}^*}) \to H^*(S)$ the map induced by $\widetilde{\mu}$ on cohomology.

Proof of Proposition 2.1. Being $\nu_{S'} : H^*(|S|) = H^*(|S'|) \to H^*(S')$ injective and $|g|$ homotopic to the identity, in order to prove the proposition it is sufficient to show that $\nu_{S'} \circ \eta = \nu_{S'} \circ |g|^*$. Now recall that $\eta = \gamma^{-1} \circ \zeta \circ \nu_S$, hence $\nu_{S'} \circ \eta = \mu \circ \zeta \circ \nu_S$. Hence it is enough to prove that $\mu(\zeta(\nu_S(c))) = \nu_{S'}(|g|^*(c))$ for all $c \in H^p(|S|)$ or, equivalently, that

$$\widetilde{\mu}(\widetilde{\zeta}(\widetilde{\nu}_S(\psi)))(\sigma') = \widetilde{\nu}_{S'}(|g|^*(\psi))(\sigma')$$

where $\psi \in C^p(|S|)$ is a cocycle and σ' is any algebraic simplex of S'. Morover, as observed above we can choose ψ to be alternating. However, if we set $\varphi = \widetilde{\nu}_S(\psi)$, then

$$\mu(\widetilde{\zeta}(\widetilde{\nu}_S(\psi)))(\sigma') = \mu(\widetilde{\zeta}(\varphi))(\sigma') = \big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle) \ ,$$
$$\widetilde{\nu}_{S'}(|g|^*(\psi))(\sigma') = (|g|^*(\psi))(\langle\sigma'\rangle) = \psi(|g|_*(\langle\sigma'\rangle)) = \varphi(g_*(\sigma')) \ ,$$

hence the conclusion follows from Lemma 2.2. □

3. Proof of Theorem 1

We can now prove the Theorem stated in the introduction. We first notice that the construction of η is compatible with continuous maps in the following sense.

Lemma 3.1. *Let $h\colon Y \to Z$ be a continuous map, and let $\mathcal{V} = \{V_i\}_{i\in I}$, $\mathcal{W} = \{W_i\}_{i\in I}$ be open covers of Y, Z, respectively, such that $h(V_i) \subseteq W_i$ for every $i \in I$. The identity of the set I extends to a simplicial map $N(h)\colon N(\mathcal{V}) \to N(\mathcal{W})$, and in particular it induces a continuous map $\hat{h} : |N(\mathcal{V})| \to |N(\mathcal{W})|$. Then the following diagram commutes:*

$$\begin{array}{ccc} H^*(|N(\mathcal{W})|) & \xrightarrow{\hat{h}^*} & H^*(|N(\mathcal{V})|) \\ \downarrow{\scriptstyle \eta_{\mathcal{W}}} & & \downarrow{\scriptstyle \eta_{\mathcal{V}}} \\ H^*(Z) & \xrightarrow{h^*} & H^*(Y) \ . \end{array}$$

Proof. By considering the restriction of h to the open subset V_i the map h induces a morphism $\{h^{p,q}\}$ between the double complex associated to $\mathcal{W}$ and the double complex associated to $\mathcal{V}$ and between their augmentations. Hence we have $\zeta_{\mathcal{V}} \circ N(h)^* = h^* \circ \zeta_{\mathcal{W}} : H^*(N(\mathcal{W})) \to H^*(C^*_{\mathcal{V}})$. We also have $\widetilde{\gamma}_{\mathcal{V}} \circ h^* = h^{-1,*} \circ \widetilde{\gamma}_{\mathcal{W}}$ and by the definition of the map ν we have $\nu_{\mathcal{V}} \circ h^* = N(h)^* \circ \nu_{\mathcal{W}}$. By the definition of η, these three commutations imply the commutativity claimed in the lemma. □

We can now conclude the proof of our main theorem. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of the space X, let $N(\mathcal{U})$ be the nerve of $\mathcal{U}$, and let $\mathcal{U}^* = \{U_i^*\}_{i\in I}$ be the open cover of $|N(\mathcal{U})|$ given by the open stars of the vertices of $N(\mathcal{U})$. Let $f_\Phi\colon X \to |N(\mathcal{U})|$ be the map associated to a partition of unity subordinate to $\mathcal{U}$ as described in the introduction. We would like to apply the previous lemma to the covers $\mathcal{U}$ of X and $\mathcal{U}^*$ of $|N(\mathcal{U})|$ and to the map $h = f_\Phi$, but the containment $f_\Phi(U_i) \subseteq U_i^*$ does not hold in general. Therefore, we consider the cover $\widetilde{\mathcal{U}} = \{\widetilde{U}_i\}_{i\in I}$ of X defined by $\widetilde{U}_i = f_\Phi^{-1}(U_i^*)$ for every $i \in I$.

We can now apply Lemma 3.1 to the map $h = f_\Phi$ and to the covers $\mathcal{V} = \widetilde{\mathcal{U}}$ and $\mathcal{W} = \mathcal{U}^*$. Since $\widetilde{U}_i \subseteq U_i$ for every $i \in I$, Lemma 3.1 also applies to the case when $h = i_X$ is the identity map of X, and to the covers $\mathcal{V} = \widetilde{\mathcal{U}}$ and $\mathcal{W} = \mathcal{U}$. Hence we obtain the following commutative diagrams:

$$\begin{array}{ccc} H^*(|N(\mathcal{U}^*)|) & \xrightarrow{\hat{f}_\Phi^*} & H^*(|N(\widetilde{\mathcal{U}})|) \\ \downarrow{\scriptstyle \eta_{\mathcal{U}^*}} & & \downarrow{\scriptstyle \eta_{\widetilde{\mathcal{U}}}} \\ H^*(|N(\mathcal{U})|) & \xrightarrow{f_\Phi^*} & H^*(X) \end{array} \qquad \begin{array}{ccc} H^*(|N(\mathcal{U})|) & \xrightarrow{\hat{i}_X^*} & H^*(|N(\widetilde{\mathcal{U}})|) \\ \downarrow{\scriptstyle \eta_{\mathcal{U}}} & & \downarrow{\scriptstyle \eta_{\widetilde{\mathcal{U}}}} \\ H^*(X) & = & H^*(X) \end{array} .$$

As already noticed in the previous section the simplicial complexes $N(\mathcal{U})$ and $N(\mathcal{U}^*)$ with set of vertices I are equal and, by construction, so are the simplicial maps $N(i_X)$ and $N(f_\Phi)$ from $N(\widetilde{\mathcal{U}})$ to $N(\mathcal{U}^*) = N(\mathcal{U})$. In particular $\hat{f}_\Phi^* = \hat{i}_X^*$. Finally by Proposition 2.1 $\eta_{\mathcal{U}^*}$ is the identity. Hence

$$f_\Phi^* = f_\Phi^* \circ \eta_{\mathcal{U}^*} = \eta_{\widetilde{\mathcal{U}}} \circ \hat{f}_\Phi^* = \eta_{\widetilde{\mathcal{U}}} \circ \hat{i}_X^* = \eta_{\mathcal{U}} \ ,$$

which proves the theorem.

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