

# Global regularity properties of steady shear thinning flows

Luigi C. Berselli and Michael Růžička

**ABSTRACT.** In this paper we study the regularity of weak solutions to systems of  $p$ -Stokes type, describing the motion of some shear thinning fluids in certain steady regimes. In particular we address the problem of regularity up to the boundary improving previous results especially in terms of the allowed range for the parameter  $p$ .

**Keywords.** Regularity of weak solutions,  $p$ -Stokes type problem.

## 1. Introduction

In this paper we study regularity for weak solution the steady Stokes approximation for flows of shear thinning fluids which is given by

$$\begin{aligned} -\operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{u}) + \nabla \pi &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here  $\Omega \subset \mathbb{R}^3$  is a bounded domain with a  $C^{2,1}$  boundary  $\partial\Omega$  (we restrict ourselves to the most interesting problem from the physical point of view, even if results can be easily transferred to the problem in  $\mathbb{R}^d$  for all  $d \geq 2$ ). The unknowns are the velocity vector field  $\mathbf{u} = (u^1, u^2, u^3)^\top$  and the scalar pressure  $\pi$ , while the external body force  $\mathbf{f} = (f^1, f^2, f^3)^\top$  is given. The extra stress tensor  $\mathbf{S}$  depends only on  $\mathbf{D}\mathbf{u} := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)$ , the symmetric part of the velocity gradient  $\nabla\mathbf{u}$ . Physical interpretation and discussion of some non-Newtonian fluid models can be found, e.g., in [13, 24, 25]. The relevant example which we will study is the following

$$\mathbf{S}(\mathbf{D}\mathbf{u}) = \mu_0 \mathbf{D}\mathbf{u} + \mu_1 (\delta + |\mathbf{D}\mathbf{u}|)^{p-2} \mathbf{D}\mathbf{u}, \tag{1.2}$$

with  $p \in (1, 2]$ ,  $\delta > 0$ , and  $\mu_0, \mu_1 \geq 0$  satisfying  $\mu_1 > 0$ .

We study global regularity properties of second order derivatives of weak solutions to (1.1) for sufficiently smooth bounded domains  $\Omega$ . If  $\mu_0 > 0$  we obtain the optimal result, namely  $\mathbf{F}(\mathbf{D}\mathbf{u}) \in W^{1,2}(\Omega)$ , where the nonlinear tensor-valued function  $\mathbf{F}$  is defined in (2.13). This is the same result as for the  $p$ -Stokes problem in the periodic setting. For  $\mu_0 = 0$  we prove (among other results) that  $\mathbf{F}(\mathbf{D}\mathbf{u}) \in W^{1,q}(\Omega)$ , for some  $q = q(p) \in [1, 2[$ . The precise results are formulated in Theorem 2.28 and Theorem 2.29, where we also write the regularity results in terms of Sobolev spaces. We treat here the case without convective term, since this quantity can be handled in a more or less standard way, by Sobolev estimates, once the precise regularity in terms of the right-hand side is proved.

Our main interest is to handle the case of a non-flat boundary and to consider the full range  $p \in ]1, 2[$ . We recall that regularity results in the flat have been obtained in [14] and under various conditions by in [6] (respectively in the case  $\mu_0 > 0$  and  $\mu_0 = 0$ ). The situation in the non-flat case becomes much more

technical and we refer to the paper [22] for an early treatment in the case  $p > 2$ . Further results in the case  $p > 2$  are present in [2, 4, 5] while some results in the case  $p < 2$  are given in [7]. For a similar problem without pressure, strong results are proved in [28] in the flat case. For a treatment of simpler problems in the non flat case (nonlinear elliptic problems without the divergence constraint) see [11].

In all previous studies of the  $p$ -Stokes problem for  $p < 2$  a technical restriction  $p > \frac{3}{2}$  occurs, which is due to the presence of some algebraic systems to recover certain derivatives in the normal direction. We are now able to remove this restriction by deriving an algebraic system for a more intrinsic quantity related to the stress tensor. Here we are giving a self-contained treatment of the steady problem in the whole range  $p \in ]1, 2[$ , and we also recover all previously known results in the range  $\frac{3}{2} < p < 2$ . We point out that the main difficulty is that of treating at the same time the difficulties arising from: i) a non-linear stress tensor; ii) the divergence-free condition (with the related pressure); iii) a non-flat domain.

**Plan of the paper** The paper is organized as follows. In Section 2 we recall the notation used throughout the paper. Moreover, we recall some basic facts related to the difference quotient in tangential directions and to the extra stress tensor  $\mathbf{S}$ . In Section 3 we prove Theorem 2.28. In particular, we treat in detail the regularity in tangential directions in Section 3.1 and in normal directions in Section 3.2. Moreover, we prove some regularity properties of the pressure. The same procedure is carried out for the proof of Theorem 2.29 in Section 4.

## 2. Preliminaries and main results

In this section we introduce the notation we will use, state the precise assumptions on the extra stress tensor  $\mathbf{S}$ , and formulate the main results of the paper.

**2.1. Function spaces.** We use  $c, C$  to denote generic constants, which may change from line to line, but are independent of the crucial quantities. Moreover, we write  $f \sim g$  if and only if there exists constants  $c, C > 0$  such that  $c f \leq g \leq C f$ .

In addition to the classical space  $(C^{k,\lambda}(\Omega), \|\cdot\|_{C^{k,\lambda}})$  of Hölder continuous functions for  $0 < \lambda < 1$  (and Lipschitz-continuous when  $\lambda = 1$ ) we use standard Lebesgue spaces  $(L^p(\Omega), \|\cdot\|_p)$  and Sobolev spaces  $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ , where  $\Omega \subset \mathbb{R}^3$ , is a sufficiently smooth bounded domain. When dealing with functions defined only on some open subset  $\omega \subset \Omega$ , we denote the norm in  $L^p(\omega)$  by  $\|\cdot\|_{p,\omega}$ . The symbol  $\text{spt } f$  denotes the support of the function  $f$ . We do not distinguish between scalar, vector-valued or tensor-valued function spaces. However, we denote vectors by boldface lower-case letter as e.g.  $\mathbf{u}$  and tensors by boldface upper case letters as e.g.  $\mathbf{S}$ . If  $\mathbf{u} \in \mathbb{R}^m$  and  $\mathbf{v} \in \mathbb{R}^n$  then the tensor product  $\mathbf{u} \otimes \mathbf{v} \in \mathbb{R}^{m \times n}$  is defined as  $(\mathbf{u} \otimes \mathbf{v})_{ij} := u_i v_j$ . If  $m = n$  then  $\mathbf{u} \otimes \mathbf{v} := \frac{1}{2}(\mathbf{u} \otimes \mathbf{v} + (\mathbf{u} \otimes \mathbf{v})^\top)$ . The Euclidean scalar product is denoted by  $\mathbf{u} \cdot \mathbf{v} = u_i v_i$  and the scalar product  $\mathbf{A} \cdot \mathbf{B} := A_{ij} B_{ij}$  of second order tensor is denoted also by the same symbol. Here and in the sequel we use the summation convention over repeated Latin indices. The space  $W_0^{1,p}(\Omega)$  is the closure of the compactly supported, smooth functions  $C_0^\infty(\Omega)$  in  $W^{1,p}(\Omega)$ . Thanks to the Poincaré inequality we equip  $W_0^{1,p}(\Omega)$  with the gradient norm  $\|\nabla \cdot\|_p$ . We denote by  $W_{0,\text{div}}^{1,p}(\Omega)$  the subspace of  $W_0^{1,p}(\Omega)$  consisting of divergence-free vector fields  $\mathbf{u}$ , i.e., such that  $\text{div } \mathbf{u} = 0$ . We denote by  $|M|$  the 3-dimensional Lebesgue measure of a measurable set  $M$ . The mean value of a locally integrable function  $f$  over a measurable set  $M \subset \Omega$  is denoted by  $\langle f \rangle_M := \frac{1}{|M|} \int_M f \, d\mathbf{x}$ . By  $L_0^p(\Omega)$  we denote the subspace of  $L^p(\Omega)$  consisting of functions  $f$  with vanishing mean value, i.e.,  $\langle f \rangle_\Omega = 0$ . For a normed space  $X$  we denote its topological dual space by  $X^*$ .

We will also use Orlicz and Sobolev–Orlicz spaces (cf. [26]). We use N-functions  $\psi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ . We always assume that  $\psi$  and the conjugate N-function  $\psi^*$  satisfy the  $\Delta_2$ -condition. We denote the smallest constant such that  $\psi(2t) \leq K\psi(t)$  by  $\Delta_2(\psi)$ . We denote by  $L^\psi(\Omega)$  and  $W^{1,\psi}(\Omega)$  the classical Orlicz and Sobolev–Orlicz spaces, i.e,  $f \in L^\psi(\Omega)$  if the modular  $\rho_\psi(f) := \int_\Omega \psi(|f|) d\mathbf{x}$  is finite and  $f \in W^{1,\psi}(\Omega)$  if  $f$  and  $\nabla f$  belong to  $L^\psi(\Omega)$ . When equipped with the Luxembourg norm  $\|f\|_\psi := \inf \{\lambda > 0 \mid \int_\Omega \psi(|f|/\lambda) d\mathbf{x} \leq 1\}$  the space  $L^\psi(\Omega)$  becomes a Banach space. The same holds for the space  $W^{1,\psi}(\Omega)$  if it is equipped with the norm  $\|\cdot\|_\psi + \|\nabla \cdot\|_\psi$ . Note that the dual space  $(L^\psi(\Omega))^*$  can be identified with the space  $L^{\psi^*}(\Omega)$ . By  $W_0^{1,\psi}(\Omega)$  we denote the closure of  $C_0^\infty(\Omega)$  in  $W^{1,\psi}(\Omega)$  and equip it with the gradient norm  $\|\nabla \cdot\|_\psi$ . By  $L_0^\psi(\Omega)$  and  $C_{0,0}^\infty(\Omega)$  we denote the subspace of  $L^\psi(\Omega)$  and  $C_0^\infty(\Omega)$ , respectively, consisting of functions  $f$  such that  $\langle f \rangle_\Omega = 0$ .

We need the following refined version of the Young inequality: for all  $\varepsilon > 0$  there exists  $c_\varepsilon > 0$ , depending only on  $\Delta_2(\psi), \Delta_2(\psi^*) < \infty$ , such that for all  $s, t \geq 0$  it holds

$$\begin{aligned} ts &\leq \varepsilon \psi(t) + c_\varepsilon \psi^*(s), \\ t \psi'(s) + \psi'(t) s &\leq \varepsilon \psi(t) + c_\varepsilon \psi(s). \end{aligned} \quad (2.1)$$

**2.2.  $(p, \delta)$ -structure.** We now define what it means that a tensor field  $\mathbf{S}$  has  $(p, \delta)$ -structure. A detailed discussion and full proofs of the results cited can be found in [17, 27]. For a tensor  $\mathbf{P} \in \mathbb{R}^{3 \times 3}$  we denote its symmetric part by  $\mathbf{P}^{\text{sym}} := \frac{1}{2}(\mathbf{P} + \mathbf{P}^\top) \in \mathbb{R}_{\text{sym}}^{3 \times 3} := \{\mathbf{P} \in \mathbb{R}^{3 \times 3} \mid \mathbf{P} = \mathbf{P}^\top\}$ . We use the notation  $|\mathbf{P}|^2 = \mathbf{P} \cdot \mathbf{P}^\top$ .

It is convenient to define for  $t \geq 0$  a special N-function  $\varphi(\cdot) = \varphi(p, \delta; \cdot)$ , for  $p \in (1, \infty), \delta \geq 0$ , by

$$\varphi(t) := \int_0^t \gamma(s) ds \quad \text{with} \quad \gamma(t) := (\delta + t)^{p-2} t.$$

The function  $\varphi$  satisfies, uniformly in  $t$  and independent of  $\delta$ , the important equivalence<sup>1</sup>

$$\varphi''(t)t \sim \varphi'(t) \quad (2.2)$$

since

$$\min\{1, p-1\}(\delta + t)^{p-2} \leq \varphi''(t) \leq \max\{1, p-1\}(\delta + t)^{p-2}. \quad (2.3)$$

Moreover, the function  $\varphi$  satisfies the  $\Delta_2$ -condition with  $\Delta_2(\varphi) \leq c 2^{\max\{2, p\}}$  (hence independent of  $\delta$ ). This implies that, uniformly in  $t$  and independent of  $\delta$ , we have

$$\varphi'(t)t \sim \varphi(t). \quad (2.4)$$

The conjugate function  $\varphi^*$  satisfies  $\varphi^*(t) \sim (\delta^{p-1} + t)^{p'-2} t^2$  where  $\frac{1}{p} + \frac{1}{p'} = 1$ . Also  $\varphi^*$  satisfies the  $\Delta_2$ -condition with  $\Delta_2(\varphi^*) \leq c 2^{\max\{2, p'\}}$ . Using the structure of  $\varphi$  we get for all  $t, \lambda \geq 0$

$$\varphi(\lambda t) \leq \max(\lambda^p, \lambda^2)\varphi(t) \leq \max(1, \lambda^2)\varphi(t) \quad (2.5)$$

if  $p \leq 2$ .

For a given N-function  $\psi$  we define the shifted N-functions  $\{\psi_a\}_{a \geq 0}$ , cf. [17, 18, 27], for  $t \geq 0$  by

$$\psi_a(t) := \int_0^t \psi'_a(s) ds \quad \text{with} \quad \psi'_a(t) := \psi'(a+t) \frac{t}{a+t}. \quad (2.6)$$

<sup>1</sup>Note that if  $\varphi''(0)$  does not exist, the left-hand side in (2.2) is continuously extended by zero for  $t = 0$ .

REMARK 2.7. (i) Defining  $\omega(t) = \omega(q; t) := \frac{1}{q}t^q$ ,  $q \in (1, \infty)$  we have for the above defined N-function

$$\varphi(t) = \omega_\delta(p; t).$$

(ii) Note that  $\varphi_a(t) \sim (\delta + a + t)^{p-2}t^2$  and also  $(\varphi_a)^*(t) \sim ((\delta + a)^{p-1} + t)^{p'-2}t^2$ . The families  $\{\varphi_a\}_{a \geq 0}$  and  $\{(\varphi_a)^*\}_{a \geq 0}$  satisfy the  $\Delta_2$ -condition uniformly with respect to  $a \geq 0$ , with  $\Delta_2(\varphi_a) \leq c 2^{\max\{2, p\}}$  and  $\Delta_2((\varphi_a)^*) \leq c 2^{\max\{2, p'\}}$ , respectively. The equivalences (2.2) and (2.4) are satisfied for the families  $\{\varphi_a\}_{a \geq 0}$  and  $\{(\varphi_a)^*\}_{a \geq 0}$ , uniformly in  $a \geq 0$ .

DEFINITION 2.8 ( $(p, \delta)$ -structure). *We say that a tensor field  $\mathbf{S}: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_{\text{sym}}^{3 \times 3}$  belonging to  $C^0(\mathbb{R}^{3 \times 3}, \mathbb{R}_{\text{sym}}^{3 \times 3}) \cap C^1(\mathbb{R}^{3 \times 3} \setminus \{\mathbf{0}\}, \mathbb{R}_{\text{sym}}^{3 \times 3})$ , satisfying  $\mathbf{S}(\mathbf{P}) = \mathbf{S}(\mathbf{P}^{\text{sym}})$ , and  $\mathbf{S}(\mathbf{0}) = \mathbf{0}$  possesses  $(p, \delta)$ -structure, if for some  $p \in (1, \infty)$ ,  $\delta \in [0, \infty)$ , and the N-function  $\varphi = \varphi_{p, \delta}$  (cf. (2.2)) there exist constants  $\kappa_0, \kappa_1 > 0$  such that*

$$\sum_{i,j,k,l=1}^3 \partial_{kl} S_{ij}(\mathbf{P}) Q_{ij} Q_{kl} \geq \kappa_0 \varphi''(|\mathbf{P}^{\text{sym}}|) |\mathbf{Q}^{\text{sym}}|^2, \quad (2.9)$$

$$|\partial_{kl} S_{ij}(\mathbf{P})| \leq \kappa_1 \varphi''(|\mathbf{P}^{\text{sym}}|).$$

are satisfied for all  $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3 \times 3}$  with  $\mathbf{P}^{\text{sym}} \neq \mathbf{0}$ . The constants  $\kappa_0, \kappa_1$ , and  $p$  are called the characteristics of  $\mathbf{S}$ .

REMARK 2.10. *The above assumption is motivated by the typical examples for the extra stress tensor in mathematical fluid mechanics. For example constitutive relations of power-law type, Carreau type, Cross-type or (1.2) satisfy this assumption. We refer the reader to [10, 19, 21, 24] for a more detailed discussion leading to Definition 2.8.*

REMARK 2.11. (i) Assume that  $\mathbf{S}$  has  $(p, \delta)$ -structure for some  $\delta \in [0, \delta_0]$ . Then, if not otherwise stated, the constants in the estimates depend only on the characteristics of  $\mathbf{S}$  and on  $\delta_0$  but are independent of  $\delta$ . This dependence comes from the difference between the modular and the norm in the case of Orlicz spaces.

(ii) An important example of an extra stress  $\mathbf{S}$  having  $(p, \delta)$ -structure is given by  $\mathbf{S}(\mathbf{P}) = \varphi'(|\mathbf{P}^{\text{sym}}|) |\mathbf{P}^{\text{sym}}|^{-1} \mathbf{P}^{\text{sym}}$ . In this case the characteristics of  $\mathbf{S}$ , namely  $\kappa_0, \kappa_1$ , and  $p$ , depend only on  $p$  and are independent of  $\delta \geq 0$ .

REMARK 2.12. For the family  $(\varphi_a)$ , with  $a \in [0, a_0]$ ,  $\delta \in [0, \delta_0]$  and  $p \in (1, \infty)$ , we get  $L^{\varphi_a^*}(\Omega) = L^{p'}(\Omega)$  and  $W^{1, \varphi_a}(\Omega) = W^{1, p}(\Omega)$  with uniform equivalence of the corresponding norms depending on  $p, a_0$  and  $\delta_0$ , since  $\Omega$  is bounded and  $\varphi_a$  and  $\omega(p; \cdot)$  are equivalent at infinity (cf. [26]).

To a tensor field  $\mathbf{S}$  with  $(p, \delta)$ -structure we associate the tensor field  $\mathbf{F}: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_{\text{sym}}^{3 \times 3}$  defined through

$$\mathbf{F}(\mathbf{P}) := (\delta + |\mathbf{P}^{\text{sym}}|)^{\frac{p-2}{2}} \mathbf{P}^{\text{sym}}. \quad (2.13)$$

The connection between  $\mathbf{S}$ ,  $\mathbf{F}$ , and  $\{\varphi_a\}_{a \geq 0}$  is best explained by the following proposition (cf. [17, 27]).

PROPOSITION 2.14. *Let  $\mathbf{S}$  has  $(p, \delta)$ -structure, and let  $\mathbf{F}$  be defined in (2.13). Then*

$$(\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}) \sim |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2 \quad (2.15a)$$

$$\sim |\varphi_{|\mathbf{P}^{\text{sym}}|}(|\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|), \quad (2.15b)$$

$$\sim \varphi''(|\mathbf{P}^{\text{sym}}| + |\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|) |\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|^2 \quad (2.15c)$$

$$|\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})| \sim \varphi'_{|\mathbf{P}^{\text{sym}}|}(|\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|). \quad (2.15d)$$

uniformly in  $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3 \times 3}$ . Moreover, uniformly in  $\mathbf{Q} \in \mathbb{R}^{3 \times 3}$ ,

$$\mathbf{S}(\mathbf{Q}) \cdot \mathbf{Q} \sim |\mathbf{F}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}^{\text{sym}}|). \quad (2.15e)$$

The constants depend only on the characteristics of  $\mathbf{S}$ .

For a detailed discussion of the properties of  $\mathbf{S}$  and  $\mathbf{F}$  and their relation to Orlicz spaces and N-functions we refer the reader to [27], [10]. We just want to mention that

$$|\mathbf{F}(\mathbf{Q})| + \delta^{\frac{p}{2}} \sim |\mathbf{Q}^{\text{sym}}|^{\frac{p}{2}} + \delta^{\frac{p}{2}}, \quad (2.16)$$

with constants depending only on  $p$ .

REMARK 2.17 (Natural distance). In view of the previous lemma we have, for all  $\mathbf{u}, \mathbf{w} \in W^{1,\varphi}(\Omega)$ ,

$$\langle \mathbf{S}(\mathbf{D}\mathbf{u}) - \mathbf{S}(\mathbf{D}\mathbf{w}), \mathbf{D}\mathbf{u} - \mathbf{D}\mathbf{w} \rangle \sim \|\mathbf{F}(\mathbf{D}\mathbf{u}) - \mathbf{F}(\mathbf{D}\mathbf{w})\|_2^2 \sim \int_{\Omega} \varphi_{|\mathbf{D}\mathbf{u}|}(|\mathbf{D}\mathbf{u} - \mathbf{D}\mathbf{w}|) dx.$$

The constants depend only on the characteristics of  $\mathbf{S}$ . The last expression equals the quasi-norm introduced in [1] raised to the power  $\rho = \max\{p, 2\}$ . We refer to all three equivalent quantities as the *natural distance*.

Since in the following we shall insert into  $\mathbf{S}$  and  $\mathbf{F}$  only symmetric tensors, we can drop in the above formulas the superscript “sym” and restrict the admitted tensors to symmetric ones.

**2.3. Description and properties of the boundary.** We assume that the boundary  $\partial\Omega$  is of class  $C^{2,1}$ , that is for each point  $P \in \partial\Omega$  there are local coordinates such that in these coordinates we have  $P = 0$  and  $\partial\Omega$  is locally described by a  $C^{2,1}$ -function, i.e., there exist  $R_P, R'_P, r_P \in (0, \infty)$  and a  $C^{2,1}$ -function  $a_P : B_{R_P}^2(0) \rightarrow B_{R'_P}^1(0)$  such that

- (b1)  $\mathbf{x} \in \partial\Omega \cap (B_{R_P}^2(0) \times B_{R'_P}^1(0)) \iff x_3 = a_P(x_1, x_2)$ ,
- (b2)  $\Omega_P := \{(x, x_3) : x = (x_1, x_2) \in B_{R_P}^2(0), a_P(x) < x_3 < a_P(x) + R'_P\} \subset \Omega$ ,
- (b3)  $\nabla a_P(0) = \mathbf{0}$ , and  $\forall x = (x_1, x_2) \in B_{R_P}^2(0) \quad |\nabla a_P(x)| < r_P$ ,

where  $B_r^k(0)$  denotes the  $k$ -dimensional open ball with center 0 and radius  $r > 0$ . By a slightly abuse of notation, to simplify further formulas, we set  $R_P := \min\{R_P, R'_P\}$ . Note also that  $r_P$  can be made arbitrarily small if we make  $R_P$  small enough.

In the sequel we will also use, for  $0 < \lambda < 1$ , the following scaled open sets,  $\lambda\Omega_P \subset \Omega_P$  defined as follows

$$\lambda\Omega_P := \{(x, x_3) : x = (x_1, x_2) \in B_{\lambda R_P}^2(0), a_P(x) < x_3 < a_P(x) + \lambda R'_P\}. \quad (2.18)$$

To prove our global estimates we first show local estimates near the boundary in  $\Omega_P$ , for every  $P \in \partial\Omega$ . To this end we fix smooth functions  $\xi_P : \mathbb{R}^3 \rightarrow \mathbb{R}$  such that  $0 \leq \xi_P \leq 1$  and

$$(\ell 1) \quad \chi_{\frac{1}{2}\Omega_P}(\mathbf{x}) \leq \xi_P(\mathbf{x}) \leq \chi_{\frac{3}{4}\Omega_P}(\mathbf{x}),$$

where  $\chi_A(\mathbf{x})$  is the indicator function of the measurable set  $A$ . For the remaining interior estimate we also localize by a smooth function  $0 \leq \xi_{00} \leq 1$  such that  $\text{spt } \xi_{00} \subset \Omega_{00}$ , where  $\Omega_{00} \subset \Omega$  is an open set such that  $\text{dist}(\partial\Omega_{00}, \partial\Omega) > 0$ . The local estimates near the boundary are obtained in two steps. In the first one (see Sections 3.1 and 4.1) we estimate in  $\Omega_P$  only tangential derivatives as defined below. In the second one we use the new obtained information and compute the normal derivatives from the system. Since the boundary  $\partial\Omega$  is compact, we can use an appropriate finite covering of it which, together with the interior estimate, yields the global estimate.

Let us now introduce the tangential derivatives near the boundary and related concepts. To simplify the notation we fix  $P \in \partial\Omega$ ,  $h \in (0, \frac{R_P}{16})$ , and simply write  $\xi := \xi_P$ ,  $a := a_P$ . We use the standard notation  $\mathbf{x} = (x', x_3)^\top$  and denote by  $\mathbf{e}^i, i = 1, 2, 3$  the canonical orthonormal basis in  $\mathbb{R}^3$ . In the following lower-case Greek letters take values 1, 2.

A crucial technicality to handle non-flat boundaries is to define a proper way of differentiation (and approximate partial derivatives) in directions that are tangential to the boundary, at least in a tubular neighborhood of  $\partial\Omega$ . For a function  $g$  (when  $g$  is vector or tensor valued the same is applied to all components) with  $\text{spt } g \subset \text{spt } \xi$  we define positive and negative tangential translations:

$$\begin{aligned} g_\tau(x', x_3) &:= g(x' + h \mathbf{e}^\alpha, x_3 + a(x' + h \mathbf{e}^\alpha) - a(x')), \\ g_{-\tau}(x', x_3) &:= g(x' - h \mathbf{e}^\alpha, x_3 + a(x' - h \mathbf{e}^\alpha) - a(x')); \end{aligned}$$

tangential differences

$$\Delta^+ g := g_\tau - g, \quad \Delta^- g := g_{-\tau} - g;$$

and tangential divided differences

$$d^+ g := h^{-1} \Delta^+ g, \quad d^- g := h^{-1} \Delta^- g.$$

It holds

$$d^+ g \rightarrow \partial_\tau g := \partial_\alpha g + \partial_\alpha a \partial_3 g \quad \text{as } h \rightarrow 0, \quad (2.19)$$

almost everywhere in  $\text{spt } \xi$ , if  $g \in W^{1,1}(\Omega)$  (cf. [23, Sec. 3]). Moreover, we have for all  $1 < q < \infty$ ,  $g \in W^{1,q}(\Omega)$  and all sufficiently small  $h > 0$ , that

$$\exists c(a) > 0: \quad \|d^+ g\|_{q, \text{spt } \xi} \leq c(a) \|\nabla g\|_q. \quad (2.20)$$

Conversely, if  $\|d^+ g\|_{q, \text{spt } \xi} \leq C$  for all sufficiently small  $h > 0$ , then

$$\|\partial_\tau g\|_{q, \text{spt } \xi} \leq C. \quad (2.21)$$

Now we formulate some auxiliary lemmas related to these objects. The first lemma clarifies the fact that tangential translations and tangential differences do not non commute with partial derivatives. Also the explicit expressions can be used to quantitatively estimate the so called commutation terms, as called in turbulence theory [12]. For simplicity we denote  $\nabla a := (\partial_1 a, \partial_2 a, 0)^\top$  and use the operations  $(\cdot)_\tau$ ,  $(\cdot)_{-\tau}$ ,  $\Delta^+(\cdot)$ ,  $\Delta^-(\cdot)$ ,  $d^+(\cdot)$  and  $d^-(\cdot)$  also for vector-valued and tensor-valued functions, intended as acting component-wise.

LEMMA 2.22. *Let  $\mathbf{v} \in W^{1,1}(\Omega)$  such that  $\text{spt } \mathbf{v} \subset \text{spt } \xi$ . Then*

$$\begin{aligned} \nabla d^\pm \mathbf{v} &= d^\pm \nabla \mathbf{v} + (\partial_3 \mathbf{v})_\tau \otimes d^\pm \nabla a, \\ \mathbf{D} d^\pm \mathbf{v} &= d^\pm \mathbf{D} \mathbf{v} + (\partial_3 \mathbf{v})_\tau \overset{\text{s}}{\otimes} d^\pm \nabla a, \\ \text{div } d^\pm \mathbf{v} &= d^\pm \text{div } \mathbf{v} + (\partial_3 \mathbf{v})_{\pm\tau} d^\pm \nabla a \\ \nabla \mathbf{v}_{\pm\tau} &= (\nabla \mathbf{v})_{\pm\tau} + (\partial_3 \mathbf{v})_{\pm\tau} d^\pm \nabla a, \end{aligned}$$

where  $\overset{\text{s}}{\otimes}$  is defined component-wise also for scalar and tensor-valued functions.

The second lemma is devoted to the relation between tangential differences and tangential translations, provided that  $h$  is small enough.

LEMMA 2.23. *Let  $\text{spt } g \subset \text{spt } \xi$ . Then*

$$(d^- g)_\tau = -d^+ g, \quad (d^+ g)_{-\tau} = -d^- g, \quad d^- g_\tau = -d^+ g.$$

The following variant of integration per parts will be often used.

LEMMA 2.24. *Let  $\text{spt } g \cup \text{spt } f \subset \text{spt } \xi$  and  $h$  small enough. Then*

$$\int_{\Omega} f g_{-\tau} \, d\mathbf{x} = \int_{\Omega} f_{\tau} g \, d\mathbf{x}.$$

Consequently,  $\int_{\Omega} f d^+ g \, d\mathbf{x} = \int_{\Omega} (d^- f) g \, d\mathbf{x}$ .

Also the following variant of the product rule will be used.

LEMMA 2.25. *Let  $\text{spt } g \cup \text{spt } f \subset \text{spt } \xi$ . Then*

$$d^{\pm}(fg) = f_{\pm\tau} d^{\pm}g + (d^{\pm}f)g.$$

If  $\mathbf{S}$  has  $(p, \delta)$ -structure we easily obtain from Lemma 2.14 the following equivalences

$$\begin{aligned} |d^+ \mathbf{S}(\mathbf{D}\mathbf{u})| &\sim (\delta + |\mathbf{D}\mathbf{u}| + |\Delta^+ \mathbf{D}\mathbf{u}|)^{p-2} |d^+ \mathbf{D}\mathbf{u}| \\ &\sim \varphi''(|\mathbf{D}\mathbf{u}| + |\Delta^+ \mathbf{D}\mathbf{u}|) |d^+ \mathbf{D}\mathbf{u}| \\ &\sim (\varphi''(|\mathbf{D}\mathbf{u}| + |\Delta^+ \mathbf{D}\mathbf{u}|))^{\frac{1}{2}} |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})| \\ &\sim (\delta + |\mathbf{D}\mathbf{u}| + |\Delta^+ \mathbf{D}\mathbf{u}|)^{\frac{p-2}{2}} |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|, \end{aligned} \tag{2.26}$$

$$\begin{aligned} d^+ \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot d^+ \mathbf{D}\mathbf{u} &\sim |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \\ &\sim (\delta + |\mathbf{D}\mathbf{u}| + |\Delta^+ \mathbf{D}\mathbf{u}|)^{p-2} |d^+ \mathbf{D}\mathbf{u}|^2 \\ &\sim \varphi''(|\mathbf{D}\mathbf{u}| + |\Delta^+ \mathbf{D}\mathbf{u}|) |d^+ \mathbf{D}\mathbf{u}|^2, \end{aligned} \tag{2.27}$$

with constants depending only on the characteristics of  $\mathbf{S}$  and  $p$ . All assertions from this section may be proved by easy manipulations of definitions and we drop their proofs.

**2.4. Main Results.** Now we can formulate our main results concerning the regularity properties of weak solutions to problems (1.1), with different assumptions on the stress tensor. We especially focus on the two different cases in which there is a part associated with the quadratic growth or in which this is lacking.

THEOREM 2.28. *Let  $\mathbf{S}$  the extra stress tensor in (1.1) be given by  $\mathbf{S} = \mathbf{S}^0 + \mathbf{S}^1$ , where  $\mathbf{S}^0$  satisfies Assumption 2.8 with  $p = 2$  and  $\mathbf{S}^1$  satisfies Assumption 2.8 for some  $p \in (1, 2)$ , and  $\delta \in (0, \infty)$ . Let  $\mathbf{F}$  be the associated tensor field to  $\mathbf{S}^1$ . Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with boundary  $\partial\Omega$  of class  $C^{2,1}$ , and let  $\mathbf{f} \in L^2(\Omega)$ . Then, the unique weak solution  $(\mathbf{u}, \pi) \in W_{0,\text{div}}^{1,2}(\Omega) \times L_0^2(\Omega)$  of the problem (1.1) satisfies*

$$\int_{\Omega} |\nabla^2 \mathbf{u}|^2 + |\nabla \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + |\nabla \pi|^2 \, d\mathbf{x} \leq c,$$

where the constant  $c$  depends on  $\|\mathbf{f}\|_2$ , on the characteristics of  $\mathbf{S}^0$ ,  $\mathbf{S}^1$ , on  $\delta^{-1}$ , and on  $\partial\Omega$ .

A more precise dependence of various quantities in terms of  $\delta$  is given in the proof of the theorem. Note that we obtain the same regularity on  $(\mathbf{u}, \pi)$  as in the case of the (linear) Stokes system, namely

$$\mathbf{u} \in W^{2,2}(\Omega) \quad \text{and} \quad \pi \in W^{1,2}(\Omega).$$

Let us consider now the case in which there is only the nonlinear part of the stress tensor.

**THEOREM 2.29.** *Let the extra stress tensor  $\mathbf{S}$  in (1.1) satisfy Assumption 2.8 for some  $p \in (1, 2)$ , and  $\delta \in (0, \infty)$ , and let  $\mathbf{F}$  be the associated tensor field to  $\mathbf{S}$ . Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with  $C^{2,1}$  boundary and let  $\mathbf{f} \in L^{p'}(\Omega)$ . Then, the unique weak solution  $\mathbf{u} \in W_{0,\text{div}}^{1,p}(\Omega)$  of the problem (1.1) satisfies*

$$\begin{aligned} \int_{\Omega} \xi_0^2 |\nabla \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \xi_0^2 |\nabla \pi|^2 \, d\mathbf{x} &\leq c(\|\mathbf{f}\|_{p'}, \|\xi_0\|_{2,\infty}, \delta^{-1}), \\ \int_{\Omega} \xi^2 |\partial_{\tau} \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \xi^2 |\partial_{\tau} \pi|^2 \, d\mathbf{x} &\leq c(\|\mathbf{f}\|_{p'}, \|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}, \delta^{-1}), \\ \int_{\Omega} \xi^{\frac{8p-4}{3p}} |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u})|^{\frac{8p-4}{3p}} + \xi^{\frac{8p-4}{p}} |\mathbf{F}(\mathbf{D}\mathbf{u})|^{\frac{8p-4}{p}} \, d\mathbf{x} &\leq c(\|\mathbf{f}\|_{p'}, \|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}, \delta^{-1}), \\ \int_{\Omega} \xi^p |\partial_3 \pi|^{\frac{4p-2}{p+1}} \, d\mathbf{x} &\leq c(\|\mathbf{f}\|_{p'}, \|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}, \delta^{-1}). \end{aligned}$$

Here  $\xi_0$  is a cut-off function with support in the interior of  $\Omega$ , while  $\xi = \xi_P$  is a cut-off function with support near to the boundary  $\partial\Omega$ , as defined in Sec. 2.3. The tangential derivative  $\partial_{\tau}$  is defined locally in  $\Omega_P$  by (2.19). This in particular implies that  $\mathbf{F}(\mathbf{D}\mathbf{u}) \in L^{\frac{8p-4}{p}}(\Omega)$  and, in terms of derivatives of  $\mathbf{u}$ , that

$$\xi \partial_{\tau} \nabla \mathbf{u} \in L^{\frac{8p-4}{3p}}(\Omega), \quad \xi \partial_3 \nabla \mathbf{u} \in L^{\frac{4p-2}{p+1}}(\Omega), \quad \nabla \mathbf{u} \in L^{4p-2}(\Omega), \quad \nabla^2 \mathbf{u} \in L^{\frac{4p-2}{p+1}}(\Omega),$$

while the partial derivatives of the pressure satisfy

$$\xi \partial_{\tau} \pi \in L^2(\Omega), \quad \xi \partial_3 \pi \in L^{\frac{4p-2}{p+1}}(\Omega), \quad \pi \in W^{1, \frac{4p-2}{p+1}}(\Omega).$$

We observe that the regularity results in Theorems 2.28-2.29 are the same as in the flat-case, which are proved (for the restricted range  $\frac{3}{2} < p < 2$ ) in [14] and [9], respectively. Nevertheless, comprehensive analysis of the changes deriving from presence/absence of the stress tensor with the quadratic part is given here with full detail. This suggests that our results are quite sharp. (Note also that in the flat case the estimates obtained are better behaved in terms of  $\delta$ .) The main obstacle to the proof –and to possible improvements– of the above theorems is given by the coupling of the boundary condition prescribed on the non-flat boundary  $\partial\Omega$ , together with the solenoidal constraint, which results in the appearance of the pressure term  $\nabla \pi$  in the equations (1.1).

The case of systems where the extra stress tensor depends on the symmetric velocity gradient, but without a solenoidal constraint, is completely solved in the case of a flat boundary and  $p < 2$  in [28].

**2.5. Auxiliary results.** Here we collect some auxiliary results needed in the sequel of the paper.

**LEMMA 2.30.** *For all  $\varepsilon > 0$ , there exists a constant  $c_{\varepsilon} > 0$  depending only on  $\varepsilon > 0$  and the characteristics of  $\mathbf{S}$  such that for all sufficiently smooth vector fields  $\mathbf{u}, \mathbf{v}$ , and  $\mathbf{w}$  we have*

$$\langle \mathbf{S}(\mathbf{D}\mathbf{u}) - \mathbf{S}(\mathbf{D}\mathbf{v}), \mathbf{D}\mathbf{w} - \mathbf{D}\mathbf{v} \rangle \leq \varepsilon \|\mathbf{F}(\mathbf{D}\mathbf{u}) - \mathbf{F}(\mathbf{D}\mathbf{v})\|_2^2 + c_{\varepsilon} \|\mathbf{F}(\mathbf{D}\mathbf{w}) - \mathbf{F}(\mathbf{D}\mathbf{v})\|_2^2.$$

**PROOF.** This is proved in [16, Lemma 2.3].  $\square$

**LEMMA 2.31.** *Let  $\psi$  be an  $N$ -function satisfying the  $\Delta_2$ -condition. Then, for all  $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n \times n}$  and all  $t \geq 0$  there holds*

$$\psi'_{|\mathbf{P}|}(t) \leq 2\Delta_2(\psi') \psi'_{|\mathbf{Q}|}(t) + \psi'_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|), \quad (2.32)$$

$$\psi'_{|\mathbf{P}|}(t) \leq 2\Delta_2(\psi') (\psi'_{|\mathbf{Q}|}(t) + \psi'_{|\mathbf{Q}|}(|\mathbf{P} - \mathbf{Q}|)). \quad (2.33)$$



PROOF. This is proved in [27, Lemma 5.13, Remark 5.14].  $\square$

LEMMA 2.34 (Change of shift). *Let  $\psi$  be an  $N$ -function such that  $\psi$  and  $\psi^*$  satisfy the  $\Delta_2$ -condition. Then for all  $\delta \in (0, 1)$  there exists  $c_\varepsilon = c_\varepsilon(\Delta_2(\psi'))$  such that all  $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n \times n}$ , and all  $t \geq 0$*

$$\psi_{|\mathbf{P}|}(t) \leq c_\varepsilon \psi_{|\mathbf{Q}|}(t) + \varepsilon \psi_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|) \quad (2.35)$$

$$\psi_{|\mathbf{P}|}(t) \leq c_\varepsilon \psi_{|\mathbf{Q}|}(t) + \varepsilon \psi_{|\mathbf{Q}|}(|\mathbf{P} - \mathbf{Q}|). \quad (2.36)$$

$$(\psi_{|\mathbf{P}|})^*(t) \leq c_\varepsilon (\psi_{|\mathbf{Q}|})^*(t) + \varepsilon \psi_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|) \quad (2.37)$$

$$(\psi_{|\mathbf{P}|})^*(t) \leq c_\varepsilon (\psi_{|\mathbf{Q}|})^*(t) + \varepsilon \psi_{|\mathbf{Q}|}(|\mathbf{P} - \mathbf{Q}|). \quad (2.38)$$

PROOF. This is proved in [27, Lemma 5.15, Lemma 5.18].  $\square$

PROPOSITION 2.39 (Divergence equation in Orlicz spaces). *Let  $G \subset \mathbb{R}^n$  be a bounded John domain. Then, there exists a linear operator  $\mathbf{B} : C_{0,0}^\infty(G) \rightarrow C_0^\infty(G)$  which extends uniquely for all  $N$ -functions  $\psi$  with  $\Delta_2(\psi), \Delta_2(\psi^*) < \infty$  to an operator  $\mathbf{B} : L_0^\psi(G) \rightarrow W_0^{1,\psi}(G)$ , satisfying  $\operatorname{div} \mathbf{B}f = f$ , and*

$$\begin{aligned} \|\nabla \mathbf{B}f\|_{L^\psi(G)} &\leq c \|f\|_{L_0^\psi(G)}, \\ \int_G \psi(|\nabla \mathbf{B}f|) \, d\mathbf{x} &\leq c \int_G \psi(|f|) \, d\mathbf{x}. \end{aligned}$$

The constant  $c$  depends on  $\Delta_2(\psi)$ ,  $\Delta_2(\psi^*)$ , and the John constant of  $G$ .

PROOF. This is Theorem 4.2 in [8].  $\square$

PROPOSITION 2.40 (Korn's inequality in Orlicz spaces). *Let  $\psi$  be an  $N$ -function with  $\Delta_2(\psi), \Delta_2(\psi^*) < \infty$  and let  $G \subset \mathbb{R}^n$  be a bounded John domain. Then, for all  $\mathbf{w} \in W_0^{1,\psi}(G)$*

$$\int_G \psi(|\nabla \mathbf{w}|) \, d\mathbf{x} \leq c \int_G \psi(|\mathbf{D}\mathbf{w}|) \, d\mathbf{x}.$$

The constant  $c$  depends only on the John constant,  $\Delta_2(\psi)$ , and  $\Delta_2(\psi^*)$ .

PROOF. This is a special case of [15, Thm. 6.10].  $\square$

PROPOSITION 2.41 (Poincaré inequality). *Let  $G \subset \mathbb{R}^n$  be open and bounded. Let  $\psi$  be an  $N$ -function with  $\Delta_2(\psi), \Delta_2(\psi^*) < \infty$ . Then, there exists  $c > 0$  only depending on  $\Delta_2(\psi)$  and  $\Delta_2(\psi^*)$  such that*

$$\int_G \psi\left(\frac{|u|}{\operatorname{diam}(G)}\right) \, d\mathbf{x} \leq c \int_G \psi(|\nabla u|) \, d\mathbf{x} \quad \forall u \in W_0^{1,\psi}(G). \quad (2.42)$$

PROOF. This is Lemma 6.3 in [8] and is based on the properties of the maximal function.  $\square$

LEMMA 2.43. *Let  $G \subset \mathbb{R}^n$  be a bounded John domain and let  $\psi$  be an  $N$ -function with  $\Delta_2(\psi), \Delta_2(\psi^*) < \infty$ . Then, for all  $q \in L_0^{\psi^*}(G)$  we have*

$$\|q\|_{L_0^{\psi^*}(G)} \leq c \sup_{\|\mathbf{v}\|_{W_0^{1,\psi}(G)} \leq 1} \langle q, \operatorname{div} \mathbf{v} \rangle,$$

and also

$$\int_G \psi^*(|q|) \, d\mathbf{x} \leq \sup_{\mathbf{v} \in W_0^{1,\psi}(G)} \left[ \int_G q \operatorname{div} \mathbf{v} \, d\mathbf{x} - \frac{1}{c} \int_G \psi(|\nabla \mathbf{v}|) \, d\mathbf{x} \right],$$

where the constants depend only on  $\Delta_2(\psi)$ ,  $\Delta_2(\psi^*)$ , and the John constant of  $G$ .

PROOF. This is Lemma 4.3 in [8] and is based on the properties of the divergence operator.  $\square$

We conclude this section by the following anisotropic embedding theorem.

THEOREM 2.44. *Let  $P \in \partial\Omega$ ,  $g \in W^{1,1}(\Omega_P)$  with  $\text{spt } g \subset \text{spt } \xi_P$ . Let all tangential derivatives  $\partial_\tau g$  satisfy  $\partial_\tau g \in L^q(\Omega_P)$ ,  $q > 1$ , and let also  $\partial_3 g \in L^r(\Omega_P)$ ,  $r > 1$ , with  $\frac{2}{q} + \frac{1}{r} > 1$ . Then  $g \in L^s(\Omega_P)$  with  $s$  given by*

$$\frac{1}{s} = \frac{1}{3} \left( \frac{2}{q} + \frac{1}{r} - 1 \right),$$

and the following inequalities hold true

$$\begin{aligned} \|g\|_{s,\Omega} &\leq c \|\partial_{\tau_1} g\|_{q,\Omega_P}^{1/3} \|\partial_{\tau_2} g\|_{q,\Omega_P}^{1/3} \|\partial_3 g\|_{r,\Omega_P}^{1/3}, \\ \|g\|_{s,\Omega} &\leq c \left( \|\partial_{\tau_1} g\|_{q,\Omega_P} + \|\partial_{\tau_2} g\|_{q,\Omega_P} + \|\partial_3 g\|_{r,\Omega_P} \right), \end{aligned} \quad (2.45)$$

where  $c$  depends only on  $q$  and  $r$ .

PROOF. The theorem is proved in [29] provided the Cartesian product of smooth open intervals (in that case one can assume that different partial derivatives belong to appropriate Lebesgue spaces). The case of the non-flat boundary can be converted to the previous one by the coordinate transformation  $\Phi : (x', x_3) \rightarrow (x', x_3 + a(x'))$ . By defining  $G := g \circ \Phi$  one obtains a function with compact support in the (closed) upper half-space and if one defines  $\tilde{G}$  by an even extension of  $G$  with respect to the  $x_3$  direction, it turns out that  $\tilde{G}$  belongs to  $W_0^{1,1}(K)$ , where  $K \subset \mathbb{R}^3$  is a compact set. We then apply [29, Theorem 1.2] to  $\tilde{G}$ , which can be approximated by functions in  $C_0^\infty(\mathbb{R}^3)$ , to obtain

$$\begin{aligned} \|G\|_{s,\mathbb{R}_+^3} &\leq \|\tilde{G}\|_s \leq c \|\partial_1 \tilde{G}\|_q^{1/3} \|\partial_2 \tilde{G}\|_q^{1/3} \|\partial_3 \tilde{G}\|_r^{1/3} \\ &\leq 2c \|\partial_1 G\|_{q,\mathbb{R}_+^3}^{1/3} \|\partial_2 G\|_{q,\mathbb{R}_+^3}^{1/3} \|\partial_3 G\|_{r,\mathbb{R}_+^3}^{1/3}. \end{aligned}$$

Since the Jacobian of the transformation  $\Phi$  is equal to one, using the reverse transformation to  $\Phi$  one gets the first statement of the theorem by a change of variables. In fact, by the definition  $G := g \circ \Phi$ , it turns out that

$$\partial_\alpha G = (\partial_\alpha g + \partial_\alpha a \partial_3 g) \circ \Phi = (\partial_\tau g) \circ \Phi,$$

which is a tangential derivative of  $g$  composed with  $\Phi$ , and also  $\partial_3 G = (\partial_3 g) \circ \Phi$ . The additive version is then proved by Young's inequality.  $\square$

### 3. Proof of Theorem 2.28

We assume that  $\mathbf{S}$  satisfies the assumption of Theorem 2.28, i.e.  $\mathbf{S} = \mathbf{S}^0 + \mathbf{S}^1$  with  $\mathbf{S}^0$  having quadratic-structure and  $\mathbf{S}^1$  having  $(p, \delta)$ -structure. Moreover,  $\mathbf{f}$  belongs to  $L^2(\Omega)$ .

From the properties of  $\mathbf{S}$  and the standard theory of monotone operators we easily obtain the existence of a unique  $\mathbf{u} \in W_{0,\text{div}}^{1,2}(\Omega)$ , satisfying for all  $\mathbf{v} \in W_{0,\text{div}}^{1,2}(\Omega)$

$$\int_{\Omega} \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}\mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}.$$

Using Proposition 2.14, the properties of  $\mathbf{S}$ , Poincaré's and Korn's inequalities as well as Young's inequality, we obtain that this solution satisfies the a-priori estimate

$$\kappa_0(2) \int_{\Omega} |\nabla \mathbf{u}|^2 \, d\mathbf{x} + \kappa_0(p) \int_{\Omega} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x} \leq c \int_{\Omega} |\mathbf{f}|^2 \, d\mathbf{x}. \quad (3.1)$$

Here and in the sequel we denote by  $\kappa_i(2)$  and  $\kappa_i(p)$ ,  $i = 0, 1$ , respectively, the constants in Definition 2.8 for 2 and  $p$ , respectively. Moreover, in the above estimate

and in the sequel all constants can depend on the characteristics of  $\mathbf{S}^0$  and  $\mathbf{S}^1$ , on  $\text{diam}(\Omega)$ ,  $|\Omega|$ , on the space dimension, and on the John constants of  $\Omega$ . Finally, the constants can also depend<sup>2</sup> on  $\delta_0$  (cf. Remark 2.11). All these dependencies will not be mentioned explicitly, while the dependence on other quantities is made explicit.

It is possible to associate to the  $\mathbf{u}$  a unique pressure  $\pi \in L_0^2(\Omega)$  satisfying for all  $\mathbf{v} \in W_0^{1,2}(\Omega)$

$$\int_{\Omega} \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}\mathbf{v} - \pi \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}. \quad (3.2)$$

Using the properties of  $\mathbf{S}$ , Lemma 2.43,  $p < 2$ , Lemma 2.34, and Young's inequality we thus obtain the following estimate

$$\int_{\Omega} |\pi|^2 \, d\mathbf{x} \leq c \left( 1 + \|\mathbf{f}\|_2^2 \right). \quad (3.3)$$

**3.1. Regularity in tangential directions and in the interior.** Let us start with the regularity in tangential directions. The interior regularity follows along the same lines of reasoning, but with several simplifications.

The main results of this section are summarized in the following proposition, which ensures local boundary estimates for tangential derivatives, that depend on  $P \in \partial\Omega$  only through  $\|\xi_P\|_{2,\infty}$  and  $\|a_P\|_{C^{2,1}}$ .

**PROPOSITION 3.4.** *Let the assumptions of Theorem 2.28 be satisfied and let the local description  $a_P$  of the boundary and the localization function  $\xi_P$  satisfy (b1)–(b3) and (ℓ1) (cf. Section 2.3). Then, there exist functions  $M_0 \in C(\mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0})$ ,  $M_1 \in C(\mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0})$  such that for every  $P \in \partial\Omega$*

$$\begin{aligned} & \int_{\Omega} \xi_P^2 |\partial_{\tau} \nabla \mathbf{u}|^2 + \xi_P^2 |\nabla \partial_{\tau} \mathbf{u}|^2 + \varphi(\xi_P |\partial_{\tau} \nabla \mathbf{u}|) + \varphi(\xi_P |\nabla \partial_{\tau} \mathbf{u}|) \, d\mathbf{x} \\ & + \int_{\Omega} \xi_P^2 |\partial_{\tau} \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \, d\mathbf{x} \leq M_0(\|\xi_P\|_{2,\infty}, \|a_P\|_{C^{2,1}}) \left( 1 + \|\mathbf{f}\|_2^2 \right), \end{aligned} \quad (3.5)$$

and

$$\int_{\Omega} \xi_P^2 |\partial_{\tau} \pi|^2 \, d\mathbf{x} \leq M_1(\delta^{p-2}, \|\xi_P\|_{2,\infty}, \|a_P\|_{C^{1,1}}) \left( 1 + \|\mathbf{f}\|_2^2 \right), \quad (3.6)$$

where  $\|a_P\|_{C^{k,1}}$  means  $\|a_P\|_{C^{k,1}(\overline{B_{3/4R_P}^2(0)})}$ , for  $k = 1, 2$ .

As usual in the study of boundary regularity we need to localize and to use appropriate test functions. Consequently, let us fix  $P \in \partial\Omega$  and in  $\Omega_P$  use  $\xi := \xi_P$ ,  $a := a_P$ , while  $h \in (0, \frac{R_P}{16})$ , as in Section 2.3. We use as test function  $\mathbf{v}$  in the weak formulation (3.2)

$$\mathbf{v} = d^-(\xi \psi),$$

(more precisely  $\xi$  is extended by zero for  $\mathbf{x} \in \Omega \setminus \Omega_P$ , in order to have a global function over  $\Omega$ ) with  $\psi \in W_0^{1,2}(\Omega)$  to get, with the help of Lemma 2.22 and Lemma 2.24, the following equality

$$\begin{aligned} & \int_{\Omega} d^+ \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}(\xi \psi) + \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot ((\partial_3(\xi \psi))_{-\tau} \otimes^s d^- \nabla a) - \pi \operatorname{div} d^-(\xi \psi) \, d\mathbf{x} \\ & = \int_{\Omega} \mathbf{f} \cdot d^-(\xi \psi) \, d\mathbf{x}. \end{aligned} \quad (3.7)$$

<sup>2</sup>This dependence occurs in most cases due to a shift change and results in an additive constant.

Due to the fact that  $\mathbf{u} \in W_{0,\text{div}}^{1,2}(\Omega)$  we can set  $\boldsymbol{\psi} = \xi d^+(\mathbf{u}|_{\tilde{\Omega}_P})$  in  $\Omega_P$  (and zero outside), hence as a test function we can consider the following vector field

$$\mathbf{v} = d^-(\xi^2 d^+(\mathbf{u}|_{\tilde{\Omega}_P})),$$

where  $\tilde{\Omega}_P := \frac{1}{2}\Omega_P$ , for the definition recall (2.18). Since  $\boldsymbol{\psi}$  has zero trace on  $\Omega_P$ , we get  $\mathbf{v} \in W_0^{1,2}(\Omega_P)$ , for small enough  $h > 0$ .

REMARK 3.8. *As a general disclaimer, we stress that in the sequel we will use the convention that functions are extended by zero off their proper set of definition, when needed. We avoid making this explicit in the sequel to avoid cumbersome expressions.*

Using Lemma 2.22–Lemma 2.24 we thus get the following identity

$$\begin{aligned} & \int_{\Omega} \xi^2 d^+ \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot d^+ \mathbf{D}\mathbf{u} \, dx \\ &= - \int_{\Omega} \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot (\xi^2 d^+ \partial_3 \mathbf{u} - (\xi_{-\tau} d^- \xi + \xi d^- \xi) \partial_3 \mathbf{u}) \overset{\circ}{\otimes} d^- \nabla a \, dx \\ & \quad - \int_{\Omega} \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \xi^2 (\partial_3 \mathbf{u})_{\tau} \overset{\circ}{\otimes} d^- d^+ \nabla a - \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot d^- (2\xi \nabla \xi \overset{\circ}{\otimes} d^+ \mathbf{u}) \, dx \\ & \quad + \int_{\Omega} \mathbf{S}((\mathbf{D}\mathbf{u})_{\tau}) \cdot (2\xi \partial_3 \xi d^+ \mathbf{u} + \xi^2 d^+ \partial_3 \mathbf{u}) \overset{\circ}{\otimes} d^+ \nabla a \, dx \\ & \quad - \int_{\Omega} \pi (\xi^2 d^- d^+ \nabla a - (\xi_{-\tau} d^- \xi + \xi d^- \xi) d^- \nabla a) \cdot \partial_3 \mathbf{u} \, dx \\ & \quad - \int_{\Omega} \pi (d^- (2\xi \nabla \xi \cdot d^+ \mathbf{u}) - \xi^2 d^+ \partial_3 \mathbf{u} \cdot d^+ \nabla a) \, dx \\ & \quad + \int_{\Omega} \pi_{\tau} (2\xi \partial_3 \xi d^+ \mathbf{u} + \xi^2 d^+ \partial_3 \mathbf{u}) \cdot d^+ \nabla a \, dx \\ & \quad + \int_{\Omega} \mathbf{f} \cdot d^-(\xi^2 d^+ \mathbf{u}) \, dx =: \sum_{j=1}^{15} I_j. \end{aligned} \tag{3.9}$$

The term providing the information concerning the regularity of the solution is the integral kept on the left-hand side. From the assumption on  $\mathbf{S}$  and Proposition 2.14 it may be estimated from below by

$$\int_{\Omega} \xi^2 d^+ \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot d^+ \mathbf{D}\mathbf{u} \, dx \geq c \int_{\Omega} \xi^2 |d^+ \mathbf{D}\mathbf{u}|^2 + \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \, dx. \tag{3.10}$$

We take advantage of the restriction  $p \leq 2$ , especially to gain further information from the right-hand side of (3.10), as in the following lemma.

LEMMA 3.11. *Let  $\mathbf{F}$  be given by (2.13) for some  $p \in (1, 2]$  and  $\delta \geq 0$ . Then, for  $\xi, a$  as above and  $\mathbf{u} \in W_0^{1,p}(\Omega)$  we have<sup>3</sup>*

$$\begin{aligned} & \int_{\Omega} \varphi(\xi |\nabla d^+ \mathbf{u}|) + \varphi(\xi |d^+ \nabla \mathbf{u}|) \, dx \leq c \int_{\Omega} \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \, dx \\ & \quad + c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, dx. \end{aligned}$$

<sup>3</sup>Note, that the constants here do not depend on  $\delta_0$ .

PROOF. Using Lemma 2.22 and the convexity of  $\varphi$  we see

$$\begin{aligned}
 & \int_{\Omega} \varphi(\xi |d^+ \nabla \mathbf{u}|) + \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x} \\
 & \leq c \int_{\Omega} \varphi(\xi |\nabla d^+ \mathbf{u}|) + \varphi(\xi |(\partial_3 \mathbf{u})_{\tau} d^+ \nabla a|) \, d\mathbf{x} \\
 & \leq c \int_{\Omega} \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x} + c(\|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x},
 \end{aligned} \tag{3.12}$$

where we also used a change of variables in the estimate of the last term. To treat the first term from the right-hand side we use the following identity

$$\xi \nabla d^+ \mathbf{u} = \nabla(\xi d^+ \mathbf{u}) - \nabla \xi \otimes d^+ \mathbf{u},$$

and consequently we get

$$\int_{\Omega} \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x} \leq c \int_{\Omega} \varphi(|\mathbf{D}(\xi d^+ \mathbf{u})|) \, d\mathbf{x} + c(\|\xi\|_{1,\infty}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x},$$

where we used Korn's inequality (cf. Proposition 2.40) and

$$\int_{\Omega \cap \text{spt } \xi} \varphi(|d^{\pm} \mathbf{u}|) \, d\mathbf{x} \leq \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}, \tag{3.13}$$

which can be proved in the same way as in the classical  $L^p$ -setting (cf. [20]). Using the identities

$$\begin{aligned}
 \mathbf{D}(\xi d^+ \mathbf{u}) &= \xi \mathbf{D}(d^+ \mathbf{u}) + \nabla \xi \overset{s}{\otimes} d^+ \mathbf{u} \\
 &= \xi d^+ \mathbf{D}\mathbf{u} + (\partial_3 \mathbf{u})_{\tau} \overset{s}{\otimes} d^+ \nabla a + \nabla \xi \overset{s}{\otimes} d^+ \mathbf{u},
 \end{aligned}$$

the properties of  $\varphi, \xi, a$  and (3.13), we obtain

$$\int_{\Omega} \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x} \leq c \int_{\Omega} \varphi(\xi |d^+ \mathbf{D}\mathbf{u}|) \, d\mathbf{x} + c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}. \tag{3.14}$$

Using Lemma 2.34, (2.2), (2.4),  $p \leq 2$  (or more generally that  $\varphi''$  is non-increasing), and Proposition 2.14, we also obtain

$$\begin{aligned}
 \varphi(\xi |d^+ \mathbf{D}\mathbf{u}|) &\leq c(\varphi_{|\mathbf{D}\mathbf{u}|+|d^+ \mathbf{D}\mathbf{u}|}(\xi |d^+ \mathbf{D}\mathbf{u}|) + \varphi(|\mathbf{D}\mathbf{u}| + |d^+ \mathbf{D}\mathbf{u}|)) \\
 &\sim (\delta + |\mathbf{D}\mathbf{u}| + |d^+ \mathbf{D}\mathbf{u}| + \xi |d^+ \mathbf{D}\mathbf{u}|)^{p-2} |\xi d^+ \mathbf{D}\mathbf{u}|^2 + \varphi(|\mathbf{D}\mathbf{u}| + |d^+ \mathbf{D}\mathbf{u}|) \\
 &\leq c(\delta + |\mathbf{D}\mathbf{u}| + |d^+ \mathbf{D}\mathbf{u}|)^{p-2} |\xi d^+ \mathbf{D}\mathbf{u}|^2 + c\varphi(|\mathbf{D}\mathbf{u}| + |d^+ \mathbf{D}\mathbf{u}|) \\
 &\sim |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \varphi(|\mathbf{D}\mathbf{u}| + |d^+ \mathbf{D}\mathbf{u}|).
 \end{aligned}$$

Inserting this into (3.14) we get

$$\int_{\Omega} \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x} \leq c \int_{\Omega} |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \, d\mathbf{x} + c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x},$$

which together with (3.12) yields the assertion.  $\square$

We can now give the proof of the regularity of tangential derivatives.

PROOF OF PROPOSITION 3.4. Using the previous lemma for the function  $\varphi$  with exponents 2 and  $p$  and observing that for any smooth enough function  $f$

$$\nabla \partial_{\tau} f = \partial_{\tau} \nabla f + \nabla \partial_{\alpha} a \partial_3 f, \tag{3.15}$$

(valid if  $a \in C^{2,1}$ ), we derive from (3.9) and (3.10) the following estimate

$$\begin{aligned} & \int_{\Omega} \xi^2 |d^+ \nabla \mathbf{u}|^2 + \xi^2 |\nabla d^+ \mathbf{u}|^2 + \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \varphi(\xi |d^+ \nabla \mathbf{u}|) + \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x} \\ & \leq c \sum_{j=1}^{15} I_j + c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\nabla \mathbf{u}|^2 + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}. \end{aligned} \quad (3.16)$$

It is very relevant that we have a full gradient  $\nabla \mathbf{u}$ , instead of  $\mathbf{D}\mathbf{u}$  in all terms on the left-hand side, except that involving  $\mathbf{F}(\mathbf{D}\mathbf{u})$ , at the price of a lower order term in the right-hand side, thanks to Lemma 3.11. We are now going to estimate the terms  $I_j$ , for  $j = 1, \dots, 15$ . By using the growth properties of  $\mathbf{S}^0$  and  $\mathbf{S}^1$ , Proposition 2.14, and Young's inequality (2.1) we obtain for the terms with  $\mathbf{S}$  the following estimates, which are valid for any given  $\varepsilon > 0$ :

$$\begin{aligned} |I_1| & \leq c(\|a\|_{C^{1,1}}) \int_{\Omega} \xi^2 (|\mathbf{D}\mathbf{u}| + \varphi'(|\mathbf{D}\mathbf{u}|)) |d^+ \partial_3 \mathbf{u}| \, d\mathbf{x} \\ & \leq c(\varepsilon^{-1}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{D}\mathbf{u}|^2 + \varphi(|\mathbf{D}\mathbf{u}|) \, d\mathbf{x} + \varepsilon \int_{\Omega} \xi^2 |d^+ \nabla \mathbf{u}|^2 + \varphi(\xi |d^+ \nabla \mathbf{u}|) \, d\mathbf{x}, \end{aligned} \quad (3.17)$$

$$\begin{aligned} |I_2 + I_3| & \leq c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} (|\mathbf{D}\mathbf{u}| + \varphi'(|\mathbf{D}\mathbf{u}|)) |\partial_3 \mathbf{u}| \, d\mathbf{x} \\ & \leq c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\nabla \mathbf{u}|^2 + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}. \end{aligned} \quad (3.18)$$

The next term requires the full regularity of  $a \in C^{2,1}$ , in fact

$$\begin{aligned} |I_4| & \leq c(\|a\|_{C^{2,1}}) \int_{\Omega \cap \text{spt } \xi} (|\mathbf{D}\mathbf{u}| + \varphi'(|\mathbf{D}\mathbf{u}|)) |(\partial_3 \mathbf{u})_{\tau}| \, d\mathbf{x} \\ & \leq c(\|a\|_{C^{2,1}}) \int_{\Omega \cap \text{spt } \xi} |\nabla \mathbf{u}|^2 + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}, \end{aligned} \quad (3.19)$$

where we also used a translation argument for  $\partial_3 \mathbf{u}$ . Also using (3.13), (2.5) we get

$$\begin{aligned} |I_5| & \leq c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{D}\mathbf{u}|^2 + \varphi(|\mathbf{D}\mathbf{u}|) \, d\mathbf{x} \\ & \quad + \frac{\varepsilon}{4} \int_{\Omega \cap \text{spt } \xi} |d^-(2\xi \nabla \xi \otimes^s d^+ \mathbf{u})|^2 + \varphi(|d^-(2\xi \nabla \xi \otimes^s d^+ \mathbf{u})|) \, d\mathbf{x} \\ & \leq c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{D}\mathbf{u}|^2 + \varphi(|\mathbf{D}\mathbf{u}|) \, d\mathbf{x} \\ & \quad + \frac{\varepsilon}{4} \int_{\Omega \cap \text{spt } \xi} |\nabla(2\xi \nabla \xi \otimes^s d^+ \mathbf{u})|^2 + \varphi(|\nabla(2\xi \nabla \xi \otimes^s d^+ \mathbf{u})|) \, d\mathbf{x} \\ & \leq c(\varepsilon^{-1}, \|\xi\|_{2,\infty}) \int_{\Omega \cap \text{spt } \xi} |\nabla \mathbf{u}|^2 + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x} \\ & \quad + \varepsilon \max(1, 4\|\xi\|_{1,\infty}^2) \int_{\Omega \cap \text{spt } \xi} \xi^2 |\nabla d^+ \mathbf{u}|^2 + \varphi(|\xi \nabla d^+ \mathbf{u}|) \, d\mathbf{x}, \end{aligned} \quad (3.20)$$

and observe that, by the definition of  $\xi = \xi_P$  we have  $\max(1, 4\|\xi\|_{1,\infty}^2) = 4\|\xi\|_{1,\infty}^2$ . Using also a translation argument for  $\mathbf{Du}$  and (3.13) yields

$$\begin{aligned} |I_6| &\leq c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} (|(\mathbf{Du})_\tau|^2 + \varphi'(|(\mathbf{Du})_\tau|)) |d^+ \mathbf{u}| \, d\mathbf{x} \\ &\leq c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\nabla \mathbf{u}|^2 + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}. \end{aligned} \quad (3.21)$$

Passing to  $I_7$  we have

$$\begin{aligned} |I_7| &\leq c(\|a\|_{C^{1,1}}) \int_{\Omega} \xi^2 (|(\mathbf{Du})_\tau| + \varphi'(|(\mathbf{Du})_\tau|)) |d^+ \partial_3 \mathbf{u}| \, d\mathbf{x} \\ &\leq c(\varepsilon^{-1}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{Du}|^2 + \varphi(|\mathbf{Du}|) \, d\mathbf{x} + \varepsilon \int_{\Omega} \xi^2 |d^+ \nabla \mathbf{u}|^2 + \varphi(\xi |d^+ \nabla \mathbf{u}|) \, d\mathbf{x}, \end{aligned} \quad (3.22)$$

where we again used a translation argument for  $\mathbf{Du}$ . The terms with the pressure are estimated, using Young's inequality, as follows

$$|I_8| \leq c(\|a\|_{C^{2,1}}) \int_{\Omega \cap \text{spt } \xi} |\pi| |\partial_3 \mathbf{u}| \, d\mathbf{x} \leq c(\|a\|_{C^{2,1}}) \int_{\Omega \cap \text{spt } \xi} |\pi|^2 + |\nabla \mathbf{u}|^2 \, d\mathbf{x}, \quad (3.23)$$

$$\begin{aligned} |I_9 + I_{10}| &\leq c(\|a\|_{C^{1,1}}, \|\xi\|_{1,\infty}) \int_{\Omega \cap \text{spt } \xi} |\pi| |\partial_3 \mathbf{u}| \, d\mathbf{x} \\ &\leq c(\|a\|_{C^{1,1}}, \|\xi\|_{1,\infty}) \int_{\Omega \cap \text{spt } \xi} |\pi|^2 + |\nabla \mathbf{u}|^2 \, d\mathbf{x}, \end{aligned} \quad (3.24)$$

$$\begin{aligned} |I_{11}| &\leq c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\pi|^2 \, d\mathbf{x} + \varepsilon \int_{\Omega \cap \text{spt } \xi} |\nabla(2\xi \nabla \xi \cdot d^+ \mathbf{u})|^2 \, d\mathbf{x} \\ &\leq c(\varepsilon^{-1}, \|\xi\|_{2,\infty}) \int_{\Omega \cap \text{spt } \xi} |\pi|^2 + |\nabla \mathbf{u}|^2 \, d\mathbf{x} + 4\varepsilon \|\xi\|_{1,\infty}^2 \int_{\Omega \cap \text{spt } \xi} \xi^2 |\nabla d^+ \mathbf{u}|^2 \, d\mathbf{x}, \end{aligned} \quad (3.25)$$

where we also used (3.13),

$$\begin{aligned} |I_{12}| &\leq c(\|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \xi^2 |\pi| |d^+ \partial_3 \mathbf{u}| \, d\mathbf{x} \\ &\leq c(\varepsilon^{-1}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\pi|^2 \, d\mathbf{x} + \varepsilon \int_{\Omega} \xi^2 |d^+ \nabla \mathbf{u}|^2 \, d\mathbf{x}, \end{aligned} \quad (3.26)$$

$$\begin{aligned} |I_{13}| &\leq c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\pi_\tau| |d^+ \mathbf{u}| \, d\mathbf{x} \\ &\leq c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\pi|^2 + |\nabla \mathbf{u}|^2 \, d\mathbf{x}, \end{aligned} \quad (3.27)$$

where we also used a translation argument for  $\pi$  and (3.13),

$$\begin{aligned} |I_{14}| &\leq c(\|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \xi^2 |\pi_\tau| |d^+ \partial_3 \mathbf{u}| \, d\mathbf{x} \\ &\leq c(\varepsilon^{-1}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\pi|^2 \, d\mathbf{x} + \varepsilon \int_{\Omega} \xi^2 |d^+ \nabla \mathbf{u}|^2 \, d\mathbf{x}, \end{aligned} \quad (3.28)$$

where we again used a translation argument for  $\pi$  and (3.13). The term with the external force is estimated as follows

$$\begin{aligned}
|I_{15}| &\leq c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{f}|^2 d\mathbf{x} + \varepsilon \int_{\Omega \cap \text{spt } \xi} |d^-(\xi^2 d^+ \mathbf{u})|^2 d\mathbf{x} \\
&\leq c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{f}|^2 d\mathbf{x} + \varepsilon \int_{\Omega \cap \text{spt } \xi} |\nabla(\xi^2 d^+ \mathbf{u})|^2 d\mathbf{x} \\
&\leq c(\varepsilon^{-1}, \|\xi\|_{1,\infty}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{f}|^2 + |\nabla \mathbf{u}|^2 d\mathbf{x} + \varepsilon \int_{\Omega \cap \text{spt } \xi} \xi^2 |\nabla d^+ \mathbf{u}|^2 d\mathbf{x}
\end{aligned} \tag{3.29}$$

where we also used (3.13).

Choosing in the estimates (3.17)–(3.29) the constant  $\varepsilon > 0$

$$\varepsilon = \frac{1}{56\|\xi\|_{1,\infty}^2},$$

we can absorb all terms involving tangential increments of  $\nabla \mathbf{u}$  in the left-hand side of (3.16), obtaining the following fundamental estimate

$$\begin{aligned}
&\int_{\Omega} \xi^2 |d^+ \nabla \mathbf{u}|^2 + \xi^2 |\nabla d^+ \mathbf{u}|^2 + \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \varphi(\xi |d^+ \nabla \mathbf{u}|) + \varphi(\xi |\nabla d^+ \mathbf{u}|) d\mathbf{x} \\
&\leq c(\|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{f}|^2 + |\pi|^2 + |\nabla \mathbf{u}|^2 + \varphi(|\nabla \mathbf{u}|) d\mathbf{x},
\end{aligned}$$

where the right-hand side is finite (and independent of  $h$ ) since (3.1) and (3.3) imply

$$\begin{aligned}
&\int_{\Omega} \xi^2 |d^+ \nabla \mathbf{u}|^2 + \xi^2 |\nabla d^+ \mathbf{u}|^2 + \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \varphi(\xi |d^+ \nabla \mathbf{u}|) + \varphi(\xi |\nabla d^+ \mathbf{u}|) d\mathbf{x} \\
&\leq M_0(\|\xi\|_{2,\infty}, \|a\|_{C^{2,1}})(1 + \|\mathbf{f}\|^2).
\end{aligned} \tag{3.30}$$

From estimate (3.30) and the properties of tangential derivatives recalled in (2.21), we thus obtain (3.5).

To prove estimate (3.6) for  $\partial_\tau \pi$  we start with

$$\int_{\Omega} \xi^2 |d^+ \pi|^2 d\mathbf{x} \leq 2 \int_{\Omega} |\xi d^+ \pi - \langle \xi d^+ \pi \rangle_{\Omega}|^2 d\mathbf{x} + \frac{2}{|\Omega|} \left| \int_{\Omega} \xi d^+ \pi d\mathbf{x} \right|^2, \tag{3.31}$$

to take advantage of the Poincaré inequality. The second term on the right-hand side is treated as follows

$$\frac{2}{|\Omega|} \left| \int_{\Omega} \xi d^+ \pi d\mathbf{x} \right|^2 = \frac{2}{|\Omega|} \left| \int_{\Omega \cap \text{spt } \xi} \pi d^- \xi d\mathbf{x} \right|^2 \leq 2 \|\xi\|_{1,\infty}^2 \int_{\Omega \cap \text{spt } \xi} |\pi|^2 d\mathbf{x}, \tag{3.32}$$

where we used Lemma 2.24. The first term on the right-hand side of (3.31) is treated with the help of Lemma 2.43. For that we re-write (3.7), using Lemma 2.22



and Lemma 2.24, and get for all  $\psi \in W_0^{1,2}(\Omega)$

$$\begin{aligned} & \int_{\Omega} \xi d^+ \pi \operatorname{div} \psi \, d\mathbf{x} \\ &= \int_{\Omega} \xi d^+ \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}\psi + \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot d^-(\nabla \xi \overset{s}{\otimes} \psi) - \mathbf{S}((\mathbf{D}\mathbf{u})_{\tau}) \cdot (\partial_3(\xi \psi) \overset{s}{\otimes} d^+ \nabla a) \, d\mathbf{x} \\ & \quad + \int_{\Omega} \pi_{\tau} \partial_3(\xi \psi) \cdot d^+ \nabla a - \pi d^-(\nabla \xi \cdot \psi) \, d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot d^-(\xi \psi) \, d\mathbf{x} =: \sum_{k=1}^6 J_k. \end{aligned}$$

Thus, we have

$$\begin{aligned} & \int_{\Omega} |\xi d^+ \pi - \langle \xi d^+ \pi \rangle_{\Omega}|^2 \, d\mathbf{x} \\ & \leq \sup_{\psi \in W_0^{1,2}(\Omega)} \left[ \int_{\Omega} (\xi d^+ \pi - \langle \xi d^+ \pi \rangle_{\Omega}) \operatorname{div} \psi \, d\mathbf{x} - \frac{1}{C_0} \int_{\Omega} |\nabla \psi|^2 \, d\mathbf{x} \right] \quad (3.33) \\ & = \sup_{\psi \in W_0^{1,2}(\Omega)} \left[ \sum_{k=1}^6 J_k - \int_{\Omega} \langle \xi d^+ \pi \rangle_{\Omega} \operatorname{div} \psi \, d\mathbf{x} - \frac{1}{C_0} \int_{\Omega} |\nabla \psi|^2 \, d\mathbf{x} \right], \end{aligned}$$

where  $C_0$  does only depend on the John constants of  $\Omega$ . Using Young's inequality, the fact that the measure of  $\Omega$  is finite, and (3.32) we have, for all  $\varepsilon > 0$ ,

$$\left| \int_{\Omega} \langle \xi d^+ \pi \rangle_{\Omega} \operatorname{div} \psi \, d\mathbf{x} \right| \leq \varepsilon \int_{\Omega} |\nabla \psi|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \|\xi\|_{1,\infty}^2 \int_{\Omega \cap \operatorname{spt} \xi} |\pi|^2 \, d\mathbf{x}.$$

The terms  $J_k$ ,  $k = 1, \dots, 6$ , are estimated, using the properties of  $\mathbf{S}^0$ ,  $\mathbf{S}^1$ , and Young's inequality, as follows. We denote by  $C_P$  the global Poincaré constant for  $W_0^{1,2}(\Omega)$ , depending only on  $|\Omega|$ . We have first

$$\begin{aligned} |J_1| & \leq c \int_{\Omega} \xi (|d^+ \mathbf{D}\mathbf{u}| + \delta^{\frac{p-2}{2}} |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|) |\nabla \psi| \, d\mathbf{x} \\ & \leq \varepsilon \int_{\Omega} |\nabla \psi|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \int_{\Omega \cap \operatorname{spt} \xi} \xi^2 |d^+ \mathbf{D}\mathbf{u}|^2 + \delta^{p-2} \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \, d\mathbf{x}, \end{aligned} \quad (3.34)$$

where we also used the equivalence (2.26) and  $p \leq 2$ . Next, we estimate

$$\begin{aligned} |J_2| & \leq \varepsilon \int_{\Omega} |d^-(\nabla \xi \overset{s}{\otimes} \psi)|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \int_{\Omega \cap \operatorname{spt} \xi} |\mathbf{D}\mathbf{u}|^2 + \delta^{p-2} \varphi(|\mathbf{D}\mathbf{u}|) \, d\mathbf{x} \\ & \leq \varepsilon \int_{\Omega} |\nabla(\nabla \xi \overset{s}{\otimes} \psi)|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \int_{\Omega \cap \operatorname{spt} \xi} |\mathbf{D}\mathbf{u}|^2 + \delta^{p-2} \varphi(|\mathbf{D}\mathbf{u}|) \, d\mathbf{x} \quad (3.35) \\ & \leq \varepsilon 2(1 + C_P) \|\xi\|_{2,\infty}^2 \int_{\Omega} |\nabla \psi|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \int_{\Omega \cap \operatorname{spt} \xi} |\mathbf{D}\mathbf{u}|^2 + \delta^{p-2} \varphi(|\mathbf{D}\mathbf{u}|) \, d\mathbf{x}, \end{aligned}$$

where we also used  $p \leq 2$ , (3.13), and Poincaré's inequality. Next,

$$\begin{aligned} |J_3| & \leq c \|a\|_{C^{1,1}} \int_{\Omega} (|(\mathbf{D}\mathbf{u})_{\tau}| + \delta^{\frac{p-2}{2}} (\varphi(|(\mathbf{D}\mathbf{u})_{\tau}|))^{\frac{1}{2}}) (|\nabla \xi| \psi + |\xi \nabla \psi|) \, d\mathbf{x} \quad (3.36) \\ & \leq \varepsilon 2(1 + C_P) \|\xi\|_{1,\infty}^2 \int_{\Omega} |\nabla \psi|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \|a\|_{C^{1,1}}^2 \int_{\Omega \cap \operatorname{spt} \xi} |\mathbf{D}\mathbf{u}|^2 + \delta^{p-2} \varphi(|\mathbf{D}\mathbf{u}|) \, d\mathbf{x}, \end{aligned}$$

where we also used a translation argument for  $\mathbf{Du}$  and Poincaré's inequality. Next, we obtain

$$\begin{aligned} |J_4| &\leq c \|a\|_{C^{1,1}} \int_{\Omega} |\pi_{\tau}| (|(\nabla \xi) \psi| + |\xi \nabla \psi|) \, d\mathbf{x} \\ &\leq \varepsilon 2(1 + C_P) \|\xi\|_{1,\infty}^2 \int_{\Omega} |\nabla \psi|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \|a\|_{C^{1,1}}^2 \int_{\Omega \cap \text{spt } \xi} |\pi|^2 \, d\mathbf{x}, \end{aligned} \quad (3.37)$$

where we used Poincaré's inequality and a translation argument for  $\pi$ . Next,

$$\begin{aligned} |J_5| &\leq \varepsilon \int_{\Omega} |d^-(\nabla \xi \cdot \psi)|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\pi|^2 \, d\mathbf{x} \\ &\leq \varepsilon \int_{\Omega} |\nabla(\nabla \xi \otimes \psi)|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\pi|^2 \, d\mathbf{x} \\ &\leq \varepsilon 2(1 + C_P) \|\xi\|_{2,\infty}^2 \int_{\Omega} |\nabla \psi|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\pi|^2 \, d\mathbf{x}, \end{aligned} \quad (3.38)$$

where we also used (3.13) and Poincaré's inequality. Finally,

$$\begin{aligned} |J_6| &\leq \varepsilon \int_{\Omega} |d^-(\xi \psi)|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{f}|^2 \, d\mathbf{x} \\ &\leq \varepsilon \int_{\Omega} |\nabla(\xi \psi)|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{f}|^2 \, d\mathbf{x} \\ &\leq \varepsilon 2(1 + C_P) \|\xi\|_{1,\infty}^2 \int_{\Omega} |\nabla \psi|^2 \, d\mathbf{x} + c(\varepsilon^{-1}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{f}|^2 \, d\mathbf{x}, \end{aligned} \quad (3.39)$$

where we used (3.13) and Poincaré's inequality. In the estimates (3.34)–(3.39) we choose the following constant  $\varepsilon > 0$

$$\varepsilon = \frac{1}{24C_0(1 + C_P) \|\xi\|_{2,\infty}^2},$$

and we absorb all terms with  $\varepsilon$  in the term with  $C_0^{-1}$  in (3.33). We thus obtain from (3.31), (3.32), (3.33), (3.39), and (3.30) that

$$\int_{\Omega} \xi^2 |d^+ \pi|^2 \, d\mathbf{x} \leq c(\delta^{p-2}, \|\xi\|_{2,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{f}|^2 + |\pi|^2 + |\nabla \mathbf{u}|^2 + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x},$$

where the right-hand side is finite, since (3.1) and (3.3) imply

$$\int_{\Omega} \xi^2 |d^+ \pi|^2 \, d\mathbf{x} \leq M_1(\delta^{p-2}, \|\xi\|_{2,\infty}, \|a\|_{C^{1,1}})(1 + \|\mathbf{f}\|^2). \quad (3.40)$$

From this and (2.21) we thus obtain the estimate (3.6).  $\square$

**REMARK 3.41.** *The same procedure as in the proof of Proposition 3.4, with many simplifications, can be done in the interior of  $\Omega$  for divided differences in all directions  $\mathbf{e}^i$ ,  $i = 1, 2, 3$ . By choosing  $h \in \left(0, \frac{1}{2} \text{dist}(\text{spt } \xi_{00}, \partial\Omega)\right)$  this leads to*

$$\int_{\Omega} \xi_{00}^2 |\nabla^2 \mathbf{u}|^2 + \xi_{00}^2 |\nabla \mathbf{F}(\mathbf{Du})|^2 + \varphi(\xi_{00} |\nabla^2 \mathbf{u}|) \, d\mathbf{x} \leq c(\|\xi_{00}\|_{2,\infty})(1 + \|\mathbf{f}\|_2^2), \quad (3.42)$$

$$\int_{\Omega} \xi_{00}^2 |\nabla \pi|^2 d\mathbf{x} \leq c(\delta^{p-2}, \|\xi_{00}\|_{2,\infty})(1 + \|\mathbf{f}\|_2^2), \quad (3.43)$$

where  $\xi_{00}$  is any cut-off function with compact support contained in  $\Omega$ .

The following fundamental result derives immediately from the interior regularity.

**COROLLARY 3.44.** *Under the same hypotheses of Theorem 2.28 we obtain that  $\mathbf{F}(\mathbf{D}) \in W_{\text{loc}}^{1,2}(\Omega)$  and we know from (3.42) and (3.43) that  $\mathbf{u} \in W_{\text{loc}}^{2,2}(\Omega)$ ,  $\pi \in W_{\text{loc}}^{1,2}(\Omega)$ . This implies –in particular– that the equations of system (1.1) hold almost everywhere in  $\Omega$*

The above corollary implies that all pointwise calculations we are going to perform in the next section to estimate the remaining derivatives are justified.

**3.2. Regularity in the normal direction.** In this section we obtain global information about the regularity of  $\nabla^2 \mathbf{u}$  and  $\nabla \pi$ , by combining the information about tangential regularity (3.5) and the fact that the couple  $(\mathbf{u}, \pi)$  satisfies almost everywhere the system (1.1). We use methods applied also in [4, 5, 22] for  $p > 2$  and which have been previously used in the case  $p < 2$  also in [11] (for the problem without pressure) and in [3]. The main result of this section is the following proposition.

**PROPOSITION 3.45.** *Let the assumptions of Theorem 2.28 be satisfied and let the local description  $a_P$  of the boundary and the localization function  $\xi_P$  satisfy (b1)–(b3) and (l1) (cf. Section 2.3). Then, there exist a constant  $C_2 > 0$  and functions  $M_2, M_3$  such that for every  $P \in \partial\Omega$*

$$\int_{\Omega} \xi_P^2 |\nabla^2 \mathbf{u}|^2 d\mathbf{x} \leq M_2 (1 + \|\mathbf{f}\|_2^2), \quad (3.46)$$

$$\int_{\Omega} \xi_P^2 |\nabla \mathbf{F}(\mathbf{D}\mathbf{u})|^2 d\mathbf{x} \leq M_2 \delta^{p-2} (1 + \|\mathbf{f}\|_2^2), \quad (3.47)$$

where  $M_2 := 1 + M_1(\delta^{p-2}, \|\xi_P\|_{2,\infty}, \|a_P\|_{C^{1,1}}) + (1 + \delta^{2(p-2)})M_0(\|\xi_P\|_{2,\infty}, \|a_P\|_{C^{2,1}})$ , and

$$\int_{\Omega} \xi_P^2 |\nabla \pi|^2 d\mathbf{x} \leq M_3 (1 + \|\mathbf{f}\|_2^2), \quad (3.48)$$

where  $M_3 := 1 + 2(1 + \kappa_1^2 \delta^{2(p-2)})M_2$ , provided that  $r_P < C_2$ .

**PROOF.** Let us consider the first two equations in (1.1). In order to better separate variables concerning tangential and normal directions we use the following convention: Greek low-case letters concern only tangential derivatives while Latin low-case concern all of them, that is  $\alpha, \beta, \gamma, \sigma = 1, 2$ , and  $i, j, k, l = 1, 2, 3$ , with the usual convention of summation over repeated indices. Moreover, we denote a two-dimensional vector with components  $(b_1, b_2)^\top$  by  $\mathbf{b}$ . Let us put in evidence the equations for the variables  $\mathbf{S}_{\alpha 3}$ , by re-writing the  $\alpha$ -th equation from (1.1) as follows:

$$-\partial_3 \mathbf{S}_{\alpha 3} = f_\alpha + \partial_\alpha \pi + \partial_\beta \mathbf{S}_{\alpha \beta} \quad \text{a.e. in } \Omega.$$

We previously estimated  $\partial_\tau \mathbf{D}\mathbf{u}$  and we need now to estimate  $\partial_3 \mathbf{D}_{\gamma 3}$ , to recover all second order derivatives of  $\mathbf{u}$ . We re-write explicitly (component-wise)  $(1.1)_\alpha$  as follows:

$$\begin{aligned} & -\partial_{\gamma 3} \mathbf{S}_{\alpha 3} \partial_3 \mathbf{D}_{\gamma 3} - \partial_{3\gamma} \mathbf{S}_{\alpha 3} \partial_3 \mathbf{D}_{3\gamma} \\ & = f_\alpha + \partial_\alpha \pi + \partial_{33} \mathbf{S}_{\alpha 3} \partial_3 \mathbf{D}_{33} + \partial_{\gamma\sigma} \mathbf{S}_{\alpha 3} \partial_3 \mathbf{D}_{\gamma\sigma} + \partial_{kl} \mathbf{S}_{\alpha\beta} \partial_\beta \mathbf{D}_{kl} =: \mathfrak{f}_\alpha, \end{aligned} \quad (3.49)$$

by keeping only the above two terms in the left-hand side. The system (3.49) can be written as the algebraic linear system

$$-2A_{\alpha\gamma}b_\gamma = f_\alpha \quad \text{a.e. in } \Omega, \quad (3.50)$$

where  $A_{\alpha\gamma} := \partial_{\gamma 3}\mathbf{S}_{\alpha 3}$ , and  $\mathbf{b}_\gamma := \partial_3\mathbf{D}_{\gamma 3}$ . This is obtained by recalling that  $\mathbf{D}$  is symmetric, hence  $\mathbf{D}_{\gamma 3} = \mathbf{D}_{3\gamma}$ , and by observing that  $\partial_{\gamma 3}\mathbf{S}(\mathbf{D}) = \partial_{3\gamma}\mathbf{S}(\mathbf{D})$ , since  $\mathbf{S}$  depends only on the symmetric tensor  $\mathbf{D}$ .

We will estimate the quantity  $\mathbf{b}_\gamma = \partial_3\mathbf{D}_{\gamma 3}$  in terms of tangential derivatives, estimated in the previous section, and then extract from it information on the normal derivatives. A similar technique was employed in the above cited references, but there immediately a system for the normal derivatives  $\partial_{33}^2\mathbf{u}$  has been derived, which led to the restriction  $p > \frac{3}{2}$ , even in the flat case.

Multiplying (3.50) pointwise a.e. by  $-\mathbf{b}_\alpha$  and summing over  $\alpha = 1, 2$  we get, also using the structure of  $\mathbf{S}$ ,

$$2(\kappa_0(2) + \kappa_0(p))\varphi''(|\mathbf{D}\mathbf{u}|)|\mathbf{b}|^2 \leq 2A_{\alpha\gamma}\mathbf{b}_\gamma\mathbf{b}_\alpha \leq |\mathbf{f}||\mathbf{b}| \quad \text{a.e. in } \Omega.$$

To handle  $\mathbf{f}$  we prove identities and estimates valid almost everywhere in  $\Omega_P$ . We start with the following identity for the pressure:

$$\partial_\alpha\pi = \partial_{\tau_\alpha}\pi - \partial_\alpha a \partial_3\pi \quad \text{a.e. in } \Omega_P.$$

Concerning the third term from the right-hand side of (3.49) we get a.e. in  $\Omega_P$ , by using the solenoidal constraint and the formula for tangential derivatives,

$$\partial_3\mathbf{D}_{33} = -\partial_3\mathbf{D}_{\alpha\alpha} = -\partial_3(\nabla\mathbf{u})_{\alpha\alpha} = -\partial_\alpha\partial_3u^\alpha = -\partial_{\tau_\alpha}\partial_3u^\alpha + \partial_\alpha a \partial_{33}^2u^\alpha.$$

Concerning the fourth term from the right-hand side of (3.49) we observe that a.e. in  $\Omega_P$

$$\partial_3\mathbf{D}_{\gamma\sigma} = \frac{1}{2}(\partial_{\gamma 3}^2u^\sigma + \partial_{\sigma 3}^2u^\gamma) = \frac{1}{2}(\partial_{\tau_\gamma}\partial_3u^\sigma - \partial_\gamma a \partial_{33}^2u^\sigma + \partial_{\tau_\sigma}\partial_3u^\gamma - \partial_\sigma a \partial_{33}^2u^\gamma).$$

The fifth term from the right-hand side of (3.49) is handled recalling that

$$\partial_\beta\mathbf{D}_{kl} = \partial_{\tau_\beta}\mathbf{D}_{kl} - \partial_\beta a \partial_3\mathbf{D}_{kl} \quad \text{a.e. in } \Omega_P.$$

Collecting all these identities we obtain that a.e. in  $\Omega_P$  the right-hand side  $\mathbf{f}$  of (3.49) can be bounded as follows :

$$|\mathbf{f}| \leq c(|\mathbf{f}| + |\partial_\tau\pi| + \|\nabla a\|_\infty|\partial_3\pi| + (1 + \varphi''(|\mathbf{D}\mathbf{u}|))(|\partial_\tau\nabla\mathbf{u}| + \|\nabla a\|_\infty|\nabla^2\mathbf{u}|)),$$

where the constant  $c$  depends only on the characteristics of  $\mathbf{S}$ . To estimate the partial derivative  $\partial_3\pi$  we use again the equations to write pointwise in  $\Omega$  that  $\partial_3\pi = -f_3 - \partial_j\mathbf{S}_{3j}$  and hence to obtain

$$|\partial_3\pi| \leq |\mathbf{f}| + c(1 + \varphi''(|\mathbf{D}\mathbf{u}|))|\nabla^2\mathbf{u}| \quad \text{a.e. in } \Omega_P. \quad (3.51)$$

Collecting the estimates and dividing both sides by  $|\mathbf{b}| \neq 0$  (when it is zero there is nothing to prove) we obtain

$$(1 + \varphi''(|\mathbf{D}\mathbf{u}|))|\mathbf{b}| \leq c(|\mathbf{f}| + |\mathbf{f}|\|\nabla a\|_\infty + |\partial_\tau\pi| + (1 + \varphi''(|\mathbf{D}\mathbf{u}|))(|\partial_\tau\nabla\mathbf{u}| + \|\nabla a\|_\infty|\nabla^2\mathbf{u}|)),$$

We now identify the ‘‘normal’’<sup>4</sup> derivative  $\partial_{33}u^\alpha$  from the left-hand side by observing that

$$\begin{aligned} \mathbf{b}_\alpha &= \frac{1}{2}(\partial_{3\alpha}^2u^3 + \partial_{33}^2u^\alpha) = \frac{1}{2}(-\partial_\alpha\mathbf{D}_{\beta\beta} + \partial_{33}^2u^\alpha) = \frac{1}{2}(-\partial_{\alpha\beta}^2u^\beta + \partial_{33}^2u^\alpha) \\ &= \frac{1}{2}(-\partial_{\tau_\alpha}\partial_\beta u^\beta + \partial_\alpha a \partial_{3\beta}^2u^\beta + \partial_{33}^2u^\alpha). \end{aligned}$$

<sup>4</sup>It is not the normal derivative at all points on  $\partial\Omega \cap \partial\Omega_P$ , but only exactly at  $\mathbf{x} = P$

Consequently, we can write (in terms of  $\tilde{\mathbf{b}}_\alpha := \partial_{33}^2 u^\alpha$ )

$$|\tilde{\mathbf{b}}| \geq 2|\tilde{\mathbf{b}}| - |\partial_\tau \nabla \mathbf{u}| - \|\nabla a\|_\infty |\nabla^2 \mathbf{u}| \quad \text{a.e. in } \Omega_P,$$

from which we can show the following fundamental estimate

$$\begin{aligned} & (1 + \varphi''(|\mathbf{D}\mathbf{u}|)) |\tilde{\mathbf{b}}| \\ & \leq c (|\mathbf{f}| + |\mathbf{f}| \|\nabla a\|_\infty + |\partial_\tau \pi| + (1 + \varphi''(|\mathbf{D}\mathbf{u}|)) (|\partial_\tau \nabla \mathbf{u}| + \|\nabla a\|_\infty |\nabla^2 \mathbf{u}|)), \end{aligned}$$

which is valid almost everywhere in  $\Omega_P$ . Next, we observe that adding to both sides, for  $\alpha = 1, 2$  and  $i, k = 1, 2, 3$  the term

$$(1 + \varphi''(|\mathbf{D}\mathbf{u}|)) (|\partial_\alpha \partial_i u^k| + |\partial_{33}^2 u^3|)$$

the left-hand side equals  $(1 + \varphi''(|\mathbf{D}\mathbf{u}|)) |\nabla^2 \mathbf{u}|$ . To handle the terms we added in the right-hand side, we observe, that  $\partial_{33} u^3 = -\partial_{3\alpha} u^\alpha$  implies (by the solenoidal constraint)

$$\begin{aligned} \partial_\alpha \partial_i u^k &= \partial_{\tau_\alpha} \partial_i u^k - \partial_\alpha a \partial_{3i}^2 u^k, \\ \partial_{33}^2 u^3 &= -\partial_{\tau_\alpha} \partial_3 u^\alpha + \partial_\alpha a \partial_{33}^2 u^\alpha. \end{aligned}$$

Hence, there exists a constant  $C_1$ , depending only on the characteristics of  $\mathbf{S}$ , such that a.e. in  $\Omega_P$  it holds

$$\begin{aligned} & (1 + \varphi''(|\mathbf{D}\mathbf{u}|)) |\nabla^2 \mathbf{u}| \\ & \leq C_1 (|\mathbf{f}| + |\mathbf{f}| \|\nabla a\|_\infty + |\partial_\tau \pi| + (1 + \varphi''(|\mathbf{D}\mathbf{u}|)) (|\partial_\tau \nabla \mathbf{u}| + \|\nabla a\|_\infty |\nabla^2 \mathbf{u}|)). \end{aligned} \quad (3.52)$$

Next, we choose the open sets  $\Omega_P$  small enough (that is we choose the radii  $R_P$  small enough) in such a way that

$$\|\nabla a_P(x)\|_{L^\infty(\Omega_P)} \leq r_P \leq \frac{1}{2C_1} =: C_2. \quad (3.53)$$

Thus, we can absorb the last term from (3.52) in the left-hand side, which yields

$$\begin{aligned} & (1 + \varphi''(|\mathbf{D}\mathbf{u}|)) |\nabla^2 \mathbf{u}| \\ & \leq c (|\mathbf{f}| + |\partial_\tau \pi| + (1 + \varphi''(|\mathbf{D}\mathbf{u}|)) |\partial_\tau \nabla \mathbf{u}|) \quad \text{a.e. in } \Omega_P. \end{aligned} \quad (3.54)$$

It is at this point that we use the special features of the stress tensor which is the sum of the quadratic part with one with  $(p, \delta)$ -structure for  $p < 2$ . Hence, neglecting the second term on the left-hand side, which is non-negative, raising the remaining inequality to the power 2, and multiplying by the cut-off function  $\xi_P^2$ , we obtain

$$|\nabla^2 \mathbf{u}|^2 \xi_P^2 \leq c \left( |\mathbf{f}|^2 + |\partial_\tau \pi|^2 + (1 + \delta^{2(p-2)}) |\partial_\tau \nabla \mathbf{u}|^2 \right) \xi_P^2 \quad \text{a.e. in } \Omega_P. \quad (3.55)$$

Corollary 3.44 implies that the left-hand side is measurable, while Proposition 3.4 implies that the right-hand side belongs to  $L^1(\Omega_P)$ . Thus, Proposition 3.4 yields estimate (3.46). Next, we observe that (see [10, Lemma 3.8])

$$|\nabla \mathbf{F}(\mathbf{D})|^2 \leq (\delta + |\mathbf{D}|)^{p-2} |\nabla \mathbf{D}\mathbf{u}|^2 \leq \delta^{p-2} |\nabla^2 \mathbf{u}|^2,$$

which together with (3.46) yields (3.47). The estimate on  $\nabla \pi$  is obtained from equation (1.1), since

$$|\nabla \pi| \leq |\mathbf{f}| + (1 + \kappa_1 \varphi''(|\mathbf{D}\mathbf{u}|)) |\nabla^2 \mathbf{u}|,$$

where we also used (2.9). This and (3.46) yields (3.48).  $\square$

**PROOF OF THEOREM 2.28.** For every  $P \in \partial\Omega$  we choose a local description  $a_P$  of the boundary satisfying (b1)–(b3) (cf. Section 2.3) with  $r_P < C_2$ . Note that

$$\partial\Omega \subset \bigcup_{P \in \partial\Omega} \tilde{\Omega}_P, \quad (3.56)$$

where  $\tilde{\Omega}_P = \frac{1}{2}\Omega_P$ , recall (2.18). Since  $\partial\Omega$  is compact there exists a finite sub-covering  $\{\tilde{\Omega}_P, P \in \Lambda\}$ . Next, we choose a set  $\Omega_0 \subset\subset \Omega$  such that  $\text{dist}(\Omega_0, \partial\Omega) < \frac{1}{16} \min\{R_P, P \in \Lambda\}$ . Associated to the covering of  $\Omega$  made by  $\{\Omega_0, \Omega_P\}_{P \in \Lambda}$  we consider a set of smooth non-negative functions  $\{\xi_0, \xi_P\}_{P \in \Lambda}$ , where  $\xi_P$  satisfy (ℓ1) and  $\xi_0$  satisfies  $\text{spt } \xi_0 \subset \Omega_0$ ,  $\text{dist}(\text{spt } \xi_0, \partial\Omega) < \frac{1}{8} \min\{R_P, P \in \Lambda\}$ , and  $\xi_0(\mathbf{x}) = 1$  for all  $\mathbf{x}$  with  $\text{dist}(\mathbf{x}, \partial\Omega) > \frac{1}{4} \min\{R_P, P \in \Lambda\}$ . Observe that, by construction the set  $\{\mathbf{x} : \xi_0 = 1\} \cup_{P \in \Lambda} \{\mathbf{x} : \xi_P = 1\}$  covers all  $\Omega$ . Since the covering  $\{\Omega_0, \Omega_P\}_{P \in \Lambda}$  is finite, the evaluation of the functions  $M_i$ ,  $i = 0, 1, 2, 3$ , yields a finite fixed constant. Thus, Proposition 3.4 and Proposition 3.45 applied to the finite covering  $\{\Omega_0, \Omega_P\}_{P \in \Lambda}$  of  $\Omega$  yields immediately the assertions of Theorem 2.28.  $\square$

#### 4. Proof of Theorem 2.29

In this section we treat the problem with the principal part having  $(p, \delta)$ -structure. Many calculations are similar to those of the previous section, hence we recall them and mainly explain the differences arising in the treatment of tangential and normal derivatives. We assume that  $\mathbf{S}$  satisfies the assumption of Theorem 2.29, i.e.,  $\mathbf{S}$  has  $(p, \delta)$ -structure. Moreover,  $\mathbf{f}$  belongs to  $L^{p'}(\Omega)$ . Due to Remark 2.12 this is equivalent to  $\mathbf{f} \in L^{\varphi^*}(\Omega)$ .

As in Section 3 we easily obtain the existence of a unique  $\mathbf{u} \in W_{0,\text{div}}^{1,p}(\Omega)$  satisfying the weak formulation, i.e. for all  $\mathbf{v} \in W_{0,\text{div}}^{1,p}(\Omega)$

$$\int_{\Omega} \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}\mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x},$$

and the a-priori estimate

$$\int_{\Omega} \varphi(|\nabla \mathbf{u}|) + |\mathbf{F}(\mathbf{D}\mathbf{u})|^2 \, d\mathbf{x} \leq c \int_{\Omega} \varphi^*(|\mathbf{f}|) \, d\mathbf{x}. \quad (4.1)$$

In this section all constants can depend on the characteristics of  $\mathbf{S}$ , on  $\text{diam}(\Omega)$ ,  $|\Omega|$ , on the space dimension, on the John constants of  $\Omega$ , and on  $\delta_0$  (cf. Remark 2.11). All these dependencies will not be mentioned explicitly, while the dependence on other quantities is made explicit.

To the weak solution  $\mathbf{u}$  there exists a unique associated pressure  $\pi \in L_0^{p'}(\Omega)$  satisfying for all  $\mathbf{v} \in W_0^{1,p}(\Omega)$

$$\int_{\Omega} \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}\mathbf{v} - \pi \text{div } \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}. \quad (4.2)$$

Using the properties of  $\mathbf{S}$ , Lemma 2.43 and Young's inequality we thus obtain

$$\int_{\Omega} \varphi^*(|\pi|) \, d\mathbf{x} \leq c \int_{\Omega} \varphi^*(|\mathbf{f}|) \, d\mathbf{x}. \quad (4.3)$$

**4.1. Regularity in tangential directions and in the interior.** We start again with the regularity in tangential directions. In the same way as in Section 3 we derive the following result.

**PROPOSITION 4.4.** *Let the assumptions of Theorem 2.29 be satisfied and let the local description  $a_P$  of the boundary and the localization function  $\xi_P$  satisfy (b1)–(b3) and (ℓ1) (cf. Section 2.3). Then, there exist functions  $M_4 \in C(\mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0})$ ,*

$M_5 \in C(\mathbb{R}^{>0} \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0})$  such that for every  $P \in \partial\Omega$

$$\begin{aligned} & \int_{\Omega} \xi_P^2 |\partial_{\tau} \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \varphi(\xi_P |\partial_{\tau} \nabla \mathbf{u}|) + \varphi(\xi_P |\nabla \partial_{\tau} \mathbf{u}|) \, d\mathbf{x} \\ & \leq M_4 (\|\xi_P\|_{2,\infty}, \|a_P\|_{C^{2,1}}) \left( 1 + \int_{\Omega} \varphi^*(|\mathbf{f}|) \, d\mathbf{x} \right), \end{aligned} \quad (4.5)$$

and

$$\int_{\Omega} \xi_P^2 |\partial_{\tau} \pi|^2 \, d\mathbf{x} \leq M_5 (\delta^{p-2}, \|\xi_P\|_{2,\infty}, \|a_P\|_{C^{2,1}}) \left( 1 + \int_{\Omega} \varphi^*(|\mathbf{f}|) \, d\mathbf{x} \right), \quad (4.6)$$

where  $\|a_P\|_{C^{k,1}}$  means  $\|a_P\|_{C^{k,1}(\overline{B_{3/4R_P}^2(0)})}$ , for  $k = 1, 2$ .

PROOF. As in the previous section we fix  $P \in \partial\Omega$  and in  $\Omega_P$  use  $\xi := \xi_P$ ,  $a := a_P$ , while  $h \in (0, \frac{R_P}{16})$ , as in Section 2.3. In the same way as in Section 3.1, replacing  $W_{0,\text{div}}^{1,2}(\Omega)$  by  $W_{0,\text{div}}^{1,p}(\Omega)$  and using Lemma 3.11 for  $p \in (1, 2)$  only, we take as test function  $\mathbf{v} = d^-(\xi^2 d^+ \mathbf{u}|_{\tilde{\Omega}_P})$ , to obtain (3.9) and

$$\begin{aligned} & \int_{\Omega} \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \varphi(\xi |d^+ \nabla \mathbf{u}|) + \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x} \\ & \leq c \sum_{j=1}^{15} I_j + c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}. \end{aligned} \quad (4.7)$$

Note, that in Section 3.1 all terms in  $I_1, \dots, I_7$  resulting from  $\mathbf{S}^0$  have been absorbed in the corresponding terms in (3.16) coming from  $\mathbf{S}^0$ . The same for the terms in  $I_1, \dots, I_7$  resulting from  $\mathbf{S}^1$  which have been absorbed in the corresponding terms in (3.16) coming from  $\mathbf{S}^1$ , only.

This implies that we can now treat  $\mathbf{S}$  in  $I_1, \dots, I_7$  here as  $\mathbf{S}^1$  in the corresponding terms  $I_1, \dots, I_7$  in Section 3 and obtain then

$$|I_1| \leq c(\varepsilon^{-1}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\mathbf{D}\mathbf{u}|) \, d\mathbf{x} + \varepsilon \int_{\Omega} \varphi(\xi |d^+ \nabla \mathbf{u}|) \, d\mathbf{x}, \quad (4.8)$$

$$|I_2 + I_3| \leq c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}, \quad (4.9)$$

$$|I_4| \leq c(\|a\|_{C^{2,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}, \quad (4.10)$$

$$|I_5| \leq c(\varepsilon^{-1}, \|\xi\|_{2,\infty}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x} + \varepsilon 4 \|\xi\|_{1,\infty}^2 \int_{\Omega \cap \text{spt } \xi} \varphi(|\xi \nabla d^+ \mathbf{u}|) \, d\mathbf{x}, \quad (4.11)$$

$$|I_6| \leq c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}, \quad (4.12)$$

$$|I_7| \leq c(\varepsilon^{-1}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\mathbf{D}\mathbf{u}|) \, d\mathbf{x} + \varepsilon \int_{\Omega} \varphi(\xi |d^+ \nabla \mathbf{u}|) \, d\mathbf{x}. \quad (4.13)$$

Also the terms with the pressure and the external force are estimated similarly to the corresponding terms in Section 3. Since we know that  $\pi \in L_0^{p'}(\Omega)$  instead of

$\pi \in L_0^2(\Omega)$  we use Young's inequality with  $\varphi$  instead of 2. In that way we obtain

$$|I_8| \leq c(\|a\|_{C^{2,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi^*(|\pi|) + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}, \quad (4.14)$$

$$|I_9 + I_{10}| \leq c(\|a\|_{C^{1,1}}, \|\xi\|_{1,\infty}) \int_{\Omega \cap \text{spt } \xi} \varphi^*(|\pi|) + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}, \quad (4.15)$$

$$\begin{aligned} |I_{11}| \leq c(\varepsilon^{-1}, \|\xi\|_{2,\infty}) \int_{\Omega \cap \text{spt } \xi} \varphi^*(|\pi|) + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x} \\ + \varepsilon \max(1, 4\|\xi\|_{1,\infty}^2) \int_{\Omega \cap \text{spt } \xi} \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x}, \end{aligned} \quad (4.16)$$

$$|I_{12}| \leq c(\varepsilon^{-1}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi^*(|\pi|) \, d\mathbf{x} + \varepsilon \int_{\Omega} \varphi(\xi |d^+ \nabla \mathbf{u}|) \, d\mathbf{x}, \quad (4.17)$$

$$|I_{13}| \leq c(\varepsilon^{-1}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi^*(|\pi|) + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}, \quad (4.18)$$

$$|I_{14}| \leq c(\varepsilon^{-1}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi^*(|\pi|) \, d\mathbf{x} + \varepsilon \int_{\Omega} \varphi(\xi |d^+ \nabla \mathbf{u}|) \, d\mathbf{x}, \quad (4.19)$$

$$|I_{15}| \leq c(\varepsilon^{-1}, \|\xi\|_{1,\infty}) \int_{\Omega \cap \text{spt } \xi} \varphi^*(|\mathbf{f}|) + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x} + \varepsilon \int_{\Omega \cap \text{spt } \xi} \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x}, \quad (4.20)$$

where we also used (2.5) for the treatment of  $I_{11}$  (cf. estimate (3.20)). Choosing in the estimates (4.8)–(4.20)  $\varepsilon$  the constant  $\varepsilon > 0$

$$\varepsilon = \frac{1}{56\|\xi\|_{1,\infty}^2},$$

we can absorb all terms involving tangential increments of  $\nabla \mathbf{u}$  in the left-hand side of (3.16), obtaining the following fundamental estimate

$$\begin{aligned} \int_{\Omega} \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \varphi(\xi |d^+ \nabla \mathbf{u}|) + \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x} \\ \leq c(\|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi^*(|\mathbf{f}|) + \varphi^*(|\pi|) + \varphi(|\nabla \mathbf{u}|) \, d\mathbf{x}, \end{aligned} \quad (4.21)$$

where the right-hand side is finite (and independent of  $h$ ) since (4.1), (4.3) imply

$$\begin{aligned} \int_{\Omega} \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \varphi(\xi |d^+ \nabla \mathbf{u}|) + \varphi(\xi |\nabla d^+ \mathbf{u}|) \, d\mathbf{x} \\ \leq M_4(\|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}) \left(1 + \int_{\Omega} \varphi^*(|\mathbf{f}|) \, d\mathbf{x}\right). \end{aligned}$$

hence, by recalling (2.21) we proved (4.5).

To estimate the tangential derivatives of the pressure we proceed as in Section 3. The only difference is that the stress tensor has no part with a 2-structure. Even in this setting with pure  $(p, \delta)$ -structure we are again able to show that  $\partial_\tau \pi \in L^2(\Omega \cap \text{spt } \xi_P)$ . Starting from (3.31) and neglecting from (3.40) the terms which



resulted from  $\mathbf{S}^0$ , we arrive at the following inequality

$$\int_{\Omega} \xi^2 |d^+ \pi|^2 d\mathbf{x} \leq c(\delta^{p-2}, \|\xi\|_{2,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} |\mathbf{f}|^2 + |\pi|^2 + \varphi(|\nabla \mathbf{u}|) d\mathbf{x}. \quad (4.22)$$

Young's inequality, Lemma 2.34 and  $p < 2$  yield

$$\int_{\Omega \cap \text{spt } \xi} |\mathbf{f}|^2 + |\pi|^2 d\mathbf{x} \leq c \left( (1 + \delta^p) |\Omega| + \int_{\Omega \cap \text{spt } \xi} \varphi^*(|\mathbf{f}|) + \varphi^*(|\pi|) d\mathbf{x} \right).$$

This and the previous estimate lead, as in Section 3, to the inequality (4.6).  $\square$

REMARK 4.23. *We can show with the same method that Proposition 4.4 holds with  $\xi_P$  replaced by  $\tilde{\xi}_P$  satisfying (l1) with  $R_P$  replaced by  $R_P/2$ , i.e.,  $\tilde{\xi}_P \in C_0^\infty(\tilde{\Omega}_P)$ , where  $\tilde{\Omega}_P := \frac{1}{2}\Omega_P$  and  $0 \leq \tilde{\xi}_P \leq 1$  with*

$$\chi_{\frac{1}{4}\Omega_P}(\mathbf{x}) \leq \tilde{\xi}_P(\mathbf{x}) \leq \chi_{\frac{3}{8}\Omega_P}(\mathbf{x}).$$

REMARK 4.24. *The same procedure, with many simplifications, can be done in the interior of  $\Omega$  for divided differences in all directions  $\mathbf{e}^i$ ,  $i = 1, 2, 3$ . By choosing  $h \in (0, \frac{1}{2} \text{dist}(\text{spt } \xi_{00}, \partial\Omega))$  and mainly with the same steps as before this leads to*

$$\int_{\Omega} \xi_{00}^2 |\nabla \mathbf{F}(\mathbf{D}\mathbf{u})|^2 + \varphi(\xi_{00} |\nabla^2 \mathbf{u}|) d\mathbf{x} \leq c(\|\xi_{00}\|_{2,\infty}) \left( 1 + \int_{\Omega} \varphi^*(|\mathbf{f}|) d\mathbf{x} \right), \quad (4.25)$$

and

$$\int_{\Omega} \xi_{00}^2 |\nabla \pi|^2 d\mathbf{x} \leq c(\delta^{p-2}, \|\xi_{00}\|_{2,\infty}) \leq c(\delta^{p-2}, \|\xi_{00}\|_{2,\infty}) \left( 1 + \int_{\Omega} \varphi^*(|\mathbf{f}|) d\mathbf{x} \right), \quad (4.26)$$

where  $\xi_{00}$  is any cut-off function with compact support contained in  $\Omega$ .

This remark proves the first estimate in Theorem 2.29. Moreover, from (4.25) and (4.26) we can infer immediately:

COROLLARY 4.27. *Under the same hypotheses of Theorem 2.29 we obtain that  $\mathbf{F}(\mathbf{D}) \in W_{\text{loc}}^{1,2}(\Omega)$ ,  $\mathbf{u} \in W_{\text{loc}}^{2,p}(\Omega)$ , and  $\pi \in W_{\text{loc}}^{1,2}(\Omega)$ . This implies –in particular– that the equations of system (1.1) hold almost everywhere in  $\Omega$*

We observe that in the interior we can extract more information about the local integrability of  $\nabla^2 \mathbf{u}$ , by using simple arguments combining Hölder and Sobolev inequalities, as for instance is done in [10] in the space periodic case. For every  $G \subset\subset \Omega$  and some  $q \in [p, 2]$  we have

$$\begin{aligned} \|\nabla^2 \mathbf{u}\|_{q,G}^q &= \int_G \left( \varphi''(|\mathbf{D}\mathbf{u}|) |\nabla^2 \mathbf{u}|^2 \right)^{\frac{q}{2}} \left( \varphi''(|\mathbf{D}\mathbf{u}|) \right)^{-\frac{q}{2}} d\mathbf{x} \\ &\leq \left( \int_G \varphi''(|\mathbf{D}\mathbf{u}|) |\nabla^2 \mathbf{u}|^2 d\mathbf{x} \right)^{\frac{q}{2}} \left( \int_G (\delta + |\mathbf{D}\mathbf{u}|)^{(2-p)\frac{q}{2-q}} d\mathbf{x} \right)^{\frac{2-q}{2}} \\ &\leq c \|\nabla \mathbf{F}(\mathbf{D}\mathbf{u})\|_{2,G}^q (1 + \|\nabla \mathbf{u}\|_{(2-p)\frac{q}{2-q},G}^{2-p}) \\ &\leq c \|\nabla \mathbf{F}(\mathbf{D}\mathbf{u})\|_{2,G}^q (1 + \|\nabla \mathbf{u}\|_{r,G}^{2-p} + \|\nabla^2 \mathbf{u}\|_{r,G}^{2-p}), \end{aligned} \quad (4.28)$$

where we used the Hölder inequality, the algebraic identity  $\partial_{jk}^2 u^i = \partial_j \mathbf{D}_{ik} + \partial_k \mathbf{D}_{ij} - \partial_i \mathbf{D}_{jk}$ , and the embedding  $W^{1,r}(G) \hookrightarrow L^{(2-p)\frac{q}{2-q}}(G)$  valid for  $r = \frac{3q(2-p)}{6-q(p+1)}$ . Requiring that  $r = q$ , this yields  $q = \frac{3p}{p+1}$  and we can absorb the last term from the

right-hand side into the left-hand side of (4.28). Note that  $\|\nabla \mathbf{u}\|_{r,G}^{2-p}$  is finite since  $\nabla \mathbf{u} \in L^{\frac{3p}{3-p}}(G)$ . This and a Sobolev embedding theorem show

$$\nabla^2 \mathbf{u} \in L^{\frac{3p}{p+1}}_{\text{loc}}(\Omega) \quad \text{and} \quad \nabla \mathbf{u} \in L^{3p}_{\text{loc}}(\Omega), \quad (4.29)$$

which is the known best regularity results also in the space-periodic case [10].

**4.2. Regularity in the normal direction.** We follow the same reasoning as in the previous Section 3.2 to prove the analogue of Proposition 3.45. Since the results are very similar to those of the previous section, we now just point out the differences going directly to the proof of the main result.

**PROOF OF THEOREM 2.29.** Let the same assumptions as in Proposition 4.4 be satisfied. Moreover, assume that  $r_P$  in (b3) satisfies (3.53). Then, we obtain for every  $P \in \partial\Omega$  the following estimate (which is the counterpart of (3.54) when the quadratic term is missing)

$$\varphi''(|\mathbf{D}\mathbf{u}|)|\nabla^2 \mathbf{u}| \leq c(|\mathbf{f}| + |\partial_\tau \pi| + \varphi''(|\mathbf{D}\mathbf{u}|)|\partial_\tau \nabla \mathbf{u}|) \quad \text{a.e. in } \Omega_P. \quad (4.30)$$

In Section 3.2 we absorbed the last term in (4.30) in the term stemming from the extra stress tensor with 2-structure. Here we have to proceed differently. Moreover, we have to deal with the problem that  $\partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u})$  in (4.5) only yields a weighted information for  $\partial_\tau \mathbf{D}\mathbf{u}$ , while in (4.30) occurs  $\partial_\tau \nabla \mathbf{u}$ . The usual approach to resolve such problems by a Korn's inequality is not applicable in the moment, since it is not known whether or not an inequality of the type

$$\int \varphi''(|\mathbf{D}\mathbf{u}|)|\nabla \partial_\tau \mathbf{u}|^2 \, d\mathbf{x} \leq c \int \varphi''(|\mathbf{D}\mathbf{u}|)|\mathbf{D} \partial_\tau \mathbf{u}|^2 \, d\mathbf{x},$$

holds true. Thus, we proceed differently and argue similarly to (4.28). Since we do not know if  $\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}) \in L^2(\Omega)$  we use the anisotropic embedding from Theorem 2.44. For technical reasons we work with the localization  $\tilde{\xi}_P$  instead of  $\xi_P$  (cf. Remark 4.23) and use Proposition 4.4 for  $\tilde{\xi}_P$ .

To have all following computations justified we use Corollary 4.27 and (4.29), restrict ourselves to  $q \in [1, \frac{3p}{p+1}]$ . We work, in the set  $\tilde{\Omega}_\varepsilon = \tilde{\Omega}_{P,\varepsilon}$  defined as follows

$$\tilde{\Omega}_{P,\varepsilon} := \{\mathbf{x} \in \tilde{\Omega}_P : a_P(x) + \varepsilon < x_3 < a_P(x) + R'_P/2, \text{ for } 0 < \varepsilon < R_P/8\}$$

for a fixed  $P \in \partial\Omega$  and abbreviate  $\tilde{\xi} = \tilde{\xi}_P$ . (As usual we call  $R_P := \min\{R_P, R'_P\}$ .) Theorem 2.44 yields

$$\begin{aligned} \|\tilde{\xi} \mathbf{F}(\mathbf{D}\mathbf{u})\|_{3q, \tilde{\Omega}_\varepsilon} &\leq c(\|\partial_3(\tilde{\xi} \mathbf{F}(\mathbf{D}\mathbf{u}))\|_{q, \tilde{\Omega}_\varepsilon} + \|\partial_{\tau_1}(\tilde{\xi} \mathbf{F}(\mathbf{D}\mathbf{u}))\|_{q, \tilde{\Omega}_\varepsilon} \\ &\quad + \|\partial_{\tau_2}(\tilde{\xi} \mathbf{F}(\mathbf{D}\mathbf{u}))\|_{q, \tilde{\Omega}_\varepsilon}) \\ &\leq c(\|\tilde{\xi}\|_{1,\infty}) (1 + \|\tilde{\xi} \partial_3 \mathbf{F}(\mathbf{D}\mathbf{u})\|_{q, \tilde{\Omega}_\varepsilon}), \end{aligned} \quad (4.31)$$

where we also used (4.1), the identity  $\partial_\nu(\tilde{\xi} \mathbf{F}(\mathbf{D}\mathbf{u})) = \tilde{\xi} \partial_\nu \mathbf{F}(\mathbf{D}\mathbf{u}) + \mathbf{F}(\mathbf{D}\mathbf{u}) \partial_\nu \tilde{\xi}$ , valid for any vector  $\mathbf{v} \in \mathbb{R}^3$ , and (4.5). To estimate the last term we recall that (cf. [10, Lemma 3.8])

$$\sqrt{\varphi''(|\mathbf{D}\mathbf{u}|)}|\nabla^2 \mathbf{u}| \sim |\nabla \mathbf{F}(\mathbf{D}\mathbf{u})|. \quad (4.32)$$

Thus, after multiplying both sides of (4.30) by  $\tilde{\xi}$ , integration over  $\tilde{\Omega}_\varepsilon$  yields

$$\begin{aligned} &\int_{\tilde{\Omega}_\varepsilon} \tilde{\xi}^q |\nabla \mathbf{F}(\mathbf{D}\mathbf{u})|^q \, d\mathbf{x} \\ &\leq c \int_{\tilde{\Omega}_\varepsilon} \varphi''(|\mathbf{D}\mathbf{u}|)^{-\frac{q}{2}} (|\mathbf{f}|^q + |\partial_\tau \pi|^q) \tilde{\xi}^q + \varphi''(|\mathbf{D}\mathbf{u}|)^{\frac{q}{2}} |\partial_\tau \nabla \mathbf{u}|^q \tilde{\xi}^q \, d\mathbf{x}. \end{aligned} \quad (4.33)$$

We know that the right-hand side is finite, but depending on  $\varepsilon$ . Now we derive estimates independent of  $\varepsilon$  for  $q \in [1, \frac{3p}{p+1}]$ , as large as possible. By Hölder inequality with exponents  $2/q$  and  $2/(2-q)$  and (2.3) we obtain, for any  $g \in L^2_{\text{loc}}(\Omega)$ ,

$$\int_{\Omega_\varepsilon} \varphi''(|\mathbf{D}\mathbf{u}|)^{-\frac{q}{2}} |g|^q \tilde{\xi}^q \, d\mathbf{x} \leq c \|g \tilde{\xi}\|_{2, \tilde{\Omega}_\varepsilon}^q \|(\delta + |\mathbf{D}\mathbf{u}|)^{(2-p)} \tilde{\xi}\|_{\frac{2-q}{2}, \tilde{\Omega}_\varepsilon}^q. \quad (4.34)$$

To get a preliminary improvement of the integrability of  $\mathbf{D}\mathbf{u}$  we apply this to  $g = \mathbf{f}$  and  $g = \partial_\tau \pi$  for  $q = p$ . We also note that

$$\int_{\tilde{\Omega}_\varepsilon} \varphi''(|\mathbf{D}\mathbf{u}|)^{\frac{p}{2}} |\partial_\tau \nabla \mathbf{u}|^p \tilde{\xi}^p \, d\mathbf{x} \leq c \delta^{\frac{(p-2)p}{2}} \int_{\tilde{\Omega}_\varepsilon} [\varphi(|\tilde{\xi} \partial_\tau \nabla \mathbf{u}|) + \delta^p] \, d\mathbf{x},$$

together with Proposition 4.4, shows that for  $q = p$  the right-hand side of (4.33) is finite and independent of  $\varepsilon$ . Thus, (4.31) and Levi's theorem on monotone convergence imply

$$\|\tilde{\xi}_P \mathbf{F}(\mathbf{D}\mathbf{u})\|_{3p, \tilde{\Omega}_P} \leq c. \quad (4.35)$$

Choosing a finite sub-covering of the covering  $\partial\Omega \subset \bigcup_{P \in \partial\Omega} \hat{\Omega}_P$ , where  $\hat{\Omega}_P = \frac{1}{4}\Omega_P$  (cf. (b2)), we get that (4.35) and Corollary 4.27 imply

$$\mathbf{F}(\mathbf{D}\mathbf{u}) \in L^{3p}(\Omega) \quad \text{and} \quad \nabla \mathbf{u} \in L^{\frac{3p^2}{2}}(\Omega), \quad (4.36)$$

where we also used (2.16) and Korn's inequality.

A better estimate of the last term on the right-hand side of (4.33) is the key to show that (4.33) is finite and independent of  $\varepsilon$ , for some  $q > p$ . To this end we observe that, by using (3.15) we have for  $\alpha = 1, 2$

$$\begin{aligned} \tilde{\xi} \partial_{\tau_\alpha} \nabla \mathbf{u} &= \tilde{\xi} (\nabla \partial_{\tau_\alpha} \mathbf{u}) - \tilde{\xi} (\nabla \partial_\alpha a \partial_3 \mathbf{u}) \\ &= \nabla (\partial_{\tau_\alpha} \mathbf{u} \tilde{\xi}) - \partial_{\tau_\alpha} \mathbf{u} \otimes \nabla \tilde{\xi} - \tilde{\xi} (\nabla \partial_\alpha a \partial_3 \mathbf{u}). \end{aligned}$$

Consequently,

$$|\tilde{\xi} \partial_\tau \nabla \mathbf{u}|^q \leq |\nabla (\partial_\tau \mathbf{u} \tilde{\xi})|^q + c(\|\tilde{\xi}\|_{1, \infty}, \|a\|_{C^{2,1}}, q) |\nabla \mathbf{u}|^q \quad \text{a.e. in } \Omega_P,$$

and, in order to estimate the last term from the right hand side of (4.33), we end up to consider the following integral

$$\mathcal{I} := \int_{\tilde{\Omega}_\varepsilon} \varphi''(|\mathbf{D}\mathbf{u}|)^{\frac{q}{2}} (|\nabla (\partial_\tau \mathbf{u} \tilde{\xi})|^q + |\nabla \mathbf{u}|^q) \, d\mathbf{x} := \mathcal{I}_1 + \mathcal{I}_2.$$

Since  $q \leq \frac{3p}{p+1} \leq \frac{3p^2}{2}$ , for all  $p \in (1, 2)$ , we get from (4.36)

$$\mathcal{I}_2 \leq \delta^{\frac{(p-2)q}{2}} \int_{\tilde{\Omega}_\varepsilon} |\nabla \mathbf{u}|^q \, d\mathbf{x} \leq c \left( \int_{\Omega} |\nabla \mathbf{u}|^{\frac{3p^2}{2}} \, d\mathbf{x} \right)^{\frac{2q}{3p^2}} < c(\delta^{-1}).$$

Due to Korn's inequality (cf. [15, Thm. 5.17]) there exists a constant depending on the John constant of  $\tilde{\Omega}$  and  $q$  such that the term  $\mathcal{I}_1$  can be estimated as follows:

$$\begin{aligned} \mathcal{I}_1 &\leq \delta^{\frac{(p-2)q}{2}} \int_{\tilde{\Omega}_\varepsilon} |\nabla (\partial_\tau \mathbf{u} \tilde{\xi})|^q \, d\mathbf{x} \\ &\leq c \delta^{\frac{(p-2)q}{2}} \int_{\tilde{\Omega}_\varepsilon} |\mathbf{D}(\partial_\tau \mathbf{u} \tilde{\xi}) - \langle \mathbf{D}(\partial_\tau \mathbf{u} \tilde{\xi}) \rangle_{\tilde{\Omega}_\varepsilon}|^q + |\partial_\tau \mathbf{u} \tilde{\xi} - \langle \partial_\tau \mathbf{u} \tilde{\xi} \rangle_{\tilde{\Omega}_\varepsilon}|^q \, d\mathbf{x} \\ &\leq c \left( 1 + \int_{\tilde{\Omega}_\varepsilon} |\mathbf{D}(\partial_\tau \mathbf{u} \tilde{\xi})|^q \, d\mathbf{x} \right), \end{aligned}$$

where we also used  $\int_{\Omega_\varepsilon} |\langle g \rangle_{\Omega_\varepsilon}|^q dx \leq \int_{\Omega_\varepsilon} |g|^q dx$ , Hölder's inequality, and (4.36). Using now the identities

$$\begin{aligned} \mathbf{D}(\partial_{\tau_\alpha} \mathbf{u} \tilde{\xi}) &= \tilde{\xi} \mathbf{D}(\partial_{\tau_\alpha} \mathbf{u}) + \partial_{\tau_\alpha} \mathbf{u} \overset{s}{\otimes} \nabla \tilde{\xi}, \\ \mathbf{D}(\partial_{\tau_\alpha} \mathbf{u}) &= \partial_{\tau_\alpha} \mathbf{D}\mathbf{u} + \mathbf{D}(\nabla a) \overset{s}{\otimes} \partial_3 \mathbf{u}, \end{aligned}$$

we finally obtain

$$\mathcal{I}_1 \leq c(\delta^{-1}, \|\tilde{\xi}\|_{1,\infty}, q, \|a\|_{C^{2,1}}, \tilde{\Omega}_P) \left(1 + \int_{\tilde{\Omega}_\varepsilon} |\partial_\tau \mathbf{D}\mathbf{u}|^q \tilde{\xi}^q dx\right).$$

From (2.27) it follows

$$\sqrt{\varphi''(|\mathbf{D}\mathbf{u}|)} |\partial_\tau \mathbf{D}\mathbf{u}| \sim |\partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u})|. \quad (4.37)$$

Proceeding similarly to (4.28) we get, also using (2.16),

$$\begin{aligned} \int_{\tilde{\Omega}_\varepsilon} |\partial_\tau \mathbf{D}\mathbf{u}|^q \tilde{\xi}^q dx &= \int_{\tilde{\Omega}_\varepsilon} \varphi''(|\mathbf{D}\mathbf{u}|)^{\frac{q}{2}} |\partial_\tau \mathbf{D}\mathbf{u}|^q \tilde{\xi}^q \varphi''(|\mathbf{D}\mathbf{u}|)^{-\frac{q}{2}} dx \\ &\leq c \left[ \int_{\tilde{\Omega}_\varepsilon} |\partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \tilde{\xi}^{\frac{4}{p'}} dx \right]^{\frac{q}{2}} \left[ \int_{\tilde{\Omega}_\varepsilon} |\tilde{\xi} \mathbf{F}(\mathbf{D}\mathbf{u})|^{\frac{2-p}{p} \frac{2q}{2-q} + \delta^{\frac{(2-p)q}{2-q}}} dx \right]^{\frac{2-q}{2}}. \end{aligned}$$

Since we want to absorb the last term in the left-hand side of (4.31), we require that  $\frac{2-p}{p} \frac{2q}{2-q} = 3q$ , which yields

$$q = \frac{8p-4}{3p}. \quad (4.38)$$

Due to the fact that the exponent of the localization function of the first term on the right-hand side is not 2 we enlarge the integration domain. Recall,  $\tilde{\xi} = \tilde{\xi}_P$  and by construction  $\xi = \xi_P \equiv 1$  on  $\tilde{\Omega} = \tilde{\Omega}_P$ . Thus, we can write

$$\begin{aligned} \int_{\tilde{\Omega}_{P,\varepsilon}} |\partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \tilde{\xi}_P^{\frac{4}{p'}} dx &\leq \int_{\tilde{\Omega}_{P,\varepsilon}} |\partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u})|^2 dx \leq \int_{\tilde{\Omega}_P} |\partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u})|^2 dx \\ &\leq \int_{\tilde{\Omega}_P} |\partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \xi_P^2 dx \leq \int_{\tilde{\Omega}_P} |\partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \xi_P^2 dx \leq c, \end{aligned}$$

which is finite by Proposition 4.4. Hence, we finally proved that

$$\mathcal{I} \leq c(\|\tilde{\xi}\|_{1,\infty}, \|a\|_{C^{2,1}}, \delta^{-1}) \left(1 + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{3q, \tilde{\Omega}_\varepsilon}^{q \frac{2-p}{p}}\right), \quad (4.39)$$

if  $q$  is given by (4.38). We estimate the term with  $\partial_\tau \pi$  in (4.33) as follows by using Hölder inequality

$$\begin{aligned} \int_{\tilde{\Omega}_\varepsilon} \varphi''(|\mathbf{D}\mathbf{u}|)^{-\frac{q}{2}} \tilde{\xi}^q |\partial_\tau \pi|^q \tilde{\xi}^q dx &\leq c \int_{\tilde{\Omega}_\varepsilon} \left(\delta^{\frac{q}{2}} + |\mathbf{F}(\mathbf{D}\mathbf{u})|\right)^{q \frac{2-p}{p}} |\partial_\tau \pi|^q \tilde{\xi}^q dx \quad (4.40) \\ &\leq c \left(1 + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{3q, \tilde{\Omega}_\varepsilon}^{q \frac{2-p}{p}}\right) \left(\int_{\tilde{\Omega}_\varepsilon} |\partial_\tau \pi|^2 \tilde{\xi}^{\frac{4}{p'}} dx\right)^{\frac{3p}{4p-2}} \\ &\leq c \left(1 + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{3q, \tilde{\Omega}_\varepsilon}^{q \frac{2-p}{p}}\right) \left(\int_{\tilde{\Omega}_P} |\partial_\tau \pi|^2 \xi^2 dx\right)^{\frac{3p}{4p-2}} \\ &\leq c \left(1 + \|\mathbf{F}(\mathbf{D}\mathbf{u})\|_{3q, \tilde{\Omega}_\varepsilon}^{q \frac{2-p}{p}}\right), \end{aligned}$$

where we increased the integration domain as above and used Proposition 4.4. The term with  $\mathbf{f}$  in (4.33) is estimated in the same way with several simplifications. Inserting this estimate, (4.39) and (4.40) into (4.33), we obtain from (4.31)

$$\|\mathbf{F}(\mathbf{Du})\tilde{\xi}\|_{3q,\tilde{\Omega}_\varepsilon} \leq c\left(1 + \|\tilde{\xi}\partial_3\mathbf{F}(\mathbf{Du})\|_{q,\tilde{\Omega}_\varepsilon}\right) \leq c\left(1 + \|\mathbf{F}(\mathbf{Du})\tilde{\xi}\|_{3q,\tilde{\Omega}_\varepsilon}^{\frac{2-p}{p}}\right).$$

Since  $p \in (1, 2)$  we can absorb the right-hand side into the left-hand side by Young's inequality. Thus, we proved for  $q$  satisfying (4.38), that there exists a constant  $c$  such that for every  $P \in \partial\Omega$  there holds

$$\|\mathbf{F}(\mathbf{Du})\tilde{\xi}_P\|_{3q,\tilde{\Omega}_\varepsilon} + \|\tilde{\xi}_P\partial_3\mathbf{F}(\mathbf{Du})\|_{q,\tilde{\Omega}_\varepsilon} \leq c(\|\xi_P\|_{2,\infty}, \|\tilde{\xi}_P\|_{1,\infty}, \|a_P\|_{C^{2,1}}, \delta^{-1}).$$

Since  $c$  is independent of  $\varepsilon > 0$ , Levi's monotone convergence theorem implies that

$$\|\mathbf{F}(\mathbf{Du})\tilde{\xi}_P\|_{\frac{8p-4}{p},\tilde{\Omega}_P} + \|\tilde{\xi}_P\partial_3\mathbf{F}(\mathbf{Du})\|_{\frac{8p-4}{3p},\tilde{\Omega}_P} \leq c \quad (4.41)$$

which proves the third estimate in Theorem 2.29. Using the same covering argument which led to (4.36) we now obtain

$$\nabla\mathbf{F}(\mathbf{Du}) \in L^{\frac{8p-4}{3p}}(\Omega) \quad \text{and} \quad \mathbf{F}(\mathbf{Du}) \in L^{\frac{8p-4}{p}}(\Omega).$$

Then, by usual manipulation of the quantity  $\mathbf{F}(\mathbf{Du})$  we obtain that

$$\nabla^2\mathbf{u} \in L^{\frac{4p-2}{p+1}}(\Omega) \quad \text{and} \quad \nabla\mathbf{u} \in L^{4p-2}(\Omega),$$

or, by considering tangential and normal derivatives of  $\nabla\mathbf{u}$ , the statements concerning the derivatives of  $\mathbf{u}$  in Theorem 2.29. This and (3.51) together with (4.6) proves the statements concerning the pressure  $\pi$  in Theorem 2.29. Theorem 2.29 is now completely proved.  $\square$

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### References

- [1] J. W. Barrett and W. B. Liu. Quasi-norm error bounds for the finite element approximation of a non-Newtonian flow. *Numer. Math.*, 68(4):437–456, 1994.
- [2] H. Beirão da Veiga. On the regularity of flows with Ladyzhenskaya shear-dependent viscosity and slip or nonslip boundary conditions. *Comm. Pure Appl. Math.*, 58(4):552–577, 2005.
- [3] H. Beirão da Veiga. Navier-Stokes equations with shear thinning viscosity. Regularity up to the boundary. *J. Math. Fluid Mech.*, 11(2):258–273, 2009.
- [4] H. Beirão da Veiga. On the Ladyzhenskaya-Smagorinsky turbulence model of the Navier-Stokes equations in smooth domains. The regularity problem. *J. Eur. Math. Soc. (JEMS)*, 11(1):127–167, 2009.
- [5] H. Beirão da Veiga, P. Kaplický, and M. Růžička. Boundary regularity of shear-thickening flows. *J. Math. Fluid Mech.*, 13:387–404, 2011.
- [6] H. Beirão da Veiga. Navier-Stokes equations with shear thinning viscosity. Regularity up to the boundary. *J. Math. Fluid Mech.*, 11(2):258–273, 2009.
- [7] H. Beirão da Veiga. On the global regularity of shear thinning flows in smooth domains. *J. Math. Anal. Appl.*, 349(2):335–360, 2009.
- [8] L. Belenki, L. C. Berselli, L. Diening, and M. Růžička. On the finite element approximation of  $p$ -stokes systems. *SIAM J. Numer. Anal.*, 50(2):373–397, 2012.
- [9] L. C. Berselli. On the  $W^{2,q}$ -regularity of incompressible fluids with shear-dependent viscosities: the shear-thinning case. *J. Math. Fluid Mech.*, 11(2):171–185, 2009.
- [10] L. C. Berselli, L. Diening, and M. Růžička. Existence of strong solutions for incompressible fluids with shear dependent viscosities. *J. Math. Fluid Mech.*, 12(1):101–132, 2010.
- [11] L. C. Berselli and C. R. Grisanti. On the regularity up to the boundary for certain nonlinear elliptic systems. *Discrete Contin. Dyn. Syst. Ser. S*, 9(1):53–71, 2016.
- [12] L. C. Berselli, C. R. Grisanti, and V. John. Analysis of commutation errors for functions with low regularity. *J. Comput. Appl. Math.*, 206(2):1027–1045, 2007.
- [13] R.B. Bird, R.C. Armstrong, and O. Hassager. *Dynamic of Polymer Liquids*. John Wiley, 1987. 2nd edition.

- [14] F. Crispo. A note on the global regularity of steady flows of generalized Newtonian fluids. *Port. Math.*, 66(2):211–223, 2009.
- [15] L. Diening, M. Růžička, and K. Schumacher. A decomposition technique for John domains. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, 35:87–114, 2010.
- [16] L. Diening, C. Ebmeyer, and M. Růžička. Optimal convergence for the implicit space-time discretization of parabolic systems with  $p$ -structure. *SIAM J. Numer. Anal.*, 45:457–472, 2007.
- [17] L. Diening and F. Ettwein. Fractional estimates for non-differentiable elliptic systems with general growth. *Forum Math.*, 20(3):523–556, 2008.
- [18] L. Diening and C. Kreuzer. Linear convergence of an adaptive finite element method for the  $p$ -Laplacian equation. *SIAM J. Numer. Anal.*, 46:614–638, 2008.
- [19] L. Diening and M. Růžička. Strong solutions for generalized Newtonian fluids. *J. Math. Fluid Mech.*, 7:413–450, 2005.
- [20] L.C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
- [21] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.
- [22] J. Málek, J. Nečas, and M. Růžička. On the non-Newtonian incompressible fluids. *Math. Models Methods Appl. Sci.*, 3:35–63, 1993.
- [23] J. Málek, J. Nečas, and M. Růžička. On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: the case  $p \geq 2$ . *Adv. Differential Equations*, 6(3):257–302, 2001.
- [24] J. Málek, K. R. Rajagopal, and M. Růžička. Existence and regularity of solutions and the stability of the rest state for fluids with shear dependent viscosity. *Math. Models Methods Appl. Sci.*, 5:789–812, 1995.
- [25] J. Málek and K.R. Rajagopal. Mathematical issues concerning the Navier-Stokes equations and some of its generalizations. In *Evolutionary equations, Vol. II*, Handb. Differ. Equ., pages 371–459. Elsevier/North-Holland, Amsterdam, 2005.
- [26] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1991.
- [27] M. Růžička and L. Diening. Non-Newtonian fluids and function spaces. In *Nonlinear Analysis, Function Spaces and Applications, Proceedings of NAFSA 2006 Prague*, volume 8, pages 95–144, 2007.
- [28] G. A. Seregin and T. N. Shilkin. Regularity of minimizers of some variational problems in plasticity theory. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 243(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funktsii. 28):270–298, 342–343, 1997.
- [29] M. Troisi. Teoremi di inclusione per spazi di Sobolev non isotropi. *Ricerche Mat.*, 18:3–24, 1969.

LUIGI C. BERSELLI, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, VIA F. BUONARROTI 1/C, I-56125 PISA, ITALY.

*E-mail address:* luigi.carlo.berselli@unipi.it

MICHAEL RŮŽIČKA, INSTITUTE OF APPLIED MATHEMATICS, ALBERT-LUDWIGS-UNIVERSITY FREIBURG, ECKERSTR. 1, D-79104 FREIBURG, GERMANY.

*E-mail address:* rose@mathematik.uni-freiburg.de