Capturing 'time': characteristics of students' written discourse on dynagraphs

Giulia Lisarelli, Antonini Samuele, Baccaglini-Frank Anna

Capturing ‘time’: characteristics of students’ written discourse on dynagraphs

Giulia Lisarelli[1], Samuele Antonini[2] and Anna Baccaglini-Frank[1]

[1]Università di Pisa, Italy; giulia.lisarelli@dm.unipi.it, anna.baccaglinifrank@unipi.it

[2]Università di Pavia, Italy; samuele.antonini@unipv.it

We study high school students’ written discourse about an experience in a dynamic interactive digital environment in which functions were represented in one dimension, as dynagraphs. In a dynagraph of a real function one variable can be acted upon while the other varies as a consequence of the movement induced on the independent variable. Students were asked to write about their experience with the dynagraphs, and their written productions were collected and analyzed using a classification, that emerged a posteriori, in “snapshots, live-photos and scenes”. This classification is presented as a tool of analysis that allows to put discourse about dynagraphs in relation with discourse about functions. Examples of excerpts analyzed through this tool are given.

Keywords: Covariation, dynagraph, dynamism, function, time.

Functions, covariation, graphs, and dynagraphs

Functions and their graphs have always had a leading role in mathematical practice, including school practices. Being able to interpret the Cartesian graph of a function and to construct a graph starting from the function’s properties are essential processes in mathematical thinking. However, these processes include an understanding of the meaning of *variable* and of the *relation between variations* of the variables, that is *covariation*; and several kinds of difficulties that students encounter when grappling with these ideas are widely reported in the literature (e.g.: Tall, 1992, 2009; Kaput, 1992; Monk & Nemirovsky, 1994; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Thompson & Carlson, 2017).

A Cartesian graph is defined as the set of points (x, f(x)) with real coordinates on the Cartesian plane, where x belongs to the domain of the function and f(x) is its image; these constitute a curve incorporating the functional relation between the two variables. Students encounter various difficulties when dealing with graphs, one being in recognizing that each point on the graph is a coordinated presentation of two pieces of information (Colacicco, Lisarelli & Antonini, 2017). This leads them to consider the curve to be the function itself and to identify a point on the curve only as its f(x) value. Both the asymmetric relation between the variables and their covariation are lost.

In this paper we focus on the use of a particular technological artifact to represent functions, realized through a dynamic interactive environment (DIE). Many research studies support the use of technology in the teaching of functions (Healy & Sinclair, 2007; Falcade, Laborde & Mariotti, 2007). Here we are interested in representing functions through DynaGraphs (Goldenberg, Lewis & O’Keefe, 1992), dynamic representations in which the domain variable is dynamically draggable on a line and it is presented separately from its image. Both the x- and y-axes are horizontal; originally, they were referred to as “the x Line” and “the f(x) Line”. DynaGraphs cannot be constructed without using a DIE, where objects can be moved on the screen.

The independence of the x variable is realized by the possibility of freely dragging a point, bound to a line (the x Line), and the resulting movement visually mediates the variation of the point within a specific domain. Whereas the dependence of the f(x) variable is realized by an indirect motion: the dragging of the independent variable along its axis causes the motion of a point, bound to another line (the f(x) Line), that cannot be directly dragged. The use of dragging can visually mediate the discourse on functional dependency, since it can help students interpret the exploration in terms of logical dependency between two different types of motion: direct and indirect motion.

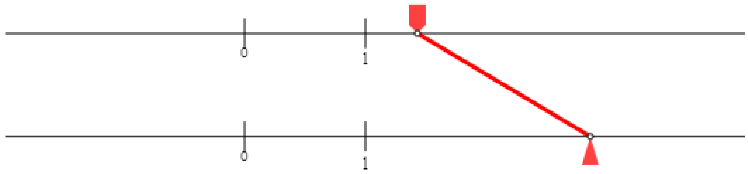

Figure 1: Screenshot of a dynagraph used in this study

The authors, together with Nathalie Sinclair, built on the original idea of DynaGraphs to realize functions as in Figure 1, within the dynamic geometry software Sketchpad (a similar design was adopted also in Lisarelli, 2018) which will be called *dynagraphs* (no capitals, to distinguish them for the original ones) in this paper. Many properties of a function can be recognized in a dynagraph. For example, looking at the movements of the two variables that either follow the same direction or move in opposite directions, an expert can recognize when a function is increasing – if both variables have the same direction of movement – or when it is decreasing – if the variables move in opposite directions. The movement and the identification of invariants are central in the exploration of dynagraphs, since the DIE enables students to deal with the dependence relation in terms of possible or impossible movements; we are interested in how this dynamic and temporal experience can be narrated on a static sheet of paper – a process that is frequent in mathematics.

Many researchers have described the relations between movement, time and mathematics, in particular in calculus. Tall (2009) describes calculus as the mathematical field that begins with the desire to quantify how things change, the function, the rate at which they change, the derivative, and the way in which they accumulate, the integral. So, this field is fundamentally dynamic: even the calculation of static quantities, such as areas or volumes, involves dynamic processes of adding up a large number of very tiny elements. At a certain historical moment, a transformation took place and calculus turned into rigorous definitions, developing into the formal theory of mathematical analysis which is used today by textbooks and by many teachers. This transformation led to the teaching of calculus based on the definition of limits, which satisfies mathematicians’ logical needs, but it proves to be rather complicated for students (Tall, 1992). However, even if formal mathematical discourse aims at eliminating time and dynamism, this does not imply that mathematicians engage in purely a-temporal modes of thinking. Indeed, they seem to frequently communicate in ways that suggest they think of mathematical objects in motion (Sinclair & Gol Tabaghi, 2010).

Theoretical background and research focus

For this study we make use of discourse analysis; in particular, we adopt the commognitive theoretical perspective (Sfard, 2008). According to Sfard, communication and cognition are two manifestations of a same phenomenon and mathematics can be considered as a special type of discourse, or communication. Mathematical learning is the process by which students become able to communicate about mathematical objects, that, unlike others, are purely discursive objects that can have different *realizations* (p. 165). Moreover, mathematical discourse, as a particular kind of discourse, involves a frequent use of *visual mediators*, visible objects realizing the object of the discourse (p. 133).

In this study, we use dynagraphs as particular realizations of the mathematical object 'function' within a DIE, in order to allow students to experience the dependency between two covarying quantities in terms of motion. This approach brings temporality into play. We intend to study how students deal with the temporal dimension that characterizes their experience in the DIE, when they are asked to describe it through a written explanation. Moreover, we are interested in the mathematical aspects expressed in these written explanations, especially in how variation of variables and covariation – the relation between variations – are conveyed.

The research question that led the design and the development of this study is the following:

What are characteristics of students' written discourse on dynagraphs related to their management of the movement and time experienced during the exploration that can be put in relation with discourse about functions (as mathematical objects)?

Methodology

We designed 7 dynagraphs of functions to be explored. The markers realizing the two variables had no labels, because we wanted to allow students to decide which words or symbols to use. We consider this to be an important process related to distinguishing the two variables and the asymmetric relation between them. The two points 0 and 1 were marked on the lines with ticks in order to provide the unit segment and to highlight that the lines visually realize two copies of the real numbers.

Students belonging to four different 10^{th} grade classes of Canadian high schools worked in pairs on as many dynagraphs as they had time for during one lesson (60 minutes). Their task was to "explore the dynagraphs on the iPad and write down their observations about them" on the whiteboard and/or on paper. We collected these written productions. This task was designed to "force" the students to describe in writing an experience which was completely time-and-motion-immersed.

Tools of analysis: snapshots, live-photos, and scenes

From analyses of the high school students' written productions we reached a characterization that was later refined through further rounds of analysis of the excerpts. Coherently with our research interest in gaining insight into how movement (the variation of position in time) enters students' written discourse we introduced a first possible characterization of written accounts of the experiences in the DIE, using a terminology that refers to images in the context of photos and videos. In Table 1 we define the main types of accounts and describe characteristics of the discourse it characterizes, both in the case in which the discourse is about the dynagraph (which may or may not be related to the notion of function for the students) and when it is about the function realized through the dynagraph.

Type of account of the DIE experience		**Discourse about dynagraph**	**Discourse about function**
snapshot	snapshot: photo at a certain position showing one or more properties	provides an instantaneous shot in which movement is stopped at a certain position	pointwise properties
	snapshot-album: set of snapshots, each showing something in particular	expresses properties in selected positions; these constitute a finite discrete set; each property is shown in one snapshot	pointwise properties at a particular set of points
	snapshot-cluster: set of snapshots showing relationships between them	one or more properties are described through a comparison of snapshots of the dynagraph; the focus is on the relationships between snapshots; it may be in a sequence coherent with the order on the line or not.	properties that refer to variation and/or to invariants in specific intervals identified by analyzing and comparing different points
live-photo	live-photo: animated photo showing what happens a bit before and a bit after a certain instant.	one or more properties that depend on the nearby positions are described at a certain position; certain positions of the dynagraph are shown close to a specific position	local properties of functions, at a specific point and the variations in any neighborhood of the point.
scene	scene: video showing movement over a period of time; it can be realized through snapshots, or live-photos, in an album or cluster.	description of one or more properties of the movement over an arc of time, in an interval of real numbers, or in the space covered by the point moving along the line; it includes various positions. The intervals can be limited or unlimited; the scene can be described statically or dynamically	properties of functions that are global, general, or present in a whole interval

Table 1: Characterization of students' written discourse

Table 1 shows the main types of accounts (snapshot, live-photo and scene) and their further characterization with respect to ways in which more than one could be used (as albums or clusters) in the excerpts of written discourse that we will now analyze.

We observe that live-photos may also be found in albums and in clusters. A live-photo album is set of live-photos, each showing something in particular; it expresses properties in selected positions, which constitute a finite discrete set; each property is shown in one live-photo. From a mathematical point of view, these correspond to local properties in neighborhoods of a particular set of points. A live-photo cluster is a set of live-photos showing relationships between them.

Analysis of students' written products

This section contains examples of analyses of students' written products using the characterization.

Excerpt 1: Scene of snapshots depicted through written words and diagrams

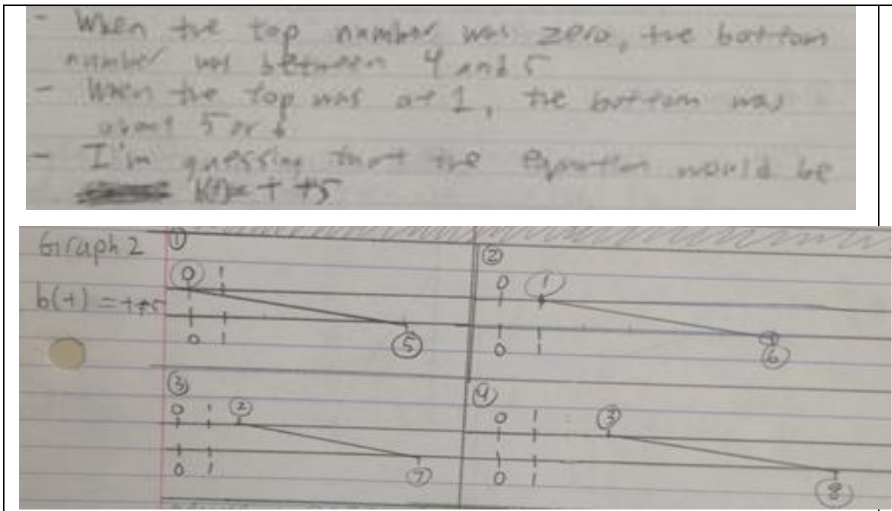

Figure 1a,b: Scene of snapshots

"When the top number was zero, the bottom number was between 4 and 5. When the top was at 1, the bottom was about 5 or 6. I'm guessing that the equation would be b(t)=t+5."

Excerpt 1 contains discourse about the dynagraph of the function f(x) = x+5; Figure 2a and 2b depict a scene through snapshots presented using written text and visual mediators. The first property in Figure 2a is described through a snapshot taken 'when the top number was zero'. The first drawing in Figure 2b presents this specific instance. The second property is described analogously as a snapshot taken "when the top number was at 1"; the second drawing in Figure 2b presents this specific instance. The third property in the scene is described through the algebraic formula "b(t) = t+5".

Excerpt 2: Live-photo depicted through two snapshots showing 'initial' and 'final' states

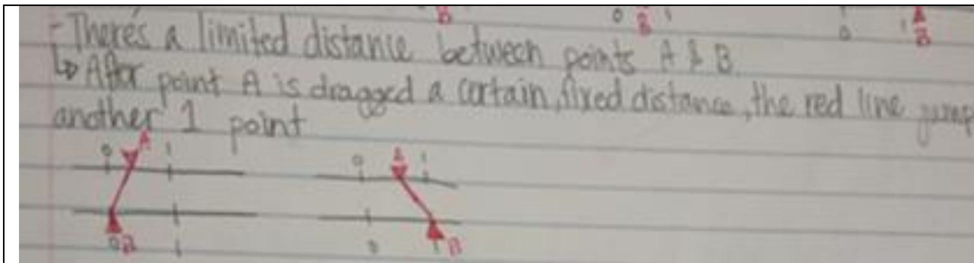

Figure 2: Live-photo

"There's a limited distance between points A & B. After point A is dragged a certain fixed distance, the red line jumps another 1 point."

The dynagraph in excerpt 2 realizes the nearest integer function. The students describe the jumping of the red line as A is dragged across the midpoint of the interval [0,1]. This part of their discourse appears to be a live-photo depicted through two visual mediators that are particular snapshots showing an 'initial' and a 'final' state of the dynagraph as A moves in a neighborhood of 0.5.

Excerpt 3: Album of live-photos

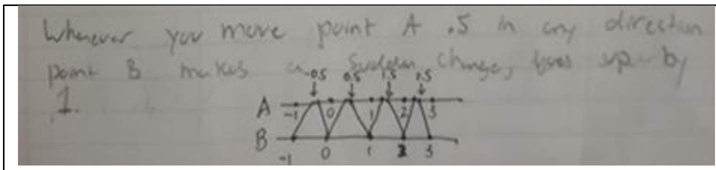

Figure 3: Album of live-photos

"Whenever you move point A .5 in any direction point B makes a sudden change, goes up by 1."

The dynagraph in excerpt 3 realizes the nearest integer function. The students produce a single visual drawing realizing the dynagraph's interval [-1,3] on both axes (Figure 4), and describe its behavior written in words. Each live photo shows A near the midpoint of a unit interval. We see these as live-photos because each 'sudden change' is shown by a position of the segment AB immediately before and immediately after the transition of A across the midpoints (the ones realized are -0.5, 0.5, 1.5, 2.5) of the unit intervals.

Excerpt 4: Scene depicted through a cluster of snapshots

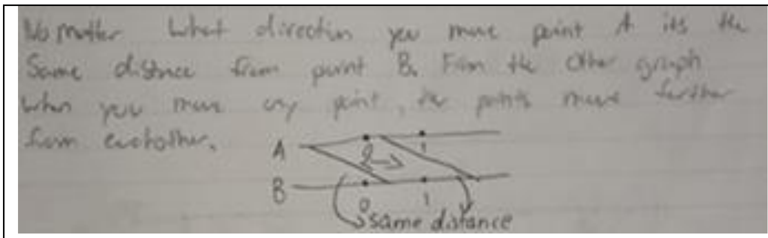

Figure 4: Scene

"No matter what direction you move point A it's the same distance from point B. From the other graph when you move any point the points move further from each other."

The dynagraph discussed in Figure 5 realizes the function f(x)=x+5. The students who author the discourse in excerpt 4 emphasize the invariance of the distance of A from B, conveyed through the words "no matter what direction" and through the visual mediation of two congruent parallel segments in the drawing with in between a double-headed arrow and two arrows pointing to the words "same distance". The written text and the visual mediators appear to be a cluster of snapshots depicting the invariant property identified.

Excerpt 5: Scene album depicting two properties of a dynagraph

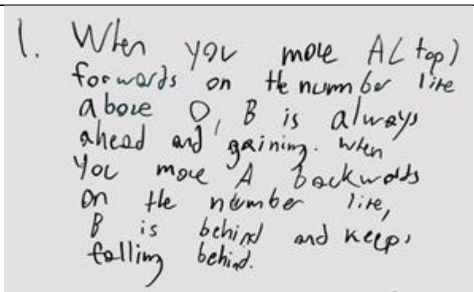

Figure 5: Scene album

" When you move A (top) forward on the number line above 0, B is always ahead and gaining. When you move A backwards on the number line, B is behind and keeps falling behind."

In excerpt 5 (Figure 6) the students use only written words to outline the behavior of dynagraph of f(x)=2x. They consider a whole interval of time (and space): this is suggested by their use of the adverb "always", and by the fact that they make several references to possible movements and to the reciprocal position of the two variables ("forwards", "ahead", "backwards", "behind").

Moreover, the students start with the present tense of the verb “to be” (“B is always ahead […] B is behind”), which gives their statements the form of an absolute observable truth, but then they add details on the quality of movement, passing to the “–ing form” of other verbs, indicating spatial dynamic features (“always […] gaining”, “keeps falling behind”). This could suggest that the students see the movement as not ending even when it cannot be seen anymore within a large interval.

Conclusions

The analytical tool we have introduced to describe written accounts of the DIE experience allows to characterize different ways used by students to realize movement and time experienced during the exploration of dynagraphs, and puts them in relation with mathematical discourse about functions. In particular, it highlights how the choice of the time interval being described, that can be identified in students’ discourse by looking at the verb tenses, adverbs and at their use of visual mediators to realize movement, corresponds from a mathematical point of view to different properties of the functions realized through the dynagraph. The classification has value at the cognitive, didactical and epistemological levels. From a cognitive point of view, the tool is important because the classification proposed was identified *a posteriori*, through empirical analysis of the data collected, and it seems to be a powerful tool for analyzing all the data collected. Epistemologically, the types of accounts can be put in relation with different mathematical properties of functions in formal mathematical discourse, as shown in Table 1.

Thinking about properties that are pointwise, local or global, the discourse – even that of experts – that can emerge about these may have characteristics such as those we have found in students’ discourses, focusing on “instants” that capture the behavior of isolated points, systems of points, sequences, neighborhoods of points, points as limits, intervals… Moreover, experts’ discourse can capture properties at certain points, variations and invariants. For example, a description of what happens in a neighborhood of a point can be conveyed through a live-photo, while the behavior at infinity requires a scene, since a neighborhood of infinity can be seen as a scene “from a certain point on”, that is, as a live-photo at infinity.

From a didactical point of view the classification is important, assuming that we value promoting various forms of students’ discourse, and therefore students’ flexibility in constructing such discourse. The teacher can interpret students’ discourse from a mathematical point of view, seeing it as a ‘mirrored image’ (with some possible distortions) of experts’ discourse about pointwise properties, local or global properties, variations, and invariants. Doing this, the teacher can then promote these forms of discourse through appropriate tasks (e.g., by using the behavior of dynagraphs in certain positions, intervals, etc.), knowing what to expect and how to gradually foster the transition to more formal mathematical discourse. One particularly fruitful type of discourse seems to be that involving transitions from pointwise to local properties, and thus discourse involving live-photos. However, this needs to be studied in future research currently under development.

Acknowledgments

We graciously thank Nathalie Sinclair for her support in the realization of the activities described in this paper, and for all the fruitful conversations she has engaged in with us. This study has been partially supported by GNSAGA of INdAM.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Colacicco, G., Lisarelli, G., & Antonini, S. (2017). Funzioni e grafici in ambienti digitali dinamici. *Didattica della Matematica: dalla ricerca alle pratiche d'aula*, *2*, 7–25.

Falcade, R., Laborde, C., & Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, *66*(3), 317–333.

Goldenberg, E. P., Lewis, P., & O'Keefe, J. (1992). Dynamic representation and the development of an understanding of functions. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*. *MAA Notes*, 25.

Healy, L., & Sinclair, N. (2007). If this is your mathematics, what are your stories? *International Journal of Computers for Mathematics Learning*, *12*(1), 3–21.

Kaput, J. (1992). Patterns in students' formalization of quantitative patterns. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*. *MAA Notes*, 25.

Lisarelli, G. (2018). How dragging mediates discourse about functions. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the IGPME* (Vol. 3, pp. 323–330). Umeå, Sweden: PME.

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education. I. CBMS Issues in Mathematics Education* (pp. 139–68). Providence, RI: American Mathematical Society.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1–36.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

Sinclair, N., & Gol Tabaghi, S. (2010). Drawing space: mathematicians' kinetic conceptions of eigenvectors. *Educational Studies in Mathematics, 74*(3), 223–240.

Tall, D. (1992). The Transition to Advanced Mathematical Thinking: functions, limits, infinity and proofs. In D. A. Grouws (Eds.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 495–511). Macmillan, New York.

Tall, D. (2009). Dynamic mathematics and the blending of knowledge structures in the calculus. *ZDM – The International Journal on Mathematics Education, 41*(4), 481–492.

Thompson, P. W., & Carlson, M. P. (2017). Variation, Covariation, and Functions: Foundational Ways of Thinking Mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp.421–456). Reston, VA: NCTM.