Introduction to the work of TWG24: Representations in mathematics teaching and learning

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Introduction

This Thematic Working Group explicitly welcomed papers from a variety of different theoretical approaches and methodological frameworks addressing the role of representations of different types in teaching and learning processes, in particular those involving visualization (considered here as defined by Arcavi (2003)). From its first appearance at CERME10 when it was made up of 24 participants from 13 countries with 16 accepted papers and 2 posters, at CERME11 TWG24 grew: there were 31 participants (authors, co-authors, and some other participants), from 16 countries with 18 accepted papers and 4 accepted posters. The structure of the working sessions allocated for discussion of each paper or poster was designed to stimulate interaction and collaboration among participants: each paper or poster was allocated to a working session in which a same or similar theme was addressed (to the extent possible); all participants were asked to read all papers ahead of time, concentrating on those they found most appealing; presenting authors were asked to prepare a short presentation, followed by open questions and a brief analysis of how the work presented contributed to the open questions in the TWG's call for proposals. After the short presentation by the presenting author, a general discussion was initiated and conducted for a total of 30 minutes for papers and 15 minutes for posters. The last session was completely devoted to summing up the main issues that had emerged from the group discussions. The four themes are: 1) theories, language and categorizations used to study representations (theories, language and categorization), 2) embodied and enactive perspectives on the use of representations in mathematics education (embodied and enactive perspectives), 3) forms of representation implemented through technological approaches (technological approaches), 4) pedagogical implications in the choice and use of representations in the teaching and learning of mathematics (pedagogical implications).

Theories, language, categorization

The researchers in TWG24 did not share a common theoretical framework for the use of the term representations. We see the diversity as a source of strength of our community, as different theoretical emphasis may illuminate various aspects of teaching and learning mathematics. We do share the need for having an explicit theoretical stance towards what is representation in our research.

The role of theory for the researchers in the group varied. Some articulated a theoretical perspective for designing tasks (e.g., Günster; Johnson et al.; Tabach & Koichu), while others view theory as a tool to guide their data analysis (e.g., Miragliotta; Lisarelli, Antonini & Baccaglini-Frank).

Some of us consider as representation what is accepted as such by the mathematical community (e.g., Duijzer et al.), while others see them as a collection of inscriptions produced by students for a given mathematical situation (e.g., what Dahl called ‘self-invented representations’). We discussed tensions that might arise, during the learning process, between students’ unique-created representations and the need for students to be familiar and work with classical representations.

The group also discussed the issue of internal/external representations: we cannot directly access internal representations, hence it is difficult to assess; across papers and posters there was variation in the extent to which internal versus external representation was theorized. This methodological-theoretical issue is important to acknowledge, even if we do not see a direct way to resolve it.

Moreover, discussions highlighted a general move away from dichotomous categories of representation towards continuous spectrums, in both research design and analysis of data.

The advance of technological tools and our ability to design complex and dynamic learning environments seems to be increasing. Our theoretical language, however, is not yet well-enough developed to allow consistent “talking about” dynamism in representations. This might be a challenge to handle in the coming years. For example, do we have a theoretical language allows us to differentiate between representing a mathematical phenomenon with dragging a finger over a touch screen, versus representing the same mathematical phenomenon with ones’ whole body?

Embodied and enactive perspectives

In our call for papers we explicitly noted that ‘representations’ should be interpreted broadly to encompass pictures, gestures, sounds, stories, metaphors and more (as well as more conventional classroom formats). Several presenters included material on the embodied and enactive representations employed by diverse participants (children and adults, from very high to very low mathematical attainment, with and without disabilities) within a wide variety of mathematical topics and contexts (educational, professional and recreational).

Several papers considered the salience of how hands, in particular, are used in embodied mathematical thinking. Miragliotta’s work on geometrical predictions analysed both discursive and gestural aspects of older school and university students’ geometrical problem-solving activity, highlighting the interplay both between language and motion, and perception and reasoning processes. One of the foci of Finesilver’s microgenetic case study of a teenager with severe numeracy difficulties was the significance of the regular pattern of hand movements in ‘dealing’ out units into equal groups. In keeping with the emerging theme of continua, this motion aspect, along with mixed-mode representation, enabled transition from concrete to graphic representation. In contrast, Wille and Schreiber focused not on informal gesture but the communication of mathematics in sign language, with a comparison of how explanations of geometrical terminology function in a visual signed language versus a spoken mode of communication. The subsequent discussion clarified that while sign language should be considered distinct from gesture, it may be productive to consider a continuum between them.

The intertwining of enaction and language was also considered by Arnoux and Soto-Andrade, with a focus on the significance of metaphor in understanding and representing mathematics. Their paper on moving between concrete and abstract forms also addressed affective concerns, and highlighted the way that metaphors may be particular to communities based on local phenomena and experiences. Meanwhile, O'Brien and de Freitas presented research from one particular community; that of those engaged in textile arts who, while they may not consider themselves to be 'doing mathematics', are creating complex haptic and visual representations of mathematical patterns. Their *fibre mathematics* includes technical-aesthetic considerations of topological dimension and connectivity as well as space, mechanics and computation.

This year's working group took a particular interest in the relationships between different representational forms, and discussion of enactive aspects also arose in the various discussions on graphical representations of time (Lisarelli et al.), motion (Duijzer, Van den Heuvel-Panhuizen, Veldhuis & Doorman), and relationships between quantities (Johnson, McClintock & Gardner). The discussions considered the different characteristics, affordances and limitations of creating moving representations with one's hands, manipulating external physical objects, and elements displayed on a screen. There was particular interest in how changing the subject of operation (objects, the environment, oneself) could affect perception of mathematical relationships.

The embodiment and enaction of mathematics can work in many different ways. However, being so fundamental to human experience, it may particularly help learners to ground concepts and form connections between mathematical ideas that they tend to see as isolated.

Technological approaches

A number of contributions presented studies in which different forms of technology were used to create or interact with representations of mathematical concept. Indeed, through technological artifacts, representations can become both dynamic and interactive. For example, Duijzer, Van den Heuvel-Panhuizen, Veldhuis and Doorman studied students creating distance-time graphs by describing their own movements in front of a motion sensor; Lisarelli, Antonini and Baccaglini-Frank analyzed students' written discourse about an experience in a dynamic interactive digital environment in which functions were represented in one dimension, as dynagraphs; and Johnson McClintock and Gardner explore students' transfer of covariational reasoning intertwined with their creation and interpretation of dynamic graphs on the Cartesian plane. Dynamism – seen as interactive, controlled motion – can be very useful for learning about particular mathematical concepts related to functions: covariation, input-output relationship, effect of parameters.

The studies also suggested that dynamical software influence students' practice and, afterwards, their drawings and verbal representations. This seems to be the case both when the whole body in involved and when only some parts of the body are used to interact with the technological artifact. For example, when a student walks in front of motion sensors and software captures and represents whole body movement, new expressions emerge such as "walking the graph" (e.g., Duijzer et al.). However, also after experiencing covariation in one-dimensional dynagraphs students use arrows in their drawing to describe movement, and words such as 'motion', 'then', 'before', 'dragging' (e.g., Lisarelli et al.). Futhermore, studies suggested that interacting with technological artifacts that produce dynamical interactive representations may also change the way students think of mathematical concepts (Duijzer et al.; Lisarelli et al.; Johnson et al.; Miragliotta; Tabach & Koichu). The group discussed how these could eventually become psychological tools as in the case of "dragging" (e.g. Baccaglini-Frank & Antonini, 2016).

The use of dynamic representations was also discussed within the proposal of a theoretical framework providing the foundation to design tasks aimed at promoting students' functional thinking (Günster). Finally, a different technological approach to representations was provided by O'Brien and de Freitas who used the loom to unpack the relationship between making, mathematics, and technology.

Much work remains to be done regarding how to employ technology's full potential in learning trajectories (including those for other topics than the ones discussed), and on how to foster the development of these experiences into formal mathematics.

Pedagogical implications

Representations play a vital role in mathematics teaching and learning. However, as researchers, we experience a constraint in communicating our findings in 'snapshots' – we noticed how many studies begin as small scale endeavors. Thus, researchers need ways to scale up from small scale studies and to bring what we are learning to teachers and their students. During the presentations and discussions, the group attempted to establish connections between representations and the pedagogical implications in the choice and use of such representations. For example, Milinkovic, Mihajlovic and Dejic found that students who managed to solve mathematical problems were able to connect different mathematical representations. However, in mathematics classrooms, teachers generally stimulate algebraic/symbolic representations to solve mathematical problems more than other types. Pressures may exist for teachers to move quickly to help students have swift ways to solve problems and represent processes promoting one of the representations more than others, and these pressures may prematurely curtail students' intuitive approaches when engaging in problem solving.

Some discussions focused on how teachers should be able to use incorrect answers as an opportunity for mathematical investigation. While some teachers may prefer to ignore or dismiss an incorrect answer and focus on teaching correct ways to represent mathematical problems, others decide to use an incorrect strategy to engage in a class discussion. Tabach and Koichu used the potential of incorrect strategies for the solution of a problem with a novel method, namely in a "who is right" form in which students are given scenarios and asked to pick their sides (if any) for the correct solution of a problem. Okumus and Dede introduced students misleading representations without letting them know that graphs tweaked information. Alternatively, some may prefer to be more explicit and provide specific representations (e.g., Hough et al.; Böcherer-Linder et al.). Teachers pedagogical decisions will influence how these tasks move students' thinking forward or how they will hinder their learning.

References

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