On the existence of weak solutions for the steady Baldwin-Lomax model and generalizations

Luigi C. Berselli

Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti 1/c, I-56127 Pisa, Italy

Dominic Breit

Heriot-Watt University, Department of Mathematics, Riccarton, EH14 4AS Edinburgh

Abstract

In this paper we consider the steady Baldwin-Lomax model, which is a rotational model proposed to describe turbulent flows at statistical equilibrium. The Baldwin-Lomax model is specifically designed to address the problem of a turbulent motion taking place in a bounded domain, with Dirichlet boundary conditions at solid boundaries. The main features of this model are the degeneracy of the operator at the boundary and a formulation in velocity/vorticity variables. The principal part of the operator is non-linear and it is degenerate, due to the presence (as a coefficient) of a power of the distance from the boundary: This fact makes the existence theory naturally set in the framework of appropriate weighted-Sobolev spaces.

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1. Introduction

In this paper we study the model (and some of its variants which are interesting from the mathematical point of view) introduced by Balwdin and Lomax [7]

$$\begin{cases} -\nu_0 \operatorname{div} \mathbf{D}\mathbf{v} + (\nabla \mathbf{v})\mathbf{v} + \operatorname{curl}\left(d^2|\operatorname{curl}\mathbf{v}|\operatorname{curl}\mathbf{v}\right) + \nabla\pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Email addresses: luigi.carlo.berselli@unipi.it (Luigi C. Berselli), d.breit@hw.ac.uk (Dominic Breit)

Dedicated to the memory of Christian G. Simader to describe turbulent fluids at the statistical equilibrium, where d is the distance from the boundary. We recall that, starting from the work of O. Reynolds in the 19th Century, a classical paradigm is that of decomposing the velocity into the sum of a mean part and (turbulent) fluctuations, see [8]. One basic question is how to model the effect of the smaller scales on the larger ones. The Boussinesq assumption suggests that –in average–this produces an additional turbulent viscosity $\nu_T \geq 0$, which is proportional to the mixing length and to the kinetic energy of fluctuations (at least in the Kolmogorov-Prandtl approximation). In the analysis of Baldwin and Lomax, this leads to a turbulent viscosity of the form

$$\nu_T \sim \ell^2(\mathbf{x})|\operatorname{curl}\mathbf{v}(\mathbf{x})|,$$

where ℓ is a multiple of the distance from the boundary and $\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v}$, hence arriving to the model (1.1), or to (3.1) when the equations for the turbulent flow are considered in the rotational formulation. (Mean velocities denoted from now on as $\mathbf{v}$).

The Baldwin-Lomax model (1.1) has been recently revisited –in the unsteady case– by Rong, Layton, and Zhao [36], in order to take into account also of the effects of back-scatter. This involves, in addition to the usual time derivative $\frac{\partial \mathbf{v}}{\partial t}$, a dispersive term of the form

$$\operatorname{curl}\Big(\ell^2(\mathbf{x})\operatorname{curl}\frac{\partial \mathbf{v}}{\partial t}(t,\mathbf{x})\Big),$$

resembling that appearing in Kelvin-Voigt materials. Also in this case the problem has some degeneracy at the boundary. Different mathematical tools are required to handle the above term: being of the Kelvin-Voigt type, the latter differential operator is linear and not dissipative, but instead it is dispersive. Further details, and its analysis in connection with Turbulent-Kinetic-Energy (TKE) models are studied in [4], in the case of a turbulent viscosity depending only on the turbulent kinetic energy, but not on $\operatorname{curl}\mathbf{v}$. Related results involving a selective anisotropic turbulence model can be also found in [17].

Here, we consider –as a starting point– the problem at statistical equilibrium. We study just the steady case, which contains nevertheless several peculiar properties; the methods and techniques involved are rather different than those used in the previous mathematical theory of unsteady Baldwin-Lomax type models in [4, 36].

The class of problems we study is that of finding a velocity field $\mathbf{v} : \Omega \to \mathbb{R}^3$ and a pressure function $\pi : \Omega \to \mathbb{R}$ such that the following boundary value problem for a nonlinear system of partial differential equations is satisfied

$$\begin{cases} -\nu_0 \operatorname{div}\mathbf{D}\mathbf{v} + \operatorname{curl}\mathbf{S} + (\nabla\mathbf{v})\,\mathbf{v} = -\nabla\pi + \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

Here Ω denotes a bounded smooth domain in $\mathbb{R}^3$, and $\mathbf{f} : \Omega \to \mathbb{R}^3$ is the volume force and $\nu_0 \geq 0$ is the kinematic viscosity.

As a generalized Baldwin-Lomax model, we will also consider the stress tensor $\mathbf{S} : \Omega \to \mathbb{R}^{3\times 3}$ given by

$$\mathbf{S} = \mathbf{S}(\mathbf{x}, \operatorname{curl}\mathbf{v}) = d(\mathbf{x})^\alpha (\kappa + |\operatorname{curl}\mathbf{v}|)^{p-2} \operatorname{curl}\mathbf{v}, \tag{1.2}$$

where $d(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \partial\Omega)$, and $\alpha > 0$, $p > 1$, $\kappa \geq 0$ are given constants.

For technical reasons we will need to assume that $p > \frac{6}{5}$ and $\alpha < p - 1$, cf. Theorem 5.6. While the restriction on p is needed to give a proper meaning to the convective term, the limiting value of the power of the weight function (which is a length) deserves some comments, which are linked with technical arguments of analysis, but that are also connected with modeling.

Modeling and a suggested exponent of the distance function

If one thinks of a flow as composed of eddies of different sizes in different places, as in Large Eddy Simulation (LES), then in a region of large eddies the velocity and its curl changes are both $\mathcal{O}(1)$ of the typical distance. In a region of smaller eddies the velocity changes over a distance of $\mathcal{O}$(eddy length scale), so the local deformation is $\mathcal{O}(1/$eddy length scale$)$, cf. [8, § 3.3.2]. Hence, the Baldwin-Lomax model introduces a turbulent viscosity $\nu_T = (C\delta)^2 |\operatorname{curl}\mathbf{w}|$, where δ is the (local) smallest resolved scale, such that

$$\nu_T = \begin{cases} \mathcal{O}(\delta^2) & \text{in regions where } |\operatorname{curl}\mathbf{w}| = \mathcal{O}(1) \\ \mathcal{O}(\delta) & \text{in the smallest resolved scale where } |\operatorname{curl}\mathbf{w}| = \mathcal{O}(\delta^{-1}). \end{cases}$$

By extrapolation motivated by experiments with central difference approximations to linear convection diffusion problems the following alternate scaling is also proposed (cf. again [8] and Layton [32])

$$\nu_T = (C\delta)^{p-1} |\mathbf{D}\mathbf{w}|^{p-2}.$$

It satisfies

$$\nu_T = \begin{cases} \mathcal{O}(\delta^p) & \text{in regions where } |\operatorname{curl}\mathbf{w}| = \mathcal{O}(1) \\ \mathcal{O}(\delta) & \text{in the smallest resolved scale where } |\operatorname{curl}\mathbf{w}| = \mathcal{O}(\delta^{-1}), \end{cases}$$

which corresponds to the critical value $\alpha = p - 1$ we consider.

In the following we also give a justification of the critical value $p-1$, based directly on dimensional arguments, rather than on numerical experiments or analogies as in [32].

Both the $\nabla\mathbf{v}$ and $\boldsymbol{\omega}$ have dimensions T^{-1}, where T is a time, hence in the classical Baldwin-Lomax model the turbulent viscosity has the correct dimensions of a viscosity $\nu_T = d^2 |\operatorname{curl}\boldsymbol{\omega}| \sim L^2 T^{-1}$, where L is a length. This is the only way to identify (by using only a typical length and the vorticity) a quantity with the dimensions of a viscosity. A possible choice is that of using a third parameter and in turbulence modeling –especially in the presence of boundary layers– it is common to introduce the so called friction velocity v_* (cf. [4]) which has the dimensions of a velocity, that is $v_* \sim LT^{-1}$.

We propose to find a turbulent viscosity of the following form

$$\nu_T = v_*^\theta d^\alpha |\boldsymbol{\omega}|^{p-2},$$

modulo multiplication by some non-dimensional constant C, for some constants θ, α, p. It turns out that the dimensions of this quantity are $\nu_T \sim L^{\theta+\alpha} T^{2-\theta-p}$, hence to be dimensionally consistent one has to solve the following system

$$\begin{cases} \theta + \alpha = 2, \\ 2 - \theta - p = -1, \end{cases}$$

which has a single solution

$$\theta = 3 - p \qquad \text{and} \qquad \alpha = p - 1. \tag{1.3}$$

In conclusion, is turns out again that the "correct" exponent in terms of dimensional analysis is the critical one, that is $\alpha = p - 1$ and the dimensionally correct generalization of the Baldwin-Lomax model is the one with stress tensor

$$S(v_*, d(\mathbf{x}), \boldsymbol{\omega}) = C v_*^{3-p} d(\mathbf{x})^{p-1} |\boldsymbol{\omega}|^{p-2},$$

and, after re-scaling, one can assume $C v_*^{3-p} = 1$.

We will prove the existence of weak solutions for various models with different parameters, and highlight the role of the parameters p, α, and ν_0. The analysis requires substantial changes in the mathematical approach depending on the range of these constants.

The main result we prove is the existence of weak solution in appropriate (weighted) function spaces. The results are obtained by using a classical Galerkin approximation procedure and the passage to the limit is done by means of monotonicity and truncation methods typical of the analysis of non-Newtonian fluids, see for instance the reviews in [12, 35].

As far as the classical Baldwin-Lomax model is concerned (that is $p = 3$ and $\alpha = 2$) the following is our main result.

Theorem 1.1. *Let be given $\nu_0 > 0$ and $\mathbf{f} \in W^{-1,2}(\Omega) = (W_0^{1,2}(\Omega))'$. Then, there exists $\mathbf{v} \in W_{0,\sigma}^{1,2}(\Omega)$, with $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v} \in L^3(\Omega, d^2) \cap L^3_{loc}(\Omega)$, which is a weak solution Baldwin-Lomax model in the sense that*

$$\int_\Omega \nu_0 \, \mathbf{D}\mathbf{v} : \mathbf{D}\boldsymbol{\varphi} + d^2 |\boldsymbol{\omega}| \boldsymbol{\omega} \cdot \operatorname{curl} \boldsymbol{\varphi} + (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\varphi} \, d\mathbf{x} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle \qquad \forall\, \boldsymbol{\varphi} \in C_{0,\sigma}^\infty(\Omega).$$

The function spaces will be introduced in Section 2 and the proof of the above theorem can be found in Section 3. Since the parameter ν_0 is typically very small in applications it is of interest also to consider the case $\nu_0 = 0$. This will be done in Section 4. In Section 5 we finally consider a general constitutive relation of the form (1.2), together with different values for α.

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2. Functional setting

In the sequel $\Omega \subset \mathbb{R}^3$ will be a smooth and bounded open set, as usual we write $\mathbf{x} = (x_1, x_2, x_3) = (x', x_3)$ for all $\mathbf{x} \in \mathbb{R}^3$. In particular, we assume that the boundary $\partial\Omega$ is of class $C^{0,1}$, such that the normal unit vector $\mathbf{n}$ at the boundary is well defined and other relevant properties hold true. We recall a domain is of class $C^{k,1}$ if for each point $P \in \partial\Omega$ there are local coordinates such that in these coordinates we have $P = 0$ and $\partial\Omega$ is locally described by a $C^{k,1}$-function, i.e., there exist R_P, $R'_P \in (0, \infty)$, $r_P \in (0, 1)$ and a $C^{k,1}$-function $a_P : B^2_{R_P}(0) \to B^1_{R'_P}(0)$ such that

i) $\mathbf{x} \in \partial\Omega \cap (B^2_{R_P}(0) \times B^1_{R'_P}(0)) \iff x_3 = a_P(x')$,

ii) $\Omega_P := \{(x \in \mathbb{R}^3 \,\big|\, x' \in B^2_{R_P}(0),\ a_P(x') < x_3 < a_P(x') + R'_P\} \subset \Omega$,

iii) $\nabla a_P(0) = \mathbf{0}$, and $\forall\, x' \in B^2_{R_P}(0) \quad |\nabla a_P(x')| < r_P$,

where $B^k_r(0)$ denotes the k-dimensional open ball with center 0 and radius $r > 0$.

We also define the distance $d(\mathbf{x}, A)$ of a point from a closed set $A \subset \mathbb{R}^3$ as follows

$$d(\mathbf{x}, A) := \inf_{\mathbf{y}\in A} |\mathbf{x} - \mathbf{y}|,$$

and we denote by $d(\mathbf{x})$ the distance of $\mathbf{x}$ from the boundary of Ω

$$d(\mathbf{x}) := d(\mathbf{x}, \partial\Omega).$$

We recall a well-known lemma about the distance function $d(\mathbf{x})$, see for instance Kufner [30].

Lemma 2.1. *Let Ω be a domain of class $C^{0,1}$, then there exist constants $0 < c_0, c_1 \in \mathbb{R}$ such that*

$$c_0\, d(\mathbf{x}) \le |a(x') - x_3| \le c_1\, d(\mathbf{x}) \qquad \forall\, \mathbf{x} = (x', x_3) \in \Omega_P.$$

For our analysis we will use the customary Lebesgue $(L^p(\Omega), \|\,.\,\|_p)$ and Sobolev spaces $(W^{k,p}(\Omega), \|\,.\,\|_{k,p})$ of integer index $k \in \mathbb{N}$ and with $1 \le p \le \infty$. As usual we denote by $\to$ the strong (norm) convergence, and by $\rightharpoonup$ the weak convergence. We do not distinguish scalar and vector valued spaces, we just use boldface for vectors and tensors. We recall that $L^p_0(\Omega)$ denotes the subspace with zero mean value, while $W^{1,p}_0(\Omega)$ is the closure of the smooth and compactly supported functions with respect to the $\|\,.\,\|_{1,p}$-norm. If Ω is bounded and if $1 < p < \infty$, the following two relevant inequalities hold true:
1) the Poincaré inequality

$$\exists\, C_P(p, \Omega) > 0: \qquad \|\mathbf{u}\|_p \le C_P \|\nabla \mathbf{u}\|_p \qquad \forall\, \mathbf{u} \in W^{1,p}_0(\Omega); \tag{2.1}$$

2) the Korn inequality

$$\exists\, C_K(p, \Omega) > 0: \qquad \|\nabla \mathbf{u}\|_p \le C_K \|\mathbf{D}\mathbf{u}\|_p \qquad \forall\, \mathbf{u} \in W^{1,p}_0(\Omega), \tag{2.2}$$

where $\mathbf{Du}$ denotes the symmetric part of the matrix of derivatives $\nabla \mathbf{u}$.

As a combination of (2.1)-(2.2) we also have that for $1 \le p < 3$ the Sobolev-type inequality

$$\exists\, C_S > 0: \qquad \|\mathbf{u}\|_{p^*} \le C_S \|\mathbf{D}\mathbf{u}\|_p, \tag{2.3}$$

holds true for all $\mathbf{u} \in W^{1,p}_0(\Omega)$, where $p^* := \frac{3p}{3-p}$.

The Korn inequality allows to control the full gradient in L^p by its symmetric part, for functions which are zero at the boundary. Classical results (cf. Bourguignon and Brezis [11]) concern controlling the full gradient with curl & divergence. The following inequality holds true: For all $s \ge 1$ and $1 < p < \infty$, there exists a constant $C = C(s, p, \Omega)$ such that,

$$\|\mathbf{u}\|_{s,p} \le C \Big[\|\operatorname{div} \mathbf{u}\|_{s-1,p} + \|\operatorname{curl} \mathbf{u}\|_{s-1,p} + \|\mathbf{u} \cdot \mathbf{n}\|_{s-1/p,p,\partial\Omega} + \|\mathbf{u}\|_{s-1,p} \Big],$$

for all $\mathbf{u} \in (W^{s,p}(\Omega))^3$, where $\|\,.\,\|_{s-1/p,p,\partial\Omega}$ is the trace norm as explained below. This same result has been later improved by von Wahl [40] obtaining, under geometric conditions on the domain, the following estimate without lower order terms: Let Ω be such that $b_1(\Omega) = b_2(\Omega) = 0$, where $b_i(\Omega)$ denotes the i-th Betti number, that is the the dimension of the i-th homology group $H^i(\Omega, \mathbb{Z})$. Then, there exists $C = C(p, \Omega)$ such that

$$\|\nabla \mathbf{u}\|_p \le C \big(\|\operatorname{div} \mathbf{u}\|_p + \|\operatorname{curl} \mathbf{u}\|_p \big), \tag{2.4}$$

for all $\mathbf{u} \in (W^{1,p}(\Omega))^3$ satisfying either $(\mathbf{u} \cdot \mathbf{n})_{|\partial\Omega} = 0$ or $(\mathbf{u} \times \mathbf{n})_{|\partial\Omega} = 0$. For more recent results see also Amrouche and Seloula [5].

In the trace-norm the fractional derivative appear in a natural way. Nevertheless, we need also to handle fractional spaces $W^{r,p}(\Omega)$, which are defined by means of the semi-norm

$$[u]_{s,p}^p := \int_\Omega \int_\Omega \frac{|u(\mathbf{x}) - u(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{3+sp}} \, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \qquad \text{for } 0 < s < 1,$$

as made by functions $u \in W^{[r],p}(\Omega)$, such that $[D^\alpha u]_{r-[r],p} = [D^\alpha u]_{s,p} < \infty$, for all multi-indices α such that $|\alpha| = [r]$ (for the trace norm one has to integrate instead with respect to the 2-dimensional Hausdorff measure). The main result we need is the following generalization of the classical Hardy inequality: Let $u \in L^p(\Omega)$, then

$$\frac{u}{d^s} \in L^p(\Omega) \Longleftrightarrow u \in W_0^{s,p}(\Omega) \quad \text{for all } 0 < s < 1, \text{ with } s - \frac{1}{p} \neq \frac{1}{2}. \tag{2.5}$$

2.1. Weighted spaces

Since we have a boundary value problem with an operator which is space dependent, a natural functional setting would be that of weighted Sobolev spaces. For this reason we define now the relevant spaces we will use. We follow the notation from the classical book of Kufner [30] and also refer further to [6] for questions related to unbounded domains and to [27] for applications more specific to fluid flows.

We start by defining weighted Sobolev spaces. Let $w(\mathbf{x}) : \Omega \to \mathbb{R}^+$ be given a function (weight) which is non-negative and almost everywhere positive. We define, for $1 \le p < \infty$, the weighted space $L^p(\Omega, w)$ as follows

$$L^p(\Omega, w) := \left\{ \mathbf{f} : \ \Omega \to \mathbb{R}^n \text{ measurable: } \int_\Omega |\mathbf{f}(\mathbf{x})|^p \, w(\mathbf{x}) \, \mathrm{d}\mathbf{x} < \infty \right\}.$$

The definition is particularly relevant if it allows to work in the standard setting of distributions $\mathcal{D}'(\Omega)$: for $p > 1$ we have

$$w^{-1/(p-1)} \in L^1_{loc}(\Omega) \quad \Rightarrow \quad L^p(\Omega, w) \subset L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega).$$

It turns out that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega, w)$ if the weight satisfies at least $w \in L^1_{loc}(\Omega)$, see [30]. In addition, $L^p(\Omega, w)$ is a Banach space when equipped with the norm

$$\|\mathbf{f}\|_{p,w} := \left(\int_\Omega |\mathbf{f}(\mathbf{x})|^p w(\mathbf{x}) \, \mathrm{d}\mathbf{x} \right)^{1/p}.$$

Clearly if $w(\mathbf{x}) \equiv 1$ then $L^p(\Omega, w) = L^p(\Omega)$.

Next, we define weighted Sobolev spaces

$$W^{k,p}(\Omega, w) := \{\mathbf{f} :\ \Omega \to \mathbb{R}^n : D^\alpha \mathbf{f} \in L^p(\Omega, w) \text{ for all } \alpha \text{ s.t. } |\alpha| \le k\},$$

equipped with the norm

$$\|\mathbf{f}\|_{k,p,w} := \Big(\sum_{|\alpha|\le k} \|D^\alpha \mathbf{f}\|^p_{p,w}\Big)^{1/p}.$$

As expected, we define $W^{k,p}_0(\Omega, w)$ as follows

$$W^{k,p}_0(\Omega, w) := \overline{\{\boldsymbol{\varphi} \in C^\infty_0(\Omega)\}}^{\|\cdot\|_{k,p,w}}.$$

In our application the weight $w(\mathbf{x})$ will a power of the distance $d(\mathbf{x})$ from the boundary. Consequently, we specialize to this setting and give specific notions regarding these so-called *power-type weights*, see Kufner [30]. First, it turns out that $W^{k,p}(\Omega, d^\alpha)$ is a separable Banach spaces provided $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ and $1 \le p < \infty$. In this special setting, several results are stronger or more precise due to the inclusion $L^p(\Omega, d^\alpha) \subset L^p_{loc}(\Omega)$ for all $\alpha \in \mathbb{R}$.

Probably one of the most relevant properties is the embedding

$$L^p(\Omega, d^\alpha) \subset L^1(\Omega) \quad \text{if} \quad \alpha < p - 1. \tag{2.6}$$

It follows directly from Hölder's inequality as follows

$$\int_\Omega |f|\,\mathrm{d}\mathbf{x} = \int_\Omega d^{\alpha/p}|f|d^{-\alpha/p}d\mathbf{x} \le \Big(\int_\Omega d^\alpha |f|^p d\mathbf{x}\Big)^{1/p}\Big(\int_\Omega d^{-\alpha p'/p}d\mathbf{x}\Big)^{1/p'},$$

using that the latter integral is finite if and only if

$$\frac{\alpha\, p'}{p} = \frac{\alpha}{p-1} < 1$$

by Lemma 2.1. Moreover, as in [30, Prop. 9.10] it follows that the quantity $\int_\Omega d^\alpha |\nabla \mathbf{f}|^p\,\mathrm{d}\mathbf{x}$ is an equivalent norm in $W^{1,p}_0(\Omega, d^\alpha)$, provided that $\alpha < p-1$. In this case functions from $W^{1,p}_0(\Omega, d^\alpha)$ are zero on $\partial\Omega$.

Remark 2.2. *The above results explain the critical role of the power $\alpha = p-1$ and highlight the fact that the original Balwdin-Lomax model is exactly that corresponding to the critical exponent. For the applications we have in mind the value of α is not so strictly relevant and in fact, following the same procedure as in [4], it also makes sense to consider turbulent viscosity as follows*

$$\nu_T(\mathbf{v}(\mathbf{x})) = \ell_0\, \ell(\mathbf{x})|\mathrm{curl}\, \mathbf{v}(\mathbf{x})|, \tag{2.7}$$

for some $\ell_0 \in \mathbb{R}^+$.

Appropriate versions of the Sobolev inequality (2.3) are valid also for weighted Sobolev spaces:

Lemma 2.3. *There exists a constant $C = C(\Omega, \delta, p)$, such that*

$$\Big\|u(\mathbf{x}) - \fint_\Omega u(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big\|_q \leq C\|d^\delta(\mathbf{x})\nabla u(\mathbf{x})\|_p = \|\nabla u\|_{p,d^{p\delta}}, \tag{2.8}$$

for all $u \in W^{1,p}(\Omega, d^{\delta p})$, where $q \leq \frac{3p}{3-p(1-\delta)}$.

For a proof see Hurri-Syrjänen [29]. Note that this inequality is formulated removing constants by means of subtracting averages and that the exponent q equals to p^* if $\delta = 0$. This will be used later on to make a proper sense of the quadratic term in the Navier–Stokes equations, cf. Definitions 4.1 and 5.2.

In addition to (2.6) and the Hardy inequality, the critical role of the exponent $p-1$ is also reflected in results about general weights and their relation with the maximal function.

Definition 2.4. *We say that $w \in L^1_{loc}(\mathbb{R}^3)$, which is $w \geq 0$ a.e., belongs to the Muckenhoupt class A_p, for $1 < p < \infty$, if there exists C such that*

$$\sup_{Q\subset\mathbb{R}^n}\left(\fint_Q w(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\left(\fint_Q w(\mathbf{x})^{1/(1-p)}\,\mathrm{d}\mathbf{x}\right)^{p-1} \leq C,$$

where Q denotes a cube in $\mathbb{R}^3$.

The role of the power α will be crucial in the sequel and we recall the following result, which allows us to embed the results within a classical framework and also to use fundamental tools of harmonic analysis. The powers of the distance function belong to the class A_p according to the following well-known result (For a proof see for instance Durán, Sammartino, and Toschi [22, Thm. 3.1])

Lemma 2.5. *The function $w(\mathbf{x}) = \big(d(\mathbf{x})\big)^\alpha$ is a Muckenhoupt weight of class A_p if and only if $-1 < \alpha < p-1$.*

The main result which we will use about singular integrals, which follows from the pioneering work of Muckenhoupt on maximal functions, is the following.

Lemma 2.6. *Let $CZ : C^\infty_0(\mathbb{R}^n) \to C^\infty_0(\mathbb{R}^n)$ be a standard Calderón-Zygmund singular integral operator in the sense of [38, Chapter II]. Let $w \in A_p$, for $1 < p < \infty$. Then, the operator CZ is continuous from $L^p(\Omega, w)$ into itself.*

We will use this result mainly on the operators related to the solution of the Poisson equation, to reconstruct a vector field from its divergence and its curl.

2.2. Solenoidal spaces

As usual in fluid mechanics, when working with incompressible fluids, it is natural to incorporate the divergence-free constraint directly in the function spaces. These spaces are built upon completing the space of solenoidal smooth functions with compact support, denoted as $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)$, in an appropriate topology. For $\alpha > 0$ define

$$L^p_\sigma(\Omega, d^\alpha) := \overline{\left\{\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)\right\}}^{\|\cdot\|_{p,d^\alpha}},$$
$$W^{1,p}_{0,\sigma}(\Omega, d^\alpha) := \overline{\left\{\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)\right\}}^{\|\cdot\|_{1,p,d^\alpha}}.$$

For $\alpha = 0$ they reduce to the classical spaces $L^p_\sigma(\Omega)$ and $W^{1,p}_{0,\sigma}(\Omega)$. Next, we will extensively use the following extension of inequality (2.4).

Lemma 2.7. *Let $1 < p < \infty$ and assume that the weight w belongs to the class A_p. Then there exists*[2] *a constant C depending on the weight $w \in A_p$ such that*

$$\|\nabla \mathbf{u}\|_{p,w} \le C(\|\operatorname{div} \mathbf{u}\|_{p,w} + \|\operatorname{curl} \mathbf{u}\|_{p,w}) \qquad \forall\, \mathbf{u} \in W^{1,p}_0(\Omega, w).$$

Proof. Let us initially assume that $\mathbf{u} \in C^\infty_0(\Omega)$. We can extend $\mathbf{u}$ by zero to an element of $C^\infty_0(\mathbb{R}^n)$ such that the boundary of Ω plays no role, anymore. We have the well-known identity

$$\operatorname{curl}\operatorname{curl} \mathbf{u}(\mathbf{x}) = -\Delta \mathbf{u}(\mathbf{x}) + \nabla \operatorname{div} \mathbf{u}(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \mathbb{R}^n.$$

By use of the Newtonian potential this implies

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi}\nabla_{\mathbf{x}} \int_{\mathbb{R}^n} \frac{\operatorname{div}_{\mathbf{y}} \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} + \frac{1}{4\pi}\operatorname{curl}_{\mathbf{x}} \int_{\mathbb{R}^n} \frac{\operatorname{curl}_{\mathbf{y}} \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y},$$

where the integral can be considered as performed only on Ω, which contains the support of $\mathbf{u}$, see e.g. von Wahl [40, Sec. 0, Introduction]. Hence, we obtain for all $\mathbf{x} \in \mathbb{R}^n$

$$\begin{aligned}\nabla \mathbf{u}(\mathbf{x}) &= -\frac{1}{4\pi}\nabla_{\mathbf{x}}\nabla_{\mathbf{x}} \int_{\mathbb{R}^n} \frac{\operatorname{div}_{\mathbf{y}} \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} + \frac{1}{4\pi}\nabla_{\mathbf{x}}\operatorname{curl}_{\mathbf{x}} \int_{\mathbb{R}^n} \frac{\operatorname{curl}_{\mathbf{y}} \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y},\\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^n} \nabla_{\mathbf{x}}\nabla_{\mathbf{x}} \frac{\operatorname{div}_{\mathbf{y}} \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} + \frac{1}{4\pi}\int_{\mathbb{R}^n} \nabla_{\mathbf{x}}\operatorname{curl}_{\mathbf{x}} \frac{\operatorname{curl}_{\mathbf{y}} \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y},\\ &= C Z_1[\operatorname{div} \mathbf{u}](\mathbf{x}) + C Z_2[\operatorname{curl} \mathbf{u}](\mathbf{x}),\end{aligned}$$

[2] The space $W^{1,p}_0(\Omega, w)$ can be replaced by other function spaces, where $C^\infty_0(\Omega)$ functions are dense.

where both terms CZ_i from the right-hand side are Calderon-Zygmund type singular integrals. Applying the Muckenhoupt result from Lemma 2.6, the claim follows for $\mathbf{u} \in C_0^\infty(\Omega) \subseteq C_0^\infty(\mathbb{R}^n)$. The general case follows by approximation in spaces such that smooth and compactly supported functions are dense, as the ones we consider. $\square$

In particular, we will use the latter result in the following special form

Corollary 2.8. *For $-1 < \alpha < p-1$ there exists a constant $C = C(\Omega, \alpha, p)$ such that*

$$\int_\Omega d^\alpha |\nabla \mathbf{v}|^p \, \mathrm{d}\mathbf{x} \leq C \int_\Omega d^\alpha |\operatorname{curl} \mathbf{v}|^p \, \mathrm{d}\mathbf{x} \qquad \forall\, \mathbf{v} \in W^{1,p}_{0,\sigma}(\Omega, d^\alpha). \tag{2.9}$$

A basic tool in mathematical fluid mechanics is the construction of a continuous right inverse of the divergence operator with zero Dirichlet conditions. This problem has infinitely many solutions and an explicit construction is that due to Bogovskiĭ [10], which is is reviewed in Galdi [26, Ch. 3] (for earlier results about the inversion of the divergence operator with homogeneous Dirichlet conditions, especially in the Hilbertian case, we refer to Tartar [39] and Ladyžhenskaya [31], but many other author contributed to this problem in various functional settings). The results we need in the sequel is the following.

Theorem 2.9. *Let $\omega \subset \mathbb{R}^3$ be a bounded smooth domain and let $f \in L^p_0(\omega)$ there exists at least one $\mathbf{u} = \mathrm{Bog}_\omega(f) \in W^{1,p}_0(\omega)$ which solves the boundary value problem*

$$\begin{cases} \operatorname{div} \mathbf{u} = f & \text{in } \omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\omega. \end{cases}$$

Among other spaces, the operator Bog_ω is linear and continuous from $L^p(\omega)$ to $W^{1,p}_0(\omega)$, for all $p \in (1, \infty)$

2.3. Solenoidal Lipschitz truncation

We recall that the nonlinear operator defined as follows $\mathcal{A}_p$

$$\mathcal{A}_p \mathbf{w} = -\operatorname{div}\left(|\mathbf{D}\mathbf{w}|^{p-2}\mathbf{D}\mathbf{w}\right),$$

is strongly monotone in $W^{1,p}_{0,\sigma}(\Omega)$, for $1 < p < \infty$. In fact

$$(|\mathbf{D}\mathbf{w}_1|^{p-2}\mathbf{D}\mathbf{w}_1 - |\mathbf{D}\mathbf{w}_2|^{p-2}\mathbf{D}\mathbf{w}_2) : (\mathbf{D}\mathbf{w}_1 - \mathbf{D}\mathbf{w}_2) \geq 0,$$

with equality if and only if $\mathbf{D}\mathbf{w}_1 = \mathbf{D}\mathbf{w}_2$. A crucial point in the classical Minty-Browder argument relies on analyzing, for $\mathbf{v}_n, \mathbf{v} \in W^{1,p}(\Omega)$ the non-negative quantity

$$\int_\Omega (|\mathbf{D}\mathbf{v}_n|^{p-2}\mathbf{D}\mathbf{v}_n - |\mathbf{D}\mathbf{v}|^{p-2}\mathbf{D}\mathbf{v}) : (\mathbf{D}\mathbf{v}_n - \mathbf{D}\mathbf{v}) \, \mathrm{d}\mathbf{x} \geq 0.$$

Here $\mathbf{v}_n$ is a Galerkin approximation and $\mathbf{v}$ its weak $W^{1,p}_{0,\sigma}$-limit. Using the weak formulation for both $\mathbf{v}_n$ and its limit $\mathbf{v}$ one can show (using the monotonicity argument) that

$$\mathcal{A}_p(\mathbf{v}_n) \to \mathcal{A}_p(\mathbf{v}) \quad \text{at least in} \quad (C^\infty_{0,\sigma}(\Omega))'.$$

Two main points of the classical argument are 1) being allowed to use $\mathbf{v}_n$ as test function and 2) showing that

$$\int_\Omega (\nabla \mathbf{v}_n)\, \mathbf{v}_n \cdot \mathbf{v}_n \, \mathrm{d}\mathbf{x} \to \int_\Omega (\nabla \mathbf{v})\, \mathbf{v} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}.$$

In general item 1) trivially follows for all $1 < p < \infty$, due to the continuity of the operator $\mathcal{A}_p$. We will see that this point is not satisfied with the degenerate operators we handle in Section 3-4 and appropriate localization/regularization/truncation must to be introduced, see below. Hence, we are using here some known technical tools in a new and non-standard context: the use of local techniques is not motivated by the presence of the convective term, but by the character of the nonlinear stress-tensor. Probably our analysis can be extended also to other degenerate fractional operator as those studied by Abdellaoui, Attar, and Bentifour [1].

Note also that it is for the request 2) that a limitation on the exponent arises, since $\mathbf{v}_n \to \mathbf{v}$ in L^q for $q < p^* = \frac{3p}{3-p}$ and this enforces a lower bound on the allowed values of p. In the analysis of non-Newtonian fluid this classical monotonicity argument is not applicable when $p \leq \frac{9}{5}$ (in the steady case). To overcome this problem and to solve the system also for smaller values of p (up to $\frac{6}{5}$) one needs test functions which are Lipschitz continuous, hence one needs to properly truncate $\mathbf{v}^m - \mathbf{v}$. This is the point where the Lipschitz truncation, originally developed by Acerbi and Fusco [2, 3] in the context of quasi-convex variational problems, comes into play. In fluid mechanics this tool has been firstly used in [19, 23], for a review we refer to [12, 35]. Being strongly nonlinear and also non-local, the Lipschitz truncation destroys the solenoidal character of a given function. Consequently, the pressure functions has to be introduced. Another approach is that of constructing a divergence-free version of the Lipschitz truncation - extending a solenoidal Sobolev function by a solenoidal Lipschitz function. This approach has been developed in [13, 14] and it completely avoids the appearance of the pressure function and highly simplifies the proofs avoiding results obtained in Simader and Sohr [37] (as done in Diening, Růžička, and Wolf [21]) to associate a pressure to the weak solution. We report the following version which can be found in [14, Thm. 4.2].

Theorem 2.10. *Let $1 < s < \infty$ and $B \subset \mathbb{R}^3$ a ball. Let $(\mathbf{u}^m) \subset W^{1,s}_{0,\sigma}(B)$ be a weak $W^{1,s}_{0,\sigma}(B)$ null sequence extended by zero to $\mathbb{R}^3$. Then, there exist $j_0 \in \mathbb{N}$ and a double sequence $(\lambda^{m,j}) \subset \mathbb{R}$ with $2^{2^j} \leq \lambda^{m,j} \leq 2^{2^{j+1}-1}$ a sequence* *of functions* $(\mathbf{u}^{m,j})$ *and open sets*[3] $(\mathcal{O}^{m,j})$ *with the following properties for* $j \geq j_0$.

(a) $\mathbf{u}^{m,j} \in W^{1,\infty}_{0,\sigma}(2B)$ *and* $\mathbf{u}^{m,j} = \mathbf{u}^m$ *on* $\mathbb{R}^3 \setminus \mathcal{O}^{m,j}$ *for all* $m \in \mathbb{N}$*;*

(b) $\|\nabla \mathbf{u}^{m,j}\|_\infty \leq c\lambda^{m,j}$ *for all* $m \in \mathbb{N}$*;*

(c) $\mathbf{u}^{m,j} \to 0$ *for* $m \to \infty$ *in* $L^\infty(\Omega)$*;*

(d) $\nabla \mathbf{u}^{m,j} \overset{*}{\rightharpoonup} 0$ *for* $m \to \infty$ *in* $L^\infty(\Omega)$*;*

(e) *For all* $m, j \in \mathbb{N}$ *it holds* $\|\lambda^{m,j}\chi_{\mathcal{O}^{m,j}}\|_s \leq c(s)\, 2^{-\frac{j}{s}} \|\nabla \mathbf{u}^m\|_s$.

As usual we denote by χ_A the indicator function of the measurable set $A \subset \mathbb{R}^3$.

We remark that the key idea in the recovery of the nonlinear stress tensor (and especially to treat convergence issues) is cutting away the singular boundary. The Lipschitz truncation will be applied locally, where the effect of the weight function is not seen. For a version of the solenoidal Lipschitz truncation in weighted spaces, not needed in our case, we refer to [15, 16] (see also [34]).

3. Existence of weak solutions for the Baldwin-Lomax model in the steady case

In this section we consider the model for the average of turbulent fluctuations attributed to Baldwin and Lomax (1.1). By using a standard notation we denote the curl of $\mathbf{v}$ by $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = \operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v}.$$

Since we consider the equations in a rotational setting, we write the convective term as follows

$$(\nabla \mathbf{v})\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v} + \frac{1}{2}\nabla |\mathbf{v}|^2.$$

By redefining the pressure we can consider the following steady system for a turbulent flow at statistical equilibrium

$$\begin{cases} -\nu_0 \operatorname{div} \mathbf{D}\mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} + \operatorname{curl}\big(d^2 |\boldsymbol{\omega}| \boldsymbol{\omega}\big) + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

in the case $\nu_0 > 0$. We have the following result, which does not follow by the standard theory of monotone operator.

[3] The set $\mathcal{O}^{m,j}$ is explicitly given by $\mathcal{O}^{m,j} := \{\mathcal{M}(\nabla^2(\operatorname{curl}^{-1}\mathbf{u}^m)) > \lambda^{m,j}\}$, where $\mathcal{M}$ is the Hardy-Littlewood maximal operator and $\operatorname{curl}^{-1} = \operatorname{curl}\Delta^{-1}$.

Theorem 3.1. *Let be given* $\nu_0 > 0$ *and* $\mathbf{f} \in W^{-1,2}(\Omega) = (W_0^{1,2}(\Omega))'$*. Then, there exists* $\mathbf{v} \in W^{1,2}_{0,\sigma}(\Omega)$*, with* $\boldsymbol{\omega} \in L^3(\Omega, d^2) \cap L^3_{loc}(\Omega)$*, which is a weak solution to* (3.1)*, that is such that*

$$\int_\Omega \nu_0 \, \mathbf{D}\mathbf{v} : \mathbf{D}\boldsymbol{\varphi} + d^2 |\boldsymbol{\omega}| \boldsymbol{\omega} \cdot \operatorname{curl} \boldsymbol{\varphi} + (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle \qquad \forall \, \boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega).$$

Here $\langle \cdot, \cdot \rangle$ denotes generically a duality pairing. By density it is enough to consider test functions $\boldsymbol{\varphi} \in W^{1,2}_{0,\sigma}(\Omega)$, with $\operatorname{curl} \boldsymbol{\varphi} \in L^3_{loc}(\Omega)$.

Remark 3.2. *It is possible to recover the pressure (in the sense of distributions) by the classical theorem due to De Rham. Using a version of the negative norm theorem in weighted spaces [20], the pressure inherits – by comparison– the integrability properties from the remaining terms of the equation.*

In this case with $\nu_0 > 0$ *uniqueness in general is not expected, but it follows in the case of small data, exactly as for the classical Navier-Stokes equations. The case with* $\nu_0 = 0$ *seems completely open, see also the remark concerning uniqueness in the final section.*

The proof of Theorem 3.1 is based on a Galerkin approximation and monotonicity arguments (beyond the classical Minty-Browder trick) to pass to the limit

We observe that the term coming from Baldwin-Lomax approach is monotone too. We prove for a general $p \in (1, \infty)$ and a general non-negative weight the following inequality.

Lemma 3.3. *For smooth enough* $\boldsymbol{\omega}_i$ *(it is actually enough that* $d^{\frac{\alpha}{p}} \boldsymbol{\omega}_i \in L^p(\Omega)$*, with* $1 < p < \infty$*) and for* $\alpha \in \mathbb{R}^+$ *it holds that*

$$\int_\Omega (d^\alpha |\boldsymbol{\omega}_1|^{p-2} \boldsymbol{\omega}_1 - d^\alpha |\boldsymbol{\omega}_2|^{p-2} \boldsymbol{\omega}_2) \cdot (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \, \mathrm{d}\mathbf{x} \geq 0,$$

for any (not necessarily the distance) bounded function such that $d : \Omega \to \mathbb{R}^+$ *for a.e.* $\mathbf{x} \in \Omega$.

Proof. We have

$$\int_\Omega (d^\alpha |\boldsymbol{\omega}_1|^{p-2} \boldsymbol{\omega}_1 - d^\alpha |\boldsymbol{\omega}_2|^{p-2} \boldsymbol{\omega}_2) \cdot (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \, \mathrm{d}\mathbf{x}$$
$$= \int_\Omega (|d^{\frac{\alpha}{p}} \boldsymbol{\omega}_1|^{p-2} d^{\frac{\alpha}{p}} \boldsymbol{\omega}_1 - |d^{\frac{\alpha}{p}} \boldsymbol{\omega}_2|^{p-2} d^{\frac{\alpha}{p}} \boldsymbol{\omega}_2) : (d^{\frac{\alpha}{p}} \boldsymbol{\omega}_1 - d^{\frac{\alpha}{p}} \boldsymbol{\omega}_2) \, \mathrm{d}\mathbf{x},$$

where the last inequality derives from the same monotonicity/convexity argument used classically for the operator $\mathcal{A}_p$. □

Proof of Theorem 3.1. The proof is based on the construction of an approximate sequence $(\mathbf{v}^m) \subset W^{1,3}_{0,\sigma}(\Omega)$ which solves the following regularized problem

$$-\frac{1}{m}\mathrm{div}\left(|\mathbf{D}\mathbf{v}^m|\mathbf{D}\mathbf{v}^m\right) - \nu_0\,\mathrm{div}\,\mathbf{D}\mathbf{v}^m + (\nabla\mathbf{v}^m)\mathbf{v}^m \\ + \mathrm{curl}\left(d^2|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m\right) + \nabla\pi = \mathbf{f} \quad \text{in } \Omega$$

subject to divergence-free constraint and homogeneous boundary conditions. In the weak formulation this reads as follows

$$\int_\Omega \frac{1}{m}|\mathbf{D}\mathbf{v}^m|\mathbf{D}\mathbf{v}^m : \mathbf{D}\boldsymbol{\varphi} + \nu_0\,\mathbf{D}\mathbf{v}^m : \mathbf{D}\boldsymbol{\varphi} + d^2|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m\cdot\mathrm{curl}\,\boldsymbol{\varphi} \\ + (\boldsymbol{\omega}^m\times\mathbf{v}^m)\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = \langle\mathbf{f},\boldsymbol{\varphi}\rangle, \qquad \forall\,\boldsymbol{\varphi}\in W^{1,3}_{0,\sigma}(\Omega), \tag{3.2}$$

The regularization is a technical step necessary to have a continuous problem, approximating (3.1) and for which the difference $\mathbf{v}^m - \mathbf{v}$ can be localized to produce a legitimate test function (This is not easy to be done at the finite dimensional level).

The construction of the solution $\mathbf{v}^m$ goes through a Galerkin approximation $\mathbf{v}^m_n \in V_n$,

$$\frac{1}{m}\int_\Omega |\mathbf{D}\mathbf{v}^m_n|\mathbf{D}\mathbf{v}^m_n : \mathbf{D}\boldsymbol{\varphi}_j + \nu_0\,\mathbf{D}\mathbf{v}^m_n : \mathbf{D}\boldsymbol{\varphi}_j + d^2|\boldsymbol{\omega}^m_n|\boldsymbol{\omega}^m_n\cdot\mathrm{curl}\,\boldsymbol{\varphi}_j \\ +(\boldsymbol{\omega}^m_n\times\mathbf{v}^m_n)\cdot\boldsymbol{\varphi}_j\,\mathrm{d}\mathbf{x} = \langle\mathbf{f},\boldsymbol{\varphi}_j\rangle \qquad \text{for } j=1,\dots,n,$$

where $V_n = \mathrm{Span}\{\boldsymbol{\varphi}_1,\dots,\boldsymbol{\varphi}_n\}$ and $\boldsymbol{\omega}^m_n = \mathrm{curl}\,\mathbf{v}^m_n$. The functions $(\boldsymbol{\varphi}_i)_i$ are a Galerkin basis made by smooth and solenoidal functions. Since only the third term on the left-hand side of (3.2) is effected by the weight, and the basis functions are smooth, the classical arguments are not spoiled by the singular weight. The existence of $\mathbf{v}^m_n$ can be proved as solution of standard nonlinear algebraic system, with a compact perturbation (leading also to a non-negative contribution in the energy estimates, which allows us to prove the a priori bounds, which are the core step also for the proof of existence of finite dimensional approximation).

Using $\mathbf{v}^m_n$ as test function gives the (uniform in $n\in\mathbb{N}$) estimate

$$\int_\Omega \frac{1}{m}|\mathbf{D}\mathbf{v}^m_n|^3 + \frac{\nu_0}{2}|\mathbf{D}\mathbf{v}^m_n|^2 + d^2|\boldsymbol{\omega}^m_n|^3\,\mathrm{d}\mathbf{x} \le \frac{C_K^2}{2\nu_0}\|\mathbf{f}\|^2_{-1,2},$$

where C_K is the constant in Korn's inequality (2.2). Hence, using Korn inequality, we have (up to a sub-sequence) that for fixed $m\in\mathbb{N}$

$$\mathbf{v}^m_n \overset{n}{\rightharpoonup} \mathbf{v}^m \qquad \text{in } W^{1,3}_{0,\sigma}(\Omega), \tag{3.3}$$

$$\mathbf{v}^m_n \overset{n}{\to} \mathbf{v} \qquad \text{in } L^q(\Omega), \qquad \forall\,q<\infty, \tag{3.4}$$

This regularity is enough to apply the classical monotonicity argument (cf. [33, p. 171,p. 216]). In particular, from (3.3)-(3.4) it follows that

$$\int_\Omega (\boldsymbol{\omega}_n^m \times \mathbf{v}_n^m)\cdot \mathbf{v}_n^m \,\mathrm{d}\mathbf{x} \stackrel{n}{\to} \int_\Omega (\boldsymbol{\omega}^m \times \mathbf{v}^m)\cdot \mathbf{v}^m \,\mathrm{d}\mathbf{x},$$

Next, the function $\mathbf{v}^m \in W^{1,3}_{0,\sigma}(\Omega)$ is a weak solution in the sense of (3.2). This can be proved by observing that if we define the following operator

$$\mathcal{B}^{1/m}(\mathbf{w}) := -\frac{1}{m}\operatorname{div}|\mathbf{D}\mathbf{w}|\mathbf{D}\mathbf{w} - \nu_0 \operatorname{div}\mathbf{D}\mathbf{w} + \operatorname{curl}(d^2\,|\operatorname{curl}\mathbf{w}|\operatorname{curl}\mathbf{w}),$$

it holds that

$$0 \le \int_\Omega \big(\mathcal{B}^{1/m}(\mathbf{v}_n^m) - \mathcal{B}^{1/m}(\mathbf{w})\big) : (\mathbf{v}_n^m - \mathbf{w})\,\mathrm{d}\mathbf{x} \qquad \forall\,\mathbf{w}\in W^{1,3}_{0,\sigma}(\Omega),$$

(the later inequality holds not only formally, but rigorously, since integral is well-defined). Moreover, being $\mathbf{v}_n^m$ a legitimate test function in the Galerkin formulation, it is possible to pass to the limit (for fixed $m \in \mathbb{N}$) as $n \to \infty$, showing that (exactly as in [33], where the tools for generalized Navier-Stokes equations have been developed)

$$0 \le \int_\Omega \big(\mathcal{B}^{1/m}(\mathbf{v}^m) - \mathcal{B}^{1/m}(\mathbf{w})\big) : (\mathbf{v}^m - \mathbf{w})\,\mathrm{d}\mathbf{x} \qquad \forall\,\mathbf{w}\in W^{1,3}_{0,\sigma}(\Omega).$$

Choosing $\mathbf{w} = \mathbf{v}^m - \lambda\boldsymbol{\varphi}$, with $\lambda > 0$ and arbitrary $\boldsymbol{\varphi} \in W^{1,3}_{0,\sigma}(\Omega)$, this is enough to infer that $\lim_{n\to+\infty}\mathcal{B}^{1/m}(\mathbf{v}_n^m) = \mathcal{B}^{1/m}(\mathbf{v}^m)$.

To study the limit $m \to +\infty$ for the sequence $(\mathbf{v}^m)$ a technique beyond the classical monotonicity is needed.

First, taking $\mathbf{v}^m$ as test function in (3.2) we get

$$\int_\Omega \frac{1}{m}|\mathbf{D}\mathbf{v}^m|^3 + \frac{\nu_0}{2}\,|\mathbf{D}\mathbf{v}^m|^2 + d^2|\boldsymbol{\omega}^m|^3\,\mathrm{d}\mathbf{x} \le \frac{C_K^2}{2\nu_0}\|\mathbf{f}\|^2_{-1,2}. \tag{3.5}$$

Hence, using Korn inequality, we have (up to a sub-sequence)

$$\frac{1}{m}|\mathbf{D}\mathbf{v}^m|\mathbf{D}\mathbf{v}^m \rightharpoonup \mathbf{0} \qquad \text{in } L^{3/2}(\Omega), \tag{3.6}$$

$$\mathbf{v}^m \rightharpoonup \mathbf{v} \qquad \text{in } W^{1,2}_{0,\sigma}(\Omega), \tag{3.7}$$

$$\mathbf{v}^m \to \mathbf{v} \qquad \text{in } L^q(\Omega), \qquad \forall\, q < 6, \tag{3.8}$$

$$d^{4/3}|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m \rightharpoonup \chi \qquad \text{in } L^{3/2}(\Omega). \tag{3.9}$$

This implies in particular that, as $m \to +\infty$,

$$\int_\Omega (\boldsymbol{\omega}^m \times \mathbf{v}^m) \cdot \mathbf{v}^m \, \mathrm{d}\mathbf{x} \to \int_\Omega (\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{v} \, \mathrm{d}\mathbf{x},$$

$$\int_\Omega d^2 |\boldsymbol{\omega}^m| \boldsymbol{\omega}^m \cdot \boldsymbol{\psi} \, \mathrm{d}\mathbf{x} = \int_\Omega d^{4/3} |\boldsymbol{\omega}^m| \boldsymbol{\omega}^m \cdot d^{2/3} \boldsymbol{\psi} \, \mathrm{d}\mathbf{x}$$

$$\to \int_\Omega \chi \cdot d^{2/3} \boldsymbol{\psi} \, \mathrm{d}\mathbf{x} = \int_\Omega d^{2/3} \chi \cdot \boldsymbol{\psi} \, \mathrm{d}\mathbf{x},$$

for all $\boldsymbol{\psi} \in L^3(\Omega)$. Passing to the limit in the weak formulation we have

$$\int_\Omega \nu_0 \, \mathbf{D}\mathbf{v} : \mathbf{D}\boldsymbol{\varphi} + d^{2/3} \chi \cdot \operatorname{curl} \boldsymbol{\varphi} + (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle \tag{3.10}$$

for all $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)$. If we formally rewrite now the inequality

$$0 \leq \int_\Omega (d^2 |\boldsymbol{\omega}^m| \boldsymbol{\omega}^m - d^2 |\boldsymbol{\omega}| \boldsymbol{\omega}) \cdot (\boldsymbol{\omega}^m - \boldsymbol{\omega}) \, \mathrm{d}\mathbf{x},$$

coming from the monotonicity and express the same quantity by means of the weak formulation, we can observe that the classical monotonicity argument will work since the convergence of the generally troubling term

$$\int_\Omega \boldsymbol{\omega}^m \times \mathbf{v}^m \cdot (\mathbf{v}^m - \mathbf{v}) \, \mathrm{d}\mathbf{x} \to 0,$$

trivially follows from the uniform bound $\|\mathbf{v}^m\|_{W^{1,2}} \leq C$.

The crucial point is now that the integral

$$\int_\Omega d^2 |\boldsymbol{\omega}| \boldsymbol{\omega} \cdot (\boldsymbol{\omega}^m - \boldsymbol{\omega}) \, \mathrm{d}\mathbf{x},$$

is not defined. In fact, for $\mathbf{v} \in W^{1,2}_{0,\sigma}(\Omega)$ we only have $d^2 |\boldsymbol{\omega}| \boldsymbol{\omega} \in L^1(\Omega)$ and also $\boldsymbol{\omega}^m - \boldsymbol{\omega} \in L^2(\Omega)$. To overcome this problem we observe that for each compact set $K \Subset \Omega$

$$\Big(\min_{\mathbf{x} \in K} d(\mathbf{x})^2 \Big) \int_K |\boldsymbol{\omega}^m|^3 \, \mathrm{d}\mathbf{x} \leq \int_\Omega d^2 |\boldsymbol{\omega}^m|^3 \, \mathrm{d}\mathbf{x} \leq \frac{C_K^2}{2\nu_0} \|\mathbf{f}\|^2_{-1,2},$$

hence a completely local argument may work, being $\boldsymbol{\omega}$ in $L^3(K)$.

Remark 3.4. *Since the function* $(d(\mathbf{x}))^2$ *is not in the Muckenhoupt* A_3 *class, we cannot recover global bounds on the sequence* $(\nabla \mathbf{v}^m)$ *from the a priori estimate* (3.5) *and Lemma 2.7. This is a mathematical peculiarity of the Baldwin-Lomax stress tensor.*

To use a local argument we consider the family of compact sets

$$K_n := \left\{ \mathbf{x} \in \Omega : \; d(\mathbf{x}) \geq \frac{1}{n} \right\} \Subset \Omega,$$

which are nested and invading, that is $K_n \subset K_{n+1}$ and $\cup_{n \in \mathbb{N}} K_n = \Omega$. Hence, by a diagonal argument, up to a further sub-sequence, we can write that for each $K \Subset \Omega$

$$\boldsymbol{\omega}^m \rightharpoonup \boldsymbol{\omega} \qquad \text{in } L^3(K),$$

where $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$, by uniqueness of the weak limit.

Next, we fix an open ball $B \Subset \Omega$ such that $\overline{2B} \Subset \Omega$ and localize with a bump function $\eta \in C_0^\infty(2B)$ such that

$$\chi_B(\mathbf{x}) \leq \eta(\mathbf{x}) \leq \chi_{2B}(\mathbf{x}), \tag{3.11}$$

and $|\nabla \eta| \leq c\, R^{-1}$, where $R > 0$ is the radius of B. We define the following divergence-free function with support in $\overline{2B}$:

$$\mathbf{w}^m := \eta\, (\mathbf{v}^m - \mathbf{v}) - \operatorname{Bog}_{2B}(\nabla \eta \cdot (\mathbf{v}^m - \mathbf{v})),$$

where Bog_{2B} is the Bogovskiĭ operator on $2B$, acting linearly from $L_0^p(2B)$ to $W_0^{1,p}(2B)$, cf. Theorem 2.9. We introduce the function $\mathbf{w}^m$ to localise the arguments and thus to avoid problems with the singularity of the weight at the boundary. The multiplication with a cut-off function destroys the solenoidal character of the functions. This is corrected by means of the Bogovskiĭ operator, which results in an additional term of lower order. Since $\nabla \eta \cdot (\mathbf{v}^m - \mathbf{v})$ is bounded in $L_0^6(2B)$ by (3.7), we have that $\mathbf{w}^m$ is bounded in $W_{0,\sigma}^{1,6}(2B)$. Moreover, $\mathbf{v}^m \to \mathbf{v}$ in $L^3(\Omega)$ and the continuity of the Bogovskiĭ operator Bog_{2B} implies

$$\mathbf{w}^m \to 0 \quad \text{in} \quad L^3(2B), \tag{3.12}$$

$$\mathbf{w}^m \rightharpoonup 0 \quad \text{in} \quad W^{1,3}(2B), \tag{3.13}$$

$$\operatorname{Bog}_{2B}(\nabla \eta \cdot (\mathbf{v}^m - \mathbf{v})) \to 0 \quad \text{in} \quad W_0^{1,3}(2B). \tag{3.14}$$

The functions $\mathbf{w}^m \in W_0^{1,3}(2B)$ and their extensions by zero on $\Omega \backslash 2B$ (which still belong to $W_0^{1,3}(\Omega)$ and which we denote by a slight abuse of notation with the same symbol) are then legitimate test functions, since $|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m$ and $|\boldsymbol{\omega}|\boldsymbol{\omega}$ both belong to $L_{loc}^{3/2}(\Omega)$.

We subtract the weak formulation (3.2) of the regularized problem from its limit version (3.10) and test with the function $\mathbf{w}^m$ introduced above. After rearranging terms we obtain the following equality

$$\begin{aligned}
&\int_\Omega \eta\big(d^2|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m - d^2|\boldsymbol{\omega}|\boldsymbol{\omega}\big)\cdot\big(\boldsymbol{\omega}^m-\boldsymbol{\omega}\big)\,\mathrm{d}\mathbf{x}\\
&\quad= -\int_\Omega \big(d^2|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m - d^2|\boldsymbol{\omega}|\boldsymbol{\omega}\big)\cdot\nabla\eta\times\big(\mathbf{v}^m-\mathbf{v}\big)\,\mathrm{d}\mathbf{x}\\
&\quad+\int_\Omega \big(d^2|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m - d^2|\boldsymbol{\omega}|\boldsymbol{\omega}\big)\cdot\operatorname{curl}\Big[\mathrm{Bog}_{2B}(\nabla\eta\cdot(\mathbf{v}^m-\mathbf{v}))\Big]\,\mathrm{d}\mathbf{x}\\
&\quad-\nu_0\int_\Omega \mathbf{D}(\mathbf{v}^m-\mathbf{v}):\mathbf{D}\mathbf{w}^m\,\mathrm{d}\mathbf{x}+\int_\Omega\big(\boldsymbol{\omega}\times\mathbf{v}-\boldsymbol{\omega}^m\times\mathbf{v}^m\big)\cdot\mathbf{w}^m\,\mathrm{d}\mathbf{x}\\
&\quad+\int_\Omega\big(d^{2/3}\chi-d^2|\boldsymbol{\omega}|\boldsymbol{\omega}\big)\cdot\operatorname{curl}\mathbf{w}^m\,\mathrm{d}\mathbf{x}-\frac{1}{m}\int_\Omega|\mathbf{D}\mathbf{v}^m|\mathbf{D}\mathbf{v}^m:\mathbf{D}\mathbf{w}^m\,\mathrm{d}\mathbf{x}\\
&\quad=: (I)+(II)+(III)+(IV)+(V)+(VI).
\end{aligned}$$

Due to the strong L^3 convergence of $\mathbf{v}^m$ and (3.14) we see that (I) and (II) vanish as $m\to+\infty$ (We also used that the function d is uniformly bounded). We write the following equality

$$\begin{aligned}
(III) &= -\nu_0\int_\Omega \eta|\mathbf{D}(\mathbf{v}^m-\mathbf{v})|^2-\nu_0\int_\Omega\mathbf{D}(\mathbf{v}^m-\mathbf{v}):\nabla\eta\otimes(\mathbf{v}^m-\mathbf{v})\,\mathrm{d}\mathbf{x}\\
&\quad+\nu_0\int_\Omega\mathbf{D}(\mathbf{v}^m-\mathbf{v}):\mathbf{D}\big[\mathrm{Bog}_{2B}(\nabla\eta\cdot(\mathbf{v}^m-\mathbf{v}))\big]\,\mathrm{d}\mathbf{x},
\end{aligned}$$

where the first term is non-positive and the second and third one vanish on account of (3.8) and (3.14) respectively. The convergence of (IV) follows trivially from the uniform bounds in $W^{1,2}(\Omega)$ and (3.12). The term $(V)\to 0$ due to (3.14) and the bound in $L^{3/2}(B)$ of χ and $|\boldsymbol{\omega}|\boldsymbol{\omega}$. Finally, $(VI)\to 0$, due the $W^{1,3}(B)$ bound of $\mathbf{v}^m-\mathbf{v}$ and (3.6) .

In conclusion, since $\eta\geq 0$, the integrand is non-negative by Lemma 2.7, and from $\eta\equiv 1$ on B, it follows

$$\begin{aligned}
0&\leq\int_B\big(d^2\,|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m-d^2|\boldsymbol{\omega}|\boldsymbol{\omega}\big)\cdot\big(\boldsymbol{\omega}^m-\boldsymbol{\omega}\big)\,\mathrm{d}\mathbf{x}\\
&\leq\int_\Omega\eta\big(d^2\,|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m-d^2|\boldsymbol{\omega}|\boldsymbol{\omega}\big)\cdot\big(\boldsymbol{\omega}^m-\boldsymbol{\omega}\big)\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Consequently, we obtain

$$\lim_{m\to\infty}\int_B\big(d^2\,|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m-d^2|\boldsymbol{\omega}|\boldsymbol{\omega}\big)\cdot\big(\boldsymbol{\omega}^m-\boldsymbol{\omega}\big)\,\mathrm{d}\mathbf{x}=0,$$

and so,

$$d^{2/3}\boldsymbol{\omega}^m \to d^{2/3}\boldsymbol{\omega} \quad \text{a.e in } B.$$

Finally, we use $d(B,\partial\Omega) > R$ and the fact that the distance $d(\mathbf{x})$ is strictly positive for each $\mathbf{x} \in \Omega$. The arbitrariness of B implies

$$\boldsymbol{\omega}^m \to \boldsymbol{\omega} \quad \text{a.e in } \Omega.$$

Next, the limit function $\boldsymbol{\omega}$ belongs to $L^2(\Omega)$ and it is finite almost everywhere. The hypotheses of Vitali's convergence theorem are satisfied since

$$\begin{aligned}
&d^{4/3}|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m \qquad \text{uniformly bounded in } L^{3/2}(\Omega),\\
&d^{4/3}|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m \to d^{4/3}|\boldsymbol{\omega}|\boldsymbol{\omega} \quad \text{a.e. in } \Omega,\\
&\boldsymbol{\omega} \text{ finite a.e.},
\end{aligned}$$

ensuring that $d^{2/3}\chi = d^2|\boldsymbol{\omega}|\boldsymbol{\omega}$ and also that the limit $\mathbf{v}$ is a weak solution to (3.1). □

4. On generalised Baldwin-Lomax models

In the proof of the result from the previous section it was essential to have ν_0 positive and fixed, to derive a uniform bound of $(\mathbf{v}^m)_{m\in\mathbb{N}}$ in $W_0^{1,2}(\Omega)$. This allows us to make sense of the boundary conditions, among the other relevant properties. On the other hand, in applications ν_0 is generally an extremely small number. The K41-Kolmogorov theory for turbulence is in fact valid in the vanishing viscosity limit, and predicts (still in a statistical sense) a non zero turbulent dissipation, see Frisch [24]. To capture the properties which are still valid in the limit $\nu_0 = 0$ we study now the following steady system

$$\begin{cases}
(\nabla\mathbf{v})\mathbf{v} + \operatorname{curl}\left(d^\alpha(\kappa + |\boldsymbol{\omega}|)^{p-2}\boldsymbol{\omega}\right) + \nabla\pi = \mathbf{f} & \text{in } \Omega,\\
\operatorname{div}\mathbf{v} = 0 & \text{in } \Omega,\\
\mathbf{v} = \mathbf{0} & \text{on } \partial\Omega.
\end{cases}$$

Here $\kappa \geq 0$ and the most *interesting* case is the following one

$$\kappa = 0, \qquad p = 3, \qquad \alpha = 1,$$

where the exponent $p = 3$ is exactly that from the turbulence theory (as a generalization of the classical Smagorinsky theory), while $\alpha = 1$ is the same as suggested in (2.7) from the model introduced in [4]. Without loss of generality we also set $\ell_0 = 1$ and $\ell(\mathbf{x}) = d(\mathbf{x})$, as in the turbulent viscosity described in Remark 2.2.

Remark 4.1. *The critical value (coming from both LES and the Muckenhoupt theory, cf. Section 2.1) for the power of the distance is* $\alpha = p - 1 = 3 - 1 = 2$. *In this case certain bounds on the first derivatives of the velocity can be still inferred from weighted estimates of the gradient, as in* (2.9).

We start our analysis focusing on the following boundary value problem still written in rotational form

$$\begin{cases} \boldsymbol{\omega} \times \mathbf{v} + \operatorname{curl}\left(d\,|\boldsymbol{\omega}|\boldsymbol{\omega}\right) + \nabla\pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Definition 4.2. *We say that* $\mathbf{v} \in W^{1,3}_{0,\sigma}(\Omega, d)$ *is a weak solution to* (4.1) *if the following equality is satisfied*

$$\int_\Omega (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\varphi} + d\,|\boldsymbol{\omega}|\boldsymbol{\omega} \cdot \operatorname{curl}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} \qquad \forall\,\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega).$$

The main result we will prove in this section is the following.

Theorem 4.3. *Let be given* $\mathbf{f} = \operatorname{div}\mathbf{F}$ *with* $\mathbf{F} \in L^{3/2}(\Omega, d^{-1/2})$ *then there exists a weak solution* $\mathbf{v} \in W^{1,3}_{0,\sigma}(\Omega, d)$ *of the problem* (4.1). *In addition, the solution satisfies the energy-type equality*

$$\int_\Omega d\,|\boldsymbol{\omega}|^3\,\mathrm{d}\mathbf{x} = -\int_\Omega \mathbf{F} \cdot \nabla\mathbf{v}\,\mathrm{d}\mathbf{x}.$$

Remark 4.4. *By using fractional spaces we have that the same theorem holds for instance if* $\partial\Omega$ *is of class* C^2 *and if*

$$\mathbf{f} \in \widehat{W}^{-2/3,3/2}(\Omega) := (W^{2/3,3}(\Omega) \cap L^3_0(\Omega))'.$$

In fact, by using Thm. 3.4 from Geißert, Heck, and Hieber [28] there exists a bounded linear operator $R : \widehat{W}^{-2/3,3/2}(\Omega) \to W^{1/3,3/2}(\Omega)$, *such that* $\operatorname{div} R(\mathbf{f}) = \mathbf{f}$. *Next, observe that* $W^{1/3,3/2}(\Omega) = W^{1/3,3/2}_0(\Omega)$, *and consequently it follows for* $\boldsymbol{\varphi} \in C^\infty_0(\Omega)$

$$\langle \mathbf{f}, \boldsymbol{\varphi}\rangle = \langle \operatorname{div} R(\mathbf{f}), \boldsymbol{\varphi}\rangle = -\langle R(\mathbf{f}), \nabla\boldsymbol{\varphi}\rangle = -\langle d^{-1/3}R(\mathbf{f}), d^{1/3}\nabla\boldsymbol{\varphi}\rangle,$$

and –with the characterization of fractional spaces from (2.5)*–*

$$\Big|\langle \mathbf{f}, \boldsymbol{\varphi}\rangle\Big| \leq c\|R(\mathbf{f})\|_{1/3,3/2}\|\nabla\boldsymbol{\varphi}\|_{3,d} \leq \|\mathbf{f}\|_{-2/3,3/2}\|\nabla\boldsymbol{\varphi}\|_{3,d}.$$

Then, the estimates follow in the same manner as before.

Due to the fact that we have a problem without a principal part of standard p-Stokes type, we need to properly approximate (4.1) in order to construct weak solutions. As in the previous section we consider, for $\varepsilon > 0$, the following approximate system

$$\begin{cases} -\varepsilon \operatorname{div}(|\mathbf{D}\mathbf{v}_\varepsilon|\mathbf{D}\mathbf{v}_\varepsilon) + \boldsymbol{\omega}_\varepsilon \times \mathbf{v}_\varepsilon + \operatorname{curl}\big(d\,|\boldsymbol{\omega}_\varepsilon|\boldsymbol{\omega}_\varepsilon\big) + \nabla\pi_\varepsilon = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{v}_\varepsilon = 0 & \text{in } \Omega, \\ \mathbf{v}_\varepsilon = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

which falls within in the classical setting as studied starting with the work of Ladyžhenskaya [31] and Lions [33].

Remark 4.5. *At this stage (existence of weak solutions for the approximate problem) the power of $d(\mathbf{x})$ entering in the equations does not play any specific role.*

With the same tools already used, we have the following result.

Theorem 4.6. *For any $\varepsilon > 0$ and for $\mathbf{f} = \operatorname{div}\mathbf{F}$ with $\mathbf{F} \in L^{3/2}(\Omega)$ there exists a weak solution $\mathbf{v}_\varepsilon \in W^{1,3}_{0,\sigma}(\Omega)$ which satisfies*

$$\int_\Omega \varepsilon|\mathbf{D}\mathbf{v}_\varepsilon|\mathbf{D}\mathbf{v}_\varepsilon : \mathbf{D}\boldsymbol{\varphi} + (\boldsymbol{\omega}_\varepsilon \times \mathbf{v}_\varepsilon)\cdot\boldsymbol{\varphi} + d\,|\boldsymbol{\omega}_\varepsilon|\boldsymbol{\omega}_\varepsilon \cdot \operatorname{curl}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = -\int_\Omega \mathbf{F}\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}, \tag{4.3}$$

for all $\boldsymbol{\varphi} \in W^{1,3}_{0,\sigma}(\Omega)$. The function $\mathbf{v}_\varepsilon$ satisfies the energy-type estimate

$$\varepsilon\|\mathbf{v}_\varepsilon\|^3_{W^{1,3}_0} + \int_\Omega d\,|\boldsymbol{\omega}_\varepsilon|^3 \le \frac{C}{\sqrt{\varepsilon}}\|\mathbf{F}\|^{3/2}_{3/2}. \tag{4.4}$$

Moreover, if $\mathbf{F} \in L^{3/2}(\Omega, d^{-1/2})$, then

$$\varepsilon\|\mathbf{v}_\varepsilon\|^3_{W^{1,3}} + \int_\Omega d\,|\boldsymbol{\omega}_\varepsilon|^3 \le C\int_\Omega \frac{|\mathbf{F}|^{3/2}}{d^{1/2}}\,\mathrm{d}\mathbf{x} = C\|\mathbf{F}\|^{3/2}_{3/2,d^{-1/2}}, \tag{4.5}$$

for some constant C independent of ε.

Remark 4.7. *The approximation in (4.2) is introduced only as a mathematical tool, no modeling is hidden inside the choice for the perturbation.*

The regularization can be also done in the following way, respecting the rotational structure of the equation:

$$\boldsymbol{\omega}_\varepsilon \times \mathbf{v}_\varepsilon + \operatorname{curl}\big((\varepsilon + d)|\boldsymbol{\omega}_\varepsilon|\boldsymbol{\omega}_\varepsilon\big) + \nabla\pi_\varepsilon = \mathbf{f} \qquad \text{in } \Omega.$$

For this approximation one can use the fact that $d + \varepsilon \ge \varepsilon > 0$ and $\|\boldsymbol{\omega}_\varepsilon\|_p \sim \|\nabla\mathbf{v}\|_p$ for functions which are divergence-free and zero at the boundary by (2.4). We preferred the more classical way in order to use directly known results, being completely equivalent in terms of existence theorems.

Proof of Theorem 4.6. We do not give the easy proof of this result we just show the basic a priori estimates. The first ε-dependent estimate (4.4) is obtained by using as test function $\mathbf{v}_\varepsilon$ itself, integrating by parts, and using Hölder inequality to estimate the right-hand side.

In the following we also need estimates which are independent of $\varepsilon > 0$ and choosing again $\boldsymbol{\varphi} = \mathbf{v}_\varepsilon$ in (4.3) the right-hand side can be estimated by

$$\int_\Omega d^{-1/2}\mathbf{F} \cdot d^{1/3}\nabla \mathbf{v}_\varepsilon \,\mathrm{d}\mathbf{x} \leq C \left(\int_\Omega \frac{|\mathbf{F}|^{3/2}}{d^{1/2}} \,\mathrm{d}\mathbf{x} \right)^{2/3} \left(\int_\Omega d\,|\nabla \mathbf{v}_\varepsilon|^3 \,\mathrm{d}\mathbf{x} \right)^{1/3},$$

using Hölder's inequality. On account of (2.9) and Young's inequality we obtain further

$$\varepsilon \|\mathbf{v}_\varepsilon\|_{W^{1,3}}^3 + \int_\Omega d\,|\boldsymbol{\omega}_\varepsilon|^3 + d\,|\nabla \mathbf{v}_\varepsilon|^3 \,\mathrm{d}\mathbf{x} \leq C \int_\Omega \frac{|\mathbf{F}|^{3/2}}{d^{1/2}} \,\mathrm{d}\mathbf{x}, \tag{4.6}$$

hence (4.5) with a constant C independent of ε.

Finally, for $q < 3/2$ we have by Hölder's inequality

$$\begin{aligned} \int_\Omega |\nabla \mathbf{v}_\varepsilon|^q \,\mathrm{d}\mathbf{x} &= \int_\Omega d^{-q/3}\, d^{q/3} |\nabla \mathbf{v}_\varepsilon|^q \,\mathrm{d}\mathbf{x} \\ &\leq \left(\int_\Omega d^{-\frac{q}{3-q}} \,\mathrm{d}\mathbf{x} \right)^{(3-q)/3} \left(\int_\Omega d\,|\nabla \mathbf{v}_\varepsilon|^3 \,\mathrm{d}\mathbf{x} \right)^{q/3} \\ &\leq c \left(\int_\Omega d\,|\nabla \mathbf{v}_\varepsilon|^3 \,\mathrm{d}\mathbf{x} \right)^{q/3}, \end{aligned}$$

such that

$$\left(\int_\Omega |\nabla \mathbf{v}_\varepsilon|^q \,\mathrm{d}\mathbf{x} \right)^{3/q} \leq c \int_\Omega d\,|\boldsymbol{\omega}_\varepsilon|^3 \,\mathrm{d}\mathbf{x} \leq C \|\mathbf{F}\|_{3/2,d^{-1/2}}^{3/2},$$

using (4.6). This proves then that the solution to (4.2) satisfies also the estimate

$$\|\nabla \mathbf{v}_\varepsilon\|_{L^q} \leq C(q, \Omega, \|\mathbf{F}\|_{3/2,d^{-1/2}}). \tag{4.7}$$

for all $q < \frac{3}{2}$. □

Collecting all estimates we can give now the main existence result for the generalized Baldwin-Lomax model (4.1), passing to the limit as $\varepsilon \to 0$.

Proof of Theorem 4.3. Using the existence result from Theorem 4.6 we obtain a sequence of solutions $(\mathbf{v}_\varepsilon) \subset W^{1,3}_{0,\sigma}(\Omega)$ to (4.2). From the uniform estimates (4.5)-(4.7) we infer the existence of a limit function $\mathbf{v} \in W^{1,q}_{0,\sigma}(\Omega)$ such that along a sequence $\varepsilon_m \to 0$ and for $\mathbf{v}^m := \mathbf{v}_{\varepsilon_m}$ it holds

$$
\begin{aligned}
\mathbf{v}^m &\rightharpoonup \mathbf{v} && \text{in } W^{1,q}_{0,\sigma}(\Omega) \qquad \forall\, q < \frac{3}{2}, && (4.8)\\
\mathbf{v}^m &\to \mathbf{v} && \text{in } L^r_\sigma(\Omega) \qquad \forall\, r < 3, && (4.9)\\
\mathbf{v}^m &\to \mathbf{v} && \text{a.e. in } \Omega, && (4.10)\\
\varepsilon_m |\mathbf{D}\mathbf{v}^m| \mathbf{D}\mathbf{v}^m &\to \mathbf{0} && \text{in } L^{3/2}_0(\Omega). && (4.11)
\end{aligned}
$$

At this point we observe that it is not possible to pass to the limit as $\varepsilon \to 0$ in the equations directly by monotonicity arguments since $\frac{3}{2} < \frac{9}{5}$. Hence the difficulty will be again proving that $\mathbf{v}$ is a weak solution to (4.1). We will employ a local argument similar to the previous section. For all compact sets $K \Subset \Omega$ it holds that

$$
\begin{aligned}
c_0 \big(\min_{\mathbf{x} \in K} d(\mathbf{x})\big) \int_K |\nabla \mathbf{v}^m|^3 \,\mathrm{d}\mathbf{x} &\le c_0 \int_K d\, |\nabla \mathbf{v}^m|^3 \,\mathrm{d}\mathbf{x} \\
&\le c_0 \int_\Omega d\, |\nabla \mathbf{v}^m|^3 \,\mathrm{d}\mathbf{x} \le C(\Omega, \|\mathbf{F}\|_{3/2, d^{-1/2}}),
\end{aligned}
$$

using (4.5). This shows that (up to possibly another sub-sequence)

$$
\begin{aligned}
(\nabla \mathbf{v}^m)_{|K} &\rightharpoonup \nabla \mathbf{v}_{|K} \qquad \text{in } L^3(K) \qquad \forall\, K \Subset \Omega,\\
(\mathbf{v}^m)_{|K} &\to \mathbf{v}_{|K} \qquad \text{in } L^r(K) \qquad \forall\, r < \infty.
\end{aligned}
\tag{4.12}
$$

This proves that

$$
\int_\Omega (\boldsymbol{\omega}^m \times \mathbf{v}^m) \cdot \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x} \xrightarrow{m \to \infty} \int_\Omega (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\varphi} \,\mathrm{d}\mathbf{x} \qquad \forall\, \boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega),
$$

while passing to the limit in the nonlinear term requires again a local approach, as developed in the previous section.

Based on the previous observations if $\overline{\mathbf{S}}$ denotes the $L^{3/2}_{loc}(\Omega)$-weak limit of $d\,|\boldsymbol{\omega}_\varepsilon|\boldsymbol{\omega}_\varepsilon$, which exists by using the uniform bound coming from (4.5), we obtain the limit system

$$
\begin{cases}
\boldsymbol{\omega} \times \mathbf{v} + \operatorname{curl} \overline{\mathbf{S}} + \nabla \pi = \operatorname{div} \mathbf{F} & \text{in } \Omega,\\
\operatorname{div} \mathbf{v} = 0 & \text{in } \Omega,\\
\mathbf{v} = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{4.13}
$$

where the first equation is satisfied in the sense of distributions over Ω. The remaining effort is to show that $\overline{\mathbf{S}} = d\,|\boldsymbol{\omega}|\boldsymbol{\omega}$.

Observe also that at this point we have that $\boldsymbol{\omega}_\varepsilon \times \mathbf{v}_\varepsilon \in L^s_{loc}(\Omega) \subset L^1_{loc}(\Omega)$ for all $s < 3$, but not uniformly in ε.

The uniform estimates imply that $\mathbf{v}^m \in W^{1,q}_0(\Omega)$, for all $q < 3/2$, hence $\mathbf{v}^m \in L^r(\Omega)$, for all $r < 3$. This is not enough to show $\boldsymbol{\omega}^m \times \mathbf{v}^m \in L^1(\Omega)$, hence testing with $\mathbf{v}$ itself seems not possible.

First, we improve the known summability of the solutions, by observing that applying (2.8) to our case ($p = 3$, $\delta = 1/3$) implies

$$\Big\|\mathbf{v}^m(\mathbf{x}) - \fint_\Omega \mathbf{v}^m(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big\|_9^3 \le C\|d^{1/3}\nabla\mathbf{v}^m\|_3^3 = \int_\Omega d\,|\nabla\mathbf{v}^m|^3\,\mathrm{d}\mathbf{x} \le C\|\mathbf{F}\|^{3/2}_{3/2,d^{-1/2}},$$

uniformly in ε. Next we recall that by Hölder inequality $\left\|\fint_\Omega f\,\mathrm{d}\mathbf{x}\right\|_p \le \|f\|_p$, such that

$$\|f\|_p - \Big\|\fint_\Omega f(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big\|_p \le \Big\|f(\mathbf{x}) - \fint_\Omega f(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big\|_p,$$

for any $f \in L^p(\Omega)$. This yields, due to the embedding into $L^r(\Omega) \subset L^1(\Omega)$ for $r < 3$, the following

$$\begin{aligned}\|\mathbf{v}^m\|_9 &\le \Big\|\fint_\Omega \mathbf{v}^m\,\mathrm{d}\mathbf{y}\Big\|_9 + C\|\mathbf{F}\|^{1/2}_{3/2,d^{-1/2}}\\ &\le \frac{1}{|\Omega|^{8/9}}\|\mathbf{v}^m\|_1 + C\|\mathbf{F}\|^{1/2}_{3/2,d^{-1/2}} \le c(|\Omega|, \|\mathbf{F}\|_{3/2,d^{-1/2}}).\end{aligned}$$

Finally, we obtain

$$\boldsymbol{\omega}^m \times \mathbf{v}^m \in L^s(\Omega) \qquad \forall\, s < \frac{9}{7},$$

uniformly in $m \in \mathbb{N}$. We can also improve (4.9) to

$$\mathbf{v}^m \to \mathbf{v} \qquad \text{in } L^r_\sigma(\Omega) \qquad \forall\, r < 9.$$

Now we consider the difference of (4.2) and (4.13) and localize as in Section 3, taking into account (4.12). Given the bump function as in (3.11) we define

$$\mathbf{w}^m := \eta\,(\mathbf{v}^m - \mathbf{v}) - \mathrm{Bog}_{2B}(\nabla\eta\cdot(\mathbf{v}^m - \mathbf{v})) \in W^{1,3}_{0,\sigma}(2B) \subset W^{1,3}_{0,\sigma}(\Omega),$$

and we have, due to the $W^{1,3}_{loc}(\Omega)$-bounds from cf. (4.12), that the same convergence as in (3.12)-(3.13)-(3.14) holds true. Now we test the difference between the ε_m-regularized system and the original one with $\mathbf{w}^m \in W^{1,3}_{0,\sigma}(\Omega)$ and, by using the same argument as before, we get

$$\lim_{m\to+\infty}\int_B \big(d\,|\boldsymbol{\omega}^m|\boldsymbol{\omega}^m - d|\boldsymbol{\omega}|\boldsymbol{\omega}\big)\cdot\big(\boldsymbol{\omega}^m - \boldsymbol{\omega}\big)\,\mathrm{d}\mathbf{x} = 0.$$

This can be used to show that

$$\boldsymbol{\omega}^m \to \boldsymbol{\omega} \quad \text{in} \quad L^3(B),$$

and since the ball $B \Subset \Omega$ is arbitrary, this implies $\overline{\mathbf{S}} = d\,|\boldsymbol{\omega}|\boldsymbol{\omega}$.

We finally prove the energy-type balance. We observe that the equality

$$\int_\Omega (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\varphi} + d\,|\boldsymbol{\omega}|\boldsymbol{\omega} \cdot \operatorname{curl} \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} = -\int_\Omega \mathbf{F} \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x},$$

by density makes sense also for $\boldsymbol{\varphi} \in W^{1,3}_{0,\sigma}(\Omega, d)$, being the integrals well defined by the following estimates for $q = \frac{9}{7} < \frac{3}{2}$

$$\left| \int_\Omega (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \right| \leq \|\nabla \mathbf{v}\|_q \|\mathbf{v}\|_9 \|\boldsymbol{\varphi}\|_9 \leq c \|\mathbf{v}\|^2_{W^{1,3}_0(\Omega,d)} \|\boldsymbol{\varphi}\|_{W^{1,3}_0(\Omega,d)},$$

$$\left| \int_\Omega d\,|\boldsymbol{\omega}|\boldsymbol{\omega} \cdot \operatorname{curl} \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \right| = \left| \int_\Omega d^{2/3}\,|\boldsymbol{\omega}|\boldsymbol{\omega} \cdot d^{1/3} \operatorname{curl} \boldsymbol{\varphi} \right| \leq c \|\mathbf{v}\|^2_{W^{1,3}_0(\Omega,d)} \|\boldsymbol{\varphi}\|_{W^{1,3}_0(\Omega,d)},$$

$$\left| \int_\Omega \mathbf{F} \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \right| \leq \|\mathbf{F}\|_{3/2, d^{-1/2}} \|\boldsymbol{\varphi}\|_{W^{1,3}_0(\Omega,d)}.$$

Note that we used again (2.8) with $p = 3$ and $\delta = \frac{1}{3}$. Hence, by setting $\boldsymbol{\varphi} = \mathbf{v}$ and by observing that

$$\int_\Omega (\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} = 0,$$

once it is well-defined, we get the claimed energy equality. □

Remark 4.8. *Since the convergence is based on local $W^{1,3}$-estimates, the convergence of the stress tensor does not depend on the power of the distance, while the range of α is crucial to handle the convective term and to give a proper meaning to the equations in the sense of distributions.*

5. Extension to more general cases

In this section we consider the same problem as in (4.1) but we consider different values of both the exponent p and of the weight α. Some results follow in a straightforward way since $p = 3$ (the main argument of monotonicity requires in fact $p > \frac{9}{5}$, while others for smaller values of p require a more technical argument with a Lipschitz truncation of the test functions).

5.1. Generalization to other values of the parameter α, but still with $p = 3$.

We consider now the possible extension to larger values of the parameter $1 \leq \alpha < 2$. As explained before the value $\alpha = 2 = 3 - 1$ is critical as it does not allow to bound the weighted gradient by the weighted curl. We study now the system

$$\begin{cases} \boldsymbol{\omega} \times \mathbf{v} + \operatorname{curl}\big(d^\alpha |\boldsymbol{\omega}|\boldsymbol{\omega}\big) + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

We write just the a priori estimates, since the approximation and the passage to the limit is exactly the same as in Theorem 4.6 being based on local estimates for the gradient in $L^3(K)$.

From the Hölder inequality we get for $1 \leq \alpha < 2$ and if $\frac{\alpha q}{3-q} < 1$ (which holds if $1 \leq q < \frac{3}{1+\alpha}$) that

$$\|\nabla \mathbf{v}\|_q^3 \leq c \int_\Omega d^\alpha \, |\nabla \mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \qquad \forall\, \mathbf{v} \in W^{1,3}(\Omega, d^\alpha),$$

Next, the Sobolev embedding from Lemma 2.3 yields

$$\Big\| \mathbf{v}(\mathbf{x}) - \fint_\Omega \mathbf{v}(\mathbf{y}) \, \mathrm{d}\mathbf{y} \Big\|_{9/\alpha}^3 \leq C \int_\Omega d^\alpha |\nabla \mathbf{v}|^3 \, \mathrm{d}\mathbf{x} \qquad \forall\, \mathbf{v} \in W^{1,3}(\Omega, d^\alpha).$$

At this point the convective term satisfies

$$(\nabla \mathbf{v})\mathbf{v} \in L^s(\Omega) \qquad \forall\, s < \frac{9}{3 + 4\alpha},$$

and $s \geq 1$ if $\alpha < \frac{3}{2}$. Under these assumptions the proof follows as before and we can prove the following result where we distinguish two cases depending if α is small enough to allow the solution to have a proper sense. A different formulation for the larger values of α. We write results in the terms of $\mathbf{F}$ such that $\mathbf{f} = \operatorname{div} \mathbf{F}$, but this can be translated in terms of $\mathbf{f}$ only, again using [28] and (2.5).

Theorem 5.1. *(a) Let $\alpha < \frac{6}{5}$ and suppose that $\mathbf{f} = \operatorname{div} \mathbf{F}$ for some $\mathbf{F} \in L^{3/2}(\Omega, d^{-\alpha/2})$. Then, there exists a weak solution $\mathbf{v} \in W^{1,3}_{0,\sigma}(\Omega, d^\alpha)$ of the problem (4.1) such that*

$$\int_\Omega (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\varphi} + d^\alpha \, |\boldsymbol{\omega}| \boldsymbol{\omega} \cdot \operatorname{curl} \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} = -\int_\Omega \mathbf{F} \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \qquad \forall\, \boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega),$$

and

$$\int_\Omega d^\alpha \, |\boldsymbol{\omega}|^3 \, \mathrm{d}\mathbf{x} = -\int_\Omega \mathbf{F} \cdot \nabla \mathbf{v} \, \mathrm{d}\mathbf{x}.$$

(b) *Let* $\frac{6}{5} \leq \alpha < \frac{3}{2}$ *and suppose that* $\mathbf{f} = \operatorname{div} \mathbf{F}$ *with* $\mathbf{F} \in L^{3/2}(\Omega, d^{-\alpha/2})$. *Then, there exists a weak solution* $\mathbf{v} \in W^{1,3}_{0,\sigma}(\Omega, d^\alpha)$ *of the problem* (4.1) *such that*

$$\int_\Omega (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\varphi} + d^\alpha \, |\boldsymbol{\omega}| \boldsymbol{\omega} \cdot \operatorname{curl} \boldsymbol{\varphi} \, d\mathbf{x} = -\int_\Omega \mathbf{F} \cdot \nabla \boldsymbol{\varphi} \, d\mathbf{x} \qquad \forall \, \boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega).$$

Proof. The proof follows exactly the same lines of that of Theorem 4.3. We observe that in order to use $\mathbf{v}$ itself as test function, hence to cancel the convective term, we need for instance the estimate

$$\Big| \int_\Omega (\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{v} \, d\mathbf{x} \Big| \leq \|\boldsymbol{\omega}\|_{3/(1+\alpha)-\varepsilon} \|\mathbf{v}\|^2_{9/\alpha} \quad \text{for some } \varepsilon > 0,$$

which holds true if $\frac{1+\alpha}{3} + \frac{2\alpha}{9} < 1$ or, equivalently, if $\alpha < \frac{6}{5}$.

In the other case, the convective term is still in $L^1(\Omega)$, but the function $\mathbf{v}$ is not regular enough to be used globally as test function and to write the energy-type estimate. □

We consider now even larger values of α and we observe that for all $0 < \alpha < 2$ it holds true that,

$$\mathbf{v} \otimes \mathbf{v} \in L^{\frac{9}{2\alpha}}(\Omega) \subset L^{\frac{9}{4}}(\Omega) \subset L^1(\Omega),$$

hence, we can reformulate the problem with the convective term written as follows

$$(\nabla \mathbf{v})\mathbf{v} = \operatorname{div} (\mathbf{v} \otimes \mathbf{v}),$$

and consider the following notion of weak solution

Definition 5.2. *We say that* $\mathbf{v} \in W^{1,3}_{0,\sigma}(\Omega, d^\alpha)$ *is a weak solution to* (5.1) *if*

$$-\int_\Omega \mathbf{v} \otimes \mathbf{v} : \nabla \boldsymbol{\varphi} + d^\alpha \, |\boldsymbol{\omega}| \boldsymbol{\omega} \cdot \operatorname{curl} \boldsymbol{\varphi} \, d\mathbf{x} = -\int_\Omega \mathbf{F} \cdot \nabla \boldsymbol{\varphi} \, d\mathbf{x} \qquad \forall \, \boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega).$$

A similar argument can be used also to prove the following result, changing the notion of weak solution.

Theorem 5.3. *Let* $0 \leq \alpha < 2$ *and suppose that* $\mathbf{f} = \operatorname{div} \mathbf{F}$ *with* $\mathbf{F} \in L^{3/2}(\Omega, d^{-\alpha/2})$. *Then, there exists a weak solution* $\mathbf{v} \in W^{1,3}_{0,\sigma}(\Omega, d^\alpha)$ *of the problem* (4.1) *in the sense of Definition 5.2.*

Remark 5.4. *The same reasoning can be used to handle the problem* (5.2) *below with* $\frac{9}{5} < p < 3$ *and any* $\alpha < p - 1$. *The important observation is that we still have* $\mathbf{v} \in W^{1,p}_\sigma(K)$ *for all* $K \Subset \Omega$ *and hence* $\mathbf{v} \otimes \mathbf{v} \in L^{p^*/2}_{Loc}(\Omega)$. *The convergence of the nonlinear stress tensor follows in the same way as before as well.*

5.2. Extension to values of p smaller than $\frac{9}{5}$

We now study what happens in the case of a model with smaller values of p, hence we consider the generic system

$$
\begin{cases}
\operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \operatorname{curl}\left(d^\alpha|\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}\right) + \nabla\pi = \operatorname{div}\mathbf{F} & \text{in } \Omega,\\
\operatorname{div}\mathbf{v} = 0 & \text{in } \Omega,\\
\mathbf{v} = \mathbf{0} & \text{on } \partial\Omega,
\end{cases}
\tag{5.2}
$$

with $1 < p < 3$ and $0 \le \alpha < p - 1$.

Definition 5.5. *We say that* $\mathbf{v} \in W^{1,p}_{0,\sigma}(\Omega, d^\alpha)$ *is a weak solution to* (5.2) *if*

$$
-\int_\Omega \mathbf{v}\otimes\mathbf{v} : \nabla\boldsymbol{\varphi} + d^\alpha\,|\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}\cdot\operatorname{curl}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = -\int_\Omega \mathbf{F}\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} \qquad \forall\,\boldsymbol{\varphi}\in C^\infty_{0,\sigma}(\Omega).
$$

We obtain the following result

Theorem 5.6. *Let* $p > \frac{6}{5}$, $0 \le \alpha < p - 1$, *and suppose that* $\mathbf{f} = \operatorname{div}\mathbf{F}$ *with* $\mathbf{F} \in L^{p'}(\Omega, d^{-\alpha/(p-1)})$. *Then, there exists a weak solution* $\mathbf{v} \in W^{1,p}_{0,\sigma}(\Omega, d^\alpha)$ *of the problem* (5.2) *in the sense of Definition 5.5.*

Proof. As before in the previous proofs we regularize (5.2) and consider the system

$$
\begin{cases}
-\varepsilon\operatorname{div}|\mathbf{D}\mathbf{v}_\varepsilon|^{p-2}\mathbf{D}\mathbf{v}_\varepsilon + \operatorname{div}(\mathbf{v}_\varepsilon\otimes\mathbf{v}_\varepsilon) & \\
\qquad +\operatorname{curl}\left(d^\alpha|\boldsymbol{\omega}_\varepsilon|^{p-2}\boldsymbol{\omega}_\varepsilon\right) + \nabla\pi = \operatorname{div}\mathbf{F} & \text{in } \Omega,\\
\operatorname{div}\mathbf{v}_\varepsilon = 0 & \text{in } \Omega,\\
\mathbf{v}_\varepsilon = \mathbf{0} & \text{on } \partial\Omega,
\end{cases}
\tag{5.3}
$$

and we can follow the same procedure to prove existence of the approximate system, at least for $p > 6/5$, following the approach from Málek and Steinhauer *et al.* [19, 23]. Also, we obtain uniform estimate

$$
\varepsilon\|\mathbf{v}_\varepsilon\|^3_{W^{1,p}} + \int_\Omega d^\alpha\,|\boldsymbol{\omega}_\varepsilon|^p\,\mathrm{d}\mathbf{x} \le C(\Omega, \mathbf{F}),
$$

which yields

$$
\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v} \qquad \text{in } W^{1,q}_{0,\sigma}(\Omega) \qquad \forall\, q < \frac{p}{\alpha+1} \tag{5.4}
$$

$$
\mathbf{v}_\varepsilon \to \mathbf{v} \qquad \text{in } L^r_\sigma(\Omega) \qquad \forall\, r < \frac{3p}{3\alpha+3-p} \tag{5.5}
$$

$$
\mathbf{v}_\varepsilon \to \mathbf{v} \qquad \text{a.e. in } \Omega, \tag{5.6}
$$

$$
\varepsilon|\mathbf{D}\mathbf{v}_\varepsilon|^{p-2}\mathbf{D}\mathbf{v}_\varepsilon \to \mathbf{0} \qquad \text{in } L^{p'}(\Omega). \tag{5.7}
$$

$$
(\nabla\mathbf{v}_\varepsilon)_{|K} \rightharpoonup \nabla\mathbf{v}_{|K} \qquad \text{in } L^p(K) \qquad \forall\, K \Subset \Omega, \tag{5.8}
$$

$$
(\mathbf{v}_\varepsilon)_{|K} \to \mathbf{v}_{|K} \qquad \text{in } L^r(K) \qquad \forall\, r < \frac{3p}{3-p}. \tag{5.9}
$$

Based on the previous observations we obtain the limit system

$$\begin{cases} \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \operatorname{curl}\overline{\mathbf{S}} + \nabla\pi = \operatorname{div}\mathbf{F} & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega, \end{cases}$$

where the first equation has to be understood in the sense of distributions. Here the limit is taken along some sequence $\varepsilon_m \to 0$ and for simplicity we set

$$\mathbf{v}^m := \mathbf{v}_{\varepsilon_m} \qquad \text{and} \qquad \boldsymbol{\omega}^m := \boldsymbol{\omega}_{\varepsilon_m}.$$

Here $\overline{\mathbf{S}}$ denotes the weak limit of $d^\alpha|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m$ which exists in $L^{p'}_{loc}(\Omega)$. The remaining effort is to show that $\overline{\mathbf{S}} = d^\alpha|\boldsymbol{\omega}|\boldsymbol{\omega}$, i.e.

$$\big\langle d^\alpha|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m, \operatorname{curl}(\boldsymbol{\varphi})\big\rangle \to \big\langle d^\alpha|\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}, \operatorname{curl}(\boldsymbol{\varphi})\big\rangle \quad \forall\,\boldsymbol{\varphi}\in C^\infty_{0,\sigma}(\Omega). \tag{5.10}$$

It suffices to prove that $\boldsymbol{\omega}^m \to \boldsymbol{\omega}$ almost everywhere. This follows from the strict monotonicity of the operator $\boldsymbol{\xi} \mapsto |\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi}$ provided that for a certain $\theta \in (0,1]$ and every ball $B \subset \Omega$ with $4B \subset \Omega$

$$\limsup_{m\to\infty} \int_B \Big(\big(|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m - |\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}\big)\cdot(\boldsymbol{\omega}^m - \boldsymbol{\omega})\Big)^\theta \, d\mathbf{x} = 0\,. \tag{5.11}$$

To verify equation (5.11), let $\eta \in C^\infty_0(2B)$ be as in (3.11), with B now such that $4B \Subset \Omega$. Define

$$\mathbf{w}^m := \eta\,(\mathbf{v}^m - \mathbf{v}) - \operatorname{Bog}_{2B}(\nabla\eta\cdot(\mathbf{v}^m - \mathbf{v})),$$

where Bog_{2B} is the Bogovskiĭ operator on $2B$ from $L^p_0(2B)$ to $W^{1,p}_0(2B)$. Since $\nabla\eta\cdot(\mathbf{v}^m - \mathbf{v})$ is bounded in $L^p_0(2B)$ by (5.9), we have that $\mathbf{w}^m$ is bounded in $W^{1,p}_{0,\sigma}(2B)$. Moreover, $\mathbf{v}^m \to \mathbf{v}$ in $L^2(2B)$ and the continuity of Bog_{2B} implies $\mathbf{w}^m \to \mathbf{0}$ at least in $L^1(2B)$. In particular, we can apply the solenoidal Lipschitz truncation of Theorem 2.10 to construct a suitable double sequence $\mathbf{w}^{m,j} \in W^{1,\infty}_{0,\sigma}(4B)$.

We use now $\mathbf{w}^{m,j}$ as a test function in (5.3) and obtain

$$\begin{aligned} \langle d^\alpha|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m - d^\alpha|\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}, \operatorname{curl}(\mathbf{w}^{m,j})\rangle &= -\langle d^\alpha|\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}, \operatorname{curl}(\mathbf{w}^{m,j})\rangle \\ &\quad - \varepsilon_m\langle|\mathbf{D}\mathbf{v}^m|^{p-2}\mathbf{D}\mathbf{v}^m, \mathbf{D}\mathbf{w}^{m,j})\rangle + \langle\mathbf{F}, \nabla\mathbf{w}^{m,j}\rangle \\ &\quad + \langle\mathbf{v}^m\otimes\mathbf{v}^m, \nabla\mathbf{w}^{m,j}\rangle. \end{aligned}$$

It follows from the properties of $\mathbf{w}^{m,j}$ and $\mathbf{v}^m$ that the right-hand side converges for fixed j to zero as $m \to \infty$. So we get

$$\lim_{m\to\infty} \langle d^\alpha|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m - d^\alpha|\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}, \operatorname{curl}(\mathbf{w}^{m,j})\rangle = 0.$$

We decompose the set $4B$ into $\{\mathbf{w}^m \neq \mathbf{w}^{m,j}\}$ and $4B \cap \{\mathbf{w}^m = \mathbf{w}^{m,j}\}$ to get

$$\begin{aligned}
(I) :=& \left| \int\limits_{4B\cap\{\mathbf{w}^n=\mathbf{w}^{m,j}\}} \eta\, d^\alpha \left(|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m - |\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}\right) \cdot (\boldsymbol{\omega}^m - \boldsymbol{\omega})\, \mathrm{d}\mathbf{x} \right| \\
=& \left| \int\limits_{\{\mathbf{w}^n\neq\mathbf{w}^{m,j}\}} d^\alpha \left(|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m - |\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}\right) \cdot \mathrm{curl}\,(\mathbf{w}^{m,j})\, \mathrm{d}\mathbf{x} \right| \\
&+ \left| \int\limits_{4B\cap\{\mathbf{w}^n=\mathbf{w}^{m,j}\}} d^\alpha \left(|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m - |\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}\right) \cdot \big(\nabla\eta \times (\mathbf{v}^m - \mathbf{v})\big)\, \mathrm{d}\mathbf{x} \right| \\
&+ \left| \int\limits_{4B\cap\{\mathbf{w}^n=\mathbf{w}^{m,j}\}} d^\alpha \left(|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m - |\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}\right) \cdot \mathrm{curl}\,\big(\mathrm{Bog}_{2B}(\nabla\eta \cdot (\mathbf{v}^m - \mathbf{v}))\big)\, \mathrm{d}\mathbf{x} \right| \\
=:& \ (II) + (III) + (IV).
\end{aligned}$$

Since $\nabla\eta \otimes (\mathbf{v}^m - \mathbf{v}) \overset{m}{\to} 0$ in $L^p(2B)$, we have $(III) + (IV) \overset{m}{\to} 0$, recall (5.8) and (5.9). Note that we also used the continuity of Bog_{2B} from $L^p_0(2B)$ to $W^{1,p}_0(2B)$.

By Hölder's inequality, (5.8) and Theorem 2.10-(e)

$$\begin{aligned}
(II) &\leq \limsup_{m\to+\infty} \left(\|\boldsymbol{\omega}^m\|_{p'} + \|\boldsymbol{\omega}\|_{p'}\right) \|\chi_{\{\mathbf{w}^n\neq\mathbf{w}^{m,j}\}} \nabla\mathbf{w}^{m,j}\|_p \\
&\leq c 2^{-j/p} \|\nabla\mathbf{w}^m\|_p \leq c 2^{-j/p}.
\end{aligned}$$

Overall we get

$$\limsup_{m\to+\infty} \left| \int\limits_{4B\cap\{\mathbf{w}^m=\mathbf{w}^{m,j}\}} \eta\, d^\alpha \left(|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m - |\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}\right) \cdot (\boldsymbol{\omega}^m - \boldsymbol{\omega})\, \mathrm{d}\mathbf{x} \right| \leq c\, 2^{-j/p}.$$

This implies

$$\limsup_{m\to+\infty} \int\limits_{4B} \left(\eta\, d^\alpha \left(|\boldsymbol{\omega}^m|^{p-2}\boldsymbol{\omega}^m - |\boldsymbol{\omega}|^{p-2}\boldsymbol{\omega}\right) \cdot (\boldsymbol{\omega}^m - \boldsymbol{\omega})\right)^\theta \mathrm{d}\mathbf{x} = 0$$

for any $\theta \in (0,1)$ as a consequence of (5.8) and Theorem 2.10-(e). Now, (5.11) follows form $\eta \geq \chi_B$ and $d \geq C_B > 0$ in B. So we obtain (5.10) as desired, which finishes the proof. □

Remark 5.7. *We are not considering here problems of regularity of the weak solutions and also of less regular weight functions as in the recent studies by Cirmi, D'Asero, and Leonardi [18]. Moreover, as it is the case for similar problems, uniqueness for the system (4.1) is not known, even for small enough solutions. Uniqueness of small solutions to (3.1) follows* *directly by the same results for the Navier-Stokes equations, as explained in Galdi [26]. On the other hand uniqueness of small solutions –even for the regularized system* (5.3)*– is not known for $p > 2$ or for $p < \frac{9}{5}$, see Blavier and Mikelić [9] and the review in Galdi [25].*

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