On global well-posedness for nonlinear semirelativistic equations in some scaling subcritical and critical cases

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Abstract

In this paper, the global well-posedness of semirelativistic equations with a power type nonlinearity on Euclidean spaces is studied. In two dimensional H^s scaling subcritical case with $1 \leq s \leq 2$, the local well-posedness follows from a Strichartz estimate. In higher dimensional H^1 scaling subcritical case, the local well-posedness for radial solutions follows from a weighted Strichartz estimate. Moreover, in three dimensional H^1 scaling critical case, the local well-posedness for radial solutions follows from a uniform bound of solutions which may be derived by the corresponding one dimensional problem. Local solutions may be extended by a priori estimates.

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1. Introduction

$$\begin{cases} i\partial_t u - (-\Delta)^{1/2}u = -i|u|^{p-1}u, & t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \\ u(0) = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $n \geq 1$, $p > 1$, Δ is the Laplacian, and $(-\Delta)^{1/2} = \mathfrak{F}^{-1}|\xi|\mathfrak{F}$ with the Fourier transform $\mathfrak{F}$.

Similar models can be connected with the simulations of neuroscience processes. Typical one is the cyclical alternation of REM (rapid eye movement) and NREM (non-rapid eye movement) sleep. See [1]. The model of the alternation of REM and NREM sleep is starting from the classical Kuramoto model [17], having its origin in special type of Landau - Ginzburg model

$$i\partial_t u - \mathcal{H}u = -iQ(u), \tag{2}$$

where $\mathcal{H}$ is appropriate Hamiltonian operator and $Q(u)$ is appropriate cubic type nonlinearity. In this work we substitute the specific cubic nonlinearity $Q(u)$ by a self - interacting nonlinear term $|u|^{p-1}u$ and our goal is to implement the recent development of fractional quantum mechanical approach (see [18]) based on the choice of $\mathcal{H} = D = (-\Delta)^{1/2}$ as a Hamiltonian of the process.

We shall observe some new interesting phenomena. On one hand, the contraction of some Sobolev norms of the solutions to (1) is manifested only for positive time $t > 0$, and therefore we have a similarity to a diffusion type process.

The Cauchy problem for (1) has different conserved (or bounded) quantities that can be compared with the classical NLS with self interaction term

$$\begin{cases} i\partial_t u - (-\Delta)^{1/2}u = -|u|^{p-1}u, & t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \\ u(0) = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{3}$$

Indeed, natural Sobolev norm that can be controlled for (3) is $H^{1/2}(\mathbb{R}^n)$, while (1) enables one to control $H^1(\mathbb{R}^n)$ norm but only in the future time instants $t > 0$.

For negative time some blow up phenomena are discussed in [9], [7].

To state our main result, we turn to the introduction of the notations used below. For a Banach space X and $1 \leq p \leq \infty$ let $L^p(\mathbb{R}^n; X)$ be a X-valued Lebesgue space of p-th power. We abbreviate $L^p(\mathbb{R}^n; \mathbb{C})$ as $L^p(\mathbb{R}^n)$. For $f, g \in L^2(\mathbb{R}^n)$, we define a inner product as

$$\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x)\overline{g}(x)dx.$$

For $s \in \mathbb{R}$, let $H^s(\mathbb{R}^n)$ be the usual inhomogeneous Sobolev space defined as $H^s(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^2(\mathbb{R}^n)$. Let $\dot{H}^s(\mathbb{R}^n)$ be the usual homogeneous Sobolev space defined as $\dot{H}^s(\mathbb{R}^n) = (-\Delta)^{-s/2}L^2(\mathbb{R}^n)$. For $f, g : A \to [0, \infty)$ with a set A, $f \lesssim g$ means there exists $C > 0$ for any $a \in A$ such that $f(a) \le Cg(a)$. For Banach spaces X, Y, $Y \hookrightarrow X$ means $Y \subset X$ with continuous embedding. Moreover, we say a Cauchy problem is locally well-posed in X, if for any X-valued initial data, there exists $T > 0$ and a Banach space $Y \hookrightarrow C([0,T]; X)$ such that there is a unique solution to the Cauchy problem in Y and $\|u_n - u\|_Y \to 0$ as $\|u_{0,n} - u_0\|_X \to 0$, where u_n and u are solutions for the Cauchy problem for initial data u_0 and $u_{0,n}$, respectively. We also say a Cauchy problem is globally well-posed in X if the Cauchy problem is locally well-posed for any $T > 0$. Moreover, we also say a Cauchy problem is globally well-posed in X with sufficiently small data, if we have the property above for sufficiently small X-valued data.

The equation (1) is invariant under the scale transformation

$$u_\lambda(t,x) = \lambda^{1/(p-1)} u(\lambda t, \lambda x)$$

with $\lambda > 0$. Then

$$\|u_{0,\sigma}\|_{\dot{H}^s(\mathbb{R}^n)} = \sigma^{1/(p-1)+s-n/2} \|u_0\|_{\dot{H}^s(\mathbb{R}^n)}$$

and with

$$s = s_{n,p} := n/2 - 1/(p-1) < n/2,$$

$\dot{H}^s$ norm of initial data is also invariant, for this $s_{n,p}$ is called scale critical exponent. We also call $p_{n,s} = 1 + 2/(n-2s)$ the $H^s(\mathbb{R}^n)$ scaling critical power. For any s, in the scaling subcritical case where $p < p_{n,s}$, (1) is expected to have local solution for any $H^s(\mathbb{R}^n)$ initial data on the analogy of scaling invariant Schrödinger equation. For instance, we refer the reader [3, 5, 4, 13, 14]. However, with power type nonlinearity without gauge invariance, semirelativistic equations may not be locally well-posed even in scaling subcritical case, see [8].

Proposition 1.1. *Let* $n = 1$. *For* $p > 1$ *the Cauchy problem* (1) *is globally well-posed in* $H^1(\mathbb{R}^1)$. *Moreover, for* $1 < s < \min(2, p)$ *or* $p = 3$, $1 < s \le 2$, *the Cauchy problem* (1) *is globally well-posed in* $H^s(\mathbb{R}^1)$.

Proposition 1.2. *Let* $n = 2$. *For* $p > 1$ *and* $3/4 < s < p < p_{2,s}$, *the Cauchy problem* (1) *is locally well-posed in* $H^s(\mathbb{R}^2)$. *Moreover, for* $p > 1$, *the Cauchy problem* (1) *is globally well-posed in* $H^1(\mathbb{R}^2)$. *For* $p = 3$, *the Cauchy problem* (1) *is globally well-posed in* $H^s(\mathbb{R}^2)$ *with* $1 < s \le 2$.

Proposition 1.3. *Let* $n \ge 3$ *and* u_0 *be radial. For* $1 < p < p_{n,1} = 1 + \frac{2}{n-2}$, *the Cauchy problem* (1) *is globally well-posed in* $H^1_{\mathrm{rad}}(\mathbb{R}^n)$.

Proposition 1.4. *Let* $n = 3$ *and* u_0 *be radial. For* $p = p_{3,1} = 3$, *the Cauchy problem of* (1) *is globally well-posed in* $H^1_{\mathrm{rad}}(\mathbb{R}^3)$ *with sufficiently small* $H^1_{\mathrm{rad}}(\mathbb{R}^3)$ *data.*

For three dimensional case $p = 3$ is a critical value in view of the result in [15]. However, the result in [15] treats nongauge invariant nonlinearities having constant sign, for which the test function method works. The question of the existence of local and global solutions for $n \geq 3$ and $p \geq 1 + 2/(n-2)$ seems still open.

This paper is organized as follows: In section 2, we collect a priori estimates for (1). In section 3, we prove Propositions 1.1, 1.2, 1.3, and 1.4. In one dimensional case, local well-posedness follows from a standard contraction argument. In the case where $n = 2$, local well-posedness follows from the Strichartz estimate derived by Nakamura and one of the authors in [19]. However, with this Strichartz estimate, we may control solutions uniformly only in the $H^s(\mathbb{R}^n)$ setting with $s > (n+1)/4$. We remark that it seems difficult to obtain the local well-posedness if $s \leq (n+1)/4$ by a simple application of an improved Strichartz estimate for radial solutions derived by Guo and Wang in [12]. In the case where $n \geq 3$, a weighted Sobolev space derived by Bellazzini, Visciglia, and one of the author in [2], and uniform controls derived by Sickel and Skrzypczak in [21] (see also [6]) play an critical role to prove local well-posedness. Moreover, in the 3 dimensional scaling critical case where $p = 3$, we obtain a uniform control of solutions by transforming (1) into the corresponding 1 dimensional problem.

2. A priori estimates

$$u(t) = U(t)u_0 - \int_0^t U(t-t')|u(t')|^{p-1}u(t')dt', \tag{4}$$

where $U(t) = e^{-itD}$ and $D = (-\Delta)^{1/2}$.

Proposition 2.1. *Let $n \in \mathbb{N}$ and $p > 1$. Let $u_0 \in L^2(\mathbb{R}^n)$ and $T > 0$. Let $u \in L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^p(0,T;L^{2p}(\mathbb{R}^n))$ be a solution to the integral equation (4) for the initial data u_0. Then, for any t_1, t_2 with $0 < t_1 < t_2 < T$,*

$$\|u(t_2)\|^2_{L^2(\mathbb{R}^n)} + 2\|u\|^{p+1}_{L^{p+1}(t_1,t_2;L^{p+1}(\mathbb{R}^n))} = \|u(t_1)\|^2_{L^2(\mathbb{R}^n)}.$$

Proof. A formal computation yields immediately the proposition. However, actual proof requires some regularization procedure to justify the formal calculation. Here we give a direct proof based on the integral equation on the basis of the method in [20].

$$
\begin{aligned}
&\langle u(t_2), u(t_2)\rangle_{L^2(\mathbb{R}^n)} \\
&= \left\langle U(t_2-t_1)u(t_1) - \int_{t_1}^{t_2} U(t_2-t)|u(t)|^{p-1}u(t)dt, u(t_2)\right\rangle_{L^2(\mathbb{R}^n)} \\
&= \|u(t_1)\|^2_{L^2(\mathbb{R}^n)} - 2\mathrm{Re}\left\langle U(t_2-t_1)u(t_1), \int_{t_1}^{t_2} U(t_2-t)|u(t)|^{p-1}u(t)dt\right\rangle_{L^2(\mathbb{R}^n)} \\
&+ \left\langle \int_{t_1}^{t_2} U(t_2-t)|u(t)|^{p-1}u(t)dt, \int_{t_1}^{t_2} U(t_2-t')|u(t')|^{p-1}u(t')dt'\right\rangle_{L^2(\mathbb{R}^n)} \\
&= \|u(t_1)\|^2_{L^2(\mathbb{R}^n)} - 2\mathrm{Re}\left\langle U(t_2-t_1)u(t_1), \int_{t_1}^{t_2} U(t_2-t)|u(t)|^{p-1}u(t)dt\right\rangle_{L^2(\mathbb{R}^n)} \\
&+ 2\mathrm{Re}\int_{t_1}^{t_2} \left\langle |u(t)|^{p-1}u(t), \int_{t_1}^{t} U(t-t')|u(t')|^{p-1}u(t')dt'\right\rangle_{L^2(\mathbb{R}^n)} dt \\
&= \|u(t_1)\|^2_{L^2(\mathbb{R}^n)} - 2\mathrm{Re}\left\langle U(t_2-t_1)u(t_1), \int_{t_1}^{t_2} U(t_2-t)|u(t)|^{p-1}u(t)dt\right\rangle_{L^2(\mathbb{R}^n)} \\
&+ 2\mathrm{Re}\int_{t_1}^{t_2} \left\langle |u(t)|^{p-1}u(t), U(t-t_1)u(t_1) - u(t)\right\rangle_{L^2(\mathbb{R}^n)} dt \\
&= \|u(t_1)\|^2_{L^2(\mathbb{R}^n)} - 2\|u\|^{p+1}_{L^{p+1}(t_1,t_2;L^{p+1}(\mathbb{R}^n))}.
\end{aligned}
$$

□

Proposition 2.2. *Let $n \in \mathbb{N}$ and $p > 1$. Let $u_0 \in H^1(\mathbb{R}^n)$ and $T > 0$. Let $u \in L^\infty(0,T;H^1(\mathbb{R}^n)) \cap L^{p-1}(0,T;L^\infty(\mathbb{R}^n))$ be a solution to the integral equation (4) for the initial data u_0. Then, for any t_1, t_2 with $0 \le t_1 < t_2 \le T$,*

$$
\begin{aligned}
&\|\nabla u(t_2)\|^2_{L^2(\mathbb{R}^n)} + 2\||u|^{\frac{p-1}{2}}\nabla u\|^2_{L^2(t_1,t_2;L^2(\mathbb{R}^n))} \\
&+ \frac{p-1}{2}\||u|^{\frac{p-3}{2}}\nabla|u|^2\|^2_{L^2(t_1,t_2;L^2(\mathbb{R}^n))} \\
&= \|\nabla u(t_1)\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{5}
$$

Proof. Since $|u|^{p-1}u \in L^1(0,T;H^1(\mathbb{R}^n))$,

$$
\begin{aligned}
&\|\nabla u(t_2)\|_{L^2(\mathbb{R}^n)}^2 \\
&= \|\nabla u(t_1)\|_{L^2(\mathbb{R}^n)}^2 - 2\mathrm{Re}\int_{t_1}^{t_2} \left\langle \nabla(|u(t)|^{p-1}u(t)), \nabla u(t)\right\rangle_{L^2(\mathbb{R}^n)} dt \\
&= \|\nabla u(t_1)\|_{L^2(\mathbb{R}^n)}^2 - 2\mathrm{Re}\int_{t_1}^{t_2} \left\langle \nabla|u(t)|^{p-1}, \overline{u(t)}\nabla u(t)\right\rangle_{L^2(\mathbb{R}^n)} dt \\
&- 2\int_{t_1}^{t_2} \left\langle |u(t)|^{p-1}\nabla u(t), \nabla u(t)\right\rangle_{L^2(\mathbb{R}^n)} dt \\
&= \|\nabla u(t_1)\|_{L^2(\mathbb{R}^n)}^2 - \frac{p-1}{2}\int_{t_1}^{t_2} \left\langle |u(t)|^{p-3}\nabla|u(t)|^2, \nabla|u(t)|^2\right\rangle_{L^2(\mathbb{R}^n)} dt \\
&- 2\int_{t_1}^{t_2} \left\langle |u(t)|^{p-1}\nabla u(t), \nabla u(t)\right\rangle_{L^2(\mathbb{R}^n)} dt.
\end{aligned}
$$

□

Proposition 2.3. *Let $n = 1, 2$, $p > 1$, $n/2 < s < \min(2, p)$, and $T > 0$. Let $u_0 \in H^s(\mathbb{R}^n)$ and $u \in L^\infty(0,T;H^s(\mathbb{R}^n)) \cap L^2(0,T;L^\infty(\mathbb{R}^n))$ be a solution to (4) for the initial data u_0. Then for any t_1, t_2 with $0 < t_1 < t_2 < T$,*

$$
\|u(t_2)\|_{\dot H^s(\mathbb{R}^n)}^2 \le \|u(t_1)\|_{\dot H^s(\mathbb{R}^n)}^2 + C\int_{t_1}^{t_2} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{p-1}\|u(t)\|_{\dot H^s(\mathbb{R}^n)}^2 dt.
$$

Proof.

$$
\begin{aligned}
&\|u(t_2)\|_{\dot H^s(\mathbb{R}^n)}^2 \\
&= \|u(t_1)\|_{\dot H^s(\mathbb{R}^n)}^2 - 2\mathrm{Re}\int_{t_1}^{t_2} \langle D^s(|u(t)|^{p-1}u(t)), D^s u(t)\rangle_{L^2(\mathbb{R}^n)} dt \\
&\le \|u(t_1)\|_{\dot H^s(\mathbb{R}^n)}^2 + 2\int_{t_1}^{t_2} \|D^s(|u(t)|^{p-1}u(t))\|_{L^2(\mathbb{R}^n)}\|u(t)\|_{\dot H^s(\mathbb{R}^n)} dt \\
&\le \|u(t_1)\|_{\dot H^s(\mathbb{R}^n)}^2 + C\int_{t_1}^{t_2} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{p-1}\|u(t)\|_{\dot H^s(\mathbb{R}^n)}^2 dt,
\end{aligned}
$$

where we use the nonlinear estimate

$$
\||f|^{p-1}f\|_{\dot H^s(\mathbb{R}^n)} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}^{p-1}\|f\|_{\dot H^s(\mathbb{R}^n)}
$$

(see [10, Lemma 3.4]). □

Proposition 2.4. *Let $1 \le n \le 3$, $u_0 \in H^2(\mathbb{R}^n)$ and $T > 0$. Let $u \in C((0,T);H^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ be a solution to the integral equation (4) for the* *initial data u_0. Then, for any t_1, t_2 with $0 < t_1 < t_2 < T$,*

$$\begin{aligned}
&\|u(t_2)\|^2_{\dot{H}^2(\mathbb{R}^n)} + 2\sum_{j,k=1}^{n}\int_{t_1}^{t_2}\|u(t)\partial_j\partial_k u(t)\|^2_{L^2(\mathbb{R}^n)}dt \\
&\leq \|u(t_1)\|^2_{\dot{H}^2(\mathbb{R}^n)} + 2n^2(n+1)\int_{t_1}^{t_2}\|u(t)\|^{4-n}_{\dot{H}^1(\mathbb{R}^n)}\|u(t)\|^{n}_{\dot{H}^2(\mathbb{R}^n)}dt. \qquad (6)
\end{aligned}$$

Proof. Since $|u|^2u \in C((0,T); H^2(\mathbb{R}^n))$, the following calculation is justified by the Plancherel identity:

$$\begin{aligned}
&\|u(t_2)\|^2_{\dot{H}^2(\mathbb{R}^n)} \\
&= \|u(t_1)\|^2_{\dot{H}^2(\mathbb{R}^n)} - 2\mathrm{Re}\int_{t_1}^{t_2}\left\langle \Delta|u(t)|^2u(t), \Delta u(t)\right\rangle_{L^2(\mathbb{R}^n)}dt \\
&= \|u(t_1)\|^2_{\dot{H}^2(\mathbb{R}^n)} - 2\mathrm{Re}\sum_{j,k=1}^{n}\int_{t_1}^{t_2}\left\langle |u(t)|^2\partial_j\partial_k u(t), \partial_j\partial_k u(t)\right\rangle_{L^2(\mathbb{R}^n)}dt \\
&- 4\mathrm{Re}\sum_{j,k=1}^{n}\int_{t_1}^{t_2}\left\langle \partial_k u(t)\partial_j|u(t)|^2, \partial_j\partial_k u(t)\right\rangle_{L^2(\mathbb{R}^n)}dt \\
&- 2\mathrm{Re}\sum_{j,k=1}^{n}\int_{t_1}^{t_2}\left\langle \partial_j\partial_k|u(t)|^2, \overline{u(t)}\partial_j\partial_k u(t)\right\rangle_{L^2(\mathbb{R}^n)}dt \\
&= \|u(t_1)\|^2_{\dot{H}^2(\mathbb{R}^n)} - 2\sum_{j,k=1}^{n}\int_{t_1}^{t_2}\|u(t)\partial_j\partial_k u(t)\|^2_{L^2(\mathbb{R}^n)}dt \\
&+ 2\sum_{j,k=1}^{n}\int_{t_1}^{t_2}\left\langle \partial_j^2|u(t)|^2, |\partial_k u(t)|^2\right\rangle_{L^2(\mathbb{R}^n)}dt \\
&- \sum_{j,k=1}^{n}\int_{t_1}^{t_2}\left\langle \partial_j\partial_k|u(t)|^2, \partial_j\partial_k|u(t)|^2 - 2\mathrm{Re}(\overline{\partial_j u(t)}\partial_k u(t))\right\rangle_{L^2(\mathbb{R}^n)}dt.
\end{aligned}$$

By the Young inequality,

$$\begin{aligned}
&\sum_{j,k=1}^{n}\left\langle \partial_j^2|u(t)|^2, |\partial_k u(t)|^2\right\rangle_{L^2(\mathbb{R}^n)} \\
&\leq \sum_{j,k=1}^{n}\|\partial_j^2|u(t)|^2\|_{L^2(\mathbb{R}^n)}\|\partial_k u(t)\|^2_{L^4(\mathbb{R}^n)} \\
&\leq \frac{1}{4}\sum_{j=1}^{n}\|\partial_j^2|u(t)|^2\|^2_{L^2(\mathbb{R}^n)} + n^2\sum_{k=1}^{n}\|\partial_k u(t)\|^4_{L^4(\mathbb{R}^n)} \\
&\leq \frac{1}{4}\sum_{j,k=1}^{n}\|\partial_j\partial_k|u(t)|^2\|^2_{L^2(\mathbb{R}^n)} + n^2\sum_{k=1}^{n}\|\partial_k u(t)\|^4_{L^4(\mathbb{R}^n)}.
\end{aligned}$$

Similarly,

$$\sum_{j,k=1}^{n} \left\langle \partial_j \partial_k |u(t)|^2, \mathrm{Re}(\overline{\partial_j u(t)} \partial_k u(t)) \right\rangle_{L^2(\mathbb{R}^n)}$$
$$\le \frac{1}{4} \sum_{j,k=1}^{n} \|\partial_j \partial_k |u(t)|^2\|_{L^2(\mathbb{R}^n)}^2 + \sum_{j,k=1}^{n} \|\partial_j u(t)\|_{L^4(\mathbb{R}^n)}^2 \|\partial_k u(t)\|_{L^4(\mathbb{R}^n)}^2.$$

Therefore, by the Sobolev inequality,

$$\|u(t_2)\|_{\dot{H}^2(\mathbb{R}^n)}^2$$
$$\le \|u(t_1)\|_{\dot{H}^2(\mathbb{R}^n)}^2 - 2 \sum_{j,k=1}^{n} \int_{t_1}^{t_2} \|u(t)\partial_j \partial_k u(t)\|_{L^2(\mathbb{R}^n)}^2 dt$$
$$+ 2n^2 \sum_{k=1}^{n} \int_{t_1}^{t_2} \|\partial_k u(t)\|_{L^4(\mathbb{R}^n)}^4 dt + 2 \sum_{j,k=1}^{n} \int_{t_1}^{t_2} \|\partial_j u(t)\|_{L^4(\mathbb{R}^n)}^2 \|\partial_k u(t)\|_{L^4(\mathbb{R}^n)}^2 dt$$
$$\le \|u(t_1)\|_{\dot{H}^2(\mathbb{R}^n)}^2 - 2 \sum_{j,k=1}^{n} \int_{t_1}^{t_2} \|u(t)\partial_j \partial_k u(t)\|_{L^2(\mathbb{R}^n)}^2 dt$$
$$+ 2n^2(n+1) \int_{t_1}^{t_2} \|u(t)\|_{\dot{H}^1(\mathbb{R}^n)}^{4-n} \|u(t)\|_{\dot{H}^2(\mathbb{R}^n)}^{n} dt.$$

□

3. Proof of the Propositions

3.1. 1 dimensional case

Since $H^2(\mathbb{R}) \hookrightarrow H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, Proposition 1.1 is obtained by a standard contraction argument with the Sobolev inequality and a priori estimates Propositions 2.1, 2.2, 2.3, and 2.4.

3.2. 2 dimensional case

Lemma 3.1 ([19, Lemma 2.1], [11, Remark 3.2])**.** *Let (q_1, r_1) and (q_2, r_2) satisfy*

$$\frac{1}{r_j} = \frac{1}{2} - \frac{2}{q_j}, \quad 2 \le r_j \le \infty, \quad 4 \le q_j \le \infty \tag{7}$$

for $j = 1, 2$. Then for $s \in \mathbb{R}$,

$$\|U(t)\phi\|_{L^{q_1}(0,T;B_{r_1}^{s-\frac{3}{q_1}}(\mathbb{R}^2))} \lesssim \|\phi\|_{H^s(\mathbb{R}^2)},$$
$$\left\| \int_0^t U(t-t')h(t')dt' \right\|_{L^{q_1}(0,T;B_{r_1}^{s-\frac{3}{q_1}}(\mathbb{R}^2))} \lesssim \|h\|_{L^{q_2'}(0,T;B_{r_2'}^{s+\frac{3}{q_2}}(\mathbb{R}^2))},$$

where $B_p^s(\mathbb{R}^2) = B_{p,2}^s(\mathbb{R}^2)$ is the usual inhomogeneous Besov space.

Lemma 3.2. *Let $r > 2$, and $T > 0$. If*

$$s > \frac{3}{4} + \frac{1}{2r},$$

then $B_r^{s-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$.

Proof. The lemma follows from the Sobolev embedding and

$$s - \frac{3}{2}\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{2}{r} > 0 \Longleftrightarrow s > \frac{3}{4} + \frac{1}{2r}.$$

□

Proof of Proposition 1.2.

Local well-posedness Let (q_1, r_1) satisfy the condition of Lemma 3.2, (7), and $q_1 > p-1$. We remark that there is such a pair under the assumption: $p < p_{2,s}$. Let $X^s(0,T) = L^\infty(0,T; H^s(\mathbb{R}^2)) \cap L^{q_1}(0,T; B_{r_1}^{s-3/2\cdot\alpha(r_1)}(\mathbb{R}^2))$. Let

$$\Phi(u)(t) = U(t)u_0 - \int_0^t U(t-t')|u(t')|^{p-1}u(t')dt'.$$

Then, for $0 < T < 1$,

$$\begin{aligned} \|\Phi(u)\|_{X^s(0,T)} &\leq \|u_0\|_{H^s(\mathbb{R}^2)} + C\||u|^{p-1}u\|_{L^1(0,T;H^s(\mathbb{R}^2))} \\ &\leq \|u_0\|_{H^s(\mathbb{R}^2)} + CT^{1-(p-1)/q_1}\|u\|_{X^s(0,T)}^p, \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\|\Phi(u) - \Phi(v)\|_{X^s(0,T)} \\ &\leq C\||u|^{p-1}u - |v|^{p-1}v\|_{L^1(0,T;H^s(\mathbb{R}^2))} \\ &\leq CT^{1-(p-1)/q_1}(\|u\|_{X^s(0,T)} + \|v\|_{X^s(0,T)})^{p-1}\|u-v\|_{X^s(0,T)} \\ &+ CT^{1-(p-1)/q_1}(\|u\|_{X^s(0,T)} + \|v\|_{X^s(0,T)})^{\max(1,p-1)}\|u-v\|_{X^s(0,T)}^{\min(1,p-1)}. \end{aligned}$$

This means if $T < T_0 := \left(2^{1+\max(1,p-1)}C(\|u_0\|_{H^s(\mathbb{R}^2)}^{p-1} + \|u_0\|_{H^s(\mathbb{R}^2)}^{\max(1,p-1)})\right)^{-q_1/(q_1-p+1)}$, then Φ is a map from

$$B_{X^s(0,T)}(2\|u_0\|_{H^s(\mathbb{R}^2)}) := \{f \in X^s(0,T) \mid \|f\|_{X^s(0,T)} \leq 2\|u_0\|_{H^s(\mathbb{R}^2)}\}.$$

to itself. Moreover, if $p \geq 2$, Φ is a contraction map in $X^s(0,T)$. If $p < 2$, since for $z_1, z_0 \in \mathbb{C}$ with $|z_1| > |z_0|$,

$$\begin{aligned} &\left||z_1|^{p-1}z_1 - |z_0|^{p-1}z_0\right| \\ &\leq |z_1|^{p-1}|z_1 - z_0| + \frac{1}{p-1}\int_0^1 (\theta|z_1| + (1-\theta)|z_0|)^{p-2}|z_0||z_1 - z_0| \\ &\leq |z_1|^{p-1}|z_1 - z_0| + \frac{1}{p-1}|z_0|^{p-1}|z_1 - z_0|, \end{aligned}$$

then

$$
\begin{aligned}
&\|\Phi(u)-\Phi(v)\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \qquad (9)\\
&\le C\||u|^{p-1}u-|v|^{p-1}v\|_{L^1(0,T;L^2(\mathbb{R}^2))}\\
&\le CT^{1-(p-1)/q_1}(\|u\|_{X^s(0,T)}+\|v\|_{X^s(0,T)})^{p-1}\|u-v\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}
\end{aligned}
$$

and therefore Φ is a contraction map in $L^\infty(0,T;L^2(\mathbb{R}^2))$. Let $u_1 \in B_{X^s(0,T)}(2\|u_0\|_{H^s(\mathbb{R}^2)})$ and $u_k = \Phi(u_{k-1})$ for $k \ge 2$. Then $(u_k)_{k=1}^\infty \subset B_{X^s(0,T)}(2\|u_0\|_{H^s(\mathbb{R}^2)})$ is a Cauchy sequence in $L^\infty(0,T;L^2(\mathbb{R}^2))$. Let u^* be the limit of $(u_k)_{k=1}^\infty$ in $L^\infty(0,T;L^2(\mathbb{R}^2))$. Since

$$\||f|^{p-1}g\|_{L^{2/p}(\mathbb{R}^n)} \le \|f\|_{L^2(\mathbb{R}^n)}^{p-1}\|g\|_{L^2(\mathbb{R}^n)}$$

for any $n\in\mathbb{N}$ and $f,g\in L^2(\mathbb{R}^n)$, $\Phi(u_k)\to\Phi(u^*)$ in $L^\infty(0,T;L^2(\mathbb{R}^2))$. Therefore u^* is a solution of (4). Moreover, since $X^s(0,T)\hookrightarrow L^\infty(0,T;H^s(\mathbb{R}^2))$, u^* is also in $L^\infty(0,T;H^s(\mathbb{R}^2))$. By (12),

$$u^* \in L^{q_1}(0,T;B_{r_1}^{s-\frac{3}{q_1}}(\mathbb{R}^2)).$$

If $s>1$, by the Sobolev embedding, for some $0<\theta<1$,

$$\|u-v\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} \lesssim \|u-v\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}^{\theta}\|u-v\|_{L^\infty(0,T;H^s(\mathbb{R}^2))}^{1-\theta}$$

and therefore the solution map depends continuously on the initial data continuously in $H^s(\mathbb{R}^2)$. In the case where $s\le 1$, the continuous dependence of Φ may be shown as follows. By (9), the solution map depends continuously on the initial data continuously in $L^2(\mathbb{R}^2)$. We define $s_3, s_4>0$ so that they satisfy the following: $\max\left(\frac{3}{4}+\frac{1}{2r_1}, s_4-\frac{3}{4}(p-1)\right) < s_3 < s_4 < \min(s, s_3+\frac{3}{4})$, $r_3 = \frac{3}{2}\left(s_3-s_4+\frac{3}{4}\right)^{-1}$, and $q_3 = \frac{3}{s_4-s_3}$, where (q_3,r_3) satisfy (7). Then $B_{r_1}^{s_3-\frac{3}{q_1}}(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$, $B_{r_3'}^{s_3+\frac{3}{q_3}}(\mathbb{R}^2) = B_{r_3'}^{s_4}(\mathbb{R}^2)$, $2<r_3<\frac{3}{2}\frac{4}{3(2-p)}$, and $\frac{r_3(p-1)}{r_3-2}>1$. Let u and v be solutions of (1) for initial data u_0 and v_0, respectively. Then

$$
\begin{aligned}
&\|u-v\|_{L^{q_1}(0,T;B_{r_1}^{s_3-\frac{3}{q_1}}(\mathbb{R}^2))}\\
&\le \|u_0-v_0\|_{H^{s_3}(\mathbb{R})} + C\||u|^{p-1}u-|v|^{p-1}v\|_{L^{q_3'}(0,T;B_{r_3'}^{s_4}(\mathbb{R}^2))}.
\end{aligned}
$$

For $z_j \in \mathbb{C}$ with $j = 1, 2, 3, 4$, with $w_1 = z_2 - z_1$ and $w_2 = z_4 - z_3$,

$$
\begin{aligned}
&|z_4|^{p-1}z_4 - |z_3|^{p-1}z_3 - |z_2|^{p-1}z_2 + |z_1|^{p-1}z_1 \\
&= \frac{p+1}{2}\int_0^1 |z_3 + \theta w_2|^{p-1} d\theta w_2 - \frac{p+1}{2}\int_0^1 |z_1 + \theta w_1|^{p-1} d\theta w_1 \\
&+ \frac{p-1}{2}\int_0^1 |z_3 + \theta w_2|^{p-3}(z_3 + \theta w_2)^2 d\theta \overline{w_2} \\
&- \frac{p-1}{2}\int_0^1 |z_1 + \theta w_1|^{p-3}(z_1 + \theta w_1)^2 d\theta \overline{w_1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left| \int_0^1 |z_3 + \theta w_2|^{p-1} d\theta w_2 - \int_0^1 |z_1 + \theta w_1|^{p-1} d\theta w_1 \right| \\
&\le \int_0^1 |z_3 + \theta w_2|^{p-1} d\theta |w_2 - w_1| \\
&+ \int_0^1 \big| |z_3 + \theta w_2|^{p-1} - |z_1 + \theta w_1|^{p-1} \big| d\theta |w_1| \\
&\le (|z_3|^{p-1} + |z_4|^{p-1})|w_2 - w_1| + |z_3 - z_1|^{p-1}|w_1| + \frac{1}{p}|w_1||w_2 - w_1|^{p-1} \\
&\le (|z_3|^{p-1} + |z_4|^{p-1})|w_2 - w_1| + \frac{p+1}{p}|w_1||z_3 - z_1|^{p-1} + \frac{1}{p}|w_1||z_4 - z_2|^{p-1}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\left| \int_0^1 |z_3 + \theta w_2|^{p-3}(z_3 + \theta w_2)^2 d\theta \overline{w_2} - \int_0^1 |z_1 + \theta w_1|^{p-3}(z_1 + \theta w_1)^2 d\theta \overline{w_1} \right| \\
&\lesssim (|z_3|^{p-1} + |z_4|^{p-1})|w_2 - w_1| + |w_1||z_3 - z_1|^{p-1} + |w_1||z_4 - z_2|^{p-1},
\end{aligned}
$$

since

$$
\begin{aligned}
||z_2|^{p-3}z_2^2 - |z_1|^{p-3}z_1^2| &\lesssim \int_0^1 |z_1^2 + \theta(z_2 - z_1)|^{p-2} d\theta |z_2 - z_1| \\
&= \int_0^1 |z_1^2 + \theta(z_2 - z_1)|^{p-2} d\theta |z_2 - z_1| \\
&\le \int_0^1 ||z_1| - \theta|z_2 - z_1||^{p-2} d\theta |z_2 - z_1| \\
&= \frac{1}{2-p}(|z_1|^{p-1} - ||z_1| - |z_2 - z_1||^{p-1}) \\
&\le |z_2 - z_1|^{p-1}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\big\| |u(t,\cdot+h)|^{p-1}u(t,\cdot+h) - |v(t,\cdot+h)|^{p-1}v(t,\cdot+h) \\
&\quad - |u(t)|^{p-1}u(t) + |v(t)|^{p-1}v(t)\big\|_{L^{r_3'}(\mathbb{R}^2)} \\
&= \big\| |u(t,\cdot+h)|^{p-1}u(t,\cdot+h) - |u(t)|^{p-1}u(t) \\
&\quad - |v(t,\cdot+h)|^{p-1}v(t,\cdot+h) + |v(t)|^{p-1}v(t)\big\|_{L^{r_3'}(\mathbb{R}^2)} \\
&\le 4\|u(t)\|^{p-1}_{L^{\frac{2r_3(p-1)}{r_3-2}}(\mathbb{R}^2)} \|u(t,\cdot+h) - v(t,\cdot+h) - u(t) + v(t)\|_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{2(p+2)}{p}\|v(t,\cdot+h) - v(t)\|_{L^2(\mathbb{R}^2)}\|u(t)-v(t)\|^{p-1}_{L^{\frac{2r_3(p-1)}{r_3-2}}(\mathbb{R}^2)}
\end{aligned}$$

and this means

$$\begin{aligned}
&\||u|^{p-1}u - |v|^{p-1}v\|_{L^{q_3'}(0,T;B^{s_4}_{r_3'}(\mathbb{R}^2))} \\
&\lesssim \big\|\|u\|^{p-1}_{L^{\frac{2r_3(p-1)}{r_3-2}}(\mathbb{R}^2)}\|u-v\|_{H^{s_4}(\mathbb{R}^2)} + \|v\|_{H^{s_4}(\mathbb{R}^2)}\|u-v\|^{p-1}_{L^{\frac{2r_3(p-1)}{r_3-2}}(\mathbb{R}^2)}\big\|_{L^{q_3'}(0,T)} \\
&\le \big\|\|u\|^{p-1-\frac{r_3-2}{r_3}}_{L^\infty(\mathbb{R}^2)}\|u\|^{\frac{r_3-2}{r_3}}_{L^2(\mathbb{R}^2)}\|u-v\|_{H^{s_4}(\mathbb{R}^2)}\big\|_{L^{q_3'}(0,T)} \\
&+ \big\|\|v\|_{H^{s_4}(\mathbb{R}^2)}\|u-v\|^{p-1-\frac{r_3-2}{r_3}}_{L^\infty(\mathbb{R}^2)}\|u-v\|^{\frac{r_3-2}{r_3}}_{L^2(\mathbb{R}^2)}\big\|_{L^{q_3'}(0,T)} \\
&\le \|u\|^{p-1-\frac{r_3-2}{r_3}}_{L^{q_3'(p-1-\frac{r_3-2}{r_3})}(0,T;L^\infty(\mathbb{R}^2))}\|u\|^{\frac{r_3-2}{r_3}}_{L^\infty(0,T;L^2(\mathbb{R}^2))}\|u-v\|_{L^\infty(0,T;H^{s_4}(\mathbb{R}^2))} \\
&+ \|v\|_{L^\infty(0,T;H^{s_4}(\mathbb{R}^2))}\|u-v\|_{L^{q_3'(p-1-\frac{r_3-2}{r_3})}(0,T;L^\infty(\mathbb{R}^2))} u-v\|^{\frac{r_3-2}{r_3}}_{L^\infty(0,T;L^2(\mathbb{R}^2))} \\
&\le \|u\|_{L^{q_1}(0,T;L^\infty(\mathbb{R}^2))}\|u\|^{\frac{r_3-2}{r_3}}_{L^\infty(0,T;L^2(\mathbb{R}^2))}\|u-v\|_{L^\infty(0,T;H^{s_4}(\mathbb{R}^2))} \\
&+ \|v\|_{L^\infty(0,T;H^{s_4}(\mathbb{R}^2))}\|u-v\|_{L^{q_1}(0,T;L^\infty(\mathbb{R}^2))}\|^{\frac{r_3-2}{r_3}}_{L^\infty(0,T;L^2(\mathbb{R}^2))},
\end{aligned}$$

where $q_1, q_3 > 4 > q_3' > q_3'(p-1-\frac{r_3-2}{r_3})$. Since

$$\begin{aligned}
&\|u-v\|_{L^\infty(0,T;H^s(\mathbb{R}^2))} \\
&\lesssim \|u_0-v_0\|_{H^s(\mathbb{R}^2)} + (\|u_0\|_{H^s(\mathbb{R}^2)} + \|v_0\|_{H^s(\mathbb{R}^2)})\|u-v\|^{p-1}_{L^{p-1}(0,T;L^\infty(\mathbb{R}^2))} \\
&\lesssim \|u_0-v_0\|_{H^s(\mathbb{R}^2)} + (\|u_0\|_{H^s(\mathbb{R}^2)} + \|v_0\|_{H^s(\mathbb{R}^2)})\|u-v\|^{p-1}_{L^{q_1}(0,T;B^{s_3-\frac{3}{q_1}}_{r_1}(\mathbb{R}^2))},
\end{aligned}$$

the solution map is also continuously dependent in $L^\infty(0,T;H^s(\mathbb{R}^2))$.

Global well-posedness When $s=1$ and when $s=2$ and $p=3$, a priori estimates shows the global well-posedness by the blow-up alternative argument. Here we consider the case where $p=3$ and $1<s<2$. Let $[a]$ be the highest integer which is less than or equal to a. Let $T_1 = \min(1, T_0)$. By using the H^1

a priori estimate, for any $t > 0$,

$$\begin{aligned}\|u\|_{L^4(0,t;L^\infty(\mathbb{R}^2))} &\le \sum_{k=0}^{[t/T_1]+1} \|u\|_{L^4(kT_1,(k+1)T_1;L^\infty(\mathbb{R}^2))}\\ &\le \sum_{k=0}^{[t/T_1]+1} \|u\|_{X^1(kT_1,(k+1)T_1)}\\ &\le 2T_1^{-1}(1+t)\|u_0\|_{H^1(\mathbb{R}^2)}.\end{aligned}$$

Then by using Proposition 2.3,

$$\begin{aligned}\|u(t)\|^2_{\dot{H}^s(\mathbb{R}^2)} &\lesssim \|u_0\|^2_{H^s(\mathbb{R}^2)} + \int_0^t \|u(t')\|^2_{L^\infty(\mathbb{R}^n)}\|u(t')\|^2_{\dot{H}^s(\mathbb{R}^2)}dt\\ &\lesssim \|u_0\|^2_{H^s(\mathbb{R}^2)} + \|u(t')\|^2_{L^4(0,t;L^\infty(\mathbb{R}^2))}\|u\|^2_{L^4(0,t;\dot{H}^s(\mathbb{R}^2))}\\ &\lesssim \|u_0\|^2_{H^s(\mathbb{R}^2)} + \|u_0\|^2_{H^1(\mathbb{R}^2)}(1+t)^2\|u\|^2_{L^4(0,t;\dot{H}^s(\mathbb{R}^2))}.\end{aligned}$$

This shows

$$\|u(t)\|^4_{\dot{H}^s(\mathbb{R}^2)} \lesssim \|u_0\|^4_{H^s(\mathbb{R}^2)} + \|u_0\|^4_{H^1(\mathbb{R}^2)}(1+t)^4\|u\|^4_{L^4(0,t;\dot{H}^s(\mathbb{R}^2))}.$$

By the Gronwall inequality,

$$\|u(t)\|_{H^s(\mathbb{R}^2)} \lesssim \|u_0\|_{H^s(\mathbb{R}^2)} \exp\{C\|u_0\|^4_{H^1(\mathbb{R}^2)}(1+t)^4\}.$$

This shows the global well-posedness in $H^s(\mathbb{R}^2)$ setting. □

3.3. The case where $n \ge 3$, global H^1 existence result

control of solutions in the $H^1(\mathbb{R}^3)$ setting. So here, we consider radial data and use the following Strauss lemma.

Lemma 3.3 ([21, Theorems 1,2], [6, Proposition 1])**.** *Let $n \ge 2$ and let $1/2 < s < n/2$. Then*

$$\||x|^{\frac{n}{2}-s} f\|_{L^\infty_{\mathrm{rad}}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s_{\mathrm{rad}}(\mathbb{R}^n)}.$$

apply the following weighted Strichartz estimate:

Lemma 3.4 ([2, Propositions 1.2 and 1.3])**.** *Let $n \in \mathbb{N}$. Let $\delta > 0$ and $[x]_\delta = |x|^{1-\delta} + |x|^{1+\delta}$. The for any $q_1 \in [2,\infty)$ and $q_2 \in (2,\infty)$,*

$$\|[x]_\delta^{-1/q_1} U(t)f\|_{L^{q_1}(\mathbb{R};L^2(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)},$$

$$\left\|[x]_\delta^{-1/q_1}\int_0^t U(t-t')F(t')dt'\right\|_{L^{q_1}(0,T;L^2(\mathbb{R}^n))} \lesssim \|[x]_\delta^{1/q_2} F\|_{L^{q_2'}(0,T;L^2(\mathbb{R}^n))}.$$

Proof of Proposition 1.3. By using the uniform $H^1(\mathbb{R}^n)$ control obtained in (5), we reduce the proof to the local well-posedness in $H^1(\mathbb{R}^n)$. Let $\delta > 0$, $1/2 < s < 1$, and $2 < q_1, q_2 < \infty$ satisfy

$$-(p-1)\left(\frac{n}{2} - s\right) + \frac{1-\delta}{q_2} = -\frac{1-\delta}{q_1}. \tag{10}$$

We remark that there exist δ, q_1, q_2, r if $1 < p < 1 + 2/(n-2)$ since,

$$(p-1)\left(\frac{n}{2} - s\right) < 1 \Longrightarrow p < 1 + \frac{2}{n-2s} < 1 + \frac{2}{n-2}.$$

We define the norm $Y^1(T)$ as

$$\begin{aligned}\|u\|_{Y^1(T)} &= \|u\|_{L^\infty(0,T;H^1_{\mathrm{rad}}(\mathbb{R}^n))} \\ &\quad + \|[x]_\delta^{-1/q_1} u\|_{L^{q_1}(0,T;L^2_{\mathrm{rad}}(\mathbb{R}^n))} + \|[x]_\delta^{-1/q_1} \nabla u\|_{L^{q_1}(0,T;L^2_{\mathrm{rad}}(\mathbb{R}^n))}.\end{aligned}$$

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\psi(x) = \begin{cases} 1 & \text{if} \quad |x| \le 1, \\ 0 & \text{if} \quad |x| \ge 2. \end{cases}$$

Then by Lemmas 3.3 and 3.4 and (10),

$$\begin{aligned}&\|\Phi(u)\|_{Y^1(T)} \\ &\lesssim \|u_0\|_{H^1_{\mathrm{rad}}(\mathbb{R}^n)} + \left\| \int_0^t U(t-t')\big(\psi |u(t')|^{p-1} u(t')\big) dt' \right\|_{Y^1(T)} \\ &+ \left\| \int_0^t U(t-t')\big((1-\psi) |u(t')|^{p-1} u(t')\big) dt' \right\|_{Y^1(T)} \\ &\lesssim \|u_0\|_{H^1_{\mathrm{rad}}(\mathbb{R}^n)} \\ &+ \||x|^{-(p-1)(\frac{n}{2}-s)+\frac{1-\delta}{q_2}} ||x|^{\frac{n}{2}-s} u|^{p-1} u\|_{L^{q_2'}(0,T;L^2_{\mathrm{rad}}(|x|\le 2))} \\ &+ \||x|^{-(p-1)(\frac{n}{2}-s)+\frac{1-\delta}{q_2}} ||x|^{\frac{n}{2}-s} u|^{p-1} \nabla u\|_{L^{q_2'}(0,T;L^2_{\mathrm{rad}}(|x|\le 2))} \\ &+ \||u|^{p-1} u\|_{L^1(0,T;L^2_{\mathrm{rad}}(|x|>1))} + \|\nabla(|u|^{p-1} u)\|_{L^1(0,T;L^2_{\mathrm{rad}}(|x|>1))} \\ &\lesssim \|u_0\|_{H^1_{\mathrm{rad}}(\mathbb{R}^n)} + T^{1-\frac{1}{q_1}-\frac{1}{q_2}} \|u\|^p_{Y^1(T)}\end{aligned}$$

and therefore for some T and R, Φ is a map from $B_{Y^1(T)}(R)$ into itself. Moreover,

$$\begin{aligned}&\|\Phi(u) - \Phi(v)\|_{Y^1(T)} \qquad (11) \\ &\lesssim \||x|^{-\frac{1-\delta}{q_1}} (||x|^{\frac{n}{2}-s} u|^{p-1} - ||x|^{\frac{n}{2}-s} v|^{p-1})(|\nabla u| + |u|)\|_{L^{q_2'}(0,T;L^2_{\mathrm{rad}}(|x|\le 2))} \\ &+ \||x|^{-\frac{1-\delta}{q_1}} ||x|^{\frac{n}{2}-s} v|^{p-1} (|\nabla(u-v)| + |u-v|)\|_{L^{q_2'}(0,T;L^2_{\mathrm{rad}}(|x|\le 2))} \\ &+ \|(||x|^{\frac{n}{2}-s} u|^{p-1} - ||x|^{\frac{n}{2}-s} v|^{p-1})|x|^{-\frac{1+\delta}{q_1}} (|\nabla u| + |u|)\|_{L^1(0,T;L^2_{\mathrm{rad}}(|x|>1))} \\ &+ \|||x|^{\frac{n}{2}-s} v|^{p-1} |x|^{-\frac{1+\delta}{q_1}} (|\nabla(u-v)| + |u-v|)\|_{L^1(0,T;L^2_{\mathrm{rad}}(|x|>1))}.\end{aligned}$$

Then for $p \geq 2$, Φ is a contraction map on $B_{Y^1(T)}(R)$. Similarly, for $1 < p < 2$, we define the auxiliary norm $Y^0(T)$ as

$$\|u\|_{Y^0(T)} = \|u\|_{L^\infty(0,T;L^2_{\rm rad}(\mathbb{R}^n))} + \|[x]_\delta^{-1/q_1} u\|_{L^{q_1}(0,T;L^2_{\rm rad}(\mathbb{R}^n))}.$$

Then for $1 < p < 2$,

$$\begin{aligned}
&\|(\Phi(u) - \Phi(v))\|_{Y^0(T)} \\
&\lesssim \Big\|[x]_\delta^{-1/q_1} \big(\big||x|^{\frac{n}{2}-s} v\big| + \big||x|^{\frac{n}{2}-s} v\big|\big)^{p-1} |u-v|\Big\|_{L^{q_2'}(0,T;L^2_{\rm rad}(|x|\leq 2))} \\
&+ \Big\|\big(\big||x|^{\frac{n}{2}-s} v\big| + \big||x|^{\frac{n}{2}-s} v\big|\big)^{p-1} |u-v|\Big\|_{L^1(0,T;L^2_{\rm rad}(|x|>1))} \\
&\lesssim T^{1-\frac{1}{q_1}-\frac{1}{q_2}} (\|u\|_{Y^1(T)} + \|v\|_{Y^1(T)})^{p-1} \|u-v\|_{Y^0(T)}.
\end{aligned}$$

and therefore Φ is a contraction map in $Y^0(T)$ for some T and R and therefore we have a unique solution to (1) in $Y^1(T)$. Moreover, by Lemma 3.3 and (11), with some $0 < \theta < 1$, for solutions u and v of (4) for initial data u_0 and v_0, respectively,

$$\begin{aligned}
&\|u - v\|_{Y^1(T)} \\
&\lesssim \|u_0 - v_0\|_{H^1_{\rm rad}(\mathbb{R}^n)} + T^{1-\frac{1}{q_1}-\frac{1}{q_2}} (\|u\|_{Y^1(T)} + \|v\|_{Y^1(T)})^{p-1} \|u-v\|_{Y^1(T)} \\
&+ T^{1-\frac{1}{q_1}-\frac{1}{q_2}} (\|u\|_{Y^1(T)} + \|v\|_{Y^1(T)}) \big\||x|^{\frac{n}{2}-s}(u-v)\big\|^{p-1}_{L^\infty(0,T;L^\infty_{\rm rad}(\mathbb{R}^n))} \\
&\lesssim \|u_0 - v_0\|_{H^1_{\rm rad}(\mathbb{R}^n)} + T^{1-\frac{1}{q_1}-\frac{1}{q_2}} (\|u\|_{Y^1(T)} + \|v\|_{Y^1(T)})^{p-1} \|u-v\|_{Y^1(T)} \\
&+ T(\|u\|_{Y^1(T)} + \|v\|_{Y^1(T)}) \big\|u-v\big\|^{p-1}_{Y^0(T)}
\end{aligned}$$

and therefore $\|u - v\|_{Y^1(T)} \to 0$ as $\|u_0 - v_0\|_{H^1_{\rm rad}(\mathbb{R}^n)} \to 0$. □

Remark 3.1. *In order to show Φ is a contraction map, Y_0 is sufficient but we dare to use Y_1 since $Y_1 \hookrightarrow Y_0$.*

3.4. 3 dimensional case, small H^1 data solutions for $p = 3$

sufficient to control solutions uniformly. So here, we transform (1) into the corresponding wave equation.

following:

$$\begin{aligned}
\Box u &= -i(-i\partial_t - D)|u|^{p-1}u \\
&= i\frac{p+1}{2}|u|^{p-1}(Du - i|u|^{p-1}u) \\
&- i\frac{p-1}{2}|u|^{p-3}u^2\overline{(Du - i|u|^{p-1}u)} + iD(|u|^{p-1}u) \\
&- i\left(D(|u|^{p-1}u) + \frac{p+1}{2}|u|^{p-1}Du - \frac{p-1}{2}|u|^{p-3}u^2 D\overline{u}\right) + p|u|^{2p-2}u \\
&=: F_p(u).
\end{aligned}$$

Then the corresponding integral equation is the following:

$$u(t) = \cos(tD)u_0 + \frac{\sin(tD)}{D}(-iDu_0 - |u_0|^{p-1}u_0) \qquad (12)$$
$$+ \int_0^t \frac{\sin((t-t')D)}{D} F_p(u)(t')dt'.$$

For any radially symmetric function f, we define $\widetilde{f}$ as $\widetilde{f}(|x|) = f(x)$. Then for any radial data, (12) is rewritten as

$$\widetilde{u}(t) \qquad (13)$$
$$= \partial_t J[u_0](t) + J[-iDu_0 - |u_0|^{p-1}u_0](t) + \int_0^t J[F_p(u)(t')](t-t')dt'$$

where

$$J[f](t,r) = \frac{1}{2r}\int_{|r-t|}^{r+t} \lambda \widetilde{f}(\lambda)d\lambda.$$

This transformation is justified as follows:

Lemma 3.5. *Let* $1 < p \le 3$ *and* $u_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ *and* $u \in C(0,T; H^1_{\mathrm{rad}}(\mathbb{R}^3))$ *be the solution of* (4). *Then* u *is also the solution of* (13)

Proof. Since $H^1(\mathbb{R}^3) \hookrightarrow L^{2p}(\mathbb{R}^3)$, $u \in C^1(0,T; L^2_{\mathrm{rad}}(\mathbb{R}^3))$ and $|u|^{p-1}u \in C^1(0,T; H^s_{\mathrm{rad}}(\mathbb{R}^3))$ with $s < -3/2$. Then $-i(-i\partial_t - D)|u|^{p-1}u = F_p(u)$ holds in the $H^s_{\mathrm{rad}}(\mathbb{R}^3)$ setting. Moreover,

$$U(t)u_0 = \cos(tD)u_0 - i\sin(tD)u_0$$

and in the $H^s_{\mathrm{rad}}(\mathbb{R}^3)$ setting, the following calculation is also justified:

$$-\int_0^t U(t-t')(|u(t')|^{p-1}u(t'))dt'$$
$$= -\int_0^t \cos((t-t')D)(|u(t')|^{p-1}u(t'))dt'$$
$$-\int_0^t -i\sin((t-t')D)(|u(t')|^{p-1}u(t'))dt'$$
$$= \left[\frac{\sin((t-t')D)}{D}(|u(t')|^{p-1}u(t'))\right]_{t'=0}^{t}$$
$$-\int_0^t \frac{\sin((t-t')D)}{D}\partial_t(|u(t')|^{p-1}u(t'))dt'$$
$$-\int_0^t -i\sin((t-t')D)(|u(t')|^{p-1}u(t'))dt'$$
$$= -\frac{\sin(tD)}{D}(|u_0|^{p-1}u_0) - i\int_0^t \frac{\sin((t-t')D)}{D}(-i\partial_t - D)(|u(t')|^{p-1}u(t'))dt'.$$

Therefore u is also a solution of (13). □

define $A[f]:\mathbb{R}\to\mathbb{C}$ as $A[f](\lambda)=f(|\lambda|)$. See also [16].

Lemma 3.6. *Let $f:[0,\infty)\to\mathbb{C}$. Then*

$$\left\|\frac{1}{2\cdot}\int_{|\cdot-t|}^{\cdot+t} f(\lambda)d\lambda\right\|_{L^\infty(0,\infty)} \leq M[A[f]](t).$$

Proof. If $r\leq t$, then

$$\frac{1}{2r}\int_{t-r}^{t+r}|f(\lambda)|d\lambda \leq M[f](t).$$

If $r>t$, then

$$\frac{1}{2r}\int_{r-t}^{t+r}|f(\lambda)|d\lambda \leq \frac{1}{2r}\int_{t-r}^{t+r}|A[f](\lambda)|d\lambda \leq M[A[f]](t).$$

□

Corollary 3.7. *Let $f:\mathbb{R}^3\to\mathbb{C}$ be radial. Then*

$$\|J[f]\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\leq C\|f\|_{L^2_{\mathrm{rad}}(\mathbb{R}^3)}.$$

Proof. Let $g(\lambda)=\lambda\widetilde{f}(\lambda)$. Then

$$\|J[f]\|_{L^2(0,T;L^\infty(0,\infty))}\leq \|M[A[g]]\|_{L^2(0,T)}\leq C\|g\|_{L^2(0,\infty)} = C\|f\|_{L^2_{\mathrm{rad}}(\mathbb{R}^3)}.$$

□

Corollary 3.8. *Let $h:[0,\infty)\times\mathbb{R}^3\to\mathbb{C}$ be radial. Then*

$$\left\|\int_0^t J[h(t')](t-t')dt'\right\|_{L^2(0,T;L^\infty(0,\infty))}\leq C\|h\|_{L^1(0,T;L^2_{\mathrm{rad}}(\mathbb{R}^3))}.$$

Proof. Let $H(t,\lambda)=\lambda\widetilde{h}(t,\lambda)$. Then

$$\begin{aligned}\left\|\int_0^t J[h(t')](t-t')dt'\right\|_{L^2(0,T;L^\infty(0,\infty))} &\leq \left\|\int_0^t M[A[H(t')]](t-t')dt'\right\|_{L^2(0,T)}\\ &\leq \int_0^T \|M[A[H(t')]]\|_{L^2(t',T)}dt'\\ &\leq C\int_0^T \|h(t')\|_{L^2_{\mathrm{rad}}(\mathbb{R}^3)}dt'.\end{aligned}$$

□

Corollary 3.9 (Hardy)**.** *Let $f\in C^1(\mathbb{R};\mathbb{C})$. Then*

$$\left\|\frac{d}{dt}\left(\frac{1}{2r}\int_{|r-t|}^{r+t}\lambda f(\lambda)d\lambda\right)\right\|_{L^2(0,\infty;L^\infty(0,\infty))}\leq C\|rf'\|_{L^2(0,\infty)}.$$

Proof of Proposition 1.4. For $0 < T < 1$ and $p = 3$, By Corollaries 3.7, 3.8, and 3.9,

$$\begin{aligned}
&\|u\|_{L^2(0,T;L^\infty_{\rm rad}(\mathbb{R}^3))} \\
&\lesssim \|u_0\|_{H^1_{\rm rad}(\mathbb{R}^3)} + \|u_0\|^3_{H^1_{\rm rad}(\mathbb{R}^3)} + \|F_3\|_{L^1(0,T;L^2_{\rm rad}(\mathbb{R}^3))} \\
&\lesssim \|u_0\|_{H^1_{\rm rad}(\mathbb{R}^3)} + \|u_0\|^3_{H^1_{\rm rad}(\mathbb{R}^3)} \\
&+ \|u\|^2_{L^2(0,T;L^\infty_{\rm rad}(\mathbb{R}^3)}\|u\|_{L^\infty(0,T;H^1_{\rm rad}(\mathbb{R}^3))} + \||u|^5\|_{L^1(0,T;L^2_{\rm rad}(\mathbb{R}^3))}.
\end{aligned} \tag{14}$$

Since

$$\||u|^5\|_{L^1(0,T;L^2_{\rm rad}(\mathbb{R}^3))} \leq \|u\|^2_{L^2(0,T;L^\infty_{\rm rad}(\mathbb{R}^3))}\|u\|^3_{L^\infty(0,T;H^1_{\rm rad}(\mathbb{R}^3))},$$

by the unitarity of $U(t)$,

$$\|u\|_{L^\infty(0,T;H^1_{\rm rad}(\mathbb{R}^3))} \lesssim \|u_0\|_{H^1_{\rm rad}(\mathbb{R}^3)} + \|u\|^{p-1}_{L^{p-1}(0,T;L^\infty_{\rm rad}(\mathbb{R}^3))}\|u_0\|_{H^1_{\rm rad}(\mathbb{R}^3)}.$$

Let $X^1_{\rm rad}(0,T) = L^\infty(0,T;H^1_{\rm rad}(\mathbb{R}^3)) \cap L^2(0,T;L^\infty_{\rm rad}(\mathbb{R}^3))$. Then, for sufficiently small initial data u_0, Φ maps from $B_{X^1_{\rm rad}(0,T)}(R)$ into itself with some T and R. Moreover,

$$\begin{aligned}
&\||u|^2u - |v|^2v\|_{L^1(0,T;H^1_{\rm rad}(\mathbb{R}^3))} \\
&\lesssim (\|u\|_{X_{\rm rad}(0,T)} + \|v\|_{X_{\rm rad}(0,T)})^2\|u - v\|_{L^\infty(0,T;H^1_{\rm rad}(\mathbb{R}^3))}.
\end{aligned}$$

Since

$$\begin{aligned}
&|F(u) - F(v)| \\
&= \Big|i\big(D(|u|^{2p-1}u) - 2|u|^2Du - u^2D\overline{u}\big) + 3|u|^4u \\
&- i\big(D(|v|^2v) - 2|v|^2Dv - v^2D\overline{v}\big) - 3|v|^4v\Big| \\
&\leq |D(|u|^2u - |v|^2v)| + |u|^2|D(u-v)| \\
&+ \Big(\big||u|^2 - |v|^2\big| + \big|u^2 - v^2\big|\Big)|Dv| + 3\big||u|^4u - |v|^4v\big|,
\end{aligned}$$

we have

$$\begin{aligned}
&\|F(u) - F(v)\|_{L^1(0,T;L^2_{\rm rad}(\mathbb{R}^3))} \\
&\lesssim (\|u\|_{X^1_{\rm rad}(0,T)} + \|v\|_{X^1_{\rm rad}(0,T)})^2\|u - v\|_{X^1_{\rm rad}(0,T)} \\
&+ (\|u\|_{X^1_{\rm rad}(0,T)} + \|v\|_{X^1_{\rm rad}(0,T)})^4\|u - v\|_{X^1_{\rm rad}(0,T)}.
\end{aligned}$$

This means for sufficiently small u_0, Φ is a contraction map on $B_{X^1_{\rm rad}(0,T)}(R)$. □

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