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Research Article

Luigi C. Berselli* and Michael Růžička

Global regularity for systems with *p*-structure depending on the symmetric gradient

Abstract: In this paper we study on smooth bounded domains the global regularity (up to the boundary) for weak solutions to systems having *p*-structure depending only on the symmetric part of the gradient.

Keywords: Regularity of weak solutions, symmetric gradient, boundary regularity, natural quantities

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1 Introduction

In this paper we study regularity of weak solutions to the boundary value problem

$$\begin{cases} -\operatorname{div} \mathbf{S}(\mathbf{Du}) = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathbf{Du} := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)$ denotes the symmetric part of the gradient $\nabla\mathbf{u}$ and $\Omega \subset \mathbb{R}^3$ is a bounded domain with a $C^{2,1}$ boundary $\partial\Omega$.[1] Our interest in this system comes from the *p*-Stokes system

$$\begin{cases} -\operatorname{div} \mathbf{S}(\mathbf{Du}) + \nabla\pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In both problems the typical example for $\mathbf{S}$ we have in mind is

$$\mathbf{S}(\mathbf{Du}) = \mu(\delta + |\mathbf{Du}|)^{p-2}\mathbf{Du},$$

where $p \in (1, 2]$, $\delta \geq 0$ and $\mu > 0$. In previous investigations of (1.2), only suboptimal results for the regularity up to the boundary have been proved. Here we mean suboptimal in the sense that the results are weaker than the results known for *p*-Laplacian systems, cf. [1, 13, 14]. Clearly, system (1.1) is obtained from (1.2) by dropping the divergence constraint and the resulting pressure gradient. Thus, system (1.1) lies in between system (1.2) and *p*-Laplacian systems, which depend on the full gradient $\nabla\mathbf{u}$.

We would like to stress that system (1.1) is of its own independent interest, since it is studied within plasticity theory, when formulated in the framework of deformation theory (cf. [11, 24]). In this context the unknown is the displacement vector field $\mathbf{u} = (u^1, u^2, u^3)^\top$, while the external body force $\mathbf{f} = (f^1, f^2, f^3)^\top$

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[1] We restrict ourselves to the problem in three space dimensions, however the results can be easily transferred to the problem in $\mathbb{R}^d$ for all $d \geq 2$.

*Corresponding author: Luigi C. Berselli, Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti 1/c, 56127 Pisa, Italy, e-mail: luigi.carlo.berselli@unipi.it. https://orcid.org/0000-0001-6208-9934
Michael Růžička, Institute of Applied Mathematics, Albert-Ludwigs-University Freiburg, Eckerstr. 1, 79104 Freiburg, Germany, e-mail: rose@mathematik.uni-freiburg.de

is given. The stress tensor **S**, which is the tensor of small elasto-plastic deformations, depends only on **Du**. Physical interpretation and discussion of both systems (1.1) and (1.2) and the underlying models can be found, e.g., in [5, 11, 15, 19, 20].

We study global regularity properties of weak solutions to (1.1) in sufficiently smooth and bounded domains Ω; we obtain, for all $p \in (1, 2]$, the optimal result, namely, that $\mathbf{F}(\mathbf{Du})$ belongs to $W^{1,2}(\Omega)$, where the nonlinear tensor-valued function **F** is defined in (2.5). This result has been proved near a flat boundary in [24] and is the same result as for p-Laplacian systems (cf. [1, 13, 14]). The situation is quite different for (1.2). There the optimal result, i.e., $\mathbf{F}(\mathbf{Du}) \in W^{1,2}(\Omega)$, is only known for

(i) two-dimensional bounded domains (cf. [16], where even the p-Navier–Stokes system is treated);

(ii) the space-periodic problem in $\mathbb{R}^d$, $d \geq 2$, which follows immediately from interior estimates, i.e., $\mathbf{F}(\mathbf{Du}) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, which are known in all dimensions, and the periodicity of the solution;

(iii) if the no-slip boundary condition is replaced by perfect slip boundary conditions (cf. [17]);

(iv) in the case of small **f** (cf. [6]).

We also observe that the above results for the p-Stokes system (apart those in the space periodic setting) require the stress tensor to be non-degenerate, that is, $\delta > 0$. In the case of homogeneous Dirichlet boundary conditions and three- and higher-dimensional bounded, sufficiently smooth domains only suboptimal results are known. To our knowledge, the state of the art for general data is that $\mathbf{F}(\mathbf{Du}) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, tangential derivatives of $\mathbf{F}(\mathbf{Du})$ near the boundary belong to L^2, while the normal derivative of $\mathbf{F}(\mathbf{Du})$ near the boundary belongs to some L^q, where $q = q(p) < 2$ (cf. [2, 4] and the discussion therein). We would also like to mention a result for another system between (1.2) and p-Laplacian systems, namely, if (1.2) is considered with **S** depending on the full velocity gradient $\nabla \mathbf{u}$. In this case, it is proved in [7] that $\mathbf{u} \in W^{2,r}(\mathbb{R}^3) \cap W^{1,p}_0(\mathbb{R}^3)$ for some $r > 3$, provided $p < 2$ is very close to 2.

In the present paper we extend the optimal regularity result for (1.1) of Seregin and Shilkin [24] in the case of a flat boundary to the general case of bounded sufficiently smooth domains and to possibly degenerate stress tensors, that is, the case $\delta = 0$. The precise result we prove is the following:

Theorem 1.1. *Let the tensor field* **S** *in* (1.1) *have* (p, δ)*-structure (cf. Definition 2.1) for some* $p \in (1, 2]$ *and* $\delta \in [0, \infty)$, *and let* **F** *be the associated tensor field to* **S** *defined in* (2.5). *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with* $C^{2,1}$ *boundary, and let* $\mathbf{f} \in L^{p'}(\Omega)$. *Then the unique weak solution* $\mathbf{u} \in W^{1,p}_0(\Omega)$ *of problem* (1.1) *satisfies*

$$\int_\Omega |\nabla \mathbf{F}(\mathbf{Du})|^2 \, d\mathbf{x} \leq c,$$

where c *denotes a positive function which is non-decreasing in* $\|\mathbf{f}\|_{p'}$ *and* δ, *and which depends on the domain through its measure* $|\Omega|$ *and the* $C^{2,1}$*-norms of the local description of* $\partial\Omega$. *In particular, the above estimate implies that* $\mathbf{u} \in W^{2, \frac{3p}{p+1}}(\Omega)$.

2 Notations and preliminaries

In this section we introduce the notation we will use, state the precise assumptions on the extra stress tensor **S**, and formulate the main results of the paper.

2.1 Notation

We use c, C to denote generic constants which may change from line to line, but they are independent of the crucial quantities. Moreover, we write $f \sim g$ if and only if there exists constants $c, C > 0$ such that $cf \leq g \leq Cf$. In some cases we need to specify the dependence on certain parameters, and consequently we denote by $c(\cdot)$ a positive function which is non-decreasing with respect to all its arguments. In particular, we denote by $c(\delta^{-1})$ a possibly critical dependence on the parameter δ as $\delta \to 0$, while $c(\delta)$ only indicates that the constant c depends on δ and will satisfy $c(\delta) \leq c(\delta_0)$ for all $\delta \leq \delta_0$.

We use standard Lebesgue spaces $(L^p(\Omega), \|\cdot\|_p)$ and Sobolev spaces $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$, where $\Omega \subset \mathbb{R}^3$ is a sufficiently smooth bounded domain. The space $W_0^{1,p}(\Omega)$ is the closure of the compactly supported, smooth functions $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Thanks to the Poincaré inequality, we equip $W_0^{1,p}(\Omega)$ with the gradient norm $\|\nabla\cdot\|_p$. When dealing with functions defined only on some open subset $G \subset \Omega$, we denote the norm in $L^p(G)$ by $\|\cdot\|_{p,G}$. As usual, we use the symbol $\rightharpoonup$ to denote weak convergence, and $\to$ to denote strong convergence. The symbol $\mathrm{spt}\, f$ denotes the support of the function f. We do not distinguish between scalar, vector-valued or tensor-valued function spaces. However, we denote vectors by boldface lower-case letter as, e.g., $\mathbf{u}$ and tensors by boldface upper case letters as, e.g., $\mathbf{S}$. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we denote $\mathbf{u} \overset{s}{\otimes} \mathbf{v} := \frac{1}{2}(\mathbf{u} \otimes \mathbf{v} + (\mathbf{u} \otimes \mathbf{v})^\top)$, where the standard tensor product $\mathbf{u} \otimes \mathbf{v} \in \mathbb{R}^{3\times 3}$ is defined as $(\mathbf{u} \otimes \mathbf{v})_{ij} := u_i v_j$. The scalar product of vectors is denoted by $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^3 u_i v_i$ and the scalar product of tensors is denoted by $\mathbf{A} \cdot \mathbf{B} := \sum_{i,j=1}^3 A_{ij} B_{ij}$.

Greek lower-case letters take only the values 1, 2, while Latin lower-case ones take the values 1, 2, 3. We use the summation convention over repeated indices only for Greek lower-case letters, but not for Latin lower-case ones.

2.2 (p, δ)-structure

We now define what it means that a tensor field $\mathbf{S}$ has (p, δ)-structure, see [8, 23]. For a tensor $\mathbf{P} \in \mathbb{R}^{3\times 3}$, we denote its symmetric part by $\mathbf{P}^{\mathrm{sym}} := \frac{1}{2}(\mathbf{P} + \mathbf{P}^\top) \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} := \{\mathbf{P} \in \mathbb{R}^{3\times 3} \mid \mathbf{P} = \mathbf{P}^\top\}$. We use the notation $|\mathbf{P}|^2 = \mathbf{P} \cdot \mathbf{P}^\top$.

It is convenient to define for $t \geq 0$ a special N-function[2] $\phi(\cdot) = \phi_{p,\delta}(\cdot)$, for $p \in (1, \infty)$, $\delta \geq 0$, by

$$\phi(t) := \int_0^t (\delta + s)^{p-2} s\, ds. \tag{2.1}$$

The function ϕ satisfies, uniformly in t and independently of δ, the important equivalences

$$\phi''(t)t \sim \phi'(t), \tag{2.2}$$
$$\phi'(t)t \sim \phi(t),$$
$$t^p + \delta^p \sim \phi(t) + \delta^p. \tag{2.3}$$

We use the convention that if $\phi''(0)$ does not exist, the left-hand side in (2.2) is continuously extended by zero for $t = 0$. We define the shifted N-functions $\{\phi_a\}_{a\geq 0}$ (cf. [8, 9, 23]), for $t \geq 0$, by

$$\phi_a(t) := \int_0^t \frac{\phi'(a + s)s}{a + s}\, ds$$

Note that the family $\{\phi_a\}_{a\geq 0}$ satisfies the Δ_2-condition uniformly with respect to $a \geq 0$, i.e., $\phi_a(2t) \leq c(p)\phi_a(t)$ holds for all $t \geq 0$.

Definition 2.1 $((p, \delta)$-structure). We say that a tensor field $\mathbf{S} \colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$, belonging to $C^0(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \cap C^1(\mathbb{R}^{3\times 3} \setminus \{\mathbf{0}\}, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, satisfying $\mathbf{S}(\mathbf{P}) = \mathbf{S}(\mathbf{P}^{\mathrm{sym}})$ and $\mathbf{S}(\mathbf{0}) = \mathbf{0}$, possesses (p, δ)-structure if for some $p \in (1, \infty)$, $\delta \in [0, \infty)$, and the N-function $\phi = \phi_{p,\delta}$ (cf. (2.1)), there exist constants $\kappa_0, \kappa_1 > 0$ such that

$$\sum_{i,j,k,l=1}^3 \partial_{kl} S_{ij}(\mathbf{P}) Q_{ij} Q_{kl} \geq \kappa_0 \phi''(|\mathbf{P}^{\mathrm{sym}}|)|\mathbf{Q}^{\mathrm{sym}}|^2, \quad |\partial_{kl} S_{ij}(\mathbf{P})| \leq \kappa_1 \phi''(|\mathbf{P}^{\mathrm{sym}}|) \tag{2.4}$$

for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}$, with $\mathbf{P}^{\mathrm{sym}} \neq \mathbf{0}$, and all $i, j, k, l = 1, 2, 3$. The constants κ_0, κ_1 and p are called the *characteristics* of $\mathbf{S}$.

Remark 2.2. (i) Assume that $\mathbf{S}$ has (p, δ)-structure for some $\delta \in [0, \delta_0]$. Then, if not otherwise stated, the constants in the estimates depend only on the characteristics of $\mathbf{S}$ and on δ_0, but they are independent of δ.

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2 For the general theory of N-functions and Orlicz spaces, we refer to [21].

(ii) An important example of a tensor field $\mathbf{S}$ having (p, δ)-structure is given by $\mathbf{S}(\mathbf{P}) = \phi'(|\mathbf{P}^{\mathrm{sym}}|)|\mathbf{P}^{\mathrm{sym}}|^{-1}\mathbf{P}^{\mathrm{sym}}$. In this case, the characteristics of $\mathbf{S}$, namely, κ_0 and κ_1, depend only on p and are independent of $\delta \geq 0$.
(iii) For a tensor field $\mathbf{S}$ with (p, δ)-structure, we have $\partial_{kl}S_{ij}(\mathbf{P}) = \partial_{kl}S_{ji}(\mathbf{P})$ for all $i, j, k, l = 1, 2, 3$ and all $\mathbf{P} \in \mathbb{R}^{3\times3}$, due to its symmetry. Moreover, from $\mathbf{S}(\mathbf{P}) = \mathbf{S}(\mathbf{P}^{\mathrm{sym}})$, it follows that $\partial_{kl}S_{ij}(\mathbf{P}) = \frac{1}{2}\partial_{kl}S_{ij}(\mathbf{P}^{\mathrm{sym}}) + \frac{1}{2}\partial_{lk}S_{ij}(\mathbf{P}^{\mathrm{sym}})$ for all $i, j, k, l = 1, 2, 3$ and all $\mathbf{P} \in \mathbb{R}^{3\times3}$, and consequently $\partial_{kl}S_{ij}(\mathbf{P}) = \partial_{lk}S_{ij}(\mathbf{P})$ for all $i, j, k, l = 1, 2, 3$ and all $\mathbf{P} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$.

To a tensor field $\mathbf{S}$ with (p, δ)-structure, we associate the tensor field $\mathbf{F}\colon \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ defined through

$$\mathbf{F}(\mathbf{P}) := (\delta + |\mathbf{P}^{\mathrm{sym}}|)^{\frac{p-2}{2}}\mathbf{P}^{\mathrm{sym}}. \tag{2.5}$$

The connection between $\mathbf{S}$, $\mathbf{F}$, and $\{\phi_a\}_{a\geq0}$ is best explained in the following proposition (cf. [8, 23]).

Proposition 2.3. *Let $\mathbf{S}$ have (p, δ)-structure, and let $\mathbf{F}$ be as defined in* (2.5)*. Then*

$$(\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}) \sim |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2 \sim \phi_{|\mathbf{P}^{\mathrm{sym}}|}(|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|)$$
$$\sim \phi''(|\mathbf{P}^{\mathrm{sym}}| + |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|)|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|^2,$$
$$|\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})| \sim \phi'_{|\mathbf{P}^{\mathrm{sym}}|}(|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|) \tag{2.6}$$

uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times3}$. Moreover, uniformly in $\mathbf{Q} \in \mathbb{R}^{3\times3}$,

$$\mathbf{S}(\mathbf{Q}) \cdot \mathbf{Q} \sim |\mathbf{F}(\mathbf{Q})|^2 \sim \phi(|\mathbf{Q}^{\mathrm{sym}}|).$$

The constants depend only on the characteristics of $\mathbf{S}$.

For a detailed discussion of the properties of $\mathbf{S}$ and $\mathbf{F}$ and their relation to Orlicz spaces and N-functions, we refer the reader to [3, 23]. Since in the following we shall insert into $\mathbf{S}$ and $\mathbf{F}$ only symmetric tensors, we can drop in the above formulas the superscript "sym" and restrict the admitted tensors to symmetric ones.

We recall that the following equivalence, which is proved in [3, Lemma 3.8], is valid for all smooth enough symmetric tensor fields $\mathbf{Q} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$:

$$|\partial_i\mathbf{F}(\mathbf{Q})|^2 \sim \phi''(|\mathbf{Q}|)|\partial_i\mathbf{Q}|^2. \tag{2.7}$$

The proof of this equivalence is based on Proposition 2.3. This proposition and the theory of divided differences also imply (cf. [4, Equation (2.26)]) that

$$|\partial_\tau\mathbf{F}(\mathbf{Q})|^2 \sim \phi''(|\mathbf{Q}|)|\partial_\tau\mathbf{Q}|^2 \tag{2.8}$$

for all smooth enough symmetric tensor fields $\mathbf{Q} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$.

A crucial observation in [24] is that the quantities in (2.7) are also equivalent to several further quantities. To formulate this precisely, we introduce, for $i = 1, 2, 3$ and for sufficiently smooth symmetric tensor fields $\mathbf{Q}$, the quantity

$$\mathcal{P}_i(\mathbf{Q}) := \partial_i\mathbf{S}(\mathbf{Q}) \cdot \partial_i\mathbf{Q} = \sum_{k,l,m,n=1}^{3} \partial_{kl}S_{mn}(\mathbf{Q})\partial_iQ_{kl}\partial_iQ_{mn}. \tag{2.9}$$

Recall that in the definition of $\mathcal{P}_i(\mathbf{Q})$ there is no summation convention over the repeated Latin lower-case index i in $\partial_i\mathbf{S}(\mathbf{Q}) \cdot \partial_i\mathbf{Q}$. Note that if $\mathbf{S}$ has (p, δ)-structure, then $\mathcal{P}_i(\mathbf{Q}) \geq 0$ for $i = 1, 2, 3$. The following important equivalences hold, first proved in [24].

Proposition 2.4. *Assume that $\mathbf{S}$ has (p, δ)-structure. Then the following equivalences are valid, for all smooth enough symmetric tensor fields $\mathbf{Q}$ and all $i = 1, 2, 3$:*

$$\mathcal{P}_i(\mathbf{Q}) \sim \phi''(|\mathbf{Q}|)|\partial_i\mathbf{Q}|^2 \sim |\partial_i\mathbf{F}(\mathbf{Q})|^2, \tag{2.10}$$

$$\mathcal{P}_i(\mathbf{Q}) \sim \frac{|\partial_i\mathbf{S}(\mathbf{Q})|^2}{\phi''(|\mathbf{Q}|)}, \tag{2.11}$$

with the constants only depending on the characteristics of $\mathbf{S}$.

Proof. The assertions are proved in [24] using a different notation. For the convenience of the reader, we sketch the proof here. The equivalences in (2.10) follow from (2.7), (2.9) and the fact that $\mathbf{S}$ has (p, δ)-structure. Furthermore, using (2.10), we have

$$|\mathcal{P}_i(\mathbf{Q})|^2 \leq |\partial_i \mathbf{S}(\mathbf{Q})|^2 |\partial_i \mathbf{Q}|^2 \leq c|\partial_i \mathbf{S}(\mathbf{Q})|^2 \frac{\mathcal{P}_i(\mathbf{Q})}{\phi''(|\mathbf{Q}|)},$$

which proves one inequality of (2.11). The other one follows from

$$|\partial_i \mathbf{S}(\mathbf{Q})|^2 \leq c \sum_{k,l=1}^{3} |\partial_{kl} \mathbf{S}(\mathbf{Q}) \partial_i Q_{kl}|^2 \leq c(\phi''(|\mathbf{Q}|))^2 |\partial_i \mathbf{Q}|^2 \leq c\phi''(|\mathbf{Q}|)\mathcal{P}_i(\mathbf{Q}),$$

where we used (2.4) and (2.10). □

2.3 Existence of weak solutions

In this section we define weak solutions of (1.1), recall the main results of existence and uniqueness and discuss a perturbed problem, which is used to justify the computations that follow. From now on, we restrict ourselves to the case $p \leq 2$.

Definition 2.5. We say that $\mathbf{u} \in W_0^{1,p}(\Omega)$ is a weak solution to (1.1) if for all $\mathbf{v} \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \mathbf{S}(\mathbf{Du}) \cdot \mathbf{Dv}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x}.$$

We have the following standard result.

Proposition 2.6. *Let the tensor field $\mathbf{S}$ in (1.1) have (p, δ)-structure for some $p \in (1, 2]$ and $\delta \in [0, \infty)$. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2,1}$ boundary, and let $\mathbf{f} \in L^{p'}(\Omega)$. Then there exists a unique weak solution $\mathbf{u}$ to (1.1) such that*

$$\int_\Omega \phi(|\mathbf{Du}|)\, d\mathbf{x} \leq c(\|\mathbf{f}\|_{p'}, \delta).$$

Proof. The assertions follow directly from the assumptions by using the theory of monotone operators. □

In order to justify some of the following computations, we find it convenient to consider a perturbed problem, where we add to the tensor field $\mathbf{S}$ with (p, δ)-structure a linear perturbation. Using again the theory of monotone operators one can easily prove the following proposition.

Proposition 2.7. *Let the tensor field $\mathbf{S}$ in (1.1) have (p, δ)-structure for some $p \in (1, 2]$ and $\delta \in [0, \infty)$, and let $\mathbf{f} \in L^{p'}(\Omega)$ be given. Then there exists a unique weak solution $\mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega)$ of the problem*

$$\begin{cases} -\operatorname{div} \mathbf{S}^\varepsilon(\mathbf{Du}_\varepsilon) = \mathbf{f} & in\ \Omega, \\ \mathbf{u}_\varepsilon = \mathbf{0} & on\ \partial\Omega, \end{cases} \tag{2.12}$$

where

$$\mathbf{S}^\varepsilon(\mathbf{Q}) := \varepsilon\mathbf{Q} + \mathbf{S}(\mathbf{Q}), \quad with\ \varepsilon > 0,$$

i.e., $\mathbf{u}_\varepsilon$ satisfies, for all $\mathbf{v} \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \mathbf{S}^\varepsilon(\mathbf{Du}_\varepsilon) \cdot \mathbf{Dv}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x}.$$

The solution $\mathbf{u}_\varepsilon$ satisfies the estimate

$$\int_\Omega \varepsilon|\nabla\mathbf{u}_\varepsilon|^2 + \phi(|\mathbf{Du}_\varepsilon|)\, d\mathbf{x} \leq c(\|\mathbf{f}\|_{p'}, \delta). \tag{2.13}$$

Remark 2.8. In fact, one could already prove more at this point. Namely, that for $\varepsilon \to 0$, the unique solution $\mathbf{u}_\varepsilon$ converges to the unique weak solution $\mathbf{u}$ of the unperturbed problem (1.1). Let us sketch the argument only, since later we get the same result with different easier arguments. From (2.13) and the properties of $\mathbf{S}$ follows that

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } W_0^{1,p}(\Omega),$$
$$\mathbf{S}(\mathbf{Du}_\varepsilon) \rightharpoonup \chi \quad \text{in } L^{p'}(\Omega).$$

Passing to the limit in the weak formulation of the perturbed problem, we get

$$\int_\Omega \chi \cdot \mathbf{Dv}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} \quad \text{for all } \mathbf{v} \in W_0^{1,p}(\Omega).$$

One can not show directly that $\lim_{\varepsilon \to 0} \int_\Omega \varepsilon \mathbf{Du}_\varepsilon \cdot (\mathbf{Du}_\varepsilon - \mathbf{Du})\, d\mathbf{x} = 0$, since $\mathbf{Du}$ belongs to $L^p(\Omega)$ only. Instead one uses the Lipschitz truncation method (cf. [10, 22]). Denoting by $\mathbf{v}^{\varepsilon,j}$ the Lipschitz truncation of $\xi(\mathbf{u}_\varepsilon - \mathbf{u})$, where $\xi \in C_0^\infty(\Omega)$ is a localization, one can show, using the ideas from [10, 22], that

$$\limsup_{\varepsilon \to 0}\left| \int_\Omega (\mathbf{S}(\mathbf{Du}_\varepsilon) - \mathbf{S}(\mathbf{Du})) \cdot \mathbf{Dv}^{\varepsilon,j}\, d\mathbf{x} \right| = 0,$$

which implies $\mathbf{Du}_\varepsilon \to \mathbf{Du}$ almost everywhere in Ω. Consequently, we have $\chi = \mathbf{S}(\mathbf{Du})$, since weak and a.e. limits coincide.

2.4 Description and properties of the boundary

We assume that the boundary $\partial\Omega$ is of class $C^{2,1}$, that is, for each point $P \in \partial\Omega$, there are local coordinates such that, in these coordinates, we have $P = 0$ and $\partial\Omega$ is locally described by a $C^{2,1}$-function, i.e., there exist $R_P, R_P' \in (0, \infty)$, $r_P \in (0, 1)$ and a $C^{2,1}$-function $a_P \colon B_{R_P}^2(0) \to B_{R_P'}^1(0)$ such that
(b1) $\mathbf{x} \in \partial\Omega \cap (B_{R_P}^2(0) \times B_{R_P'}^1(0)) \Leftrightarrow x_3 = a_P(x_1, x_2)$,
(b2) $\Omega_P := \{(x, x_3) \mid x = (x_1, x_2)^\top \in B_{R_P}^2(0),\ a_P(x) < x_3 < a_P(x) + R_P'\} \subset \Omega$,
(b3) $\nabla a_P(0) = \mathbf{0}$, and for all $x = (x_1, x_2)^\top \in B_{R_P}^2(0)$, $|\nabla a_P(x)| < r_P$,
where $B_r^k(0)$ denotes the k-dimensional open ball with center 0 and radius $r > 0$. Note that r_P can be made arbitrarily small if we make R_P small enough. In the sequel, we will also use, for $0 < \lambda < 1$, the following scaled open sets:

$$\lambda\Omega_P := \left\{(x, x_3) \mid x = (x_1, x_2)^\top \in B_{\lambda R_P}^2(0),\ a_P(x) < x_3 < a_P(x) + \lambda R_P'\right\} \subseteq \Omega_P.$$

To localize near to $\partial\Omega \cap \partial\Omega_P$ for $P \in \partial\Omega$, we fix smooth functions $\xi_P \colon \mathbb{R}^3 \to \mathbb{R}$ such that
(ℓ1) $\chi_{\frac{1}{2}\Omega_P}(\mathbf{x}) \leq \xi_P(\mathbf{x}) \leq \chi_{\frac{3}{4}\Omega_P}(\mathbf{x})$,
where $\chi_A(\mathbf{x})$ is the indicator function of the measurable set A. For the remaining interior estimate, we localize by a smooth function $0 \leq \xi_{00} \leq 1$, with $\mathrm{spt}\, \xi_{00} \subset \Omega_{00}$, where $\Omega_{00} \subset \Omega$ is an open set such that $\mathrm{dist}(\partial\Omega_{00}, \partial\Omega) > 0$. Since the boundary $\partial\Omega$ is compact, we can use an appropriate finite sub-covering which, together with the interior estimate, yields the global estimate.

Let us introduce the tangential derivatives near the boundary. To simplify the notation, we fix $P \in \partial\Omega$, $h \in (0, \frac{R_P}{16})$, and simply write $\xi := \xi_P$, $a := a_P$. We use the standard notation $\mathbf{x} = (x', x_3)^\top$ and denote by $\mathbf{e}^i$, $i = 1, 2, 3$, the canonical orthonormal basis in $\mathbb{R}^3$. In the following, lower-case Greek letters take the values $1, 2$. For a function g, with $\mathrm{spt}\, g \subset \mathrm{spt}\, \xi$, we define, for $\alpha = 1, 2$,

$$g_\tau(x', x_3) = g_{\tau_a}(x', x_3) := g(x' + h\mathbf{e}^\alpha, x_3 + a(x' + h\mathbf{e}^\alpha) - a(x')),$$

and if $\Delta^+ g := g_\tau - g$, we define tangential divided differences by $d^+ g := h^{-1}\Delta^+ g$. It holds that, if $g \in W^{1,1}(\Omega)$, then we have for $\alpha = 1, 2$

$$d^+ g \to \partial_\tau g = \partial_{\tau_a} g := \partial_\alpha g + \partial_\alpha a \partial_3 g \quad \text{as } h \to 0, \tag{2.14}$$

almost everywhere in spt ξ (cf. [18, Section 3]). Conversely, uniform L^q-bounds for d^+g imply that $\partial_\tau g$ belongs to $L^q(\text{spt}\,\xi)$. For simplicity, we denote $\nabla a := (\partial_1 a, \partial_2 a, 0)^\top$. The following variant of integration by parts will be often used.

Lemma 2.9. *Let* spt $g \cup$ spt $f \subset$ spt ξ, *and let* h *be small enough. Then*

$$\int_\Omega fg_{-\tau}\,d\mathbf{x} = \int_\Omega f_\tau g\,d\mathbf{x}.$$

Consequently, $\int_\Omega fd^+g\,d\mathbf{x} = \int_\Omega (d^-f)g\,d\mathbf{x}$. *Moreover, if in addition* f *and* g *are smooth enough and at least one vanishes on* $\partial\Omega$, *then*

$$\int_\Omega f\partial_\tau g\,d\mathbf{x} = -\int_\Omega (\partial_\tau f)g\,d\mathbf{x}.$$

3 Proof of the main result

In the proof of the main result we use finite differences to show estimates in the interior and in tangential directions near the boundary, and for calculations involving directly derivatives in "normal" directions near the boundary. In order to justify that all occurring quantities are well posed, we perform the estimate for the approximate system (2.12).

The first intermediate step is the following result for the approximate problem.

Proposition 3.1. *Let the tensor field* $\mathbf{S}$ *in* (1.1) *have* (p, δ)-*structure for some* $p \in (1, 2]$ *and* $\delta \in (0, \infty)$, *and let* $\mathbf{F}$ *be the associated tensor field to* $\mathbf{S}$. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with* $C^{2,1}$ *boundary and let* $\mathbf{f} \in L^{p'}(\Omega)$. *Then the unique weak solution* $\mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega)$ *of the approximate problem* (2.12) *satisfies*[3]

$$\begin{cases} \displaystyle\int_\Omega \varepsilon\xi_0^2|\nabla^2\mathbf{u}_\varepsilon|^2 + \xi_0^2|\nabla\mathbf{F}(\mathbf{Du}_\varepsilon)|^2\,d\mathbf{x} \le c(\|\mathbf{f}\|_{p'}, \|\xi_0\|_{2,\infty}, \delta), \\[4pt] \displaystyle\int_\Omega \varepsilon\xi_P^2|\partial_\tau\mathbf{Du}_\varepsilon|^2 + \xi_P^2|\partial_\tau\mathbf{F}(\mathbf{Du}_\varepsilon)|^2\,d\mathbf{x} \le c(\|\mathbf{f}\|_{p'}, \|\xi_P\|_{2,\infty}, \|a_P\|_{C^{2,1}}, \delta). \end{cases} \tag{3.1}$$

Here ξ_0 *is a cut-off function with support in the interior of* Ω, *while for arbitrary* $P \in \partial\Omega$ *the function* ξ_P *is a cut-off function with support near to the boundary* $\partial\Omega$, *as defined in Section 2.4. The tangential derivative* ∂_τ *is defined locally in* Ω_P *by* (2.14). *Moreover, there exists a constant* $C_1 > 0$ *such that*[4]

$$\int_\Omega \varepsilon\xi^2|\partial_3\mathbf{Du}_\varepsilon|^2 + \xi^2|\partial_3\mathbf{F}(\mathbf{Du}_\varepsilon)|^2\,d\mathbf{x} \le c(\|\mathbf{f}\|_{p'}, \|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}, \delta^{-1}, \varepsilon^{-1}, C_1), \tag{3.2}$$

provided that in the local description of the boundary, we have $r_P < C_1$ *in* (b3).

In particular, these estimates imply that $\mathbf{u}_\varepsilon \in W^{2,2}(\Omega)$ *and that* (2.12) *holds pointwise a.e. in* Ω.

The two estimates in (3.1) are uniform with respect to ε and could be also proved directly for problem (1.1). However, the third estimate (3.2) depends on ε and it is needed to justify all subsequent steps, which will give the proof of an estimate uniformly in ε, by using a different technique.

Proof of Proposition 3.1. The proof of estimate (3.1) is very similar, being in fact a simplification (due to the fact that there is no pressure term involved) to the proof of the results in [4, Theorems 2.28 and 2.29]. On the other hand, the proof of (3.2) is different from the one in [4] due to the missing divergence constraint. In fact, it adapts techniques known from nonlinear elliptic systems. For the convenience of the reader, we recall the main steps here.

---

[3] Recall that $c(\delta)$ only indicates that the constant c depends on δ and will satisfy $c(\delta) \le c(\delta_0)$ for all $\delta \le \delta_0$.
[4] Recall that $c(\delta^{-1})$ indicates a possibly critical dependence on δ as $\delta \to 0$.

Fix $P \in \partial\Omega$ and use in Ω_P

$$\mathbf{v} = d^-(\xi^2 d^+(\mathbf{u}_\varepsilon|_{\frac{1}{2}\Omega_P})),$$

where $\xi := \xi_P$, $a := a_P$, and $h \in (0, \frac{R_P}{16})$, as a test function in the weak formulation of (2.12). This yields

$$\int_\Omega \xi^2 d^+ \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot d^+ \mathbf{D}\mathbf{u}_\varepsilon \, d\mathbf{x} = -\int_\Omega \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot (\xi^2 d^+ \partial_3 \mathbf{u}_\varepsilon - (\xi_{-\tau} d^- \xi + \xi d^- \xi) \partial_3 \mathbf{u}_\varepsilon) \overset{s}{\otimes} d^- \nabla a \, d\mathbf{x}$$

$$- \int_\Omega \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot \xi^2 (\partial_3 \mathbf{u}_\varepsilon)_\tau \overset{s}{\otimes} d^- d^+ \nabla a - \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot d^- (2\xi \nabla \xi \overset{s}{\otimes} d^+ \mathbf{u}_\varepsilon) \, d\mathbf{x}$$

$$+ \int_\Omega \mathbf{S}^\varepsilon((\mathbf{D}\mathbf{u}_\varepsilon)_\tau) \cdot (2\xi \partial_3 \xi d^+ \mathbf{u}_\varepsilon + \xi^2 d^+ \partial_3 \mathbf{u}_\varepsilon) \overset{s}{\otimes} d^+ \nabla a \, d\mathbf{x}$$

$$+ \int_\Omega \mathbf{f} \cdot d^-(\xi^2 d^+ \mathbf{u}_\varepsilon) \, d\mathbf{x} =: \sum_{j=1}^8 I_j.$$

From the assumption on $\mathbf{S}$, Proposition 2.3 and [4, Lemma 3.11], we have the following estimate:

$$\int_\Omega \varepsilon \xi^2 |d^+ \nabla \mathbf{u}_\varepsilon|^2 + \varepsilon \xi^2 |\nabla d^+ \mathbf{u}_\varepsilon|^2 + |d^+ \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 + \phi(\xi |\nabla d^+ \mathbf{u}|) + \phi(\xi |d^+ \nabla \mathbf{u}|) \, d\mathbf{x}$$

$$\le c \int_\Omega \xi^2 d^+ \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot d^+ \mathbf{D}\mathbf{u}_\varepsilon \, d\mathbf{x} + c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt}\,\xi} \phi(|\nabla \mathbf{u}_\varepsilon|) \, d\mathbf{x}.$$

The terms $I_1 - I_7$ are estimated exactly as in [4, Equations (3.17)–(3.22)], while I_8 is estimated as the term I_{15} in [4, (4.20)]. Thus, we get

$$\int_\Omega \varepsilon \xi^2 |d^+ \nabla \mathbf{u}_\varepsilon|^2 + \varepsilon \xi^2 |\nabla d^+ \mathbf{u}_\varepsilon|^2 + \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 + \phi(\xi |d^+ \nabla \mathbf{u}_\varepsilon|) + \phi(\xi |\nabla d^+ \mathbf{u}_\varepsilon|) \, d\mathbf{x}$$

$$\le c(\|\mathbf{f}\|_{p'}, \|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}, \delta).$$

This proves the second estimate in (3.1) by standard arguments. The first estimate in (3.1) is proved in the same way with many simplifications, since we work in the interior where the method works for all directions. This estimate implies that $\mathbf{u}_\varepsilon \in W^{2,2}_{\text{loc}}(\Omega)$ and that the system (2.12) is well-defined point-wise a.e. in Ω.

To estimate the derivatives in the x_3 direction, we use equation (2.12) and it is at this point that we have changes with respect to the results in [4]. In fact, as usual in elliptic problems, we have to recover the partial derivatives with respect to x_3 by using the information on the tangential ones. In this problem the main difficulty is that the leading order term is nonlinear and depends on the symmetric part of the gradient. Thus, we have to exploit the properties of (p, δ)-structure of the tensor $\mathbf{S}$ (cf. Definition 2.1). Denoting, for $i = 1, 2, 3$,[5] $\mathfrak{f}_i := -f_i - \partial_{\gamma\sigma} S_{i3}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 D_{\gamma\sigma} \mathbf{u}_\varepsilon - \sum_{k,l=1}^3 \partial_{kl} S_{i\beta}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_\beta D_{kl} \mathbf{u}_\varepsilon$, we can re-write the equations in (2.12) as follows:

$$\sum_{k=1}^3 \partial_{k3} S_{i3}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 D_{k3} \mathbf{u}_\varepsilon + \partial_{3\alpha} S_{i3}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 D_{3\alpha} \mathbf{u}_\varepsilon = \mathfrak{f}_i \quad \text{a.e. in } \Omega.$$

Contrary to the corresponding equality [4, Equation (3.49)], here we use directly all the equations in (1.1), and not only the first two. Now we multiply these equations not by $\partial_3 D_{i3} \mathbf{u}_\varepsilon$ as expected, but by $\partial_3 \widehat{D}_{i3} \mathbf{u}_\varepsilon$, where $\widehat{D}_{\alpha\beta} \mathbf{u}_\varepsilon = 0$ for $\alpha, \beta = 1, 2$, $\widehat{D}_{\alpha3} \mathbf{u}_\varepsilon = \widehat{D}_{3\alpha} \mathbf{u}_\varepsilon = 2D_{\alpha3} \mathbf{u}_\varepsilon$ for $\alpha = 1, 2$, $\widehat{D}_{33} \mathbf{u}_\varepsilon = D_{33} \mathbf{u}_\varepsilon$. Summing over $i = 1, 2, 3$, we get, by using the symmetries in Remark 2.2 (iii), that

$$4\partial_{\alpha3} S^\varepsilon_{\beta3}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 D_{\alpha3} \mathbf{u}_\varepsilon \partial_3 D_{\beta3} \mathbf{u}_\varepsilon + 2\partial_{\alpha3} S^\varepsilon_{33}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 D_{\alpha3} \mathbf{u}_\varepsilon \partial_3 D_{33} \mathbf{u}_\varepsilon$$

$$+ 2\partial_{33} S^\varepsilon_{\beta3}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 D_{33} \mathbf{u}_\varepsilon \partial_3 D_{\beta3} \mathbf{u}_\varepsilon + \partial_{33} S^\varepsilon_{33}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 D_{33} \mathbf{u}_\varepsilon \partial_3 D_{33} \mathbf{u}_\varepsilon = \sum_{i=1}^3 \mathfrak{f}_i \partial_3 \widehat{D}_{i3} \mathbf{u}_\varepsilon \quad \text{a.e. in } \Omega. \quad (3.3)$$

---

5 Recall that we use the summation convention over repeated Greek lower-case letters from 1 to 2.

To obtain a lower bound for the left-hand side, we observe that the terms on the left-hand side of (3.3) containing $\mathbf{S}$ are equal to

$$\sum_{i,j,k,l=1}^{3} \partial_{kl} S_{ij}(\mathbf{Du}_\varepsilon) Q_{ij} Q_{kl}$$

if we choose $\mathbf{Q} = \partial_3 \overline{\mathbf{D}}\mathbf{u}_\varepsilon$, where $\overline{D}_{\alpha\beta}\mathbf{u}_\varepsilon = 0$ for $\alpha, \beta = 1, 2$, $\overline{D}_{\alpha 3}\mathbf{u}_\varepsilon = \overline{D}_{3\alpha}\mathbf{u}_\varepsilon = D_{\alpha 3}\mathbf{u}_\varepsilon$ for $\alpha = 1, 2$, and $\overline{D}_{33}\mathbf{u}_\varepsilon = D_{33}\mathbf{u}_\varepsilon$. Thus, it follows from the coercivity estimate in (2.4) that these terms are bounded from below by $\kappa_0 \phi''(|\mathbf{Du}_\varepsilon|)|\partial_3 \overline{\mathbf{D}}\mathbf{u}_\varepsilon|^2$. Similarly, we see that the remaining terms on the left-hand side of (3.3) are equal to $\varepsilon |\partial_3 \overline{\mathbf{D}}\mathbf{u}_\varepsilon|^2$. Denoting $\mathfrak{b}_i := \partial_3 D_{i3}\mathbf{u}_\varepsilon$, $i = 1, 2, 3$, we see that $|\mathfrak{b}| \sim |\widehat{\mathbf{D}}\mathbf{u}_\varepsilon| \sim |\overline{\mathbf{D}}\mathbf{u}_\varepsilon|$. Consequently, we get from (3.3) the estimate

$$(\varepsilon + \phi''(|\mathbf{Du}_\varepsilon|))|\mathfrak{b}| \le |\mathfrak{f}| \quad \text{a.e. in } \Omega.$$

By straightforward manipulations (cf. [4, Sections 3.2 and 4.2]), we can estimate the right-hand side as follows:

$$|\mathfrak{f}| \le c(|\mathbf{f}| + (\varepsilon + \phi''(|\mathbf{Du}_\varepsilon|))(|\partial_\tau \nabla \mathbf{u}_\varepsilon| + \|\nabla a\|_\infty |\nabla^2 \mathbf{u}_\varepsilon|)).$$

Note that we can deduce from $\mathfrak{b}$ information about $\widetilde{\mathfrak{b}}_i := \partial_{33}^2 u_\varepsilon^i$, $i = 1, 2, 3$, because

$$|\mathfrak{b}| \ge 2|\widetilde{\mathfrak{b}}| - |\partial_\tau \nabla \mathbf{u}_\varepsilon| - \|\nabla a\|_\infty |\nabla^2 \mathbf{u}_\varepsilon|$$

holds a.e. in Ω_P. This and the last two inequalities imply

$$(\varepsilon + \phi''(|\mathbf{Du}_\varepsilon|))|\widetilde{\mathfrak{b}}| \le c(|\mathbf{f}| + (\varepsilon + \phi''(|\mathbf{Du}_\varepsilon|))(|\partial_\tau \nabla \mathbf{u}_\varepsilon| + \|\nabla a\|_\infty |\nabla^2 \mathbf{u}_\varepsilon|)) \quad \text{a.e. in } \Omega_P.$$

Adding on both sides, for $\alpha = 1, 2$ and $i, k = 1, 2, 3$, the term

$$(\varepsilon + \phi''(|\mathbf{Du}_\varepsilon|))|\partial_\alpha \partial_i u_\varepsilon^k|,$$

and using on the right-hand side the definition of the tangential derivative (cf. (2.14)), we finally arrive at

$$(\varepsilon + \phi''(|\mathbf{Du}_\varepsilon|))|\nabla^2 \mathbf{u}_\varepsilon| \le c(|\mathbf{f}| + (\varepsilon + \phi''(|\mathbf{Du}_\varepsilon|))(|\partial_\tau \nabla \mathbf{u}_\varepsilon| + \|\nabla a\|_\infty |\nabla^2 \mathbf{u}_\varepsilon|)),$$

which is valid a.e. in Ω_P. Note that the constant c only depends on the characteristics of $\mathbf{S}$. Next, we can choose the open sets Ω_P in such a way that $\|\nabla a_P(x)\|_{\infty,\Omega_P}$ is small enough, so that we can absorb the last term from the right-hand side, which yields

$$(\varepsilon + \phi''(|\mathbf{Du}_\varepsilon|))|\nabla^2 \mathbf{u}_\varepsilon| \le c(|\mathbf{f}| + (\varepsilon + \phi''(|\mathbf{Du}_\varepsilon|))|\partial_\tau \nabla \mathbf{u}_\varepsilon|) \quad \text{a.e. in } \Omega_P,$$

where again the constant c only depends on the characteristics of $\mathbf{S}$. By neglecting the second term on the left-hand side (which is non-negative), raising the remaining inequality to the power 2, and using that $\mathbf{S}$ has (p, δ)-structure for $p < 2$, we obtain

$$\int_\Omega \varepsilon \xi_P^2 |\nabla^2 \mathbf{u}_\varepsilon|^2 \, d\mathbf{x} \le c \int_\Omega |\mathbf{f}|^2 \, d\mathbf{x} + \frac{(\varepsilon + \delta^{2(p-2)})}{\varepsilon} \left(\varepsilon \int_\Omega \xi_P^2 |\partial_\tau \nabla \mathbf{u}_\varepsilon|^2 \, d\mathbf{x} \right).$$

The already proven results on tangential derivatives and Korn's inequality imply that the last integral from the right-hand side is finite. Thus, the properties of the covering imply the estimate in (3.2). □

3.1 Improved estimates for normal derivatives

In the proof of (3.2), we used system (2.12) and obtained an estimate that is not uniform with respect to ε. In this section, by following the ideas in [24], we proceed differently and estimate $\mathcal{P}_3$ in terms of the quantities occurring in (3.1). The main technical step of the paper is the proof of the following result.

Proposition 3.2. *Let the hypotheses in Theorem 1.1 be satisfied with $\delta > 0$, and let the local description a_P of the boundary and the localization function ξ_P satisfy (b1)–(b3) and (ℓ1) (cf. Section 2.4). Then there exists a constant $C_2 > 0$ such that the weak solution $\mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega)$ of the approximate problem (2.12) satisfies,[6] for every $P \in \partial\Omega$,*

$$\int_\Omega \varepsilon\xi_P^2|\partial_3\mathbf{Du}_\varepsilon|^2 + \xi_P^2|\partial_3\mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x} \le C(\|\mathbf{f}\|_{p'}, \|\xi_P\|_{2,\infty}, \|a_P\|_{C^{2,1}}, \delta, C_2),$$

provided $r_P < C_2$ in (b3).

Proof. Let us fix an arbitrary point $P \in \partial\Omega$ and a local description $a = a_P$ of the boundary and the localization function $\xi = \xi_P$ satisfying (b1)–(b3) and (ℓ1). In the following we denote by C constants that depend only on the characteristics of $\mathbf{S}$. First we observe that, by the results of Proposition 2.4, there exists a constant C_0, depending only on the characteristics of $\mathbf{S}$, such that

$$\frac{1}{C_0}|\partial_3\mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \le \mathcal{P}_3(\mathbf{Du}_\varepsilon) \quad \text{a.e. in } \Omega.$$

Thus, using also the symmetry of $\mathbf{Du}_\varepsilon$ and $\mathbf{S}$, we get

$$\int_\Omega \varepsilon\xi^2|\partial_3\mathbf{Du}_\varepsilon|^2 + \frac{1}{C_0}\xi^2|\partial_3\mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x} \le \int_\Omega \xi^2(\varepsilon\partial_3\mathbf{Du}_\varepsilon + \partial_3\mathbf{S}(\mathbf{Du}_\varepsilon)) \cdot \partial_3\mathbf{Du}_\varepsilon \, d\mathbf{x}$$

$$= \int_\Omega \sum_{i,j=1}^3 \xi^2(\varepsilon\partial_3 D_{ij}\mathbf{u}_\varepsilon + \partial_3 S_{ij}(\mathbf{Du}_\varepsilon))\partial_3\partial_j u^i \, d\mathbf{x}$$

$$= \int_\Omega \xi^2(\varepsilon\partial_3 D_{\alpha\beta}\mathbf{u}_\varepsilon + \partial_3 S_{\alpha\beta}(\mathbf{Du}_\varepsilon))\partial_3 D_{\alpha\beta}\mathbf{u}_\varepsilon \, d\mathbf{x}$$

$$+ \int_\Omega \xi^2(\varepsilon\partial_3 D_{3\alpha}\mathbf{u}_\varepsilon + \partial_3 S_{3\alpha}(\mathbf{Du}_\varepsilon))\partial_\alpha D_{33}\mathbf{u}_\varepsilon \, d\mathbf{x}$$

$$+ \int_\Omega \sum_{j=1}^3 \xi^2\partial_3(\varepsilon D_{j3}\mathbf{u}_\varepsilon + S_{j3}(\mathbf{Du}_\varepsilon))\partial_3^2 u_\varepsilon^j \, d\mathbf{x}$$

$$=: I_1 + I_2 + I_3.$$

To estimate I_2, we multiply and divide by the quantity $\sqrt{\phi''(|\mathbf{Du}_\varepsilon|)} \ne 0$, and use Young's inequality and Proposition 2.4. This yields that, for all $\lambda > 0$, there exists $c_\lambda > 0$ such that

$$|I_2| \le \sum_{\alpha=1}^2 \int_\Omega \xi^2|\partial_3\mathbf{S}(\mathbf{Du}_\varepsilon)||\partial_\alpha\mathbf{Du}_\varepsilon|\frac{\sqrt{\phi''(|\mathbf{Du}_\varepsilon|)}}{\sqrt{\phi''(|\mathbf{Du}_\varepsilon|)}} \, d\mathbf{x} + \lambda\int_\Omega \varepsilon\xi^2|\partial_3\mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + c_{\lambda^{-1}}\sum_{\alpha=1}^2 \int_\Omega \varepsilon\xi^2|\partial_\alpha\mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}$$

$$\le \lambda\int_\Omega \xi^2\frac{|\partial_3\mathbf{S}(\mathbf{Du}_\varepsilon)|^2}{\phi''(|\mathbf{Du}_\varepsilon|)} \, d\mathbf{x} + c_{\lambda^{-1}}\sum_{\alpha=1}^2 \int_\Omega \xi^2\phi''(|\mathbf{Du}_\varepsilon|)|\partial_\alpha\mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}$$

$$+ \lambda\int_\Omega \varepsilon\xi^2|\partial_3\mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + c_{\lambda^{-1}}\sum_{\alpha=1}^2 \int_\Omega \varepsilon\xi^2|\partial_\alpha\mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}$$

$$\le C\lambda\int_\Omega \xi^2|\partial_3\mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x} + c_{\lambda^{-1}}\sum_{\alpha=1}^2 \int_\Omega \xi^2|\partial_\alpha\mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x}$$

$$+ \lambda\int_\Omega \varepsilon\xi^2|\partial_3\mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + c_{\lambda^{-1}}\sum_{\alpha=1}^2 \int_\Omega \varepsilon\xi^2|\partial_\alpha\mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}.$$

Here and in the following we denote by $c_{\lambda^{-1}}$ constants that may depend on the characteristics of $\mathbf{S}$ and on λ^{-1}, while C denotes constants that may depend on the characteristics of $\mathbf{S}$ only.

---

[6] Recall that $c(\delta)$ only indicates that the constant c depends on δ and will satisfy $c(\delta) \le c(\delta_0)$ for all $\delta \le \delta_0$.

To treat the third integral I_3, we proceed as follows: We use the following well-known algebraic identity, valid for smooth enough vectors $\mathbf{v}$ and $i, j, k = 1, 2, 3$:

$$\partial_j \partial_k v^i = \partial_j D_{ik}\mathbf{v} + \partial_k D_{ij}\mathbf{v} - \partial_i D_{jk}\mathbf{v}, \tag{3.4}$$

and equations (2.12) point-wise, which can be written, for $j = 1, 2, 3$, as

$$\partial_3(\varepsilon D_{j3}\mathbf{u}_\varepsilon + S_{j3}(\mathbf{Du}_\varepsilon)) = -f^j - \partial_\beta(\varepsilon D_{j\beta}\mathbf{u}_\varepsilon + S_{j\beta}(\mathbf{Du}_\varepsilon)) \quad \text{a.e. in } \Omega.$$

This is possible due to Proposition 3.1. Hence, we obtain

$$|I_3| \leq \sum_{j=1}^{3} \left| \int_\Omega \xi^2 \left(- f^j - \partial_\beta S_{j\beta}(\mathbf{Du}_\varepsilon) - \varepsilon \partial_\beta D_{j\beta}\mathbf{u}_\varepsilon \right)(2\partial_3 D_{j3}\mathbf{u}_\varepsilon - \partial_j D_{33}\mathbf{u}_\varepsilon) \, d\mathbf{x} \right|.$$

The right-hand side can be estimated in a way similar to I_2. This yields that, for all $\lambda > 0$, there exists $c_\lambda > 0$ such that

$$|I_3| \leq \int_\Omega \xi^2 \left(|\mathbf{f}| + \sum_{\beta=1}^{2} |\partial_\beta \mathbf{S}(\mathbf{Du}_\varepsilon)| \right) \left(2|\partial_3 \mathbf{Du}_\varepsilon| + \sum_{\alpha=1}^{2} |\partial_\alpha \mathbf{Du}_\varepsilon| \right) \frac{\sqrt{\phi''(|\mathbf{Du}_\varepsilon|)}}{\sqrt{\phi''(|\mathbf{Du}_\varepsilon|)}} \, d\mathbf{x}$$

$$+ \lambda \int_\Omega \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \sum_{\beta=1}^{2} \int_\Omega \varepsilon \xi^2 |\partial_\beta \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}$$

$$\leq \lambda C \int_\Omega \xi^2 |\partial_3 \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \sum_{\beta=1}^{2} \int_\Omega \xi^2 |\partial_\beta \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x} + \lambda \int_\Omega \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}$$

$$+ c_{\lambda^{-1}} \sum_{\beta=1}^{2} \int_\Omega \varepsilon \xi^2 |\partial_\beta \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \int_\Omega \frac{\xi^2 |\mathbf{f}|^2}{\phi''(|\mathbf{Du}_\varepsilon|)} \, d\mathbf{x}$$

$$\leq \lambda C \int_\Omega \xi^2 |\partial_3 \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \sum_{\beta=1}^{2} \int_\Omega \xi^2 |\partial_\beta \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x} + \lambda \int_\Omega \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}$$

$$+ c_{\lambda^{-1}} \sum_{\beta=1}^{2} \int_\Omega \varepsilon \xi^2 |\partial_\beta \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + c_{\lambda^{-1}}(\|\mathbf{f}\|_{p'}^{p'} + \|\mathbf{Du}_\varepsilon\|_p^p + \delta^p).$$

Observe that we used $p \leq 2$ to estimate the term involving $\mathbf{f}$.

To estimate I_1, we employ the algebraic identity (3.4) to split the integral as follows:

$$I_1 = \int_\Omega \xi^2(\varepsilon \partial_3 D_{\alpha\beta}\mathbf{u}_\varepsilon + \partial_3 S_{\alpha\beta}(\mathbf{Du}_\varepsilon))(\partial_\alpha D_{3\beta}\mathbf{u}_\varepsilon + \partial_\beta D_{3\alpha}\mathbf{u}_\varepsilon) \, d\mathbf{x} - \int_\Omega \xi^2(\varepsilon \partial_3 D_{\alpha\beta}\mathbf{u}_\varepsilon + \partial_3 S_{\alpha\beta}(\mathbf{Du}_\varepsilon))\partial_\beta \partial_\alpha u_\varepsilon^3 \, d\mathbf{x}$$

$$=: A + B.$$

The first term is estimated in a way similar to I_2, yielding, for all $\lambda > 0$,

$$|A| \leq C\lambda \int_\Omega \xi^2 |\partial_3 \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \sum_{\beta=1}^{2} \int_\Omega \xi^2 |\partial_\beta \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x}$$

$$+ \lambda \int_\Omega \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \sum_{\beta=1}^{2} \int_\Omega \varepsilon \xi^2 |\partial_\beta \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}.$$

To estimate B we observe that by the definition of the tangential derivative, we have

$$\partial_\alpha \partial_\beta u_\varepsilon^3 = \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 - (\partial_\alpha \partial_\beta a)D_{33}\mathbf{u}_\varepsilon - (\partial_\beta a)\partial_\alpha D_{33}\mathbf{u}_\varepsilon,$$

and consequently the term B can be split into the following three terms:

$$- \int_\Omega \xi^2(\varepsilon \partial_3 D_{\alpha\beta}\mathbf{u}_\varepsilon + \partial_3 S_{\alpha\beta}(\mathbf{Du}_\varepsilon))(\partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 - (\partial_\alpha \partial_\beta a)D_{33}\mathbf{u}_\varepsilon - (\partial_\beta a)\partial_\alpha D_{33}\mathbf{u}_\varepsilon) \, d\mathbf{x} =: B_1 + B_2 + B_3.$$

We estimate B_2 as follows:

$$|B_2| \leq \int_\Omega \xi^2 |\partial_3 \mathbf{S}(\mathbf{Du}_\varepsilon)||\nabla^2 a||\mathbf{Du}_\varepsilon| \frac{\sqrt{\phi''(|\mathbf{Du}_\varepsilon|)}}{\sqrt{\phi''(|\mathbf{Du}_\varepsilon|)}} + \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon||\nabla^2 a||\mathbf{Du}_\varepsilon| \, d\mathbf{x}$$

$$\leq \lambda \int_\Omega \xi^2 \frac{|\partial_3 \mathbf{S}(\mathbf{Du}_\varepsilon)|^2}{\phi''(|\mathbf{Du}_\varepsilon|)} \, d\mathbf{x} + c_{\lambda^{-1}} \|\nabla^2 a\|_\infty^2 \int_\Omega \xi^2 |\mathbf{Du}_\varepsilon|^2 \phi''(|\mathbf{Du}_\varepsilon|) \, d\mathbf{x}$$

$$+ \lambda \int_\Omega \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \|\nabla^2 a\|_\infty^2 \int_\Omega \varepsilon \xi^2 |\mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}$$

$$\leq \lambda C \int_\Omega \xi^2 |\partial_3 \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \|\nabla^2 a\|_\infty^2 \rho_\phi(|\mathbf{Du}_\varepsilon|) + \frac{1}{8} \int_\Omega \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + 2\varepsilon \|\nabla^2 a\|_\infty^2 \|\mathbf{Du}_\varepsilon\|_2^2.$$

The term B_3 is estimated in a way similar to I_2, yielding, for all $\lambda > 0$,

$$|B_3| \leq \lambda C \int_\Omega \xi^2 |\partial_3 \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \|\nabla a\|_\infty^2 \sum_{\beta=1}^2 \int_\Omega \xi^2 |\partial_\beta \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x}$$

$$+ \lambda \int_\Omega \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + c_{\lambda^{-1}} |\nabla a\|_\infty^2 \sum_{\beta=1}^2 \sum_{\beta=1}^2 \int_\Omega \varepsilon \xi^2 |\partial_\beta \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}.$$

Concerning the term B_1, we would like to perform some integration by parts, which is one of the crucial observations we are adapting from [24]. Neglecting the localization ξ in B_1, we would like to use that

$$\int_\Omega \partial_3 S_{\alpha\beta}^\varepsilon(\mathbf{Du}_\varepsilon) \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} = \int_\Omega \partial_\alpha S_{\alpha\beta}^\varepsilon(\mathbf{Du}_\varepsilon) \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x}.$$

This formula can be justified by using an appropriate approximation that exists for $\mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, since $\partial_\tau \mathbf{u}_\varepsilon = \mathbf{0}$ on $\partial\Omega$. More precisely, to treat the term B_1, we use that the solution $\mathbf{u}_\varepsilon$ of (2.12) belongs to $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. Thus, $\partial_\tau(\mathbf{u}_{\varepsilon|\Omega_P}) = \mathbf{0}$ on $\partial\Omega_P \cap \partial\Omega$, hence $\xi_P \partial_\tau(\mathbf{u}_\varepsilon^3) = \mathbf{0}$ on $\partial\Omega$. This implies that we can find a sequence $\{(\mathcal{S}_n, \mathcal{U}_n)\} \subset C^\infty(\Omega) \times C_0^\infty(\Omega)$ such that $(\mathcal{S}_n, \mathcal{U}_n) \to (\mathbf{S}^\varepsilon, \partial_\tau \mathbf{u}_\varepsilon)$ in $W^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, and perform calculations with $(\mathcal{S}_n, \mathcal{U}_n)$, showing then that all formulas of integration by parts are valid. The passage to the limit as $n \to +\infty$ is done only in the last step. For simplicity, we drop the details of this well-known argument (sketched also in [24]), and we write directly formulas without this smooth approximation. Thus, performing several integrations by parts, we get

$$\int_\Omega \xi^2 \partial_3 S_{\alpha\beta}(\mathbf{Du}_\varepsilon) \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} = \int_\Omega (\partial_\alpha \xi^2) S_{\alpha\beta}(\mathbf{Du}_\varepsilon) \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} - \int_\Omega (\partial_3 \xi^2) S_{\alpha\beta}(\mathbf{Du}_\varepsilon) \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x}$$

$$+ \int_\Omega \xi^2 \partial_\alpha S_{\alpha\beta}(\mathbf{Du}_\varepsilon) \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x}$$

and

$$\varepsilon \int_\Omega \xi^2 \partial_3 D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} = \varepsilon \int_\Omega (\partial_\alpha \xi^2) D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} - \varepsilon \int_\Omega (\partial_3 \xi^2) D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x}$$

$$+ \varepsilon \int_\Omega \xi^2 \partial_\alpha D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x}.$$

This shows that

$$B_1 = \int_\Omega 2\xi \partial_\alpha \xi S_{\alpha\beta}(\mathbf{Du}_\varepsilon) \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} - \int_\Omega 2\xi \partial_3 \xi S_{\alpha\beta}(\mathbf{Du}_\varepsilon) \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x}$$

$$+ \int_\Omega \xi^2 \partial_\alpha S_{\alpha\beta}(\mathbf{Du}_\varepsilon) \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} + \varepsilon \int_\Omega 2\xi \partial_\alpha \xi D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x}$$

$$- \varepsilon \int_\Omega 2\xi \partial_3 \xi D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} + \varepsilon \int_\Omega \xi^2 \partial_\alpha D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x}$$

$$=: B_{1,1} + B_{1,2} + B_{1,3} + B_{1,4} + B_{1,5} + B_{1,6}.$$

To estimate $B_{1,1}$, $B_{1,3}$, $B_{1,4}$, $B_{1,6}$, we observe that

$$\partial_3 \partial_{\tau_\beta} u_\varepsilon^3 = \partial_{\tau_\beta} \partial_3 u_\varepsilon^3 = \partial_{\tau_\beta} D_{33} \mathbf{u}_\varepsilon.$$

By using Young's inequality, the growth properties of $\mathbf{S}$ in (2.6) and (2.8), we get

$$|B_{1,1}| \leq \|\nabla\xi\|_\infty^2 \int_\Omega \frac{|\mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon)|^2}{\phi''(|\mathbf{D}\mathbf{u}_\varepsilon|)}\, d\mathbf{x} + C \sum_{\beta=1}^{2} \int_\Omega \xi^2 \phi''(|\mathbf{D}\mathbf{u}_\varepsilon|)|\partial_{\tau_\beta}\mathbf{D}\mathbf{u}_\varepsilon|^2\, d\mathbf{x}$$

$$\leq \|\nabla\xi\|_\infty^2 \rho_\phi(|\mathbf{D}\mathbf{u}_\varepsilon|) + C \sum_{\beta=1}^{2} \int_\Omega \xi^2 |\partial_{\tau_\beta}\mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2\, d\mathbf{x}$$

and

$$|B_{1,3}| \leq \sum_{\beta=1}^{2} \int_\Omega \xi^2 \frac{|\partial_\beta \mathbf{S}_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon)|^2}{\phi''(|\mathbf{D}\mathbf{u}_\varepsilon|)}\, d\mathbf{x} + \sum_{\beta=1}^{2} \int_\Omega \xi^2 \phi''(|\mathbf{D}\mathbf{u}_\varepsilon|)|\partial_{\tau_\beta}\mathbf{D}\mathbf{u}_\varepsilon|^2\, d\mathbf{x}$$

$$\leq C \sum_{\beta=1}^{2} \int_\Omega \xi^2 |\partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 + \xi^2 |\partial_{\tau_\beta}\mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2\, d\mathbf{x}.$$

Similarly, we get

$$|B_{1,4}| \leq C\varepsilon\|\nabla\xi\|_\infty^2\|\mathbf{D}\mathbf{u}_\varepsilon\|_2^2 + C \sum_{\beta=1}^{2} \int_\Omega \varepsilon\xi^2 |\partial_{\tau_\beta}\mathbf{D}\mathbf{u}_\varepsilon|^2\, d\mathbf{x}$$

and

$$|B_{1,6}| \leq C\varepsilon \sum_{\beta=1}^{2} \int_\Omega \xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2 + \xi^2 |\partial_{\tau_\beta}\mathbf{D}\mathbf{u}_\varepsilon|^2\, d\mathbf{x}.$$

To estimate $B_{1,2}$ and B_{15}, we observe that, using the algebraic identity (3.4) and the definition of the tangential derivative,

$$\begin{aligned}
\partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 &= \partial_\alpha(\partial_\beta \mathbf{u}_\varepsilon^3 + \partial_\beta a \partial_3 u_\varepsilon^3) \\
&= \partial_\alpha \partial_\beta u_\varepsilon^3 + \partial_\alpha \partial_\beta a D_{33}\mathbf{u}_\varepsilon + \partial_\beta a\, \partial_\alpha D_{33}\mathbf{u}_\varepsilon \\
&= \partial_\alpha D_{\beta3}\mathbf{u}_\varepsilon + \partial_\beta D_{\alpha3}\mathbf{u}_\varepsilon - \partial_3 D_{\alpha\beta}\mathbf{u}_\varepsilon + \partial_\alpha \partial_\beta a D_{33}\mathbf{u}_\varepsilon + \partial_\beta a \partial_\alpha D_{33}\mathbf{u}_\varepsilon.
\end{aligned}$$

Hence, by substituting and again the same inequalities as before, we arrive to the following estimates:

$$|B_{1,2}| \leq \lambda C \int_\Omega \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2\, d\mathbf{x} + C(1 + \|\nabla a\|_\infty^2) \sum_{\beta=1}^{2} \int_\Omega \xi^2 |\partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2\, d\mathbf{x} + c_{\lambda^{-1}}(1 + \|\nabla^2 a\|_\infty)\|\nabla\xi\|_\infty^2 \rho_\phi(|\mathbf{D}\mathbf{u}_\varepsilon|),$$

$$|B_{1,5}| \leq \lambda \int_\Omega \varepsilon\xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2\, d\mathbf{x} + c_{\lambda^{-1}}(1 + \|\nabla a\|_\infty^2) \sum_{\beta=1}^{2} \int_\Omega \varepsilon\xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2\, d\mathbf{x} + c_{\lambda^{-1}}(1 + \|\nabla^2 a\|_\infty)\|\nabla\xi\|_\infty^2 \varepsilon\|\mathbf{D}\mathbf{u}_\varepsilon\|_2^2.$$

Collecting all estimates and using that $\|\nabla a\|_\infty \leq r_P \leq 1$, we finally obtain

$$\int_\Omega \varepsilon\xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 + \frac{1}{C_0}\xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2\, d\mathbf{x}$$

$$\leq \lambda \int_\Omega \varepsilon\xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2\, d\mathbf{x} + \lambda C \int_\Omega \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2\, d\mathbf{x}$$

$$+ c_{\lambda^{-1}} \sum_{\beta=1}^{2} \int_\Omega \xi^2 |\partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 + \xi^2 |\partial_{\tau_\beta}\mathbf{F}(\mathbf{u}_\varepsilon)|^2\, d\mathbf{x} + c_{\lambda^{-1}} \sum_{\beta=1}^{2} \int_\Omega \varepsilon\xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2\, d\mathbf{x}$$

$$+ c_{\lambda^{-1}}\big(1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2)\|\nabla\xi\|_\infty^2\big)\big(\|\mathbf{f}\|_{p'}^{p'} + \rho_\phi(|\mathbf{D}\mathbf{u}_\varepsilon|) + \rho_\phi(\delta)\big)$$

$$+ c_{\lambda^{-1}}\big(1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2)\|\nabla\xi\|_\infty^2\big)\|\mathbf{D}\mathbf{u}_\varepsilon\|_2^2.$$

The quantities that are bounded uniformly in $L^2(\Omega_P)$ are the tangential derivatives of $\varepsilon \mathbf{Du}_\varepsilon$ and of $\mathbf{F}(\mathbf{Du}_\varepsilon)$. By definition, we have

$$\partial_\beta \mathbf{Du}_\varepsilon = \partial_{\tau_\beta} \mathbf{Du}_\varepsilon - \partial_\beta a \partial_3 \mathbf{Du}_\varepsilon,$$
$$\partial_\beta \mathbf{F}(\mathbf{Du}_\varepsilon) = \partial_{\tau_\beta} \mathbf{F}(\mathbf{Du}_\varepsilon) - \partial_\beta a \partial_3 \mathbf{F}(\mathbf{Du}_\varepsilon),$$

and if we substitute, we obtain

$$\int_\Omega \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon|^2 + \frac{1}{C_0} \xi^2 |\partial_3 \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x}$$

$$\leq (\lambda + 4\|\nabla a\|_\infty^2) \int_\Omega \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x} + (\lambda C + c_{\lambda^{-1}} \|\nabla a\|_\infty^2) \int_\Omega \xi^2 |\partial_3 \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x}$$

$$+ c_{\lambda^{-1}} \sum_{\beta=1}^2 \int_\Omega \xi^2 |\partial_{\tau_\beta} \mathbf{F}(\mathbf{u}_\varepsilon)|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \sum_{\beta=1}^2 \int_\Omega \varepsilon \xi^2 |\partial_{\tau_\beta} \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}$$

$$+ c_{\lambda^{-1}} \big(1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2) \|\nabla \xi\|_\infty^2\big) \big(\|\mathbf{f}\|_{p'}^{p'} + \rho_\phi(|\mathbf{Du}_\varepsilon|) + \rho_\phi(\delta)\big)$$

$$+ c_{\lambda^{-1}} \big(1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2) \|\nabla \xi\|_\infty^2\big) \|\mathbf{Du}_\varepsilon\|_2^2.$$

By choosing first $\lambda > 0$ small enough such that $\lambda C < 4^{-1} C_0$ and then choosing in the local description of the boundary $R = R_P$ small enough such that $c_\lambda \|\nabla a\|_\infty < 4^{-1} C_0$, we can absorb the first two terms from the right-hand side into the left-hand side to obtain

$$\int_\Omega \varepsilon \xi^2 |\partial_3 \mathbf{Du}_\varepsilon|^2 + \frac{1}{C_0} \xi^2 |\partial_3 \mathbf{F}(\mathbf{Du}_\varepsilon)|^2 \, d\mathbf{x}$$

$$\leq c_{\lambda^{-1}} \sum_{\beta=1}^2 \int_\Omega \xi^2 |\partial_{\tau_\beta} \mathbf{F}(\mathbf{u}_\varepsilon)|^2 \, d\mathbf{x} + c_{\lambda^{-1}} \sum_{\beta=1}^2 \int_\Omega \varepsilon \xi^2 |\partial_{\tau_\beta} \mathbf{Du}_\varepsilon|^2 \, d\mathbf{x}$$

$$+ c_{\lambda^{-1}} \big(1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2) \|\nabla \xi\|_\infty^2\big) \big(\|\mathbf{f}\|_{p'}^{p'} + \rho_\phi(|\mathbf{Du}_\varepsilon|) + \rho_\phi(\delta)\big)$$

$$+ c_{\lambda^{-1}} \big(1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2) \|\nabla \xi\|_\infty^2\big) \|\mathbf{Du}_\varepsilon\|_2^2,$$

where now c_λ depends on the fixed paramater λ, the characteristics of $\mathbf{S}$ and on C_2. The right-hand side is bounded uniformly with respect to $\varepsilon > 0$, due to Proposition 3.1, proving the assertion of the proposition. □

Choosing now an appropriate finite covering of the boundary (for the details, see also [4]), Propositions 3.1–3.2 yield the following result.

Theorem 3.3. *Let the hypotheses in Theorem 1.1 with $\delta > 0$ be satisfied. Then[7]*

$$\varepsilon \|\nabla \mathbf{Du}_\varepsilon\|_2^2 + \|\nabla \mathbf{F}(\mathbf{Du}_\varepsilon)\|_2^2 \leq C(\|\mathbf{f}\|_{p'}, \delta, \partial\Omega).$$

3.2 Passage to the limit

Once this has been proved, by means of appropriate limiting process, we can show that the estimate is inherited by $\mathbf{u} = \lim_{\varepsilon \to 0} \mathbf{u}_\varepsilon$, since $\mathbf{u}$ is the unique solution to the boundary value problem (1.1). We can now give the proof of the main result.

Proof of Theorem 1.1. Let us firstly assume that $\delta > 0$. From Propositions 2.3 and 2.7, and Theorem 3.3, we know that $\mathbf{F}(\mathbf{Du}_\varepsilon)$ is uniformly bounded with respect to ε in $W^{1,2}(\Omega)$. This also implies (cf. [3, Lemma 4.4]) that $\mathbf{u}_\varepsilon$ is uniformly bounded with respect to ε in $W^{2,p}(\Omega)$. The properties of $\mathbf{S}$ and Proposition 2.7 also yield

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7 Recall that $c(\delta)$ only indicates that the constant c depends on δ and will satisfy $c(\delta) \leq c(\delta_0)$ for all $\delta \leq \delta_0$.

that $\mathbf{S}(\mathbf{Du}_\varepsilon)$ is uniformly bounded with respect to ε in $L^{p'}(\Omega)$. Thus, there exists a subsequence $\{\varepsilon_n\}$ (which converges to 0 as $n \to +\infty$), $\mathbf{u} \in W^{2,p}(\Omega)$, $\tilde{\mathbf{F}} \in W^{1,2}(\Omega)$, and $\chi \in L^{p'}(\Omega)$ such that

$$\mathbf{u}_{\varepsilon_n} \rightharpoonup \mathbf{u} \quad \text{in } W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$
$$\mathbf{Du}_{\varepsilon_n} \to \mathbf{Du} \quad \text{a.e. in } \Omega,$$
$$\mathbf{F}(\mathbf{Du}_{\varepsilon_n}) \rightharpoonup \tilde{\mathbf{F}} \quad \text{in } W^{1,2}(\Omega),$$
$$\mathbf{S}(\mathbf{Du}_{\varepsilon_n}) \rightharpoonup \chi \quad \text{in } L^{p'}(\Omega).$$

The continuity of $\mathbf{S}$ and $\mathbf{F}$, and the classical result stating that the weak limit and the a.e. limit in Lebesgue spaces coincide (cf. [12]) imply that

$$\tilde{\mathbf{F}} = \mathbf{F}(\mathbf{Du}) \quad \text{and} \quad \chi = \mathbf{S}(\mathbf{Du}).$$

These results enable us to pass to the limit in the weak formulation of the perturbed problem (2.12), which yields

$$\int_\Omega \mathbf{S}(\mathbf{Du}) \cdot \mathbf{Dv}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} \quad \text{for all } \mathbf{v} \in C_0^\infty(\Omega),$$

where we also used that $\lim_{\varepsilon_n \to 0} \int_\Omega \varepsilon_n \mathbf{Du}_{\varepsilon_n} \cdot \mathbf{Dv}\, d\mathbf{x} = 0$. By density, we thus know that $\mathbf{u}$ is the unique weak solution of problem (1.1). Finally, the lower semi-continuity of the norm implies that

$$\int_\Omega |\nabla \mathbf{F}(\mathbf{Du})|^2\, d\mathbf{x} \le \liminf_{\varepsilon_n \to 0} \int_\Omega |\nabla \mathbf{F}(\mathbf{Du}_{\varepsilon_n})|^2\, d\mathbf{x} \le c.$$

Note that in [3, Section 4] it is shown that

$$\|\mathbf{u}\|^p_{W^{2,\frac{3p}{p+1}}(\Omega)} \le c(\|\mathbf{F}(\mathbf{Du})\|_2^2 + \delta^p),$$

which implies the Sobolev regularity stated in Theorem 1.1. This finishes the proof in the case $\delta > 0$.

Let us now assume that $\delta = 0$. Propositions 3.1 and 3.2 are valid only for $\delta > 0$ and thus cannot be used directly for the case that $\mathbf{S}$ has (p, δ)-structure with $\delta = 0$. However, it is proved in [3, Section 3.1] that for any stress tensor with $(p, 0)$-structure $\mathbf{S}$, there exist stress tensors $\mathbf{S}^\kappa$, having (p, κ)-structure with $\kappa > 0$, approximating $\mathbf{S}$ in an appropriate way.[8] Thus, we approximate (2.12) by the system

$$\begin{cases} -\operatorname{div} \mathbf{S}^{\varepsilon,\kappa}(\mathbf{Du}_{\varepsilon,\kappa}) = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

where

$$\mathbf{S}^{\varepsilon,\kappa}(\mathbf{Q}) := \varepsilon \mathbf{Q} + \mathbf{S}^\kappa(\mathbf{Q}), \quad \text{with } \varepsilon > 0, \kappa \in (0, 1).$$

For fixed $\kappa > 0$, we can use the above theory and the fact that the estimates are uniformly in κ to pass to the limit as $\varepsilon \to 0$. Thus, we obtain that for all $\kappa \in (0, 1)$, there exists a unique $\mathbf{u}_\kappa \in W_0^{1,p}(\Omega)$ satisfying, for all $\mathbf{v} \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \mathbf{S}^\kappa(\mathbf{Du}_\kappa) \cdot \mathbf{Dv}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x}$$

and

$$\int_\Omega |\mathbf{F}^\kappa(\mathbf{Du}_\kappa)|^2 + |\nabla \mathbf{F}^\kappa(\mathbf{Du}_\kappa)|^2\, d\mathbf{x} \le c(\|\mathbf{f}\|_{p'}, \partial\Omega), \tag{3.5}$$

where the constant is independent of $\kappa \in (0, 1)$ and $\mathbf{F}^\kappa \colon \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}_{\mathrm{sym}}$ is defined through

$$\mathbf{F}^\kappa(\mathbf{P}) := (\kappa + |\mathbf{P}^{\mathrm{sym}}|)^{\frac{p-2}{2}} \mathbf{P}^{\mathrm{sym}}.$$

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8 The special case $\mathbf{S}(\mathbf{D}) = |\mathbf{D}|^{p-2}\mathbf{D}$ could be approximated by $\mathbf{S}^\delta = (\delta + |\mathbf{D}|)^{p-2}\mathbf{D}$. However, for a general extra stress tensor $\mathbf{S}$ having only (p, δ)-structure, this is not possible.

Now we can proceed as in [3]. Indeed, from (3.5) and the properties of $\phi_{p,\kappa}$ (in particular (2.3)), it follows that $\mathbf{F}^{\kappa}(\mathbf{Du}_{\kappa})$ is uniformly bounded in $W^{1,2}(\Omega)$, that $\mathbf{u}_{\kappa}$ is uniformly bounded in $W^{1,p}_{0}(\Omega)$ and that $\mathbf{S}^{\kappa}(\mathbf{Du}_{\kappa})$ is uniformly bounded in $L^{p'}(\Omega)$. Thus, there exist $\mathbf{A} \in W^{1,2}(\Omega)$, $\mathbf{u} \in W^{1,p}_{0}(\Omega)$, $\chi \in L^{p'}(\Omega)$, and a subsequence $\{\kappa_n\}$, with $\kappa_n \to 0$, such that

$$\mathbf{F}(\mathbf{Du}_{\kappa_n}) \rightharpoonup \mathbf{A} \quad \text{in } W^{1,2}(\Omega),$$
$$\mathbf{F}^{\kappa_n}(\mathbf{Du}_{\kappa_n}) \to \mathbf{A} \quad \text{in } L^{2}(\Omega) \text{ and a.e. in } \Omega,$$
$$\mathbf{u}_{\kappa_n} \rightharpoonup \mathbf{u} \quad \text{in } W^{1,p}_{0}(\Omega),$$
$$\mathbf{S}^{\kappa}(\mathbf{Du}_{\kappa}) \rightharpoonup \chi \quad \text{in } L^{p'}(\Omega).$$

Setting $\mathbf{B} := (\mathbf{F}^0)^{-1}(\mathbf{A})$, it follows from [3, Lemma 3.23] that

$$\mathbf{Du}_{\kappa_n} = (\mathbf{F}^{\kappa_n})^{-1}(\mathbf{F}^{\kappa_n}(\mathbf{Du}_{\kappa_n})) \to (\mathbf{F}^0)^{-1}(\mathbf{A}) = \mathbf{B} \quad \text{a.e. in } \Omega.$$

Since weak and a.e. limit coincide, we obtain that

$$\mathbf{Du}_{\kappa_n} \to \mathbf{Du} = \mathbf{B} \quad \text{a.e. in } \Omega.$$

From [3, Lemma 3.16] and [3, Corollary 3.22], it now follows that

$$\mathbf{F}(\mathbf{Du}_{\kappa_n}) \rightharpoonup \mathbf{F}^0(\mathbf{Du}) \quad \text{in } W^{1,2}(\Omega),$$
$$\mathbf{S}^{\kappa_n}(\mathbf{Du}_{\kappa_n}) \to \mathbf{S}(\mathbf{Du}) \quad \text{a.e. in } \Omega.$$

Since weak and a.e. limit coincide, we obtain that

$$\mathbf{Du} = \chi \quad \text{a.e. in } \Omega.$$

Now we can finish the proof in the same way as in the case $\delta > 0$.   □

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