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Dragging, instrumented abduction and evidence in processes of conjecture generation in a dynamic geometry environment

1 Introduction

Several studies on the teaching and learning of geometry over the past decades (for an extensive review of the literature from the first two decades see Battista 2007; after that, see, e.g., Mariotti 2015; Sinclair and Robutti 2013; Baccaglini-Frank et al. 2018) have shown that a Dynamic Geometry Environment (DGE) can foster learners' processes of conjecture generation and argumentation. Especially in open problem situations, the dragging tool plays a crucial role (e.g., Arzarello et al. 2002; Leung et al. 2013; Sinclair and Robutti 2013; Mariotti 2015). However, in a DGE, phenomena are experienced dynamically and interactively, while in the Euclidean world everything is axiomatic and deductive. The discussion is still open on how to link phenomena experienced in a DGE with their interpretations in the Euclidean world. Previous studies have addressed questions such as the following: How can we make geometrical sense of 'dragging phenomena' in a DGE? How can dragging lead to identifying invariants that potentially correspond to geometrical properties? How is perception involved when particular ways of dragging are used during explorations in a DGE, and how are invariants and their relationships identified and geometrically interpreted? (e.g., Hölzl 1996, 2001; Sträßer 2001; Lopez-Real and Leung 2006; Leung et al. 2013).

With this paper I wish to contribute to this research field and, in particular, extend the discussion by looking at the issue of *evidence* in the context of conjecture generation in a DGE. To accomplish this purpose, I will refer to the definition of evidence contained in the Oxford Dictionary, that is "the available body of facts or information indicating whether a belief or proposition is true or valid." I will avoid using the word "belief" because of its use in mathematics education in studies on affect; I will speak of *propositions*. In the domain of Euclidean Geometry, the validity of a proposition is given by a body of facts, such that subsets of them can be organized into deductive chains that prove the proposition. This validation process has two implications: 1) validity is referred to the underlying theory (here, Euclidean Geometry)—which facts to consider and how to link them in deductive chains depend on such theory; 2) different sets of facts (of course any of these sets will include the facts in the premise of the proposition) from the overarching body of facts may be linked together to constitute different proofs of the same proposition. So the whole body of facts, constituting evidence that guarantees validity of a certain proposition in Euclidean Geometry, includes all possible proofs of such a proposition. However, the definition also implicitly relates such a 'body of facts' to a person, an actor, to whom the facts are available and for whom they indicate *validity* or *truth* of the proposition. Such a body of facts available to the actor may not include any complete proof of the proposition. This situation could happen either because the facts, including the theorems available to the actor, are not sufficient to be re-organized into a deductive chain constituting a proof, or in the case that they are sufficient, because the actor is still not able to organize them into a proof.

Whenever evidence is collected to infer validity of a proposition—i.e., whenever there is a theory of reference—I will refer to it as *theoretical evidence*. In particular, given the considerations above, theoretical evidence of a proposition may be partial, but it may still indicate to the actor that the proposition is likely to be valid (in this case the proposition is a conjecture, not a valid proposition, or *theorem*, until at least one proof is reached).

A key feature of the definition of evidence presented and discussed above is that it allows for another interpretation, which is quite insightful in the context of a DGE. Indeed, within the phenomenological domain of a DGE, the situation is somewhat different from that of Euclidean Geometry: propositions may be perceived as true based on the facts that the actor collects as feedback from the system. I will call this body of facts, which are not referred to a mathematical theory, *phenomenological evidence*.

What makes the relationship between the phenomenological domain of a DGE and the theoretical domain of Euclidean Geometry so complex? While the latter cannot be directly accessed, experiences within a DGE are concrete, figures on the screen have a physical nature, they can be interacted with through physical movement (though this movement is not always direct); while Euclidean Geometry is static and abstract, explorations in a DGE are immersed in time, figures are dynamic, they change over time. An important aspect of this complex relationship has been described in terms of an implicit "dragging exploration principle" (Leung et al. 2013):

> [D]uring dragging, a figure maintains all the properties according to which it was constructed and all the consequences that the construction properties entail within the axiomatic world of Euclidean geometry. (Leung et al. 2013, p. 458)

So, a DGE provides a context where geometrical properties, which appear as invariants under dragging, can be visually and kinesthetically perceived and linked back to the theoretical grounds that underlie them, embedded in the system. The dragging principle lies at the heart of the design of a typical DGE and it embraces variation, invariants and relationships between invariants, explicitly linking the manipulability of DGE figures to geometrical properties in Euclidean Geometry.

However, research has shown that students can encounter difficulties in trying to restructure and re-interpret theoretically their dynamic DGE experiences. For example, Goldenberg and Cuoco (1998) provide an insightful glimpse into this complexity.

> We hypothesize that when an endpoint of a stretchy segment is moved, and the segment is the only object present, the user perceives the movement as a *translation* of the point. That is, dragging A to A' may feel psychologically like a translation. The display may also tend to be seen more as a mapping of A (in its various positions) to C (the midpoint of AB), than as a mapping of A and C to A' and C' respectively. But other situations may lead to very different perceptions. For example consider the same construction with a perpendicular to AB at B. A comparable movement of A now appears to rotate the system; the sense that A is being translated is now considerably diminished…What students make out of this we don't yet fully know. (p. 352)

In Goldenberg and Cuoco's example above, of the construction in a DGE of a segment AB with a perpendicular line l through its endpoint B, the following two propositions could be stated: l moves around B (DGE proposition); $l \perp AB$ is an invariant of the construction, or, simply $l \perp AB$ (Euclidean Geometry proposition). The DGE proposition is supported by phenomenological evidence in the form of visual feedback from the DGE as A is dragged. On the other hand, the proposition $l \perp AB$ is supported by theoretical evidence, which is the construction property that establishes the geometrical relationship between AB and l. The phenomenological evidence of the DGE proposition lies in the construction's interactivity and visual feedback, while the theoretical evidence of the Euclidean Geometry proposition can be viewed as completely independent of any feedback from the DGE, including dynamism.

A major difficulty that students face is how to interpret DGE propositions and phenomenological evidence and transfer them into Euclidean Geometry propositions with theoretical evidence for them. In this paper I concentrate on a particular context in which such an interpretation seems to occur, supported by an abductive inference intertwined with the manipulation of a DGE figure through dragging. In particular, I look at the key role played by such an abductive inference that occurs when maintaining dragging is used to generate a conjecture. I argue that this particular abductive inference is what allows the interpreting of phenomenological evidence in terms of theoretical evidence supporting the conjecture, although in the case in focus such theoretical evidence is not yet sufficient for the students to reach a proof. In other words, I show how abductive reasoning can provide a cognitive key for transitioning from phenomenological experiences to theoretical domains. This transitioning has been discussed in other domains, for example by Abrahamson (2012), which increases its significance in this paper.

2 Theoretical background

Making sense of the feedback from a DGE in open problem explorations is not always a straightforward process. Research has shown how different students may interpret the same feedback (e.g., Goldenberg and Cuoco 1998; Hölzl 2001); in particular, *invariants* and their identification have been central in many studies (e.g, Baccaglini-Frank et al. 2009; Leung et al. 2013; Mariotti 2015). One aspect that contributes to the complexity of interpreting DGE feedback, which is particularly relevant here, is the following.

In a DGE a student can perceive invariants, i.e., properties of a figure that persist during dragging, but she may also perceive invariant relationships between invariants (Leung et al. 2013). For example, if the perpendicular bisectors, *r* and *l*, of two parallel segments, AB and CD, are constructed, the following properties appear as invariants, as they are embedded into the construction: $AB \parallel CD$; the midpoint of AB belongs to *r*, the midpoint of CD belongs to *l*; $r \perp AB$; $l \perp CD$. However, another invariant can also be identified: $r \parallel l$, which is a consequence, within the theory of Euclidean Geometry, of the figure's construction properties. The appearance of derived invariants such as this one, make it possible to also perceive invariant relationships between invariants within a DGE; in this case, for example, this invariant relationship can be stated as a proposition such as "if $AB \parallel CD$, then $r \parallel l$", or as "when $AB \parallel CD$, $r \parallel l$".

Table 1 shows possible facts that could be collected as phenomenological and theoretical evidence, respectively, for such propositions. The two different versions of the proposition are used in this example to indicate whether there is reference to the theory of Euclidean Geometry (version with "if…then…") or not (version with "when"). Notice that the facts in rows 1, 2, 3 of Table 1 alone do not constitute sufficient theoretical evidence for a proof of the proposition. On the other hand, if the theoretical evidence collected, includes, facts 1, 2, 3 and 4 in this column, then the theoretical evidence available to the actor can include a proof of the proposition. Indeed, if the theorem in fact 4 is used with facts 1, 2, and 3 in its premise, it proves that $r \parallel l$. The phenomenological evidence, in this example, includes invariants perceived by the actor, their simultaneous appearance, and awareness of the order in which the elements of the figure were constructed.

Table 1: phenomenological evidence and theoretical evidence that could be collected for the proposition stated in two versions

Proposition	
Version without reference to Eucl. Geo.: When $AB \parallel CD$, $r \parallel l$	Version with reference to Eucl. Geo.: If $AB \parallel CD$, then $r \parallel l$
Phenomenological evidence	*Theoretical evidence*
1. The two invariants $AB \parallel CD$ and $r \parallel l$ appear simultaneously	1. $AB \parallel CD$ (premise of the proposition)
2. $AB \parallel CD$ was imposed voluntarily, at the beginning, in the act of construction	2. $r \perp AB$ by construction, because of the definition of perpendicular bisector of AB (implicit premise of the proposition)
3. $r \parallel l$ was observed once the construction was complete	3. $l \perp CD$ by construction, because of the definition of perpendicular bisector of CD (implicit premise of the proposition)
	4. If two lines are perpendicular to parallel lines, then they, too, are parallel (theorem known by the actor)

Invariant relationships between invariants, how these can be identified through abduction and stated through propositions, and what evidence supports such propositions, are the main foci of sections 2 and 3. The theoretical constructs I introduce in this section of the paper are summarized in Table 5.

2.1 Evidence for propositions stating a relationship between invariants properties of a figure

It has been extensively discussed in the literature how we can distinguish in a DGE between *direct invariants*, those determined by the geometrical relations defined by the commands used to accomplish the construction, and *indirect invariants*, those that are derived as a consequence within Euclidean Geometry (Laborde and Sträßer 1990). Indeed, as discussed by Mariotti (2015), the relationship of logical dependency between the two types of invariants corresponds to an asymmetry between the two types of invariants, an asymmetry that can also be recognized in the relative movement of the different elements of a figure (Mariotti 2015, p. 159).

Indeed, the motion of an element constructed in a DGE can be direct or indirect: it is *direct* if the variation of this element occurs directly under the direct control of the mouse (or finger on a touch screen); it is *indirect* if the variation of this element of the construction occurs as a consequence of a direct motion of another element. Through the action of dragging the actor can ‘feel’ motion dependency: *direct control* is exercised over an element that can be moved directly, while over the other variations of the construction *indirect control* is exercised. A focus on the simultaneity of two invariants together with an asymmetry in the control exercised over each of them can lead to the perception of *causality* within the DGE. In other words, the actor may perceive a causal link between the two invariants, inv(A) and inv(B), that she is observing while acting directly on inv(B), which can lead to a proposition such as ‘inv(A) because inv(B)’.

In the case of maintaining dragging analyzed in this paper, typically an actor within a DGE will notice that dragging a point P along a certain path *C*, an invariant inv(A) appears. Her proposition might state: ‘When I drag P on *C*, inv(A)’. The movement of P along the path *C* can be interpreted geometrically as $P \in C$. In any case, however it is described, inv(B) (dragging P on *C* or $P \in C$) is perceived as what causes inv(A). The phenomenological evidence the actor may consider for a proposition like ‘inv(A) because inv(B)’ is the simultaneity of the two invariants and the fact that she controls directly inv(B) and only indirectly inv(A). Such a proposition can turn into a conditional statement such as: 'If inv(B), then inv(A)', a conjecture. How can a causal relationship between two invariants perceived in a DGE be interpreted in terms of *conditionality* (Mariotti 2015) linking two geometrical properties in the theoretical domain of Euclidean Geometry? And what evidence will emerge for the new conjecture? The following sections answer these questions in a case in which the actor uses maintaining dragging, described in the next section.

2.2 Maintaining dragging and abduction in DGE-based processes of conjecture generation

Arzarello et al. (2002) published a classification of various types of dragging, stemming from students' objectives that had been observed during explorations in which the main task was to generate conjectures. This classification led to a model that describes processes through which students generate and validate conjectures. Such a model assigns a key role to the transition from the exploratory phase, in which conjectures are generated, to the proving phase. This transition can be led by an *abduction*, a form of reasoning corresponding to the selection of appropriate fragments of theory for an identified case, in response to the (usually implicit) question 'Which rule is this the case of?'. The rules are usually theorems familiar to the students. The process culminates with the formulation of a conjecture.

The abduction described here refers to Peirce's (1960) definition in a general form: (*fact*) a fact A is observed; (*rule*) if C were true, then A would certainly be true; (*hypothesis*) so, it is reasonable to assume C is true[1]. Comparing this description of abduction to those proposed by Eco, it seems quite similar to what Eco describes as an "undercoded abduction"—the type that occurs when the rule is selected among a series of equiprobable alternatives—and the type Thagard calls abduction *stricto sensu* (Eco 1986, pp. 41–43).

Moreover, in Arzarello and his colleagues' model, the abductive inference corresponds to the use of a particular type of dragging, during which the students try to maintain a selected property that they consider interesting, inducing it as a soft invariant (Healy 2000). This dragging modality,

[1] Peirce's classical example for this general form is given using as fact "These beans [oddly] are white", as rule "All the beans from this bag are white" and as hypothesis "These beans are from this bag".

later called *maintaining dragging* (Baccaglini-Frank and Mariotti 2010), and conjecture generation processes in which it is used, became the focus of a study conducted between 2008 and 2010 (Baccaglini-Frank 2010b; Baccaglini-Frank and Mariotti 2010) leading to a new model that clarifies the relationships between abductive inferences, dragging and the conjecture generation process in focus, introducing the notion of *instrumented abduction* (Baccaglini-Frank, 2010a). I explain how, in fact, instrumented abduction can be seen as a type of *manipulative abduction*, an abduction that "happens when we are thinking through doing and not only, in a pragmatic sense, about doing" (Magnani 2004, p. 880), supported by maintaining dragging in a DGE, and leading to the formulation of a conjecture.

2.2.1 Using maintaining dragging to generate conjectures: an example

At this point an example is useful to clarify how maintaining dragging can be used in the process of conjecture generation.

> Construct the quadrilateral ABCD (see Figure 1) following the steps below and make conjectures about the possible types of quadrilaterals. Describe all the ways in which you can obtain a particular type of quadrilateral. *Construction steps*: a point P and a line *r* through P, the perpendicular line to *r* through P, C on the perpendicular line, a point A symmetrical to C with respect to P, a point D on the side of *r* containing A, the circle with center C and radius CP, point B as the second intersection between the circle and the line through P and D.

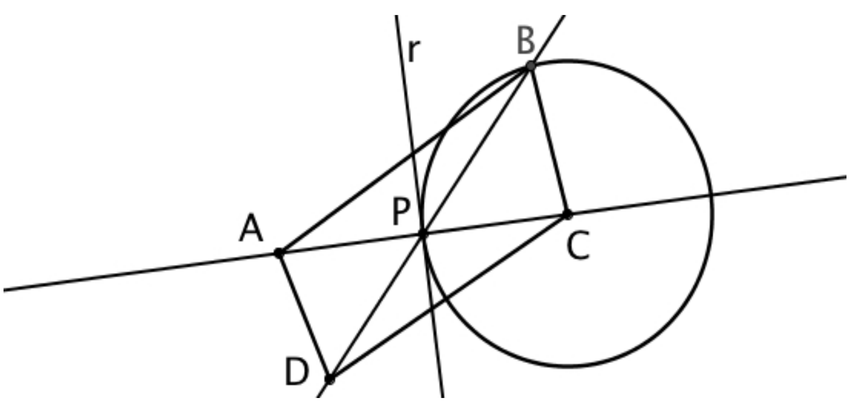


Fig. 1: **a possible result of the construction in the situation described above.**

The figure can be acted upon by dragging points (let us think about dragging D), and some geometrical properties appear as invariants no matter how the point is dragged (e.g., 'CP = PA') while others can become soft invariants if they are induced through maintaining dragging (e.g., 'DA = CB', 'CD || BA', 'ABCD parallelogram').

Let us concentrate on the event in which the actor decides to drag D maintaining the property 'ABCD parallelogram', that is inducing such a property as a soft invariant, and to search for a specific condition under which such a property occurs. New invariants, such as 'D lies on a circle C_{AP} with its center in A and the radius AP' (not visible in Fig. 1) can be perceived while the intentionally induced invariant is visually verified.

The emergence of new soft invariant geometrical properties during the use of maintaining dragging can be supported by the use of the trace mark, a functionality present in most DGEs. The actor may perceive the newly observed invariants as causally linked to the invariant she was inducing. In this process, causality stems from an intentional action: the direct movement of D on the circle C_{AP} causes the induced invariant, resulting in ABCD being a parallelogram. A DGE proposition such as 'Dragging D along C_{AP} causes ABCD to be a parallelogram' is supported by phenomenological evidence: the simultaneity of the invariants' appearance and the direct control perceived over the regularity in the movement of D.

Research has suggested that high school students who learn to use maintaining dragging for generating conjectures re-interpret DGE propositions such as the one above (though usually implicit) into conditional statements such as: 'if D belongs to C_{AP}, then ABCD is a parallelogram' (Baccaglini-Frank and Mariotti 2011; Mariotti and Baccaglini-Frank 2011; Baccaglini-Frank 2010a, 2010b; Baccaglini-Frank and Mariotti 2010; Leung et al. 2013; Antonini and Baccaglini-Frank 2016). The following sections shed light on how such re-interpretation might be explained in terms of a particular kind of abduction, and on the evidence supporting the final conjecture at the end of the abductive process. This way of looking at the conjecturing process is proposed here for the first time.

2.2.2 Abduction in conjecture generation through maintaining dragging

Attempts to use Peirce's construct to analyze the main abductive process that takes place when maintaining dragging is used for conjecture generation, do not work (Baccaglini-Frank and Mariotti, 2011). The situation can be represented schematically as shown in Table 2. In tasks of conjecture generation, the *hypothesis*, in Peirce's terms, is not what we consider to be the product of the abduction. Instead, the overall product in our case is much closer to Peirce's *rule* "if C were true then A would be true". Moreover, such a rule comes from the actor's experience[2]; it is a piece of information from the actor's bag of acquired knowledge. In the case of our conjecture-generation example, what is the equivalent of such a rule? I mark this question with a question mark in the corresponding cell of Table 2. The hypothesis that emerges as the product of the abduction in the conjecture-generation example is a conjecture: 'if inv(C), then inv(A)'. So the rule we are looking for has to deal with finding phenomenological evidence for a proposition related to the conjecture—a proposition such as 'inv(C) causes inv(A)'—and interpreting it in conditional terms with respect to the theory of Euclidean Geometry. Indeed, as the actor uses maintaining dragging she is prepared to observe that the dragged point moves along a path, which she interprets as a new invariant, inv(C); she also interprets her perception of inv(C) causing inv(A) in conditional terms. The conjecture which is the product of the abduction, expresses a conditional relationship between inv(C) and inv(A).

Table 2: A comparison of abduction according to Peirce and abduction in particular tasks of conjecture-generation in a DGE.

	Peirce's example	**Conjecture-generation example**
Fact	A is observed	Inv(A) is induced
Rule	If C was true, then A would be true	?
Hypothesis	C is true	If inv(C), then inv(A)

The issue of the nature of the 'rule' is actually quite delicate and has been explored by philosophers. For example, Eco (1986) distinguishes between three types of abduction: a *hypothesis* or *overcoded abduction* "when the law is given automatically or quasi-automatically"; an *undercoded abduction* "when the rule must be selected among a series of equiprobable alternatives […]. Thagard calls this kind of reasoning an abduction *stricto sensu*: the rule selected can be, in a given co-text, the most plausible one, but it is not certain whether it is the most correct or the only correct one"; a *creative abduction*, "in which the rule acting as an explanation has to be invented *ex novo*" (pp. 41–43).

When maintaining dragging is used to generate conjectures, among the three types described by Eco, the abductive process seems to be closest to the *creative* type. Indeed, the rule, too, needs to be found. Moreover, manipulation within the DGE, through maintaining dragging plays an

[2] In his famous example the rule is "all the beans from this bag are white".

essential role in the abductive process in focus. A form of abduction described by Magnani (2004) seems to suit our case quite well. Magnani describes abduction in general as:

> the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated. (p. 879)

Indeed, the product of our abduction is an explanatory hypothesis in that it explains a phenomenological experience with respect to a theory, that of Euclidean Geometry. Moreover, in the case of maintaining dragging, an important manipulation occurs because of the kinesthetic-haptic dimension of the exploration. When maintaining dragging is used to generate a conjecture, discovery happens "through doing"; indeed, Magnani introduces *manipulative abduction*, as

> a kind of 'discovering through doing', cases in which new and still unexpressed information is codified by means of manipulations of some external objects […] Manipulative abduction happens when we are thinking through doing and not only, in a pragmatic sense, about doing. It refers to an extra-theoretical behavior that aims at creating communicable accounts of new experiences to integrate them into previously existing systems of experimental and linguistic (theoretical) practices. (p. 880)

Indeed, the phenomenological experience in which causality is perceived between two invariants of a figure in a DGE is 'extra-theoretical' with respect to the Theory of Euclidean Geometry, and the manipulative abduction that takes place creates a communicable account of this experience to integrate it into the existing system of Euclidean Geometry (a specific theoretical practice).

According to both Magnani's and Eco's definitions of creative abduction, the product of the abductive process we are studying is the conditional relationship between inv(A) and inv(C). This conditional relationship is by all means an 'explanatory hypothesis' developed to describe a complex observed phenomenon, as discussed above. Before being expressed as a conjecture, explicit hints of establishment of this conditional relationship frequently appear in students' words in expressions like: "Since/given that/because of/whenever this is true/can be seen/there is [a new property observed during dragging], there is also this [the original interesting property induced through maintaining dragging]"; or "In order for this to be seen/true [the interesting property induced through maintaining dragging], also this [a new property observed during dragging] needs to happen/be seen/true."

3 Instrumented abduction as a bridge from phenomenological to theoretical evidence

As we have established, the product of abduction best fits the definition of the outcome of a manipulative abduction according to Magnani. We still have something to gain from returning to Peirce's idea of creative abduction, because it guides us in clarifying the *rule* involved in the abduction we are studying. Considerations on such a rule will shed light on the evidence that our abduction brings to the final conjecture. In previous studies we noticed that the rule seems to develop and stabilize for many students after their first few uses of maintaining dragging (Baccaglini-Frank and Mariotti 2011). These expert students seem to make a theoretical 'leap' from the maintained inv(A) to the newly perceived inv(C). However, the transition from inv(A) to inv(C) and the conditional proposition expressing their conjecture appears to be extremely smooth, and to occur through a seemingly automatic process. Indeed, in these cases it is no longer possible to identify key moments in the abductive process, which instead were so evident in the explorations analyzed by Arzarello and his colleagues:

> […] the process of conjecture-generation […] seems to become 'automatic', and the solver proceeds through steps that lead smoothly to the discovery of invariants and to the generation of a conjecture, with no apparent abductive ruptures in the process. (Baccaglini-Frank and Mariotti 2011, p. 105)

I explain this absence of "abductive ruptures" using the notions of manipulative abduction and of a special rule, a *meta-theorem-in-action*[3]. The prefix *meta* indicates that the theorem-in-action does strictly not belong either to the phenomenological domain of the DGE or to the theoretical domain of Euclidean Geometry (it is "extra-theoretical" in Magnani's terms). This rule is outside of each domain, but it constitutes a bridge between them. Indeed, within the phenomenological domain of the DGE, expert students are likely to be aware of the *simultaneous perception* of inv(A) and inv(C) and of their *asymmetry* determined by the different types of control they

[3] The name echoes Vergnaud's construct of *theorem-in-action* (2009).

exercise over them: *direct control* over C and *indirect control* over A. Expert students establish a causal relationship between these invariants, which could be stated as ‘inv(C) causes inv(A)’, and for which they have collected phenomenological evidence.

Thanks to the meta-theorem-in-action (presented in Table 3) the expert student can also re-interpret this proposition into a conditional proposition such as ‘if inv(C), then inv(A)’.

Table 3: a possible meta-theorem-in-action associated to maintaining dragging

Meta-theorem-in-action (associated to maintaining dragging)		
The conditional proposition: If C, then A	implies (in the DGE)	perception of causality between inv(C) and inv(A), through: • Perception of simultaneity of soft invariants inv(A) and inv(C) • Awareness of different control: direct over inv(C), indirect over inv(A)

The meta-theorem-in-action is now ready to become the rule of the manipulative abduction that allows expert students to produce a conditional proposition, replacing the ‘?’ in Table 2. The structure of such manipulative abduction is shown in Table 4.

Table 4: Structure of the manipulative abduction associated with use of maintaining dragging

Fact	perception of causality: simultaneous perception of inv(C) and inv(A); direct control over inv(C); indirect control over inv(A)
Rule	meta-theorem-in-action (associated to maintaining dragging)
Hypothesis	conditional proposition: if inv(C), then inv(A)

The transition from inv(A) to inv(C) may be so smooth for expert students because the meta-theorem-in-action has been constructed as they learned to use maintaining dragging for generating conjectures, and so it guides the production of the explanatory hypothesis. Moreover, the meta-theorem-in-action acts as a bridge between the phenomenological domain of the DGE and the theoretical domain of Euclidean Geometry. Once it has been constructed, on one hand, it can guide the student in searching for specific phenomenological evidence for a causal proposition, which usually remains implicit; and, on the other hand, it allows the student to predict the perceptive effects of a relationship of a logical dependence (a conditional proposition) between geometrical properties identifiable in the DGE. So, through this meta-theorem-in-action phenomenological evidence for a causal proposition is re-interpreted as theoretical evidence for the conjecture, a conditional proposition (e.g., ‘if D belongs to C_{AP}, then ABCD is a parallelogram’).

Instrumented abductions are quite different from abductions *stricto sensu*, in that their rule is not a theorem in Euclidean Geometry (indeed, their rule is “extra-theoretical” with respect to Euclidean Geometry, as discussed), whereas in the case of abductions *stricto sensu* the rule is a theorem in Euclidean Geometry. This seems to leave a ‘theoretical gap’ for the student who uses instrumented abduction between the premise and the conclusion of the conjecture (Baccaglini-Frank 2010a; Baccaglini-Frank and Antonini 2016), a lack of theoretical evidence to actually prove the validity of the conjecture. This gap hinders a successive proving phase. Instead, in the case of abductions *stricto sensu*, the theorem constituting the rule in the abduction can be used to flip the abduction into a deduction, thanks to the rule, one (or more) known theorem(s) in Euclidean Geometry. I show examples of this in section 4.

Below is a table (Table 5) summarizing the main theoretical constructs presented, which will be used to analyze students’ explorations in the next section.

Table 5: Key notions with respect to the phenomenological domain and the theoretical domain

	Phenomenological domain (DGE)	**Theoretical domain of Euclidean Geometry**
dependency relationships	*causality* (inducing inv(C) through maintaining dragging causes inv(A) to happen)	*conditionality* (inv(C) implies inv(A); or ‘if C, then A’)
manipulation	*dragging*: a physical process involving haptic control (hand-eye coordination)	(manipulation with respect to the theory) occurs through conceptual control coherent with Euclidean Geometry
evidence	*Phenomenological e.*: information collected as feedback from the DGE indicating truth of a proposition	*Theoretical e.:* facts referred to a mathematical theory; they can be sufficient to be organized into deductive chains proving a proposition in Euclidean Geometry; or they can be insufficient for reaching a proof
abductive processes	*manipulative* abduction: a kind of discovering through doing aimed at creating communicable accounts of	*abduction stricto sensu*: abduction that makes use of a known theorem in Euclidean Geometry

	new experiences to integrate them into previously existing systems of practices	as its rule
	instrumented abduction: manipulative abduction or creative abduction supported by maintaining dragging in a DGE, and leading to the formulation of a conjecture in Euclidean Geometry	

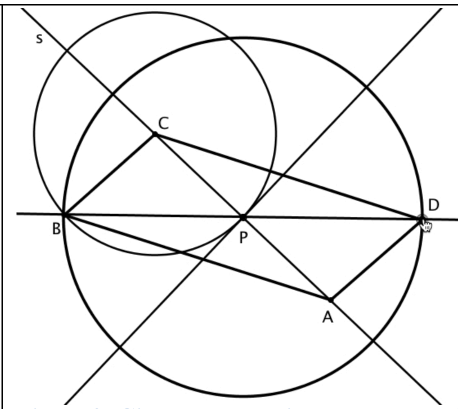
4 The case of Giu and Ste

This section presents a process of conjecture generation through maintaining dragging carried out by two high school students. In the analyses I focus on the abductions that took place and on instrumented abduction's bridging role between phenomenological evidence and theoretical evidence. The actors in the following excerpts are Giu and Ste, 15-year-old students in 10th grade at an Italian scientific high school; they have used dynamic geometry through the 9th grade. These excerpts come from a study (Baccaglini-Frank 2010b) in which about thirty students between the ages of 15 and 17 in Italian scientific high schools were introduced to four ways of dragging a point, including maintaining dragging, and then observed as they worked in pairs on a set of open problems in a DGE. Data were collected in the form of screen recordings of the students' activity, videos, and students' written productions.

In excerpt 1 (see below), the students have decided to try to maintain the property 'BP = PD' in order to explore "when ABCD is a parallelogram" (see Figure 2). This excerpt begins after almost 13 minutes from the beginning of the exploration (times marked in the excerpts correspond to the time elapsed from the beginning of the exploration). The bold writing of a student's name indicates that the student is holding the mouse. The excerpts have been translated from Italian.

Excerpt 1

time	*who*	*what is said*	*what is done*
12:40	Giu	For us it is good only if it [point D] is also at an equal distance [as B from P].	He moves D so that BP appears congruent to PD.
12:46	**Giu**	Since I like circles very much… […] Uh, let's do something really ugly.	He builds a circle with its center at P and radius PD, C_{PD}. (see Figure 2).
13:00	Ste	No, I was verifying…	His gaze is fixed on the screen.
13:08	**Giu**	Because, see, doing like this…the thing…[laughing]	He drags D.

			 **Figure 2: Giu drags D having constructed C_{PD}.**
13:16	Ste	I mean, it can come out to be a parallelogram, but…	
13:18	**Giu**	But you can do it like this, see. I mean you can see that like this it comes out only when …no, see…[…] you try and see so that this thing stays.	He drags D so that C_{PD} passes through the intersection defining B.

The words “good only if it is also” suggest that for Giu BP=PD constitutes a necessary and sufficient condition for ABCD to be a parallelogram. The property of D ‘at an equal distance’ as B is from P, puts D in relationship with two geometrical objects: the parallelogram with intersecting diagonals at P, and a new circle C_{PD}. Giu seems to be using his theoretical knowledge to produce an abduction *stricto sensu* leading to the circle C_{PD}. Such abduction *stricto sensu* can be described as follows:

Abduction (1)

Fact: BP = PD (discerned),

Rule: all radii of a circle have the same length, which applied to this figure can be stated as: ‘If a circle with center at P and radius PD passes through B, then BP = PD (known theorem),

Hypothesis: C_{PD} passes through B (a property that can be easily controlled perceptually in the phenomenological domain).

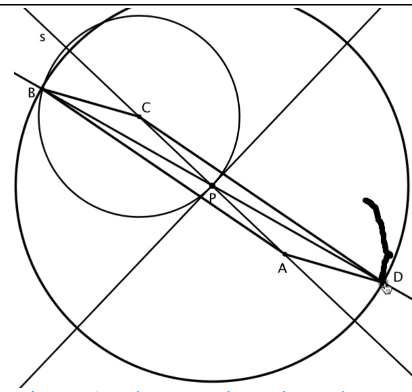
The inferred hypothesis is a proposition supported by theoretical evidence: if B belongs to C_{PD} (or C_{PD} passes through B) BP and PD are radii of the same circle, and so they are equal. This shows an example of how abductions *stricto sensu* can easily be flipped into deductive chains constituting theoretical evidence of the proposition in their hypothesis. In this case the theoretical evidence collected by Giu and Ste for the proposition ‘if B belongs to C_{PD}, then ABCD is a parallelogram’, which includes the known theorem used as a rule in the abduction *stricto sensu*, is sufficient for reaching a proof of the proposition; indeed the abduction needs only to be flipped into a deduction to reach a correct proof validating the proposition.

However, the students seem to be searching for a proposition with a different premise, one that involves D explicitly, because it is D, not B, that they can control directly. Indeed, (at times 13:00 and 13:08) Giu drags D so that the circle C_{PD} passes through B, and he describes his perception as: "doing like this" (time 13:08), the points are at "an equal distance" (time 12:40). Interestingly, Giu's idea of using the geometrical object C_{PD} seems to be triggered by his perceptual need to gain better control on the property "BP=PD" as he drags D (for a deeper analysis of this phenomenon see Antonini and Baccaglini-Frank 2016). The students' dissatisfaction with the proposition for which they have found evidence (both phenomenological and theoretical) is further suggested by Giu's statement that "it comes out" to be a parallelogram "only when […] this thing stays" and by his insisting to "try and see so that this thing stays" (time 13:18).

From what follows in excerpt 2, we can infer that "this thing" refers to the passing of C_{PD} through B (which immediately implies ABCD parallelogram), and to "try and see" refers to the beginning of an instrumented abduction that will lead to a premise of a new proposition. This process will be supported by maintaining dragging of D with the trace active, to maintain C_{PD} passing through B ("this thing").

Excerpt 2

time	*who*	*what is said*	*what is done*
13:51	**Ste**	and let's do trace of D.	
[…]			
14:20	Giu	Try to maintain these things here.	He points to B at the corresponding intersection of C_{PD} and C_{CP}.
14:20	**Ste**		He performs maintaining dragging.

			 Figure 3: The trace is activated on D.
14:39	Giu	It looks like a curve…unless it is him who is not able to do anything…	
14:43	**Ste**	But it’s really hard!	
14:44	Giu	I know, I can only imagine…but I think it is a circle…with center at A.	
[…]			
14:54	Giu	…and maybe with …with radius P. You hadn’t thought of that!	
15:00	**Ste**	What do you mean with center at A and radius P?	
15:03	Giu	AP.	
15:04	**Ste**	Ah!	
[…]			
15:09	**Ste**	No, uh, the radius is necessarily AD! In any case you should have AP equal to AD.	He is holding the mouse but not dragging anything.
15:15	Giu	Maybe I even understand why.	
[…]			
16:16	Giu	So, uh, why, why does it always happen when it is along there?	
[…]			
16:22	Giu and Ste		As Giu dictates Ste writes the conjecture: ‘If PA=AD, then ABCD is a parallelogram.’

Excerpt 2 starts with Ste dragging and setting out to “do the trace of D” (time 13:51); for him (and Giu, who watches and does not object) this seems to mean to activate the trace on D and use maintaining dragging, moving D to “try to maintain these things here” (time 14:20). These words, followed by the use of maintaining dragging and the search for regularity in the movement of D, suggest that Giu has developed a meta-theorem-in-action: he acts as if he knows that if there were a conditional proposition linking a new invariant involving D and the property of ABCD being a parallelogram, this would imply perception of causality between such a new invariant and 'ABCD is a parallelogram' in the DGE. In agreement with what happened in the immediately preceding excerpt 1, "these things here" probably refers to the belonging of B to C_{PD}, or to the property 'BP equals PD', which the students may already consider as logically equivalent to the property 'ABCD is a parallelogram' (see discussion of excerpt 1), the property they will write as conclusion of their conjecture. For 19 seconds the students are silent and watch Ste perform maintaining dragging. During this process a trace mark slowly appears on the screen showing the positions successively occupied by D.

Giu notices that "it looks like a curve" (time 14:39). This is a first proposition that he then proceeds to refine theoretically, with the help of Ste, identifying new properties of such a curve (see Figure 3):

- Circle – "I think it is a circle" (time 14:44),
- Circle with center at A – "with its center at A" (time 14:44),
- Circle with radius AP (times 14:54-15:04).

Ste seems to recognize the equivalence between the properties: "D belongs to C_{AP}" and "AP=AD" (time 15:09). Indeed, earlier the students had used the rule: all radii of a circle have the same length. Table 6 summarizes the phenomenological and theoretical evidence collected by Giu and Ste for a proposition relating the newly identified invariant involving D to the belonging of B to C_{PD} or, possibly, to ABCD being a rectangle.

Table 6: Phenomenological evidence and theoretical evidence that was collected by Giu and Ste for an implicit proposition stated in two forms in the shaded rows of the table

Proposition:	
Version without reference to Eucl. Geo.: D on circle C_{AP} (or AP=AD) causes B to lie at an intersection of the circles C_{PD} and C_{CP}	Version with reference to Eucl. Geo.: If D belongs to C_{AP} (or AP=AD), then B belongs to C_{PD}
Phenomenological evidence	*Theoretical evidence*
1. it is possible to maintain the following: B on C_{PD}, or B on an intersection of the two circles	1. B on C_{PD} implies ABCD is a parallelogram
2. D moves along a curve	2. D belongs to C_{AP} if and only if AP=AD

3. simultaneous appearance of fact 2 and AP=AD	3. the meta-theorem-in-action applied to the two invariants, D belongs to C_{AP} and B belongs to C_{PD}, supports the conjecture stated in the proposition 'if D belongs to C_{AP}, then B belongs to C_{PD}', or 'if AP=AD, then ABCD is a parallelogram'
4. simultaneous perception of fact 1 and fact 2	
5. direct control over fact 2	
6. indirect control over fact 1	

Giu's final words "maybe I even understand why" indicate an intention to search for theoretical evidence for Ste's proposition "AP equal to AD [in order for ABCD to be a parallelogram]", which I see as the partial statement of the final conjecture that the students write at time 16:22. Giu's words also suggest that he realizes that he does not yet have enough theoretical evidence to prove such a conjecture. Indeed, the right column of Table 6 shows the theoretical evidence collected for a yet implicit proposition that is closely related to the students' final conjecture, but such evidence is not sufficient for a proof. The meta-theorem-in-action is used to interpret the causal relationship between two invariants as a conditional relationship: in this way phenomenological evidence joins the theoretical evidence supporting the conjecture, convincing the students of the potential validity of the proposition.

Overall, the students appear to be experts in using maintaining dragging and they proceed smoothly from the identification of a property to maintain during dragging (B belongs to C_{PD}) to inducing a regularity in the movement of D, leading to the identification of a new invariant. The manipulative abduction that has taken place can be described as shown in Table 7:

Table 7: Structure of the manipulative abduction associated with use of maintaining dragging

(Observed) Fact	perception of causality: simultaneous perception of (1) 'B on C_{PD}' and (2) 'D moving along a circle', or 'AP=AD'; direct control over (2); indirect control over (1)
(Manipulative) Rule	meta-theorem-in-action associated with maintaining dragging–applied to this exploration: the conditional proposition 'If D belongs to C_{AP} (or AP=AD), then B belongs to C_{PD}' implies perception of causality in the DGE
(Explanatory) Hypothesis	conditional proposition: 'If D belongs to C_{AP} (or AP=AD), then

	B belongs to C_{PD}' (conjecture)

The students then construct C_{AP} and drag D along it, seemingly, in search for additional evidence for their conjecture. Indeed, Giu's words "why, why does it always happen when it is along there" (time 16:16) suggest that he feels the need to continue searching for theoretical evidence, working in the direction of a proof that completely explains the conjecture. These words also suggest that the conjecture resulting from the manipulative abduction is not supported by sufficient theoretical evidence for it to be proved. Indeed, Table 6 also shows that the theoretical evidence collected is not sufficient for the students to construct a proof of the proposition in the top row of the table, or of the conjecture.

In fact, this appears to be the case also in the other data collected from expert students' explorations (Baccaglini-Frank and Mariotti 2010). Conjectures generated through maintaining dragging seem to come with a strong theoretical rupture between their premise and their conclusion, due to the lack of theoretical evidence found during the process. We have hypothesized that this is due to a cognitive 'offloading' of the abduction onto the use of maintaining dragging[4].

5 Conclusions

The construct of instrumented abduction emerged from the reflections I presented upon a particular kind of process of conjecture generation in a DGE, based on the use of maintaining dragging. This construct sheds light on key aspects involved in unravelling the complex relationship students need to tackle between the phenomenology of a DGE and the theoretical domain of Euclidean Geometry. In particular, it allowed us to analyze the "delicate cognitive point" identified by Arzarello et al. (2002), now describable in terms of a particular kind of manipulative abduction, whose rule resides at a meta-level with respect to the DGE and the theoretical domain of Euclidean Geometry. Instrumented abduction guides students in identifying a property to maintain during dragging while searching for a regularity in the movement of the dragged point, leading to the identification of a new property. Expert students

[4] Indeed, this is what led to the name *instrumented* abduction in the first place, since maintaining dragging can be seen as an instrument for generating conjectures.

expect a relationship between these properties thanks to their construction of a meta-theorem-in-action, according to which a conditional proposition implies a perception of causality in the DGE. Such a meta-theorem-in-action plays the role of the rule in the instrumented abduction, and it leads the students to reach a conditional proposition linking two properties as a product of their abduction.

In this paper, I have shown how instrumented abduction can foster expert students' re-interpretation of phenomenological evidence as theoretical evidence. Evidence that supports claims made within the phenomenology of the DGE is in general of a different nature from evidence that supports statements in the theoretical domain if Euclidean Geometry. Therefore, I introduced a distinction between phenomenological evidence, that can have physical, haptic, kinesthetic and perceptive components, and theoretical evidence, which is, instead, achieved through conceptual control with a clear reference to Euclidean Geometry. Phenomenological evidence in the case of instrumented abduction can be constituted by a set of facts guaranteeing the conditions for application of the meta-theorem-in-action; this evidence allows an instrumented abduction to take place and a conjecture to be generated. Moreover, thanks to the concept of a 'meta-theorem-in-action', phenomenological evidence can be re-interpreted as theoretical evidence of a conjecture. The perceived invariants and relationships are seen as geometrical properties in a relationship of conditionality with one another.

Finally, I wish to remark that when instrumented abduction is used, from a phenomenological-perceptive point of view, the process proceeds smoothly, without ruptures. On the other hand, we have seen in the case of Giu and Ste—data which are consistent with those of other expert students—that at a theoretical level there can be a strong rupture given by an 'offloading' of the abduction onto the use of maintaining dragging. This implies that theoretical evidence generated during the process of conjecture-generation is not sufficient for proving the conjecture. This has educational implications. Instrumented abduction allows us to produce a conjecture that is supported by theoretical evidence, convincing the student of its validity within Euclidean Geometry. However, data suggest that the price is an incompleteness of theoretical evidence with respect to what is needed to prove the conjecture. From an educational point of view, fostering conjecture generation as a process of mathematical discovery is highly desirable, but teaching interventions focused on the use of maintaining dragging for generating conjectures should keep in mind the potential fragility that such processes may bring to later argumentation processes initiated to prove conjectures.

Taking a step back and looking at the overall abductive process, it is the instrumented abduction that fosters a transition from propositions within the phenomenological domain of the DGE to propositions in Euclidean Geometry. The phenomenon, described in these terms, may present analogies with other situations in which phenomenological experiences are re-interpreted in theoretical terms. For example, in a context of this sort, Abrahamson (2012) speaks of "guided mediated abduction". This is an interesting direction for future research in mathematics education and the learning sciences in general.

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