# REPRESENTATION AND CODING OF RATIONAL PAIRS ON A TRIANGULAR TREE AND DIOPHANTINE APPROXIMATION IN $\mathbb{R}^{2}$ 

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#### Abstract

In this paper we study the properties of the Triangular tree, a complete tree of rational pairs introduced in [7], in analogy with the main properties of the Farey tree (or Stern-Brocot tree). To our knowledge the Triangular tree is the first generalisation of the Farey tree constructed using the mediant operation. In particular we introduce a two-dimensional representation for the pairs in the tree, a coding which describes how to reach a pair by motions on the tree, and its description in terms of $S L(3, \mathbb{Z})$ matrices. The tree and the properties we study are then used to introduce rational approximations of non-rational pairs.


## 1. Introduction

The theory of multidimensional continued fractions has received increasing attention in the last years from researchers both in number theory and in ergodic theory. The origin of multidimensional continued fraction expansions may be traced back to a letter that Hermite sent to Jacobi asking for a generalisation of Lagrange's Theorem for quadratic irrationals to algebraic irrationals of higher degree. It was for this reason that Jacobi developed what is now called the Jacobi-Perron algorithm. Unfortunately, despite numerous attempts and the introduction of many different algorithms, Hermite's question remains unanswered. We refer the reader to 5 for a geometric description of the theory of multidimensional continued fractions. On the other hand, it is well-known that (regular) continued fraction expansions are related to the theory of dynamical systems as the expansion of a real number can be obtained by the symbolic representation of its orbit under the action of the Gauss map on the unit interval. The dynamical systems approach has led to new proofs of the Gauss-Kuzmin Theorem, Khinchin's weak law and other metric results first obtained by Khinchin and Lévy, also thanks to the modern results of ergodic theory (see for example [11, 13]). In recent years the methods of ergodic theory have been applied also to maps related to multidimensional continued fraction algorithms, we refer to [15 for the first results in this research area. It is also interesting to mention that, in the opposite direction, number theoretical properties of real numbers have led to new results in ergodic theory, as the Three Gaps Theorem for instance, which can be interpreted in terms of return times for irrational rotations on the circle.

In this paper we continue the work initiated by the authors with Sara Munday in 7, where we have studied the properties of a tree of rational pairs, here called the Triangular tree, which was introduced as a two-dimensional version of the well-known Farey tree (or Stern-Brocot tree). The aim of this paper is to show that all the structures of the Farey tree can be found also in the Triangular tree and to construct approximations of real pairs using the tree. We believe that the Triangular tree will be useful for the study of interesting phenomena related to two-parameter systems, as this is the case for the Farey tree and one-parameter systems. For recent investigations on higher-dimensional phenomena we refer to [3, 4, 10.

The paper is structured as follows. In Section 2 we recall the construction of the Farey tree and its main properties based on the continued fraction expansions of real numbers, in particular the $\{L, R\}$ coding of real numbers which is based on the relations of the Farey tree with the group $S L(2, \mathbb{Z})$. In Section 3 we recall

[^0]the main results of [7] concerning the Triangular tree. In particular we recall that it can be constructed in a dynamical way, by using the Triangle map defined in [8] which acts on the triangle $\triangle:=\left\{(x, y) \in \mathbb{R}^{2}\right.$ : $1 \geq x \geq y>0\}$ and its slow version $S$ introduced in [7], and in a geometric way, by using the notion of mediant of two pairs of fractions. The existence of these two possible constructions is the first basic property of the Farey tree that has been proved for the Triangular tree in [7]. Afterwards, Section 4 contains some preparatory results on the triangle sequences, that is the symbolic representation of the orbits of real pairs for the Triangle map. These sequences are the analogous of the continued fraction expansions for real numbers and the Gauss map, and are thus the two-dimensional continued fraction expansions of real pairs generated by the Triangle map. It is known that there are cases in which different real pairs have the same triangle sequence, this is discussed in details in Section 4 ,

Sections 5 and 6 contain the main results of the paper. First we use the slow map $S$ and its local inverses to introduce in Definition 5.3 a unique two-dimensional representation for rational pairs in $\bar{\triangle}$, thus improving the expansions obtained by using the Triangle map. For non-rational pairs the two-dimensional representation is that obtained by the Triangle map. Then in Theorem6.6 we introduce a coding for rational pairs in the Triangular tree in terms of possible motions in the tree, in analogy with the coding of the Farey tree. This coding can be described also in terms of $S L(3, \mathbb{Z})$ matrices defined in (6.1). This coding and its descriptions are useful also to introduce approximations of real pairs, and of non-rational pairs in particular, in terms of the rational pairs on the Triangular tree. The definition of the approximations together with some examples are described in Section 7 Finally, in Section 8 we study the speed of the approximations introduced in the sense of the simultaneous approximations of couples of real numbers. We give results only for two classes of non-rational pairs, those with finite triangle sequence and those corresponding to fixed points of the Triangle map, leaving further developments for future research.

## 2. The Farey coding

In this section we recall the construction of the Farey tree, the coding it induces for the rationals in $(0,1)$, and its connection with the continued fraction expansion of real numbers.

The Farey tree is a binary tree which contains all the rational numbers in the interval $(0,1)$ and can be generated in a dynamical way using the Farey map, and in an arithmetic way using the notion of mediant between two fractions.

The Farey map is the map $F:[0,1] \rightarrow[0,1]$ defined to be

$$
F(x):= \begin{cases}\frac{x}{1-x}, & \text { if } 0 \leq x \leq \frac{1}{2} \\ \frac{1-x}{x}, & \text { if } \frac{1}{2} \leq x \leq 1\end{cases}
$$

Denoting with $F_{0}:\left[0, \frac{1}{2}\right] \rightarrow[0,1]$ and $F_{1}:\left[\frac{1}{2}, 1\right] \rightarrow[0,1]$ its two branches, $F$ admits two local inverses, $\psi_{0}:=F_{0}^{-1}$ and $\psi_{1}:=F_{1}^{-1}$, given by

$$
\begin{equation*}
\psi_{0}(x)=\frac{x}{1+x} \quad \text { and } \quad \psi_{1}(x)=\frac{1}{1+x} \tag{2.1}
\end{equation*}
$$

The Farey tree is then generated using the Farey map by setting $\mathcal{L}_{0}:=\left\{\frac{1}{2}\right\}$ for the root of the tree, and setting recursively $\mathcal{L}_{n}:=F^{-n}\left(\frac{1}{2}\right)$ for all $n \geq 1$. Each level of the tree is written by ascending order as shown in Figure 1. The connections between the fractions of different levels are explained below. It is known that the Farey tree contains all the rational numbers in the interval $(0,1)$, that is $\bigcup_{n=0}^{+\infty} \mathcal{L}_{n}=\bigcup_{n=0}^{+\infty} F^{-n}\left(\frac{1}{2}\right)=\mathbb{Q} \cap(0,1)$, and each rational number appears in the tree exactly once.

We now describe the second way to construct the levels of the Farey tree. Let us consider the Stern-Brocot sets $\left(\mathcal{F}_{n}\right)_{n \geq-1}$, with $\mathcal{F}_{-1}:=\left\{\frac{0}{1}, \frac{1}{1}\right\}$, and for all $n \geq 0$ let $\mathcal{F}_{n}$ be obtained from $\mathcal{F}_{n-1}$ by inserting the mediant of each pair of neighbouring fractions. We recall that the mediant of two fractions $\frac{p}{q}$ and $\frac{r}{s}$ with $\frac{p}{q}<\frac{r}{s}$ is

$$
\frac{p}{q} \oplus \frac{r}{s}:=\frac{p+r}{q+s}
$$



Figure 1. The first four levels of the Farey tree.
and that $\frac{p}{q}<\frac{p}{q} \oplus \frac{r}{s}<\frac{r}{s}$. We say that $\frac{p}{q} \oplus \frac{r}{s}$ is the child of $\frac{p}{q}$ and $\frac{r}{s}$, which are its left and right parent, respectively. It can be easily shown that two fractions $\frac{p}{q}<\frac{r}{s}$ are neighbours in a Stern-Brocot set if and only if $q r-p s=1$ (see, for instance, [9). Moreover, since the ancestors $\frac{0}{1}$ and $\frac{1}{1}$ are in lowest terms, it follows that all the fractions obtained through the mediant operation appear in lowest terms. The first sets $\mathcal{F}_{n}$ are

$$
\mathcal{F}_{0}=\left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}, \quad \mathcal{F}_{1}=\left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}, \quad \mathcal{F}_{2}=\left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\} .
$$

The levels of the Farey tree are then given by $\mathcal{L}_{n}=\mathcal{F}_{n} \backslash \mathcal{F}_{n-1}$ for all $n \geq 0$. Two fractions in Figure 1 are connected if one of the fractions is a parent of the other. Notice that a fraction in a level $\mathcal{L}_{n}$ has a parent in the level $\mathcal{L}_{n-1}$ and the other parent in a level $\mathcal{L}_{m}$ with $m<n-1$.

For later use we recall the notion of rank of a rational number $x$ in the open unit interval: $\operatorname{rank}(x)=r$ if and only if $x \in \mathcal{L}_{r}$. For more details on the Farey tree, we refer to [6].
2.1. The Farey coding. We now recall the $\{L, R\}$ coding of rational numbers (see also [12, 6]). Let $\frac{p}{q} \in \mathbb{Q} \cap(0,1)$ be a fraction reduced in lowest terms. Since $\frac{p}{q}$ belongs to a unique level of the Farey tree, we can write in a unique way

$$
\frac{p}{q}=\frac{l}{m} \oplus \frac{r}{s},
$$

with $r m-l s=1$. That is, $\frac{p}{q}$ is generated by taking the mediant between $\frac{l}{m}$ and $\frac{r}{s}$, which are thus its parents in the tree. We associate to the fraction $\frac{p}{q}$ the matrix

$$
\mathfrak{m}\left(\frac{p}{q}\right):=\left(\begin{array}{cc}
r & l \\
s & m
\end{array}\right) \in S L(2, \mathbb{Z})
$$

Introducing the two $S L(2, \mathbb{Z})$ matrices

$$
L:=\left(\begin{array}{ll}
1 & 0  \tag{2.2}\\
1 & 1
\end{array}\right) \quad \text { and } \quad R:=\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right)
$$

the left and right children of $\frac{p}{q}=\frac{l+r}{m+s}$ on the tree, that are $\frac{l}{m} \oplus \frac{l+r}{m+s}=\frac{2 l+r}{2 m+s}$ and $\frac{l+r}{m+s} \oplus \frac{r}{s}=\frac{l+2 r}{m+2 s}$ respectively, can be obtained by right multiplication with the matrices $L$ and $R$ respectively, since

$$
\mathfrak{m}\left(\frac{2 l+r}{2 m+s}\right)=\mathfrak{m}\left(\frac{p}{q}\right) L=\left(\begin{array}{cc}
l+r & l \\
m+s & m
\end{array}\right) \quad \text { and } \quad \mathfrak{m}\left(\frac{l+2 r}{m+2 s}\right)=\mathfrak{m}\left(\frac{p}{q}\right) R=\left(\begin{array}{cc}
r & l+r \\
s & m+s
\end{array}\right)
$$

In other words, the action of $L$ and $R$ given by the right matrix multiplication corresponds to travelling downward along the Farey tree by moving to the left or to the right, respectively. We finally notice that $L=\mathfrak{m}\left(\frac{1}{2}\right)$ is the matrix of the root of the Farey tree.

Proposition $2.1(12])$. Let $x \in \mathbb{Q} \cap(0,1)$ with $\operatorname{rank}(x)=r$, and let $x=\left[a_{1}, \ldots, a_{n}\right]$ be its continued fraction expansion. Then

$$
\operatorname{rank}(x)=\sum_{i=1}^{n} a_{i}-2
$$

and the matrix associated to $x$ is

$$
\mathfrak{m}(x)=\mathfrak{m}\left(\frac{1}{2}\right) \prod_{i=1}^{r} M_{i}= \begin{cases}L L^{a_{1}-1} R^{a_{2}} \cdots L^{a_{n-1}} R^{a_{n}-1} & \text { if } n \text { is even }  \tag{2.3}\\ L L^{a_{1}-1} R^{a_{2}} \cdots R^{a_{n-1}} L^{a_{n}-1} & \text { if } n \text { is odd }\end{cases}
$$

where $M_{i}=L$ if the $i$-th turn along the path joining $\frac{1}{2}$ with $x$ on the Farey tree goes left, and $M_{i}=R$ if it goes right1.

The digits of the continued fraction expansion also have a dynamical meaning in terms of the Gauss map, hence of the Farey map. In fact we recall that the Gauss map is a fast version of the Farey map, precisely the Gauss map is the jump transformation of $F$ on the interval $\left(\frac{1}{2}, 1\right]$. If $x=\left[a_{1}, \ldots, a_{n}\right]$ we have

$$
x=\psi_{0}^{a_{1}-1} \psi_{1} \cdots \psi_{0}^{a_{n}-1} \psi_{1}(0)
$$

and $\left(a_{i}\right)_{i=1, \ldots, n}$ is the sequence of return times to $\left(\frac{1}{2}, 1\right]$ of the orbit of $x$ under $F$. Since $\psi_{0} \psi_{1}(0)=\frac{1}{2}$, we can also express explicitly every rational number $x \in(0,1)$ as a backward image of $\frac{1}{2}$ under the Farey map. If $a_{n}>1$ we have $x=\psi_{0}^{a_{1}-1} \psi_{1} \cdots \psi_{0}^{a_{n}-2}\left(\frac{1}{2}\right)$, and if $a_{n}=1$ we have $x=\psi_{0}^{a_{1}-1} \psi_{1} \cdots \psi_{0}^{a_{n-1}-1}\left(\frac{1}{2}\right)$.
Example. Let $x=\frac{7}{12}$. This rational number appears at the fourth level of the Farey tree, so $x=\frac{7}{12} \in \mathcal{L}_{4}$ and $\operatorname{rank}(x)=4$. Starting from the root $\frac{1}{2}$, the path to reach $x$ on the Farey tree is $R L L R$, thus $\mathfrak{m}\left(\frac{7}{12}\right)=$ $L R L L R=L R L^{2} R$. Indeed

$$
L R L^{2} R=\left(\begin{array}{ll}
3 & 4 \\
5 & 7
\end{array}\right)
$$

and $\frac{4}{7} \oplus \frac{3}{5}=\frac{7}{12}$. Moreover we have $\frac{7}{12}=[1,1,2,2]$, so that $\frac{7}{12}=\psi_{1} \psi_{1} \psi_{0} \psi_{1}\left(\frac{1}{2}\right)$.
The coding for all the real numbers in the closed unit interval extends the above construction and is given by a map $\pi:[0,1] \rightarrow\{L, R\}^{\mathbb{N}}$ which associate to each $x \in[0,1]$ an infinite sequence $\pi(x)$ over the alphabet $\{L, R\}$. First we set

$$
\pi\left(\frac{0}{1}\right)=L^{\infty} \quad \text { and } \quad \pi\left(\frac{1}{1}\right)=R^{\infty}
$$

Let $x \in \mathbb{Q} \cap(0,1)$, so that it has a finite continued fraction expansion, say $x=\left[a_{1}, \ldots, a_{n}\right]$. Then note that there exists two infinite paths on the Farey tree which agree down to the node of $x$. Both starts with the finite sequence coding the path from the root $\frac{1}{2}$ to reach $x$, according to (2.3) and terminating with either $R L^{\infty}$ or $L R^{\infty}$. We let the infinite sequence terminate with $R L^{\infty}$ or $L R^{\infty}$ according to whether the number of partial quotients of $x$ is even or odd. Thus for $x=\left[a_{1}, \ldots, a_{n}\right]$ we set

$$
\pi(x)= \begin{cases}L^{a_{1}} R^{a_{2}} \cdots L^{a_{n-1}} R^{a_{n}} L^{\infty} & \text { if } n \text { is even } \\ L^{a_{1}} R^{a_{2}} \cdots R^{a_{n-1}} L^{a_{n}} R^{\infty} & \text { if } n \text { is odd }\end{cases}
$$

In case $x \in(\mathbb{R} \backslash \mathbb{Q}) \cap(0,1)$ the continued fraction expansion is infinite, say $x=\left[a_{1}, a_{2}, \ldots\right]$, thus in this case we simply define

$$
\pi(x)=L^{a_{1}} R^{a_{2}} L^{a_{3}} R^{a_{4}} \cdots
$$

[^1]
## 3. Triangle maps and the Triangular tree

3.1. The setting. The Triangle Map has been introduced in [8] to define a two-dimensional analogue of the continued fraction algorithm. Let us consider the triangle

$$
\triangle:=\left\{(x, y) \in \mathbb{R}^{2}: 1 \geq x \geq y>0\right\}
$$

and the pairwise disjoint subtriangles $\triangle_{k}:=\{(x, y) \in \triangle: 1-x-k y \geq 0>1-x-(k+1) y\}$, with $k \geq 0$, and the line segment $\Lambda:=\{(x, 0): 0 \leq x \leq 1\}$. Note that $\bar{\triangle}=\bigcup_{k \geq 0} \triangle_{k} \cup \Lambda$ (see Figure 21). We also introduce

$$
\Sigma:=\bar{\triangle} \cap\{x=y\} \quad \text { and } \quad \Upsilon:=\bar{\triangle} \cap\{x=1\}
$$

the slanting and the vertical side of $\triangle$, respectively. The Triangle Map $T: \Delta \rightarrow \bar{\triangle}$ is then defined to be

$$
T(x, y):=\left(\frac{y}{x}, \frac{1-x-k y}{x}\right) \quad \text { for }(x, y) \in \triangle_{k}
$$

The map $T$ generates an expansion associated to each point of $\triangle$, the so-called triangle sequence. In particular, to a point $(x, y) \in \triangle$ we associate the sequence of non-negative integers $\left[\alpha_{0}, \alpha_{1}, \alpha_{2}, \ldots\right]$ if and only if $T^{k}(x, y) \in \triangle_{\alpha_{k}}$ for all $k \geq 0$. In case $T^{k_{*}}(x, y) \in \Lambda$ for some $k_{*}>0$ then we say that the triangle sequence terminates. An important result of [8] is that pairs of rational numbers have a finite triangle sequence. However, the converse is not true: also non-rational points can have a finite triangle sequence and actually there are entire line segments with every point having the same triangle sequence. As it is clear from the definition, note that if $(x, y)$ has triangle sequence $\left[\alpha_{0}, \alpha_{1}, \alpha_{2}, \ldots\right]$ then $T(x, y)$ has triangle sequence $\left[\alpha_{1}, \alpha_{2}, \ldots\right]$. In other words, the Triangle Map acts on triangle sequences as the left shift, exactly as the Gauss map does for the continued fraction expansions.


Figure 2. Partition of $\triangle$ into $\left\{\triangle_{k}\right\}_{k \geq 0}$.
The analogous of the Farey map in this two-dimensional setting has been introduced in [7]. Let $\left\{\Gamma_{0}, \Gamma_{1}\right\}$ be the partition of $\bar{\triangle}$ such that $\Gamma_{0}:=\triangle_{0}$ and $\Gamma_{1}:=\bar{\triangle} \backslash \Gamma_{0}$. The map $S: \bar{\triangle} \rightarrow \bar{\triangle}$ is then defined to be

$$
S(x, y):= \begin{cases}\left(\frac{y}{x}, \frac{1-x}{x}\right), & (x, y) \in \Gamma_{0}  \tag{3.1}\\ \left(\frac{x}{1-y}, \frac{y}{1-y}\right), & (x, y) \in \Gamma_{1}\end{cases}
$$

Notice that $S$ maps $\triangle_{k}$ onto $\triangle_{k-1}$ for all $k \geq 1$ and that $S\left(\Gamma_{0}\right) \cup \Sigma=S\left(\Gamma_{1}\right)=\bar{\triangle}$. As a consequence, if $(x, y) \in \triangle_{k}$ for some $k \geq 0$ then

$$
\begin{equation*}
T(x, y)=S^{k+1}(x, y)=\left.\left.S\right|_{\Gamma_{0}} \circ S\right|_{\Gamma_{1}} ^{k}(x, y) \tag{3.2}
\end{equation*}
$$

In other words, the Triangle Map $T$ is the jump transformation of $S$ on the set $\Gamma_{0}$, as the Gauss map is the jump transformation of the Farey map on $\left(\frac{1}{2}, 1\right]$. Thus the map $S$ can be thought of as a "slow version" of the Triangle Map $T$.

The map $S$ also induces a coding for the points of the triangle $\triangle$. In particular, if the triangle sequence of $(x, y)$ is $\left[\alpha_{0}, \alpha_{1}, \ldots\right]$ then the itinerary under $S$ of a point $(x, y) \in \triangle$ with respect to the partition $\left\{\Gamma_{0}, \Gamma_{1}\right\}$ is

$$
(\underbrace{1, \ldots, 1}_{\alpha_{0}}, 0, \underbrace{1, \ldots, 1}_{\alpha_{1}}, 0, \ldots)
$$

Many properties of the map $S$ have been proved in [7]: $S$ is ergodic with respect to the Lebesgue measure, it preserves the infinite Lebesgue-absolutely continuous measure with density $\frac{1}{x y}$, and it is pointwise dual ergodic. Finally, the role of the map $S$ as a two-dimensional version of the Farey map is confirmed by the construction of a complete tree of rational pairs, the Triangular tree, by using the inverse branches of $S$, in the same way as the Farey tree is generated by the Farey map, and then, equivalently, by a generalised mediant operation. In Section 3.2 we recall the main steps of this construction.
3.2. Construction of the Triangular tree. We now briefly recap the construction of the Triangular tree and its main properties, following [7, Section 5]. The two inverse branches of the map $S$ are

$$
\begin{equation*}
\phi_{0}:=\left(\left.S\right|_{\Gamma_{0}}\right)^{-1}: \bar{\triangle} \backslash \Sigma \rightarrow \Gamma_{0}, \quad \phi_{0}(x, y)=\left(\frac{1}{1+y}, \frac{x}{1+y}\right) \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\phi_{1}:=\left(\left.S\right|_{\Gamma_{1}}\right)^{-1}: \bar{\triangle} \rightarrow \Gamma_{1}, \quad \phi_{1}(x, y)=\left(\frac{x}{1+y}, \frac{y}{1+y}\right) \tag{3.4}
\end{equation*}
$$

We then introduce the map

$$
\begin{equation*}
\phi_{2}: \Sigma \rightarrow \Lambda, \quad \phi_{2}(x, x):=(x, 0) \tag{3.5}
\end{equation*}
$$

and restrict $\phi_{1}$ to the set $\bar{\triangle} \backslash \Lambda=\triangle$. The maps $\phi_{0}$ and $\phi_{1}$ so modified, and the map $\phi_{2}$ form all together the set of local inverses of a map $\tilde{S}: \bar{\triangle} \rightarrow \bar{\triangle}$ which coincides with $S$ on $\triangle$, and satisfies $\tilde{S}(x, 0)=(x, x)$ on $\Lambda$. Thus the maps $S$ and $\tilde{S}$ coincide up to a zero-measure set.

The levels of the Triangular tree will be denoted by $\left(\mathcal{T}_{n}\right)_{n \geq-1}$. We also use the notation $\mathcal{B}_{n}:=\mathcal{T}_{n} \cap \partial \triangle$ and $\mathcal{I}_{n}:=\mathcal{T}_{n} \cap \triangle$ for the boundary points and the interior points of the $n$-th level of the tree, respectively. We start by setting

$$
\mathcal{T}_{-1}:=\{(0,0),(1,0),(1,1)\} \quad \text { and } \quad \mathcal{T}_{0}:=\left\{\left(\frac{1}{2}, 0\right),\left(1, \frac{1}{2}\right),\left(\frac{1}{2}, \frac{1}{2}\right)\right\}
$$

We now describe precisely how the levels of the tree are generated, by showing all the possibilities for taking counterimages depending on the location of the point in $\bar{\triangle}$ (see Figure 8 for reference).
(R1) An interior point $\left(\frac{p}{q}, \frac{r}{q}\right) \in \mathcal{I}_{n}$ generates the two interior points $\left(\frac{q}{r+q}, \frac{p}{r+q}\right)$ and $\left(\frac{p}{r+q}, \frac{r}{r+q}\right)$ in $\mathcal{I}_{n+1}$, through the application of $\phi_{0}$ and $\phi_{1}$, respectively.
(R2) A boundary point $\left(\frac{p}{q}, \frac{p}{q}\right) \in \mathcal{B}_{n}$ generates the point $\left(\frac{p}{q}, 0\right) \in \mathcal{B}_{n}$ through the application of $\phi_{2}$ and the boundary point $\left(\frac{p}{p+q}, \frac{p}{p+q}\right) \in \mathcal{B}_{n+1}$ through the application of $\phi_{1}$.
(R3) A boundary point $\left(\frac{p}{q}, 0\right) \in \mathcal{B}_{n}$ generates the point $\left(1, \frac{p}{q}\right) \in \mathcal{B}_{n}$ through the application of $\phi_{0}$.
(R4) A boundary point $\left(1, \frac{p}{q}\right) \in \mathcal{B}_{n}$ generates the boundary point $\left(\frac{q}{p+q}, \frac{q}{p+q}\right) \in \mathcal{B}_{n+1}$ and the interior point $\left(\frac{q}{p+q}, \frac{p}{p+q}\right) \in \mathcal{I}_{n+1}$, through the application of $\phi_{0}$ and $\phi_{1}$, respectively.

Note that, conversely to the Farey tree, taking a counterimage does not necessarily imply a change in the level of the tree.

We now describe the geometric way to obtain the same two-dimensional tree of rational pairs constructed above by counterimages (see Figure 3 for reference). We define the mediant of two couples of fractions $\left(\frac{p}{q}, \frac{r}{q}\right)$ and $\left(\frac{p^{\prime}}{q^{\prime}}, \frac{r^{\prime}}{q^{\prime}}\right)$ as

$$
\left(\frac{p}{q}, \frac{r}{q}\right) \oplus\left(\frac{p^{\prime}}{q^{\prime}}, \frac{r^{\prime}}{q^{\prime}}\right):=\left(\frac{p+p^{\prime}}{q+q^{\prime}}, \frac{r+r^{\prime}}{q+q^{\prime}}\right)
$$

Note that we require that the two fractions of each couple have the same denominator, so that the mediant lies on the line segment joining the two points it is computed from. We further assume that the two fractions of each couple are reduced to their least common denominator.


Figure 3. The first four levels of the Triangular tree as points in $\bar{\triangle}$ with the sides of the triangles of the partitions.

Definition 3.1. Consider a set $\mathfrak{R}=\left\{\mathfrak{r}_{i}: i=1, \ldots, r\right\}$ of rational points on a line segment, consisting of at least two points, and in ascending lexicographic order. The Farey sum of $\mathfrak{R}$ is obtained by adding to $\mathfrak{R}$ the mediant between each pair of neighbouring points, that is

$$
\mathfrak{R}^{\oplus}:=\left\{\mathfrak{r}_{i} \oplus \mathfrak{r}_{i+1}: i=1, \ldots, r-1\right\} \cup \Re
$$

To define the levels of the tree in this second, geometric way, we start from the set $\mathcal{S}_{-1}:=\mathcal{T}_{-1}$ of the vertices of $\triangle$ and then we will define a sequence $\left(\mathcal{S}_{n}\right)_{n \geq-1}$ of sets such that $\mathcal{S}_{-1}$ are the three vertices of $\triangle$ and $\mathcal{S}_{n} \supseteq \mathcal{S}_{n-1}$ for all $n \geq 0$. In particular, we introduce a sequence $\left(\mathscr{P}_{n}\right)_{n \geq 0}$ of measurable partitions of $\triangle$,
each refining the previous one and such that the points of $\mathcal{S}_{n}$ lie on the sides of the partition $\mathscr{P}_{n}$. Then the recursive construction is the following: given the set of points $\mathcal{S}_{n}$ up to a certain level $n$, we obtain $\mathcal{S}_{n+1}$ by inserting the mediant between each pair of neighbouring points along each side of the triangles of the partition $\mathscr{P}_{n+1}$. More formally, let $\mathscr{P}_{0}=\{\bar{\triangle}\}$ and let $v_{0}=(0,0), v_{1}=(1,0), v_{2}=(1,1)$ be the three vertices of the triangle $\triangle$. We partition $\bar{\triangle}$ into two subtriangles by the line segment joining $v_{1}$ and $v_{0} \oplus v_{2}=\left(\frac{1}{2}, \frac{1}{2}\right)$. This determines the partition $\mathscr{P}_{1}$. Additionally, we label the vertices of the two subtriangles according to the rule shown in Figure 4 We now proceed inductively. Each triangle of $\mathscr{P}_{n}$ is partitioned into two subtriangles by the line segment joining the vertex labelled " 1 " with the mediant of the vertex " 0 " and the vertex " 2 " and this gives us the next partition $\mathscr{P}_{n+1}$. Then, for $n \geq-1$,

$$
\mathcal{S}_{n+1}:=\bigcup_{\mathfrak{S} \in \mathscr{S}_{n+1}}\left(\mathfrak{S} \cap \mathcal{S}_{n}\right)^{\oplus}
$$

where $\mathscr{S}_{n}, n \geq 0$, is the set of sides of the triangles of the partition $\mathscr{P}_{n}$. To better understand this construction, we recall the conclusions of [7, Lemma 5.8] (for simplicity of notation, we continuously extend the maps $\phi_{0}$ and $\phi_{1}$ defined in (3.3) and (3.4) to $\left.\bar{\triangle}\right)$ : for any finite binary word $\omega \in\{0,1\}^{*}$ of length $n$, let $\phi_{\omega}$ denote the composition $\phi_{\omega_{0}} \circ \phi_{\omega_{1}} \circ \cdots \circ \phi_{\omega_{n-1}}$, then
(i) the triangles of $\mathscr{P}_{n}$ are given by all the possible counterimages $\triangle_{\omega}:=\phi_{\omega}(\bar{\triangle})$ with $|\omega|=n$;
(ii) $\mathscr{S}_{n}=\mathscr{S}_{0} \cup\left\{\overline{\phi_{\omega}(\ell)}:|\omega| \leq n-1\right\}$, where $\ell$ is the open line segment joining $(1,0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$.


Figure 4. Partition of a triangle of $\mathscr{P}_{n}$ into two subtriangles and relabelling of the vertices.
The main properties concerning the triangular tree are contained in Theorem 5.4 and Theorem 5.8 of [7]. The first result states that the tree is complete, that is

$$
\bigcup_{n \geq-1} \mathcal{T}_{n}=\mathbb{Q}^{2} \cap \bar{\triangle}
$$

with every rational pair appearing exactly once in the tree. The second result establishes the level-by-level equivalence between the counterimages tree and the geometric tree defined above, that is $\mathcal{T}_{n}=\mathcal{S}_{n} \backslash \mathcal{S}_{n-1}$ for all $n \geq 0$.

## 4. Triangle sequences: Convergence and non-Convergence

It is known that a triangle sequence does not necessarily represent a unique pair of real numbers, but could correspond to an entire line segment. If the triangle sequence terminates, we do not have uniqueness and Lemma 5.1 characterises the points having a given finite triangle sequence. Uniqueness is not guaranteed even when the triangle sequence is infinite: [8] gives a sufficient condition to have uniqueness and a criterion, equivalent to uniqueness is proved in the later work [1]. In this section we discuss this problem.

We start by introducing some notation. Let $X=\left(\begin{array}{l}q \\ p \\ r\end{array}\right)$ be a three-dimensional vector with integer components and $q \neq 0$. We then define the correspondent rational pair

$$
\hat{X}:=\left(\frac{p}{q}, \frac{r}{q}\right)
$$

for which both components have the same denominator. For instance, the vertices $v_{0}, v_{1}$ and $v_{2}$ of $\triangle$ are represented by $\left(\begin{array}{l}1 \\ 0 \\ 0\end{array}\right),\left(\begin{array}{l}1 \\ 1 \\ 0\end{array}\right)$, and $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$, respectively. Note that the sum of two three-dimensional vectors corresponds to the mediant between the two correspondent two-dimensional vectors, that is

$$
\widehat{X+Y}=\hat{X} \oplus \hat{Y}
$$

For a sequence $\left(\alpha_{0}, \alpha_{1}, \ldots\right)$ of non-negative integers and for an integer $k \geq 0$ we define

$$
\triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right):=\left\{(x, y) \in \triangle: T^{j}(x, y) \in \triangle_{\alpha_{j}} \text { for all } j=0, \ldots, k\right\} .
$$

The set $\triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right)$ is a triangle and consists of all those points whose first $k+1$ triangle sequence digits are precisely $\alpha_{0}, \ldots, \alpha_{k}$ and thus these triangles are nested, that is

$$
\triangle \supset \triangle\left(\alpha_{0}\right) \supset \triangle\left(\alpha_{0}, \alpha_{1}\right) \supset \cdots
$$

Let $\left(X_{k}\right)_{k \geq-3}$ be the sequence of three-dimensional vectors defined as follows:

$$
X_{-3}=\left(\begin{array}{l}
0  \tag{4.1}\\
0 \\
1
\end{array}\right), \quad X_{-2}=\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right), X_{-1}=\left(\begin{array}{l}
1 \\
1 \\
0
\end{array}\right), \quad X_{k}=X_{k-3}+X_{k-1}+\alpha_{k} X_{k-2} \text { for all } k \geq 0
$$

then the vertices of $\triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right)$ are $\hat{X}_{k-1}, \hat{X}_{k}$ and $\hat{X}_{k-2} \oplus \hat{X}_{k}$ (see [1, Theorem 3]). Figure 5] shows the recursive construction of the triangles $\triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right)$.

When the triangle sequence is infinite, the infinite intersection $\bigcap_{k \geq 0} \triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right)$ can be either a point or a line segment.
(1) In the first case the nested triangles shrink to a point, which means that the triangle sequence $\left(\alpha_{k}\right)_{k \geq 0}$ denotes a unique pair of real numbers $(\alpha, \beta)$. As a consequence $\operatorname{diam} \triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right) \rightarrow 0$ and the sequence $\left(\hat{X}_{k}\right)_{k \geq-3}$ converges to $(\alpha, \beta)$. We will refer to this case as the convergent case.
(2) In the second case the triangle sequence does not uniquely describe a point but instead identifies a line segment $\mathfrak{L}$ of length $l>0$, such that all the points of $\mathfrak{L}$ have the same triangle sequence $\left[\alpha_{0}, \alpha_{1}, \ldots\right]$. In this case diam $\triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right) \rightarrow l$ and the sequence $\left(\hat{X}_{k}\right)_{k \geq-3}$ does not admit a limit. More precisely, we have that the odd and even terms of $\left(\hat{X}_{k}\right)_{k \geq-3}$ converge to the two endpoints of $\mathfrak{L}$ [1, Theorem 6]. In particular, $d\left(\hat{X}_{k-1}, \hat{X}_{k}\right) \rightarrow l$ and it also holds that $d\left(\hat{X}_{k}, \hat{X}_{k-1} \oplus \hat{X}_{k}\right) \rightarrow 0$, where $d(\cdot, \cdot)$ is the Euclidean distance in $\mathbb{R}^{2}$. We will refer to this case as the non-convergent case.

The main result of [1, Section 6] is a criterion of uniqueness, which we now state. Let

$$
\lambda_{k}:=\frac{d\left(\hat{X}_{k-1}, \hat{X}_{k+1}\right)}{d\left(\hat{X}_{k-1}, \hat{X}_{k} \oplus \hat{X}_{k-2}\right)},
$$

and refer again to Figure 5 for the geometric interpretation of this quantity. The triangle sequence $\left[\alpha_{0}, \alpha_{1}, \ldots\right]$ does not correspond to a unique pair of real numbers if and only if it contains only a finite number of zeroes and $\prod_{k \geq N}\left(1-\lambda_{k}\right)>0$, where $N$ is such that $\alpha_{k}>0$ for all $k \geq N \sqrt{2}$. The convergence of the infinite product to a non-zero number is equivalent to the convergence of $\sum_{k \geq 0} \lambda_{k}$, which is in turn equivalent to $\lambda_{k} \rightarrow 0$ sufficiently fast: the geometric meaning of $\lambda_{k}$ suggests that this condition is equivalent to a sufficiently fast growth of the triangle sequence digits $\alpha_{k}$.

We now prove some results of convergence for the points of the closed triangle $\bar{\triangle}$ in the non-convergent case. For two rational pairs $\hat{X}$ and $\hat{Y}$ and a non-negative integer $s$ we introduce the notation

$$
\hat{Y} \oplus_{s} \hat{X}:=\hat{Y} \oplus \underbrace{\hat{X} \oplus \cdots \oplus \hat{X}}_{s \text { times }}
$$

and recall that the maps $\phi_{0}$ and $\phi_{1}$ commute with the mediant operation for rational pairs whose components have the same denominator, that is

$$
\phi_{i}(\hat{X} \oplus \hat{Y})=\phi_{i}(\hat{X}) \oplus \phi_{i}(\hat{Y}) \quad \text { for } i=0,1 .
$$

[^2]

Figure 5. Construction of the triangle $\triangle\left(\alpha_{0}, \ldots, \alpha_{k}, \alpha_{k+1}\right)$ starting from $\triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right)$
Also, in the following we denote by $\phi_{0}$ and $\phi_{1}$ the continuous extension to $\bar{\triangle}$ of the maps defined in (3.3) and (3.4).

Lemma 4.1. Let $k \geq 0$ and let $\left(\alpha_{j}\right)_{j=0, \ldots, k}$ be a sequence of non-negative integers. It holds that

$$
\triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right)=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k}} \phi_{0}(\bar{\triangle} \backslash \Sigma)
$$

and more precisely $\hat{X}_{k}=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k}} \phi_{0}(1,0)$.
Proof. Let $d$ be a non-negative integer. We have $\triangle_{d}=\phi_{1}^{d} \phi_{0}(\bar{\triangle} \backslash \Sigma)$, which can be verified by explicitly computing the vertices of the two triangles. This in turn proves the result in the case $k=0$.
For $k \geq 1$, by definition we have $(x, y) \in \triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right)$ if and only if

$$
T^{j}(x, y) \in \triangle_{\alpha_{j}} \quad \text { for all } j=0, \ldots, k
$$

We can thus write

$$
T^{k}(x, y)=\left.\left.\left.\left.\left.\left.T\right|_{\Delta_{\alpha_{k-1}}} \circ \cdots \circ T\right|_{\Delta_{\alpha_{0}}}(x, y) \stackrel{(3.2)}{=} S\right|_{\Gamma_{0}} S\right|_{\Gamma_{1}} ^{\alpha_{k-1}} \circ \cdots \circ S\right|_{\Gamma_{0}} \circ S\right|_{\Gamma_{1}} ^{\alpha_{0}}(x, y) \in \triangle_{\alpha_{k}}
$$

which holds if and only if $(x, y) \in \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-1}} \phi_{0}\left(\triangle_{\alpha_{k}}\right)$. Since $\triangle_{\alpha_{k}}=\phi_{1}^{\alpha_{k}} \phi_{0}(\bar{\triangle} \backslash \Sigma)$, the first part of the result follows.
For the second part of the lemma we argue by strong induction on $k \geq 0$. The case $k=0$ follows from the first part of this proof, by the explicit computation of the vertices of $\triangle\left(\alpha_{0}\right)$. Now let $k \geq 1$ and consider the triangle $\triangle\left(\alpha_{0}, \ldots, \alpha_{k-1}\right)=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-1}} \phi_{0}(\bar{\triangle} \backslash \Sigma)$. By inductive hypothesis two of its vertices are $\hat{X}_{k-1}=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-1}} \phi_{0}(1,0)$ and $\hat{X}_{k-2}=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-2}} \phi_{0}(1,0)$. Since $\phi_{0}(0,0)=(1,0)$ and $\phi_{1}(1,0)=(1,0)$, we can also write $\hat{X}_{k-2}=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-1}} \phi_{0}(0,0)$. Thus, from $\triangle\left(\alpha_{0}, \ldots, \alpha_{k-1}\right)=$ $\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-1}} \phi_{0}(\bar{\triangle} \backslash \Sigma)$, it follows that the other vertex of $\triangle\left(\alpha_{0}, \ldots, \alpha_{k-1}\right)$ has to be the backward image of $(1,1)$, that is $\hat{X}_{k-3} \oplus \hat{X}_{k-1}=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-1}} \phi_{0}(1,1)$. Since the maps $\phi_{0}$ and $\phi_{1}$ commute with the mediant, we have

$$
\begin{aligned}
\hat{X}_{k} & =\left(\hat{X}_{k-3} \oplus \hat{X}_{k-1}\right) \oplus_{\alpha_{k}} \hat{X}_{k-2}=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-1}} \phi_{0}(1,1) \oplus_{\alpha_{k}} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-1}} \phi_{0}(0,0)= \\
& =\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-1}} \phi_{0}\left(\frac{1}{\alpha_{k}+1}, \frac{1}{\alpha_{k}+1}\right)=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k-1}} \phi_{0} \phi_{1}^{\alpha_{k}} \phi_{0}(1,0)
\end{aligned}
$$

and this completes the proof.
Lemma 4.2. Let $\hat{X}=\left(\frac{p}{q}, \frac{r}{q}\right)$ and $\hat{Y}=\left(\frac{p^{\prime}}{q^{\prime}}, \frac{r^{\prime}}{q^{\prime}}\right)$, and let $s$ be a non-negative integer. Then

$$
d\left(\hat{Y} \oplus_{s} \hat{X}, \hat{X}\right)=d(\hat{X}, \hat{Y}) \frac{q^{\prime}}{q^{\prime}+s q} \quad \text { and } \quad d\left(\hat{Y} \oplus_{s} \hat{X}, \hat{Y}\right)=d(\hat{X}, \hat{Y}) \frac{s q}{q^{\prime}+s q}
$$

Proof. It is a straightforward computation involving $\hat{X}, \hat{Y}$ and $\hat{Y} \oplus_{s} \hat{X}=\left(\frac{p^{\prime}+s p}{q^{\prime}+s q}, \frac{r^{\prime}+s r}{q^{\prime}+s q}\right)$.
For $k \geq-3$ we introduce the notation

$$
X_{k}=\left(\begin{array}{l}
q_{k} \\
p_{k} \\
r_{k}
\end{array}\right)
$$

so that $\hat{X}_{k}=\left(\frac{p_{k}}{q_{k}}, \frac{r_{k}}{q_{k}}\right)$. Note that (4.1) implies that the denominators satisfy the recurrence

$$
q_{k}=q_{k-3}+q_{k-1}+\alpha_{k} q_{k-2}
$$

where $q_{-3}=0, q_{-2}=1$ and $q_{-1}=1$. This remark and Lemma 4.2 imply that for all $k \geq 0$ we have

$$
\lambda_{k}=\frac{d\left(\hat{X}_{k-1}, \hat{X}_{k+1}\right)}{d\left(\hat{X}_{k-1}, \hat{X}_{k} \oplus \hat{X}_{k-2}\right)}=\frac{q_{k-2}+q_{k}}{q_{k-2}+q_{k}+\alpha_{k+1} q_{k-1}}=\frac{q_{k-2}+q_{k}}{q_{k+1}} .
$$

It also easily follows that

$$
\begin{equation*}
1-\lambda_{k}=\alpha_{k+1} \frac{q_{k-1}}{q_{k+1}} . \tag{4.2}
\end{equation*}
$$

Lemma 4.3. Let $\left(\alpha_{k}\right)_{k \geq 0}$ be a non-convergent triangle sequence describing a line segment of length $l>0$ (as described in point (2) above).
(i) The ratio of consecutive denominators of the $X_{k}$ diverges, that is $\lim _{k \rightarrow+\infty} \frac{q_{k}}{q_{k-1}}=+\infty$.
(ii) It holds

$$
\lim _{k \rightarrow+\infty} d\left(\hat{X}_{k-2} \oplus \hat{X}_{k}, \hat{X}_{k-1}\right)=l
$$

(iii) For any non-negative integer s, it holds

$$
\lim _{k \rightarrow+\infty} d\left(\hat{X}_{k} \oplus_{s} \hat{X}_{k-1}, \hat{X}_{k}\right)=0 \quad \text { and } \quad \lim _{k \rightarrow+\infty} d\left(\left(\hat{X}_{k-2} \oplus \hat{X}_{k}\right) \oplus_{s} \hat{X}_{k-1}, \hat{X}_{k-2} \oplus \hat{X}_{k}\right)=0
$$

Proof. (i) From the recurrence for the denominators we have

$$
\frac{q_{k}}{q_{k-1}} \geq 1+\alpha_{k} \frac{q_{k-2}}{q_{k-1}} \stackrel{(4.2)}{=} 1+\left(1-\lambda_{k-1}\right) \frac{q_{k}}{q_{k-1}}
$$

so that $\frac{q_{k}}{q_{k-1}} \geq \frac{1}{\lambda_{k}}$. Since $\lambda_{k} \rightarrow 0^{+}$as $k \rightarrow+\infty$, the thesis follows.
(ii) For $k$ large enough we have

$$
d\left(\hat{X}_{k-1}, \hat{X}_{k}\right) \leq d\left(\hat{X}_{k-2} \oplus \hat{X}_{k}, \hat{X}_{k-1}\right) \leq d\left(\hat{X}_{k-2} \oplus \hat{X}_{k}, \hat{X}_{k}\right)+d\left(\hat{X}_{k-1}, \hat{X}_{k}\right)
$$

where the first inequality is shown in [1, Theorem 6] and the second inequality is the triangle inequality applied to $\triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right)$. This is enough to conclude because we already know that $d\left(\hat{X}_{k-1}, \hat{X}_{k}\right) \rightarrow l$ and $d\left(\hat{X}_{k-2} \oplus \hat{X}_{k}, \hat{X}_{k}\right) \rightarrow 0$ (see point (2) above).
(iii) From Lemma 4.2 we have

$$
d\left(\hat{X}_{k} \oplus_{s} \hat{X}_{k-1}, \hat{X}_{k}\right)=d\left(\hat{X}_{k}, \hat{X}_{k-1}\right) \frac{s}{s+\frac{q_{k}}{q_{k-1}}}
$$

and

$$
d\left(\left(\hat{X}_{k-2} \oplus \hat{X}_{k}\right) \oplus_{s} \hat{X}_{k-1}, \hat{X}_{k-2} \oplus \hat{X}_{k}\right)=d\left(\hat{X}_{k-2} \oplus \hat{X}_{k}, \hat{X}_{k-1}\right) \frac{s}{s+\frac{q_{k}}{q_{k-1}}+\frac{q_{k-2}}{q_{k-1}}}
$$

Using $d\left(\hat{X}_{k-1}, \hat{X}_{k}\right) \rightarrow l$, (i), and (ii), the result easily follows.

Proposition 4.4. Let $\left(\alpha_{k}\right)_{k \geq 0}$ be a non-convergent triangle sequence and denote by $\mathfrak{p}_{\mathfrak{L}}$ and $\mathfrak{q}_{\mathfrak{L}}$ the two endpoints of the line segment $\mathfrak{L}=\bigcap_{k \geq 0} \triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right)$, such that $\left(\hat{X}_{2 k}\right)_{k \geq 0}$ and $\left(\hat{X}_{2 k+1}\right)_{k \geq 0}$ converge respectively to $\mathfrak{p}_{\mathfrak{L}}$ and $\mathfrak{q}_{\mathfrak{L}}$. Then for all $(\bar{x}, y) \in \bar{\triangle} \backslash\{(0,0)\}$ it holds

$$
\lim _{k \rightarrow+\infty} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(x, y)=\mathfrak{p}_{\mathfrak{L}}, \quad \lim _{k \rightarrow+\infty} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k+1}} \phi_{0}(x, y)=\mathfrak{q}_{\mathfrak{L}}
$$

and

$$
\lim _{k \rightarrow+\infty} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(0,0)=\mathfrak{q}_{\mathfrak{L}}, \quad \lim _{k \rightarrow+\infty} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k+1}} \phi_{0}(0,0)=\mathfrak{p}_{\mathfrak{L}}
$$

Proof. We give the proof just for the case of even indices, the odd case is analogous. By notation we have that $\hat{X}_{2 k}=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(1,0)$ converges to $\mathfrak{p}_{\mathfrak{R}}$ and that $\hat{X}_{2 k+1}=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k+1}} \phi_{0}(1,0)$ converges to $\mathfrak{q}_{\mathfrak{L}}$. Using that $(1,0)=\phi_{1} \phi_{0}(0,0)$, we can write $\hat{X}_{2 k+1}=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k+2}} \phi_{0}(0,0)$, so that $\lim _{k \rightarrow+\infty} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(0,0)=\mathfrak{q}_{\mathfrak{L}}$.
Let now $(x, y) \in \bar{\triangle} \backslash\{(0,0)\}$ and notice that if $\lim _{k \rightarrow+\infty} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(x, y)$ exists then it must lie on the line segment $\mathfrak{L}$. We start considering the case when $(x, y)$ is on the boundary of $\bar{\triangle} \backslash\{(0,0)\}$. Suppose that $(x, y) \in \Lambda \backslash\{(0,0)\}$, so that $(x, y)=(\xi, 0)$ for a certain $\xi \in(0,1]$, and let $s$ be a positive integer such that $\frac{1}{s}<\xi$. Thus $\left(\frac{1}{s}, 0\right)=(1,0) \oplus_{s-1}(0,0)$. Writing $\hat{X}_{2 k}$ and $\hat{X}_{2 k-1}$ as above, we have

$$
\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}\left(\frac{1}{s}, 0\right)=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(1,0) \oplus_{s-1} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(0,0)=\hat{X}_{2 k} \oplus_{s-1} \hat{X}_{2 k-1}
$$

where we have used that $\phi_{0}$ and $\phi_{1}$ commute with the mediant operation. Moreover $\phi_{0}$ and $\phi_{1}$ are monotonic along line segments (with respect to the lexicographic order) and $\left(\frac{1}{s}, 0\right) \leq_{l e x}(\xi, 0) \leq_{l e x}(1,0)$, hence

$$
0 \leq d\left(\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(\xi, 0), \hat{X}_{2 k}\right) \leq d\left(\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}\left(\frac{1}{s}, 0\right), \hat{X}_{2 k}\right)=d\left(\hat{X}_{2 k} \oplus_{s-1} \hat{X}_{2 k-1}, \hat{X}_{2 k}\right)
$$

Lemma 4.3 (iii) gives $d\left(\hat{X}_{2 k} \oplus_{s-1} \hat{X}_{2 k-1}, \hat{X}_{2 k}\right) \rightarrow 0$ for $k \rightarrow+\infty$, thus $d\left(\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(\xi, 0), \hat{X}_{2 k}\right) \rightarrow 0$, which means that $\lim _{k \rightarrow+\infty} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(\xi, 0)=\mathfrak{p}_{\mathfrak{L}}$. If $(x, y) \in \Upsilon$ or $(x, y) \in \Sigma \backslash\{(0,0)\}$ we can argue as above to conclude that $\lim _{k \rightarrow+\infty} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(x, y)=\mathfrak{p}_{\mathfrak{L}}$.
Finally, let $(x, y)$ be an interior point of the triangle $\triangle$ and let $A$ be a triangle containing $(x, y)$ as interior point and having all the vertices along $\partial \triangle \backslash\{(0,0)\}$. The maps $\phi_{0}$ and $\phi_{1}$ map triangles into triangles and the same does the composite map $\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}$. Thus for all $k \geq 0$ the image $\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(A)$ is a triangle containing $\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(x, y)$ as an interior point. The thesis follows by observing that $\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{2 k}} \phi_{0}(A)$ shrinks to $\mathfrak{p}_{\mathfrak{L}}$ because its three vertices converge to $\mathfrak{p}_{\mathfrak{L}}$.

The last result will be important in Section 5 to better understand the coding of real pairs in the nonconvergent case.

## 5. A two-dimensional representation

In this section we begin to use our construction of the Triangular tree and the properties of the maps $\phi_{0}$, $\phi_{1}$ and $\phi_{2}$ defined in (3.3)-(3.5) to introduce a new representation of real pairs of numbers in $\bar{\triangle}$ by combining triangle sequences and continued fraction expansions. We recall that for any finite binary word $\omega \in\{0,1\}^{*}$ of length $n$, we let $\phi_{\omega}:=\phi_{\omega_{0}} \circ \phi_{\omega_{1}} \circ \cdots \circ \phi_{\omega_{n-1}}$.

Lemma 5.1. Let $(\alpha, \beta)$ be a point of $\bar{\triangle}$ with finite triangle sequence $\left[\alpha_{0} \ldots, \alpha_{k}\right]$. If $(\alpha, \beta)$ is an interior point, then:
(i) $\quad(\alpha, \beta) \in \phi_{\omega} \phi_{2}(\Sigma \backslash\{(0,0),(1,1)\})$, where $\omega=1^{\alpha_{0}} 0 \cdots 1^{\alpha_{k-1}} 01^{\alpha_{k}} 0$, so that there exists a unique $\xi \in(0,1)$ such that

$$
\begin{equation*}
(\alpha, \beta)=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k}} \phi_{0} \phi_{2}(\xi, \xi) \tag{5.1}
\end{equation*}
$$

(ii) $\xi$ is rational if and only if $(\alpha, \beta) \in \mathbb{Q}^{2}$.

Proof. (i) By definition of triangle sequences we have

$$
T^{j}(\alpha, \beta) \in \triangle_{\alpha_{j}} \quad \text { for all } j=0, \ldots, k \quad \text { and } \quad T^{k+1}(\alpha, \beta) \in \Lambda
$$

Thus $T^{k+1}(\alpha, \beta)=(\xi, 0)=\phi_{2}(\xi, \xi)$ for some $\xi \in(0,1)$. From (3.2) and arguing as in Lemma 4.1 the thesis follows.
(ii) From (i) we have that $(\alpha, \beta)$ can be obtained from $(\xi, 0)$ with the application of a finite number of maps between $\phi_{0}$ and $\phi_{1}$, which are linear fractional maps.

Remark 5.2. Let $(\alpha, \beta)$ be an interior point having finite triangle sequence $\left[\alpha_{0}, \ldots, \alpha_{k}\right]$. Note that $\alpha_{k}>0$ because the only points in $\bar{\triangle}$ for which the triangle sequence is defined and ends with 0 are located on $(\Sigma \cup \Upsilon) \backslash \Lambda$. In light of this simple remark we can rewrite (5.1) as

$$
(\alpha, \beta)=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k}-1}\left(\frac{1}{1+\xi}, \frac{\xi}{1+\xi}\right)
$$

so that $(\alpha, \beta) \in \phi_{\omega}(\ell)$ where $\omega=1^{\alpha_{0}} 0 \cdots 1^{\alpha_{k}-1}$.
Definition 5.3. Let $(\alpha, \beta)$ be an interior point of $\bar{\triangle}$ with finite triangle sequence $\left[\alpha_{0}, \ldots, \alpha_{k}\right]$, and let $\left[a_{1}, a_{2}, \ldots\right]$ be the continued fraction expansion of the unique number $\xi$ given by Lemma 5.1.(ii). We associate to $(\alpha, \beta)$ the representation given by the pair

$$
\left(\left[\alpha_{0}, \ldots, \alpha_{k}\right],\left[a_{1}, \ldots\right]\right)
$$

As for the second component, we write [0] and [1] to denote the continued fraction expansion of 0 and 1 , respectively.

Remark 5.4. If $(\alpha, \beta)$ is a rational pair, then $\xi$ is rational (Lemma 5.1f(iii)) and thus its continued fraction expansion is finite, say $\left[a_{1}, \ldots, a_{n}\right]$. In this case we further assume that $a_{n}>1$ when $n>1$.

We give further details for the coding of boundary rational points and, in particular, for the vertices of the triangle $\triangle$.
(1) A point on $\Sigma \backslash\{(0,0),(1,1)\}$ is of the kind $(\xi, \xi)$, for some $\xi$ in the open unit interval. If [ $\left.a_{1}, \ldots\right]$ is the continued fraction expansion of $\xi$, then $\frac{1}{a_{1}-1}<\xi \leq \frac{1}{a_{1}}$, so that $(\xi, \xi) \in \triangle_{a_{1}-1}$. In case $\xi=\frac{1}{a_{1}}$ we have $T(\xi, \xi)=(1,0) \in \Lambda$, so that the triangle sequence of $(\xi, \xi)$ is $\left[a_{1}-1\right]$. Otherwise, if $\frac{1}{a_{1}-1}<\xi<\frac{1}{a_{1}}$, we have $T(\xi, \xi) \in \Upsilon \backslash\{(1,0)\}$, so that the triangle sequence of $(\xi, \xi)$ is $\left[a_{1}-1,0\right]$. Thus we set the representation of $(\xi, \xi)$ to be respectively

$$
\left(\left[a_{1}-1\right],\left[a_{1}\right]\right) \quad \text { or } \quad\left(\left[a_{1}-1,0\right],\left[a_{1}, \ldots\right]\right)
$$

(2) A point in $\Lambda$ is of the kind $(\xi, 0)$, for some $\xi$ in the unit interval, and its triangle sequence is not defined and assumed to be empty. We thus represent $(\xi, 0)$ with

$$
\left([],\left[a_{1}, \ldots\right]\right)
$$

where $\left[a_{1}, \ldots\right]$ is the continued fraction expansion of $\xi$. In particular, the representation of the vertices $(0,0)$ and $(1,0)$ are $([],[0])$ and $([],[1])$, respectively.
(3) A point in $\Upsilon \backslash\{(1,0)\}$ is of the kind $(1, \xi)$, for some $\xi \in(0,1]$, and has triangle sequence [0]. The representation of $(1, \xi)$ is thus

$$
\left([0],\left[a_{1}, \ldots\right]\right)
$$

where $\left[a_{1}, \ldots\right]$ is the continued fraction expansion of $\xi$. In particular, the representation of the vertex $(1,1)$ is $([0],[1])$.

The representation of real pairs with infinite triangle sequences depends on the convergence of the sequence. We have seen in Section 4 that if $(\alpha, \beta)$ is a real pair in $\bar{\triangle}$ with convergent infinite triangle sequence [ $\left.\alpha_{0}, \alpha_{1}, \ldots\right]$, then

$$
\{(\alpha, \beta)\}=\bigcap_{k \geq 0} \triangle\left(\alpha_{0}, \ldots, \alpha_{k}\right)
$$

This shows that in this case it is enough to associate to $(\alpha, \beta)$ its triangle sequence, since

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k}} \phi_{0}(x, y)=(\alpha, \beta) \tag{5.2}
\end{equation*}
$$

for all $(x, y) \in \bar{\triangle}$. In the definition below we choose $(x, y)=\left(\frac{1}{2}, 0\right)$, but other choices would work as well.
Definition 5.5. Let $(\alpha, \beta)$ be a point of $\bar{\triangle}$ with convergent infinite triangle sequence $\left[\alpha_{0}, \alpha_{1}, \ldots\right]$. We associate to $(\alpha, \beta)$ the representation given by the pair

$$
\left(\left[\alpha_{0}, \alpha_{1}, \ldots\right],[2]\right)
$$

Let us now consider the case of non-convergent infinite triangle sequences. In this case a line segment $\mathfrak{L}$ is associated to such a sequence, and we refer to Proposition 4.4 for the notation of its endpoints and more properties. Using these results we give the following definition.

Definition 5.6. Let $(\alpha, \beta)$ be a point of $\bar{\triangle}$ with non-convergent infinite triangle sequence $\left[\alpha_{0}, \alpha_{1}, \ldots\right]$. Then $(\alpha, \beta)$ belongs to a line segment $\mathfrak{L}$ of real pairs having the same triangle sequence, with endpoints $\mathfrak{p}_{\mathfrak{L}}$ and $\mathfrak{q}_{\mathfrak{L}}$. Then we consider

$$
\left(\left[\alpha_{0}, \alpha_{1}, \ldots\right],[2]\right)
$$

to be the representation of the segment $\mathfrak{L}$.

## 6. The Triangular coding

In this section we use the ideas exposed in Section 5 to introduce a coding for the rational pairs in the Triangular tree in an analogous way as for the Farey coding. As recalled in Section 2 the continued fraction expansion of a rational number in $(0,1)$ is related to the path on the Farey tree to reach it starting from the root $\frac{1}{2}$. This information is contained in the $\{L, R\}$ coding, which in turn can be seen as the action by right multiplication of two matrices $L, R \in S L(2, \mathbb{Z})$ on the matrix representation of rational numbers. We now generalise this setting for the two-dimensional case by first defining a coding for the possible moves from parents to children along the Triangular tree in figure 3 Then we introduce a matrix representation for rational pairs and convert the action of the moves on the tree into the action by right multiplication of specific $S L(3, \mathbb{Z})$ matrices.

Definition 6.1. Let $\mathfrak{S}$ be a line segment in $\bar{\triangle}$ obtained as a counterimage of $\Sigma \backslash\{(0,0),(1,1)\}$ by a combination $\phi_{\tilde{\omega}}$ of the maps $\phi_{0}, \phi_{1}$ and $\phi_{2}$, where $\tilde{\omega}$ is a finite word which is either empty or is the concatenation $(\omega 2)$ with $\omega \in\{0,1\}^{*}$. We consider on $\mathfrak{S}$ the orientation induced by the lexicographic ordering on $\Sigma$ by the $\operatorname{map} \phi_{\tilde{\omega}}$.
A rational pair $\left(\frac{p}{q}, \frac{r}{q}\right)$ in $\mathfrak{S}$ is obtained in the Triangular tree as the mediant of the neighbouring pairs, its parents. Then we define two actions, $L$ and $R$, on $\left(\frac{p}{q}, \frac{r}{q}\right)$, as follows: $\left(\frac{p}{q}, \frac{r}{q}\right) L$ is the rational pair obtained as the mediant of $\left(\frac{p}{q}, \frac{r}{q}\right)$ with its left parent and $\left(\frac{p}{q}, \frac{r}{q}\right) R$ is the rational pair obtained as the mediant of $\left(\frac{p}{q}, \frac{r}{q}\right)$ with its right parent.

Lemma 6.2. Let $\frac{a}{b} \in \mathbb{Q} \cap(0,1)$ and let $\left[a_{1}, \ldots, a_{n}\right]$ be its continued fraction expansion. We have

$$
\left(\frac{a}{b}, \frac{a}{b}\right)= \begin{cases}\left(\frac{1}{2}, \frac{1}{2}\right) L^{a_{1}-1} R^{a_{2}} \cdots L^{a_{n-1}} R^{a_{n}-1}, & \text { if } n \text { is even } \\ \left(\frac{1}{2}, \frac{1}{2}\right) L^{a_{1}-1} R^{a_{2}} \cdots R^{a_{n-1}} L^{a_{n}-1} & \text { if } n \text { is odd }\end{cases}
$$

The same combination of actions sends $\left(\frac{1}{2}, 0\right)$ to $\left(\frac{a}{b}, 0\right)$ on $\Lambda$ and $\left(1, \frac{1}{2}\right)$ to $\left(1, \frac{a}{b}\right)$ on $\Upsilon$.

Proof. According to (2.3), the path on the Farey tree from $\frac{1}{2}$ to $\frac{a}{b}$ is $L^{a_{1}-1} R^{a_{2}} \cdots L^{a_{n-1}} R^{a_{n}-1}$ if $n$ is even and $L^{a_{1}-1} R^{a_{2}} \cdots R^{a_{n-1}} L^{a_{n}-1}$ if $n$ is odd. By the definition of $L$ and $R$ in Definition 6.1, this is also the path to reach $\left(\frac{a}{b}, \frac{a}{b}\right)$ from $\left(\frac{1}{2}, \frac{1}{2}\right)$. The same holds on $\Lambda$ and $\Upsilon$.
Remark 6.3. In the case $n=1$ the above formula reads $\left(\frac{a}{b}, \frac{a}{b}\right)=\left(\frac{1}{2}, \frac{1}{2}\right) L^{a_{1}-2}$. The same holds also in Theorem 6.6 below.

The previous lemma shows that the basic moves $L$ and $R$ are enough to reach every boundary rational pair starting from the midpoints of the sides of $\triangle$. We now describe how to reach interior rational pairs along the tree always starting from $\left(\frac{1}{2}, \frac{1}{2}\right)$. Recall that we denote by $\ell$ the open line segment joining $(1,0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$. Since interior rational pairs are located on backward images $\phi_{\omega}(\ell)$ with $\omega \in\{0,1\}^{*}$, our strategy is divided into two steps: first we describe a path from $\left(\frac{1}{2}, \frac{1}{2}\right)$ to the mediant between the endpoints of $\phi_{\omega}(\ell)$, and then we encode the sequence of moves along $\phi_{\omega}(\ell)$ to reach the considered rational pair.
Definition 6.4. Let $\omega \in\{0,1\}^{*}$ with $n=|\omega| \geq 0$ and consider the triangle $\triangle_{\omega}=\phi_{\omega}(\bar{\triangle})$, partitioned by $\phi_{\omega}(\ell)$ into two subtriangles. The action of the symbol $I$ on $\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right)$, which we denote as a right action by $\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right) I$, gives the mediant between $\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\phi_{\omega}(1,0)$. In other words,

$$
\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right) I:=\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right) \oplus \phi_{\omega}(1,0)=\phi_{\omega}\left(\frac{2}{3}, \frac{1}{3}\right)
$$

We now give a geometric interpretation of this definition, also clarified by Figure 6. The point $\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right)$ is the right endpoint of $\phi_{\omega}(\ell)$ and the action of $I$ is to step from $\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right)$ to the mediant between the two endpoints of $\phi_{\omega}(\ell)$, which is one of the children of $\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right)$.


Figure 6. Right action of the symbol $I$.
Lemma 6.5. Let $n \geq 0$ and let $\omega \in\{0,1\}^{*}$ with $|\omega|=n$. Then

$$
\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right)=\left(\frac{1}{2}, \frac{1}{2}\right) \prod_{i=1}^{n} W_{i}
$$

where $W_{i}=I$ if $\omega_{i}=0$ and $W_{i}=L$ if $\omega_{i}=1$.
Proof. We argue by induction on $n \geq 0$. The conclusion is trivial when $n=0$. Suppose that the thesis holds for all the finite binary words of length $n$ and let $\omega$ be a word with $|\omega|=n+1$. Thus $\omega=\tilde{\omega} 0$ or $\omega=\tilde{\omega} 1$ for some word $\tilde{\omega}$ of length $n$. By definition of $I$ we have $\phi_{\tilde{\omega}}\left(\frac{1}{2}, \frac{1}{2}\right) I=\phi_{\tilde{\omega}}\left(\frac{2}{3}, \frac{1}{3}\right)=\phi_{\tilde{\omega} 0}\left(\frac{1}{2}, \frac{1}{2}\right)$, thus the thesis holds for $\tilde{\omega} 0$. For $\tilde{\omega} 1$ note that the left and right parents of the point $\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right)$ are $\phi_{\omega}(0,0)$ and $\phi_{\omega}(1,1)$, respectively. Hence

$$
\phi_{\tilde{\omega}}\left(\frac{1}{2}, \frac{1}{2}\right) L=\phi_{\tilde{\omega}}\left(\frac{1}{2}, \frac{1}{2}\right) \oplus \phi_{\tilde{\omega}}(0,0)=\phi_{\tilde{\omega}}\left(\frac{1}{3}, \frac{1}{3}\right)=\phi_{\tilde{\omega} 1}\left(\frac{1}{2}, \frac{1}{2}\right)
$$

so that the thesis is also true for $\tilde{\omega} 1$.

Theorem 6.6. Let $\left(\frac{p}{q}, \frac{r}{q}\right)$ be the interior rational pair with representation $\left(\left[\alpha_{0}, \ldots, \alpha_{k}\right],\left[a_{1}, \ldots, a_{n}\right]\right)$ (see Definition (5.3) and let $\omega=1^{\alpha_{0}} 0 \cdots 1^{\alpha_{k-1}} 01^{\alpha_{k}-1}$, so that $\left(\frac{p}{q}, \frac{r}{q}\right) \in \phi_{\omega}(\ell)$. Then:
(i) $\quad \phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right)=\left(\frac{1}{2}, \frac{1}{2}\right) L^{\alpha_{0}} I \cdots L^{\alpha_{k-1}} I L^{\alpha_{k}-1}$;
(ii) $\left(\frac{p}{q}, \frac{r}{q}\right)=\left\{\begin{array}{ll}\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right) I L^{a_{1}-1} R^{a_{2}} \cdots L^{a_{n-1}} R^{a_{n}-1} & \text { if } n \text { is even } \\ \phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right) I L^{a_{1}-1} R^{a_{2}} \cdots R^{a_{n-1}} L^{a_{n}-1}, & \text { if } n \text { is odd }\end{array}\right.$.

Hence

$$
\left(\frac{p}{q}, \frac{r}{q}\right)= \begin{cases}\left(\frac{1}{2}, \frac{1}{2}\right) L^{\alpha_{0}} I \cdots L^{\alpha_{k-1}} I L^{\alpha_{k}-1} I L^{a_{1}-1} R^{a_{2}} \cdots L^{a_{n-1}} R^{a_{n}-1}, & \text { if } n \text { is even } \\ \left(\frac{1}{2}, \frac{1}{2}\right) L^{\alpha_{0}} I \cdots L^{\alpha_{k-1}} I L^{\alpha_{k}-1} I L^{a_{1}-1} R^{a_{2}} \cdots R^{a_{n-1}} L^{a_{n}-1}, & \text { if } n \text { is odd }\end{cases}
$$

Proof. (i) It is a straightforward consequence of Lemma 6.5
(ii) Let $\frac{a}{b}$ the unique rational number associated to $\left(\frac{p}{q}, \frac{r}{q}\right)$ according to Lemma 5.1f(ii), so that by definition $\left[a_{1}, \ldots, a_{n}\right]$ is its continued fraction expansion. Thus $\left(1, \frac{a}{b}\right)$ can be reached from $\left(1, \frac{1}{2}\right)$ with the sequence $L^{a_{1}-1} R^{a_{2}} \cdots L^{a_{n-1}} R^{a_{n}-1}$ if $n$ is even or with $L^{a_{1}-1} R^{a_{2}} \cdots R^{a_{n-1}} L^{a_{n}-1}$ if $n$ is odd, from Lemma 6.2, Since the maps $\phi_{0}$ and $\phi_{1}$ commute with the mediant operation, it is easy to prove that also the parent-child relationship is preserved. Thus by the above sequence of $L$ and $R$ moves one obtains $\left(\frac{p}{q}, \frac{r}{q}\right)$ from $\phi_{\omega 1}\left(1, \frac{1}{2}\right)=$ $\phi_{\omega}\left(\frac{2}{3}, \frac{1}{3}\right)=\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right) I$, which is the mediant between the endpoints of $\phi_{\omega}(\ell)$.

To each interior rational pair we have thus associated a finite word over the alphabet $\{L, R, I\}$, with the geometric meaning of telling how to move on the Triangular tree to reach the rational pair under consideration starting from the root $\left(\frac{1}{2}, \frac{1}{2}\right)$. In particular, this finite word is the concatenation between a word over $\{L, I\}$ ending necessarily with the symbol $I$ and a word over $\{L, R\}$ : the first word gives the path to reach $\phi_{\omega}\left(\frac{1}{2}, \frac{1}{2}\right) I$, which is the rational point on $\phi_{\omega}(\ell)$ appearing on the level of the tree with smallest index and also the mediant between its endpoints; and the second word gives the moves along the line segment $\phi_{\omega}(\ell)$ to reach the given point from the mediant of its endpoints.

To continue the analogy with the one-dimensional case, we now convert the actions of the symbols $L$, $R$, and $I$, into the actions by right multiplication of three $S L(3, \mathbb{Z})$ matrices, using a matrix representation for the rational pairs in $\bar{\triangle}$. Towards this aim we recall the correspondence between rational pairs and three-dimensional vectors we have introduced in Section 4

Definition 6.7. Let $\left(\frac{p}{q}, \frac{r}{q}\right)$ be a rational pair in $\bar{\triangle} \backslash \mathcal{T}_{-1}$. We associate to $\left(\frac{p}{q}, \frac{r}{q}\right)$ the $3 \times 3$ matrix $\mathfrak{m}\left(\frac{p}{q}, \frac{r}{q}\right)$ defined as follows. The first two columns are respectively the right and the left parents of $\left(\frac{p}{q}, \frac{r}{q}\right)$, expressed as three-dimensional vectors. The third column depends on the location of the point:
(i) if the pair is a boundary point, the third column is the vertex of $\bar{\triangle}$ which is opposite to the side containing the pair;
(ii) if $\left(\frac{p}{q}, \frac{r}{q}\right) \in \phi_{\omega}(\ell)$ for $\omega \in\{0,1\}^{*}$, the third column is the vertex " 2 " of $\triangle_{\omega}$, that is $\phi_{\omega}(1,1)$.

The three-dimensional vector associated to $\left(\frac{p}{q}, \frac{r}{q}\right)$ is obtained as the sum of the first two columns of its matrix.

Since they will take on great importance, we explicitly show the matrices representing the midpoints of the three sides of $\triangle$ :

$$
\mathfrak{m}\left(\frac{1}{2}, \frac{1}{2}\right)=\left(\begin{array}{lll}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 0
\end{array}\right), \quad \mathfrak{m}\left(\frac{1}{2}, 0\right)=\left(\begin{array}{lll}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{array}\right), \quad \mathfrak{m}\left(1, \frac{1}{2}\right)=\left(\begin{array}{lll}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{array}\right) .
$$

Let us now define

$$
L:=\left(\begin{array}{lll}
1 & 0 & 0  \tag{6.1}\\
1 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), \quad R:=\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), \quad I:=\left(\begin{array}{lll}
1 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right)
$$

Note that all the above matrices are in $S L(3, \mathbb{Z})$ and that $L$ and $R$ extend their $2 \times 2$ counterpart defined in (2.2). A straightforward computation shows that the action of each of the above matrices by right multiplication on the matrix representing a rational pair yields the matrix of the child obtained according to the move bearing the same name of the matrix. As a direct consequence of Lemma 6.2 and Theorem 6.6 we thus have the following properties.
(1) Let $\frac{a}{b} \in \mathbb{Q} \cap(0,1)$ and let $\left[a_{1}, \ldots, a_{n}\right]$ be its continued fraction expansion. Then

$$
\mathfrak{m}\left(\frac{a}{b}, \frac{a}{b}\right)= \begin{cases}\mathfrak{m}\left(\frac{1}{2}, \frac{1}{2}\right) L^{a_{1}-1} R^{a_{2}} \cdots L^{a_{n-1}} R^{a_{n}-1}, & \text { if } n \text { is even } \\ \mathfrak{m}\left(\frac{1}{2}, \frac{1}{2}\right) L^{a_{1}-1} R^{a_{2}} \cdots R^{a_{n-1}} L^{a_{n}-1}, & \text { if } n \text { is odd }\end{cases}
$$

The same right action yields $\mathfrak{m}\left(\frac{a}{b}, 0\right)$ from $\mathfrak{m}\left(\frac{1}{2}, 0\right)$ and $\mathfrak{m}\left(1, \frac{a}{b}\right)$ from $\mathfrak{m}\left(1, \frac{1}{2}\right)$.
(2) Let $\left(\frac{p}{q}, \frac{r}{q}\right)$ be the interior rational pair with representation $\left(\left[\alpha_{0}, \ldots, \alpha_{k}\right],\left[a_{1}, \ldots, a_{n}\right]\right)$. Then

$$
\mathfrak{m}\left(\frac{p}{q}, \frac{r}{q}\right)= \begin{cases}\mathfrak{m}\left(\frac{1}{2}, \frac{1}{2}\right) L^{\alpha_{0}} I \cdots L^{\alpha_{k-1}} I L^{\alpha_{k}-1} I L^{a_{1}-1} R^{a_{2}} \cdots L^{a_{n-1}} R^{a_{n}-1}, & \text { if } n \text { is even } \\ \mathfrak{m}\left(\frac{1}{2}, \frac{1}{2}\right) L^{\alpha_{0}} I \cdots L^{\alpha_{k-1}} I L^{\alpha_{k}-1} I L^{a_{1}-1} R^{a_{2}} \cdots R^{a_{n-1}} L^{a_{n}-1}, & \text { if } n \text { is odd }\end{cases}
$$

Example. Consider the point in with representation $([2,0,1,1],[2,2])$. Using $\ell=\phi_{1} \phi_{0} \phi_{2}(\Sigma \backslash\{(0,0),(1,1)\})$ and (5.1), our point is located along the open line segment $\phi_{\omega}(\ell)$ with $\omega=110010$, whose left and right endpoints can be readily computed and are respectively:

$$
\phi_{110010}(1,0)=\left(\frac{3}{8}, \frac{2}{8}\right) \quad \text { and } \quad \phi_{110010}\left(\frac{1}{2}, \frac{1}{2}\right)=\left(\frac{5}{15}, \frac{4}{15}\right)
$$

By Theorem 6.6 we have that the path along the Triangular tree to reach this point from the root $\left(\frac{1}{2}, \frac{1}{2}\right)$ is

$$
L^{2} I I L I I L R
$$

which is the concatenation of the words $L^{2} I I L I I$ and $L R$. The word $L^{2} I I L I I$ brings $\left(\frac{1}{2}, \frac{1}{2}\right)$ first to $\left(\frac{1}{2}, \frac{1}{2}\right) L^{2} I I L I=\left(\frac{5}{15}, \frac{4}{15}\right)$, the right endpoint of $\phi_{110010}(\ell)$, and then to $\left(\frac{1}{2}, \frac{1}{2}\right) L^{2} I I L I I=\left(\frac{5}{15}, \frac{4}{15}\right) I=$ $\left(\frac{8}{23}, \frac{6}{23}\right)$, the mediant between the endpoints of $\phi_{110010}(\ell)$. The word $L R$ shows how to move along $\phi_{110010}(\ell)$ starting from $\left(\frac{8}{23}, \frac{6}{23}\right)$ to obtain our point:

$$
\left(\frac{8}{23}, \frac{6}{23}\right) L R=\left(\frac{11}{31}, \frac{8}{31}\right) R=\left(\frac{19}{54}, \frac{14}{54}\right)
$$

Figure 7 shows the the path in the triangle to reach $\phi_{110010}(\ell)$ and the moves along this line segment to get to our point. The corresponding matrix multiplication yields

$$
\mathfrak{m}\left(\frac{1}{2}, \frac{1}{2}\right) L^{2} I I L I I L R=\left(\begin{array}{ccc}
23 & 31 & 11 \\
8 & 11 & 4 \\
6 & 8 & 3
\end{array}\right)
$$

which is by definition the matrix representing $\left(\frac{19}{54}, \frac{14}{54}\right)$.
The last step to complete the similarity with the one-dimensional case is a way to use the coding we have introduced to express a rational pair as a counterimage of $\left(\frac{1}{2}, \frac{1}{2}\right)$ and vice versa. The key observation is that using the local inverses $\phi_{0}, \phi_{1}$, and $\phi_{2}$, it is possible to mimic the behaviour of the one-dimensional Farey map along $\Sigma$ in terms of the inverses $\psi_{0}$ and $\psi_{1}$ defined in (2.1). In fact a straightforward computation shows that for $0 \leq x \leq 1$ it holds

$$
\begin{equation*}
\phi_{1}(x, x)=\left(\psi_{0}(x), \psi_{0}(x)\right) \quad \text { and } \quad \phi_{0}^{2} \phi_{2}(x, x)=\left(\psi_{1}(x), \psi_{1}(x)\right) \tag{6.2}
\end{equation*}
$$



Figure 7. Path to reach the right endpoint $\left(\frac{5}{15}, \frac{4}{15}\right)$ of the line segment $\phi_{110010}(\ell)$ with the moves $L^{2} I I L I$. The line segment $\phi_{110010}(\ell)$ is enhanced in blue and the remaining moves $I L R$ are represented in the bottom-right figure.

Proposition 6.8. (i) Let $\frac{a}{b}$ be a rational number with continued fraction expansion $\left[a_{1}, \ldots, a_{n}\right]$. Then

$$
\begin{aligned}
& \left(\frac{a}{b}, \frac{a}{b}\right)=\phi_{1}^{a_{1}-1} \phi_{0}^{2} \phi_{2} \cdots \phi_{1}^{a_{n-1}-1} \phi_{0}^{2} \phi_{2} \phi_{1}^{a_{n}-2}\left(\frac{1}{2}, \frac{1}{2}\right) \\
& \left(\frac{a}{b}, 0\right)=\phi_{2} \circ \phi_{1}^{a_{1}-1} \phi_{0}^{2} \phi_{2} \cdots \phi_{1}^{a_{n-1}-1} \phi_{0}^{2} \phi_{2} \phi_{1}^{a_{n}-2}\left(\frac{1}{2}, \frac{1}{2}\right), \\
& \left(1, \frac{a}{b}\right)=\phi_{0} \circ \phi_{2} \circ \phi_{1}^{a_{1}-1} \phi_{0}^{2} \phi_{2} \cdots \phi_{1}^{a_{n-1}-1} \phi_{0}^{2} \phi_{2} \phi_{1}^{a_{n}-2}\left(\frac{1}{2}, \frac{1}{2}\right) .
\end{aligned}
$$

(ii) Let $\left(\frac{p}{q}, \frac{r}{q}\right)$ be the interior rational pair with representation $\left(\left[\alpha_{0}, \ldots, \alpha_{k}\right],\left[a_{1}, \ldots, a_{n}\right]\right)$. Then

$$
\left(\frac{p}{q}, \frac{r}{q}\right)=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k}} \phi_{0} \circ \phi_{2} \circ \phi_{1}^{a_{1}-1} \phi_{0}^{2} \phi_{2} \cdots \phi_{1}^{a_{n-1}-1} \phi_{0}^{2} \phi_{2} \phi_{1}^{a_{n}-2}\left(\frac{1}{2}, \frac{1}{2}\right) .
$$

Proof. (i) From the properties of the Farey map we have

$$
\frac{a}{b}=\psi_{0}^{a_{1}-1} \psi_{1} \cdots \psi_{0}^{a_{n}-2}\left(\frac{1}{2}\right)
$$

By using the correspondence between $\psi_{0}$ and $\phi_{1}$ and between $\psi_{1}$ and $\phi_{0}^{2} \phi_{2}$ we can conclude for $\left(\frac{a}{b}, \frac{a}{b}\right)$. The expressions for $\left(\frac{a}{b}, 0\right)$ and $\left(1, \frac{a}{b}\right)$ are now a trivial consequence.
(ii) Let $\frac{a}{b}$ the unique rational number associated to $\left(\frac{p}{q}, \frac{r}{q}\right)$ according to Lemma 5.1-(i), so that by definition $\left[a_{1}, \ldots, a_{n}\right]$ is its continued fraction expansion. Equation (5.1) gives $\left(\frac{p}{q}, \frac{r}{q}\right)=\phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k}} \phi_{0} \phi_{2}\left(\frac{a}{b}, \frac{a}{b}\right)$, so that the conclusion holds directly from (i).

Remark 6.9. (i) In the statement of the above proposition we have explicitly used the composition sign to emphasise the separation between the first group of maps, the map $\phi_{2}$, and the second group, because they correspond to the different parts of the representation.
(ii) Note that the map $\phi_{2}$ separating the two groups of maps is an actual separator, meaning that if we exhibit a point as a counterimage of $\left(\frac{1}{2}, \frac{1}{2}\right)$, it can be unambiguously identified. In fact it is the only map $\phi_{2}$ which is preceded by a single occurrence of $\phi_{0}$.

Also the definition of rank can be extended to our two-dimensional setting. We say that $\operatorname{rank}\left(\frac{p}{q}, \frac{r}{q}\right)=m$ if and only if $x \in \mathcal{T}_{m} \backslash \mathcal{T}_{m-1}$.

Corollary 6.10. Let $\left(\frac{p}{q}, \frac{r}{q}\right)$ be the rational pair with representation $\left(\left[\alpha_{0}, \ldots, \alpha_{k}\right],\left[a_{1}, \ldots, a_{n}\right]\right)$. Then

$$
\operatorname{rank}\left(\frac{p}{q}, \frac{r}{q}\right)= \begin{cases}\sum_{j=0}^{k} \alpha_{j}+\sum_{i=1}^{n} a_{i}+k-2, & \text { if the triangle sequence exists and } \alpha_{k}>0 \\ \sum_{i=1}^{n} a_{i}-2, & \text { otherwise }\end{cases}
$$

Proof. By definition, $\operatorname{rank}\left(\frac{1}{2}, \frac{1}{2}\right)=0$. Consider the expression of a rational pair as a backward image of $\left(\frac{1}{2}, \frac{1}{2}\right)$ given in Proposition 6.8 and look separately to its (at most) three blocks of compositions, starting from the right to the left.
(1) In the first group of maps, the one corresponding to the continued fraction expansion, each application of $\phi_{1}$ and each block $\phi_{0}^{2} \phi_{2}$ increases the rank by $1\left(\phi_{0}^{2} \phi_{2}\right.$ does not increase the rank more because it is applied to points on $\Sigma$ ). Thus the rank of $\phi_{1}^{a_{1}-1} \phi_{0}^{2} \phi_{2} \cdots \phi_{1}^{a_{n-1}-1} \phi_{0}^{2} \phi_{2} \phi_{1}^{a_{n}-2}\left(\frac{1}{2}, \frac{1}{2}\right)$ is $\sum_{i=1}^{n} a_{i}-2$.
(2) The application of the separator $\phi_{2}$ appears if and only if the point is not along $\Sigma$ and anyway does not change the level.
(3) The third block appears if and only if the point is along $\Upsilon$ or the point is interior. Moreover, when the third block appears, its right end is exactly one application of the map $\phi_{0}$, but since it acts on a point on $\Lambda$, it does not change the level. Each of the subsequent maps increases the level by 1 , for a total increase of $\sum_{j=0}^{k} \alpha_{j}+k$. Thus note that the block related to the triangle sequence contributes to the rank if and only if the point is an interior rational pair.

Example. As in Example 6, consider the point $\left(\frac{19}{54}, \frac{14}{54}\right)$ whose representation is ( $\left.[2,0,1,1],[2,2]\right)$. By Proposition 6.8, we have

$$
\left(\frac{19}{54}, \frac{14}{54}\right)=\phi_{1}^{2} \phi_{0} \phi_{0} \phi_{1} \phi_{0} \phi_{1} \phi_{0} \circ \phi_{2} \circ \phi_{1} \phi_{0}^{2} \phi_{2}\left(\frac{1}{2}, \frac{1}{2}\right)=\phi_{1}^{2} \phi_{0} \phi_{0} \phi_{1} \phi_{0} \phi_{1} \phi_{0} \circ \phi_{2}\left(\frac{2}{5}, \frac{2}{5}\right),
$$

where we have emphasised that our point is a backward image of $\left(\frac{2}{5}, \frac{2}{5}\right)$, which corresponds to the second component of the representation since $\frac{2}{5}=[2,2]$.

## 7. Approximations for non-rational pairs

In this section we use the structure of the Triangular tree to define approximations of real pairs in $\bar{\triangle}$ by rational pairs. This is particularly important for non-rational pairs, which have an infinite number of rational approximations.

Non-rational pairs have either an infinite triangle sequence or a finite triangle sequence with infinite associated continued fraction expansion. An analogous of Proposition 6.8 holds also for non-rational pairs, apart from the non-convergent infinite case, for which Proposition 4.4 basically implies that such a result is not possible.
7.1. Finite triangle sequence. Let us first consider the case of real pairs $(\alpha, \beta)$ with representation of the form $\left(\left[\alpha_{0}, \ldots, \alpha_{k}\right],\left[a_{1}, a_{2}, \ldots\right]\right)$ (the second component being finite or infinite). By extending the ideas of Section 6, we associate to a non-rational pair $(\alpha, \beta)$ the infinite word $\mathcal{W}$ over the alphabet $\{L, R, I\}$ defined to be

$$
\mathcal{W}(\alpha, \beta)=L^{\alpha_{0}} I \cdots L^{\alpha_{k-1}} I L^{\alpha_{k}-1} I L^{a_{1}-1} R^{a_{2}} L^{a_{3}} \cdots
$$

Moreover we define the approximations of $(\alpha, \beta)$ to be the rational pairs $\left(\frac{m_{j}}{s_{j}}, \frac{n_{j}}{s_{j}}\right)$ with coding $\mathcal{W}_{(j)}$, the prefix of $\mathcal{W}$ of length $j$.

Example. We consider the rational pair $\left(\frac{19}{54}, \frac{14}{54}\right)$ with representation $([2,0,1,1],[2,2])$. As shown in Example 6 the word associated to this point is $\mathcal{W}=L L I I L I I L R$. Thus its approximations are the following:

$$
\begin{array}{lll}
\left(\frac{1}{2}, \frac{1}{2}\right) & \mathcal{W}_{(0)}=\varepsilon & ([1],[2]) \\
\left(\frac{1}{3}, \frac{1}{3}\right) & \mathcal{W}_{(1)}=L & ([2],[3]) \\
\left(\frac{1}{4}, \frac{1}{4}\right) & \mathcal{W}_{(2)}=L L & ([3],[4]) \\
\left(\frac{2}{5}, \frac{1}{5}\right) & \mathcal{W}_{(3)}=L L I & ([3],[2]) \\
\left(\frac{3}{8}, \frac{2}{8}\right) & \mathcal{W}_{(4)}=L L I I & ([2,1],[2]) \\
\left(\frac{4}{11}, \frac{3}{11}\right) & \mathcal{W}_{(5)}=L L I I L & ([2,1],[2]) \\
\left(\frac{5}{15}, \frac{4}{15}\right) & \mathcal{W}_{(6)}=\text { LLIILI } & ([2,0,2],[2]) \\
\left(\frac{8}{23}, \frac{6}{23}\right) & \mathcal{W}_{(7)}=\text { LLIILII } & ([2,0,1,1],[2]) \\
\left(\frac{11}{31}, \frac{8}{31}\right) & \mathcal{W}_{(8)}=\text { LLIILIIL } & ([2,0,1,1],[3]) \\
\left(\frac{19}{54}, \frac{14}{54}\right) & \mathcal{W}_{(9)}=\text { LLIILIILR } & ([2,0,1,1],[2,2])
\end{array}
$$

Example. We now consider the non-rational pair $(\alpha, \beta)=\left(\frac{1}{2}, \sqrt{2}-1\right)$, with triangle sequence $[1,1]$. Since $T^{2}(\alpha, \beta)=\left(\frac{\sqrt{2}-1}{2}, 0\right)$ and $\frac{\sqrt{2}-1}{2}=[\overline{4,1}]$, we have that the representation of $(\alpha, \beta)$ is $([1,1],[\overline{4,1}])$ and thus the infinite word associated to $(\alpha, \beta)$ is

$$
\mathcal{W}=L I I L^{3} \overline{R L^{4}}
$$

Thus its first ten approximations are the following:

$$
\begin{array}{lll}
\left(\frac{1}{2}, \frac{1}{2}\right) & \mathcal{W}_{(0)}=\varepsilon & ([1],[2]) \\
\left(\frac{1}{3}, \frac{1}{3}\right) & \mathcal{W}_{(1)}=L & ([2],[3]) \\
\left(\frac{2}{4}, \frac{1}{4}\right) & \mathcal{W}_{(2)}=L I & ([2],[2]) \\
\left(\frac{3}{6}, \frac{2}{6}\right) & \mathcal{W}_{(3)}=L I I & ([1,1],[2]) \\
\left(\frac{4}{8}, \frac{3}{8}\right) & \mathcal{W}_{(4)}=L I I L & ([1,1],[3]) \\
\left(\frac{5}{10}, \frac{4}{10}\right) & \mathcal{W}_{(5)}=L I I L L & ([1,1],[4]) \\
\left(\frac{6}{12}, \frac{5}{12}\right) & \mathcal{W}_{(6)}=L I I L L L & ([1,1],[5]) \\
\left(\frac{11}{22}, \frac{9}{22}\right) & \mathcal{W}_{(7)}=\text { LIILLLR } & ([1,1],[4,2]) \\
\left(\frac{17}{34}, \frac{14}{34}\right) & \mathcal{W}_{(8)}=\text { LIILLLRL } & ([1,1],[4,1,2]) \\
\left(\frac{23}{46}, \frac{19}{46}\right) & \mathcal{W}_{(9)}=\text { LIILLLLRLL } & ([1,1],[4,1,3])
\end{array}
$$

Since the triangle sequence is finite, if $(\alpha, \beta)$ is a non-rational pair, after a finite transient the approximations have the same triangle sequence as $(\alpha, \beta)$, and they are simply given by truncating the continued fraction expansion in the second component of the representation of $(\alpha, \beta)$. We are using the fact that

$$
\lim _{n \rightarrow \infty} \phi_{1}^{\alpha_{0}} \phi_{0} \cdots \phi_{1}^{\alpha_{k}} \phi_{0} \circ \phi_{2} \circ \phi_{1}^{a_{1}-1} \phi_{0}^{2} \phi_{2} \cdots \phi_{1}^{a_{n-1}-1} \phi_{0}^{2} \phi_{2} \phi_{1}^{a_{n}-2}\left(\frac{1}{2}, \frac{1}{2}\right)=(\alpha, \beta)
$$

which follows by Proposition 6.8 and the properties of the continued fraction expansion translated into our framework with (6.2).

By learning from the previous examples we can give a general way of constructing the coding associated to the approximations of a real pair $(\alpha, \beta)$ with representation $\left(\left[\alpha_{0}, \ldots, \alpha_{k}\right],\left[a_{1}, a_{2}, \ldots\right]\right)$. The correspondence between the words and the representation of the related approximation can be recovered by the following properties, which easily follows from Theorem 6.6 and Proposition 6.8,
(1) For $j$ a non-negative integer, the word $L^{j}$ corresponds to the representation $([j+1],[j+2])$.
(2) For $j, r$, and $d_{0}, \ldots, d_{r}$ non-negative integers, the word $L^{d_{0}} I \cdots L^{d_{r}} I L^{j}$ corresponds to the representation $\left(\left[d_{0}, \ldots, d_{r-1}, d_{r}+1\right],[j+2]\right)$.
(3) For $r$ and $s$ non-negative integers, and for $d_{0}, \ldots, d_{r}$, and $c_{0}, \ldots, c_{s}$ non-negative integers, the word $L^{d_{0}} I \cdots L^{d_{r}} I L^{c_{0}} R^{c_{1}} \cdots L^{c_{s}}$ if $s$ is even (or the word $L^{d_{0}} I \cdots L^{d_{r}} I L^{c_{0}} R^{c_{1}} \cdots R^{c_{s}}$ if $s$ is odd) corresponds to the representation $\left(\left[d_{0}, \ldots, d_{r-1}, d_{r}+1\right],\left[c_{0}+1, c_{1}, \ldots, c_{s-1}, c_{s}+1\right]\right)$.
7.2. Convergent infinite triangle sequence. If the real pair $(\alpha, \beta)$ has a convergent infinite triangle sequence $\left[\alpha_{0}, \alpha_{1}, \alpha_{2}, \ldots\right]$, its representation is ( $\left[\alpha_{0}, \alpha_{1}, \alpha_{2} \ldots\right],[2]$ ) and (5.2) holds. Thus we associate to $(\alpha, \beta)$ the infinite word $\mathcal{W}$ over the alphabet $\{L, R, I\}$ defined by

$$
\mathcal{W}(\alpha, \beta)=L^{\alpha_{0}} I L^{\alpha_{1}} I L^{\alpha_{2}} I \cdots
$$

and the approximations of $(\alpha, \beta)$ are again the rational pairs $\left(\frac{m_{j}}{s_{j}}, \frac{n_{j}}{s_{j}}\right)$ with coding $\mathcal{W}_{(j)}$, the prefix of $\mathcal{W}$ of length $j$. Rules (1) and (2) above still hold and show how to construct the representations of the approximations. Notice that the choice of [2] as the second component of the representation is arbitrary, and [2] can be replaced by any continued fraction expansion. However this choice does not change the form of the word $\mathcal{W}$.

Example. Let $\alpha_{0}=1$ and for $k \geq 1$ let $\alpha_{k}=p_{k}$, with $p_{k}$ being the $k$-th prime number, so that the triangle sequence we are considering is $[1,2,3,5,7,11, \ldots]$. From (4.2), for $k \geq 0$ we have

$$
1-\lambda_{k}=\alpha_{k+1} \frac{q_{k-1}}{q_{k+1}}=\frac{\alpha_{k+1}}{\alpha_{k+1}+\frac{q_{k-2}+q_{k}}{q_{k-1}}} \leq \frac{\alpha_{k+1}}{1+\alpha_{k+1}}
$$

where the last inequality holds since $\frac{q_{k-2}+q_{k}}{q_{k-1}} \geq \frac{q_{k}}{q_{k-1}} \geq 1$. Each triangle sequence digit is non-zero, thus $\lambda_{k}<1$ for all $k \geq 0$ and we thus have

$$
\prod_{k=0}^{\infty}\left(1-\lambda_{k}\right)=\prod_{p} \frac{p}{1+p}
$$

where the right-hand-side product extends over all the prime numbers. We thus have

$$
\prod_{p} \frac{p}{1+p}=\prod_{p}\left(1-\frac{1}{p+1}\right)<\prod_{p>2}\left(1-\frac{1}{p}\right)=0
$$

and the last product diverges to 0 because the sum of the reciprocals of the primes diverges. As a consequence, the given triangle sequence represents a unique point of $\bar{\triangle}$. This point is represented by the word

$$
\mathcal{W}=L I L^{2} I L^{3} I L^{5} I L^{7} I L^{11} I \ldots
$$

and its first approximations are

| $\left(\frac{1}{2}, \frac{1}{2}\right)$ | $\mathcal{W}_{(0)}=\varepsilon$ | $([1],[2])$ |
| :---: | :--- | :--- |
| $\left(\frac{1}{3}, \frac{1}{3}\right)$ | $\mathcal{W}_{(1)}=L$ | $([2],[3])$ |
| $\left(\frac{2}{4}, \frac{1}{4}\right)$ | $\mathcal{W}_{(2)}=L I$ | $([2],[2])$ |
| $\left(\frac{3}{5}, \frac{1}{5}\right)$ | $\mathcal{W}_{(3)}=L I L$ | $([2],[3])$ |
| $\left(\frac{4}{6}, \frac{1}{6}\right)$ | $\mathcal{W}_{(4)}=L I L L$ | $([2],[4])$ |
| $\left(\frac{5}{8}, \frac{2}{8}\right)$ | $\mathcal{W}_{(5)}=L I L L I$ | $([1,3],[2])$ |
| $\left(\frac{6}{10}, \frac{3}{10}\right)$ | $\mathcal{W}_{(6)}=L I L L I L$ | $([1,3],[3])$ |
| $\left(\frac{7}{12}, \frac{4}{12}\right)$ | $\mathcal{W}_{(7)}=L I L L I L L$ | $([1,3],[4])$ |
| $\left(\frac{8}{14}, \frac{5}{14}\right)$ | $\mathcal{W}_{(8)}=L I L L I L L L$ | $([1,3],[5])$ |
| $\left(\frac{11}{19}, \frac{6}{19}\right)$ | $\mathcal{W}_{(9)}=L I L L I L L L I$ | $([1,2,4],[2])$ |
| $\vdots$ |  |  |

7.3. Non-convergent infinite case. Let $(\alpha, \beta)$ have a non-convergent infinite triangle sequence $\left[\alpha_{0}, \alpha_{1}, \ldots\right]$. It means that $(\alpha, \beta)$ is in the line segment $\mathfrak{L}$ of points sharing the same triangle sequence. In this case we have seen that the representation ( $\left[\alpha_{0}, \alpha_{1}, \ldots\right],[2]$ ) encodes the whole segment, and it is impossible to distinguish different points on $\mathfrak{L}$ by using it.

Here we propose a possible way to construct approximations of $(\alpha, \beta)$ following the methods used in the other cases. This proposal is certainly not the only meaningful and not the most "natural" in any sense. For all $j \geq 0$, the point $(\alpha, \beta)$ is in $\triangle\left(\alpha_{0}, \ldots, \alpha_{j}\right)$, so that

$$
\left(\xi_{j}, \eta_{j}\right):=T^{j}(\alpha, \beta) \in \triangle\left(\alpha_{j}\right)
$$

Since $\alpha_{j} \rightarrow+\infty$, the second component $\eta_{j}$ is vanishing as $j$ increases. Let then $\frac{p_{j}(j)}{q_{j}(j)}=\left[a_{1}(j), \ldots, a_{j}(j)\right]$ be the $j$-th convergent of $\xi_{j}$, obtained from the continued fraction expansion $\left[a_{1}(j), a_{2}(j), a_{3}(j), \ldots\right]$ of $\xi_{j}$. Then for $\omega=1^{\alpha_{0}} 0 \ldots 1^{\alpha_{j-1}} 0$ it holds

$$
\left(\frac{m_{j}}{s_{j}}, \frac{n_{j}}{s_{j}}\right):=\phi_{\omega}\left(\frac{p_{j}(j)}{q_{j}(j)}, \frac{0}{q_{j}(j)}\right) \in \triangle\left(\alpha_{0}, \ldots, \alpha_{j}\right)
$$

and

$$
\lim _{j \rightarrow \infty}\left|(\alpha, \beta)-\left(\frac{m_{j}}{s_{j}}, \frac{n_{j}}{s_{j}}\right)\right|=0
$$

We then consider the approximations of $(\alpha, \beta)$ given by the rational pairs with representations

$$
\left(\left[\alpha_{0}, \ldots, \alpha_{j}\right],\left[a_{1}(j), \ldots, a_{j}(j)\right]\right)
$$

for $j \geq 0$, defined as above.

## 8. Speed of the approximations

In this final section we study the problem of the speed of the approximations introduced before. In particular given a non-rational pair $(\alpha, \beta) \in \bar{\triangle}$ we have defined a sequence $\left(\frac{m_{j}}{s_{j}}, \frac{n_{j}}{s_{j}}\right)$ of rational pairs in the Triangular tree which approximate $(\alpha, \beta)$. The speed of the approximations we consider concerns the supremum of the exponents $\eta>0$ for which

$$
\begin{equation*}
\liminf _{j \rightarrow \infty} s_{j}^{\eta}\left|\alpha-\frac{m_{j}}{s_{j}}\right|\left|\beta-\frac{n_{j}}{s_{j}}\right|=0 \tag{8.1}
\end{equation*}
$$

This problem is also known as the problem of simultaneous approximations of real numbers, and two famous problems in this research area are Dirichlet's Theorem and Littlewood's Conjecture (see [14] and [2] for more details).

We start with some notation. Let $(\alpha, \beta)$ be a point in $\bar{\triangle}$ with $\alpha$ or $\beta$ irrational. Let $\omega \in\{0,1\}^{*}$ and $\phi_{\omega}$ the correspondent concatenation of $\phi_{0}$ and $\phi_{1}$, we use the notation

$$
\begin{equation*}
\phi_{\omega}\left(\frac{p}{q}, \frac{r}{q}\right)=\left(\frac{m_{\omega}(p, r, q)}{s_{\omega}(p, r, q)}, \frac{n_{\omega}(p, r, q)}{s_{\omega}(p, r, q)}\right) \tag{8.2}
\end{equation*}
$$

for the image of rational pairs $\left(\frac{p}{q}, \frac{r}{q}\right)$ under $\phi_{\omega}$. In [7, Appendix A] we have introduced a matrix representation of the maps $\phi_{0}$ and $\phi_{1}$ by

$$
M_{0}:=M_{\phi_{0}}=\left(\begin{array}{ccc}
1 & 0 & 1  \tag{8.3}\\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right) \quad \text { and } \quad M_{1}:=M_{\phi_{1}}=\left(\begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

from which we obtain a matrix representation with non-negative integers coefficients for any combination $\phi_{\omega^{j}}$ of $\phi_{0}$ and $\phi_{1}$, which we denote by

$$
M_{\omega^{j}}:=\left(\begin{array}{ccc}
\rho & \sigma & \tau \\
\rho_{1} & \sigma_{1} & \tau_{1} \\
\rho_{2} & \sigma_{2} & \tau_{2}
\end{array}\right)
$$

not including the dependence on $j$ in the notation for the coefficients of the matrix if not necessary. Using the matrix representation we write for all $(x, y) \in \mathbb{R}^{2}$

$$
\begin{equation*}
\phi_{\omega^{j}}(x, y)=\left(\frac{\rho_{1}+\sigma_{1} x+\tau_{1} y}{\rho+\sigma x+\tau y}, \frac{\rho_{2}+\sigma_{2} x+\tau_{2} y}{\rho+\sigma x+\tau y}\right) \tag{8.4}
\end{equation*}
$$

so that in particular it holds

$$
\begin{equation*}
s_{\omega^{j}}(p, r, q)=\rho q+\sigma p+\tau r \tag{8.5}
\end{equation*}
$$

in (8.2).
In this paper we begin to study the problem of the speed of the approximations by considering the first simple classes of real pairs of numbers with at least one irrational component: the pairs with finite triangle sequence and the pairs with periodic triangle sequence $[d, d, d, \ldots]$ for $d \geq 3$.
8.1. Real pairs with finite triangle sequence. Let $(\alpha, \beta) \in \bar{\triangle}$, with $\alpha$ or $\beta$ irrational, have finite triangle sequence $\left[\alpha_{0}, \ldots, \alpha_{k}\right]$, then as proved in Lemma 5.1, in particular see (5.1), there exists $\omega \in\{0,1\}^{*}$ and $\xi \in[0,1]$ such that $(\alpha, \beta)=\phi_{\omega}(\xi, 0)$ (recall that $(\xi, 0)=\phi_{2}(\xi, \xi)$ ). As remarked in Section 7 we can definitively consider the problem (8.1) for rational pairs $\left(\frac{m_{j}}{s_{j}}, \frac{n_{j}}{s_{j}}\right)$ with the same triangle sequence of $(\alpha, \beta)$. It follows that we can consider rational pairs as obtained in (8.2) with $\phi_{\omega}$. Hence we can write

$$
\begin{aligned}
& \left|\alpha-\frac{m_{\omega}(p, r, q)}{s_{\omega}(p, r, q)}\right|=\left|\left(\phi_{\omega}(\xi, 0)\right)_{1}-\left(\phi_{\omega}\left(\frac{p}{q}, \frac{r}{q}\right)\right)_{1}\right| \\
& \left|\beta-\frac{n_{\omega}(p, r, q)}{s_{\omega}(p, r, q)}\right|=\left|\left(\phi_{\omega}(\xi, 0)\right)_{2}-\left(\phi_{\omega}\left(\frac{p}{q}, \frac{r}{q}\right)\right)_{2}\right|
\end{aligned}
$$

where the subscripts refer to the first and second component respectively, and consider $\left(\frac{p}{q}, \frac{r}{q}\right)$ as approximations of $(\xi, 0)$.

Using the matrix representation (8.4) for $\phi_{\omega}$ we can write

$$
\begin{aligned}
\alpha-\frac{m_{\omega}(p, r, q)}{s_{\omega}(p, r, q)} & =\frac{\rho_{1}+\sigma_{1} \xi}{\rho+\sigma \xi}-\frac{\rho_{1} q+\sigma_{1} p+\tau_{1} r}{\rho q+\sigma p+\tau r}= \\
& =\frac{\left(\sigma_{1} \tau-\sigma \tau_{1}\right) \xi r+\left(\sigma_{1} \rho-\sigma \rho_{1}\right)(\xi q-p)+\left(\tau \rho_{1}-\tau_{1} \rho\right) r}{(\rho+\sigma \xi)(\rho q+\sigma p+\tau r)}
\end{aligned}
$$

and analogously

$$
\beta-\frac{n_{\omega}(p, r, q)}{s_{\omega}(p, r, q)}=\frac{\left(\sigma_{2} \tau-\sigma \tau_{2}\right) \xi r+\left(\sigma_{2} \rho-\sigma \rho_{2}\right)(\xi q-p)+\left(\tau \rho_{2}-\tau_{2} \rho\right) r}{(\rho+\sigma \xi)(\rho q+\sigma p+\tau r)}
$$

Using that for all $\xi \in \mathbb{R}$ there exist two sequences $\left(p_{j}\right)_{j}$ of integers and $\left(q_{j}\right)_{j}$ of positive integers with $\left(p_{j}, q_{j}\right)=1$, such that $\left|\xi q_{j}-p_{j}\right| \leq \frac{1}{q_{j}}$ for all $j$, and letting $r_{j}=0$ for all $j$, so that it holds $\left(\frac{p_{j}}{q_{j}}, \frac{r_{j}}{q_{j}}\right) \in \bar{\triangle} \cap \mathbb{Q}^{2}$, we obtain that there exist two sequences $\left(p_{j}\right)_{j}$ of integers and $\left(q_{j}\right)_{j}$ of positive integers such that

$$
\left|\alpha-\frac{m_{\omega}\left(p_{j}, 0, q_{j}\right)}{s_{\omega}\left(p_{j}, 0, q_{j}\right)}\right| \leq c \frac{1}{q_{j} s_{\omega}\left(p_{j}, 0, q_{j}\right)} \quad \text { and } \quad\left|\beta-\frac{n_{\omega}\left(p_{j}, 0, q_{j}\right)}{s_{\omega}\left(p_{j}, 0, q_{j}\right)}\right| \leq c \frac{1}{q_{j} s_{\omega}\left(p_{j}, 0, q_{j}\right)}
$$

where the constant $c$ does not depend on $p_{j}$ and $q_{j}$. Therefore choosing the two sequences $\left(p_{j}\right)_{j}$ and $\left(q_{j}\right)_{j}$ as before, we have that for all $\varepsilon>0$

$$
\begin{aligned}
& \liminf _{s \rightarrow \infty} s^{(4-\varepsilon)}\left(\inf _{m \in \mathbb{Z}}\left|\alpha-\frac{m}{s}\right|\right)\left(\inf _{n \in \mathbb{Z}}\left|\beta-\frac{n}{s}\right|\right) \leq \liminf _{j \rightarrow \infty} s_{j}^{(4-\varepsilon)}\left|\alpha-\frac{m_{j}}{s_{j}}\right|\left|\beta-\frac{n_{j}}{s_{j}}\right| \leq \\
& \quad \leq \lim _{j \rightarrow \infty} s_{\omega}\left(p_{j}, 0, q_{j}\right)^{(4-\varepsilon)}\left|\alpha-\frac{m_{\omega}\left(p_{j}, 0, q_{j}\right)}{s_{\omega}\left(p_{j}, 0, q_{j}\right)}\right|\left|\beta-\frac{n_{\omega}\left(p_{j}, 0, q_{j}\right)}{s_{\omega}\left(p_{j}, 0, q_{j}\right)}\right| \leq \\
& \quad \leq c^{2} \lim _{j \rightarrow \infty} \frac{s_{\omega}\left(p_{j}, 0, q_{j}\right)^{(2-\varepsilon)}}{q_{j}^{2}}=0
\end{aligned}
$$

since by (8.5) holds $s_{\omega}\left(p_{j}, 0, q_{j}\right) \leq c^{\prime} q_{j}$ for a suitable constant $c^{\prime}$. This argument also implies that in this case $(\alpha, \beta)$ is is not a $\operatorname{Bad}\left(i_{1}, i_{2}\right)$ pair, for all admissible $\left(i_{1}, i_{2}\right)$ (see [14] and [2]).
8.2. Real pairs with periodic triangle sequence $[d, d, d, \ldots]$ for $d \geq 3$. Let $(\alpha, \beta) \in \bar{\triangle}$, with $\alpha$ or $\beta$ irrational, have triangle sequence $[d, d, d, \ldots]$ for $d \geq 3$. These pairs correspond to fixed points for the Triangle Map $T$ defined in Section 3 (all fixed points of $T$ are obtained by considering also the real pairs with triangle sequence $[d, d, d, \ldots]$ for $d=0,1,2)$.

It is shown in [8] that if $(\alpha, \beta)$ has triangle sequence $[d, d, d, \ldots]$ then $\beta=\alpha^{2}$ and $\alpha \in(0,1)$ is the largest root of the polynomial $P(t)=t^{3}+d t^{2}+t-1$. It follows that $\alpha$ is a cubic number, and $\alpha$ and $\beta$ are in the same cubic number field. We remark that the polynomial $P(t)$ has three real roots for $d \geq 3$, and only one real root for $d=0,1,2$. Hence here we only consider the simplest case for the roots of $P(t)$.

In Section 7 we have constructed approximations of $(\alpha, \beta)$ by rational pairs in the Triangular tree with representation $([d, d, \ldots, d],[2])$, where we recall that instead of [2] one could choose any fixed continued fraction expansion $\left[a_{1}, \ldots, a_{n}\right]$. Then using (8.3) we consider the matrix

$$
M_{d}:=M_{1}^{d} M_{0}=\left(\begin{array}{lll}
1 & d & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right)
$$

which as in (8.4) represents the map

$$
\phi_{\omega}(x, y)=\left(\frac{1}{1+d x+y}, \frac{x}{1+d x+y}\right) \quad \text { with } \quad \omega=(11 \cdots 10) \in\{0,1\}^{d+1}
$$

and use approximations of $(\alpha, \beta)$ of the form

$$
\begin{equation*}
\left(\frac{m_{k}}{s_{k}}, \frac{n_{k}}{s_{k}}\right)=\phi_{M_{d}^{k}}\left(\frac{1}{2}, 0\right)=: \phi_{\bar{\omega}^{k}}\left(\frac{1}{2}, 0\right) \tag{8.6}
\end{equation*}
$$

where $\bar{\omega}^{k} \in\{0,1\}^{k(d+1)}$ is the string obtained by concatenating $k$ copies of $\omega$. In matrix representation, (8.6) can be written as

$$
\left(\begin{array}{c}
s_{k} \\
m_{k} \\
n_{k}
\end{array}\right)=M_{d}^{k}\left(\begin{array}{l}
2 \\
1 \\
0
\end{array}\right)
$$

and we can then use linear algebra to study the properties of the approximations.
Remark 8.1. As shown in (5.2) all points in $\bar{\triangle}$ converge to $(\alpha, \beta)$ under repeated applications of $\phi_{\omega}$. Therefore different approximations of $(\alpha, \beta)$ can be constructed as in (8.6) by using a sequence $\left(\frac{p_{k}}{q_{k}}, \frac{r_{k}}{q_{k}}\right)$ of possibly different rational pairs instead of the fixed pair $\left(\frac{1}{2}, 0\right)$.

The matrix $M_{d}$ has characteristic polynomial $p_{d}(\lambda)=\lambda^{3}-\lambda^{2}-d \lambda-1$ and, for $d \geq 3$, distinct eigenvalues $\lambda_{1}, \lambda_{2}$, and $\lambda_{3}$ satisfying

$$
\begin{equation*}
\lambda_{1}=\frac{1}{\alpha_{2}}<\lambda_{2}=\frac{1}{\alpha_{1}}<0<1<\lambda_{3}=\frac{1}{\alpha} \quad \text { with } \quad \lambda_{3}>\left|\lambda_{1}\right| \geq 1>\left|\lambda_{2}\right| \tag{8.7}
\end{equation*}
$$

where $\alpha_{1}<\alpha_{2}<0<\alpha<1$ are the roots of $P(t)$, and eigenvectors

$$
v_{1}=\left(\begin{array}{c}
1 \\
\alpha_{2} \\
\alpha_{2}^{2}
\end{array}\right), \quad v_{2}=\left(\begin{array}{c}
1 \\
\alpha_{1} \\
\alpha_{1}^{2}
\end{array}\right), \quad v_{3}=\left(\begin{array}{c}
1 \\
\alpha \\
\beta
\end{array}\right)
$$

We also recall that for $d \geq 3$ fixed, and given $\alpha$ the largest root of $P(t)$, it holds

$$
\begin{equation*}
\alpha_{1}=\frac{-(d+\alpha)-\sqrt{(d+\alpha)^{2}-\frac{4}{\alpha}}}{2} \quad \text { and } \quad \alpha_{2}=\frac{-(d+\alpha)+\sqrt{(d+\alpha)^{2}-\frac{4}{\alpha}}}{2} \tag{8.8}
\end{equation*}
$$

For simplicity of notation in the following we let

$$
h(d, \alpha):=\sqrt{(d+\alpha)^{2}-\frac{4}{\alpha}}
$$

We are now ready to prove the following result.

Proposition 8.2. Let $(\alpha, \beta) \in \bar{\triangle}$, with $\alpha$ or $\beta$ irrational, have triangle sequence $[d, d, d, \ldots]$ for $d \geq 3$. There exist a constant $c(\alpha, d)$, functions $f_{\alpha, d}^{i}: \mathbb{Z}^{3} \rightarrow \mathbb{R}$ with $i=1,2$ given by

$$
\begin{gathered}
f_{\alpha, d}^{1}(q, p, r)=\frac{h(d, \alpha)(d+3 \alpha+h(d, \alpha))\left(1+\alpha^{2}(d+2 \alpha)\right)}{4 \alpha}\left[q\left(3-\alpha+\alpha^{2} h(d, \alpha)\right)+\right. \\
+p(-2-2 d \alpha-2 \alpha h(d, \alpha))+r(-3 \alpha-d+h(d, \alpha))] \\
f_{\alpha, d}^{2}(q, p, r)=-\frac{h(d, \alpha)(d+3 \alpha-h(d, \alpha))\left(1+\alpha^{2}(d+2 \alpha)\right)}{4 \alpha}\left[q\left(3-\alpha-\alpha^{2} h(d, \alpha)\right)+\right. \\
+p(-2-2 d \alpha+2 \alpha h(d, \alpha))+r(-3 \alpha-d-h(d, \alpha))]
\end{gathered}
$$

and functions $g_{\alpha, d}^{i}: \mathbb{Z}^{3} \rightarrow \mathbb{R}$ with $i=1,2,3$ given by

$$
\begin{gathered}
g_{\alpha, d}^{1}(q, p, r)=\frac{d+3 \alpha+h(d, \alpha)}{2}\left[q \frac{(-\alpha)(d+\alpha+h(d, \alpha))}{2}+p \frac{d-\alpha+h(d, \alpha)}{2}+r\right] \\
g_{\alpha, d}^{2}(q, p, r)=\frac{-d-3 \alpha+h(d, \alpha)}{2}\left[q \frac{(-\alpha)(d+\alpha-h(d, \alpha))}{2}+p \frac{d-\alpha-h(d, \alpha)}{2}+r\right] \\
g_{\alpha, d}^{3}(q, p, r)=-h(d, \alpha)\left[q \frac{1}{\alpha}+p(d+\alpha)+r\right]
\end{gathered}
$$

such that for all $k \geq 1$ and all $\left(\frac{p}{q}, \frac{r}{q}\right) \in \bar{\triangle}$ the rational pair $\left(\frac{m}{s}, \frac{n}{s}\right)=\phi_{M_{d}^{k}}\left(\frac{p}{q}, \frac{r}{q}\right)$ satisfies

$$
\begin{align*}
& \alpha-\frac{m}{s}=c(\alpha, d) \frac{f_{\alpha, d}^{1}(q, p, r) \lambda_{1}^{k}+f_{\alpha, d}^{2}(q, p, r) \lambda_{2}^{k}}{g_{\alpha, d}^{1}(q, p, r) \lambda_{1}^{k}+g_{\alpha, d}^{2}(q, p, r) \lambda_{2}^{k}+g_{\alpha, d}^{3}(q, p, r) \lambda_{3}^{k}} \\
& \beta-\frac{n}{s}=\frac{c(\alpha, d)}{2} \frac{(\alpha-d+h(d, \alpha)) f_{\alpha, d}^{1}(q, p, r) \lambda_{1}^{k}+(\alpha-d-h(d, \alpha)) f_{\alpha, d}^{2}(q, p, r) \lambda_{2}^{k}}{g_{\alpha, d}^{1}(q, p, r) \lambda_{1}^{k}+g_{\alpha, d}^{2}(q, p, r) \lambda_{2}^{k}+g_{\alpha, d}^{3}(q, p, r) \lambda_{3}^{k}} \tag{8.9}
\end{align*}
$$

where $\lambda_{1}, \lambda_{2}$, and $\lambda_{3}$ are defined in (8.7).
Proof. If $(\alpha, \beta)$ has triangle sequence $[d, d, d, \ldots]$ then it is a fixed point of the Triangle Map $T$ with $\beta=\alpha^{2}$ and, in particular using (3.2), $(\alpha, \beta)=S^{d+1}(\alpha, \beta)=\left.\left.S\right|_{\Gamma_{0}} \circ S\right|_{\Gamma_{1}} ^{d}(\alpha, \beta)$ for the map $S$ defined in (3.1). It follows that $(\alpha, \beta)=\phi_{M_{d}^{k}}(\alpha, \beta)$ for all $k \geq 1$. We can thus write for fixed $k \geq 1$ and rational pair $\left(\frac{p}{q}, \frac{r}{q}\right) \in \bar{\triangle}$

$$
\begin{align*}
\alpha-\frac{m}{s} & =\left(\phi_{M_{d}^{k}}(\alpha, \beta)\right)_{1}-\left(\phi_{M_{d}^{k}}\left(\frac{p}{q}, \frac{r}{q}\right)\right)_{1}=  \tag{8.10}\\
& =\frac{\left(\rho \sigma_{1}-\rho_{1} \sigma\right)(\alpha q-p)+\left(\tau \sigma_{1}-\tau_{1} \sigma\right)(\alpha r-\beta p)+\left(\rho \tau_{1}-\rho_{1} \tau\right)(\beta q-r)}{(\rho+\sigma \alpha+\tau \beta)(\rho q+\sigma p+\tau r)}
\end{align*}
$$

using the matrix representation (8.4). Let $\Lambda$ be the diagonal matrix $\Lambda=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$, then $M_{d}^{k}=\Pi \Lambda^{k} \Pi^{-1}$ where $\Pi$ is the matrix with columns the eigenvectors $v_{1}, v_{2}$ and $v_{3}$. We use this representation to have an explicit form for the terms in (8.10), where the dependence on $k$ appears in the power of the eigenvalues and is of course hidden in the coefficients of $M_{d}^{k}$. In what follows, all not explained steps are just straightforward computations. We have

$$
\Pi=\left(\begin{array}{ccc}
1 & 1 & 1 \\
\alpha_{2} & \alpha_{1} & \alpha \\
\alpha_{2}^{2} & \alpha_{1}^{2} & \beta
\end{array}\right) \quad \text { and } \quad \Pi^{-1}=\frac{1}{\operatorname{det}(\Pi)}\left(\begin{array}{ccc}
\alpha_{1} \beta-\alpha_{1}^{2} \alpha & \alpha_{1}^{2}-\beta & \alpha-\alpha_{1} \\
\alpha_{2}^{2} \alpha-\alpha_{2} \beta & \beta-\alpha_{2}^{2} & \alpha_{2}-\alpha \\
\alpha_{1}^{2} \alpha_{2}-\alpha_{1} \alpha_{2}^{2} & \alpha_{2}^{2}-\alpha_{1}^{2} & \alpha_{1}-\alpha_{2}
\end{array}\right)
$$

with $\operatorname{det}(\Pi)=\alpha_{1} \alpha\left(\alpha-\alpha_{1}\right)+\alpha_{2} \alpha\left(\alpha_{2}-\alpha\right)+\alpha_{1} \alpha_{2}\left(\alpha_{1}-\alpha_{2}\right)$.

We first consider the denominator of (8.10). Let us write the first row of $M_{d}^{k}$ as a function of $\alpha$ and $d$. It holds

$$
\begin{align*}
\rho & =\frac{1}{\operatorname{det}(\Pi)}\left[\lambda_{1}^{k} \alpha \alpha_{1}\left(\alpha-\alpha_{1}\right)+\lambda_{2}^{k} \alpha \alpha_{2}\left(\alpha_{2}-\alpha\right)+\lambda_{3}^{k} \alpha_{1} \alpha_{2}\left(\alpha_{1}-\alpha_{2}\right)\right] \\
\sigma & =\frac{1}{\operatorname{det}(\Pi)}\left[\lambda_{1}^{k}\left(\alpha_{1}^{2}-\alpha^{2}\right)+\lambda_{2}^{k}\left(\alpha^{2}-\alpha_{2}^{2}\right)+\lambda_{3}^{k}\left(\alpha_{2}^{2}-\alpha_{1}^{2}\right)\right]  \tag{8.11}\\
\tau & =\frac{1}{\operatorname{det}(\Pi)}\left[\lambda_{1}^{k}\left(\alpha-\alpha_{1}\right)+\lambda_{2}^{k}\left(\alpha_{2}-\alpha\right)+\lambda_{3}^{k}\left(\alpha_{1}-\alpha_{2}\right)\right]
\end{align*}
$$

then we obtain

$$
\begin{equation*}
\rho+\sigma \alpha+\tau \beta=\lambda_{3}^{k} \tag{8.12}
\end{equation*}
$$

and

$$
\begin{aligned}
r h o q+\sigma p+\tau r=\frac{1}{\operatorname{det}(\Pi)}\left[\lambda_{1}^{k}\right. & \left(q \alpha_{1} \alpha\left(\alpha-\alpha_{1}\right)+p\left(\alpha_{1}^{2}-\alpha^{2}\right)+r\left(\alpha-\alpha_{1}\right)\right)+ \\
& +\lambda_{2}^{k}\left(q \alpha_{2} \alpha\left(\alpha_{2}-\alpha\right)+p\left(\alpha^{2}-\alpha_{2}^{2}\right)+r\left(\alpha_{2}-\alpha\right)\right)+ \\
& \left.+\lambda_{3}^{k}\left(q \alpha_{1} \alpha_{2}\left(\alpha_{1}-\alpha_{2}\right)+p\left(\alpha_{2}^{2}-\alpha_{1}^{2}\right)+r\left(\alpha_{1}-\alpha_{2}\right)\right)\right]
\end{aligned}
$$

which using (8.8) gives

$$
\begin{equation*}
\rho q+\sigma p+\tau r=\frac{1}{\operatorname{det}(\Pi)}\left(g_{\alpha, d}^{1}(q, p, r) \lambda_{1}^{k}+g_{\alpha, d}^{2}(q, p, r) \lambda_{2}^{k}+g_{\alpha, d}^{3}(q, p, r) \lambda_{3}^{k}\right) \tag{8.13}
\end{equation*}
$$

with the functions $g_{\alpha, d}^{i}(q, p, r)$ defined in the statement. For the numerator of (8.10) let us write the second row of $M_{d}^{k}$ as a function of $\alpha$ and $d$. It holds

$$
\begin{aligned}
\rho_{1} & =\frac{1}{\operatorname{det}(\Pi)}\left[\lambda_{1}^{k}\left(\alpha-\alpha_{1}\right)+\lambda_{2}^{k}\left(\alpha_{2}-\alpha\right)+\lambda_{3}^{k}\left(\alpha_{1}-\alpha_{2}\right)\right] \\
\sigma_{1} & =\frac{1}{\operatorname{det}(\Pi)}\left[\lambda_{1}^{k} \alpha_{2}\left(\alpha_{1}^{2}-\alpha^{2}\right)+\lambda_{2}^{k} \alpha_{1}\left(\alpha^{2}-\alpha_{2}^{2}\right)+\lambda_{3}^{k} \alpha\left(\alpha_{2}^{2}-\alpha_{1}^{2}\right)\right] \\
\tau_{1} & =\frac{1}{\operatorname{det}(\Pi)}\left[\lambda_{1}^{k} \alpha_{2}\left(\alpha-\alpha_{1}\right)+\lambda_{2}^{k} \alpha_{1}\left(\alpha_{2}-\alpha\right)+\lambda_{3}^{k} \alpha\left(\alpha_{1}-\alpha_{2}\right)\right]
\end{aligned}
$$

and then one can write the numerator of (8.10) as

$$
\frac{1}{(\operatorname{det}(\Pi))^{2}}\left(f_{\alpha, d}^{1}(q, p, r) \lambda_{1}^{k} \lambda_{3}^{k}+f_{\alpha, d}^{2}(q, p, r) \lambda_{2}^{k} \lambda_{3}^{k}+f_{\alpha, d}^{3}(q, p, r) \lambda_{1}^{k} \lambda_{2}^{k}\right)
$$

with $f_{\alpha, d}^{3}(q, p, r) \equiv 0$, and $f_{\alpha, d}^{1}(q, p, r)$ and $f_{\alpha, d}^{2}(q, p, r)$ defined as in the statement. This gives the first of (8.9).

Now we can repeat the argument for the second term in (8.9) to get for fixed $k \geq 1$ and rational pair $\left(\frac{p}{q}, \frac{r}{q}\right) \in \bar{\triangle}$

$$
\begin{align*}
\beta-\frac{n}{s} & =\left(\phi_{M_{d}^{k}}(\alpha, \beta)\right)_{2}-\left(\phi_{M_{d}^{k}}\left(\frac{p}{q}, \frac{r}{q}\right)\right)_{2}= \\
& =\frac{\left(\rho \sigma_{2}-\rho_{2} \sigma\right)(\alpha q-p)+\left(\tau \sigma_{2}-\tau_{2} \sigma\right)(\alpha r-\beta p)+\left(\rho \tau_{2}-\rho_{2} \tau\right)(\beta q-r)}{(\rho+\sigma \alpha+\tau \beta)(\rho q+\sigma p+\tau r)} \tag{8.14}
\end{align*}
$$

Equations (8.11), (8.12) and (8.13) give the denominator of (8.14). For the numerator we write the third row of $M_{d}^{k}$ as a function of $\alpha$ and $d$, finding

$$
\frac{1}{(\operatorname{det}(\Pi))^{2}}\left(\frac{\alpha-d+h(d, \alpha)}{2} f_{\alpha, d}^{1}(q, p, r) \lambda_{1}^{k} \lambda_{3}^{k}+\frac{\alpha-d-h(d, \alpha)}{2} f_{\alpha, d}^{2}(q, p, r) \lambda_{2}^{k} \lambda_{3}^{k}\right)
$$

From this we obtain the second equation in (8.9).

Corollary 8.3. Let $(\alpha, \beta) \in \bar{\triangle}$, with $\alpha$ or $\beta$ irrational, have triangle sequence $[d, d, d, \ldots]$ for $d \geq 3$, and consider the approximations $\left(\frac{m_{k}}{s_{k}}, \frac{n_{k}}{s_{k}}\right)$ defined in (8.6). Then

$$
\lim _{k \rightarrow \infty} s_{k}^{\eta}\left|\alpha-\frac{m_{k}}{s_{k}}\right|\left|\beta-\frac{n_{k}}{s_{k}}\right|=0
$$

for all $\eta<2\left(1-\frac{\log \left|\lambda_{1}\right|}{\log \lambda_{3}}\right)$ if $d \geq 4$, where we are using (8.7), and for for all $\eta<4$ if $d=3$.
Proof. We apply Proposition 8.2 with $(q, p, r)=(2,1,0)$ together with the relations (8.7). It follows

$$
s_{k}^{\eta}\left|\alpha-\frac{m_{k}}{s_{k}}\right|\left|\beta-\frac{n_{k}}{s_{k}}\right| \leq c^{\prime}(\alpha, d)\left(g_{\alpha, d}^{3}(2,1,0) \lambda_{3}^{k}+o\left(\lambda_{3}^{k}\right)\right)^{\eta-2}\left(f_{\alpha, d}^{1}(2,1,0) \lambda_{1}^{k}+o\left(\lambda_{1}^{k}\right)\right)^{2}
$$

where $g_{\alpha, d}^{3}(2,1,0)$ and $f_{\alpha, d}^{1}(2,1,0)$ do not vanish for $d \geq 4$. The result for this case immediately follows.
The case $d=3$ is particular since $\alpha=-1+\sqrt{2}$ is a quadratic irrational. It also follows $\beta=3-2 \sqrt{2}$, $h(3, \alpha)=\sqrt{2}$, and

$$
\lambda_{1}=-1, \quad \lambda_{2}=1-\sqrt{2}, \quad \lambda_{3}=1+\sqrt{2}
$$

Moreover

$$
f_{\alpha, 3}^{1}(2,1,0)=c^{\prime \prime}(\alpha, 3)\left(4-8 \alpha+2\left(\alpha^{2}-\alpha\right) h(3, \alpha)\right)=0
$$

and $f_{\alpha, 3}^{2}(2,1,0)$ and $g_{\alpha, 3}^{3}(2,1,0)$ do not vanish. Since $\lambda_{2}=-\lambda_{3}^{-1}$, it follows that

$$
s_{k}^{\eta}\left|\alpha-\frac{m_{k}}{s_{k}}\right|\left|\beta-\frac{n_{k}}{s_{k}}\right| \leq c^{\prime \prime \prime}(\alpha, 3)\left(g_{\alpha, 3}^{3}(2,1,0) \lambda_{3}^{k}+o\left(\lambda_{3}^{k}\right)\right)^{\eta-2} \lambda_{3}^{-2 k}
$$

and the thesis follows.
The result for $d \geq 4$ can be improved if we change the construction of the approximations as explained in Remark 8.1. For the moment we leave this problem and the study of the speed of the approximations for other real pairs to future research.

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Figure 8. The levels $\mathcal{T}_{0}, \mathcal{T}_{1}$ and $\mathcal{T}_{2}$ of the Triangular tree generated through the rules (R1)-(R4).

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[^1]:    ${ }^{1}$ The leftmost symbol $L$ denotes $\frac{1}{2}$ and is not associated to a move on the Farey tree.

[^2]:    ${ }^{2}$ Note that $\lambda_{k}=1$ if and only if $\alpha_{k}=0$.

