# WEYL FORMULA FOR THE NEGATIVE DISSIPATIVE EIGENVALUES OF MAXWELL'S EQUATIONS 

FERRUCCIO COLOMBINI AND VESSELIN PETKOV


#### Abstract

Let $V(t)=e^{t G_{b}}, t \geq 0$, be the semigroup generated by Maxwell's equations in an exterior domain $\Omega \subset \mathbb{R}^{3}$ with dissipative boundary condition $E_{t a n}-\gamma(x)\left(\nu \wedge B_{t a n}\right)=0, \gamma(x)>0, \forall x \in \Gamma=\partial \Omega$. We study the case when $\Omega=\left\{x \in \mathbb{R}^{3}:|x|>1\right\}$ and $\gamma \neq 1$ is a constant. We establish a Weyl formula for the counting function of the negative real eigenvalues of $G_{b}$.


## 1. Introduction

Let $K \subset\left\{x \in \mathbb{R}^{3}:|x| \leq a\right\}$ be an open connected domain and let $\Omega=\mathbb{R}^{3} \backslash \bar{K}$ be connected domain with $C^{\infty}$ smooth boundary $\Gamma$. Consider the boundary problem

$$
\begin{align*}
& \partial_{t} E=\operatorname{curl} B, \quad \partial_{t} B=-\operatorname{curl} E \quad \text { in } \quad \mathbb{R}_{t}^{+} \times \Omega \\
& E_{t a n}-\gamma(x)\left(\nu \wedge B_{t a n}\right)=0 \quad \text { on } \quad \mathbb{R}_{t}^{+} \times \Gamma  \tag{1.1}\\
& E(0, x)=E_{0}(x), \quad B(0, x)=B_{0}(x)
\end{align*}
$$

with initial data $f=\left(E_{0}, B_{0}\right) \in\left(L^{2}(\Omega)\right)^{6}=\mathcal{H}$. Here $\nu(x)$ is the unit outward normal to $\partial \Omega$ at $x \in \Gamma$ pointing into $\Omega,\langle$,$\rangle denotes the scalar product in \mathbb{C}^{3}$, $u_{\text {tan }}:=u-\langle u, \nu\rangle \nu$, and $\gamma(x) \in C^{\infty}(\Gamma)$ satisfies $\gamma(x)>0$ for all $x \in \Gamma$. The solution of the problem (1.1) is described by a contraction semigroup

$$
(E, B)=V(t) f=e^{t G_{b}} f, t \geq 0
$$

where the generator $G_{b}$ has domain $D\left(G_{b}\right)$ which is the closure in the graph norm of functions $u=(v, w) \in\left(C_{(0)}^{\infty}\left(\mathbb{R}^{3}\right)\right)^{3} \times\left(C_{(0)}^{\infty}\left(\mathbb{R}^{3}\right)\right)^{3}$ satisfying the boundary condition $v_{t a n}-\gamma\left(\nu \wedge w_{t a n}\right)=0$ on $\Gamma$.

In [1] it was proved that the spectrum of $G_{b}$ in the open half plan $\{z \in \mathbb{C}$ : $\operatorname{Re} z<0\}$ is formed by isolated eigenvalues with finite multiplicities. Note that if $G_{b} f=\lambda f$ with $\operatorname{Re} \lambda<0$, the solution $u(t, x)=V(t) f=e^{\lambda t} f(x)$ of (1.1) has exponentially decreasing global energy. Such solutions are called asymptotically disappearing and they are very important for the inverse scattering problems (see [1]). In particular, the eigenvalues $\lambda$ with $\operatorname{Re} \lambda \rightarrow-\infty$ imply a very fast decay of the corresponding solutions. In [2] the existence of eigenvalues of $G_{b}$ has been studied for the ball $B_{3}=\left\{x \in \mathbb{R}^{3},|x|<1\right\}$ assuming $\gamma$ constant. It was proved for $\gamma=1$ there are no eigenvalues in $\{z \in \mathbb{C}: \operatorname{Re} z<0\}$, while for $\gamma \neq 1$ there is always an infinite number of real eigenvalues $\lambda_{m}$ and with exception of one they satisfy the estimate

$$
\begin{equation*}
\lambda_{m} \leq-\frac{1}{\max \left\{\left(\gamma_{0}-1\right), \sqrt{\gamma_{0}-1}\right\}}=-c_{0} \tag{1.2}
\end{equation*}
$$

[^0]where $\gamma_{0}=\max \left\{\gamma, \frac{1}{\gamma}\right\}$.
In this Note we study the distribution of the negative eigenvalues and our purpose is to obtain a Weyl formula for the counting function
$$
N(r)=\#\left\{\lambda \in \sigma_{p}\left(G_{b}\right) \cap \mathbb{R}^{-}:|\lambda| \leq r\right\}, r>r_{0}(\gamma)
$$
where every eigenvalues $\lambda_{m}$ is counted with its algebraic multiplicity given by
$$
\operatorname{mult}\left(\lambda_{m}\right)=\operatorname{rank} \frac{1}{2 \pi \mathbf{i}} \int_{\left|\lambda_{n}-z\right|=\epsilon}\left(z-G_{b}\right)^{-1} d z
$$
where $0<\epsilon \ll 1$. Our main result is the following
Theorem 1.1. Let $\gamma \neq 1$ be a constant and let $\gamma_{0}=\max \left\{\gamma, \frac{1}{\gamma}\right\}$. Then the counting function $N(r)$ for the ball $B_{3}$ has the asymptotic
\[

$$
\begin{equation*}
N(r)=\left(\gamma_{0}^{2}-1\right) r^{2}+\mathcal{O}_{\gamma}(r), r \geq r_{0}(\gamma)>c_{0} \tag{1.3}
\end{equation*}
$$

\]

The proof of Theorem 1.1 is based on a precise analysis of the roots of the equation (3.1) involving spherical Hankel functions $h_{n}^{(1)}(\lambda)$ of first kind. We show in Section 3 that for $\gamma>1$ this equation has only one real root $\lambda_{n}<0$. Moreover, we have $\lambda_{n+1}<\lambda_{n}, \forall n \in \mathbb{N}$, so we have a decreasing sequence of eigenvalues. The geometric multiplicity of $\lambda_{n}$ is $2 n+1$. Since $C_{b}$ is not a self-adjoint operator the geometric multiplicity could be less than the algebraic one. In our case these multiplicities coincide and the proof is based on a representation of $\left(G_{b}-z\right)^{-1}$. To estimate $\lambda_{n}$ as $n \rightarrow \infty$, we apply an approximation of the exterior semiclassical Dirichlet to Neumann map for the operator $\left(h^{2} \Delta+z\right)$ established in [6] (see also [8) combined with an application of Rouché theorem.

We conjecture that in the general case of strictly convex obstacles and $\min _{y \in \Gamma} \gamma(y)=$ $\gamma_{1}>1$ we have the asymptotic

$$
N(r)=\frac{1}{4 \pi}\left(\int_{\Gamma}\left(\gamma^{2}(y)-1\right) d S_{y}\right) r^{2}+\mathcal{O}_{\gamma}(r), r \geq r_{0}\left(\gamma_{0}\right)
$$

For the ball $B_{3}$ this agrees with (1.3).

## 2. Boundary problem for Maxwell system

Our purpose is to study the eigenvalues of $G_{b}$ in case the obstacle is equal to the ball $B_{3}=\left\{x \in \mathbb{R}^{3}:|x| \leq 1\right\}$. Setting $\lambda=\mathbf{i} \mu$, $\operatorname{Im} \mu>0$, an eigenfunction $(E, B) \neq 0$ of $G_{b}$ satisfies

$$
\begin{equation*}
\operatorname{curl} E=-\mathbf{i} \mu B, \quad \operatorname{curl} B=\mathbf{i} \mu E . \tag{2.1}
\end{equation*}
$$

Replacing $B$ by $H=-B$ yields for $(E, H) \in\left(H^{2}(|x| \leq 1)\right)^{6}$,

$$
\left\{\begin{array}{l}
\operatorname{curl} E=\mathbf{i} \mu H, \quad \operatorname{curl} H=-\mathbf{i} \mu E, \quad \text { for } \quad x \in B_{3},  \tag{2.2}\\
E_{t a n}+\gamma\left(\nu \wedge H_{t a n}\right)=0, \quad \text { for } \quad x \in \mathbb{S}^{2}
\end{array}\right.
$$

Expand $E(x), H(x)$ in the spherical functions $Y_{n}^{m}(\omega), n=0,1,2, \ldots,|m| \leq n, \omega \in \mathbb{S}^{2}$ and the spherical Hankel functions of first kind

$$
h_{n}^{(1)}(z):=\frac{H_{n+1 / 2}^{(1)}(z)}{\sqrt{z}}, n \geq 1
$$

An application of Theorem 2.50 in [3] (in the notation of [3] it is necessary to replace $\omega$ by $\mu \in \mathbb{C} \backslash\{0\}$ ) says that the solution of the system (2.2) for $x=|x| \omega, r=$ $|x|>0, \omega=\frac{x}{r}$ has the form

$$
\begin{gather*}
E(x)=\sum_{n=1}^{\infty} \sum_{|m| \leq n}\left[\alpha_{n}^{m} \sqrt{n(n+1)} \frac{h_{n}^{(1)}(\mu r)}{r} Y_{n}^{m}(\omega) \omega\right. \\
\left.+\frac{\alpha_{n}^{m}}{r}\left(r h_{n}^{(1)}(\mu r)\right)^{\prime} U_{n}^{m}(\omega)+\beta_{n}^{m} h_{n}^{(1)}(\mu) V_{n}^{m}(\omega)\right],  \tag{2.3}\\
H(x)=-\frac{1}{\mathbf{i} \mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n}\left[\beta_{n}^{m} \sqrt{n(n+1)} \frac{h_{n}^{(1)}(\mu r)}{r} Y_{n}^{m}(\omega) \omega\right. \\
+  \tag{2.4}\\
\left.+\frac{\beta_{n}^{m}}{r}\left(r h_{n}^{(1)}(\mu r)\right)^{\prime} U_{n}^{m}(\omega)+\mu^{2} \alpha_{n}^{m} h_{n}^{(1)}(\mu) V_{n}^{m}(\omega)\right] .
\end{gather*}
$$

Here $U_{n}^{m}(\omega)=\frac{1}{\sqrt{n(n+1)}} \operatorname{grad}_{\mathbb{S}^{2}} Y_{n}^{m}(\omega)$ and $V_{n}^{m}(\omega)=\nu \wedge U_{n}^{m}(\omega)$ for $n \in \mathbb{N},-n \leq$ $m \leq n$ form a complete orthonormal basis in

$$
L_{t}^{2}\left(\mathbb{S}^{2}\right)=\left\{u(\omega) \in\left(L^{2}\left(\mathbb{S}^{2}\right)\right)^{3}:\langle\omega, u(\omega)\rangle=0 \text { on } \mathbb{S}^{2}\right\}
$$

To find a representation of $\nu \wedge H_{\text {tan }}$, observe that $\nu \wedge\left(\nu \wedge U_{n}^{m}\right)=-U_{n}^{m}$, so for $r=1$ one has

$$
\begin{gathered}
\left(\nu \wedge H_{t a n}\right)(\omega)=-\frac{1}{\mathbf{i} \mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n}\left[\beta_{n}^{m}\left(h_{n}^{(1)}(\mu)+\left.\frac{d}{d r} h_{n}^{(1)}(\mu r)\right|_{r=1}\right) V_{n}^{m}(\omega)\right. \\
\left.-\mu^{2} \alpha_{n}^{m} h_{n}^{(1)}(\mu) U_{n}^{m}(\omega)\right]
\end{gathered}
$$

and the boundary condition in (2.2) is satisfied if

$$
\begin{align*}
& \alpha_{n}^{m}\left[h_{n}^{(1)}(\mu)+\left.\frac{d}{d r}\left(h_{n}^{(1)}(\mu r)\right)\right|_{r=1}-\gamma \mathbf{i} \mu h_{n}^{(1)}(\mu)\right]=0, \forall n \in \mathbb{N},|m| \leq n  \tag{2.5}\\
- & \frac{\beta_{n}^{m} \gamma}{\mathbf{i} \mu}\left[h_{n}^{(1)}(\mu)+\left.\frac{d}{d r}\left(h_{n}^{(1)}(\mu r)\right)\right|_{r=1}-\frac{\mathbf{i} \mu}{\gamma} h_{n}^{(1)}(\mu)\right]=0, \forall n \in \mathbb{N},|m| \leq n \tag{2.6}
\end{align*}
$$

## 3. Roots of the equation $g_{n}(\lambda)=0$

To examine the eigenvalues of $G_{b}$ it is necessary to find the roots of the equations (2.3) and (2.4). Since $h_{n}^{(1)}(\mu) \neq 0$ for $\operatorname{Im} \mu>0$, the problem is reduced to study the roots $\lambda \in \mathbb{R}^{-}$of the equation

$$
\begin{equation*}
1+\left.\frac{d}{d r} h_{n}^{(1)}(-\mathbf{i} \lambda r)\right|_{r=1}\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)^{-1}-\lambda \gamma=0 \tag{3.1}
\end{equation*}
$$

and the same equation with $\gamma$ replaced by $\frac{1}{\gamma}$. Clearly, if $\mu=-\mathbf{i} \lambda$ is such that the expressions in the brackets [...] in (2.5) and (2.6) are non-vanishing for every $n \geq 1$, we must have $\alpha_{n}^{m}=\beta_{n}^{m}=0$ which implies $E_{t a n}=B_{t a n}=0$. Hence $(E, B)=0$ because the boundary problem with $\gamma=0$ has no eigenvalues in $\{z \in \mathbb{C}: \operatorname{Re} z<0\}$. In this section we suppose that $\gamma \neq 1$ and examine the equation

$$
\begin{equation*}
g_{n}(\lambda):=\frac{1}{\lambda}+\frac{d}{d \lambda}\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)^{-1}-\gamma=0 . \tag{3.2}
\end{equation*}
$$

It is well known that (see [5])

$$
h_{n}^{(1)}(-\mathbf{i} \lambda)=(-\mathbf{i})^{n+1} \frac{e^{\lambda}}{-\mathbf{i} \lambda} R_{n}\left(\frac{\mathbf{i}}{-2 \mathbf{i} \lambda}\right)=(-\mathbf{i})^{n} \frac{e^{\lambda}}{\lambda} R_{n}\left(-\frac{1}{2 \lambda}\right)
$$

with

$$
R_{n}(z):=\sum_{m=0}^{n} a_{m, n} z^{m}, a_{m, n}=\frac{(n+m)!}{m!(n-m)!}>0
$$

We will prove the following
Proposition 3.1. For $\lambda<0$ we have

$$
\begin{equation*}
G_{n, n+1}(\lambda)=\frac{\frac{d}{d \lambda} h_{n+1}^{(1)}(-\mathbf{i} \lambda)}{h_{n+1}^{(1)}(-\mathbf{i} \lambda)}-\frac{\frac{d}{d \lambda} h_{n}^{(1)}(-\mathbf{i} \lambda)}{h_{n}^{(1)}(-\mathbf{i} \lambda)}>0 \tag{3.3}
\end{equation*}
$$

Proof. The purpose is to show that
$\left(h_{n}^{(1)}(-\mathbf{i} \lambda) \frac{d}{d \lambda} h_{n+1}^{(1)}(-\mathbf{i} \lambda)-h_{n+1}^{(1)}(-\mathbf{i} \lambda) \frac{d}{d \lambda} h_{n}^{(1)}(-\mathbf{i} \lambda)\right)\left(h_{n+1}^{(1)}(-\mathbf{i} \lambda) h_{n}^{(1)}(-\mathbf{i} \lambda)\right)^{-1}>0$.
Introduce the functions

$$
\xi_{n}(\lambda):=\frac{e^{\lambda}}{\lambda} R_{n}\left(-\frac{1}{2 \lambda}\right), \eta_{n}(\lambda):=\lambda \xi_{n}(\lambda)
$$

Then $h_{n}^{(1)}(-\mathbf{i} \lambda)=(-\mathbf{i})^{n} \xi_{n}(\lambda)$ and the above inequality is equivalent to

$$
\begin{aligned}
& \left(\xi_{n}(\lambda) \frac{d}{d \lambda} \xi_{n+1}(\lambda)-\xi_{n+1}(\lambda) \frac{d}{d \lambda} \xi_{n}(\lambda)\right)\left(\xi_{n+1}(\lambda) \xi_{n}(\lambda)\right)^{-1} \\
= & \left(\eta_{n}(\lambda) \frac{d}{d \lambda} \eta_{n+1}(\lambda)-\eta_{n+1}(\lambda) \frac{d}{d \lambda} \eta_{n}(\lambda)\right)\left(\eta_{n+1}(\lambda) \eta_{n}(\lambda)\right)^{-1}>0
\end{aligned}
$$

Since $\eta_{n}(\lambda) \eta_{n+1}(\lambda)>0$ for $\lambda<0$, it suffices to show that the function

$$
F(\lambda)=\eta_{n}(\lambda) \frac{d}{d \lambda} \eta_{n+1}(\lambda)-\eta_{n+1}(\lambda) \frac{d}{d \lambda} \eta_{n}(\lambda)
$$

has positive values for $\lambda \in(-\infty, 0)$. Consider the derivative

$$
F^{\prime}(\lambda)=\eta_{n}(\lambda) \frac{d^{2}}{d \lambda^{2}} \eta_{n+1}(\lambda)-\eta_{n+1}(\lambda) \frac{d^{2}}{d \lambda^{2}} \eta_{n}(\lambda)
$$

We have

$$
\eta_{n}(\lambda)=\mathbf{i}^{n+1} h_{n}^{(1)}(-\mathbf{i} \lambda)(-\mathbf{i} \lambda)=\mathbf{i}^{n+1} \Xi_{n}(-\mathbf{i} \lambda)=-\mathbf{i}^{n-1} \Xi_{n}(-\mathbf{i} \lambda)
$$

The function $\Xi_{n}(z)=z h_{n}^{(1)}(z)$ satisfies the equation

$$
\Xi_{n}^{\prime \prime}(z)+\left(1-\frac{n^{2}+n}{z^{2}}\right) \Xi_{n}(z)=0
$$

and

$$
\begin{gathered}
\frac{d^{2}}{d \lambda^{2}} \eta_{n}(\lambda)=\mathbf{i}^{n-1} \Xi_{n}^{\prime \prime}(-\mathbf{i} \lambda)=-\mathbf{i}^{n-1}\left(1+\frac{n^{2}+n}{\lambda^{2}}\right) \Xi_{n}(-\mathbf{i} \lambda) \\
=\left(1+\frac{n^{2}+n}{\lambda^{2}}\right) \eta_{n}(\lambda)
\end{gathered}
$$

Consequently,

$$
\begin{gathered}
F^{\prime}(\lambda)=\left[\frac{(n+1)^{2}+n+1}{\lambda^{2}}-\frac{n^{2}+n}{\lambda^{2}}\right] \eta_{n}(\lambda) \eta_{n+1}(\lambda) \\
=2(n+2) \frac{\eta_{n}(\lambda) \eta_{n+1}(\lambda)}{\lambda^{2}}>0
\end{gathered}
$$

On the other hand,

$$
\begin{gathered}
F(\lambda)=e^{\lambda} R_{n}\left(-\frac{1}{2 \lambda}\right) \frac{d}{d \lambda}\left(e^{\lambda} R_{n+1}\left(-\frac{1}{2 \lambda}\right)\right)-e^{\lambda} R_{n+1}\left(-\frac{1}{2 \lambda}\right) \frac{d}{d \lambda}\left(e^{\lambda} R_{n}\left(-\frac{1}{2 \lambda}\right)\right) \\
=\frac{e^{2 \lambda}}{2 \lambda^{2}}\left[R_{n}\left(-\frac{1}{2 \lambda}\right) R_{n+1}^{\prime}\left(-\frac{1}{2 \lambda}\right)-R_{n+1}\left(-\frac{1}{2 \lambda}\right) R_{n}^{\prime}\left(-\frac{1}{2 \lambda}\right)\right]
\end{gathered}
$$

and

$$
\lim _{\lambda \rightarrow-\infty} F(\lambda)=0, \lim _{\lambda \nearrow 0} F(\lambda)=+\infty
$$

since

$$
\lim _{w \rightarrow+\infty}\left[R_{n}(w) R_{n+1}^{\prime}(w)-R_{n+1}(w) R_{n}^{\prime}(w)\right]=+\infty
$$

Finally, the function $F(\lambda)$ in the interval $(-\infty, 0]$ is increasing from 0 to $+\infty$ and this completes the proof.

Now if $\lambda_{n}<0$ is a solution the equation

$$
\begin{equation*}
g_{n}(\lambda):=\frac{1}{\lambda}+\left(\frac{d}{d \lambda} h_{n}^{(1)}(-\mathbf{i} \lambda)\right)\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)^{-1}-\gamma=0, \tag{3.4}
\end{equation*}
$$

one has

$$
g_{n+1}\left(\lambda_{n}\right)=\frac{1}{\lambda_{n}}+\left(\frac{d}{d \lambda} h_{n+1}^{(1)}\left(-i \lambda_{n}\right)\right)\left(h_{n+1}^{(1)}\left(-i \lambda_{n}\right)\right)^{-1}-\gamma=G_{n, n+1}\left(\lambda_{n}\right)>0
$$

so $\lambda_{n}$ is not a root of the equation

$$
g_{n+1}(\lambda)=\frac{1}{\lambda}+\left(\frac{d}{d \lambda} h_{n+1}^{(1)}(-\mathbf{i} \lambda)\right)\left(h_{n+1}^{(1)}(-\mathbf{i} \lambda)\right)^{-1}-\gamma=0 .
$$

In the following we assume that $\gamma>1$. Then for $\lambda \rightarrow-\infty$ we have $g_{n+1}(\lambda) \rightarrow$ $1-\gamma<0$, and since $g_{n+1}\left(\lambda_{n}\right)>0$ the equation $g_{n+1}(\lambda)=0$ has at least one root $-\infty<\lambda_{n+1}<\lambda_{n}$.

Lemma 3.1. Let $\gamma>1$. For every $n \geq 1$ the equation $g_{n}(\lambda)=0$ in the interval $(-\infty, 0)$ has exactly one root $\lambda_{n}<0$.

Proof. Setting $w=-\frac{1}{2 \lambda}$, we write the equation (3.2) as $\mathcal{R}_{n}(w):=w^{2} R_{n}^{\prime}(w)+$ $\alpha R_{n}(w)=0$, where $\alpha=\frac{1-\gamma}{2}<0$. We will show that this equation has exactly one positive root. Since

$$
w^{2} R_{n}^{\prime}(w)=\sum_{k=1}^{n} k a_{k, n} w^{k+1}, R_{n}(w)=\sum_{k=0}^{n} a_{k, n} w^{k}
$$

the polynomial $\mathcal{R}_{n}(w)$ has the representation

$$
\mathcal{R}_{n}(w)=\sum_{k=0}^{n+1} b_{k, n} w^{k}
$$

with

$$
\left\{\begin{array}{l}
b_{k, n}=(k-1) a_{k-1, n}+\alpha a_{k, n}, 0 \leq k \leq n, a_{-1, n}=0 \\
b_{n+1, n}=\frac{(2 n)!}{(n-1)!}
\end{array}\right.
$$

Taking into account the form of $a_{k, n}$, we deduce

$$
\begin{equation*}
b_{k, n}=\frac{(n+k-1)!}{(n-k+1)!k!}(k(k-1)+\alpha(n+k)(n-k+1)), 0 \leq k \leq n+1 \tag{3.5}
\end{equation*}
$$

Thus the sign of $b_{k, n}$ depends on the sign of the function

$$
B(k):=(1-\alpha) k^{2}+(\alpha-1) k+\alpha\left(n^{2}+n\right)
$$

which for $k \geq 1$ is increasing since

$$
B^{\prime}(k)=2(1-\alpha) k+\alpha-1 \geq 1-\alpha>0 .
$$

Clearly, $b_{0, n}=\alpha<0$ and $b_{n+1, n}>0$. There are two cases:
(i) $b_{1, n} \leq 0$. Then there is only one change of sing in the Descartes' sequence $\left\{b_{n+1, n}, b_{n, n}, \ldots, b_{1, n}, b_{0, n}\right\}$.
(ii) $b_{1, n}>0$. Then $b_{k, n}>0$ for $1 \leq k \leq n+1$ and in the Descartes' sequence $\left\{b_{n+1, n}, b_{n, n}, \ldots, b_{1, n}, b_{0, n}\right\}$ one has again only one change of sign.

Applying the Descartes' rule of signs, we conclude that the number of the positive roots of $\mathcal{R}_{n}(w)=0$ is exactly one.

Combining Proposition 3.1 and Lemma 3.1, one obtain the following
Corollary 3.1. Let $\gamma>1$. Then the generator $G_{b}$ has an infinite sequence of real eigenvalues

$$
-\infty<\ldots<\lambda_{n}<\ldots<\lambda_{2}<\lambda_{1}<0
$$

and $\lambda_{n}$ has geometric multiplicity $2 n+1$.
The last statement concerns the geometric multiplicity since the functions $\left\{Y_{m, n}(\omega)\right\}_{m=-n}^{m}$ are linearly independent. The algebraic multiplicity of $\lambda_{m}$ will be discussed in Section 5.

## 4. Estimation of the roots

Throughout this section we assume $\gamma>1$. Set $\lambda=\frac{\mathbf{i} \sqrt{z}}{h}, 0<h \ll 1$ with $z=-1+\mathbf{i} \eta, 0 \leq|\eta| \leq h^{1 / 2}, \eta \in \mathbb{R}$. Consider the Dirichlet problem

$$
\left\{\begin{array}{l}
\left(h^{2} \Delta+z\right) w=0,|x|>1, w \in H^{2}(|x|>1)  \tag{4.1}\\
w=f,|x|=1
\end{array}\right.
$$

and note that $\Delta+\frac{z}{h^{2}}=\Delta-\lambda^{2}$. The solution of (4.1) has the form

$$
w(r \omega)=\sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_{n}^{(1)}(-\mathbf{i} \lambda r)\left(h_{n}^{(1)}(-\mathbf{i} \lambda)^{-1} \alpha_{n, m} Y_{n, m}(\omega)\right.
$$

where

$$
f(\omega)=\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n, m} Y_{n, m}(\omega)
$$

The semiclassical Dirichlet-to-Neumann operator $\mathcal{N}_{\text {ext }}(h, z)=\left.\frac{h}{\mathrm{i}} \frac{d}{d r} w\right|_{r=1}$ related to (4.1) becomes

$$
\begin{gathered}
\mathcal{N}_{\text {ext }}(h, z)=-\mathbf{i} \sqrt{z} \sum_{n=0}^{\infty} \sum_{m=-n}^{n}\left(h_{n}^{(1)}\right)^{\prime}(-\mathbf{i} \lambda)\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)^{-1} \alpha_{n, m} Y_{n, m} \\
=\sqrt{z} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{d}{d \lambda}\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)^{-1} \alpha_{n, m} Y_{n, m}
\end{gathered}
$$

By using the approximation of $\mathcal{N}_{\text {ext }}(h, z)$ established in [8], [6] for $z=-1+\mathbf{i} \eta$, one deduces

$$
\left\|\mathcal{N}_{e x t}(h, z) f-O p_{h}(\rho) f\right\|_{L^{2}\left(\mathbb{S}^{2}\right)} \leq C \frac{|\sqrt{z}|}{|\lambda|}\|f\|_{L^{2}\left(\mathbb{S}^{2}\right)}, 0<h \leq h_{0}
$$

with $\rho=\sqrt{z-r_{0}\left(x^{\prime}, \xi^{\prime}\right)}$ and a constant $C>0$ independent of $z, \lambda$ and $f$. Here $r_{0}\left(x^{\prime}, \xi^{\prime}\right)$ is the principal symbol of the semiclasssical Laplace-Beltrami operator $-h^{2} \Delta_{\mathbb{S}^{2}}=\frac{z}{\lambda^{2}} \Delta_{\mathbb{S}^{2}}$. Moreover, $\sqrt{z}=\mathbf{i} \sqrt{1-\mathbf{i} \eta}=\mathbf{i}\left(1-\frac{\mathbf{i} \eta}{2}+\mathcal{O}\left(\eta^{2}\right)\right)$ and

$$
\operatorname{Re} \lambda=-\frac{1}{h}+\mathcal{O}(1), \operatorname{Im} \lambda=\mathcal{O}\left(h^{-1 / 2}\right)
$$

Hence, for $0<h \leq h_{0}$ we get

$$
\lambda \in \Lambda_{0}=\left\{z \in \mathbb{C}:|\operatorname{Im} z| \leq c h_{0}^{1 / 2}|\operatorname{Re} z|, \operatorname{Re} \lambda<-\epsilon<0,|\lambda| \geq \lambda_{0}\right\}
$$

On the other hand,

$$
\left\|O p_{h}(\rho)-\sqrt{z}\left(\sqrt{1-\frac{\Delta_{\mathbb{S}^{2}}}{\lambda^{2}}}\right)\right\|_{L^{2}\left(\mathbb{S}^{2}\right) \rightarrow L^{2}\left(\mathbb{S}^{2}\right)} \leq C_{1}|\lambda|^{-1}, \lambda \in \Lambda_{0}
$$

Applying the spectral theorem, one deduces

$$
\left(\sqrt{1-\frac{\Delta_{S^{2}}}{\lambda^{2}}}\right) f=\sum_{n=0}^{\infty} \sum_{m=-n}^{n}\left(\sqrt{1+\frac{n(n+1)}{\lambda^{2}}}\right) \alpha_{n, m} Y_{n . m}
$$

and

$$
\begin{gathered}
\|\left(\left.\mathcal{N}_{e x t}(h,-z)-\sqrt{z}\left(\sqrt{1-\frac{\Delta_{S^{2}}}{\lambda^{2}}}\right) f \|_{L^{2}\left(S^{2}\right)}^{2}=|z| \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \right\rvert\, \frac{d}{d \lambda}\left(h_{n}(-i \lambda)\right)\left(h_{n}(-i \lambda)\right)^{-1}\right. \\
-\left.\sqrt{1+\frac{n(n+1)}{\lambda^{2}}}\right|^{2}\left|a_{n, m}\right|^{2}
\end{gathered}
$$

This implies

$$
\begin{equation*}
\left|\frac{d}{d \lambda}\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)^{-1}-\sqrt{1+\frac{n(n+1)}{\lambda^{2}}}\right| \leq C_{2}|\lambda|^{-1}, \forall n \in \mathbb{N}, \lambda \in \Lambda_{0} \tag{4.2}
\end{equation*}
$$

which we write as

$$
\begin{equation*}
\left|\left[\frac{1}{\lambda}+\frac{d}{d \lambda}\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)^{-1}-\gamma\right]-\left[\sqrt{1+\frac{n(n+1)}{\lambda^{2}}}-\gamma\right]\right| \leq C_{0}|\lambda|^{-1} \tag{4.3}
\end{equation*}
$$

Remark 4.1. For bounded $1 \leq n \leq N_{0}$ and sufficiently large $|\lambda|$ the estimate (4.2) follows easily from the fact that $\frac{R_{n}^{\prime}(w)}{R_{n}(w)}=n(n+1)+\mathcal{O}(|w|)$ as $|w| \rightarrow 0$.

Remark 4.2. The estimate (4.2) is similar to that in Proposition 2.1 in [7], where the function $\frac{J_{\nu}^{\prime}(\lambda)}{J_{\nu}(\lambda)}$ for $\nu \geq 0$ and $0<C \leq|\operatorname{Im} \lambda| \leq \delta|\operatorname{Re} \lambda|$, $\operatorname{Re} \lambda>C_{1}$ has been approximated. Here $J_{\nu}(z)$ is the Bessel function, while the boundary problem examined in [7] is in the domain $|x|<1$.

Put $z=\lambda$ and for $z \in \Lambda_{0}$ consider the function

$$
f_{n}(z):=\sqrt{1+\frac{n(n+1)}{z^{2}}}-\gamma
$$

with zeros

$$
z_{n}^{ \pm}= \pm \sqrt{\frac{n^{2}+n}{\gamma^{2}-1}}
$$

In the following we set $z_{n}=-\sqrt{\frac{n(n+1)}{\gamma^{2}-1}}$. Clearly,

$$
f_{n}^{\prime}(z)=-\frac{1}{z} \frac{\frac{n(n+1)}{z^{2}}}{\sqrt{1+\frac{n(n+1)}{z^{2}}}}
$$

and $\frac{n(n+1)}{z_{n}^{2}}=\gamma^{2}-1, f_{n}^{\prime}\left(z_{n}\right)=-\frac{\gamma^{2}-1}{\gamma z_{n}}$. A calculus yields the second derivative

$$
\begin{gathered}
f_{n}^{\prime \prime}(z)=\frac{1}{z^{2}}\left[\frac{3 n(n+1)}{z^{2}}\left(\sqrt{1+\frac{n(n+1)}{z^{2}}}\right)\right. \\
\left.-\frac{n^{2}(n+1)^{2}}{z^{4}}\left(\sqrt{1+\frac{n(n+1)}{z^{2}}}\right)^{-1 / 2}\right]\left(1+\frac{n(n+1)}{z^{2}}\right)^{-1} .
\end{gathered}
$$

For $n$ large enough and $a>0$ to be fixed below introduce the contour

$$
C_{n}(a):=\left\{z=z_{n}+a e^{\mathbf{i} \varphi}, 0 \leq \varphi<2 \pi\right\} \subset \Lambda_{0}
$$

Our purpose is to choose $a$ so that

$$
\begin{equation*}
\left|f_{n}(z)\right| \geq \frac{C_{0}}{|z|}, \quad \forall z \in C_{n}(a) \tag{4.4}
\end{equation*}
$$

We have

$$
z^{2}=z_{n}^{2}+2 z_{n} a e^{\mathbf{i} \varphi}+a^{2} e^{2 \mathbf{i} \varphi}
$$

and

$$
\begin{equation*}
\frac{n(n+1)}{z^{2}}=\left(\gamma^{2}-1\right)\left(1+\mathcal{O}\left(\frac{1}{n}\right) a+\mathcal{O}\left(\frac{1}{n^{2}}\right) a^{2}\right)^{-1}, z \in C_{n}(a) \tag{4.5}
\end{equation*}
$$

On the other hand,

$$
\sqrt{\frac{n(n+1)}{z^{2}}+1}=\left[\frac{\gamma^{2}+\mathcal{O}\left(\frac{1}{n}\right) a+\mathcal{O}\left(\frac{1}{n^{2}}\right) a^{2}}{1+\mathcal{O}\left(\frac{1}{n}\right) a+\mathcal{O}\left(\frac{1}{n^{2}}\right) a^{2}}\right]^{1 / 2}
$$

Clearly, one has the estimate

$$
\begin{equation*}
\left|f_{n}(z)\right| \geq \frac{\gamma^{2}-1}{\gamma\left|z_{n}\right|} a-\frac{a^{2}}{2} \sup _{z \in C_{n}(a)}\left|f_{n}^{\prime \prime}(z)\right|, z \in C_{n}(a) \tag{4.6}
\end{equation*}
$$

Set $C_{\gamma}=\frac{\gamma^{2}-1}{\gamma}>0$ and choose $a>0$ so that $C_{\gamma} a>4 C_{0}$. We fix $a$ and obtain

$$
\frac{C_{\gamma} a}{2\left|z_{n}\right|}>\frac{2 C_{0}}{\left|z_{n}\right|}>\frac{C_{0}}{\left|z_{n}\right|\left|1+\frac{a e^{\mathrm{i} \varphi}}{z_{n}}\right|}, 0 \leq \varphi<2 \pi
$$

taking $n$ large enough to satisfy the inequality

$$
\frac{1}{\left|1+\frac{a e^{i} \varphi}{z_{n}}\right|}<2 .
$$

Next we arrange the inequality

$$
\begin{equation*}
\frac{C_{\gamma} a}{2\left|z_{n}\right|}-\frac{a^{2}}{2} \sup _{z \in C_{n}(a)}\left|f_{n}^{\prime \prime}(z)\right|>0 \tag{4.7}
\end{equation*}
$$

It is clear that

$$
f_{n}^{\prime \prime}(z)=\frac{1}{z^{2}} G\left(\frac{n(n+1)}{z^{2}}\right),
$$

where

$$
G(\zeta)=\left[3 \zeta \sqrt{\zeta+1}-\zeta^{2}(\zeta+1)^{-1 / 2}\right](\zeta+1)^{-1}
$$

Note that for $z \in C_{n}(a)$ and $n$ large enough according to (4.4), the function $\left|G\left(\frac{n(n+1)}{z^{2}}\right)\right|$ is bounded by a constant $B_{\gamma, a}$ depending on $\gamma$ and $a$. Thus for large $n$ we get

$$
\sup _{z \in C_{n}(a)}\left|f_{n}^{\prime \prime}(z)\right| \leq B_{\gamma, a} \sup _{z \in C_{n}(a)} \frac{1}{|z|^{2}}=B_{\gamma, a} \frac{1}{\left|z_{n}\right|^{2}} \sup _{z \in C_{n}(a)} \frac{1}{\left|1+\frac{a e^{i \varphi}}{z_{n}}\right|^{2}} \leq 4 B_{\gamma, a} \frac{1}{\left|z_{n}\right|^{2}}
$$

and the proof of (4.7) is reduced to

$$
C_{\gamma}>4 B_{\gamma, a} \frac{a}{\left|z_{n}\right|}
$$

which is satisfied taking again $n$ large. Finally, we proved the estimate (4.3) and we can apply Rouché theorem for the functions $g_{n}(z)$ and $f_{n}(z)$ and conclude that the function $g_{n}(z)$ has exactly one simple zero $\lambda_{n}$ in $C_{n}(a)$. Since $g_{n}(z)$ has only real zeros (see Appendix in [2]), this implies the following

Lemma 4.1. There exist $n_{0}(\gamma)$ and $a(\gamma)>0$ depending on $\gamma$ such that for $n \geq$ $n_{0}(\gamma)$ the negative root $\lambda_{n}$ of the equation (3.2) satisfies the estimate

$$
\begin{equation*}
\left|\lambda_{n}+\sqrt{\frac{n(n+1)}{\gamma^{2}-1}}\right| \leq a(\gamma) \tag{4.8}
\end{equation*}
$$

Remark 4.3. According to Proposition 2.1, $n_{0}(\gamma)$ must satisfy the inequality

$$
n_{0}(\gamma) \geq \frac{\sqrt{\gamma^{2}-1}}{\max \{\gamma-1, \sqrt{\gamma-1}\}}
$$

## 5. Weyl asymptotics

We start with the analysis of the multiplicity of $\lambda_{n}$.
Lemma 5.1. For $n \geq n_{0}(\gamma)$ we have $\operatorname{mult}\left(\lambda_{n}\right)=2 n+1$.
Proof. Since the geometric multiplicity of $\lambda_{n}$ is $2 n+1$, it is sufficient to show that

$$
\begin{equation*}
\operatorname{mult}\left(\lambda_{n}\right) \leq 2 n+1 \tag{5.1}
\end{equation*}
$$

Let $\lambda \in \Lambda_{0}$, where $\Lambda_{0}$ is the set introduced in the previous section and let $\lambda \notin$ $\sigma\left(G_{b}\right)$. If $0 \neq(f, g) \in$ Image $G_{b} \cap L^{2}(\Omega)$, one has $\operatorname{div} f=\operatorname{div} g=0$ and for $(u, v)=\left(G_{b}-\lambda\right)^{-1}(f, g)$ we get $\operatorname{div} u=\operatorname{div} v=0$. Consider the skew self-adjoint operator

$$
A=\left(\begin{array}{cc}
0 & - \text { curl } \\
\text { curl } & 0
\end{array}\right)
$$

with boundary condition $\nu \wedge u=0$ on $\mathbb{S}^{2}$. Then $\sigma(A) \subset \mathbf{i} \mathbb{R}$ and let $\left(u_{0}(x ; \lambda), v_{0}(x ; \lambda)\right)=$ $(A-\lambda)^{-1}(f, g)$, that is

$$
\left\{\begin{array}{l}
(A-\lambda)\binom{u_{0}}{v_{0}}=\binom{f}{g} \text { for }|x|>1  \tag{5.2}\\
\nu \wedge u_{0}=0 \text { on } \mathbb{S}^{2}
\end{array}\right.
$$

Since $\operatorname{div} u_{0}=\operatorname{div} v_{0}=0$, the well known coercive estimates yield $\left(u_{0}, v_{0}\right) \in$ $H^{1}(\Omega)$. Moreover the resolvent $(A-\lambda)^{-1}$ is analytic in $\{z \in \mathbb{C}: \operatorname{Re} z<0\}$ and
$u_{0}(x ; \lambda), v_{0}(x ; \lambda)$ depend analytically on $\lambda$. We write $(u, v)=\left(u_{0}, v_{0}\right)+\left(u_{1}, v_{1}\right)$, where $\left(u_{1}(x ; \lambda), v_{1}(x ; \lambda)\right)$ is the solution of the problem

$$
\left\{\begin{array}{l}
(G-\lambda)\binom{u_{1}}{v_{1}}=\binom{0}{0} \text { for }|x|>1,  \tag{5.3}\\
\left(u_{1}\right)_{\tan }-\gamma\left(\nu \wedge\left(v_{1}\right)_{\tan }\right)=-\gamma\left(\nu \wedge\left(v_{0}\right)_{\tan }(x ; z)\right) \text { on } \mathbb{S}^{2} .
\end{array}\right.
$$

To solve (5.3), note that $-\gamma\left(\nu \wedge\left(v_{0}\right)_{\tan }(\omega ; z)\right)=F(\omega ; \lambda) \in L^{2}\left(\mathbb{S}^{2}\right)$ with $F(\omega ; \lambda)$ analytical in $\lambda$ for $\lambda \in \Lambda_{0}$. Thus we may write

$$
F(\omega ; \lambda)=\sum_{n=1}^{\infty} \sum_{m=-n}^{n} \tilde{\alpha}_{n}^{m}(\lambda) U_{n}^{m}(\omega)+\tilde{\beta}_{n}^{m}(\lambda) V_{n}^{m}(\omega)
$$

with analytical coefficients $\tilde{\alpha}_{n}^{m}(\lambda), \tilde{\beta}_{n}^{m}(\lambda)$. Now we can solve (2.5), (2.6) with right hand part $\left(\tilde{\alpha}_{n}^{m}(\lambda), \tilde{\beta}_{n}^{m}(\lambda)\right)$. Finally, we obtain a representation of the solution of (5.3) with meromorphic coefficients

$$
\begin{gathered}
\alpha_{n}^{m}(\lambda)=\frac{\tilde{\alpha}_{n}^{m}(\lambda)}{h_{n}^{(1)}(-\mathbf{i} \lambda)\left[1+\left.\frac{d}{d r}\left(h_{n}^{(1)}(-\mathbf{i} \lambda r)\right)\right|_{r=1}\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)^{-1}-\lambda \gamma\right]}, \\
\beta_{n}^{m}(\lambda)=-\frac{\lambda \tilde{\beta}_{n}^{m}(\lambda)}{\gamma h_{n}^{(1)}(-\mathbf{i} \lambda)\left[1+\left.\frac{d}{d r}\left(h_{n}^{(1)}(-\mathbf{i} \lambda r)\right)\right|_{r=1}\left(h_{n}^{(1)}(-\mathbf{i} \lambda)\right)^{-1}-\lambda \gamma^{-1}\right]} .
\end{gathered}
$$

If $\gamma>1$ the analysis in the previous section shows that for $\lambda \in \Lambda_{0}$ the meromorphic function $\alpha_{n}^{m}(\lambda)$ has a simple pole at $\lambda_{n}<0$, while $\beta_{n}^{m}(\lambda)$ is analytic in $\Lambda_{0}$. For $0<\gamma<1$ the function $\alpha_{n}^{m}(\lambda)$ is analytic in $\Lambda_{0}$ and $\beta_{n}^{m}(\lambda)$ is meromorphic. Next we integrate $(u(x ; \lambda), v(x ; \lambda))$ over the circle $\left|\lambda_{n}-\lambda\right|=\epsilon$, where $\epsilon$ is sufficiently small. The integral of $\left(u_{0}(x ; \lambda), v_{0}(x ; \lambda)\right)$ vanish, while for the integral of $\left(u_{1}(x ; \lambda), v_{1}(x ; \lambda)\right)$, taking into account the representation of the solution of (5.3), we will obtain a sum

$$
S_{n}=\left\{\begin{array}{l}
c_{n} \sum_{m=-n}^{m} \tilde{\alpha}_{n}^{m}\left(\lambda_{n}\right) U_{n}^{m}(\omega), c_{n} \neq 0, \gamma>1, \\
d_{n} \sum_{m=-n}^{m} \lambda_{n} \tilde{\beta}_{n}^{m}\left(\lambda_{n}\right) \gamma^{-1} V_{n}^{m}(\omega), d_{n} \neq 0,0<\gamma<1
\end{array}\right.
$$

This completes the proof of (5.1).
Passing to the analysis of $N(r)$, consider first the case $\gamma>1$. The root $\lambda_{n}$ has algebraic multiplicity $2 n+1$ and to find a lower bound of $N(r)$ we apply the estimate

$$
\left|\lambda_{n}\right| \leq \sqrt{\frac{n(n+1)}{\gamma^{2}-1}}+a(\gamma)<\frac{n+1}{\sqrt{\gamma^{2}-1}}+a(\gamma) \leq r
$$

for $r \geq a(\gamma)+\frac{n_{0}(\gamma)+1}{\sqrt{\gamma^{2}-1}}$. Then

$$
N(r) \geq \sum_{j=n_{0}(\gamma)}^{\left[(r-a(\gamma)) \sqrt{\gamma^{2}-1}-1\right]}(2 j+1)=\left(\gamma^{2}-1\right) r^{2}+\mathcal{O}_{\gamma}(r)+A_{\gamma}
$$

To get a upper bound for $N(r)$, we use the estimate

$$
\left|\lambda_{n}\right| \geq \sqrt{\frac{n(n+1)}{\gamma^{2}-1}}-a(\gamma)>\frac{n}{\sqrt{\gamma^{2}-1}}-a(\gamma) \geq r
$$

for

$$
n \geq(r+a(\gamma)) \sqrt{\gamma^{2}-1} \geq 2 a(\gamma) \sqrt{\gamma^{2}-1}+n_{0}(\gamma)+1
$$

hence

$$
N(r) \leq \sum_{j=n_{0}(\gamma)}^{\left[(r+a(\gamma)) \sqrt{\gamma^{2}-1}\right]+1}(2 j+1)+D_{\gamma}=\left(\gamma^{2}-1\right) r^{2}+\mathcal{O}_{\gamma}(r)+A_{\gamma}^{\prime}
$$

If $0<\gamma<1$, we have $\frac{1}{\gamma}>1$ and one applies our argument to the the equation (2.6). This completes the proof of theorem 1.1

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Dipartimento di Matematica, Università di Pisa, Italia
E-mail address: colombini@dm.unipi.it
Institut de Mathématiques de Bordeaux, 351, Cours de la Libération, 33405 Talence, France

E-mail address: petkov@math.u-bordeaux.fr


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