# WEYL FORMULA FOR THE NEGATIVE DISSIPATIVE EIGENVALUES OF MAXWELL'S EQUATIONS

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ABSTRACT. Let  $V(t)=e^{tG_b},\,t\geq 0$ , be the semigroup generated by Maxwell's equations in an exterior domain  $\Omega\subset\mathbb{R}^3$  with dissipative boundary condition  $E_{tan}-\gamma(x)(\nu\wedge B_{tan})=0, \gamma(x)>0, \forall x\in\Gamma=\partial\Omega.$  We study the case when  $\Omega=\{x\in\mathbb{R}^3:|x|>1\}$  and  $\gamma\neq 1$  is a constant. We establish a Weyl formula for the counting function of the negative real eigenvalues of  $G_b$ .

## 1. Introduction

Let  $K \subset \{x \in \mathbb{R}^3 : |x| \leq a\}$  be an open connected domain and let  $\Omega = \mathbb{R}^3 \setminus \bar{K}$  be connected domain with  $C^{\infty}$  smooth boundary  $\Gamma$ . Consider the boundary problem

$$\partial_t E = \operatorname{curl} B, \qquad \partial_t B = -\operatorname{curl} E \quad \text{in} \quad \mathbb{R}_t^+ \times \Omega,$$

$$E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0 \quad \text{on} \quad \mathbb{R}_t^+ \times \Gamma,$$

$$E(0, x) = E_0(x), \qquad B(0, x) = B_0(x).$$
(1.1)

with initial data  $f = (E_0, B_0) \in (L^2(\Omega))^6 = \mathcal{H}$ . Here  $\nu(x)$  is the unit outward normal to  $\partial\Omega$  at  $x \in \Gamma$  pointing into  $\Omega$ ,  $\langle , \rangle$  denotes the scalar product in  $\mathbb{C}^3$ ,  $u_{tan} := u - \langle u, \nu \rangle \nu$ , and  $\gamma(x) \in C^{\infty}(\Gamma)$  satisfies  $\gamma(x) > 0$  for all  $x \in \Gamma$ . The solution of the problem (1.1) is described by a contraction semigroup

$$(E,B) = V(t)f = e^{tG_b}f, \ t \ge 0,$$

where the generator  $G_b$  has domain  $D(G_b)$  which is the closure in the graph norm of functions  $u = (v, w) \in (C_{(0)}^{\infty}(\mathbb{R}^3))^3 \times (C_{(0)}^{\infty}(\mathbb{R}^3))^3$  satisfying the boundary condition  $v_{tan} - \gamma(\nu \wedge w_{tan}) = 0$  on  $\Gamma$ .

In [1] it was proved that the spectrum of  $G_b$  in the open half plan  $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$  is formed by isolated eigenvalues with finite multiplicities. Note that if  $G_b f = \lambda f$  with  $\operatorname{Re} \lambda < 0$ , the solution  $u(t,x) = V(t)f = e^{\lambda t}f(x)$  of (1.1) has exponentially decreasing global energy. Such solutions are called **asymptotically disappearing** and they are very important for the inverse scattering problems (see [1]). In particular, the eigenvalues  $\lambda$  with  $\operatorname{Re} \lambda \to -\infty$  imply a very fast decay of the corresponding solutions. In [2] the existence of eigenvalues of  $G_b$  has been studied for the ball  $B_3 = \{x \in \mathbb{R}^3, |x| < 1\}$  assuming  $\gamma$  constant. It was proved for  $\gamma = 1$  there are no eigenvalues in  $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ , while for  $\gamma \neq 1$  there is always an infinite number of real eigenvalues  $\lambda_m$  and with exception of one they satisfy the estimate

$$\lambda_m \le -\frac{1}{\max\{(\gamma_0 - 1), \sqrt{\gamma_0 - 1}\}} = -c_0,$$
(1.2)

where  $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}.$ 

In this Note we study the distribution of the negative eigenvalues and our purpose is to obtain a Weyl formula for the counting function

$$N(r) = \#\{\lambda \in \sigma_p(G_b) \cap \mathbb{R}^- : |\lambda| \le r\}, \ r > r_0(\gamma),$$

where every eigenvalues  $\lambda_m$  is counted with its algebraic multiplicity given by

$$\operatorname{mult}(\lambda_m) = \operatorname{rank} \frac{1}{2\pi \mathbf{i}} \int_{|\lambda_n - z| = \epsilon} (z - G_b)^{-1} dz,$$

where  $0 < \epsilon \ll 1$ . Our main result is the following

**Theorem 1.1.** Let  $\gamma \neq 1$  be a constant and let  $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$ . Then the counting function N(r) for the ball  $B_3$  has the asymptotic

$$N(r) = (\gamma_0^2 - 1)r^2 + \mathcal{O}_{\gamma}(r), \ r \ge r_0(\gamma) > c_0.$$
(1.3)

The proof of Theorem 1.1 is based on a precise analysis of the roots of the equation (3.1) involving spherical Hankel functions  $h_n^{(1)}(\lambda)$  of first kind. We show in Section 3 that for  $\gamma > 1$  this equation has only one real root  $\lambda_n < 0$ . Moreover, we have  $\lambda_{n+1} < \lambda_n$ ,  $\forall n \in \mathbb{N}$ , so we have a decreasing sequence of eigenvalues. The geometric multiplicity of  $\lambda_n$  is 2n+1. Since  $C_b$  is not a self-adjoint operator the geometric multiplicity could be less than the algebraic one. In our case these multiplicities coincide and the proof is based on a representation of  $(G_b - z)^{-1}$ . To estimate  $\lambda_n$  as  $n \to \infty$ , we apply an approximation of the exterior semiclassical Dirichlet to Neumann map for the operator  $(h^2\Delta + z)$  established in [6] (see also [8]) combined with an application of Rouché theorem.

We conjecture that in the general case of strictly convex obstacles and  $\min_{y \in \Gamma} \gamma(y) = \gamma_1 > 1$  we have the asymptotic

$$N(r) = \frac{1}{4\pi} \left( \int_{\Gamma} (\gamma^{2}(y) - 1) dS_{y} \right) r^{2} + \mathcal{O}_{\gamma}(r), \ r \ge r_{0}(\gamma_{0}).$$

For the ball  $B_3$  this agrees with (1.3).

#### 2. Boundary Problem for Maxwell system

Our purpose is to study the eigenvalues of  $G_b$  in case the obstacle is equal to the ball  $B_3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ . Setting  $\lambda = \mathbf{i}\mu$ , Im  $\mu > 0$ , an eigenfunction  $(E, B) \neq 0$  of  $G_b$  satisfies

$$\operatorname{curl} E = -\mathbf{i}\mu B, \qquad \operatorname{curl} B = \mathbf{i}\mu E.$$
 (2.1)

Replacing B by H = -B yields for  $(E, H) \in (H^2(|x| \le 1))^6$ ,

$$\begin{cases} \operatorname{curl} E = \mathbf{i}\mu H, & \operatorname{curl} H = -\mathbf{i}\mu E, & \text{for } x \in B_3, \\ E_{tan} + \gamma(\nu \wedge H_{tan}) = 0, & \text{for } x \in \mathbb{S}^2. \end{cases}$$
(2.2)

Expand E(x), H(x) in the spherical functions  $Y_n^m(\omega), n=0,1,2,...,|m| \leq n, \omega \in \mathbb{S}^2$  and the spherical Hankel functions of first kind

$$h_n^{(1)}(z) := \frac{H_{n+1/2}^{(1)}(z)}{\sqrt{z}}, n \ge 1$$

An application of Theorem 2.50 in [3] (in the notation of [3] it is necessary to replace  $\omega$  by  $\mu \in \mathbb{C} \setminus \{0\}$ ) says that the solution of the system (2.2) for  $x = |x|\omega, r = |x| > 0, \omega = \frac{x}{r}$  has the form

$$E(x) = \sum_{n=1}^{\infty} \sum_{|m| \le n} \left[ \alpha_n^m \sqrt{n(n+1)} \frac{h_n^{(1)}(\mu r)}{r} Y_n^m(\omega) \omega + \frac{\alpha_n^m}{r} (r h_n^{(1)}(\mu r))' U_n^m(\omega) + \beta_n^m h_n^{(1)}(\mu) V_n^m(\omega) \right],$$
(2.3)

$$H(x) = -\frac{1}{\mathbf{i}\mu} \sum_{n=1}^{\infty} \sum_{|m| \le n} \left[ \beta_n^m \sqrt{n(n+1)} \frac{h_n^{(1)}(\mu r)}{r} Y_n^m(\omega) \omega + \frac{\beta_n^m}{r} (r h_n^{(1)}(\mu r))' U_n^m(\omega) + \mu^2 \alpha_n^m h_n^{(1)}(\mu) V_n^m(\omega) \right]. \tag{2.4}$$

Here  $U_n^m(\omega) = \frac{1}{\sqrt{n(n+1)}} \operatorname{grad}_{\mathbb{S}^2} Y_n^m(\omega)$  and  $V_n^m(\omega) = \nu \wedge U_n^m(\omega)$  for  $n \in \mathbb{N}, -n \leq m \leq n$  form a complete orthonormal basis in

$$L_t^2(\mathbb{S}^2) = \{u(\omega) \in (L^2(\mathbb{S}^2))^3 : \langle \omega, u(\omega) \rangle = 0 \text{ on } \mathbb{S}^2 \}.$$

To find a representation of  $\nu \wedge H_{tan}$ , observe that  $\nu \wedge (\nu \wedge U_n^m) = -U_n^m$ , so for r=1 one has

$$(\nu \wedge H_{tan})(\omega) = -\frac{1}{i\mu} \sum_{n=1}^{\infty} \sum_{|m| \le n} \left[ \beta_n^m \left( h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r)|_{r=1} \right) V_n^m(\omega) \right]$$

$$-\mu^2 \alpha_n^m h_n^{(1)}(\mu) U_n^m(\omega) \bigg]$$

and the boundary condition in (2.2) is satisfied if

$$\alpha_n^m \left[ h_n^{(1)}(\mu) + \frac{d}{dr} (h_n^{(1)}(\mu r)) |_{r=1} - \gamma \mathbf{i} \mu h_n^{(1)}(\mu) \right] = 0, \ \forall n \in \mathbb{N}, \ |m| \le n,$$
 (2.5)

$$-\frac{\beta_n^m \gamma}{\mathbf{i}\mu} \left[ h_n^{(1)}(\mu) + \frac{d}{dr} (h_n^{(1)}(\mu r))|_{r=1} - \frac{\mathbf{i}\mu}{\gamma} h_n^{(1)}(\mu) \right] = 0, \ \forall n \in \mathbb{N}, \ |m| \le n.$$
 (2.6)

## 3. Roots of the equation $q_n(\lambda) = 0$

To examine the eigenvalues of  $G_b$  it is necessary to find the roots of the equations (2.3) and (2.4). Since  $h_n^{(1)}(\mu) \neq 0$  for  $\text{Im } \mu > 0$ , the problem is reduced to study the roots  $\lambda \in \mathbb{R}^-$  of the equation

$$1 + \frac{d}{dr} h_n^{(1)}(-\mathbf{i}\lambda r) \Big|_{r=1} (h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \lambda \gamma = 0$$
 (3.1)

and the same equation with  $\gamma$  replaced by  $\frac{1}{\gamma}$ . Clearly, if  $\mu = -\mathbf{i}\lambda$  is such that the expressions in the brackets [...] in (2.5) and (2.6) are non-vanishing for every  $n \geq 1$ , we must have  $\alpha_n^m = \beta_n^m = 0$  which implies  $E_{tan} = B_{tan} = 0$ . Hence (E, B) = 0 because the boundary problem with  $\gamma = 0$  has no eigenvalues in  $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ . In this section we suppose that  $\gamma \neq 1$  and examine the equation

$$g_n(\lambda) := \frac{1}{\lambda} + \frac{d}{d\lambda} \left( h_n^{(1)}(-\mathbf{i}\lambda) \right) (h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \gamma = 0.$$
 (3.2)

It is well known that (see [5])

$$h_n^{(1)}(-\mathbf{i}\lambda) = (-\mathbf{i})^{n+1} \frac{e^{\lambda}}{-\mathbf{i}\lambda} R_n \left(\frac{\mathbf{i}}{-2\mathbf{i}\lambda}\right) = (-\mathbf{i})^n \frac{e^{\lambda}}{\lambda} R_n \left(-\frac{1}{2\lambda}\right)$$

with

$$R_n(z) := \sum_{m=0}^n a_{m,n} z^m, \ a_{m,n} = \frac{(n+m)!}{m!(n-m)!} > 0.$$

We will prove the following

**Proposition 3.1.** For  $\lambda < 0$  we have

$$G_{n,n+1}(\lambda) = \frac{\frac{d}{d\lambda} h_{n+1}^{(1)}(-\mathbf{i}\lambda)}{h_{n+1}^{(1)}(-\mathbf{i}\lambda)} - \frac{\frac{d}{d\lambda} h_n^{(1)}(-\mathbf{i}\lambda)}{h_n^{(1)}(-\mathbf{i}\lambda)} > 0.$$
(3.3)

*Proof.* The purpose is to show that

$$\Big(h_n^{(1)}(-\mathbf{i}\lambda)\frac{d}{d\lambda}h_{n+1}^{(1)}(-\mathbf{i}\lambda)-h_{n+1}^{(1)}(-\mathbf{i}\lambda)\frac{d}{d\lambda}h_n^{(1)}(-\mathbf{i}\lambda)\Big)\Big(h_{n+1}^{(1)}(-\mathbf{i}\lambda)h_n^{(1)}(-\mathbf{i}\lambda)\Big)^{-1}>0.$$

Introduce the functions

$$\xi_n(\lambda) := \frac{e^{\lambda}}{\lambda} R_n\left(-\frac{1}{2\lambda}\right), \ \eta_n(\lambda) := \lambda \xi_n(\lambda).$$

Then  $h_n^{(1)}(-\mathbf{i}\lambda) = (-\mathbf{i})^n \xi_n(\lambda)$  and the above inequality is equivalent to

$$\left(\xi_{n}(\lambda)\frac{d}{d\lambda}\xi_{n+1}(\lambda) - \xi_{n+1}(\lambda)\frac{d}{d\lambda}\xi_{n}(\lambda)\right)\left(\xi_{n+1}(\lambda)\xi_{n}(\lambda)\right)^{-1}$$

$$= \left(\eta_{n}(\lambda)\frac{d}{d\lambda}\eta_{n+1}(\lambda) - \eta_{n+1}(\lambda)\frac{d}{d\lambda}\eta_{n}(\lambda)\right)\left(\eta_{n+1}(\lambda)\eta_{n}(\lambda)\right)^{-1} > 0.$$

Since  $\eta_n(\lambda)\eta_{n+1}(\lambda) > 0$  for  $\lambda < 0$ , it suffices to show that the function

$$F(\lambda) = \eta_n(\lambda) \frac{d}{d\lambda} \eta_{n+1}(\lambda) - \eta_{n+1}(\lambda) \frac{d}{d\lambda} \eta_n(\lambda)$$

has positive values for  $\lambda \in (-\infty, 0)$ . Consider the derivative

$$F'(\lambda) = \eta_n(\lambda) \frac{d^2}{d\lambda^2} \eta_{n+1}(\lambda) - \eta_{n+1}(\lambda) \frac{d^2}{d\lambda^2} \eta_n(\lambda).$$

We have

$$\eta_n(\lambda) = \mathbf{i}^{n+1} h_n^{(1)}(-\mathbf{i}\lambda)(-\mathbf{i}\lambda) = \mathbf{i}^{n+1} \Xi_n(-\mathbf{i}\lambda) = -\mathbf{i}^{n-1} \Xi_n(-\mathbf{i}\lambda).$$

The function  $\Xi_n(z) = zh_n^{(1)}(z)$  satisfies the equation

$$\Xi_n''(z) + \left(1 - \frac{n^2 + n}{z^2}\right)\Xi_n(z) = 0$$

and

$$\frac{d^2}{d\lambda^2}\eta_n(\lambda) = \mathbf{i}^{n-1}\Xi_n''(-\mathbf{i}\lambda) = -\mathbf{i}^{n-1}\left(1 + \frac{n^2 + n}{\lambda^2}\right)\Xi_n(-\mathbf{i}\lambda)$$
$$= \left(1 + \frac{n^2 + n}{\lambda^2}\right)\eta_n(\lambda).$$

Consequently,

$$F'(\lambda) = \left[\frac{(n+1)^2 + n + 1}{\lambda^2} - \frac{n^2 + n}{\lambda^2}\right] \eta_n(\lambda) \eta_{n+1}(\lambda)$$
$$= 2(n+2) \frac{\eta_n(\lambda) \eta_{n+1}(\lambda)}{\lambda^2} > 0.$$

On the other hand,

$$F(\lambda) = e^{\lambda} R_n \left( -\frac{1}{2\lambda} \right) \frac{d}{d\lambda} \left( e^{\lambda} R_{n+1} \left( -\frac{1}{2\lambda} \right) \right) - e^{\lambda} R_{n+1} \left( -\frac{1}{2\lambda} \right) \frac{d}{d\lambda} \left( e^{\lambda} R_n \left( -\frac{1}{2\lambda} \right) \right)$$
$$= \frac{e^{2\lambda}}{2\lambda^2} \left[ R_n \left( -\frac{1}{2\lambda} \right) R'_{n+1} \left( -\frac{1}{2\lambda} \right) - R_{n+1} \left( -\frac{1}{2\lambda} \right) R'_n \left( -\frac{1}{2\lambda} \right) \right]$$

and

$$\lim_{\lambda \to -\infty} F(\lambda) = 0, \lim_{\lambda \nearrow 0} F(\lambda) = +\infty$$

since

$$\lim_{w \to +\infty} \left[ R_n(w) R'_{n+1}(w) - R_{n+1}(w) R'_n(w) \right] = +\infty.$$

Finally, the function  $F(\lambda)$  in the interval  $(-\infty, 0]$  is increasing from 0 to  $+\infty$  and this completes the proof.

Now if  $\lambda_n < 0$  is a solution the equation

$$g_n(\lambda) := \frac{1}{\lambda} + \left(\frac{d}{d\lambda} h_n^{(1)}(-\mathbf{i}\lambda)\right) (h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \gamma = 0, \tag{3.4}$$

one has

$$g_{n+1}(\lambda_n) = \frac{1}{\lambda_n} + \left(\frac{d}{d\lambda}h_{n+1}^{(1)}(-i\lambda_n)\right)(h_{n+1}^{(1)}(-i\lambda_n))^{-1} - \gamma = G_{n,n+1}(\lambda_n) > 0,$$

so  $\lambda_n$  is not a root of the equation

$$g_{n+1}(\lambda) = \frac{1}{\lambda} + \left(\frac{d}{d\lambda}h_{n+1}^{(1)}(-\mathbf{i}\lambda)\right)(h_{n+1}^{(1)}(-\mathbf{i}\lambda))^{-1} - \gamma = 0.$$

In the following we assume that  $\gamma > 1$ . Then for  $\lambda \to -\infty$  we have  $g_{n+1}(\lambda) \to 1 - \gamma < 0$ , and since  $g_{n+1}(\lambda_n) > 0$  the equation  $g_{n+1}(\lambda) = 0$  has at least one root  $-\infty < \lambda_{n+1} < \lambda_n$ .

**Lemma 3.1.** Let  $\gamma > 1$ . For every  $n \ge 1$  the equation  $g_n(\lambda) = 0$  in the interval  $(-\infty, 0)$  has exactly one root  $\lambda_n < 0$ .

*Proof.* Setting  $w = -\frac{1}{2\lambda}$ , we write the equation (3.2) as  $\mathcal{R}_n(w) := w^2 R_n'(w) + \alpha R_n(w) = 0$ , where  $\alpha = \frac{1-\gamma}{2} < 0$ . We will show that this equation has exactly one positive root. Since

$$w^{2}R'_{n}(w) = \sum_{k=1}^{n} k a_{k,n} w^{k+1}, \ R_{n}(w) = \sum_{k=0}^{n} a_{k,n} w^{k},$$

the polynomial  $\mathcal{R}_n(w)$  has the representation

$$\mathcal{R}_n(w) = \sum_{k=0}^{n+1} b_{k,n} w^k$$

with

$$\begin{cases} b_{k,n} = (k-1)a_{k-1,n} + \alpha a_{k,n}, \ 0 \le k \le n, \ a_{-1,n} = 0, \\ b_{n+1,n} = \frac{(2n)!}{(n-1)!}. \end{cases}$$

Taking into account the form of  $a_{k,n}$ , we deduce

$$b_{k,n} = \frac{(n+k-1)!}{(n-k+1)!k!} \Big( k(k-1) + \alpha(n+k)(n-k+1) \Big), \ 0 \le k \le n+1.$$
 (3.5)

Thus the sign of  $b_{k,n}$  depends on the sign of the function

$$B(k) := (1 - \alpha)k^{2} + (\alpha - 1)k + \alpha(n^{2} + n)$$

which for  $k \geq 1$  is increasing since

$$B'(k) = 2(1 - \alpha)k + \alpha - 1 \ge 1 - \alpha > 0.$$

Clearly,  $b_{0,n} = \alpha < 0$  and  $b_{n+1,n} > 0$ . There are two cases:

- (i)  $b_{1,n} \leq 0$ . Then there is only one change of sing in the Descartes' sequence  $\{b_{n+1,n},b_{n,n},...,b_{1,n},b_{0,n}\}.$
- (ii)  $b_{1,n} > 0$ . Then  $b_{k,n} > 0$  for  $1 \le k \le n+1$  and in the Descartes' sequence  $\{b_{n+1,n},b_{n,n},...,b_{1,n},b_{0,n}\}$  one has again only one change of sign.

Applying the Descartes' rule of signs, we conclude that the number of the positive roots of  $\mathcal{R}_n(w) = 0$  is exactly one.

Combining Proposition 3.1 and Lemma 3.1, one obtain the following

Corollary 3.1. Let  $\gamma > 1$ . Then the generator  $G_b$  has an infinite sequence of real eigenvalues

$$-\infty < \dots < \lambda_n < \dots < \lambda_2 < \lambda_1 < 0$$

and  $\lambda_n$  has geometric multiplicity 2n+1.

The last statement concerns the geometric multiplicity since the functions  $\{Y_{m,n}(\omega)\}_{m=-n}^m$  are linearly independent. The algebraic multiplicity of  $\lambda_m$  will be discussed in Section 5.

## 4. Estimation of the roots

Throughout this section we assume  $\gamma > 1$ . Set  $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ ,  $0 < h \ll 1$  with  $z = -1 + \mathbf{i}\eta$ ,  $0 \le |\eta| \le h^{1/2}$ ,  $\eta \in \mathbb{R}$ . Consider the Dirichlet problem

$$\begin{cases} (h^2 \Delta + z)w = 0, \ |x| > 1, w \in H^2(|x| > 1), \\ w = f, \ |x| = 1 \end{cases}$$
 (4.1)

and note that  $\Delta + \frac{z}{h^2} = \Delta - \lambda^2$ . The solution of (4.1) has the form

$$w(r\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)} (-\mathbf{i}\lambda r) (h_n^{(1)} (-\mathbf{i}\lambda)^{-1} \alpha_{n,m} Y_{n,m}(\omega),$$

where

$$f(\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n,m} Y_{n,m}(\omega).$$

The semiclassical Dirichlet-to-Neumann operator  $\mathcal{N}_{ext}(h,z) = \frac{h}{\mathbf{i}} \frac{d}{dr} w|_{r=1}$  related to (4.1) becomes

$$\mathcal{N}_{ext}(h,z) = -\mathbf{i}\sqrt{z}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(h_n^{(1)})'(-\mathbf{i}\lambda)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1}\alpha_{n,m}Y_{n,m}$$
$$= \sqrt{z}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{d}{d\lambda}\Big(h_n^{(1)}(-\mathbf{i}\lambda)\Big)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1}\alpha_{n,m}Y_{n,m}.$$

By using the approximation of  $\mathcal{N}_{ext}(h,z)$  established in [8],[6] for  $z=-1+\mathbf{i}\eta$ , one deduces

$$\|\mathcal{N}_{ext}(h,z)f - Op_h(\rho)f\|_{L^2(\mathbb{S}^2)} \le C \frac{|\sqrt{z}|}{|\lambda|} \|f\|_{L^2(\mathbb{S}^2)}, \ 0 < h \le h_0$$

with  $\rho = \sqrt{z - r_0(x', \xi')}$  and a constant C > 0 independent of  $z, \lambda$  and f. Here  $r_0(x', \xi')$  is the principal symbol of the semiclassical Laplace-Beltrami operator  $-h^2\Delta_{\mathbb{S}^2} = \frac{z}{\lambda^2}\Delta_{\mathbb{S}^2}$ . Moreover,  $\sqrt{z} = \mathbf{i}\sqrt{1 - \mathbf{i}\eta} = \mathbf{i}(1 - \frac{\mathbf{i}\eta}{2} + \mathcal{O}(\eta^2))$  and

$$\operatorname{Re} \lambda = -\frac{1}{h} + \mathcal{O}(1), \operatorname{Im} \lambda = \mathcal{O}(h^{-1/2}).$$

Hence, for  $0 < h \le h_0$  we get

$$\lambda \in \Lambda_0 = \{ z \in \mathbb{C} : |\operatorname{Im} z| \le ch_0^{1/2} |\operatorname{Re} z|, \operatorname{Re} \lambda < -\epsilon < 0, |\lambda| \ge \lambda_0 \}.$$

On the other hand,

$$\left\| Op_h(\rho) - \sqrt{z} \left( \sqrt{1 - \frac{\Delta_{\mathbb{S}^2}}{\lambda^2}} \right) \right\|_{L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)} \le C_1 |\lambda|^{-1}, \ \lambda \in \Lambda_0.$$

Applying the spectral theorem, one deduces

$$\left(\sqrt{1 - \frac{\Delta_{S^2}}{\lambda^2}}\right) f = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\sqrt{1 + \frac{n(n+1)}{\lambda^2}}\right) \alpha_{n,m} Y_{n,m}$$

and

$$\left\| \left( \mathcal{N}_{ext}(h, -z) - \sqrt{z} \left( \sqrt{1 - \frac{\Delta_{S^2}}{\lambda^2}} \right) f \right\|_{L^2(S^2)}^2 = |z| \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left| \frac{d}{d\lambda} \left( h_n(-i\lambda) \right) (h_n(-i\lambda))^{-1} - \sqrt{1 + \frac{n(n+1)}{\lambda^2}} \right|^2 |a_{n,m}|^2.$$

This implies

$$\left| \frac{d}{d\lambda} \left( h_n^{(1)}(-\mathbf{i}\lambda) \right) (h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \sqrt{1 + \frac{n(n+1)}{\lambda^2}} \right| \le C_2 |\lambda|^{-1}, \, \forall n \in \mathbb{N}, \, \lambda \in \Lambda_0$$
 (4.2)

which we write as

$$\left| \left[ \frac{1}{\lambda} + \frac{d}{d\lambda} \left( h_n^{(1)} (-\mathbf{i}\lambda) \right) (h_n^{(1)} (-\mathbf{i}\lambda))^{-1} - \gamma \right] - \left[ \sqrt{1 + \frac{n(n+1)}{\lambda^2}} - \gamma \right] \right| \le C_0 |\lambda|^{-1}. \tag{4.3}$$

**Remark 4.1.** For bounded  $1 \le n \le N_0$  and sufficiently large  $|\lambda|$  the estimate (4.2) follows easily from the fact that  $\frac{R'_n(w)}{R_n(w)} = n(n+1) + \mathcal{O}(|w|)$  as  $|w| \to 0$ .

**Remark 4.2.** The estimate (4.2) is similar to that in Proposition 2.1 in [7], where the function  $\frac{J'_{\nu}(\lambda)}{J_{\nu}(\lambda)}$  for  $\nu \geq 0$  and  $0 < C \leq |\operatorname{Im} \lambda| \leq \delta |\operatorname{Re} \lambda|$ ,  $\operatorname{Re} \lambda > C_1$  has been approximated. Here  $J_{\nu}(z)$  is the Bessel function, while the boundary problem examined in [7] is in the domain |x| < 1.

Put  $z = \lambda$  and for  $z \in \Lambda_0$  consider the function

$$f_n(z) := \sqrt{1 + \frac{n(n+1)}{z^2}} - \gamma$$

with zeros

$$z_n^{\pm} = \pm \sqrt{\frac{n^2 + n}{\gamma^2 - 1}}.$$

In the following we set  $z_n = -\sqrt{\frac{n(n+1)}{\gamma^2 - 1}}$ . Clearly,

$$f'_n(z) = -\frac{1}{z} \frac{\frac{n(n+1)}{z^2}}{\sqrt{1 + \frac{n(n+1)}{z^2}}}$$

and  $\frac{n(n+1)}{z_n^2} = \gamma^2 - 1$ ,  $f'_n(z_n) = -\frac{\gamma^2 - 1}{\gamma z_n}$ . A calculus yields the second derivative

$$f_n''(z) = \frac{1}{z^2} \left[ \frac{3n(n+1)}{z^2} \left( \sqrt{1 + \frac{n(n+1)}{z^2}} \right) - \frac{n^2(n+1)^2}{z^4} \left( \sqrt{1 + \frac{n(n+1)}{z^2}} \right)^{-1/2} \right] \left( 1 + \frac{n(n+1)}{z^2} \right)^{-1}.$$

For n large enough and a > 0 to be fixed below introduce the contour

$$C_n(a) := \{ z = z_n + ae^{\mathbf{i}\varphi}, \ 0 \le \varphi < 2\pi \} \subset \Lambda_0.$$

Our purpose is to choose a so that

$$|f_n(z)| \ge \frac{C_0}{|z|}, \quad \forall z \in C_n(a). \tag{4.4}$$

We have

$$z^2 = z_n^2 + 2z_n a e^{\mathbf{i}\varphi} + a^2 e^{2\mathbf{i}\varphi}$$

and

$$\frac{n(n+1)}{z^2} = (\gamma^2 - 1)\left(1 + \mathcal{O}\left(\frac{1}{n}\right)a + \mathcal{O}\left(\frac{1}{n^2}\right)a^2\right)^{-1}, \ z \in C_n(a). \tag{4.5}$$

On the other hand,

$$\sqrt{\frac{n(n+1)}{z^2}+1} = \left[\frac{\gamma^2 + \mathcal{O}\left(\frac{1}{n}\right)a + \mathcal{O}\left(\frac{1}{n^2}\right)a^2}{1 + \mathcal{O}\left(\frac{1}{n}\right)a + \mathcal{O}\left(\frac{1}{n^2}\right)a^2}\right]^{1/2}.$$

Clearly, one has the estimate

$$|f_n(z)| \ge \frac{\gamma^2 - 1}{\gamma |z_n|} a - \frac{a^2}{2} \sup_{z \in C_n(a)} |f_n''(z)|, \ z \in C_n(a).$$

$$(4.6)$$

Set  $C_{\gamma} = \frac{\gamma^2 - 1}{\gamma} > 0$  and choose a > 0 so that  $C_{\gamma} a > 4C_0$ . We fix a and obtain

$$\frac{C_{\gamma}a}{2|z_n|} > \frac{2C_0}{|z_n|} > \frac{C_0}{|z_n||1 + \frac{ae^{i\varphi}}{z_n}|}, \ 0 \le \varphi < 2\pi,$$

taking n large enough to satisfy the inequality

$$\frac{1}{\left|1 + \frac{ae^{i\varphi}}{z_n}\right|} < 2.$$

Next we arrange the inequality

$$\frac{C_{\gamma}a}{2|z_n|} - \frac{a^2}{2} \sup_{z \in C_n(a)} |f_n''(z)| > 0.$$
(4.7)

It is clear that

$$f_n''(z) = \frac{1}{z^2} G\left(\frac{n(n+1)}{z^2}\right),$$

where

$$G(\zeta) = \left[ 3\zeta \sqrt{\zeta + 1} - \zeta^2 (\zeta + 1)^{-1/2} \right] (\zeta + 1)^{-1}.$$

Note that for  $z\in C_n(a)$  and n large enough according to (4.4), the function  $|G(\frac{n(n+1)}{z^2})|$  is bounded by a constant  $B_{\gamma,a}$  depending on  $\gamma$  and a. Thus for large n we get

$$\sup_{z \in C_n(a)} |f_n''(z)| \le B_{\gamma,a} \sup_{z \in C_n(a)} \frac{1}{|z|^2} = B_{\gamma,a} \frac{1}{|z_n|^2} \sup_{z \in C_n(a)} \frac{1}{|1 + \frac{ae^{i\varphi}}{z_n}|^2} \le 4B_{\gamma,a} \frac{1}{|z_n|^2}$$

and the proof of (4.7) is reduced to

$$C_{\gamma} > 4B_{\gamma,a} \frac{a}{|z_n|}$$

which is satisfied taking again n large. Finally, we proved the estimate (4.3) and we can apply Rouché theorem for the functions  $g_n(z)$  and  $f_n(z)$  and conclude that the function  $g_n(z)$  has exactly one simple zero  $\lambda_n$  in  $C_n(a)$ . Since  $g_n(z)$  has only real zeros (see Appendix in [2]), this implies the following

**Lemma 4.1.** There exist  $n_0(\gamma)$  and  $a(\gamma) > 0$  depending on  $\gamma$  such that for  $n \ge n_0(\gamma)$  the negative root  $\lambda_n$  of the equation (3.2) satisfies the estimate

$$\left|\lambda_n + \sqrt{\frac{n(n+1)}{\gamma^2 - 1}}\right| \le a(\gamma). \tag{4.8}$$

**Remark 4.3.** According to Proposition 2.1,  $n_0(\gamma)$  must satisfy the inequality

$$n_0(\gamma) \ge \frac{\sqrt{\gamma^2 - 1}}{\max\{\gamma - 1, \sqrt{\gamma - 1}\}}.$$

## 5. Weyl asymptotics

We start with the analysis of the multiplicity of  $\lambda_n$ .

**Lemma 5.1.** For  $n \ge n_0(\gamma)$  we have  $\operatorname{mult}(\lambda_n) = 2n + 1$ .

*Proof.* Since the geometric multiplicity of  $\lambda_n$  is 2n+1, it is sufficient to show that

$$\operatorname{mult}(\lambda_n) \le 2n + 1. \tag{5.1}$$

Let  $\lambda \in \Lambda_0$ , where  $\Lambda_0$  is the set introduced in the previous section and let  $\lambda \notin \sigma(G_b)$ . If  $0 \neq (f,g) \in \text{Image } G_b \cap L^2(\Omega)$ , one has div f = div g = 0 and for  $(u,v) = (G_b - \lambda)^{-1}(f,g)$  we get div u = div v = 0. Consider the skew self-adjoint operator

$$A = \begin{pmatrix} 0 & -\text{curl} \\ \text{curl} & 0 \end{pmatrix}$$

with boundary condition  $\nu \wedge u = 0$  on  $\mathbb{S}^2$ . Then  $\sigma(A) \subset \mathbf{i}\mathbb{R}$  and let  $(u_0(x; \lambda), v_0(x; \lambda)) = (A - \lambda)^{-1}(f, g)$ , that is

$$\begin{cases} (A - \lambda) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \text{ for } |x| > 1, \\ \nu \wedge u_0 = 0 \text{ on } \mathbb{S}^2. \end{cases}$$
 (5.2)

Since div  $u_0 = \text{div } v_0 = 0$ , the well known coercive estimates yield  $(u_0, v_0) \in H^1(\Omega)$ . Moreover the resolvent  $(A - \lambda)^{-1}$  is analytic in  $\{z \in \mathbb{C} : \text{Re } z < 0\}$  and

 $u_0(x;\lambda)$ ,  $v_0(x;\lambda)$  depend analytically on  $\lambda$ . We write  $(u,v)=(u_0,v_0)+(u_1,v_1)$ , where  $(u_1(x;\lambda),v_1(x;\lambda))$  is the solution of the problem

$$\begin{cases} (G - \lambda) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } |x| > 1, \\ (u_1)_{tan} - \gamma(\nu \wedge (v_1)_{tan}) = -\gamma(\nu \wedge (v_0)_{tan}(x; z)) \text{ on } \mathbb{S}^2. \end{cases}$$

$$(5.3)$$

To solve (5.3), note that  $-\gamma(\nu \wedge (v_0)_{tan}(\omega; z)) = F(\omega; \lambda) \in L^2(\mathbb{S}^2)$  with  $F(\omega; \lambda)$  analytical in  $\lambda$  for  $\lambda \in \Lambda_0$ . Thus we may write

$$F(\omega; \lambda) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \tilde{\alpha}_{n}^{m}(\lambda) U_{n}^{m}(\omega) + \tilde{\beta}_{n}^{m}(\lambda) V_{n}^{m}(\omega)$$

with analytical coefficients  $\tilde{\alpha}_n^m(\lambda)$ ,  $\tilde{\beta}_n^m(\lambda)$ . Now we can solve (2.5), (2.6) with right hand part  $(\tilde{\alpha}_n^m(\lambda), \tilde{\beta}_n^m(\lambda))$ . Finally, we obtain a representation of the solution of (5.3) with meromorphic coefficients

$$\alpha_n^m(\lambda) = \frac{\tilde{\alpha}_n^m(\lambda)}{h_n^{(1)}(-\mathbf{i}\lambda)\left[1 + \frac{d}{dr}(h_n^{(1)}(-\mathbf{i}\lambda r))|_{r=1}(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \lambda\gamma\right]},$$

$$\beta_n^m(\lambda) = -\frac{\lambda\tilde{\beta}_n^m(\lambda)}{\gamma h_n^{(1)}(-\mathbf{i}\lambda)\left[1 + \frac{d}{dr}(h_n^{(1)}(-\mathbf{i}\lambda r))|_{r=1}(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \lambda\gamma^{-1}\right]}.$$

If  $\gamma > 1$  the analysis in the previous section shows that for  $\lambda \in \Lambda_0$  the meromorphic function  $\alpha_n^m(\lambda)$  has a simple pole at  $\lambda_n < 0$ , while  $\beta_n^m(\lambda)$  is analytic in  $\Lambda_0$ . For  $0 < \gamma < 1$  the function  $\alpha_n^m(\lambda)$  is analytic in  $\Lambda_0$  and  $\beta_n^m(\lambda)$  is meromorphic. Next we integrate  $(u(x;\lambda),v(x;\lambda))$  over the circle  $|\lambda_n - \lambda| = \epsilon$ , where  $\epsilon$  is sufficiently small. The integral of  $(u_0(x;\lambda),v_0(x;\lambda))$  vanish, while for the integral of  $(u_1(x;\lambda),v_1(x;\lambda))$ , taking into account the representation of the solution of (5.3), we will obtain a sum

$$S_{n} = \begin{cases} c_{n} \sum_{m=-n}^{m} \tilde{\alpha}_{n}^{m}(\lambda_{n}) U_{n}^{m}(\omega), \ c_{n} \neq 0, \ \gamma > 1, \\ d_{n} \sum_{m=-n}^{m} \lambda_{n} \tilde{\beta}_{n}^{m}(\lambda_{n}) \gamma^{-1} V_{n}^{m}(\omega), \ d_{n} \neq 0, \ 0 < \gamma < 1. \end{cases}$$

This completes the proof of (5.1)

Passing to the analysis of N(r), consider first the case  $\gamma > 1$ . The root  $\lambda_n$  has algebraic multiplicity 2n+1 and to find a lower bound of N(r) we apply the estimate

$$|\lambda_n| \le \sqrt{\frac{n(n+1)}{\gamma^2 - 1}} + a(\gamma) < \frac{n+1}{\sqrt{\gamma^2 - 1}} + a(\gamma) \le r$$

for  $r \ge a(\gamma) + \frac{n_0(\gamma)+1}{\sqrt{\gamma^2-1}}$ . Then

$$N(r) \ge \sum_{j=n_0(\gamma)}^{[(r-a(\gamma))\sqrt{\gamma^2-1}-1]} (2j+1) = (\gamma^2-1)r^2 + \mathcal{O}_{\gamma}(r) + A_{\gamma}.$$

To get a upper bound for N(r), we use the estimate

$$|\lambda_n| \ge \sqrt{\frac{n(n+1)}{\gamma^2 - 1}} - a(\gamma) > \frac{n}{\sqrt{\gamma^2 - 1}} - a(\gamma) \ge r$$

for

$$n \ge (r + a(\gamma))\sqrt{\gamma^2 - 1} \ge 2a(\gamma)\sqrt{\gamma^2 - 1} + n_0(\gamma) + 1,$$

hence

$$N(r) \le \sum_{j=n_0(\gamma)}^{[(r+a(\gamma))\sqrt{\gamma^2-1}]+1} (2j+1) + D_{\gamma} = (\gamma^2-1)r^2 + \mathcal{O}_{\gamma}(r) + A'_{\gamma}.$$

If  $0 < \gamma < 1$ , we have  $\frac{1}{\gamma} > 1$  and one applies our argument to the equation (2.6). This completes the proof of theorem 1.1

#### References

- F. Colombini, V. Petkov and J. Rauch, Spectral problems for non elliptic symmetric systems with dissipative boundary conditions, J. Funct. Anal. 267 (2014), 1637-1661.
- [2] F. Colombini, V. Petkov and J. Rauch, Eigenvalues for Maxwell's equations with dissipative boundary conditions, Asymptotic Analysis, 99 (1-2) (2016), 105-124.
- [3] A. Kirsch and F. Hettlich, *The Mathematical Theory of Time-Harmonic Maxwells Equations*, vol. 190 of Applied Mathematical Sciences, Springer, Switzerland, 2015.
- [4] P. Lax and R. Phillips, Scattering theory for dissipative systems, J. Funct. Anal. 14 (1973), 172-235.
- [5] F. Olver, Asymptotics and Special Functions, Academic Press, New York, London, 1974.
- [6] V. Petkov, Location of the eigenvalues of the wave equation with dissipative boundary conditions, Inverse Problems and Imaging, 10 (4) (2016), 1111-1139.
- [7] V. Petkov and G. Vodev, Localization of the interior transmission eigenvalues for a ball, Inverse Problems and Imaging, 11 (2) (2017), 355-372.
- [8] G. Vodev, Transmission eigenvalue-free regions. Commun. Math. Phys. 336 (2015), 1141-1166.

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