High regularity of the solution to the singular elliptic $p(\cdot)$-Laplacian system.

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Abstract We study the regularity properties of solutions to the non-homogeneous singular $p(x)$-Laplacian system in a bounded domain of $\mathbb{R}^n$. Under suitable restrictions on the range of $p(x)$, we construct a $W^{2,r}$ solution, with $r > n$, that implies the Hölder continuity of the gradient. Moreover, assuming just $p(x) \in (1, 2)$ we prove that the second derivatives belong to L^2.

Keywords: $p(x)$-Laplacian system, higher integrability, global regularity, variable exponents spaces.

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1 Introduction

In this note we consider the elliptic $p(\cdot)$-Laplacian boundary value problem, namely

$$(1.1)\qquad \begin{cases} -\nabla \cdot (|\nabla u|^{p(x)-2}\nabla u) = f & \text{in } E\,, \\ u = 0 & \text{on } \partial E\,, \end{cases} \qquad 1 < p_- \leq p(x) \leq p_+ < 2$$

where $u : E \subset \mathbb{R}^n \to \mathbb{R}^N$, $n \geq 2$, $N \geq 1$, is a vector field, E is a C^2-smooth bounded domain, and $p : \overline{E} \longrightarrow (1, 2)$ is a given function belonging to $C^{0,1}(\overline{E})$.

We are interested in global regularity properties of solutions of (1.1). This issue naturally fits into the framework of the regularity of minimizers of functionals with non-standard growth. There are several regularity results on this topic, particularly in the scalar case, starting with the papers by Zhikov [33], Fan and Zhao [21, 22] and Marcellini [29] for (p-q) growth problems. Concerning the vectorial case, Coscia and Mingione [12] proved the local Hölder continuity of the gradient of the solution, and later Acerbi and Mingione [3] proved the partial Hölder continuity of the gradient of the solution for more general functionals too. We also mention that a similar kind of system governs the steady motion of electrorheological fluids (see [31, 30]). In this connection we recall paper [4] on partial $C^{1,\alpha}$-regularity of weak solutions, the paper by Diening, Ettwein and Růžička [19] for $C^{1,\alpha}_{loc}$-regularity in the two-dimensional case, the paper by the present authors [14] where $C^{1,\alpha}$-regularity up to the boundary and global $W^{2,2}$ summability is obtained for small data, and, finally, the recent paper by Sin [32], where boundary partial regularity for weak solutions is established. We

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refer to [24] for an overview on regularity questions related to this problem and for a wide list of references.

In this paper we prove the first result of global $C^{1,\alpha}$-regularity for solutions of (1.1), that is obtained as a consequence of the summability of the second derivatives. In this respect we want to recall the papers [7, 17, 8] where, in the case of constant p and with different boundary conditions, similar results are obtained. Actually, we prove the high regularity of the weak solution, in the following sense:

Definition 1.1. *Given a distribution f, by high-regular solution of system* (1.1) *we mean a field $u \in W_0^{1,p(\cdot)}(E) \cap W^{2,r}(E)$, for some $r > n$, such that*

$$\int_E |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = \int_E f \cdot \varphi \, dx, \text{ for all } \varphi \in C_0^\infty(E)\,.$$

More precisely, we show the following results.

Theorem 1.1. *Let E be a bounded domain of class C^2 and $p \in C^{0,1}(\overline{E})$, $1 < p_- \leq p(x) \leq p_+ < 2$. Set*

$$\widehat{r} \begin{cases} = \dfrac{2n}{n(p_- - 1) + 2(2 - p_-)}, & \text{if } n \geq 3, \\ \in \left(2, \dfrac{2}{p_- - 1}\right), & \text{if } n = 2\,, \end{cases} \tag{1.2}$$

and assume that $f \in L^{\widehat{r}}(E) \cap L^{p'(\cdot)}(E)$. Then, denoting by u the unique weak solution of (1.1)*, the following regularity estimates hold*

$$\|\nabla u\|_{p(\cdot)} \leq c \left(1 + \|f\|_{p'(\cdot)}^{\frac{1}{p_- - 1}}\right),$$

$$\|D^2 u\|_2 \leq c \left(1 + \|f\|_{p'(\cdot)}^{\frac{1}{p_- - 1}} + \|f\|_{p'(\cdot)}^{\frac{\beta}{p_- - 1}} + \|f\|_{\widehat{r}}^{\frac{1}{p_- - 1}}\right)$$

with β defined in (3.14).

Theorem 1.2. *Let E be a bounded domain of class C^2. Assume that $f \in L^r(E) \cap L^{p'(\cdot)}(E)$, with $r \in (n, \infty)$. Then, there exists $\overline{p} \in (1, 2)$, depending on r, such that, if $p(x) \in (\overline{p}, 2)$, the unique weak solution of* (1.1) *is a high-regular solution u, in the sense of Definition 1.1. Moreover the following estimates hold true with a constant c not depending on u*

$$\|\nabla u\|_{p(\cdot)} \leq c \left(1 + \|f\|_{p'(\cdot)}^{\frac{1}{p_- - 1}}\right), \tag{1.3}$$

$$\|D^2 u\|_r \leq c \left(\|f\|_r^{\frac{1}{p_- - 1}} + \|f\|_{p'(\cdot)}^{\frac{1}{p_- - 1}}\right). \tag{1.4}$$

Theorems 1.1 and 1.2 are the counterpart of the regularity results obtained in [15] for the classical p-Laplacian problem. On the other hand, for now we are forced to limit ourselves to consider a bounded domain, since in the case of exterior domains, also considered in [15] (see also [16]), we are not able to make the estimates independent of the size of the domain. An interesting extension would be not only to exterior domains but also to domain with minimal regularity assumptions as in [11]. Besides their own interest, we observe that Theorems 1.1 and 1.2 offer a way to handle several related parabolic problems. We mention, for instance, the high regularity of solutions to the $p(t,x)$-Laplacian systems, like in [10], where the authors prove the local Hölder continuity of the solutions, or, arguing as in [16], the extinction in a finite time of the motion of an electro-rheological fluid, extending to the $p(t,x)$-Stokes system the result obtained in [1] for the $p(x)$-Stokes system. Another interesting issue is the existence of time periodic solutions to the $p(t,x)$-Navier-Stokes problem, to fill the gap between the existence of time periodic solutions obtained in [13] only for the $p(t,x)$-Stokes problem, and the existence of time periodic solutions obtained in [6, 2] just in the case of the p-Navier-Stokes equations.

The restriction on the exponent p in Theorem 1.2 arises from some L^r-estimates for second order derivatives of solutions to the Dirichlet problem for the Poisson equation in bounded domains. We point out that, on the contrary, the L^2-regularity result for second derivatives is established without any restriction on the image of $p(x)$, which has only to be contained in the interval $(1,2)$.

Finally, we spend a few words about the technique. We begin by considering a μ-non singular approximation of the system and, employing some known $W^{2,2}$ regularity results for non singular systems, we get an L^2 estimate for the second derivatives of its solutions, uniformly in μ. Hence we may pass to the limit to get the same result for solutions to the singular problem (Theorem 1.1). This is the subject of Section 3. Starting from the $W^{2,2}$ regularity of the approximating system, in Section 4, we use a bootstrap argument to increase the summability of the second derivatives of the approximating system, until we reach L^n and, finally L^r, $r > n$. We like to point out that, while for the $W^{2,2}$ regularity we argue following the way in [15], here we need to proceed in a different way since, as far as we know, we do not have at disposal the Hölder continuity of the gradient of the solution of the non-singular approximating system with variable exponent. In the last section, we pass to the limit to cover also the singular case (Theorem 1.2).

2 Notation and some preliminary results

Throughout the paper, E will denote a bounded domain in $\mathbb{R}^n$, $n \geq 2$ with boundary of class C^2 and $|E|$ its Lebesgue measure. We assume that the exponent function $p : \overline{E} \longrightarrow \mathbb{R}$ is a $C^{0,1}$ function with

$$1 < p_- := \min p(x) \leq \max p(x) =: p_+ < 2.$$

We use the summation convention on repeated indices. For a vector valued function, by $\partial_j v_i$ we mean $\frac{\partial v_i}{\partial x_j}$. If v and w are two vector fields, by $w \cdot \nabla v$ we mean $w_j \partial_j v_i$ and, by the symbol $(\nabla v \otimes \nabla v)\, D^2 w$, we mean $\partial_j v_i\, \partial_k v_h \partial^2_{jk} w_h$.

We shall use the lower case letter c to denote a positive constant whose value can change even in the same line, since it is unessential for our aims.

By $L^r(E)$ and $W^{m,r}(E)$, m nonnegative integer and $r \in [1,\infty]$, we denote the usual Lebesgue and Sobolev spaces, with norms $\|\cdot\|_r$ and $\|\cdot\|_{m,r}$, respectively.

By $L^{p(\cdot)}(E)$ and $W^{m,p(\cdot)}(E)$ we denote the variable Lebesgue and Sobolev spaces, with norms $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|_{m,p(\cdot)}$. In most cases, when the meaning is clear, we will omit the explicit dependence on the variable x writing simply p instead of $p(\cdot)$. For the reader's convenience we quote some basic results concerning variable Lebesgue and Sobolev spaces, referring, for more specific issues, to the monographs [18] and [20], keeping in mind that in the present paper $p \in C^{0,1}(\overline{E})$ and $1 < p(x) < 2$.

For any $v \in L^{p(\cdot)}(E)$ we denote the *modular* of v by

$$\rho_p(v) = \int_E |v(x)|^{p(x)}\, dx.$$

Lemma 2.1 ([18, Corollary 2.23]). *For any $f \in L^{p(\cdot)}$ it results*

$$\|f\|_p \leq 1 + \rho_p(f)^{\frac{1}{p_-}},$$

$$\rho_p(f) \leq \left(1 + \|f\|_p^{p_+}\right).$$

Lemma 2.2 ([18, Corollary 2.50]). *If $|E| < +\infty$ then there exist constants $c_1, c_2 > 0$ such that*

$$c_1\|f\|_{p_-} \leq \|f\|_p \leq c_2\|f\|_{p_+}.$$

Theorem 2.3 (Hölder's inequality [18, Theorem 2.26]). *For any $f \in L^{p(\cdot)}(E)$ and $g \in L^{p'(\cdot)}(E)$, with $1 < p(x) < +\infty$ and $p'(x) = \frac{p(x)}{p(x)-1}$, $fg \in L^1(E)$ and*

$$\|fg\|_1 \leq \left(\frac{1}{p_-} + \frac{1}{p_+} + 1\right)\|f\|_p\|g\|_{p'}.$$

Corollary 2.4 ([18, Corollary 2.7]). *If $f \in L^{q(\cdot)}(E)$, $g \in L^{r(\cdot)}(E)$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, then $fg \in L^{p(\cdot)}(E)$ and*

$$\|fg\|_p \leq c\|f\|_q\|g\|_r.$$

We also recall some results, concerning linear elliptic systems.

Lemma 2.5. *Assume that $v \in W_0^{1,q}(E) \cap W^{2,q}(E)$ with $q > 1$. Then, we have*

$$\|D^2 v\|_q \leq K(q)\|\Delta v\|_q\,, \tag{2.1}$$

where the constant $K(q)$ depends on q and E. In the case $q = 2$ it holds

$$\|D^2 v\|_{2,E} \leq \|\Delta v\|_{2,E} + C\|\nabla v\|_{2,\partial E}. \tag{2.2}$$

with C depending on the curvature of ∂E.

Proof. For the proof of (2.1), we refer to [5], [25]. Concerning the estimate (2.2) see [26, §1.1.5] and [27, §3.8]. We remark that the result is not explicitly stated in [26, §1.1.5], but, from the proof therein, we can infer that $\|D^2 v\|_{2,E}$ can be controlled by $\|\Delta v\|_{2,E}$ and by the boundary integral of the normal derivative multiplied by a term $\kappa(\partial E)$ involving the curvature, namely

$$\|D^2 v\|_{2,E}^2 = \|\Delta v\|_{2,E}^2 + \int_{\partial E} (n \cdot \nabla v)^2 \kappa(\partial E) ds. \tag{2.3}$$

□

Lemma 2.6 ([23, Theorem 7.3])**.** *Let v be a $W^{2,2}(E)$-solution of the linear system*

$$A_{i\alpha j\beta}\partial^2_{\alpha\beta}v_j = f_i,$$

with $A_{i\alpha j\beta} \in C(\overline{E})$ satisfying the Legendre-Hadamard condition. If f belongs to $L^q(E)$, for some $q \geq 2$, then $D^2v \in L^q(E)$, with

$$\|D^2v\|_q \leq C(q,n,L,\omega)\left(\|f\|_q + \|D^2v\|_2\right),$$

where L is the constant of the Legendre-Hadamard condition and ω is the modulus of continuity of A.

For the reader's convenience, we prove the following uniqueness result for solutions of the linear elliptic problem

$$\begin{cases} (\delta_{ij}\delta_{\alpha\beta} + A_{i\alpha j\beta}(x))\partial^2_{\alpha\beta}v_j = F_i & \text{in } E \\ v = 0 & \text{on } \partial E \end{cases} \tag{2.4}$$

Lemma 2.7. *Let $A \in L^\infty(E)$ be such that system (2.4) is elliptic. Let $F \in L^2(E)$. If $K(2)\|A\|_\infty < 1$, there exists at most one function $v \in W^{1,2}_0(E) \cap W^{2,2}(E)$ satisfying problem (2.4).*

Proof. Assume $\widetilde{v} \in W^{1,2}_0(E) \cap W^{2,2}(E)$ is another solution of system (2.4), and let $V = \widetilde{v} - v$. Then V satisfies

$$\begin{cases} (\delta_{ij}\delta_{\alpha\beta} + A_{i\alpha j\beta}(x))\partial^2_{\alpha\beta}V_j = 0 & \text{in } E \\ V = 0 & \text{on } \partial E. \end{cases} \tag{2.5}$$

Multiplying by ΔV_i, integrating in E, and recalling estimate (2.1), we get

$$\|\Delta V\|_2^2 = \int_E A_{i\alpha j\beta}\partial^2_{\alpha\beta}V_j\partial^2_{hh}V_i dx \leq \|A\|_\infty\|D^2V\|_2\|\Delta V\|_2 \leq \|A\|_\infty K(2)\|\Delta V\|_2^2$$

from which $V \equiv 0$ easily follows. □

3 First step in regularity

We introduce the following auxiliary problem

$$\begin{cases} -\dfrac{\Delta u}{(\mu + |\nabla u|^2)^{\frac{2-p}{2}}} - (p-2)\dfrac{(\nabla u \otimes \nabla u)D^2u}{(\mu + |\nabla u|^2)^{\frac{4-p}{2}}} \\ \qquad\qquad -\dfrac{\log\left(\mu + |\nabla u|^2\right)\nabla u \nabla p}{(\mu + |\nabla u|^2)^{\frac{2-p}{2}}} = f \text{ in } E \\ \qquad\qquad\qquad\qquad\qquad u = 0 \text{ on } \partial E. \end{cases} \tag{3.1}$$

Proposition 3.1. *Let $\mu \in (0,1]$, $p \in C^{0,1}(\overline{E})$, $1 < p_- \leq p(x) \leq p_+ < 2$ and let $f \in L^{\widehat{r}}(E) \cap L^{p'}(E)$, with $\widehat{r}$ defined in (1.2). Then, there exists a solution $u_\mu \in W^{1,2}_0(E) \cap W^{2,2}(E)$ of system (3.1). Moreover, the following estimates hold*

$$\|\nabla u_\mu\|_p \leq c\left(1 + \|f\|_{p'}^{\frac{1}{p_- - 1}} + \mu^{\frac{1}{2}}|E|^{\frac{1}{p_-}}\right), \tag{3.2}$$

$$\|D^2u_\mu\|_2 \le c\Big(1+\mu^{\frac{2-p_+}{2}}\|f\|_{p'}+\mu^{\frac12}|E|^{\frac{1}{p_-}}+\mu^{\frac{\beta}{2}}|E|^{\frac{\beta}{p_-}} + \|f\|_{p'}^{\frac{1}{p_--1}}+\|f\|_{p'}^{\frac{\beta}{p_--1}}+\|f\|_{\widehat{r}}^{\frac{1}{p_--1}}\Big) \tag{3.3}$$

with β defined in (3.14).

Proof. Let us consider the following auxiliary problem for fixed $\varepsilon > 0$

$$\begin{aligned} -\varepsilon\Delta u - \nabla\cdot\Big((\mu+|\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u\Big) &= f \quad \text{in } E,\\ u &= 0 \quad \text{on } \partial E. \end{aligned} \tag{3.4}$$

By [9, Theorem 8.2], as $f \in L^2(E)$ we determine the solution $u_\varepsilon \in W^{2,2}(E)$ of the above problem. Multiplying (3.4) by u_ε and integrating over E we get

$$\varepsilon\|\nabla u_\varepsilon\|_2^2 + \int_E \frac{|\nabla u_\varepsilon|^2}{(\mu+|\nabla u_\varepsilon|^2)^{\frac{2-p}{2}}}\,dx = \int_E f\cdot u_\varepsilon\,dx.$$

By Lemma 2.1, the Poincaré (see [20, Theorem 8.2.4]) and Young inequalities, it follows that, for any $0<\delta\le 1$,

$$\begin{aligned} &\int_E \frac{|\nabla u_\varepsilon|^2}{(\mu+|\nabla u_\varepsilon|^2)^{\frac{2-p}{2}}}\,dx \le c\|f\|_{p'}\|u_\varepsilon\|_p \le c\|f\|_{p'}\|\nabla u_\varepsilon\|_p\\ &\le c\|f\|_{p'}\Big(1+\rho_p(\nabla u_\varepsilon)^{\frac{1}{p_-}}\Big) \le C(\delta)\Big(\|f\|_{p'}^{(p_-)'}+1\Big)+\delta\rho_p(\nabla u_\varepsilon). \end{aligned}$$

Using the above estimate together with the Young inequality, since $\mu \le 1$, we have

$$\begin{aligned} \rho_p(\nabla u_\varepsilon) &= \int_{\{|\nabla u_\varepsilon|^2\ge\mu\}} |\nabla u_\varepsilon|^p\,dx + \int_{\{|\nabla u_\varepsilon|^2<\mu\}} |\nabla u_\varepsilon|^p\,dx\\ &\le 2^{\frac{2-p_-}{2}}\int_E \frac{|\nabla u_\varepsilon|^2}{(\mu+|\nabla u_\varepsilon|^2)^{\frac{2-p}{2}}}\,dx + \mu^{\frac{p_-}{2}}|E|\\ &\le 2^{\frac{2-p_-}{2}}\Big(C(\delta)\Big(\|f\|_{p'}^{(p_-)'}+1\Big)+\delta\rho_p(\nabla u_\varepsilon)\Big)+\mu^{\frac{p_-}{2}}|E|\\ &\le \delta\rho_p(\nabla u_\varepsilon)+C(\delta)\Big(\|f\|_{p'}^{(p_-)'}+1\Big)+\mu^{\frac{p_-}{2}}|E| \end{aligned}$$

hence

$$\rho_p(\nabla u_\varepsilon) \le c\Big(1+\|f\|_{p'}^{(p_-)'}+\mu^{\frac{p_-}{2}}|E|\Big), \tag{3.5}$$

with the constant c independent of ε and μ. Hence, by Lemma 2.2 and Lemma 2.1, we have that

$$\|\nabla u_\varepsilon\|_{p_-} \le c\|\nabla u_\varepsilon\|_{p(\cdot)} \le c\Big(1+\rho_p(\nabla u_\varepsilon)^{\frac{1}{p_-}}\Big) \le c\Big(1+\|f\|_{p'}^{\frac{1}{p_--1}}+\mu^{\frac12}|E|^{\frac{1}{p_-}}\Big) \tag{3.6}$$

with c not depending on ε and μ.

Considering that $u_\varepsilon \in W_0^{1,2}(E)\cap W^{2,2}(E)$ we can write equation (3.4) in a strong form obtaining

$$\begin{aligned} -\varepsilon\Delta u_\varepsilon - \frac{\Delta u_\varepsilon}{(\mu+|\nabla u_\varepsilon|^2)^{\frac{2-p(x)}{2}}} - (p(x)-2)\frac{(\nabla u_\varepsilon\otimes\nabla u_\varepsilon)D^2u_\varepsilon}{(\mu+|\nabla u_\varepsilon|^2)^{\frac{4-p(x)}{2}}}&\\ -\frac{\log\big(\mu+|\nabla u_\varepsilon|^2\big)\nabla u_\varepsilon\nabla p}{(\mu+|\nabla u_\varepsilon|^2)^{\frac{2-p(x)}{2}}} = f \quad \text{a.e. in } E.& \end{aligned} \tag{3.7}$$

Multiplying equation (3.7) by $(\mu+|\nabla u_\varepsilon|^2)^{\frac{2-p}{2}}$, taking the $L^2(E)$-norm of both sides, keeping in mind that ∇p is bounded and that $p'>2$, we get

(3.8)
$$\begin{aligned}
&\|\Delta u_\varepsilon\|_2 \le \|\Delta u_\varepsilon(1+\varepsilon(\mu+|\nabla u_\varepsilon|^2)^{\frac{2-p}{2}})\|_2\\
&\le (2-p_-)\|D^2u_\varepsilon\|_2 + \|(\mu+|\nabla u_\varepsilon|^2)^{\frac{2-p}{2}} f\|_2 + \|\log\left(\mu+|\nabla u_\varepsilon|^2\right)\nabla u_\varepsilon \nabla p\|_2\\
&\le (2-p_-)\|D^2u_\varepsilon\|_2 + \|f|\nabla u_\varepsilon|^{2-p}\|_2 + c\mu^{\frac{2-p_+}{2}}\|f\|_{p'} + c\|\log\left(\mu+|\nabla u_\varepsilon|^2\right)\nabla u_\varepsilon\|_2.
\end{aligned}$$

To estimate the last term, we will use the following inequalities, which hold true for any $\alpha>0$:

$$|\log t| \le \begin{cases} \frac{1}{e\alpha t^\alpha}, & t\in(0,1),\\ \frac{t^\alpha}{e\alpha}, & t\ge 1.\end{cases} \tag{3.9}$$

Choosing $\alpha=\frac{2-p_-}{2}$, we have

$$\begin{aligned}
&\int\limits_{\{\mu+|\nabla u_\varepsilon|^2<1\}} |\log\left(\mu+|\nabla u_\varepsilon|^2\right)\nabla u_\varepsilon|^2\,dx\\
&\le \frac{4}{e^2(2-p_-)^2}\int\limits_{\{\mu+|\nabla u_\varepsilon|^2<1\}} |\nabla u_\varepsilon|^{p_-}\,dx \le c(1+\rho_p(\nabla u_\varepsilon)).
\end{aligned}\tag{3.10}$$

Moreover, for any $\alpha>0$ and $0<\mu\le\frac{1}{2}$, we have

$$\begin{aligned}
&\int\limits_{\{\mu+|\nabla u_\varepsilon|^2\ge1\}} |\log\left(\mu+|\nabla u_\varepsilon|^2\right)\nabla u_\varepsilon|^2\,dx\\
&\le \frac{1}{(e\alpha)^2}\int\limits_{\{\mu+|\nabla u_\varepsilon|^2\ge1\}} \left(\mu+|\nabla u_\varepsilon|^2\right)^{2\alpha}|\nabla u_\varepsilon|^2\,dx\\
&\le \frac{4^\alpha}{(e\alpha)^2}\int\limits_{\{\mu+|\nabla u_\varepsilon|^2\ge1\}} |\nabla u_\varepsilon|^{4\alpha+2}\,dx \le c\|\nabla u_\varepsilon\|_{2+4\alpha}^{2+4\alpha}.
\end{aligned}\tag{3.11}$$

Now we remark that, choosing α small enough it results $p_-<2+4\alpha<2^*$, hence, for suitable $\theta\in(0,1)$, namely $\theta=\frac{n(2+4\alpha-p_-)}{(1+2\alpha)(2n-np_-+2p_-)}$, we have

$$\|\nabla u_\varepsilon\|_{2+4\alpha} \le \|\nabla u_\varepsilon\|_{2^*}^\theta\|\nabla u_\varepsilon\|_{p_-}^{1-\theta} \le c\|D^2u_\varepsilon\|_2^\theta\|\nabla u_\varepsilon\|_{p_-}^{1-\theta}.$$

If we choose $\alpha<\frac{2p_-}{n}$ we have that $\theta(1+2\alpha)<1$ hence we can apply the Young inequality to get

$$\|\nabla u_\varepsilon\|_{2+4\alpha}^{1+2\alpha} \le \delta\|D^2u_\varepsilon\|_2 + C(\delta)\|\nabla u_\varepsilon\|_{p_-}^{\frac{(1-\theta)(1+2\alpha)}{1-\theta(1+2\alpha)}}. \tag{3.12}$$

for any $\delta>0$. Hence, by estimates (3.10), (3.11), (3.12), (3.6) and (3.5), we have

(3.13)

$$
\begin{aligned}
&\|\log\left(\mu+|\nabla u_\varepsilon|^2\right)\nabla u_\varepsilon\|_2\\
&\le c(1+\rho_p(\nabla u_\varepsilon))^{\frac12}+\delta\|D^2u_\varepsilon\|_2+C(\delta)\left(1+\|f\|_{p'}^{\frac{1}{p_--1}}+\mu^{\frac12}|E|^{\frac{1}{p_-}}\right)^{\frac{(1-\theta)(1+2\alpha)}{1-\theta(1+2\alpha)}}\\
&\qquad\qquad\le \delta\|D^2u_\varepsilon\|_2+C(\delta)\left(1+\|f\|_{p'}^{\frac{1}{p_--1}}+\mu^{\frac12}|E|^{\frac{1}{p_-}}\right)^{\beta}
\end{aligned}
$$

with

$$
\beta=\max\left\{\frac{(1-\theta)(1+2\alpha)}{1-\theta(1+2\alpha)},\frac{p_-}{2}\right\}. \tag{3.14}
$$

Substituting the above inequality in (3.8), we get

$$
\begin{aligned}
\|\Delta u_\varepsilon\|_2\le(2-p_-+\delta)\|D^2u_\varepsilon\|_2+\|f|\nabla u_\varepsilon|^{2-p}\|_2+c\mu^{\frac{2-p_+}{2}}\|f\|_{p'}\\
+C(\delta)\left(1+\|f\|_{p'}^{\frac{1}{p_--1}}+\mu^{\frac12}|E|^{\frac{1}{p_-}}\right)^{\beta}.
\end{aligned} \tag{3.15}
$$

In order to estimate the L^2-norm of D^2u_ε, we use Lemma 2.5, which yields

$$
\|D^2u_\varepsilon\|_{2,E}\le\|\Delta u_\varepsilon\|_{2,E}+C\|\nabla u_\varepsilon\|_{2,\partial E}. \tag{3.16}
$$

To estimate the boundary term we make use of a Gagliardo-Nirenberg's type inequality (see [27, Ch. 2, (2.25)]), to get

$$
\|\nabla u_\varepsilon\|_{2,\partial E}\le C(\sigma)\|\nabla u_\varepsilon\|_{2,E}+\sigma\|D^2u_\varepsilon\|_{2,E} \tag{3.17}
$$

for any $\sigma>0$. Employing once again the Gagliardo-Nirenberg inequality with $a:=\frac{n(2-p_-)}{2n+2p_--np_-}$, and successively the Young inequality, we get

$$
\begin{aligned}
\|\nabla u_\varepsilon\|_2&\le c\left(\|D^2u_\varepsilon\|_2^a\|\nabla u_\varepsilon\|_{p_-}^{1-a}+\|\nabla u_\varepsilon\|_{p_-}\right)\\
&\le\gamma\|D^2u_\varepsilon\|_2+C(\gamma)\|\nabla u_\varepsilon\|_{p_-}.
\end{aligned} \tag{3.18}
$$

Substituting estimate (3.18) in (3.17), choosing $\gamma=\gamma(\sigma)$ small enough, by (3.16), we get

$$
\|D^2u_\varepsilon\|_2\le\|\Delta u_\varepsilon\|_2+2\sigma\|D^2u_\varepsilon\|_2+C(\sigma)\|\nabla u_\varepsilon\|_{p_-}
$$

hence

$$
\|D^2u_\varepsilon\|_2\le\frac{1}{1-2\sigma}\|\Delta u_\varepsilon\|_2+C(\sigma)\|\nabla u_\varepsilon\|_{p_-}. \tag{3.19}
$$

For the term $\|f|\nabla u_\varepsilon|^{2-p}\|_2$ in (3.8) we distinguish between $n=2$ and $n\ge3$. Let be $n\ge3$. By applying Corollary 2.4 with exponents $2,\widehat{r}$, $\frac{2n}{(n-2)(2-p_-)}$, we have

(3.20)

$$
\|f\,|\nabla u_\varepsilon|^{2-p}\|_2\le c\|f\|_{\widehat{r}}\,\|\,|\nabla u_\varepsilon|^{2-p}\,\|_{\frac{2n}{(n-2)(2-p_-)}}\le c\|f\|_{\widehat{r}}\left(1+\|\nabla u_\varepsilon\|_{\frac{2n}{n-2}}^{2-p_-}\right).
$$

If $n = 2$, we set $r = \frac{2\widehat{r}(2-p_-)}{\widehat{r}-2}$ and we apply Corollary 2.4 with exponents $2, \widehat{r}, \frac{r}{2-p_-}$ to get

$$\|f\,|\nabla u_\varepsilon|^{2-p}\|_2 \le \|\,f\|_{\widehat{r}}\,\|\,|\nabla u_\varepsilon|^{2-p}\,\|_{\frac{r}{2-p_-}} \le c\|\,f\|_{\widehat{r}}\left(1+\|\nabla u_\varepsilon\|_r^{2-p_-}\right) . \tag{3.21}$$

We set $\overline{r} = \frac{2n}{n-2}$, for $n \ge 3$, and $\overline{r} = r$, for $n = 2$. Since $\widehat{r} \in (2, \frac{2}{p_- -1})$, we have that $r > 2$ hence, for any $n \ge 2$, we can apply the Sobolev embedding theorem to obtain

$$\|\,\nabla u_\varepsilon\|_{\overline{r}} \le c(\|D^2 u_\varepsilon\|_2 + \|\nabla u_\varepsilon\|_2). \tag{3.22}$$

Interpolating L^2 between L^{p_-} and $L^{\overline{r}}$ and using the Young inequality we get that, for any $\delta > 0$,

$$\|\nabla u_\varepsilon\|_2 \le \delta\|\nabla u_\varepsilon\|_{\overline{r}} + C(\delta)\|\nabla u_\varepsilon\|_{p_-}. \tag{3.23}$$

If we choose a suitable δ in (3.23), and we replace (3.23) in (3.22), we get

$$\|\,\nabla u_\varepsilon\|_{\overline{r}} \le c(\|D^2 u_\varepsilon\|_2 + \|\nabla u_\varepsilon\|_{p_-}). \tag{3.24}$$

Hence, by applying the Young inequality with exponents $\frac{1}{2-p_-}$ and $\frac{1}{p_- -1}$, (3.20) (or (3.21)) and (3.24), we finally get, for any $\eta > 0$,

$$\begin{aligned} &\|f|\nabla u_\varepsilon|^{2-p}\|_2 \le \|f\|_{\widehat{r}}\left(\|\nabla u_\varepsilon\|_{\overline{r}}^{2-p_-} + 1\right) \\ &\le \eta\|D^2 u_\varepsilon\|_2 + C(\eta)\left(\|f\|_{\widehat{r}}^{\frac{1}{p_- -1}} + \|\nabla u_\varepsilon\|_{p_-} + 1\right). \end{aligned} \tag{3.25}$$

Therefore, by using estimate (3.25), (3.19) and (3.6) in (3.15), we obtain,

$$\|\Delta u_\varepsilon\|_2 \le \left(\frac{2-p_-+\delta}{1-2\sigma} + \eta\right)\|\Delta u_\varepsilon\|_2 + C(\sigma,\eta,\delta)\Phi(f,\mu)$$

where

$$\Phi(f,\mu) := 1+\mu^{\frac{2-p_+}{2}}\|f\|_{p'}+\mu^{\frac{1}{2}}|E|^{\frac{1}{p_-}}+\mu^{\frac{\beta}{2}}|E|^{\frac{\beta}{p_-}}+\|f\|_{p'}^{\frac{1}{p_- -1}}+\|f\|_{p'}^{\frac{\beta}{p_- -1}}+\|f\|_{\widehat{r}}^{\frac{1}{p_- -1}}.$$

Choosing σ, η and δ small enough we get

$$\|\Delta u_\varepsilon\|_2 \le c\,\Phi(f,\mu). \tag{3.26}$$

Hence, by (3.19), (3.26) and (3.6), we get

$$\|D^2 u_\varepsilon\|_2 \le c\,\Phi(f,\mu) \tag{3.27}$$

for any $\varepsilon > 0$. Therefore, with the aid of the Poincaré inequality, we get that

$$\|u_\varepsilon\|_{2,2} \le c\,\Phi(f,\mu),$$

uniformly in ε. It follows that we can find a function $u_\mu \in W^{2,2}(E) \cap W_0^{1,2}(E)$ such that, up to subsequences, $u_\varepsilon \rightharpoonup u_\mu$ weakly in $W^{2,2}(E)$ as $\varepsilon \to 0$. Moreover, by the Rellich-Kondrachov embedding theorem, we can suppose that

$$\nabla u_\varepsilon \to \nabla u_\mu \quad \text{almost everywhere in } E. \tag{3.28}$$

We remark that, by (3.5),

$$\rho_{p'}\left(\left(\mu+|\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}}\nabla u_\varepsilon\right)\le \rho_p(\nabla u_\varepsilon)\le c \tag{3.29}$$

uniformly in ε. By the above inequality, Lemma 2.2 and Lemma 2.1 we get

$$\begin{aligned}\left\|\left(\mu+|\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}}\nabla u_\varepsilon\right\|_{(p')_-} &\le c\left\|\left(\mu+|\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}}\nabla u_\varepsilon\right\|_{p'}\\ &\le c\left(\rho_{p'}\left(\left(\mu+|\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}}\nabla u_\varepsilon\right)^{\frac{1}{(p')_-}}+1\right)\le c\end{aligned} \tag{3.30}$$

hence, by [28, Lemma I.1.3], we infer that

$$(\mu+|\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}\nabla u_\varepsilon \rightharpoonup (\mu+|\nabla u_\mu|^2)^{\frac{p-2}{2}}\nabla u_\mu \qquad \text{in } L^{(p')_-}(E) \text{ as } \varepsilon\to 0. \tag{3.31}$$

Writing (3.4) in a weak formulation, we get

$$\int_E \varepsilon\nabla u_\varepsilon\cdot\nabla\phi dx+\int_E\left(\mu+|\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}}\nabla u_\varepsilon\cdot\nabla\phi\,dx=\int_E f\cdot\phi\,dx \qquad \forall\phi\in C_0^\infty(E).$$

Passing to the limit in ε, with the aid of (3.31), we get

$$\int_E\left(\mu+|\nabla u_\mu|^2\right)^{\frac{p-2}{2}}\nabla u_\mu\cdot\nabla\phi\,dx=\int_E f\cdot\phi\,dx \qquad \forall\phi\in C_0^\infty(E).$$

Since $u_\mu\in W^{2,2}(E)$, we can write once again the above system in a strong formulation to get that u_μ satisfies (3.1). Passing to the limit in (3.27) we get estimate (3.3). By the convergence (3.28) and the lower semicontinuity in variable Lebesgue spaces (see [20, Theorem 2.3.17]), we can pass to the limit in (3.6) to get (3.2). □

Proof of Theorem 1.1. For any $\mu\in(0,1]$ let u_μ be the solution of (3.1). By Lemma 2.2, estimates (3.2) and (3.3), we have that

$$\|\nabla u_\mu\|_{p_-}\le c\|\nabla u_\mu\|_p\le c\left(1+\|f\|_{p'}^{\frac{1}{p_--1}}\right), \tag{3.32}$$

$$\|D^2u_\mu\|_2\le c\left(1+\|f\|_{p'}^{\frac{1}{p_--1}}+\|f\|_{p'}^{\frac{\beta}{p_--1}}+\|f\|_{\widehat{r}}^{\frac{1}{p_--1}}\right). \tag{3.33}$$

By (3.32), (3.33) and the Rellich-Kondrachov theorem we can extract a subsequence (not relabeled) and find a function $u\in W^{2,2}(E)$ such that

$$u_\mu\rightharpoonup u \qquad \text{weakly in } L^{(p_-)_*}(E),$$

$$\nabla u_\mu\to\nabla u \qquad \text{a.e. in } E,$$

$$D^2u_\mu\rightharpoonup D^2u \qquad \text{weakly in } L^2(E).$$

By (3.1) we get

$$\int_E\left(\mu+|\nabla u_\mu|^2\right)^{\frac{p-2}{2}}\nabla u_\mu\cdot\nabla\phi\,dx=\int_E f\cdot\phi\,dx \qquad \forall\phi\in C_0^\infty(E). \tag{3.34}$$

Following the computations in (3.30) and (3.29), by Lemma 2.1 and (3.32), we have the estimate

$$\left\| \left(\mu + |\nabla u_\mu|^2\right)^{\frac{p-2}{2}} \nabla u_\mu \right\|_{(p')_-} \leq c(1+\rho_p(\nabla u_\mu)) \leq c(1+\|\nabla u_\mu\|_p^{p^+}) \leq c\left(1 + \|f\|_{p'}^{\frac{p_+}{p_- -1}}\right)$$

Applying [28, Lemma I.1.3] we deduce the weak convergence

$$\left(\mu + |\nabla u_\mu|^2\right)^{\frac{p-2}{2}} \nabla u_\mu \rightharpoonup |\nabla u|^{p-2} \nabla u \qquad \text{in } L^{(p')_-}(E).$$

Passing to the limit as $\mu \to 0$ in (3.34) we get that $u \in W^{2,2}(E)$ is a weak solution of (1.1) and it is unique, thanks to the strict monotonicity of the operator. Moreover, passing to the limit in (3.32) and (3.33) we obtain the estimates of Theorem 1.1. □

4 Additional regularity of the non-singular system

Our aim is to increase the summability of the second derivatives of u.

Let us define

$$p(s) := \max\left\{\frac{3}{2}, 2 - \frac{1}{K(s)}, 2 - \frac{1}{K(2)}\right\}, \tag{4.1}$$

with $K(s)$ introduced in Lemma 2.5.

Proposition 4.1. *Let $f \in C^\infty(\overline{E})$ and u be the solution obtained in Proposition 3.1. For any $q \in [2, n]$, let be $\tilde{q} = \frac{q+q^*}{2}$ if $q < n$, or any number strictly greater than n if $q = n$. If $D^2 u \in L^q(E)$, and $p_- \in (p(\tilde{q}), 2)$, then $D^2 u \in L^{\tilde{q}}(E)$. Moreover, if $\tilde{q} > n$, then the following estimate holds*

$$\|D^2 u\|_{\tilde{q}} \leq c\left(1 + \mu^{\frac{2-p_-}{2}} \|f\|_{\tilde{q}} + \|f\|_{\tilde{q}}^{\frac{1}{p_- -1}} + \|f\|_{p'}^{\frac{1}{p_- -1}} + \|D^2 u\|_q\right). \tag{4.2}$$

Proof. We multiply system (3.1) by $(\mu + |\nabla u|^2)^{\frac{2-p}{2}}$, obtaining

$$\begin{cases} \Delta u + (p-2)\dfrac{\nabla u \otimes \nabla u}{\mu + |\nabla u|^2} D^2 u = F \text{ in } E \\ u = 0 \text{ on } \partial E \end{cases} \tag{4.3}$$

with

$$F := (\mu + |\nabla u|^2)^{\frac{2-p}{2}} f + \log\left(\mu + |\nabla u|^2\right) \nabla u \nabla p.$$

For any $\eta > 0$, let J_η be a Friedrichs mollifier. We set $A_\eta = J_\eta\left(\frac{\nabla u \otimes \nabla u}{\mu + |\nabla u|^2}\right)$ and we consider the following linear elliptic system

$$\begin{cases} \left[\nabla \cdot (\nabla v + (p-2)(A_\eta \nabla v))\right]_i = F_i + (p-2)\partial_\alpha (A_\eta)_{ij}^{\alpha\beta} \partial_\beta v_j + \left[\nabla p A_\eta \nabla v\right]_i \text{ in } E \\ v = 0 \text{ on } \partial E. \end{cases} \tag{4.4}$$

By the Lax-Milgram theorem, the above system has a unique weak solution $v =: v^\eta \in W_0^{1,2}(E)$. Let us verify that the right-hand side of $(4.4)_1$ is in $L^2(E)$.

Since $u \in W^{2,2}(E)$ and $f \in C^\infty(\overline{E})$, then $(\mu + |\nabla u|^2)^{\frac{2-p}{2}} f \in L^2(E)$. Using inequality (3.9), like in (3.10) and (3.11), and remembering that $\nabla p \in L^\infty(E)$, we get that $\log\left(\mu + |\nabla u|^2\right) \nabla u \nabla p \in L^2(E)$. Finally, the term A_η is bounded and $v \in W_0^{1,2}(E)$, hence the claim is proved. Since $(p-2)A_\eta \in C^{0,1}(\overline{E})$, by [23, Theorem 4.14] we get that $v^\eta \in W^{2,2}(E)$ and we can write system (4.4) in the form

$$\begin{cases} \Delta v^\eta + (p-2) A_\eta D^2 v^\eta = F & \text{in } E \\ v^\eta = 0 & \text{on } \partial E. \end{cases} \tag{4.5}$$

To obtain the higher power of summability for $D^2 v^\eta$ we want to use Lemma 2.6, hence it is enough to prove that $F \in L^{\tilde{q}}(E)$.
By the Hölder inequality we have

$$\begin{aligned} \int_E \left| \left(\mu + |\nabla u|^2\right)^{\frac{2-p}{2}} f \right|^{\tilde{q}} dx &\le \int_E \left| \left(1 + \mu + |\nabla u|^2\right)^{\frac{2-p_-}{2}} f \right|^{\tilde{q}} dx \\ &\le c \left((1+\mu)^{\tilde{q}} \|f\|_{\tilde{q}}^{\tilde{q}} + \|f\|^{\tilde{q}}_{\frac{\tilde{q}}{p_- - 1}} \|\nabla u\|_{\tilde{q}}^{(2-p_-)\tilde{q}} \right). \end{aligned} \tag{4.6}$$

Moreover, since $\nabla p \in L^\infty(E)$ and using (3.9) with $\alpha < \min\left\{\frac{1}{2}, \frac{q^* - \tilde{q}}{2\tilde{q}}\right\}$, we have

$$\begin{aligned} &\|\log(\mu + |\nabla u|^2) \nabla u \nabla p\|_{\tilde{q}}^{\tilde{q}} \le c \|\log(\mu + |\nabla u|^2) \nabla u\|_{\tilde{q}}^{\tilde{q}} \\ &\le c \left(\int_{\{\mu + |\nabla u|^2 < 1\}} |\nabla u|^{\tilde{q}(1-2\alpha)}\, dx + \int_{\{\mu + |\nabla u|^2 \ge 1\}} |\nabla u|^{\tilde{q}(1+2\alpha)}\, dx \right) \\ &\qquad \le c \left(1 + \|\nabla u\|_{q^*}^{\tilde{q}}\right) \le c \left(1 + \|D^2 u\|_q^{\tilde{q}}\right) \end{aligned} \tag{4.7}$$

hence $F \in L^{\tilde{q}}(E)$ and, by Lemma 2.6, $D^2 v^\eta \in L^{\tilde{q}}(E)$. Unfortunately the norm of $D^2 v^\eta$ depends on the modulus of continuity of A_η and this prevents us to pass to the limit on η. We need a new estimate uniform with respect to η. To this purpose, we multiply the system (4.5) by a generic function $w \in L^{\tilde{q}'}(E)$, and, remarking that $\|A_\eta\|_\infty \le 1$, we obtain

$$|(\Delta v^\eta, w)| \le (2 - p_-) \|D^2 v^\eta\|_{\tilde{q}} \|w\|_{\tilde{q}'} + \|F\|_{\tilde{q}} \|w\|_{\tilde{q}'} \,.$$

Hence, by duality

$$\|\Delta v^\eta\|_{\tilde{q}} \le (2 - p_-) \|D^2 v^\eta\|_{\tilde{q}} + \|F\|_{\tilde{q}}.$$

By (2.1) we have

$$\|D^2 v^\eta\|_{\tilde{q}} \le K(\tilde{q}) \|\Delta v^\eta\|_{\tilde{q}} \,. \tag{4.8}$$

Since, by assumption,

$$p_- > p(\tilde{q}) \ge 2 - \frac{1}{K(\tilde{q})}, \tag{4.9}$$

then

$$\|\Delta v^\eta\|_{\tilde{q}} \le \frac{\|F\|_{\tilde{q}}}{1 - (2 - p_-) K(\tilde{q})}. \tag{4.10}$$

By (4.8) and (4.10), we obtain

$$\|D^2 v^\eta\|_{\tilde q} \le \frac{K(\tilde q)}{1-(2-p_-)K(\tilde q)}\|F\|_{\tilde q}, \qquad \forall \eta>0. \tag{4.11}$$

Since $v^\eta \in W_0^{1,2}(E)$, by the Sobolev embedding theorem and (4.11), we have that

$$\|v^\eta\|_{2,\tilde q} \le c\frac{K(\tilde q)}{1-(2-p_-)K(\tilde q)}\|F\|_{\tilde q},$$

uniformly with respect to η. Hence we can find $v \in W^{2,\tilde q}(E)\cap W_0^{1,2}(E)$ and a subsequence of $\{v^\eta\}$ (not relabeled) weakly converging to v in $W^{2,\tilde q}(E)$. We want to pass to the limit as $\eta \to 0$ in the system (4.5). To this purpose, let us multiply (4.5) by $\varphi \in C_0^\infty(E)$ and integrate over E. Since $\frac{\nabla u\otimes\nabla u}{\mu+|\nabla u|^2} \in L^\infty(E)$ we have

$$J_\eta\left(\frac{\nabla u\otimes\nabla u}{\mu+|\nabla u|^2}\right) \longrightarrow \frac{\nabla u\otimes\nabla u}{\mu+|\nabla u|^2} \qquad \text{strongly in } L^s(E)\ \forall s\ge 1.$$

By the weak convergence of $v^\eta \rightharpoonup v$ in $W^{2,\tilde q}(E)$, it follows that

$$\left(\Delta v + (p-2)\frac{\nabla u\otimes\nabla u}{\mu+|\nabla u|^2}D^2 v, \varphi\right) = (F,\varphi)$$

for any $\varphi \in C_0^\infty(E)$. It follows that v is a solution of system (4.3), like u is. Since $p_- > 2-\frac{1}{K(2)}$ we have that $K(2)\|(p-2)\frac{\nabla u\otimes\nabla u}{\mu+|\nabla u|^2}\|_\infty < 1$ hence, from Lemma 2.7, it follows that $u=v$.

Finally, we have to prove estimate (4.2) when $\tilde q > n$. In this case, we can reconsider our estimates using the fact that $\nabla u \in L^\infty(E)$. First we observe that, in place of (4.6) we can get

$$\left\| f\left(\mu+|\nabla u|^2\right)^{\frac{2-p}{2}}\right\|_{\tilde q} \le c\left(1+\mu^{\frac{2-p_-}{2}}\|f\|_{\tilde q} + \|\nabla u\|_\infty^{2-p_-}\|f\|_{\tilde q}\right). \tag{4.12}$$

Multiplying equation (3.1) by $(\mu+|\nabla u|^2)^{2-p}$, and taking the $L^{\tilde q}$-norm of both sides, by (4.12) and (4.7), we have

$$\begin{aligned} \|\Delta u\|_{\tilde q} \quad &\le (2-p_-)\left\|\frac{\nabla u\otimes\nabla u}{(\mu+|\nabla u|^2)}D^2u\right\|_{\tilde q} + \|\log\left(\mu+|\nabla u|^2\right)\nabla u\nabla p\|_{\tilde q} \\ &\qquad + \|f(\mu+|\nabla u|^2)^{\frac{2-p}{2}}\|_{\tilde q} \\ &\le (2-p_-)\|D^2u\|_{\tilde q} + c\left(1+\mu^{\frac{2-p_-}{2}}\|f\|_{\tilde q} + \|f\|_{\tilde q}\|\nabla u\|_\infty^{2-p_-} + \|D^2u\|_q\right). \end{aligned} \tag{4.13}$$

For the L^∞ norm of ∇u, we employ the Sobolev embedding theorem, the convexity of the norm, and then the Young inequality to get

$$\begin{aligned} \|\nabla u\|_\infty &\le c(\|D^2u\|_{\tilde q} + \|\nabla u\|_{\tilde q}) \le c(\|D^2u\|_{\tilde q} + \|\nabla u\|_\infty^\theta \|\nabla u\|_{p_-}^{1-\theta}) \\ &\le c\|D^2u\|_{\tilde q} + \delta\|\nabla u\|_\infty + c(\delta)\|\nabla u\|_{p_-}, \end{aligned}$$

with $\theta = \frac{\tilde q - p_-}{\tilde q}$. Hence, choosing a small $\delta>0$,

$$\|\nabla u\|_\infty \le c\left(\|D^2u\|_{\tilde q} + \|\nabla u\|_{p_-}\right).$$

Therefore, by (2.1) and the Young inequality, we obtain

$$\begin{aligned}\|f\|_{\tilde q}\|\nabla u\|_\infty^{2-p_-} &\le c\,\|f\|_{\tilde q}(\|\Delta u\|_{\tilde q}+\|\nabla u\|_{p_-})^{2-p_-}\\ &\le \varepsilon\|\Delta u\|_{\tilde q}+c(\varepsilon)\|f\|_{\tilde q}^{\frac{1}{p_- -1}}+c\|f\|_{\tilde q}\|\nabla u\|_{p_-}^{2-p_-}\,.\end{aligned}$$

Inserting this estimate in (4.13), we get

$$\begin{aligned}\|\Delta u\|_{\tilde q}\le\ &(2-p_-)\|D^2u\|_{\tilde q}+\varepsilon\|\Delta u\|_{\tilde q}+c(\varepsilon)\|f\|_{\tilde q}^{\frac{1}{p_- -1}}\\ &+c\left(1+\mu^{\frac{2-p_-}{2}}\|f\|_{\tilde q}+\|D^2u\|_q+\|f\|_{\tilde q}\|\nabla u\|_{p_-}^{2-p_-}\right)\,.\end{aligned}\tag{4.14}$$

By (2.1), we get

$$\begin{aligned}(1-(2-p_-)K(\tilde q)-\varepsilon)\,\|\Delta u\|_{\tilde q}\le c\left(1+\mu^{\frac{2-p_-}{2}}\|f\|_{\tilde q}+\|D^2u\|_q\right)&\\ +c(\varepsilon)\|f\|_{\tilde q}^{\frac{1}{p_- -1}}+c\|f\|_{\tilde q}\|\nabla u\|_{p_-}^{2-p_-}\,,&\end{aligned}$$

whence, by (2.1), applying the Young inequality, recalling estimate (3.2) and the assumption on p, we get (4.2). □

5 High regularity of the singular system

In this section we prove Theorem 1.2.

Let be $f\in C^\infty(\overline{E})$ and u_μ the solution of Proposition 4.1. We consider first the case $n=2$.

From Proposition 3.1, $u_\mu\in W^{2,n}(E)$. Then, setting $\overline{p}:=p(r)$, with $p(r)$ defined in (4.1), we can apply Proposition 4.1, with $\tilde q=r>n=2$ and $q=2$, to find

$$\|D^2u_\mu\|_r\le c\left(1+\mu^{\frac{2-p_-}{2}}\|f\|_r+\|f\|_r^{\frac{1}{p_- -1}}+\|f\|_{p'}^{\frac{1}{p_- -1}}+\|D^2u_\mu\|_2\right).\tag{5.1}$$

Now let be $n\ge 3$. We set, recursively,

$$q_0=2,\qquad q_{j+1}=\tilde q_j,\qquad \text{if } q_j<n$$

with $\tilde q$ defined in Proposition 4.1. We can find $k\in\mathbb{N}$ such that

$$q_{k-1}<n\le q_k,$$

hence, since, by Proposition 3.1, $u_\mu\in W^{2,2}(E)$, we can apply k times Proposition 4.1 to get that if, $\forall x\in E$,

$$p(x)>\overline{p}:=\max\{p(r),p(q_j),j=1,\cdots,k-1\},$$

then $D^2u_\mu\in L^n(E)$. Since $r>n$ we can apply for the last time Proposition 4.1 with $q=n$, $\tilde q=r$ to get that $D^2u_\mu\in L^r(E)$ and estimate (5.1) is satisfied. Observing that $r>n\ge\hat r$ if $n\ge 3$, and that we can choose $\hat r\le r$ if $n=2$, we get that $\|f\|_{\hat r}\le c\|f\|_r$ hence, by (3.3), we get that

(5.2)

$$\|D^2u_\mu\|_r\le c\left(1+\mu^\alpha\left(1+\|f\|_r+\|f\|_{p'}\right)+\|f\|_r^{\frac{1}{p_- -1}}+\|f\|_{p'}^{\frac{\beta}{p_- -1}}+\|f\|_{p'}^{\frac{1}{p_- -1}}\right)$$

for a suitable $\alpha > 0$. The last step consists in showing that, in the limit as μ goes to 0, u_μ tends to a function u, which is the high-regular solution of (1.1). By (3.2) and (5.2), the sequence $\{u_\mu\}$ is bounded in $L^{(p_-)^*}(E)$, $\{\nabla u_\mu\}$ is bounded in $L^{p_-}(E)$, and $\{D^2 u_\mu\}$ is bounded in $L^r(E)$. Moreover, being E a bounded set, we can apply the Rellich-Kondrachov theorem to extract a subsequence (not relabeled) and find a function u such that

$$u_\mu \rightharpoonup u \quad \text{weakly in } L^{(p_-)^*}(E),$$

$$\nabla u_\mu \rightharpoonup \nabla u \quad \text{weakly in } L^{p_-}(E),$$

$$D^2 u_\mu \rightharpoonup D^2 u \quad \text{weakly in } L^r(E),$$

$$\nabla u_\mu \longrightarrow \nabla u \text{ a.e. in } E.$$

Replicating the estimate (3.30) with u_μ in place of u_ε we get that the sequence $\left\{\left(\mu + |\nabla u_\mu|^2\right)^{\frac{p-2}{2}} \nabla u_\mu\right\}$ is bounded in $L^{(p')_-}(E)$ and, by [28, Lemma I.1.3],

$$\left(\mu + |\nabla u_\mu|^2\right)^{\frac{p-2}{2}} \nabla u_\mu \rightharpoonup |\nabla u|^{p-2}\nabla u \qquad \text{in } L^{(p')_-}(E). \tag{5.3}$$

Multiplying system (3.1) by an arbitrary function $\varphi \in C_0^\infty(E)$, we get

$$\int_E \left(\mu + |\nabla u_\mu|^2\right)^{\frac{p-2}{2}} \nabla u_\mu \cdot \nabla\varphi \, dx = \int_E f \cdot \varphi \, dx. \tag{5.4}$$

Passing to the limit as μ goes to 0, with the aid of (5.3), we obtain that u is a solution of (1.1).

The estimates (1.3) and (1.4) follow by the semicontinuity of the norm, passing to the limit respectively in (3.2) and (5.2).

Now we remove the smoothness hypothesis on f considering $f \in L^r(E) \cap L^{p'}(E)$.

We can find a sequence of functions $f_\varepsilon \in C^\infty(\overline{E})$ such that

$$\lim_{\varepsilon \to 0} f_\varepsilon = f \qquad \text{in } L^r(E) \cap L^{p'}(E)$$

and, for any $\varepsilon > 0$, we have the corresponding high-regular solution u_ε. Passing to the limit as ε goes to 0 in the sequence $\{u_\varepsilon\}$, with the same technique employed above, we easily find the existence of a high regular solution of (1.1). The strict monotonicity of the operator ensures that it coincides with the unique weak solution of (1.1).

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