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From Pseudo-Objects in Dynamic Explorations to Proof by Contradiction

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Abstract

Proof by contradiction presents various difficulties for students relating especially to the formulation and interpretation of a negation, the managing of impossible mathematical objects, and the acceptability of the validity of the statement once a contradiction has been reached from its negation. This article discusses how a Dynamic Geometry Environment (DGE) can contribute to students' argumentation processes when trying to explain contradictions. Four cases are presented and analysed, involving students from high school, as well as undergraduate and graduate students. The approach of the analyses makes use of a symbolic logical chain and the notion of pseudo-object. Such analyses lead to a hypothesis, that experiencing a pseudo-object during an exploration can foster DGE-supported processes of argumentation culminating in geometrical proofs by contradiction, while the lack of experience of a pseudo-object may hinder such processes. If this hypothesis is confirmed by further studies, we foresee important didactical implications since it sheds light on the transition from students' DGE-based argumentations to proofs by contradiction.

Keywords

Dynamic geometry
Indirect argument
Proof by contradiction
Pseudo-object

Previous studies have highlighted how a Dynamic Geometry Environment (DGE) can mediate students' proof processes (e.g. Laborde 2000; Mariotti 2000; de Villiers 2004; Sinclair and Robutti 2013), especially when the tasks foster students' reasoning and production of conjectures (e.g. Pedemonte 2007). Indeed, as asserted by Laborde and Laborde (1992), when speaking about Cabri (a specific DGE), reasoning processes are supported by the software, which results in variation to the solving process: "the changes in the solving process brought about by the dynamic possibilities of Cabri come from an active and reasoning visualisation, from what we call an interactive process between inductive and deductive reasoning" (p. 185).

Existing literature also serves to explain how certain argumentation processes potentially contribute to student production of proofs by contradiction (e.g. Leung and Lopez-Real 2002; Baccaglini-Frank et al. 2013), which is what we focus on in this article. Indeed, research centred on proof by contradiction has pointed to various difficulties present for students (see, for example, Leron 1985; Wu Yu et al. 2003; Antonini 2004; Antonini and Mariotti 2006, 2007, 2008; Mariotti and Antonini 2009). These challenges relate especially to the formulation and interpretation of a negation, the managing of impossible mathematical objects and the acceptability of the validity of the statement once a contradiction has been reached from its negation.

In managing mathematical objects, the "active and reasoning visualisation" offered by DGEs seems to yield great potential, because it allows students to see simultaneously the consequences of all the geometrical properties from which a figure was constructed, thereby maintaining theoretical control (Mariotti 2002) over the figure for the student. This should allow the student to allocate more cognitive resources to potentially conflicting properties in the case of an impossible mathematical object – that is, properties that cannot co-exist in a robustly constructed[1] dynamic figure. How students deal with the co-existence of such properties is what this article explores.

In particular, we illustrate the potential of a specific type of open problem, namely tasks that require the construction of a geometrical object that cannot exist within the theory of Euclidean geometry, tasks with respect to proof by contradiction (Baccaglini-Frank et al. 2013, 2017). Building on previous work (Leung and Lopez-Real 2002; Baccaglini-Frank et al. 2011), we analyse such potential through the notion of *pseudo-object*, "a geometrical figure associated to another geometrical figure either by construction or by projected-perception in such a way that it contains properties that are contradictory in the Euclidean theory" (Baccaglini-Frank et al. 2013, p. 65). We further elaborate on such a notion to illustrate better a DGE's potential to help students perceive and deal with contradictions as they engage in the exploration of a non-constructability problem. We present analyses of four cases in which students produced such arguments after a phase of dynamic exploration, making use of a symbolic logical chain to illustrate the emergence of pseudo-objects as a main feature of the potential of the proposed problems.

Conceptual Framework

The relations between argumentation and proof constitute one of the main issues in research in mathematics education (see, for example, Boero 2007; Hanna and de Villiers 2012; Stylianides et al. 2016). Many articles, based on different theoretical assumptions, have proposed different approaches and, therefore, different didactical implications. Some researchers (e.g. Duval 1993) have highlighted a separation between argumentation and proof, while others have focused on the analogies between argumentation and proof, seen as two processes (Boero et al. 1996; Garuti et al. 1996). In this second case, the main didactical implication is the importance for students of engaging in generating conjectures, in order to promote certain processes relevant to developing their competence at mathematical proving.

Here, we focus on the relationship between processes of argumentation and of proof in the specific case of proof by contradiction involving DGEs. First of all, we need to clarify terminology such as 'proof by contradiction', 'indirect proof', 'proof by contraposition', 'proof by *reductio ad absurdum*', etc., since such terms are not always used consistently, either by practicing mathematicians or textbooks. Given a statement S, a *proof by contradiction* is a direct proof of the statement $\neg S \rightarrow r \wedge \neg r$, where r is a previously proven theorem, an axiom or a proposition. If the statement S can be expressed as $p \rightarrow q$, since $p \rightarrow q$ is logically equivalent to $\neg p \vee q$, then the negation of $p \rightarrow q$ can be substituted by $\neg(\neg p \vee q)$ which is equivalent to $p \wedge \neg q$: then the negation of S is $p \wedge \neg q$. In this case, a proof by contradiction of S is a direct proof of $p \wedge \neg q \rightarrow r \wedge \neg r$. A *proof by contraposition* of S is a direct proof of $\neg q \rightarrow \neg p$.

We refer to an *indirect proof* as a proof of a statement in which the premise contains the negation of the conclusion. So, both proofs by contradiction and proofs by contraposition are indirect proofs, because they refer to statements that contain a negation ($\neg q$) in their premise.

Studies in mathematics education have revealed that proof by contradiction is a very complex activity for students, as mentioned above. However, some studies have also shown that students spontaneously produce arguments very similar to proofs by contradiction:

> The indirect proof is a very common activity ('Peter is at home since otherwise the door would not be locked'). A child who is left to himself with a problem starts to reason spontaneously '... if it were not so, it would happen that ...'. (Freudenthal 1973, p. 629)

In agreement both with Freudenthal and with the characterisation of indirect proof given above, we use *indirect argumentation* to refer to an argument stemming from assumptions that contain the negation of the statement to be argued, or the negation of part of such statement: that is, an argument with a structure like: "*... if it were not so, it would happen that ...*". (For a more articulated and refined definition, see Antonini 2010.)

Open construction problems, non-constructability problems and proof by contradiction

Construction problems constitute the core of classic Euclidean geometry. The use of specific artefacts, i.e. ruler and compass, can be considered as the origin of the set of axioms defining the theoretical system of Euclid's *Elements*. Any geometrical construction corresponds to a theorem. This means that there is a proof that validates the construction procedure that solves a corresponding construction problem. Thus, in classic Euclidean geometry, the *theoretical nature* of a geometrical construction is clearly stated (e.g. Vinner 1999), in spite of the apparent practical objective, i.e. the accomplishment of a drawing following the construction procedure. We note that the "non-constructability" of a figure may become manifest in two fundamentally different ways: (1) a figure, though existing, may be non-constructible with certain (pre-defined) tools, say with a straightedge and compass; (2) a figure's non-constructability may derive from the non-existence of the geometrical object *per se*: that is, from the contradiction that follows once its existence is presumed. Historically, there have been many examples of the first case, such as the trisection of an angle, the doubling of a cube or the squaring of a circle. Non-constructability of the second type, however, does not depend on the tools used to accomplish the construction, because it is a logical consequence within the theory of Euclidean geometry itself: if the figure were to exist, there would be a contradiction. This article considers this second type of non-constructability.

The problems we are concerned with are construction problems. If the construction *is* possible, we speak of *constructability problems*, while if the construction is *not* possible, of *non-constructability problems*. Clearly, the solver does not know initially whether the construction problem s/he is addressing is a constructability or non-constructability problem. In this article, the non-constructability problems dealt with are of the following type: the solver is asked whether a figure with prescribed properties is constructible or not, and, in either case, s/he is required to provide an argument to support the response. Usually, the solver attempts to construct the requested or hypothesised geometrical object. The solution can be provided either by producing the construction procedure and its validation according the theory available (in this case, Euclidean geometry) or by proving the fact that no construction procedure can be exhibited. This latter case, because of its very nature, may lead to an indirect argument, sowing seeds that may eventually lead to a proof by contradiction. A non-constructability statement expresses the fact that it is impossible to display a valid procedure for constructing a certain figure. By assuming that the negation of a certain property is always true, the solver proves that such property is not possible.

Pseudo-Objects

Within the very little literature in this area, a study conducted by Leung and Lopez-Real (2002) describes a proof by contradiction produced by two students working in Cabri (a DGE), which triggered the development of a framework of theorem acquisition and justification in a DGE. The researchers used such framework to describe a “scheme for ‘seeing’ proof by contradiction” (p. 150). Within this framework, the idea of a *pseudo-quadrilateral* (in this case, a quadrilateral that cannot exist, unless it is degenerate, in a figure containing all the required properties) was first introduced to *visualise a proof by contradiction* in a DGE. Figures 1 and 2 visually summarise the idea. Building on such introductory work, Baccaglini-Frank et al. (2013) extended this idea introducing the notion of a *pseudo-object*:

> A geometrical figure associated to another geometrical figure either by construction or by projected-perception in such a way that it contains properties that are contradictory in the Euclidean theory. (p. 65)

Fig. 1

A learner’s projected-perception on an arbitrary quadrilateral ABCD: a *pseudo-quadrilateral* EBFD associated with ABCD is constructed visually inheriting an “impossible” Euclidean property (Leung and Lopez-Real 2002, p. 155)

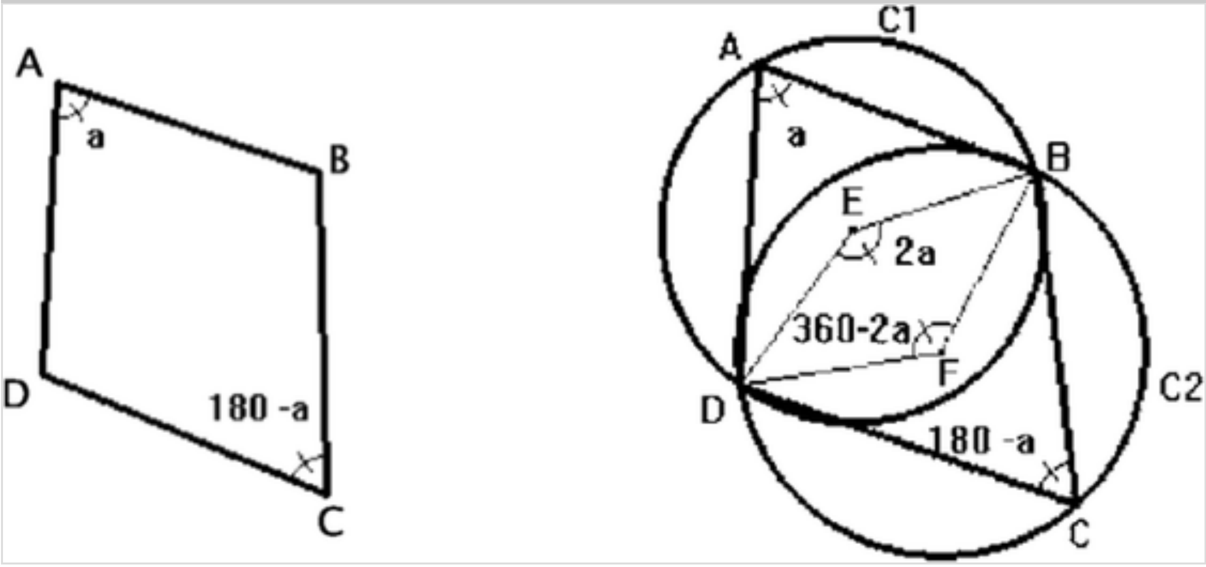
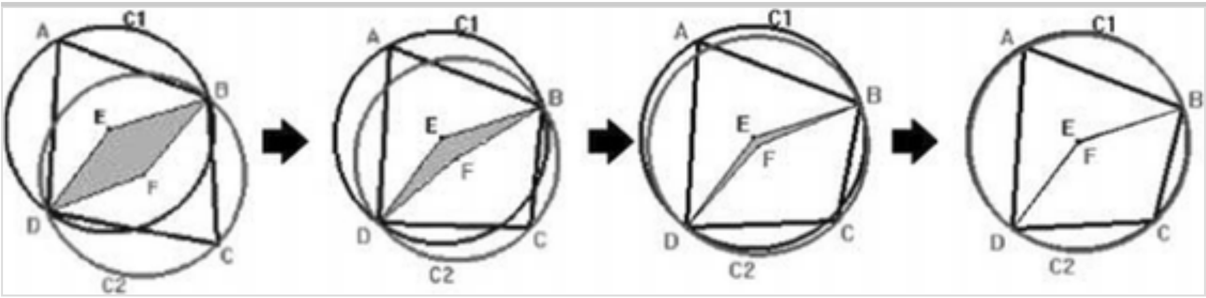


Fig. 2

By dragging the vertices of ABCD, the pseudo-quadrilateral EBFD can be made to “vanish” (i.e. degenerate into a linear object), thus realising a possible theorem associated with the imposed condition (Leung and Lopez-Real 2002, p. 157)



By “projected-perception”, we mean something visually perceived in the DGE that is affected by a mentally imposed condition (we discuss this in greater depth in the next section). Such a notion has been used to suggest that a pseudo-object can be conducive to developing a dialectic between dynamic visual reasoning in a DGE and theoretical reasoning in the Euclidean axiomatic system, eventually leading to a proof by contradiction.

Dragging and invariant properties of figures in a DGE

Any figure in a DGE that has been constructed using specific primitives can be *acted upon* through dragging, which determines the phenomenon of *moving figures*. A dragging exploration principle was proposed (Leung et al. 2013) to epitomise the DGE dragging phenomenon:

> During dragging, a figure maintains all the properties according to which it was constructed and all the consequences that the construction properties entail within the axiomatic world of Euclidean geometry. (p. 458)

As it has been previously discussed in the literature, the perception of a *moving figure* in a DGE is the phenomenon through which something about the figure changes while something else is preserved during dragging (Mariotti 2014). What is preserved during dragging (the *invariant*) becomes the identity of the object/figure, in contrast with what changes which determines its *variation* and consequently its movement.

As a figure is acted upon in a DGE, there are two kinds of invariants appearing simultaneously: the invariants determined by the geometrical relations defined by the commands used to construct the figure (which are called *direct invariants*) and the invariants that are derived consequently within Euclidean geometry (which are called *indirect invariants*) (Laborde and Sträßer 1990). The relationship of dependency between these two types of invariants constitutes a crucial point in the process of exploration using a DGE. The experience of dragging allows the user to interpret what appears on the screen in terms of logical consequences among geometrical properties: in particular, indirect invariants will be interpreted in terms of consequences of the direct invariants. Discerning invariants and invariant relations among invariants are cognitively different tasks (Leung et al. 2013). Simultaneous appearance of direct and indirect invariants during dragging leads to the possibility of perceiving the dependence of an indirect invariant (B) upon a set of direct invariants (A_1, A_2, …, A_n). We express this dependency relationship between (direct and indirect) invariants in the logical form: $A_1 \wedge A_2 \wedge \ldots \wedge A_n \Rightarrow B$. In the following section, we exemplify how direct invariants, indirect invariants and pseudo-objects come into dialectical play in the case of non-constructability problems.

Research Hypothesis and Methods

Over the last decade, we have conducted a number of studies to investigate processes of conjecture generation, argumentation and proof, within a DGE: in particular, we have collected data in Italy and in Hong Kong on students' solution processes when working on non-constructability problems. (Results have been published in: Leung and Lopez-Real 2002; Baccaglini-Frank 2010; Baccaglini-Frank and Mariotti 2010; Baccaglini-Frank et al. 2011, 2013; Leung et al. 2013; Baccaglini-Frank et al. 2017.) The research presented in this article stems from revisiting the part of this data regarding students' solution processes mobilised to solve non-constructability problems, one of which we will introduce in the next section. The practice of data revisitation has been suggested and used insightfully, for example, by Nachlieli and Tabach (2012), who revisited data collected during a project that took place over fifteen years previously. From revisiting our data in the light of the notion of pseudo-object, the following new hypothesis emerged, which lies at the heart of this article:

> *Experiencing a pseudo-object during an exploration can foster DGE-supported processes of argumentation, culminating in geometrical proofs by contradiction, while the lack of experience of a pseudo-object can hinder such processes.*

The data, revisited

We collected data in the following forms: audio and video recordings and transcriptions of the introductory lessons; Cabri and GeoGebra files worked on by the instructor and by the students during the classroom activity; audio and video recordings, screenshots of the students' explorations, transcriptions of the task-based interviews, and the students' work on paper that were produced during the interviews. For revisitation in the light of the notion of pseudo-object, we singled out the interviews of students working alone or in pairs on two non-constructability problems proposed in the context of a DGE.

A total of twelve interviews were found in the data which we revisited in the light of the notion of pseudo-object, six of which dealt with the same non-constructability problem presented in the following sub-section. All participants had been working with dynamic geometry for at least two months prior to the time that the task-based interviews were carried out. The four cases in this article were chosen because of the heterogeneity of the students involved: two pairs of students are from Italian high schools in different regions (aged 15–16); an Italian graduate student (aged 21), enrolled in a Mathematics Education Course, although part of his master's programme was in a Math Department; a pair of Hong Kong undergraduate students (aged 21) taking part in a joint Mathematics and Mathematics Education bachelor's programme.

The high school students had been introduced to different forms of dragging in a DGE in the context of conjecture-generation by the first author of this article; they had never worked with her before on non-constructability problems, but they had been using a DGE with their teacher for a year prior to the interview. The Italian graduate student had been introduced to dynamic geometry as part of the Mathematics Education Course two months prior to the interview, but had not used such software before during his education. The Hong Kong undergraduates had been introduced to dynamic geometry during their bachelor's degree at least one year prior to the interview. In terms of their previous exposure to Euclidean geometry, all of the students had worked on proof in this domain for about one year in high school. The Hong Kong students had also taken an undergraduate class on this topic.

The method we will use to explore our research hypothesis consists of: (1) presenting an *a priori* analysis of the problem assigned in the selected interviews, through which we will highlight the potential of perceiving pseudo-objects with respect to transitioning from argumentation to proof by contradiction. This is done by developing a *symbolic logical chain approach* that allows us to identify and describe pseudo-objects and their potential role in argumentation processes. (2) Then we use such a frame to analyse the four selected cases and explore our research hypothesis.

An a priori *analysis of the problem*

The problem we introduce here is an open non-constructability problem, upon which the selected student interviews that we will discuss in later sections are all based. It is formulated as follows:

> Is it possible to construct a triangle with two angle bisectors that are mutually perpendicular? If so, provide steps for a construction. If not, explain why not.

To simplify the reading of our analyses using such an approach, we have added a table summarising the properties and abbreviations we shall be using (Table 1).

Table 1

Abbreviations of the geometrical properties considered in the analyses (the letters in the first column from the left refer to labels in the figures corresponding to the cases indicated in the third column)

Geometrical property	Abbreviation used	Appearance in the analyses
$\angle CPA$ is a right angle	A_1	*a priori*, all cases
CP is the bisector of $\angle BCA$	A_2	*a priori*, all cases
AP is the bisector of $\angle CAB$	A_3	*a priori*, all cases
ABC is a triangle	A_4	*a priori*, all cases
$m \angle BCA + m \angle BAC = 180^\circ$	B_1	*a priori*, case 2
sides BC and BA coincide	C_1	*a priori*, case 2
sides BC and BA are non-coincident and parallel	C_2	*a priori*
B, the point of intersection of the triangle’s sides that are not common to the two angle bisectors, does not exist	D_1	*a priori*, case 3
A, B, and C lie on the same line	D_2	*a*, case 2, 3, 4
all sides of the figure collapse into a point	D_3	case 3
the figure is a rhombus	E_1	case 1

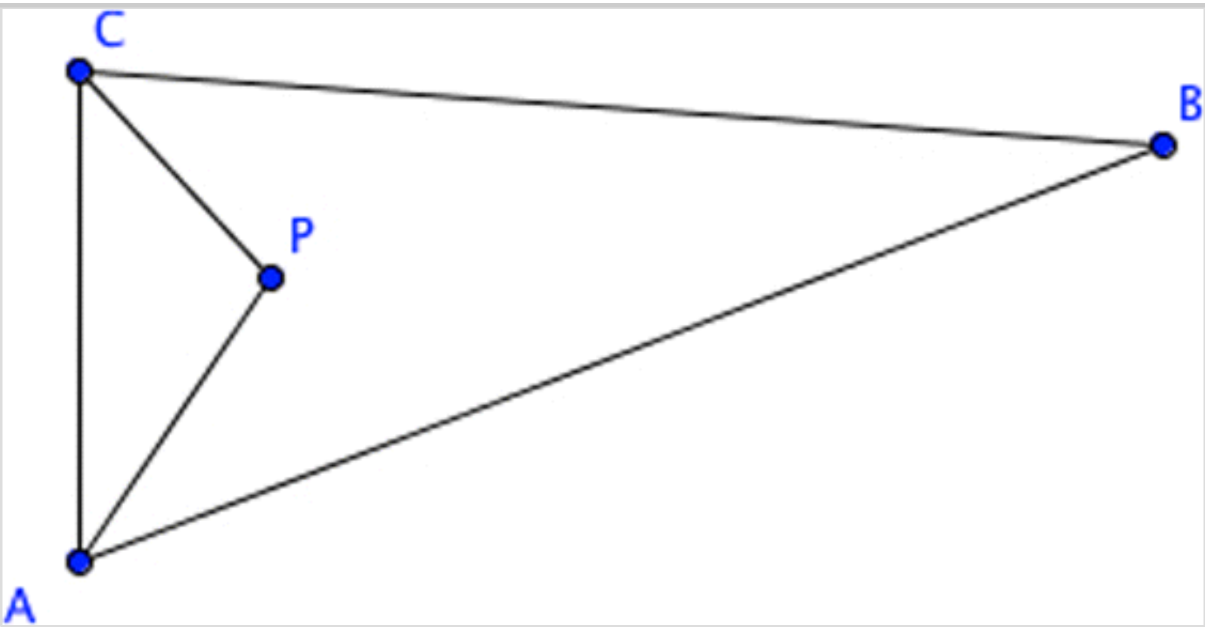
the figure is a square	E_2	case 1
EF ≅ E’F′	F_1	case 3
the figure is a parallelogram	F_2	case 3

The cases in which the DGE gives the solver feedback in the form of an indirect invariant (e.g., $A_1 \wedge A_2 \wedge A_3 \Rightarrow \neg A_4$) that conflicts with what s/he expected (e.g., $A_1 \wedge A_2 \wedge A_3 \Rightarrow \ldots \Rightarrow A_4$) are situations with a particularly high potential for fostering reasoning by contradiction in continuity with the exploration.

The answer to the question posed by the problem is: “No. A triangle with two angle bisectors that are mutually perpendicular cannot be constructed”. A proof by contradiction might go along the following lines (Fig. 3).

Fig. 3

Possible attempt at constructing a suitable triangle



Let $\angle CPA$ be a right angle (property A_1) and CP be the bisector of $\angle BCA$ (A_2) and AP the bisector of $\angle CAB$ (A_3). Then, using the angle measures, $\frac{1}{2}m\angle BCA + \frac{1}{2}m\angle BAC = 90^\circ$, so $m\angle BCA + m\angle BAC = 180^\circ$ (property B_1). On the other hand, since $m\angle BCA + m\angle BAC < 180^\circ$ then $m\angle BCA + m\angle BAC \neq 180^\circ$ (we can write this as $\neg B_1$) because $\angle BCA$ and $\angle BAC$ are two angles of the triangle ABC (property A_4). Therefore, we have a contradiction: that is, the co-existence of a proposition and its negation ($B_1 \wedge \neg B_1$). We have therefore proved that $A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow B_1 \wedge \neg B_1$.

As we have shown in previous studies (e.g. Mariotti and Antonini 2009), from a cognitive point of view, we could have the necessity to visualise the consequences of the property $m\angle BCA + m\angle BAC = 180^\circ$ (B_1): sides BC and BA either

$$\left.\begin{array}{l}\text{coincide } (C_1) \\ \text{or} \\ \text{are non-coincident and parallel } (C_2)\end{array}\right\} C_1 \vee C_2$$

vertex B does not exist, hence the initial triangle does not exist (D_1, essentially $\neg A_4$), *or* A, B, and C must lie on the same line (D_2, a different way of obtaining $\neg A_4$), hence the triangle cannot exist, at least in a non-degenerate form.

A determining difference of how this situation may be seen is how the figure degenerates. In one case, the triangle can be seen to degenerate, breaking into an open figure when CB and BA are seen as becoming parallel (see Mariotti and Antonini 2009), or in the other case it can be perceived as turning into a single line (for example, BC and BA are seen as collapsing onto the same line). So, to explain the impossibility we might need to envision a logical sequence, like:

$$A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow B_1 \Rightarrow C_2 \Rightarrow D_1 \text{ (recognised as } \neg A_4)$$

or and This line

$$A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow B_1 \Rightarrow C_1 \Rightarrow D_2 \text{ (recognised as } \neg A_4)$$

should be moved 2 lines down, under the end of the array which is now on the left, in this layout. capture a conflict between initially assumed properties and those derived (e.g. $A_4 \wedge D_1$; that is, $A_4 \wedge \neg A_4$). We will argue that recognising such a deductive chain and capturing conflicting properties within it constitutes an important step towards explaining the impossibility, and eventually constructing a proof by contradiction.

What is the potential of a DGE in supporting the development of this process? When working with paper and pencil, slight inaccuracies in the drawing allow the figure to represent properties, which a *proper* construction would not permit. For example, on paper, with no trouble, one can assume to have drawn a triangle, for which two angle bisectors intersect at a right angle. In this instance, one may easily be perceptually unaware of the presence of contradictory properties; awareness of a contradiction depends on the solver's conceptual control over the figure, which allows her/him to construct a deductive chain such as $A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow B_1 \Rightarrow C_2 \Rightarrow D_1$ and thereby perceive conflicting properties. In a DGE, solvers can make different choices concerning which properties to use to construct the figure robustly (Healy 2000). The choice determines the type of guidance that the DGE can provide for the concomitant reasoning, as shown by the authors previously (Baccaglini-Frank et al. 2013).

For example, a similar situation to that described involving paper and pencil occurs when the solver constructs a figure with robust properties while mentally imposing on it a soft, contradictory property. We could start with a robust triangle (A_4) and robust bisectors ($A_2 \wedge A_3$), and try to obtain bisectors that are perpendicular (A_1) through dragging. This allows the solver to use the DGE (only) as a sort of "amplified paper-and-pencil drawing", in that it allows the exploration of many cases without having to redraw the figure. An exploration in this situation might lead to experiencing that *when* engaging with $A_2 \wedge A_3 \wedge A_4$, subsequently imposing A_1 leads *every time* to D_2. To solve the problem correctly, the observation "Given $A_2 \wedge A_{3,}$ a conflict between A_1 and A_4 may be perceived" needs to be transformed into a chain of deductive links (e.g., $A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow B_1 \Rightarrow C_1 \Rightarrow D_2$ or $A_1 \wedge A_2 \wedge A_3 \wedge A_4 \Rightarrow \ldots \Rightarrow \neg A_4$), thereby exposing a contradiction (e.g. … $A_4 \Rightarrow \neg A_4$). The fact that a DGE can potentially "show" the solver conflicting soft properties simultaneously is a first aspect of its potential with respect to transitioning from argumentation to proof by contradiction.

What happens if, instead, the solver attempts to construct the three properties $A_1 \wedge A_2 \wedge A_3$ robustly, expecting to find the third vertex of the triangle and, thus, also obtain A_4 robustly? Since a DGE immediately generates all properties that are logical consequences of the construction properties, the figure obtained by imposing $A_1 \wedge A_2 \wedge A_3$ will also show the robust property D_1 (and so $\neg A_4$). That is, the solver can face a surprising situation in which s/he expected A_4, but robustly obtains $\neg A_4$. This unexpected feedback may generate uncertainty about the possibility of the construction and require an interpretation. The crucial point for the solver is to realise that impossibility emerging as feedback can be related to the properties constructed robustly and to properties deriving from them. In other words, in this case the DGE gives the solver feedback in the form of an indirect invariant $A_1 \wedge A_2 \wedge A_3 \Rightarrow \neg A_4$, one that conflicts with the expected $A_1 \wedge A_2 \wedge A_3 \Rightarrow \ldots \Rightarrow A_4$. We see this situation as having a particularly high potential with respect to transitioning from argumentation to proof by contradiction.

Perception of Pseudo-Objects

In the case we examined above, where we had a robust construction of the triangle and of the angle bisectors (properties: A_4, A_2, A_3), the notion of pseudo-object – introduced in the previous section (Baccaglini-Frank et al. 2013, p. 65) – can be used to describe the non-degenerate triangle (A_4) along with the property $m \angle BCA + m \angle BAC = 180^\circ$ (B_1) derived from the projected perception that $\angle CPA$ is a right angle (A_1). By *projected-perception*, we mean what is visually perceived in the DGE is affected by a mentally imposed condition. Analogously, in the case of robust construction of the two angle bisectors that are to be perpendicular to one another (A_1), together with two bisectors (properties: A_2, A_3), a pseudo-object could be perceived in the projected perception of the figure being a triangle (A_4), together with the property of having two parallel sides (C_2). Table 2 presents a summary of conditions to perceive possible pseudo-objects.

Table 2

Conditions to perceive possible pseudo-objects

Robust construction (direct invariant)	Projected perception or imposed condition	Indirect invariant	Contradiction
A_2, A_3, A_4	A_1	D_2 (as $\neg A_4$)	$A_4 \wedge \neg A_4$
A_1, A_2, A_3	A_4	D_1 (as $\neg A_4$)	$A_4 \wedge \neg A_4$

In the process of dragging the pseudo-object (mentally or within a DGE), contradiction can arise or disappear (when the pseudo-object is made to degenerate) to obtain configurations in which the imposed condition is visually verified. The arising and disappearing of contradictions may guide the argument supporting the claim of an impossibility in the case of a non-constructability problem. In the perception of a pseudo-object, it is not a single property that is important, but a *conflicting relationship* between a particular geometrical property and other geometrical properties of the figure. We can interpret a *pseudo-object* (for a solver) as a geometrical figure in which the solver simultaneously perceives a *conflict* between types of invariant (direct or indirect). The main reason we are interested in using the construct of pseudo-object is because its presence can give rise to contradiction in the process of argumentation, and we seek to explore its educational potential with respect to proof by contradiction.

Student Cases

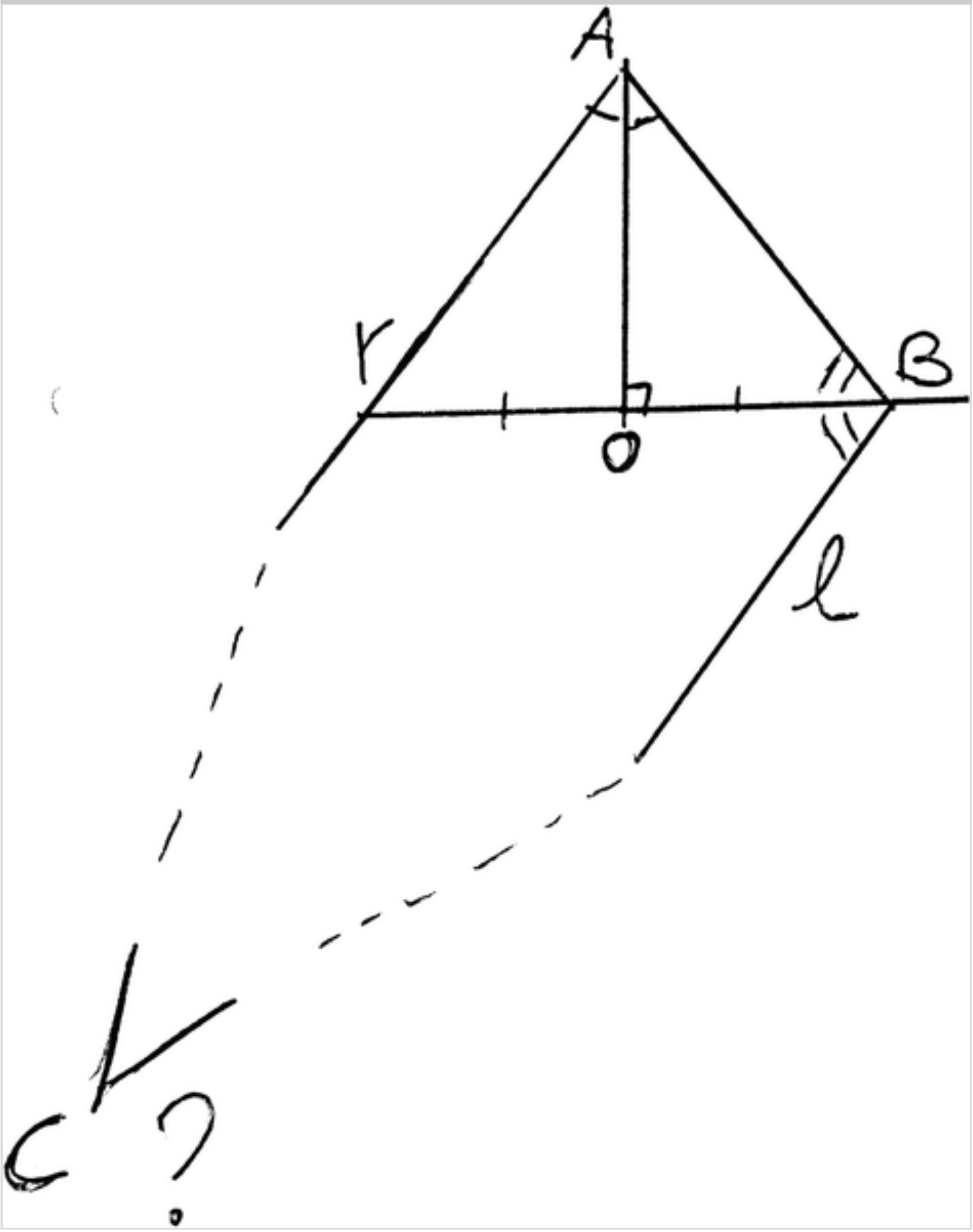
In this section, we first present analyses of two cases in which we argue that the students did encounter pseudo-objects during their explorations, and, indeed, successfully transformed their arguments into geometrical proofs by contradiction. Subsequently, we present two cases in which we argue that the students failed to encounter pseudo-objects, and no proof by contradiction stemming from the exploration was reached.

Case 1: Matteo

Matteo, an Italian graduate student, had become quite fluent in using a DGE during a course of mathematics education. He seemed to have extremely strong conceptual control over the figure he was thinking about, claiming to be thinking about the figure "as if he had constructed it in a dynamic file", and drew the following figure (Fig. 4[2]).

Fig. 4

Matteo's drawing of a triangle with two perpendicular angle bisectors



Matteo seemed to visualise a figure with properties $A_1 \wedge A_2 \wedge A_3 \wedge A_4$ in which he questioned the existence of the point C. He seemed to be expecting a contradiction, because he drew *r* and *l* (initially) parallel to one another, then had them deviating towards C. Matteo then opened *GeoGebra* and constructed a figure like the one he drew: starting from a right triangle AOB, he extended BO past O by a segment of the same length as BO, and then constructed the ray through this point and A, and the ray through the symmetric image of AB over BO. The two rays appeared to be parallel. Matteo looked at the figure and said: "the triangle's sides *r* and *l* just pop open". We can hypothesise the appearance of a pseudo-object for Matteo in ABC (the figure he drew and the "open triangle" that appeared on the screen), because Matteo seemed to perceive $A_1 \wedge A_2 \wedge A_3 \Rightarrow \neg A_4$ and, simultaneously $A_1 \wedge A_2 \wedge A_3 \Rightarrow A_4$.

Matteo wrote:

– Construct two angle bisectors that are perpendicular

– Visualise ABO right triangle ⇒ ∠OAB + ∠OBA = 90°

– Construct sides of the "triangle"

– From theory:

Alternate interior angles supplementary ⇒ $r \parallel l$ ⇒ ABC is NOT a triangle.

In his written argument, Matteo seemed to be logically working out the relationship between the given properties of the triangle AOB and the existence of C and ∠AOB, *r* and *l*. He correctly derived the property $r \parallel l$ (D_1) which can "resolve" the contradiction, thereby eliminating the existence of the triangle ABC.

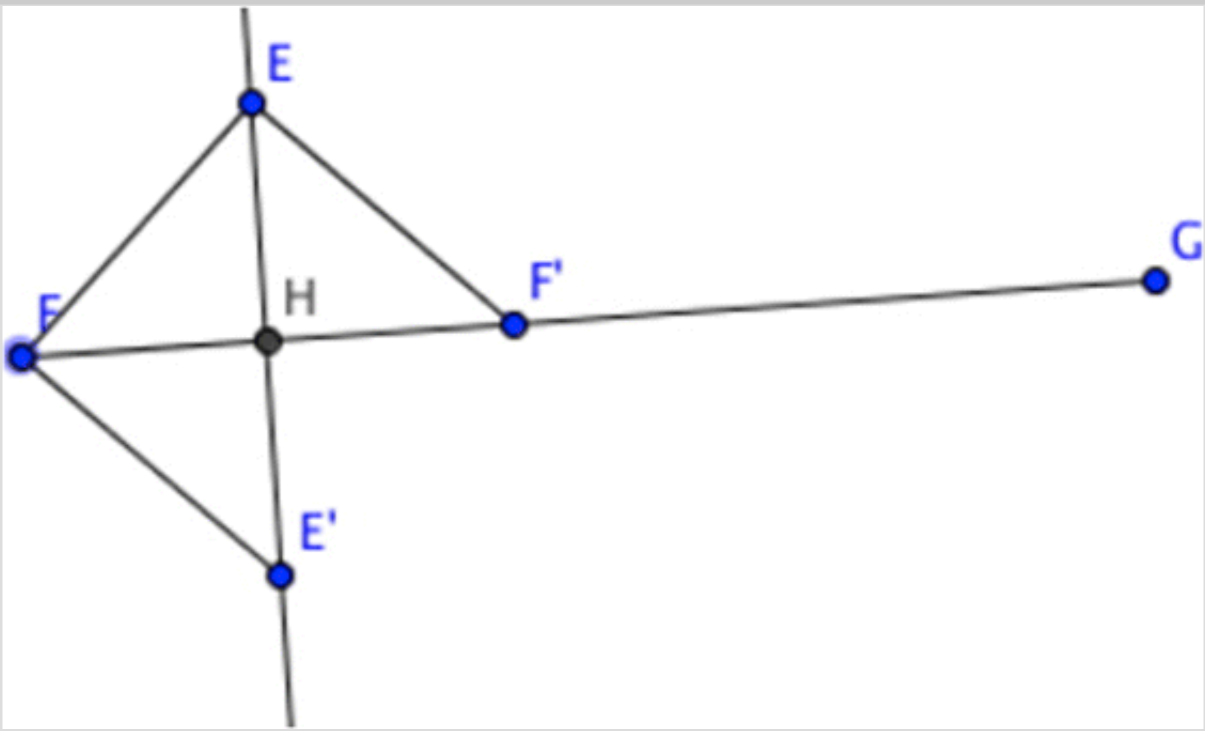
Matteo's high conceptual control over the drawing allowed him to construct and explore his figure, even before acting in the DGE; indeed he did not seem surprised by the feedback he received from it, since he seemed already to have imagined it and envisioned the contradiction. This way of reasoning allowed him to perceive a pseudo-object through which he could directly elaborate a proof by contradiction. We see this example as strongly supporting the part of our working hypothesis that states that experiencing a pseudo-object during an exploration can foster processes of indirect argumentation.

Case 2: Gille and Bernard

The two Hong Kong students, Gille (G) and Bernard (B), in the following excerpts, robustly constructed two bisectors that are perpendicular to one another, EE′ and FF′ (properties $A_2 \wedge A_3 \wedge A_1$) as follows: they constructed the segment EF, and a second segment FG; they then constructed a robust line perpendicular to FG through E, that intersects FG at H; lastly, they reflected points E and F across the bisectors, obtaining E′ and F′ (respectively) and connected E to F′ and F to E′, to produce segments along which the sides of the triangle-to-be should lie (Fig. 5).

Fig. 5

Initial construction by G and B



Excerpt 1.

What was said and done	Our interpretation
1 G (exclaims): There are 3 sides in total, but they do not stick together.[3]	G seemed to have the expectation of seeing sides EF′ and FE′ converge. That is, she seemed to have projected the following indirect invariant onto the figure: $A_2 \wedge A_3 \wedge A_1$ (in her construction) $\Rightarrow A_4$. However, the software's feedback guided her to perceive the invariant D_2, seen as $\neg A_4$, (or possibly an indirect invariant: $A_2 \wedge A_3 \wedge A_1 \Rightarrow D_2$), which she would discover to be robust and to cause the quadrilateral to degenerate into a point whenever she tried to 'close' the triangle, by bringing E′ and F′ together. These conflicting indirect invariants ($A_2 \wedge A_3 \wedge A_1 \Rightarrow A_4$ and $A_2 \wedge A_3 \wedge A_1 \Rightarrow D_2$) seemed to be perceived simultaneously, giving rise to a pseudo-object for her.

The rest of the exploration was dedicated to explaining this surprising feedback.

The exploration proceeded as follows (the text is translated from Chinese) (Fig. 6).

Fig. 6

Degeneration of the triangle-to-be

AQ3

AQ4

Excerpt 2.

What was said and done	Our interpretation
2 G: [She drags E so that the "vertexes" converge.] We need to have E′ and F′ stick together. They stick together when they become one point.	G seemed to be looking for a case in which her construction also would have the property A_4. As she attempted to impose the soft property A_4 on the figure, she experienced the indirect invariant: $A_2 \wedge A_3 \wedge A_1 \wedge A_4 \Rightarrow$ all sides of the triangle-to-be collapsing into a point (D_3).
3 G: If E′ and F′ stick together, E′ and F′ will have to move towards the intersection point [H].	
4. G: … and E will tend to F then. So all points will stick together [Fig. 6].	
5 B: The distance EF is equal to the distance E'F′. Therefore, if we want E′ and F′ to become one point, E and F need to become one point also.	B tried to explain the figure's behaviour, identifying a new invariant property: EF ≅ E'F′ (property F_1), which implies degeneration of the triangle-to-be in the desired case.

6 G: There should be a line right there [she points to a line that seems to connect E′ and F′] ...	Possibly G was remembering the construction accomplished using symmetry, and she related it to a known property of parallelograms.
7 B: Because this is [pointing to EF’E’F] is a parallelogram.	
8 G: Because here FF′ and here EE′ … are all mirrors.	
9 B: Yes, this is a parallelogram. So there is no triangle.	What became crucial was the perception of a robust parallelogram (F_2) in place of the triangle-to-be that conflicted with A_4.
10 G: Not possible to make the triangle with angle bisectors perpendicular to each other.	

The exploration seemed to lead the students to identify a chain of invariants:

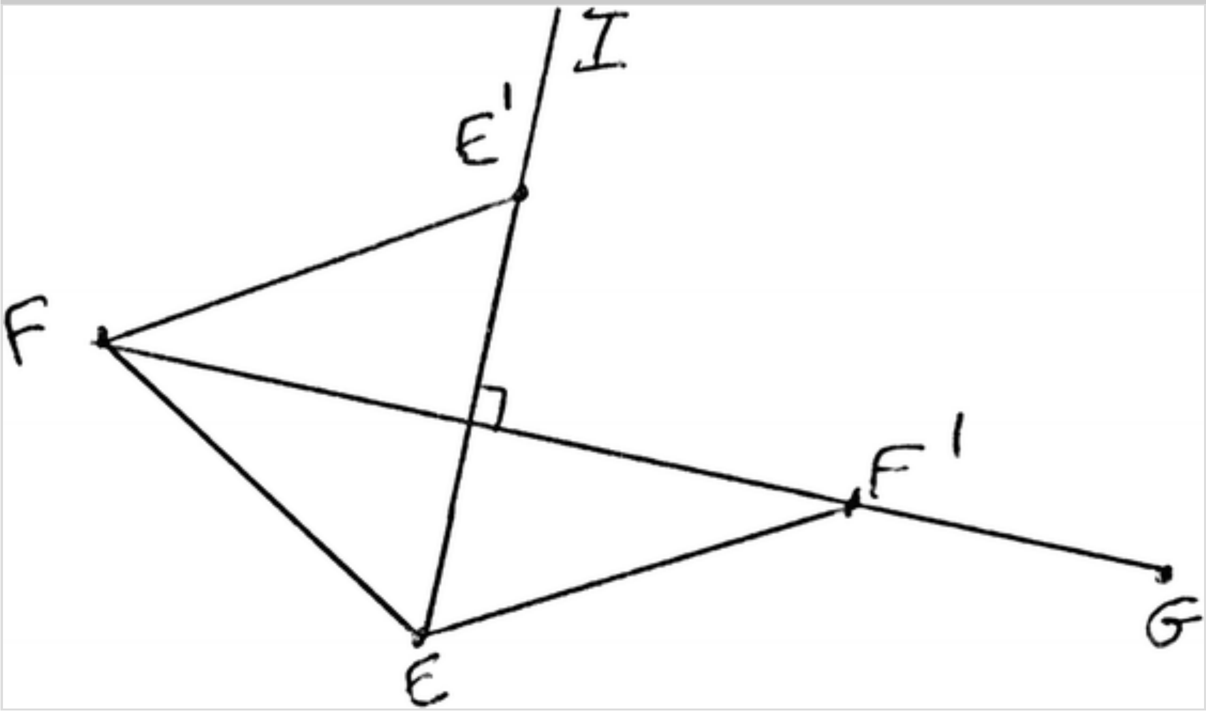
$$A_2 \wedge A_3 \wedge A_1 \Rightarrow F_1 \Rightarrow \dots F_2 \Rightarrow D_2 \text{ (seen as} \neg A_4\text{)},$$

exposing the impossibility of obtaining A_4 from the constructed figure. Moreover, this explains the unexpected behaviour experienced when the students perceived $A_2 \wedge A_3 \wedge A_1 \wedge A_4 \Rightarrow D_3$ as all sides of the triangle-to-be collapsing into a point from trying to impose A_4 softly. In fact, this “explanation” is what seemed to eliminate the contradiction overcoming the pseudo-object: in turns 9 and 10, the students did not seem to be projecting A_4 onto the figure any more nor perceiving it as an invariant relationship between invariants. The triangle simply did not exist and it had been replaced by the parallelogram.

The proof produced by the students went as follows and was accompanied by a figure (Fig. 7).[4] The written text (originally in English) is transcribed literally.

Fig. 7

The figure G and B drew for the proof



1. Construct a point E and a line segment FG [omitted in written text: “construct line segment EI so that”] EI⊥FG.

2. Reflect E along FG to E′ such that FG bisects ∠E’FE.

3. Reflect F along EI to F′ such that EI bisects ∠F′EF.

4. Drag E onto F to make E′ and F′ be the same point.

Result: All the points E, F, E′, F′ intersect [they seem to use this as a synonym of “coincide”] together at the same point.

Reason: EFF’E′ is a parallelogram (diag. ⊥).

∴ EF = F’E′ (prop. of parallelogram)

∴ E′ and F′ won’t intersect [i.e. coincide].

∴ Contradict with the statement, no triangles can be formed under this situation.

In the students’ written argument, they described the construction created in the

DGE, reproduced it on paper, and argued that E′ and F′ cannot coincide unless the whole figure degenerates, because "EFF'E′ is a parallelogram" (F_2). Though a correct derivation of the claim "EFF'E′ is a parallelogram" is not actually given (perpendicular diagonals is not a sufficient condition, and the students did not prove that E′ lies on to EI and F′ lies on FG), in the proof we can find an attempt at directly deriving property F_2 and expressing the conflict between this condition and the coincidence of points E′ and F′, which would give rise to a triangle. Although the argument is not a proper proof by contradiction, and the derivations are not entirely correct, it explains the perceived contradiction in the figure assumed to be a triangle but turning out necessarily to be a parallelogram. We suggest that the argument could have become a proof by contradiction if property A_4 had been explicitly added to the students' initial assumptions. According to this analysis, the case of Gille and Bernard supports the part of our working hypothesis that states that experiencing a pseudo-object during an exploration can foster DGE-supported processes of argumentation culminating in a proof by contradiction.

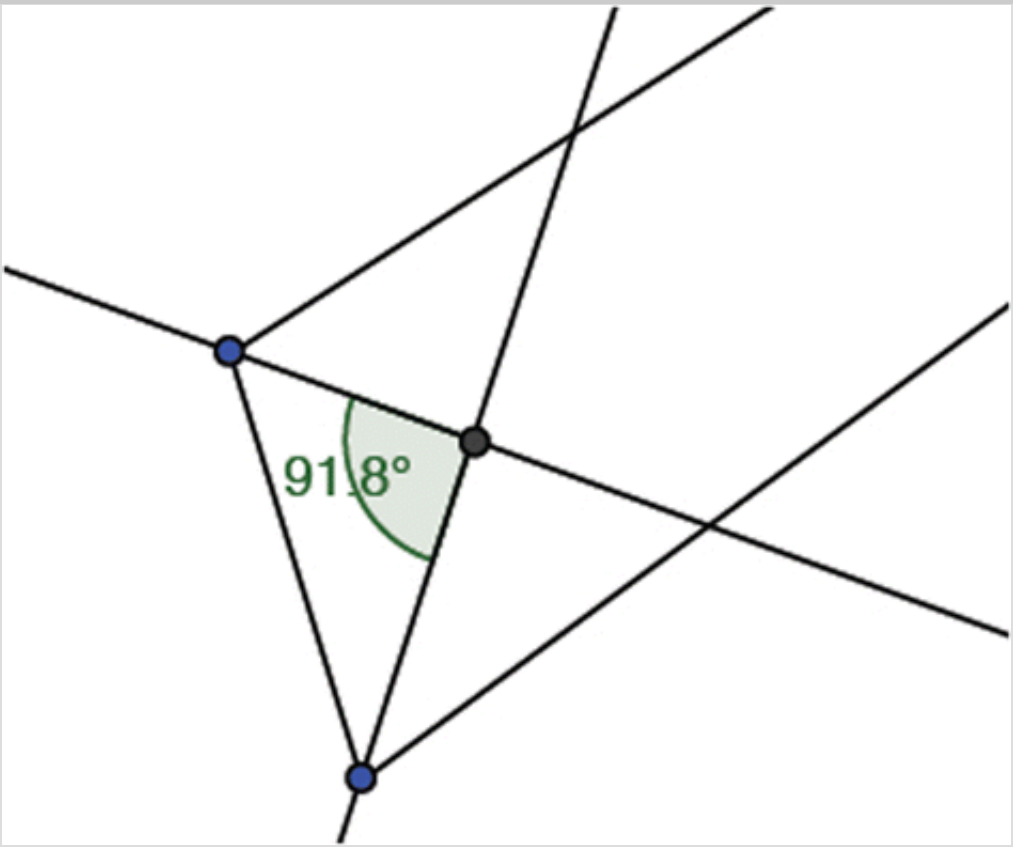
We now introduce two cases supporting the part of our working hypothesis in which we claim that lack of experiencing of a pseudo-object may hinder processes culminating in the production of a proof by contradiction. In these cases, we argue, the students fail to encounter pseudo-objects and as a result they do not construct indirect arguments or proofs by contradiction as a culmination of their dynamic exploration process.

Case 3: Simone

Simone was a 16-year-old Italian high school student. He proceeded by constructing a proper triangle (A_4) and two of its angle bisectors ($A_2 \wedge A_3$). Then he marked an angle formed by the bisectors and started dragging one vertex of the triangle in the attempt to get the measure to say "90°" (A_1) (Fig. 8). So, in the situation of having constructed the robust properties $A_4 \wedge A_2 \wedge A_3$ he was trying to induce the soft property A_1, and possibly investigate the conditions under which A_1 was visually verified.

Fig. 8

Simone's attempt at obtaining a right angle at the intersection of the two bisectors: the third vertex of the triangle is too "far away" to be shown here



Excerpt 3.

What was said and done	Our interpretation
1 *Sim*: It’s endless! [dragging “up” the third vertex of the triangle.]	Sim seemed to be convinced that it was possible to obtain the (soft) property A_1.
2 *Sim*: 91.2 [reading the measure of the angle between the bisectors.]	
3 *Sim*: Well, yes, in any case it will come out!	
4 *Sim:* Well, of course! It’s not like it can go on forever! At the end it will make it to be 90!	
5 Either insert row in the table, containing "[...]" to indicate that some lines were skipped, or add it above this "5", still leaving an empty line. *Sim:* Eh, it is impossible to construct it! Because … I only have these two bisectors.	Sim claims that it is not possible (in general) to construct such a figure, however he does not seem to belive this. Indeed, what

	follows is not consistent with this claim.
6 *Interviewer:* Hmm.	
7 *Sim:* How can I …	
8 *Sim:* Since … the bisectors are perpendicular … it means here there is a rhombus … or a square.	Although he could not obtain the desired property visually, Sim projected onto the figure the property of bisectors being perpendicular (A_1), and expressed two implied properties: rhombus (E_1) or square (E_2). A relationship between invariants seemed to be perceived: $A_4 \wedge A_2 \wedge A_3 \wedge A_1 \Rightarrow E_1 \vee E_2$.
9 *Sim:* If like here … [he draws a segment] Here … there were … a rhombus … this would be 90, 90 … or a square. And therefore … then …Eh, I mean, if this is like a rhombus, no? Here there is 90 and here there is 90, and these are the bisectors.	
10 *Sim:* And then… and then I bring these up [pointing to the "open" looking sides of the triangle (see Fig. 8)] and I find their point of … of intersection.	His new argument seemed to lead to the conclusion that it was, in fact, possible to construct a triangle with the desired properties. At this point, Sim seemed to have forgotten that A_4 is already a property imposed on the figure; that is, the point of intersection of the seemingly open sides existed robustly. No conflicting invariants seemed to be perceived.

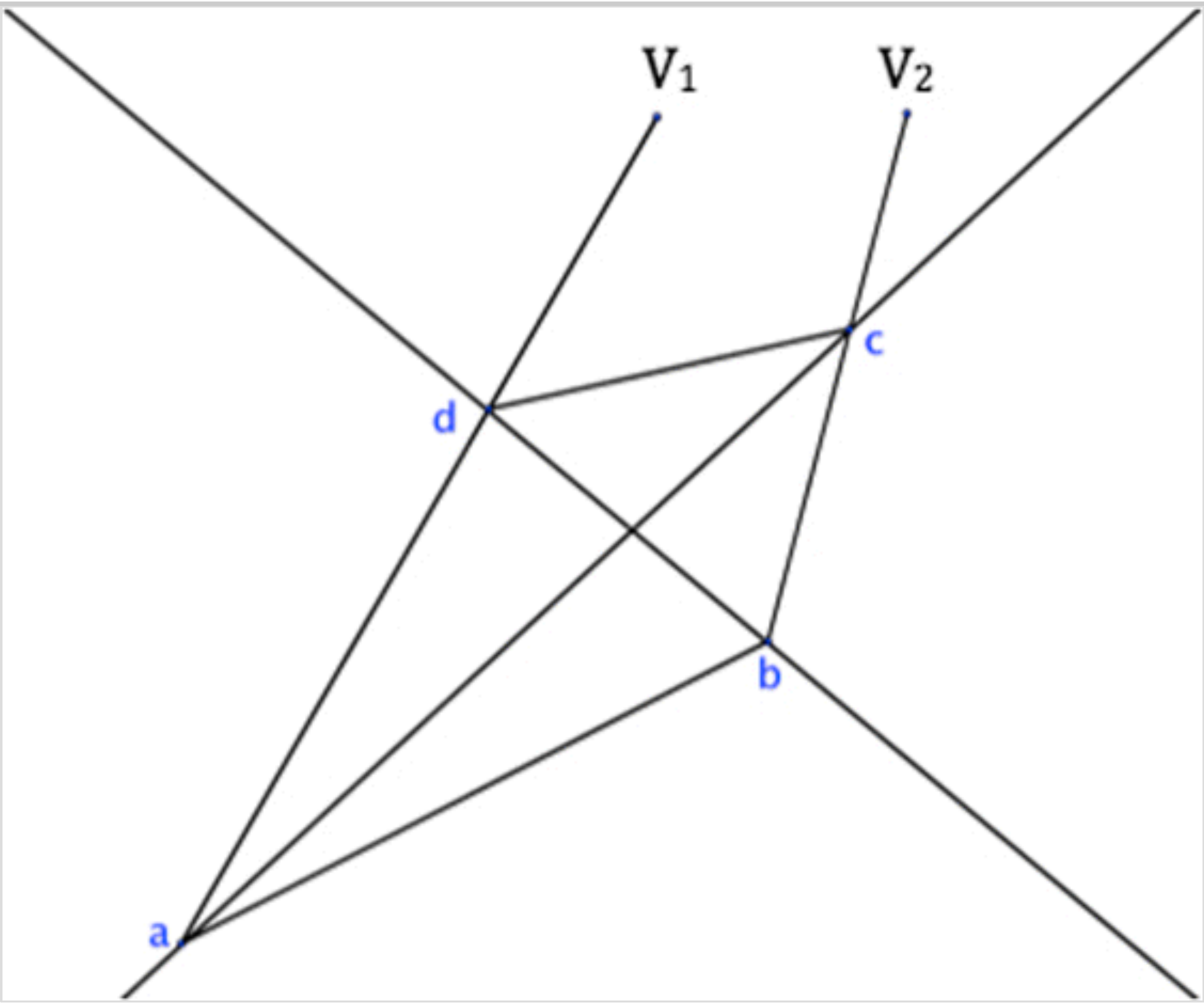
Although in turn 5 Simone claimed that it was not possible to construct the triangle, the behaviour that followed was not consistent with this claim. On the contrary, he acted as if it were possible to construct the triangle, and his exclamation could be interpreted as a sign of distress in that he might have be having trouble conceiving a set of steps that would lead to the desired construction. There seemed to be three possible indirect invariants that Simone perceived (of course, this is our interpretation):

$$A_4 \wedge A_2 \wedge A_3 \wedge A_1 \Rightarrow E_1 \vee E_2$$
$$A_2 \wedge A_3 \wedge A_1 \Rightarrow E_1 \vee E_2$$
$$A_4 \wedge A_2 \wedge A_3 \wedge (E_1 \vee E_2) \Rightarrow A_1$$

Consistent with this interpretation, he proceeded to construct a new figure in which the triangle was not closed (Fig. 9).

Fig. 9

Simone's new construction in which A_4 was no longer robust: that is, the points V_1 and V_2 could be dragged to merge (the labels V_1 and V_2 were inserted by the authors)



However, he did not seem to be aware of any conflict between his perceived invariant and other invariants he might have perceived simultaneously. Therefore, we can claim that Simone had not perceived any pseudo-object. He was not able to continue his argument, nor did he discover the impossibility of the construction within the DGE. This case and its analysis support the part of our working hypothesis claiming that the lack of experience of a pseudo-object can hinder DGE-supported processes of indirect argumentation.

Case 4: Emiliana and Ilaria

We conclude with the case of two Italian high school students, Emiliana and Ilaria (aged 16). They started by constructing the triangle and the two bisectors robustly ($A_4 \wedge A_2 \wedge A_3$) (Fig. 10).

Fig. 10

The constructed figure upon which Ila was acting

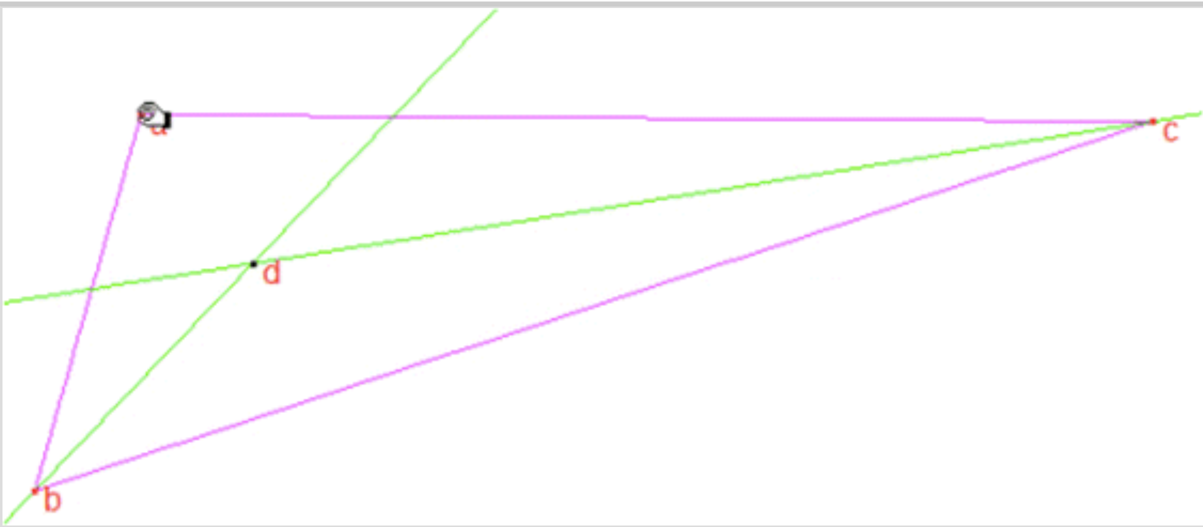
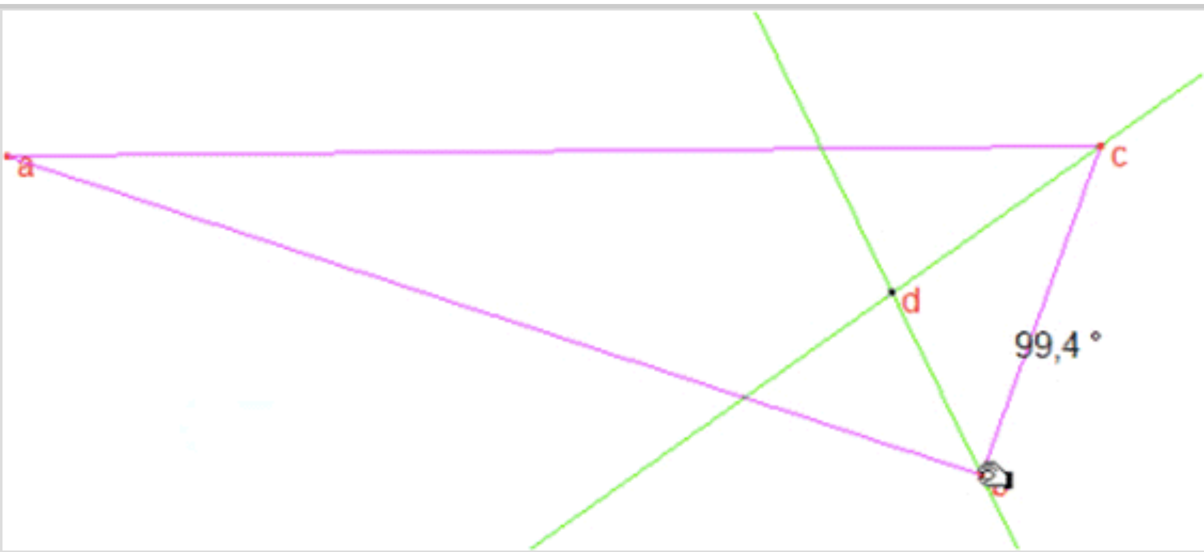


Fig. 11

The measure of the angle at ‘d’ changes as Ila drags ‘b’



Excerpt 4.

What was said and done	Our interpretation
1 *Ila*: Do the angle bisectors and try to move the two angles until we get a right angle and put 'mark the angle'. [she takes control of the mouse]	Ila was proposing to explore what happened trying to impose the soft condition A_1 on the figure.
2 *Ila*: We have to move 'a'[5]…	The students realised that, when vertex 'a' is collinear with vertices 'b' and 'c', the bisectors appeared to be perpendicular. The students initially rejected this case, because 'abc' was no longer a triangle.
3 *Emi*: Why 'a'?	
4 *Ila*: Uh, all, uh, we have to try *all*, … uh… [she stops in a degenerate situation, with a, b, c on a same line. The bisectors appear to be perpendicular]	
5 *Emi*: No, that's a line. [they giggle]	
6 *Ila*: So … [moving away from the degenerate configuration]	
…	
7 *Emi*: It is not useful to move 'a', because then, uh, then … move it … the angle keeps the same measure, and we have to try to make it so that *this* one [pointing to 'd'[6]] is 90.	However 'a' is dragged, the students did not seem to be happy with what they found, so they decided to drag a different vertex.
8 *Ila*: Right.	
9 *Emi*: So move 'b'.	
10 *Emi*: Measure the angle, and do 'b', 'd', 'c' … OK … see that it changes? (Fig. 11)	Again, the students experienced the necessity of the degeneration that occurred whenever they obtained the desired property. This would correspond to an indirect invariant such as: $A_4 \wedge A_2 \wedge A_3 \wedge A_1 \Rightarrow D_2$ (which they seemed to recognise as $\neg A_4$) or like $A_4 \wedge A_2 \wedge A_3 \wedge D_2 \Leftrightarrow A_1$ (see turn 13).
11 *Ila*: Yes, but, excuse me, only when it is a line [referring to Fig. 12].… then it's impossible!	
12 *Emi*: OK, it's impossible. Then I guess it is impossible.	The students seemed to conceive the impossibility of the construction, but limited their
13 *Ila*: See? [she shows Emi	

other cases in which the three vertices are collinear and the measure of the angle between the bisectors is 90°] … It's impossible.	attention to the invariant relationship $A_4 \wedge A_2 \wedge A_3 \wedge D_2 \Leftrightarrow A_1$.
14 *Emi*: No, because, if … [she starts writing what will become an algebraic proof by contradiction and no longer pays attention to the screen]	Emi does not produce a geometrical proof by contradiction.

The perceived indirect invariant did contain a contradiction (if D_2 is interpreted as $\neg A_4$): however, simultaneous perception of the contradictory invariants (A_4 and D_2) seemed to be lacking. Indeed, what Ila seemed to be perceiving was: "$A_4 \wedge A_2 \wedge A_3 \Leftrightarrow \neg A_1$" or (strictly) "$A_2 \wedge A_3 \wedge A_1 \Leftrightarrow D_2$", seen as "$A_2 \wedge A_3 \wedge A_1 \Leftrightarrow \neg A_4$". She seemed unable to see these invariants simultaneously projected onto the figure; thus, according to this interpretation, a pseudo-object was not perceived. The students' argumentation processes in this exploration did not culminate in a geometrical proof by contradiction, so this case supports the part of our working hypothesis claiming that the lack of experience of a pseudo-object can hinder DGE-supported processes of argumentation culminating in proof by contradiction.

Conclusion

This article presents a revisiting of some data through the lens of a new tool of analysis: the notion of pseudo-object. The analysis leads to the emergence of a research hypothesis concerning the relationship between argumentation and proof, specifically proof by contradiction. We highlighted key elements in the argumentation and reasoning process through which the solvers seemed to be trying to find harmony among conflicting dynamic phenomena experienced through dragging (what is seen via dragging) and geometric properties of figures (what is expected to be seen happening according to Euclidean geometry). We suggest that perceiving a pseudo-object can be key to reaching such harmony: perceivable conflicts are related to one or more indirect invariant(s) (invariant relationships between properties). Furthermore, we developed a symbolic logical chain approach to identify and describe the emergence of pseudo-objects and their role in argumentation processes.

The notion of pseudo-object allowed us to advance the hypothesis that experiencing a pseudo-object during an exploration can foster DGE-supported processes of argumentation, culminating in proof by contradiction, while the lack of experience of a pseudo-object may hinder such processes. In the analyses of the four cases presented, we identified the invariants which we could recognise in the argumentation processes and wrote them as deductive chains, highlighting dependency relationships between perceived invariants. We note that, in the two excerpts in which we inferred the presence of pseudo-objects in the students' arguments, such pseudo-objects emerged as a conflict between two indirect invariants, one expected and one perceived as feedback from the DGE. Indeed, as noted in the *a priori* analysis, the cases in which the appearance of an indirect invariant (e.g., $A_1 \wedge A_2 \wedge A_3 \Rightarrow \neg A_4$) conflicts with what is expected (e.g., $A_1 \wedge A_2 \wedge A_3 \Rightarrow \ldots \Rightarrow A_4$) are situations with a particularly high potential for fostering reasoning by contradiction in continuity with the exploration. In the third and fourth cases presented, it seems that neither the phenomenon of the emergence of a pseudo-object occurred, nor did indirect argumentations stem from the explorations.

The theoretical tool offered by the notion of pseudo-objects and the symbolic logical chain approach allowed us to notice missing steps in each argument that could have led to perceiving contradiction. It seems that the appearance of the phenomenon of the pseudo-object occurring during an exploration could be related to elaborating arguments supporting the answer in terms of existence of the requested figure. Moreover, the conflicting nature of the pseudo-object may make a contradiction evident and, in this way, support the production of a proof culminating with a contradiction.

A pseudo-object, as discussed in this article, can be thought of as a virtual object that exists in the interface between our cognitive world and the DGE micro-world. It withholds the *potential* that, when actualised visually in a DGE (a projected perception), it can lead to realising the Euclidean possibility or impossibility. Therefore, even though one cannot construct "impossible" figures in a DGE, coupled with their mental world, DGEs have the potential to create uncertain dynamic geometrical phenomena which can be perceived (visually) leading to argumentation and proof. This learner–DGE coupling expands the epistemic dimension of a DGE where dragging, in particular dragging a pseudo-object, becomes simultaneously a mental and a physical activity. We have shown that perceiving and resolving conflict, uncertainty, surprise and contradiction motivate students to engage in meaningful (successful or not) mathematical reasoning and argumentation. Thus, pseudo-objects can be invested in the design of DGE tasks with the pedagogical aim of fostering geometrical proof by contradiction. Such a goal might eventually be achieved by a transitional phase in which students produce DGE-based arguments, as in the case of Gille and Bernard. Figure 2 and Table 2 show two types of pseudo-object representations that are conducive to mathematical reasoning: one is visual and dynamic, while the other is logical and linguistic. Their combination in pedagogical settings and task design opens a new direction for DGE and geometrical proof research.

Fig. 12 Can this figure be moved closer to the excerpt in which it is referred to?

Ila makes the triangle degenerate on purpose

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[1] By robust construction in a DGE, we mean construction that can maintain the desired properties of a figure invariant under dragging; we use *robust* in contrast to *soft* (Healy 2000).

[2] This figure was redrawn by the authors because the scan of the original was too light.

[3] This is a translation from Chinese: another possible translation might be, "they do not close".

[4] This figure was redrawn by the authors because the scan of the original was too light.

[5] This point corresponds to vertex B in Figure 3

[6] This point corresponds to point P in Figure 3