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On the Existence of Leray-Hopf Weak Solutions to the Navier-Stokes Equations

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Abstract: We give a rather short and self-contained presentation of the global existence for Leray-Hopf weak solutions to the three dimensional incompressible Navier-Stokes equations, with constant density. We give a unified treatment in terms of the domains and the relative boundary conditions and in terms of the approximation methods. More precisely, we consider the case of the whole space, the flat torus, and the case of a general bounded domain with a smooth boundary (the latter supplemented with homogeneous Dirichlet conditions). We consider as approximation schemes the Leray approximation method, the Faedo-Galerkin method, the semi-discretization in time and the approximation by adding a Smagorinsky-Ladyžhenskaya term. We mainly focus on developing a unified treatment especially in the compactness argument needed to show that approximations converge to the weak solutions.

Keywords: Navier-Stokes equations; Leray-Hopf weak solutions; existence

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1. Introduction

Let $T > 0$ be an arbitrary finite number representing the time, $\Omega \subset \mathbb{R}^3$ be a domain to be specified later, and $\nu > 0$ be a positive number representing the kinematic viscosity. The incompressible Navier-Stokes equations model the dynamic of a viscous and incompressible fluid at constant temperature and with constant density. They are given by the following system of PDE's posed in $(0,T) \times \Omega$:

$$\begin{cases} \partial_t u + (u \cdot \nabla)\, u + \nabla p - \nu \Delta u = f & \text{in } (0,T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega. \end{cases} \tag{1}$$

The vector field $u \in \mathbb{R}^3$ is the velocity, $p \in \mathbb{R}$ is the scalar pressure, and to avoid inessential complications, we set the external force $f = 0$ (but all results presented here can be easily extended to the case of a non vanishing external force, see Remark 4). The first equation is the conservation of linear momentum and the second equation, also called the incompressibility constraint, can be considered as the conservation of the mass, since the density is assumed to be constant. The system (1) has to be supplemented with initial and boundary conditions. Regarding the initial condition we impose that

$$u|_{t=0} = u_0, \qquad \text{in } \Omega,$$

with u_0 satisfying the compatibility condition $\operatorname{div} u_0 = 0$ in Ω. For the boundary conditions we need to specify the assumptions on the domain. We consider three cases, $\Omega = \mathbb{R}^3$, $\Omega = \mathbb{T}^3$ with $\mathbb{T}^3$ being the three-dimensional flat torus, and $\Omega \subset \mathbb{R}^3$ being a bounded domain, whose boundary will be denoted by $\partial\Omega$; we refer to Assumption 1 for the precise hypotheses on Ω.

For each of the three different cases we impose the different and natural boundary conditions:

$$
\begin{aligned}
&(i) \quad u \to 0 \text{ for } |x| \to \infty, \quad \text{if } \Omega = \mathbb{R}^3;\\
&(ii) \quad u \text{ is periodic, } \text{if } \Omega = \mathbb{T}^3;\\
&(iii) \quad u = 0 \text{ on } (0,T) \times \partial\Omega, \quad \text{if } \Omega \text{ is a bounded domain.}
\end{aligned}
\tag{2}
$$

Note that the initial datum will be requested to be tangential to the boundary in the case (i), and to satisfy the condition (ii) and (iii) in the other cases. Contrary to the system of compressible Navier-Stokes equations, the pressure p, instead of being obtained through a state equation, is an unknown of the system. This is a consequence of the incompressibility conditions and indeed the pressure can be interpreted as Lagrange multiplier associated with the incompressibility constraint. Note that there are no initial/boundary conditions imposed on the pressure, which (since it appears only as a gradient in the momentum equation) is always determined up to an arbitrary function of time.

Generally speaking, it is very difficult to prove existence and uniqueness of smooth solutions to nonlinear PDE's. Here, with existence we always mean global in time existence, namely existence on any given time interval $(0, T)$, for arbitrary $T > 0$. The available theories for weak solutions provide a framework to give a proper meaning to PDE's, without requiring too much regularity on the solutions and they rely on the theory of generalized functions and distributions. In particular, the landmark idea in the theory of weak solutions is to give up on solving the equations point-wise but trying to solve them in an averaged sense, which is meaningful also from a physical point of view. In the case of fluid mechanics, we expect a very complex behavior by (turbulent) flows appearing in real life, hence we expect to be able to capture only averages of the velocity and pressure, see Reference [1].

The problem of global existence generally becomes easier since the class of available solution is enlarged and several functional analysis tools can be now used. However, the price to pay to have such a relatively simple existence theory is that the uniqueness problem becomes a very difficult one and many calculation which are obvious when dealing with smooth solutions are not possible or hard to be justified. The three-dimensional incompressible Navier-Stokes is a paradigmatic example of a such situation and the introduction of weak solutions dates back about 100 years ago. In fact, in a series of celebrated papers (the time evolution is treated in Reference [2]) Jean Leray introduced the notion of weak solution as a mathematical tool, but also with a strong understanding of the physics behind the equations. The theory of weak solutions is also strictly linked with the name of Eberhard Hopf [3] who gave the first contribution to the problem of existence of weak solutions in a bounded domain, by means of the Faedo-Galerkin method.

It is interesting to observe that many methods and techniques of functional analysis (which are now a common background of graduate students in mathematics) originated from the study of PDE's and especially from those arising in fluid mechanics. In this note we are trying to explain an extremely limited part of the theory: the existence of (Leray-Hopf) weak solutions. This is a topic at the level of most undergraduate students, with a minimal knowledge of Sobolev spaces and functional analysis (mainly weak convergence and weak compactness), as for instance in the widely used (text)books by Brezis [4], just to name one. Note also that we try to present a minimal spot in the abstract theory of Navier-Stokes equations, which can be an "appetizer" for students trying to start a serious understanding of (part of) the mathematical fluid mechanics. It is impossible to review what is done on the subject, even only for the mathematical analysis side. Nevertheless many information, at an introductory or more advanced level, can be found in several books, see for instance, just to name a few in alphabetical order [5–11].

We think that we will not discourage any reader unfolding the (many) mathematical difficulties of the topic, but –instead– we hope that highlighting the challenges which are typical of mathematical fluid mechanics further interest could be stimulated; To this end we quote the following coming from an interview reported in Reference [12] in a essay in memory of Jacob Schwartz:

> *When I asked him [Jacob Schwartz] if there was a subject he had trouble learning, he admitted that there was, namely, fluid dynamics. "It is not a subject that can be expressed in terms of theorems and their proofs," he said.*

Following the above point of view, the first step even in the mathematical analysis of the Navier-Stokes equations is that of giving an appropriate definition of weak solutions, which take into account the functional spaces where it is reasonable find weak solutions, the initial and the boundary conditions. Usually, the functional space to be considered are hinted by the a priori estimate available for the system under consideration. The informal notion of an a priori estimate may be a quantitative bound depending only on the data of the problem, which holds for smooth solutions of the system under consideration, regardless their existence. In particular, for system arising from physics, the a priori estimates usually have a deep physical interpretation.

In the context of the three-dimensional incompressible Navier-Stokes equations the main a priori estimate is indeed the conservation of the energy of the system and is given by the following integral equality:

$$\int |u(t,x)|^2\,dx + 2\nu \int_0^t \int |\nabla u(s,x)|^2\,dxds = \int |u_0(x)|^2\,dx \qquad t \in [0,T], \tag{3}$$

where the space integral is over the domain under consideration.

The equality (3) has a very simple *formal* proof. Indeed, let (u,p) be a smooth solution of (1) and (2). By multiplying the momentum equation by u and integrating over Ω we get

$$\int \partial_t u \cdot u - \nu \Delta u \cdot u + (u \cdot \nabla)\, u \cdot u + \nabla p \cdot u\,dx = 0.$$

By integrating by parts and using the divergence free condition and (2) we get

$$-\int \Delta u \cdot u\,dx = \int |\nabla\, u|^2\,dx, \qquad \int (u \cdot \nabla)\, u \cdot u\,dx = 0, \qquad \text{and} \qquad \int \nabla p \cdot u\,dx = 0.$$

Then, after integration in time on $(0,t)$ with $t \in (0,T)$ we get (3). Note that (3) gives a quantitative bound depending only on T, and u_0 of square integrals of the velocity field u and its gradient ∇u. The energy equality (3) will serve as motivation for the definition of Leray-Hopf weak solution we will give in Section 3.

Once a reasonable definition of weak solution is given, to prove global existence one usually exploits what it is know as a *compactness argument*, which consists in (1) proving the existence of a sequence of relatively smooth approximating solutions satisfying appropriate uniform estimates; (2) proving that limits of these approximating solutions are effectively weak solution of the problem under consideration. We remark that usually the uniform bounds obtained on the sequence of approximating solutions are the same inferred by the a priori estimates available for the system under consideration; These bounds are then hopefully inherited by weak solutions obtained with a passage to the limit. To be more precise, in the case of the Navier-Stokes equations, the approximation method should be chosen such that the approximate solutions satisfy the energy (in)equality. Due to the limited regularity which can be generally inferred on weak solutions, the validity of any energy balance on the weak solutions to the 3D Navier-Stokes equations is obtained with a limiting process on the approximate solutions and not using the solution u itself as a test function as done to obtain (3), since this argument is only formal and not justified when dealing with genuine Leray-Hopf weak solutions.

In this short note we provide a rather self-contained account on the global existence of weak solutions for the three-dimensional incompressible Navier-Stokes equations and some of the (several) approximation methods used in the literature. Since the convergence argument is essentially the same for every approximation methods and for every choice of the domains and boundary conditions mentioned above, we introduce (for the purpose of the exposition) a notion of *approximating solution* for which we will prove the convergence to a Leray-Hopf weak solution of the problem (1) and (2). This is not the historical path, but is a way we identify to have a unified treatment, which can describe the existence theory within the notion of *approximating solutions*.

Then, we show how several and well-known approximations fit in the framework introduced and, therefore, we recover the existence of Leray-Hopf weak solution by using those methods. In particular, we will consider the most common techniques available for the construction. Further results based on the energy type methods, concerning uniqueness, regularity and the connection with applied analysis of turbulent flows, can be found in the forthcoming monograph [1], which is also written in the spirit of being an introduction for undergraduate students, interested in applied analysis of the Navier-Stokes equations.

Organization of the Paper

The paper is organized as follows: In Section 2 we introduce the functional spaces that we use. Then, in Section 3 we define of Leray-Hopf weak solutions and study their main properties. In Section 4 we give the definition of *approximating solution* and we prove the convergence to a Leray-Hopf weak solution. Finally, in Section 5 we prove that certain approximating schemes fit in the framework of *approximating solution*.

2. Preliminaries

In this section we fix some notations and we recall some basic preliminaries we will need for the analysis. We start by fixing the assumptions on the domain Ω.

Assumption 1. *The domain $\Omega \subset \mathbb{R}^3$ will be of the following type:*

$(A1)$ *the whole space, $\Omega = \mathbb{R}^3$;*
$(A2)$ *the flat torus, $\Omega = \mathbb{T}^3$;*
$(A3)$ *a bounded connected open set $\Omega \subset \mathbb{R}^3$, locally situated on one side of the boundary $\partial\Omega$, which is at least locally Lipschitz.*

2.1. Notation

We will never distinguish between scalar and vector functions unless it is not clear from the context. We will denote by $C_c^\infty(\Omega)$ the space of compactly supported functions which are infinitely differentiable and $\mathcal{D}'(\Omega)$ its dual, which is the space of distributions over Ω. In the case $\Omega = \mathbb{T}^3$ the subscript "$_c$" is not needed and we set $C_c^\infty(\mathbb{T}^3) = C^\infty(\mathbb{T}^3)$. With an abuse of notation we will use $C_c^\infty(\Omega)$ for all the three choices of the domain Ω satisfying Assumption 1. We recall that for any vector $f \in C_c^\infty(\Omega;\mathbb{R}^3)$ the Helmholtz decomposition holds true: there exists two function $g \in C_c^\infty(\Omega;\mathbb{R}^3)$ and $q \in C_c^\infty(\Omega;\mathbb{R})$ such that $f = g + \nabla q$, and g is divergence-free. Given a Banach space E, we denote with $\|\cdot\|_E$ its norm. However, for the classical Lebesgue spaces $L^p(\Omega)$, with $p \in [1,\infty]$, we shall denote their norms with $\|\cdot\|_p$. Finally, we recall that the space $H_0^1(\Omega)$ is the classical Sobolev space obtained as a closure of $C_c^\infty(\Omega)$ in the norm

$$\|v\|_{H^1} := \left(\int_\Omega |v|^2 + |\nabla v|^2 \, dx\right)^{\frac{1}{2}}, \qquad v : \Omega \mapsto \mathbb{R}^k.$$

The subscript "$_0$" is needed only when Ω is a bounded domain. In the case of $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3$ we have $H_0^1(\mathbb{R}^3) = H^1(\mathbb{R}^3)$ and $H_0^1(\mathbb{T}^3) = H^1(\mathbb{T}^3)$, but as before, with an abuse of notation, we will use $H_0^1(\Omega)$ for each one of the three choices of the domain Ω satisfying Assumption 1. Moreover, we recall that $H^1(\mathbb{R}^3)$ and $H^1(\mathbb{T}^3)$ can also be characterized in terms of the Fourier Transform and the Fourier Series, respectively. When dealing with a Banach space $(E, \|\,.\,\|_E)$ we denote by $x_n \to x$, $x_n \rightharpoonup x$ and $x_n \overset{*}{\rightharpoonup} x$, the strong, weak and weak* convergence, respectively.

Next, let E be a Banach space, then $L^p(0,T;E)$, with $1 \leq p < \infty$, and $L^\infty(0,T;E)$ denote the classical Bochner spaces of strongly measurable (classes of) functions $u : (0,T) \to E$ such that

$$\|u\|_{L^p(E)} := \left(\int_0^T \|u(s)\|_E^p \, ds\right)^{\frac{1}{p}} < \infty,$$

$$\|u\|_{L^\infty(E)} := \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_E < \infty.$$

Finally, the space of weakly continuous functions in E, which is denoted by $C_w([0,T];E)$, consists of functions $u : [0,T] \mapsto E$ such that for any $f \in E^*$ the real function of real variable

$$\langle f, u\rangle_{E^*\times E} : \ [0,T] \ni t \mapsto \langle f, u(t)\rangle_{E^*\times E},$$

is continuous.

Finally, when we write $A \lesssim B$, this means that there exists a constant $c > 0$ (independent on the relevant parameters of the problem) such that $A \leq cB$.

2.2. The Spaces H and V

In the analysis of solutions of the Navier-Stokes equations is useful to consider spaces of divergence-free functions. We start by defining the space

$$\mathcal{V}(\Omega) := \{\phi \in C_c^\infty(\Omega) : \operatorname{div}\phi = 0\},$$

Then, we define the spaces

$$H := \overline{\mathcal{V}(\Omega)}^{\|\cdot\|_2}, \qquad V := \overline{\mathcal{V}(\Omega)}^{\|\cdot\|_{1,2}}.$$

We start by noticing that H and V are closed subspace of $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. Therefore, they are Hilbert space themselves with the inherited scalar products, which are

$$(u,v) := \int_\Omega u \cdot v \, dx \qquad ((u,v)) := \int_\Omega u \cdot v + \nabla u : \nabla v \, dx.$$

Next, although H and V are Hilbert space, hence reflexive, we will not identify them with their duals. We will instead denote by H' and V' the topological dual of H and V endowed with the classical dual norms

$$\|f\|_{H'} := \sup_{\phi\in\mathcal{V}(\Omega), \|\phi\|_2\leq 1} |\langle f, \phi\rangle_{H'\times H}|,$$

$$\|f\|_{V'} := \sup_{\phi\in\mathcal{V}(\Omega), \|\phi\|_{1,2}\leq 1} |\langle f, \phi\rangle_{V'\times V}|.$$

We stress that H' and V' are not subset of the space of distributions $\mathcal{D}'(\Omega)$ since $\mathcal{D}(\Omega) \not\subset H$.

Finally, we recall that by Sobolev embedding theorem and the interpolation inequality for the L^p-norm, there exists a constant $C > 0$ such that for any $2 \leq p \leq 6$, $\theta = \frac{6-p}{2p} \in [0,1]$, and any $u \in H_0^1(\Omega)$ it holds that

$$\|u\|_p \leq C\|u\|_2^\theta \|\nabla u\|_2^{1-\theta}. \tag{4}$$

The inequality (4) is a particular case of the well-known Gagliardo-Nirenberg-Sobolev inequality, see Reference [13].

3. Definition of Leray-Hopf Weak Solutions

In this section we give the definition of Leray-Hopf weak solutions and we prove some related properties. The definition is the following.

Definition 1. *A measurable vector field $u : (0,T) \times \Omega \mapsto \mathbb{R}^3$ is a Leray-Hopf weak solution of the Navier-Stokes Equations (1) and (2) if the following conditions are satisfied.*

1. *It holds that*
$$u \in C_w([0,T]; H) \cap L^2(0,T;V); \tag{5}$$
2. *For any $\phi \in \mathcal{V}(\Omega)$ and any $\chi \in C^\infty([0,T))$, it holds that*
$$\int_0^T \Big((u(t),\phi)\dot{\chi}(t) - ((u\cdot\nabla)\,u,\phi)\chi(t) - \nu(\nabla u, \nabla\phi)\chi(t) \Big)\, dt - (u_0,\phi)\chi(0) = 0; \tag{6}$$
3. *For any $t \in [0,T]$*
$$\|u(t)\|_2^2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2\, ds \le \|u_0\|_2^2. \tag{7}$$

Remark 1. *It is important to point out that it is an open problem whether or not condition (7) can be deduced from the conditions (5) and (6). Note also that in the definition we have (7) which is the so-called global energy inequality and not the equality (3).*

Remark 2. *In literature Leray-Hopf weak solutions are often defined in the space $L^\infty(0,T;H)$ rather than $C_w([0,T];H)$ and satisfying (7) for a.e. everywhere $t \in (0,T)$ instead that for any $t \in (0,T)$. This is equivalent to Definition 1, because in that case the velocity field can redefined on a set of measure zero in time in order to lie in $C_w([0,T];H)$ and satisfying (7) for any $t \in (0,T)$, see Reference [14]. We preferred to start with a solution already weakly continuous, to avoid the technical step of redefinition.*

We want to show that once we have proved the existence of a vector field satisfying the conditions in the Definition 1, we are actually solving the initial value boundary problem (1) and (2) in the sense of distributions. First of all we notice that from the condition (1), we can deduce that u is divergence-free and satisfies the boundary conditions (2) in the appropriate weak sense. The following lemma guarantee that u attains the initial datum u_0.

Lemma 1. *Let $u_0 \in H$ and u a Leray-Hopf weak solution. Then,*
$$u(t) \to u_0 \qquad \text{strongly in } H.$$

Proof. For $k \in \mathbb{N}$ and $\bar{t} \in (0,T)$, we consider the following function
$$\chi_k^{\bar{t}}(t) = \begin{cases} 1, & t \in [0,\bar{t}) \\ k(\bar{t}-t)+1, & t \in [\bar{t}, \bar{t}+\frac{1}{k}) \\ 0, & t \in [\bar{t}+\frac{1}{k}, T). \end{cases}$$

Then, by using $\chi_k^{\bar{t}}$, after sending $k \to \infty$ and using that $u \in C_w([0,T];H)$ we arrive to the following estimate:
$$\begin{aligned} |(u(t),\phi) - (u_0,\phi)| &\le \int_0^t |(\nabla u(s),\phi)| + |(u(s)\cdot\nabla)\,u(s),\phi)|\, ds \\ &\le t^{\frac{1}{2}} \left(\int_0^T \|\nabla u(s)\|_2^2 \right)^{\frac{1}{2}} \left(\|\phi\|_2^{\frac{1}{2}} + \|\phi\|_\infty \sup_{s\in[0,T]} \|u(s)\|_2 \right). \end{aligned}$$

Then, for any fixed $\phi \in \mathcal{V}(\Omega)$ we can send $t \to 0^+$ and we can conclude that $(u(0),\phi) = (u_0,\phi)$. By using the Helmholtz decomposition we deduce that this is true for any $\phi \in C_c^\infty(\Omega)$) and therefore $u(0) = u_0$ a.e. on Ω. Moreover, the previous calculations also show that

$$u(t) \to u_0 \qquad \text{in } C_w([0,T];H) \qquad \text{as } t \to 0^+.$$

and, again by using the Helmholtz decomposition, the same result is valid also for $\phi \in L^2(\Omega)$. By weak lower semi-continuity of norms in weak convergence we get

$$\|u_0\|_2 \leq \liminf_{t\to 0^+} \|u(t)\|_2^2.$$

Next, by using the energy inequality (7) we also get, by disregarding the non-negative dissipative term and taking the superior limit that

$$\limsup_{t\to 0^+} \|u(t)\|_2^2 \leq \|u_0\|_2^2.$$

This shows that $\|u(t)\|_2 \to \|u_0\|_2$, which combined with the weak convergence implies the strong convergence, since we are in an Hilbert space. Since the norm induced on H is the same as in $L^2(\Omega)$, this proves the strong convergence also in H. □

Finally, we show that to any Leray-Hopf weak solution u it is possible to associate a pressure p such that (u, p) solves the momentum equation in (1) in the sense of distributions.

Lemma 2. *Let u be a Leray-Hopf weak solution of (1) and (2). Then, there exists $p \in \mathcal{D}'((0,T);\times\Omega))$ such that*

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0 \quad \textit{in } \mathcal{D}'((0,T)\times\Omega).$$

and, for any $t \in (0,T)$, we have $p(t) \in L^2_{loc}(\Omega)$ and $\int_\Omega p(t)\,dx = 0$.

In the case of a general bounded domain Ω satisfying the Assumption 3, the proof of the Lemma 2 is very technical and requires several preliminaries of operator theory. We refer to References [7,10,11,15,16] for the proof. On the other hand in the case of Ω has no physical boundary the proof is straightforward. We consider here the case $\Omega = \mathbb{T}^3$.

Proof. Let u be a Leray-Hopf weak solution in the sense of Definition 1. For a.e. $t \in (0,T)$ consider the elliptic problem

$$\begin{aligned} -\Delta p(t) &= \mathrm{div}(u(t)\cdot\nabla u(t)) \qquad \text{in } \mathbb{T}^3, \\ \int_{\mathbb{T}^3} p(t)\,dx &= 0. \end{aligned} \tag{8}$$

Note that by (5), Gagliardo-Nirenberg Sobolev inequality (4), and standard elliptic regularity we can infer that there exists a unique solution of (8) satisfying $p \in L^{\frac{5}{3}}((0,T)\times\mathbb{T}^3)$. Next, we show that (u, p) solve the Navier-Stokes equations in the sense of distributions. Let $\psi(t,x) = \chi(t)\phi(x)$ with $\chi \in C_c^\infty(0,T)$ and $\phi \in C^\infty(\mathbb{T}^3)$. Let $\phi = P\phi + Q\phi$ be the Helmholtz decomposition, where we denote by $P\phi$ the divergence-free part of ϕ. Then, since P and Q commute with derivatives because there are no physical boundaries, we have that

$$\begin{aligned} &\int_0^T (u(t),\phi)\dot\chi(t) - ((u\cdot\nabla)u,\phi)\chi(t) - \nu(\nabla u,\nabla\phi)\chi(t) + (p(t),\mathrm{div}\,Q\phi)\chi(t)\,dt \\ &= \int_0^T (u(t),P\phi)\dot\chi(t) - ((u\cdot\nabla)u,P\phi)\chi(t) - \nu(\nabla u,\nabla P\phi)\chi(t) \\ &- \int_0^T ((u\cdot\nabla)u,Q\phi)\chi(t) - (p(t),\mathrm{div}\,\phi)\chi(t)\,dt \\ &= -\int_0^T ((u\cdot\nabla)u,Q\phi)\chi(t) - (p(t),\mathrm{div}\,Q\phi)\chi(t)\,dt = 0, \end{aligned} \tag{9}$$

where we have used (6) in the second equality and (8) together with the fact that $Q\phi = \nabla q$ for some $q \in C^\infty(\mathbb{T}^3)$ in the last equality. Finally, by an approximation argument, we have that (9) holds for any $\phi \in C_c^\infty((0,T) \times \mathbb{T}^3)$ and we conclude. □

4. Approximate Solutions of the Incompressible Navier-Stokes Equations

In this section, we define the notion of *approximate sequence of solutions* to the Navier-Stokes equations and we prove the convergence to Leray-Hopf weak solutions. We use an approach which is a little different from the one usual used. Our choice, which does not follows the historical path, is motivated by the pedagogical purpose of having a unified treatment for several different methods.

Definition 2. *Let $n \in \mathbb{N}$. We say that $\{u^n\}_n \subset C(0,T;L^2(\Omega))$ is an* approximate sequence of solutions *with divergence-free initial datum u_0^n if*

1. *It holds that*

$$\{u^n\}_n \text{ is a bounded sequence in } L^\infty(0,T;H) \cap L^2(0,T;V); \tag{10}$$

2. *For any $n \in \mathbb{N}$ and any $\phi \in \mathcal{V}(\Omega)$ there exists $R_\phi^n \in L^1(0,T)$ such that for any $\chi \in C_c^\infty([0,T))$*

$$\begin{aligned} \int_0^T \Big((u^n(t),\phi)\dot{\chi}(t) + ((u^n \cdot \nabla)\, u^n, \phi)\chi(t) + \nu(\nabla u^n, \nabla\phi)\chi(t) \Big)\, dt \\ - (u_0^n, \phi)\chi(0) = \int_0^T R_\phi^n(t)\chi(t)\, dt; \end{aligned} \tag{11}$$

3. *It holds*

$$R_\phi^n \rightharpoonup 0 \qquad \text{weakly in } L^1(0,T) \text{ as } n \to \infty; \tag{12}$$

4. *For any $n \in \mathbb{N}$ and $t \in (0,T)$ it holds that*

$$\|u^n(t)\|_2^2 + 2\nu \int_0^t \|\nabla u^n(s)\|_2^2\, ds \le \|u_0^n\|_2^2. \tag{13}$$

Since generally the existence of (smooth) approximating sequences is *rather easy* to be proved, the advantage of this definition is that one has just to check a condition on the data and condition (12) on the remainder (commutator) to show that the approximate solutions converge to a Leray-Hopf weak solution, as is done in the next theorem.

Theorem 1. *Let $u_0 \in H$ and $\{u^n\}_n$ be a sequence of approximate solutions with initial data $\{u_0^n\}_n$ such that*

$$u_0^n \to u_0 \qquad \text{strongly in } H. \tag{14}$$

Then, up to a sub-sequence not relabelled, there exists u such that if Ω satisfies $(A1)$ and $(A2)$ then

$$u^n \to u \qquad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{15}$$

and if $\Omega = \mathbb{R}^3$

$$u^n \to u \qquad \text{strongly in } L^2(0,T;L^2_{loc}(\mathbb{R}^3)). \tag{16}$$

Moreover, u is a Leray-Hopf weak solution of (1) and (2).

Remark 3. *We stress that because of Remark 1 requiring condition (13) (that is a good energy balance already on the approximate functions) is fundamental in order to obtain the energy inequality (7). Moreover, by inspecting the proof below it will be clear that given $\{u^n\}_n$ satisfying* (1) *and* (2) *in Definition 2 then there exists u satisfying* (1) *and* (2) *in Definition 1 such that the convergences (15) and (16) hold. This remark will be important in the analysis of the Implicit Euler Scheme in Section 5.3, because the scheme will not fully fit in the framework of Definition 2.*

Remark 4. *We point out that Theorem 1 is not limited to the case of vanishing external force, but the result holds also in the presence of an external body force $f \in L^2(0, T; (H_0^1(\Omega))')$. This can be obtained with minor changes in the proof. In this case Definition 2 must also be integrated with an approximating sequence f^n, and adding the term*

$$\int_0^T (f^n, u^n)\chi(t)\,dt$$

to the left-hand side of (11). This is due to the fact that some of the approximation methods in Section 5 require the body force to be smooth and it is enough to require $f^n \to f$ in $L^2(0, T; (H_0^1(\Omega))')$.

We also note that requiring the body force to be in $L^2(0, T; (H_0^1(\Omega))')$ (and not only in $L^2(0, T; V')$) is needed in order to remain inside the space of distributions and to have an associated pressure as in Lemma 2; We refer to Reference [17] for more details on this issue.

We start with the following straightforward corollary of the classical Arzelà-Ascoli theorem for real functions of a real variable.

Lemma 3. *Let E be a separable Banach space and let $\mathcal{E} \subset E$ be a dense subset. Let $\{F_n\}_n$ be a sequence of measurable functions such that $F_n : [0, T] \mapsto E^*$. Assume that*

1. *the sequence $\{F_n\}_n$ is equi-bounded in E^*,*
2. *for any fixed $\phi \in \mathcal{E}$ the sequence of real functions $\langle F_n, \phi\rangle : [0, T] \ni t \mapsto \langle F_n(t), \phi\rangle$, $n \in \mathbb{N}$, is equi-continuous.*

Then, $F_n \in C_w([0, T]; E^*)$ and there exists $F \in C_w([0, T]; E^*)$ such that, up to a sub-sequence,

$$F_n \to F \quad \text{in } C_w([0, T]; E^*).$$

A fundamental step in the proof of existence for nonlinear partial differential equations is the proof of certain compactness which allows to get strong convergence in suitable norms. Observe that the a-priori bounds are useful to get weak or weak-* convergences, by means of application of the Riesz representation theorem and –more generally– of Banach-Alaoglu-Bourbaki theorem. On the other hand, since $T(x_n) \rightharpoonup T(x)$ for a linear operator T, weak convergence allows to consider linear equations, or more precisely, the linear terms in the equations. On the other hand weak convergence is in general not enough to prove that

$$\int_0^T ((u^n \cdot \nabla)\, u^n, \phi)\chi(t)\,dt \to \int_0^T ((u \cdot \nabla)\, u, \phi)\chi(t)\,dt, \qquad \text{as } n \to \infty.$$

Hence, by the a priori estimates we can construct a limit object u, but we still have to show that u is a weak solution of the limiting problem.

To address this point several results have been used. Leray used Helly's theorem on monotone functions and an ingenious application of Riesz theorem with multiple Cantor diagonal arguments. Hopf used an inequality by Friederichs to handle the Galerkin case. Starting from the work of J.L. Lions [18] it became common to use the approach by the so-called Aubin-Lions lemma, which is borrowed from the general theory of abstract equations and is based on obtaining some estimates on the time derivative (at least in negative space) of the solution. This latter approach is very flexible, but it requires some non-trivial functional analysis preliminaries to estimate the time-derivative, since instead one can use directly some properties coming from the proper definition of the approximation. Note that in the Definition 1 of weak solution there is no mention to the time-derivative. We will show how to obtain compactness in a elementary way, directly from the weak formulation and thus avoiding the use of time derivatives in Bochner spaces. We believe this may be a simpler approach, at least for presentation to students. We also point out that in certain applications to more complex fluid problems as for instance fluids in a moving domain or non-Newtonian fluids with rheology with time-dependent constitutive law, the proper definition of the time derivative is technically complicated and an approach avoiding the use of this notion becomes particularly welcome.

The next lemma provides a general criterion for strong convergence, which has the advantage to avoid assumptions on the time derivative. Of course, the lemma holds only on bounded domains and therefore we exclude the whole space case, since in the latter one has to work locally. We also stress that the hypothesis are not optimal since we do not prove an *if and only if*, but the hypotheses are easily verifiable for general nonlinear evolution problems.

The lemma below is very similar to the one proved by Landes and Mustonen in Reference [19] and for an application to the Navier-Stokes equations see Landes [20]. For an optimal version (at least in general Hilbert spaces) we refer to Rakotoson and Temam [21].

Lemma 4. *Let $U \subset \mathbb{R}^3$ be any bounded domain or $U = \mathbb{T}^3$. Let $1 < p < \infty$ and assume that $g \in L^\infty(0,T;L^1(U)) \cap L^p(0,T;W^{1,p}_0(U))$ and $\{g^n\}_n$ is a sequence such that*

$$\{g^n\}_n \text{ is bounded in } L^\infty(0,T;L^1(U)) \cap L^p(0,T;W^{1,p}_0(U)),$$

and $g^n(t) \rightharpoonup g(t)$ weakly in $L^1(U)$ for a.e. $t \in [0,T]$. Then, it holds that

$$g^n \to g \qquad \text{in } L^p(0,T;L^p(U)).$$

Proof. We prove the lemma only in the case of $U \subset \mathbb{R}^3$ being a bounded domain with smooth boundary. First, since $g^n(t,\cdot)$ and $g(t,\cdot)$ are in $W^{1,p}_0(U)$, their extensions to zero off U are both in $W^{1,p}(\mathbb{R}^3)$. We denote by $\bar{g}^n$ and $\bar{g}$ these extensions. a.e. $t \in (0,T)$.

Let ρ_ε be a standard spatial mollifier and set $g^n_\varepsilon := \rho_\varepsilon * \bar{g}^n$ and $g_\varepsilon := \rho_\varepsilon * \bar{g}$. Next, we have that

$$\begin{aligned}
|g^n_\varepsilon(t,x) - \bar{g}^n(t,x)| &\le \varepsilon \int_{B_1} \rho(y) \int_0^1 |\nabla \bar{g}^n(t, x - \varepsilon\tau y)|\, d\tau dy \\
&\le \varepsilon \Big(\int_{B_1} \rho^{\,p'}(y)\, dy \Big)^{\frac{1}{p'}} \Big(\int_{B_1} \Big(\int_0^1 |\nabla \bar{g}^n(t, x - \varepsilon\tau y)|\, d\tau \Big)^p dy \Big)^{\frac{1}{p}} \\
&\le \varepsilon \Big(\int_{B_1} \rho^{\,p'}(y)\, dy \Big)^{\frac{1}{p'}} \Big(\int_{B_1} \int_0^1 |\nabla \bar{g}^n(t, x - \varepsilon\tau y)|^p\, d\tau dy \Big)^{\frac{1}{p}},
\end{aligned}$$

and the same estimate holds also for $\bar{g}$. Therefore, the following estimates hold

$$\begin{aligned}
\int_0^T \|g^n_\varepsilon - \bar{g}^n\|_p^p\, dt \lesssim \varepsilon^p \int_0^T \|\nabla \bar{g}^n\|_p^p\, dt, \\
\int_0^T \|g_\varepsilon - \bar{g}\|_p^p\, dt \lesssim \varepsilon^p \int_0^T \|\nabla \bar{g}\|_p^p\, dt,
\end{aligned}$$

with bounds depending only on ρ.

Next, by triangular inequality we have

$$\begin{aligned}
\int_0^T \|g^n - g\|^p_{L^p(U)}\, dt &\le \int_0^T \|g^n - g^n_\varepsilon\|^p_{L^p(U)}\, dt + \int_0^T \|g^n_\varepsilon - g_\varepsilon\|^p_{L^p(U)}\, dt + \int_0^T \|g_\varepsilon - g\|^p_{L^p(U)}\, dt \\
&\le \int_0^T \|\bar{g}^n - g^n_\varepsilon\|^p_{L^p(\mathbb{R}^3)}\, dt + \int_0^T \|g^n_\varepsilon - g_\varepsilon\|^p_{L^p(U)}\, dt + \int_0^T \|g_\varepsilon - \bar{g}\|^p_{L^p(\mathbb{R}^3)}\, dt \qquad (17) \\
&\le \varepsilon^p \int_0^T \big(\|\nabla \bar{g}^n\|_p^p + \|\nabla \bar{g}\|_p^p \big)\, dt + \int_0^T \int_U |g^n_\varepsilon(t,x) - g_\varepsilon(t,x)|^p\, dx dt.
\end{aligned}$$

The first term from the right-hand side can be made arbitrarily small by choosing ε small enough.

To conclude, we first note that by definition of convolution, there exists $C = C(\varepsilon, U)$ such that

$$|g_\varepsilon(t,x)| + |g_\varepsilon^n(t,x)| \leq C.$$

Next, since clearly it holds that for the extended functions it holds

$$\bar{g}^n(t) \rightharpoonup \bar{g}(t) \quad \text{weakly in } L^1(\mathbb{R}^3),$$

we also have that

$$g_\varepsilon^n(t,x) \to g_\varepsilon(t,x), \quad \text{a.e. in } t \in (0,T) \text{ and for all } x \in \mathbb{R}^3.$$

This follows by fixing $\epsilon > 0$, a time t such that $\bar{g}^n(t) \rightharpoonup \bar{g}(t)$, $x \in \mathbb{R}^3$, and noticing that

$$\bar{g}_\varepsilon^n(t,x) - \bar{g}_\varepsilon(t,x) = \int_{\mathbb{R}^3} (\bar{g}^n(t,y) - \bar{g}(t,y))\rho_\epsilon(x-y)\,dy \to 0,$$

as $n \to \infty$.

This shows that, for any fixed $\varepsilon > 0$, the last term in last inequality in (17) goes to zero as $n \to \infty$, by using Dominated Convergence Theorem. The proof is concluded since we showed that $\|g^n - g\|_{L^p(0,T;L^p(U))}$ can be made arbitrarily small. $\square$

The following theorem is the main result of this section.

Proof of the Theorem 1. Let $\{u^n\}_n$ be a sequence of approximate solutions. By condition (10) of Definition 2 we can infer that up to a sub-sequence (not relabelled) there exists $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ such that

$$u^n \overset{*}{\rightharpoonup} u \qquad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)), \tag{18}$$

$$u^n \rightharpoonup u \qquad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{19}$$

$$\nabla u^n \rightharpoonup \nabla u \qquad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{20}$$

For $k \in \mathbb{N}$ and $\bar{t} \in (0,T)$, we consider the following function

$$\chi_k^{\bar{t}}(t) = \begin{cases} 1, & t \in [0,\bar{t}) \\ k(\bar{t}-t)+1, & t \in [\bar{t}, \bar{t}+\frac{1}{k}) \\ 0, & t \in [\bar{t}+\frac{1}{k}, T), \end{cases} \tag{21}$$

Let $\phi \in C_c^\infty(\Omega)$ with $\operatorname{div}\phi = 0$ and $s,t \in (0,T)$. By using the function $\chi_k^{\bar{t}}(t)$ with first with $\bar{t} = t$ and then with $\bar{t} = s$, together with the fact that $u^n \in C(0,T;L^2(\Omega))$ we can infer that

$$(u^n(t),\phi) - (u^n(s),\phi) + \int_s^t \Big(((u^n(\tau)\cdot\nabla)\,u^n(\tau),\phi) + \nu(\nabla u^n(\tau),\nabla\phi) + R_\phi^n(\tau) \Big)\,d\tau = 0.$$

Next, for $\phi \in C_c^\infty(\Omega)$, let $F^n(t) := ((u^n(t)\cdot\nabla)\,u^n(t),\phi) + \nu(\nabla u^n(t),\nabla\phi) + R_\phi^n(t)$. Then, condition (10), the Gagliardo-Nirenberg-Sobolev inequality (4), and the hypothesis on R_ϕ^n in (11) imply that the family $\{F^n\}_n$ is equi-integrable and then the function $t \mapsto (u^n(t),\phi)$ is equi-continuous. Since $\mathcal{V}(\Omega)$ is dense in H and H is reflexive, we can conclude by using Lemma 3 that

$$u^n \to u \quad \text{in } C_w([0,T];H). \tag{22}$$

By using (22) and (20) we can prove that u satisfies the energy inequality (7). Indeed, for any $t \in (0,T)$ we have that

$$\|u(t)\|_2^2 \leq \liminf_{n\to\infty} \|u^n(t)\|_2^2,$$

$$\int_0^t \|\nabla u(s)\|_2^2\,ds \leq \liminf_{n\to\infty} \int_0^t \|\nabla u^n(s)\|_2^2\,ds.$$

Then,

$$\begin{aligned}\|u(t)\|_2^2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2\,ds &\leq \liminf_{n\to\infty} \|u^n(t)\|_2^2 + \liminf_{n\to\infty} 2\nu \int_0^t \|\nabla u^n(s)\|_2^2\,ds \\ &\leq \liminf_{n\to\infty}\Big(\|u^n(t)\|_2^2 + 2\nu \int_0^t \|\nabla u^n(s)\|_2^2\,ds\Big) \\ &\leq \lim_{n\to\infty} \|u_0^n\|_2^2 = \|u_0\|_2^2.\end{aligned}$$

where we have used (14). In order to conclude it remains only to prove (15) and (16). If Ω is the flat torus or a bounded domain, then (15) follows directly by Lemma 4. If $\Omega = \mathbb{R}^3$ we need a localization argument. We first note that by the Helmholtz decomposition (22) holds also in $C_w([0,T];L^2(\mathbb{R}^3))$. Next, for $k \in \mathbb{N}$ let $\psi \in C_c^\infty(B_{k+1}(0))$ such that $\psi = 1$ on B_k and define $g^n := u^n\,\psi$. The sequence $\{g^n\}_n$ satisfies the hypothesis of Lemma 4, and therefore, after a diagonal argument, it follows that there exists a sub-sequence not relabelled such that

$$g^n \to g = u\,\psi \qquad \text{strongly in } L^2(0,T;L^2(B_{k+1})). \tag{23}$$

Then, condition (23) easily implies (16). □

5. Approximation Methods

After the general result of the previous section, we are now going to show that a general class of methods used to construct weak solutions will fit the in the framework of Theorem 1, as described in Section 4.

5.1. Leray Approximation Scheme

We start describing the original scheme introduced by Leray in Reference [2] (even if we use a completely different compactness argument to show the convergence of approximations). In this case we consider $\Omega = \mathbb{R}^3$. We fix a sequence $\{\varepsilon_n\}$ of positive numbers going to zero and let ρ_{ε_n} be a standard mollifier (only) in the space variables. For $v : (0,T)\times\mathbb{R}^3 \mapsto \mathbb{R}^3$ we set $\Psi_n(v) := \rho_{\varepsilon_n} * v$, where the convolution is only in the space variables. Let $u_0 \in H$ and let $n \in \mathbb{N}$. Define $u_0^n = \Psi_n(u^0)$ and u^n as the solution of the following Cauchy problem:

$$\begin{cases} \partial_t u^n - \nu\Delta u^n + \big(\Psi_n(u^n)\cdot\nabla\big)\,u^n + \nabla p^n = 0 & \text{in } (0,T)\times\mathbb{R}^3, \\ \operatorname{div} u^n = 0 & \text{in } (0,T)\times\mathbb{R}^3, \\ u^n|_{t=0} = u_0^n & \text{on } \{t=0\}\times\mathbb{R}^3. \end{cases} \tag{24}$$

We want to prove that for any $n \in \mathbb{N}$ the function u^n exists, is smooth, and $\{u^n\}_n$ is an approximate sequence of solutions in the sense of Definition 2.

Theorem 2. *Let $u_0 \in H$. Then, it holds that*

1. *for any fixed $n \in \mathbb{N}$ there exists a unique $u^n \in C([0,T);H^3(\mathbb{R}^3))$ solution of (24);*
2. *there exists a Leray-Hopf weak solution u and a possible sub-sequence of $\{u^n\}_n$ such that*

$$u^n \to u \qquad \textit{strongly in } L^2(0,T;L^2_{loc}(\mathbb{R}^3)).$$

Proof. Let us prove (1). The proof is very classical so we only sketch it. By using a fixed point argument we can prove that there exists a time $T_1 = T_1(\|u_0^n\|_{H^3}) > 0$ such that there exists a unique $u^n \in C([0,T^*);H^3(\mathbb{R}^3))$ solution of (24) for $T^* \geq T_1$. Let us suppose that T^* is the maximal time of existence of u^n and, if $T^* < T$ then $\lim_{t\to T^{*-}} \|u^n(s)\|_{H^3} = \infty$.

To obtain a global solution we exploit a standard energy estimates argument. Indeed, we first note that by multiplying (24) by u^n and integrating by parts we get

$$\|u^n(t)\|_2^2 + 2\nu \int_0^t \|\nabla u^n(s)\|_2^2\, ds = \|u_0^n\|_2^2, \tag{25}$$

with an equality which is valid for all $t < T^*$. Note that this is exactly the same calculation we have done formally to obtain the energy inequality (3) in the introduction. In particular, from (25) we obtain

$$\sup_{t\in(0,T)} \|u^n(t)\|_2 \le \|u_0^n\|_2. \tag{26}$$

Next, by using that $H^3(\mathbb{R}^3)$ is an algebra and by using the standard properties of mollifiers, it is easy to prove that

$$\begin{aligned}
\frac{d}{dt}\|u^n(t)\|_{H^3}^2 + \nu\|\nabla u^n(t)\|_{H^3}^2 &\lesssim \left|\left(\left(\Psi_n(u^n(t))\cdot\nabla\right) u^n(t), u^n(t)\right)_{H^3}\right| \\
&\lesssim \|\Psi_n(u^n(t))\|_{H^3}\|\nabla u^n(t)\|_{H^3}\|u^n(t)\|_{H^3} \\
&\lesssim \|\Psi_n(u^n(t))\|_{H^3}^2\|u^n(t)\|_{H^3}^2 + \frac{\nu}{2}\|\nabla u^n(t)\|_{H^3}^2 \\
&\lesssim \frac{1}{\epsilon_n^6}\|u^n(t)\|_2^2\|u^n(t)\|_{H^3}^2 + \frac{\nu}{2}\|\nabla u^n(t)\|_{H^3}^2 \\
&\lesssim \frac{1}{\epsilon_n^6}\|u_0^n\|_2^2\|u^n(t)\|_{H^3}^2 + \frac{\nu}{2}\|\nabla u^n(t)\|_{H^3}^2.
\end{aligned}$$

where in the last inequality we have used (26). Therefore, we have that

$$\frac{d}{dt}\|u^n(t)\|_{H^3}^2 \le C_n\|u_0^n\|_2^2\|u^n(t)\|_{H^3}^2$$

and, by using the Gronwall Lemma, we conclude that necessarily $T^* = T$ (this argument shows that in fact u^n is defined for all $t > 0$, for any fixed $n \in \mathbb{N}$).

Next, to show (2) it is enough to prove that $\{u^n\}_n$ satisfies the conditions in Definition 2. Clearly, from(1) we have that $\{u^n\}_n \subset C([0,T];L^2(\mathbb{R}^3))$. By using the standard property of mollifiers $\|\Psi_n(u^0)\|_2 \le \|u_0\|_2$, and from the energy estimate (25) we get that $\{u^n\}_n$ is bounded uniformly in $L^\infty(0,T;H) \cap L^2(0,T;H)$. Moreover, (25) is exactly (13) and then it remains only to verify (11). For $\phi \in C_c^\infty(\Omega)$ and $t \in (0,T)$ the function $t \mapsto R_\phi^n(t)$ is defined as follows

$$R_\phi^n(t) := \Big(\left(\left[\Psi_n(u^n(t)) - u^n(t)\right)\right]\cdot\nabla\big)\, u^n(t), \phi\Big).$$

With this choice of R_ϕ^n, the equation (11) is satisfied and it remains only to prove that convergence stated in condition (12) of Definition 2. First, note that by Hölder inequality

$$|R_\phi^n(t)| \le \|\nabla u^n(t)\|_2\|\phi\|_\infty\|\Psi_n(u^n(t)) - u^n(t)\|_2. \tag{27}$$

Then, for any fixed $(t,x) \in (0,T)\times\mathbb{R}^3$, by a direct calculation (using again the properties of mollifiers) we have

$$\begin{aligned}
|\Psi_n(u^n(t,x)) - u^n(t,x)| &\le \varepsilon_n \int_{B_1} \rho(y) \int_0^1 |\nabla u^n(t, x-\varepsilon\tau y)|\, d\tau dy \\
&\le \varepsilon_n \left(\int_{B_1} \rho^2(y)\, dy\right)^{\frac{1}{2}} \left(\int_{B_1}\left(\int_0^1 |\nabla u^n(t,x-\varepsilon\tau y)|\, d\tau\right)^2 dy\right)^{\frac{1}{2}} \\
&\le \varepsilon_n \left(\int_{B_1} \rho^2(y)\, dy\right)^{\frac{1}{2}} \left(\int_{B_1}\int_0^1 |\nabla u^n(t,x-\varepsilon\tau y)|^2\, d\tau dy\right)^{\frac{1}{2}}.
\end{aligned}$$

Then, by a further integration

$$\int_{\mathbb{R}^3} |\Psi_n(u^n(t,x)) - u^n(t,x)|^2 \, dx \leq \varepsilon_n^2 \int_{\mathbb{R}^2} |\nabla u^n(t,x)|^2 \, dx,$$

and going back to (27) we have that

$$|R^n_\phi(t)| \leq \varepsilon_n \|\nabla u^n(t)\|_2^2 \|\phi\|_\infty.$$

Since the sequence $\{\nabla u^n\}_n$ is bounded uniformly with respect to n in $L^2(0,T;L^2(\mathbb{R}^3))$, we have that $R^n_\phi \to 0$ in $L^1(0,T)$. □

5.2. Faedo-Galerkin Method

The next scheme we consider is the Faedo-Galerkin method. The variant we present is close to the one considered by Hopf and is at the basis of several computational methods, which are used also in fields different from fluid dynamics. In particular, we will see that the unified treatment is possible under the assumption of having a basis which is orthogonal in both L^2 and H^1, as is the case of the spectral basis made by eigenfunctions of the Stokes operator. Observe that in the space-periodic case this basis is explicitly constructed by considering complex exponentials, while in the case of a smooth bounded domain, the existence is obtained via the standard theory of compact operators, showing existence of countable non-decreasing positive $\{\lambda_j\}$ and smooth $\{\psi_j\}$ such that it holds for all $j \in \mathbb{N}$

$$\begin{aligned} -\Delta\psi_j + \nabla\pi_j &= \lambda_j\psi_j && \text{in } \Omega, \\ \operatorname{div} \psi_j &= 0 && \text{in } \Omega, \\ \psi_j &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We consider $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$ or the three-dimensional flat torus. Let be given an orthonormal basis $\{\psi_m\}_{m\in\mathbb{N}}$ of H, such that $\psi_m \in \mathcal{V}(\Omega)$. The Faedo-Galerkin method is based on the construction of approximate solutions of the type

$$u^n(t,x) = \sum_{j=1}^{n} c_j^n(t)\psi_j(x) \qquad n \in \mathbb{N}, \tag{28}$$

which solve the Navier-Stokes equations projected equations over the finite dimensional space $V_n = \mathrm{Span}(\psi_1, \ldots, \psi_n) \subset V$. This means that for $n \in \mathbb{N}$, the approximate problem to be solved is given by

$$\begin{cases} \dfrac{d}{dt}(u^n, \psi_m) + \nu(\nabla u^n, \nabla\psi_m) + ((u^n \cdot \nabla)\, u^n, \psi_m) = 0 & t \in (0,T), \\ (u^n(0), \boldsymbol{\psi}_m) = (u_0, \psi_m) & t = 0, \end{cases} \tag{29}$$

for $m = 1, \ldots, n$, which is a Cauchy problem for a system of n ODE's in the coefficients $\{c_j^n(t)\}_{j=1}^n$. Let P^n be projection operator from H into V_n:

$$P^n : f \in H \mapsto P^n f := \sum_{m=1}^{n} (f, \psi_m)\psi_m.$$

Then, the ODE's (29) reduce to the following system of PDE's:

$$\begin{cases} \partial_t u^n + P^n\big((u^n \cdot \nabla)\, u^n\big) - \nu\Delta u^n = 0 & \text{in } (0,T) \times \Omega, \\ u^n|_{t=0} = P^n\, u^0 & \text{in } \Omega. \end{cases} \tag{30}$$

In the next theorem we prove that u^n is smooth and exists on $(0,T)$, and that $\{u^n\}_n$ is an approximate sequence of solutions.

Theorem 3. *Let $u_0 \in H$. Then, it holds that*

1. *For any fixed $n \in \mathbb{N}$ there exists a unique $u^n \in C^1((0,T);C^\infty(\Omega)) \cap C^0([0,T);C^\infty(\Omega))$ solution of (30);*
2. *There exists a Leray-Hopf weak solution u and a possible sub-sequence of $\{u^n\}_n$ such that*

$$u^n \to u \qquad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Proof. We prove (1). By the theory of ordinary differential equations one easily obtains that there exists a unique solution $c_j^n(t) \in C^1(0,T^n)$, for some $0 < T^n \leq T$, being (29) a nonlinear (quadratic) system in the coefficients $c_j^n(t)$. Moreover, u^n is defined through (28) and satisfies (30). Then, by multiplying (30) by u^n and integrating by parts we get

$$\|u^n(t)\|_2^2 + 2\nu \int_0^t \|\nabla u^n(s)\|_2^2\, ds = \|u_0^n\|_2^2 \leq \|u_0\|_2^2, \tag{31}$$

where we have used that $\|u_0^n\|_2 = \|P^n u_0\|_2 \leq \|u_0\|_2$. Therefore, for any $n \in \mathbb{N}$ we have that

$$\sum_{j=1}^n |c_j^n(t)|^2 = \|u^n(t)\|_2^2 \leq \|u_0\|_2^2,$$

which easily implies that necessarily $T^n = T$.

To prove (2) we show that $\{u^n\}_n$ satisfy the conditions in Definition 2. Clearly, the sequence $\{u^n\}_n$ is in $C([0,T];L^2(\Omega))$ and by (31) it verifies the condition (1) and the energy inequality (13). To check that (11) is verified, let $\phi \in \mathcal{V}(\Omega)$ and, for any $t \in (0,T)$, define

$$R_\phi^n(t) := \big(P^n((u^n(t)\cdot\nabla)\,u^n(t)) - (u^n(t)\cdot\nabla)\,u^n(t), \phi\big).$$

Note that we have that

$$\begin{aligned}|R_\phi^n(t)| &\lesssim \|u^n(t)\|_3 \|\nabla u^n(t)\|_2 \|\phi - P^n\phi\|_6 \\ &\lesssim \|u^n(t)\|_2^{\frac{1}{2}} \|u^n(t)\|_{H^1}^{\frac{3}{2}} \|\phi - P^n\phi\|_{H^1},\end{aligned}$$

where we have used the Gagliardo-Nirenberg-Sobolev inequality (4) and that P^n is a projection in both H and V, since in this case it holds

$$\|\nabla[f - \sum_{m=1}^n (f,\psi_m)\psi_m]\|^2 = \sum_{m=n+1}^\infty \lambda_m |(f,\psi_m)|^2 \|\psi_m\|^2 \overset{n\to+\infty}{\longrightarrow} 0,$$

for all $f \in H_0^1(\Omega)$. Then, by using Hölder inequality, and Gagliardo-Nirenberg Sobolev inequality and taking into account that $T < \infty$ we have that $R_\phi^n \to 0$ in $L^1(0,T)$. □

As already specified if $\Omega = \mathbb{T}^3$, then one can take $\{\psi_m\}_{m\in\mathbb{N}}$ to be the Fourier basis. Then, the Faedo-Galerkin method consists in finding the approximated sequence of type (28) solving the Navier-Stokes equations projected over the first n Fourier modes. On the other hand, in the case $\Omega = \mathbb{R}^3$ one possible choice is to use the method of invading domains, that is to consider the problem in the ball $B(0,R)$ with zero boundary conditions on $\partial B(0,R)$ and to construct a solution u_R by the Galerkin method. It turns out that the energy estimate (3) is valid for u_R, providing uniform estimates (on u_R which is considered as a function over the whole space, after extension by zero off of Ω); this allows to pass to the limit as $R \to +\infty$, more or less in the same way as before.

5.3. Implicit Euler Scheme

The scheme we consider in the present subsection deals with the time-discretization and represents a first step also in the numerical analysis of the Navier-Stokes equations.

We consider the case of Ω being a bounded domain satisfying the hypothesis $(A3)$. Let $n \in \mathbb{N}$ and define the time-step $\kappa_n := T/n$ and the net $I^M = \{t_m\}_{m=0}^n$, such that $t_i - t_{i-1} = \kappa_n$, for any $i = 1, ..., n$.

Moreover, given $u_0 \in H$, consider a sequence (In the space periodic setting this can be obtained simply with a mollification with kernel ρ_ϵ, with $\epsilon = \frac{1}{\sqrt{n}}$.) of initial data $\{u_0^n\}_n \subset V$ such that

$$\|\nabla u_0^n\|_2 \lesssim \sqrt{n}\|u_0^n\|_2, \qquad \text{and} \qquad u_0^n \to u_0 \qquad \text{strongly in } H. \tag{32}$$

For $m \in \{1, ..., n\}$, given $\widetilde{u}_n^{m-1}$ the iterate $\widetilde{u}_n^m$ is obtained by solving the boundary value problem

$$\begin{cases} \dfrac{\widetilde{u}_n^m - \widetilde{u}_n^{m-1}}{\kappa_n} - \nu\Delta\widetilde{u}_n^m + (\widetilde{u}_n^m \cdot \nabla)\,\widetilde{u}_n^m + \nabla\widetilde{p}_n^m = 0 & \text{in } \Omega, \\ \nabla \cdot \widetilde{u}_n^m = 0 & \text{in } \Omega, \\ \widetilde{u}_n^m = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\widetilde{u}_n^0 = u_0^n$. (In the case of a non-zero force one has to set $\widetilde{f}_n^m = \kappa_n^{-1}\int_{t_{m-1}}^{t_m} f(t)\,dt$ in the right-hand side of the momentum equation which defines $\widetilde{u}_n^m$.)

For any fixed $n \in \mathbb{N}$, we define the following sequences of functions defined on $[0, T]$ with values in V and in $L^2(\Omega)$:

$$\begin{aligned} u^n(t) &= \sum_{m=1}^n \chi_{[t_{m-1},t_m)}(t)\left(\widetilde{u}_n^{m-1} + \frac{(t - t_{m-1})}{\kappa_n}(\widetilde{u}_n^m - \widetilde{u}_n^{m-1})\right), \quad u^n(t_n) = \widetilde{u}_n^n, \\ v^n(t) &= \sum_{m=1}^n \chi_{(t_{m-1},t_m]}(t)\,\widetilde{u}_n^m, \quad v^n(t_0) = u_0^n. \\ p^n(t) &= \sum_{m=1}^n \chi_{(t_{m-1},t_m]}(t)\,\widetilde{p}_n^m. \end{aligned} \tag{33}$$

We are now ready to prove the following theorem, which is referred in literature as an "alternate proof" by semi-discretization, see Reference [11].

Theorem 4. *Let $u_0 \in H$. Then, it holds that*

1. *For any fixed $n \in \mathbb{N}$, there exist $\{\widetilde{u}_n^m\}_{m=1}^n \subset H_0^1(\Omega)$ such that for any $m = 1, ..., n$ and any $\psi \in V$*

$$(\widetilde{u}_n^m, \psi) - (\widetilde{u}_n^{m-1}, \psi) + \kappa_n\,\nu(\nabla\widetilde{u}_n^m, \nabla\psi) + \kappa_n((\widetilde{u}_n^m \cdot \nabla)\,\widetilde{u}_n^m, \psi) = 0; \tag{34}$$

2. *There exists a Leray-Hopf weak solution u such that the sequence $\{u^n\}_n$ and $\{v^n\}_n$, defined in (33), satisfy*

$$\begin{aligned} u^n &\to u & &\textit{strongly in } L^2(0, T; L^2(\Omega)), \\ u^n - v^n &\to 0 & &\textit{strongly in } L^2(0, T; L^2(\Omega)). \end{aligned}$$

Proof. For the proof of *(1)* we refer to Reference [11]. The idea is the following: For any fixed $n \in \mathbb{N}$ and any $m = 1, ..., n$, the existence of $\widetilde{u}_n^m \in V$ solution of (34) is obtained by applying the Brouwer fixed point theorem to the following modified version of the steady Navier-Stokes equations, where the given iterate $\widetilde{u}_n^{m-1}$ is considered an external force:

$$\begin{aligned} \frac{\widetilde{u}_n^m}{\kappa_n} - \nu\Delta\widetilde{u}_n^m + (\widetilde{u}_n^m \cdot \nabla)\,\widetilde{u}_n^m + \nabla\widetilde{p}_n^m - \frac{\widetilde{u}_n^{m-1}}{\kappa_n} &= 0 \quad \text{in } \Omega, \\ \nabla \cdot \widetilde{u}_n^m &= 0 \quad \text{in } \Omega, \\ \widetilde{u}_n &= 0 \quad \text{in } \partial\Omega. \end{aligned}$$

In particular, by the definitions (33), we have that $\{u^n\}_n \subset C(0,T;L^2(\Omega))$ and $\{v^n\}_n \subset L^2(0,T;L^2(\Omega))$.

Next, we prove part (2).

By taking $\psi = \widetilde{u}_n^m$ in (34) and by using the elementary inequality $(a-b,a) = \frac{a^2-b^2}{2} + \frac{(a-b)^2}{2}$ valid for all $a,\ b \in \mathbb{R}$, we have that

$$\|\widetilde{u}_n^m\|_2^2 - \|\widetilde{u}_n^{m-1}\|_2^2 + \|\widetilde{u}_n^m - \widetilde{u}_n^{m-1}\|_2^2 + \kappa_n \nu \|\nabla \widetilde{u}_n^m\|_2^2 = 0. \tag{35}$$

Then, for any fixed $m \in \{1, ..., n\}$ we have that

$$\|\widetilde{u}_n^m\|_2^2 \le \|u_0^n\|_2^2 \le \|u_0\|_2^2, \tag{36}$$

$$\kappa\, \nu \sum_{i=1}^{m} \|\nabla \widetilde{u}_n^i\|_2^2 \le \|u_0^n\|_2^2 \le \|u_0\|_2^2, \tag{37}$$

$$\sum_{i=1}^{m} \|\widetilde{u}_n^i - \widetilde{u}_n^{i-1}\|_2^2 \le \|u_0^n\|_2^2 \le \|u_0\|_2^2. \tag{38}$$

By using (36)–(38) and (33) we easily have that

$$\{u^n\}_n \text{ is bounded in } L^\infty(0,T;H), \tag{39}$$

$$\{v^n\}_n \text{ is bounded in } L^\infty(0,T;H) \cap L^2(0,T;V). \tag{40}$$

We want to prove a uniform bound in $L^2(0,T;V)$ also for $\{u^n\}_n$. By a direct calculation we have that

$$\begin{aligned}
\int_0^T \|\nabla u^n(t)\|_2^2\, dt &= \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \left(1 - \frac{(t-t_{m-1})}{\kappa_n}\right)^2 \|\nabla \widetilde{u}_n^{m-1}\|_2^2\, dt \\
&\quad + 2\sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \left(1 - \frac{(t-t_{m-1})}{\kappa_n}\right)\left(\frac{(t-t_{m-1})}{\kappa_n}\right) (\nabla \widetilde{u}_n^{m-1}, \nabla \widetilde{u}_n^m)\, dt \\
&\quad + \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \left(\frac{(t-t_{m-1})}{\kappa_n}\right)^2 \|\nabla \widetilde{u}_n^m\|_2^2\, dt \\
&\le \frac{\kappa_n}{2} \sum_{m=1}^{n} \|\nabla \widetilde{u}_n^{m-1}\|_2^2 + \frac{\kappa_n}{2} \sum_{m=1}^{n} \|\nabla \widetilde{u}_n^m\|_2^2 \\
&\le \frac{\kappa_n}{2} \|\nabla \widetilde{u}_n^0\|_2^2 + \kappa_n \sum_{m=1}^{n} \|\nabla \widetilde{u}_n^m\|_2^2.
\end{aligned}$$

By using (32) we obtain

$$\begin{aligned}
\int_0^T \|\nabla u^n(t)\|_2^2\, dt &\lesssim \kappa_n \sum_{m=1}^{n} \|\nabla \widetilde{u}_n^m\|_2^2 + \kappa_n \|\nabla u_0^n\|_2^2 \\
&\lesssim \kappa_n \sum_{m=1}^{n} \|\nabla \widetilde{u}_n^m\|_2^2 + \|u_0\|_2^2 \lesssim \|u_0\|_2^2.
\end{aligned}$$

where we have also used that $\kappa_n = T/n$ and (37). Therefore we have that $\{u^n\}_n$ is bounded in $L^2(0,T;V)$ and then, taking into account (39), $\{u^n\}_n$ satisfies the condition *(1)* in Definition 2. Next, we show that $\{u^n\}_n$ satisfies the condition *(2)* of Definition 2. First, for all $\phi \in \mathcal{V}(\Omega)$ and $\chi \in C_c^\infty([0,T))$ we have, by using (33) and (34), that

$$\int_0^T \Big((u^n(t), \phi)\dot{\chi}(t) + ((v^n(t) \cdot \nabla) v^n(t), \phi)\chi(t) + \nu (\nabla v^n(t), \nabla \phi)\chi(t) \Big)\, dt - (u_0^n, \phi)\chi(0) = 0.$$

If we define

$$R^n_\phi := (v^n(t) - u^n(t), \nu\Delta\phi) + ((v^n(t) - u^n(t)) \otimes v^n(t) + u^n(t) \otimes (v^n(t) - u^n(t)), \nabla\phi),$$

then $\{u^n\}_n$ satisfies the formulation (11) and we only need to prove that (12). To this end we note that

$$\int_0^T |R^n_\phi(t)|\,dt \le c\|\nabla\phi\|_\infty \Big(\int_0^T \|u^n(s) - v^n(s)\|_2^2\,ds\Big)^{\frac12}\Big(\int_0^T \|u^n(s)\|^2 + \|v^n(s)\|_2^2\,ds\Big)^{\frac12}$$
$$+ \nu\,T^{\frac12}\|\nabla^2\phi\|_2\Big(\int_0^T \|u^n(s) - v^n(s)\|_2^2\,ds\Big)^{\frac12}.$$

By a direct calculation, we have that

$$\int_0^T \|u^n(t) - v^n(t)\|_2^2\,dt = \frac{\kappa_n}{3}\sum_{m=1}^{n} \|\widetilde{u}^m_n - \widetilde{u}^{m-1}_n\|_2^2 \le C\kappa_n, \tag{41}$$

and therefore

$$\int_0^T |R^n_\phi(t)|\,dt \le C\|\nabla\phi\|_\infty\,\kappa_n,$$

and (12) follows.

In conclusion, we have proved that $\{u^n\}_n$ satisfies the conditions (1) and (2) of Definition 2 and thanks to Theorem 1 and Remark 3, there exists u satisfying the condition (1) and (2) in Definition 1. Then, in order to conclude, we only need to prove that u satisfies also the energy inequality. First, we note that by using (35) and (33), a direct calculation implies that for any $t \in (0,T)$

$$\|v^n(t)\|_2^2 + 2\nu\int_0^t \|\nabla\,v^n(s)\|_2^2\,ds \le \|u^n_0\|_2^2. \tag{42}$$

By using (40) and (41) we can infer that v^n converges to the same limit of u^n, namely that

$$\begin{aligned} v^n &\to u && \text{strongly in } L^2(0,T;L^2(\Omega)),\\ \nabla v^n &\rightharpoonup \nabla u && \text{weakly in } L^2(0,T;L^2(\Omega)). \end{aligned} \tag{43}$$

For $k \in \mathbb{N}$ and $t \in (0,T)$, let χ^t_k be the same function already defined in (21). Noticing that $-\dot{\chi}^t_k$ is positive, after multiplying (42) and integrating in time we get that, for any $t \in (0,T)$, it holds

$$\frac{1}{k}\int_t^{t+k} \|v^n(s)\|_2^2\,ds + 2\nu\int_0^T \chi^t_k(s)\|\nabla v^n(s)\|_2^2\,ds \le \|u^n_0\|_2^2\int_0^T(-\dot{\chi}^t_k(s))\,ds = \|u^n_0\|_2^2.$$

By using (43) we get

$$\frac{1}{k}\int_t^{t+k} \|u(s)\|_2^2\,ds + 2\nu\int_0^T \chi^t_k(s)\|\nabla u(s)\|_2^2\,ds \le \|u_0\|_2^2,$$

and by Lebesgue differentiation and dominated convergence theorems we obtaine that for a.e. $t \in (0,T)$

$$\|u(t)\|_2^2 + 2\nu\int_0^t \|\nabla u(s)\|_2^2\,ds \le \|u_0\|_2^2. \tag{44}$$

Let $\mathcal{N} \subset (0,T)$ the set of measure zero where (44) does not hold and fix $t \in \mathcal{N}$. Then, there exists $\{t_k\}_k \subset (0,T)\setminus\mathcal{N}$ such that $t_k \to t$ and

$$\|u(t_k)\|_2^2 + 2\nu\int_0^{t_k} \|\nabla u(s)\|_2^2\,ds \le \|u_0\|_2^2.$$

Since $u \in C_w(0,T;H)$ and $\|\nabla u(\cdot)\|_2^2 \in L^1(0,T)$ it follows that

$$\|u(t)\|_2^2 + 2\nu \int_0^t \|\nabla u(s)\|_2^2\, ds \leq \liminf_{k\to\infty} \|u(t_k)\|_2^2 + \lim_{k\to\infty} 2\nu \int_0^{t_k} \|\nabla u(s)\|_2^2\, ds \leq \|u_0\|_2^2,$$

and therefore (44) holds for any $t \in (0,T)$. □

5.4. Smagorinsky-Ladyžhenskaya Model

In this section we show how the approximation by adding a nonlinear stress tensor produce weak solutions. We consider for $n \in \mathbb{N}$ the following boundary initial value problem

$$\begin{cases} \partial_t u^n + (u^n \cdot \nabla)\, u^n + \nabla p^n - \nu \Delta u^n - \dfrac{1}{n} \operatorname{div}(|Du^n| Du^n) = 0 & \text{in } (0,T) \times \Omega, \\ \operatorname{div} u^n = 0 & \text{in } (0,T) \times \Omega, \\ u^n = 0 & \text{on } (0,T) \times \partial\Omega, \\ u^n(0) = u_0 & \text{in } \Omega, \end{cases} \tag{45}$$

where $Du^n = \frac{\nabla u^n + (\nabla u^n)^T}{2}$. This system has been introduced for numerical approximation of turbulent flows by Smagorinsky [22] and its analysis as a possible approximation for the Navier-Stokes equations started with the studies by Ladyženskaya [23], cf. also Reference [24] for the role of this method in the analysis of Large Eddy Simulation models. For the analysis also of related models, with general stress tensor given by $S(v) = S(Dv) = |Dv|^{p-2}Dv$, with various values of p, see References [18,25] and also the more recent Reference [26].

Theorem 5. *Let $u_0 \in H$. Then, it holds that*

1. *For any fixed $n \in \mathbb{N}$ there exists a unique $u^n \in C([0,T);L^2(\Omega))$ solution of (45);*
2. *There exists a Leray-Hopf weak solution u and a possible sub-sequence of $\{u^n\}_n$ such that*

$$u^n \to u \qquad \textit{strongly in } L^2(0,T;L^2(\Omega)).$$

Proof. By using the theory of monotone operators (cf. References [8,18]) there exists a unique $u^n \in C(0,T;L^2(\Omega))$ weak solution of (45) with $u^n \in L^\infty(0,T;H) \cap L^2(0,T;V)$ and $Du^n \in L^3(0,T;L^3(\Omega))$ such that it holds

$$\|u^n(t)\|_2^2 + 2\nu \int_0^t \|\nabla u^n(s)\|_2^2\, ds + \frac{2}{n}\int_0^t \|Du^n(s)\|_3^3\, ds \leq \|u_0^n\|_2^2.$$

Observe that by Korn inequality $\|Du^n\|_3 \sim \|\nabla u^n\|_3$.

To prove the second part of Theorem 5 we show that $\{u^n\}_n$ satisfy the conditions in Definition 2. Define the remainder $R^n_\phi(t)$ by

$$R^n_\phi(t) := -\frac{1}{n}\int_\Omega |Du^n(t)| Du^n(t) \cdot D\phi\, dx.$$

By means of the Hölder inequality we get

$$|R^n_\phi(t)| \leq \frac{1}{n}\int_\Omega |Du^n(t)|^2 |D\phi|\, dx \leq \frac{1}{n}\|Du^n\|_3^2 \|D\phi\|_3.$$

Consequently, it also holds

$$\int_0^T |R^n_\phi(t)|\, dt \leq \frac{T^{1/3}}{n^{1/3}} \left(\frac{1}{n}\int_0^T \|Du^n\|_3^3\, dt\right)^{2/3} \|D\phi\|_3,$$

showing that $R_\phi^n \to 0$ in $L^1(0,T)$. Since the other conditions in Definition 2 are trivially satisfied, an application of Theorem 1 finally ends the proof. □

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