Exponential Pseudo-Splines: looking beyond Exponential B-splines

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Abstract

Pseudo-splines are a rich family of functions that allows the user to meet various demands for balancing polynomial reproduction (i.e., approximation power), regularity and support size. Such a family includes, as special members, B-spline functions, universally known for their usefulness in different fields of application. When replacing polynomial reproduction by exponential polynomial reproduction, a new family of functions is obtained. This new family is here constructed and called the family of exponential pseudo-splines. It is the nonstationary counterpart of (polynomial) pseudo-splines and includes exponential B-splines as a special subclass. In this work we provide a computational strategy for deriving the explicit expression of the Laurent polynomial sequence that identifies the family of exponential pseudo-spline nonstationary subdivision schemes. For this family we study its symmetry properties and perform its convergence and regularity analysis. Finally, we also show that the family of primal exponential pseudo-splines fills in the gap between exponential B-splines and interpolatory cardinal functions. This extends the analogous property of primal pseudo-spline stationary subdivision schemes.

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1. Introduction

This paper deals with the family of exponential pseudo-splines and the nonstationary subdivision schemes used to generate them. Exponential pseudo-splines are a generalization of exponential B-splines (see [8, 41]) and, like these can be obtained as limits of nonstationary subdivision schemes. Subdivision schemes are efficient iterative methods for the generation of graphs of functions, curves and surfaces via the specification of an initial set of discrete data and a set of local refinement rules. Over the past 20 years subdivision schemes have shown their usefulness in several application contexts ranging from Computer-Aided Geometric Design and Signal/Image Processing to Computer Graphics and Animation. Recently, subdivision schemes have become of interest also in biomedical imaging applications (see, e.g., [38]) and Isogeometric Analysis (IgA), a modern computational approach that integrates Finite Element Analysis into conventional CAD systems (see, e.g., [1, 4, 9, 10]).
From a mathematical point of view, any linear subdivision scheme is identified by a sequence of Laurent polynomials, also called subdivision symbols, which describe the linear rules determining the successive refinements of the initial set of discrete data. The most popular examples of subdivision schemes are B-spline subdivision schemes and their nonstationary counterparts, namely exponential B-spline subdivision schemes (see, e.g., [16, 43]), characterized by the property of representing polynomials and exponential polynomials, respectively. Since in many applicative areas the capability of representing shapes described by exponential polynomial functions is fundamental, interpolating and approximating subdivision schemes based on exponential B–splines and inheriting their generation properties, have been recently introduced (see, e.g., [2, 3, 5, 7, 13, 14, 16, 17, 28, 35, 39]). We recall that, while the term generation usually refers to the subdivision scheme capability of providing specific types of limit functions, with reproduction we mean the capability of a subdivision scheme to reproduce in the limit exactly the same function from which the data are sampled. The property of reproduction of exponential polynomials is also important since strictly connected to the approximation order of subdivision schemes and to their regularity (see [18]). In fact, the higher is the number of exponential polynomials reproduced, the higher is the approximation order and the possible regularity of the scheme. This is why, in applications, we require subdivision schemes with exponential polynomial reproduction properties, that allow to meet various demands for balancing approximation order, regularity and support size. Such kind of schemes turn out to constitute the family of exponential pseudo-splines, the nonstationary counterpart of polynomial pseudo-splines. In the primal case, the latter family neatly fills in the gap between odd-degree B-spline and interpolatory subdivision schemes, both extreme cases of pseudo-splines: while B-splines stand out due to their high smoothness and short support, they provide a rather poor approximation order; in contrast, the limit functions of interpolatory subdivision schemes have optimal approxima-

tion order but low smoothness and large support.

In the binary, stationary case, primal pseudo-splines were originally presented in [25], whereas their dual analogues were successively discovered in [26]. Binary, primal and dual (polynomial) pseudo-splines have been more recently generalized to any arity and to arbitrary parametrizations by [15]. Both primal and dual (polynomial) pseudo-splines are obtained by means of stationary subdivision schemes whose symbols can be read as a suitable polynomial "correction" of the polynomial B-spline symbol. Indeed, denoting by $B_N(z)$ the order-N polynomial B-spline symbol, we can write all pseudo-spline symbols as $a_{M,N}(z) = B_N(z)c_M(z)$. The polynomial correction $c_M(z)$ is such that the subdivision schemes with symbols $a_{M,N}(z)$ are the ones of minimal support that, besides generating polynomials of degree $N-1$, satisfy the conditions for reproduction of polynomials of degree $M-1$, with $M \leq N$. Similarly to the stationary case, we here define exponential pseudo-spline subdivision schemes by means of k-level subdivision symbols which are a suitable "correction" of the k-level subdivision symbols $B^{(k)}_{N,\Gamma}(z)$ of exponential B-spline schemes, i.e. of the form $a^{(k)}_{M,N,\Gamma}(z) = B^{(k)}_{N,\Gamma}(z)c^{(k)}_{M,\Gamma}(z)$. Here Γ is a set identifying the particular space of exponential polynomials EP_Γ we deal with, while N and M are related to the number of exponential polynomials that are being generated and reproduced, respectively. Again, $c^{(k)}_{M,\Gamma}(z)$ is such that the symbols $a^{(k)}_{M,N,\Gamma}(z)$ are of minimal support and satisfy the conditions for reproduction of the space EP_Γ (or a subset of it), generated by the exponential B-spline schemes with symbols $B^{(k)}_{N,\Gamma}(z)$.

The main contribution of this paper consists in showing how the symbols of exponential pseudo-spline subdivision schemes can be explicitly derived. Indeed, we provide the expressions of the inverse matrices of the linear systems arising by imposing the algebraic conditions for exponential polynomial reproduction. Such conditions were first given in [17] and successively extended to any arbitrary arity in [7]. We also show that, under the symmetry assumption on Γ (or on a subset of it), the symbol $a^{(k)}_{M,N,\Gamma}(z)$ has the same symmetry as $B^{(k)}_{N,\Gamma}(z)$. To prove the latter we also discover remarkable algebraic properties, never highlighted so far, of symmetric nonstationary subdivision symbols. As a minor contribution, we show how the k-level normalization factor of the exponential B-spline symbol can be selected in accordance with the shift parameter in order to ensure that the exponential B-spline is correctly normalized: namely, besides generating the space EP_Γ, it reproduces a specific pair of exponential polynomials $\{e^{\theta\, x},\ e^{-\theta\, x}\} \in EP_\Gamma$. Finally, we additionally provide a convergence and regularity analysis of the nonstationary subdivision schemes corresponding to the exponential pseudo-spline symbols here derived. This is possible by first showing that exponential pseudo-spline schemes are asymptotically similar to polynomial pseudo-spline schemes, and then combining recent advances on convergence and regularity of nonstationary subdivision schemes presented in [6] and in [18].

The remainder of the paper is organized as follows. In Section 2 we recall basic notions on nonstationary subdivision schemes reproducing exponential polynomials. Then, in Section 3 we discuss new important results concerning symmetry properties of such subdivision schemes. Symmetric exponential B-spline symbols are recalled in Section 4 where accordance between their parameter shift and their normalization factor is also considered with respect to their reproduction capabilities. The derivation of the symbols of exponential pseudo-spline subdivision schemes is provided in Section 5 where the symmetry properties of such symbols are also discussed. Convergence and regularity of the new family of (nonstationary) exponential pseudo-spline subdivision schemes are then investigated in Section 6. As an example of application of the presented theoretical results, the expression of the subdivision symbols of a new family of exponential pseudo-spline schemes is also explicitly derived in Section 7, where pictures of the corresponding basic limit functions are also given. The closing Section 8 is to draw conclusions.

2. Non-stationary subdivision schemes and exponential polynomial reproduction

The interest in nonstationary subdivision schemes arose in the last ten years after it was pointed out that they can be equipped with tension parameters that allow us to get as close as desired to the original mesh and to obtain considerable variations of shape (see [16, 30, 39, 40, 42, 43]). Indeed, differently from stationary subdivision schemes, nonstationary subdivision schemes are capable of reproducing conic sections, spirals or, in general, of generating exponential polynomials $x^r e^{\theta x}$, $x \in \mathbb{R}$, $r \in \mathbb{N} \cup \{0\}$, $\theta \in \mathbb{C}$. This generation property is important not only in geometric design (see, e.g., [30, 32, 37, 42, 43]), but also in many other applications, e.g., in biomedical imaging (see, e.g., [20, 21]) and in Isogeometric Analysis (see, e.g., [19, 31]). However, the use of nonstationary subdivision schemes in IgA is nowadays limited to the case of exponential B-splines since they are the only functions that have been shown to be able to overcome the NURBS limits while preserving their useful properties. Precisely, exponential B-spline functions are able to accurately reproduce a greater variety of geometric shapes than traditional NURBS, may be exploited to achieve shape-preserving approximations and, differently from rational functions, they can also be differentiated and integrated with the same ease with which polynomial B-splines are treated. In this work we show that exponential B-splines are just a special subclass of the rich family of exponential pseudo-splines, which can all be generated as limits of nonstationary subdivision schemes.

Following the notation in [27, 29], in the remainder of this paper, for any $k \geq 0$ we denote by $\mathbf{a}^{(k)} := \{\mathrm{a}_j^{(k)} \in \mathbb{R},\ j \in \mathbb{Z}\}$ the finite set of real coefficients corresponding to the so called *k-level mask* of a nonstationary subdivision scheme. We here assume that all masks have a common support independent of k, namely, for all $k \geq 0$, $\mathrm{supp}(\mathbf{a}^{(k)}) := \{i \in \mathbb{Z} : \mathrm{a}_i^{(k)} \neq 0\} \subseteq [-S, S]$, $S \in \mathbb{N}$.

Moreover, we define by $a^{(k)}(z) := \sum_{j\in\mathbb{Z}} \mathrm{a}_j^{(k)} z^j,\ z \in \mathbb{C}\backslash\{0\}$, the Laurent polynomial whose coefficients are exactly the entries of $\mathbf{a}^{(k)}$. This polynomial is commonly known as the *k-level symbol* of the nonstationary subdivision scheme. With any mask $\mathbf{a}^{(k)}$ comes a linear subdivision operator identifying a refinement process, that is the process which transforms a set of real data at level k, $\mathbf{f}^{(k)} = \{f_i^{(k)} \in \mathbb{R},\ i \in \mathbb{Z}\}$, into the denser set $\mathbf{f}^{(k+1)}$ given by

$$\mathbf{f}^{(k+1)} := S_{\mathbf{a}^{(k)}}\mathbf{f}^{(k)}, \quad \text{where} \quad (S_{\mathbf{a}^{(k)}}\mathbf{f}^{(k)})_i := \sum_{j\in\mathbb{Z}} \mathrm{a}_{i-2j}^{(k)} f_j^{(k)}, \qquad \forall\, k \geq 0. \tag{2.1}$$

The subdivision scheme consists in the repeated application of the subdivision operators starting from any initial data sequence $\mathbf{f}^{(0)} \equiv \mathbf{f} := \{f_i \in \mathbb{R},\ i \in \mathbb{Z}\}$, and therefore is shortly denoted by $\{S_{\mathbf{a}^{(k)}},\ k \geq 0\}$.

Since the subdivision process generates denser and denser sequences of data, attaching the data $f_i^{(k)}$ generated at the k-th step to the parameter values $t_i^{(k)}$ with $t_i^{(k)} < t_{i+1}^{(k)}$ and $t_{i+1}^{(k)} - t_i^{(k)} = 2^{-k},\ k \geq 0$, a notion of convergence can be established by taking into account the piecewise linear function $F^{(k)}$ that interpolates the data (namely $F^{(k)}(t_i^{(k)}) = f_i^{(k)},\ F^{(k)}|_{[t_i^{(k)}, t_{i+1}^{(k)}]} \in \Pi_1,\ i \in \mathbb{Z},\ k \geq 0$). If the sequence of continuous functions $\{F^{(k)},\ k \geq 0\}$ converges uniformly, we denote its limit by

$$g_{\mathbf{f}} := \lim_{k\to+\infty} S_{\mathbf{a}^{(k)}} S_{\mathbf{a}^{(k-1)}} \cdots S_{\mathbf{a}^{(0)}}\mathbf{f} = \lim_{k\to+\infty} F^{(k)},$$

and say that $g_{\mathbf{f}}$ is the limit function of the nonstationary subdivision scheme based on the rules in (2.1) for the data $\mathbf{f}$.

If the nonstationary subdivision scheme is convergent, and $g_{\mathbf{f}} \equiv 0$ if and only if $\mathbf{f} \equiv 0$, then the subdivision scheme is termed non-singular. In the forthcoming discussion we restrict ourselves to non-singular schemes only.

As will be clarified later on, with respect to the subdivision capability of reproducing specific classes of functions, the standard parametrization (corresponding to the choice $t_i^{(k)} := \frac{i}{2^k}$, $i \in \mathbb{Z}$) is not always the optimal one. Indeed, the choice

$$t_i^{(k)} := \frac{i+p}{2^k}, \quad i \in \mathbb{Z}, \quad p \in \mathbb{R}, \quad k \geq 0\,, \tag{2.2}$$

with p suitably set, turns out to be a better selection. In particular, when $p \in \mathbb{Z}$ the parametrization is termed *primal*, whereas if $p \in \frac{\mathbb{Z}}{2}$ it is called *dual*. For a complete discussion concerning the choice of the parametrization in the analysis of the polynomial reproduction properties of stationary subdivision schemes, we refer the reader to [5, 15, 26].

In consideration of the fact that the main goal of this work is the construction of a special class of nonstationary subdivision symbols capable of generating as limit functions exponential polynomials, we continue by recalling the following definitions (see, e.g, [7, 17, 39]).

Definition 2.1 (Exponential polynomials). *Let $n \in \mathbb{N}$ and let $\Gamma := \{(\theta_1, \tau_1), \ldots, (\theta_n, \tau_n)\}$ with $\theta_i \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$, $\theta_i \neq \theta_j$ if $i \neq j$ and $\tau_i \in \mathbb{N}$, $i = 1, \cdots, n$. We define the space of exponential polynomials EP_Γ as*

$$EP_\Gamma := span\{ \; x^{r_i} \, e^{\theta_i x}, \; r_i = 0, \cdots, \tau_i - 1, \; i = 1, \cdots, n\}.$$

Remark 2.2. *For each $i = 1, \cdots, n$, τ_i denotes the multiplicity of the value $\theta_i \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$.*

For a fixed set Γ, and for the corresponding space EP_Γ, we recall the following definition.

Definition 2.3 (E-Generation and E-Reproduction). *The subdivision scheme associated with the symbols $\{a^{(k)}(z), \; k \geq 0\}$ is said to be EP_Γ-*generating *if it is convergent and for all initial sequences $\mathbf{f}^{(0)} := \{f(t_i^{(0)}), \; i \in \mathbb{Z}\}$, $f \in EP_\Gamma$, it is verified that $\lim_{k \to +\infty} S_{\mathbf{a}^{(k)}} S_{\mathbf{a}^{(k-1)}} \cdots S_{\mathbf{a}^{(0)}} \mathbf{f}^{(0)} \in EP_\Gamma$. We say it is EP_Γ-*reproducing *if it is convergent and for all initial sequences $\mathbf{f}^{(0)} := \{f(t_i^{(0)}), \; i \in \mathbb{Z}\}$, $f \in EP_\Gamma$, it is verified that $\lim_{k \to +\infty} S_{\mathbf{a}^{(k)}} S_{\mathbf{a}^{(k-1)}} \cdots S_{\mathbf{a}^{(0)}} \mathbf{f}^{(0)} = f$.*

In the following theorem we recall the algebraic conditions on the k-level symbol $a^{(k)}(z)$ that fully identify the generation and reproduction properties of a non-singular, univariate, binary, nonstationary subdivision scheme. A more general version of these conditions, holding for nonstationary subdivision schemes of arbitrary arity, has recently appeared in [7]. According to [11], here and hereafter for real z and $k \geq 0$ integer we use the notation

$$(z)_k := z(z-1) \cdots (z-k+1), \quad (z)_0 = 1,$$

to denote the shifted falling factorial of z of order k.

Theorem 2.4. *[17, Theorem 1] Let $n \in \mathbb{N}$ and let $\Gamma := \{(\theta_1, \tau_1), \ldots, (\theta_n, \tau_n)\}$ with $\theta_\ell \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$, $\theta_\ell \neq \theta_j$ if $\ell \neq j$ and $\tau_\ell \in \mathbb{N}$, $\ell = 1, \cdots, n$. Let also $z_\ell^{(k)} := e^{\frac{-\theta_\ell}{2^{k+1}}}$, $\ell = 1, \ldots, n$. A non-singular, nonstationary subdivision scheme associated with the symbols $\{a^{(k)}(z), \; k \geq 0\}$ is EP_Γ-generating if and only if, for each $k \geq 0$, the following conditions are satisfied*

$$\frac{d^r \, a^{(k)}(-z_\ell^{(k)})}{dz^r} = 0, \qquad \ell = 1, \ldots, n, \quad r = 0, \ldots, \tau_\ell - 1. \tag{2.3}$$

Furthermore, it is EP_Γ-reproducing if and only if, for each $k \geq 0$, in addition to (2.3) the conditions

$$\frac{d^r \, a^{(k)}(z_\ell^{(k)})}{dz^r} = 2(p)_r \left(z_\ell^{(k)}\right)^{p-r}, \qquad \ell = 1, \ldots, n, \quad r = 0, \ldots, \tau_\ell - 1 \tag{2.4}$$

are also satisfied where $p \in \mathbb{R}$ is the shift parameter identifying the parametrization in (2.2).

3. Symmetric subdivision symbols reproducing exponential polynomials

Symmetric subdivision schemes are considered of remarkable interest in several applications. In this section, we thus analyze in detail the case of EP_Γ-reproducing symmetric subdivision schemes featured by symmetric symbols. To this purpose, we first introduce the definition of k-level symmetric symbol, then we point out the symmetric structure we require on the set Γ that identifies the space of exponential polynomials EP_Γ reproduced by a symmetric subdivision scheme.

Definition 3.1 (Symmetric k-level symbol). *A k-level subdivision symbol $a^{(k)}(z)$ is called* odd-symmetric *if $a^{(k)}(z) = a^{(k)}(z^{-1})$ and* even-symmetric *if $z\, a^{(k)}(z) = a^{(k)}(z^{-1})$. In terms of k-level masks the odd/even symmetry translates into the condition $\mathrm{a}_{-i}^{(k)} = \mathrm{a}_{i}^{(k)}$, $i \in \mathbb{Z}$, and $\mathrm{a}_{-i}^{(k)} = \mathrm{a}_{i-1}^{(k)}$, $i \in \mathbb{Z}$, respectively.*

Remark 3.2. *It is worth mentioning that a subdivision scheme has to be considered symmetric even if its k-level symbol satisfies the above condition after a suitable shift, i.e. after multiplication by z^s, $s \in \mathbb{Z}$. Note that, as shown in [17], the shift s does affect the value of the parameter p in a well-known way: the parameter p_s, characterizing the parametrization of the shifted scheme, is simply $p_s = p + s$.*

From now on, we consider symmetric sets Γ of the form specified in the following definition.

Definition 3.3 (Symmetric set Γ). *Let Γ be the set in Definition 2.1. If*

$$\Gamma := \begin{cases} \text{(a)} \quad \{(\theta_\ell, \tau_\ell), (-\theta_\ell, \tau_\ell)\}_{\ell=1,\ldots,\frac{n}{2}}, & \textit{when } n \textit{ is even}, \\ \\ \text{(b)} \quad \{(\theta_\ell, \tau_\ell), (-\theta_\ell, \tau_\ell)\}_{\ell=1,\ldots,\frac{n-1}{2}} \cup \{(0,1)\}, & \textit{when } n \textit{ is odd}, \end{cases}$$

$$\textit{with} \quad \theta_\ell \in \mathbb{R}^+ \cup \mathrm{i}[0,\pi), \quad \theta_\ell \neq \theta_j \;\; \textit{if} \;\; \ell \neq j, \quad \textit{and} \quad \mathbb{R}^+ := \{x \in \mathbb{R} \,:\, x > 0\}, \tag{3.1}$$

then the set Γ is said to be symmetric*. The space of exponential polynomials EP_Γ, associated to a symmetric set Γ, is also said to be* symmetric*.*

In the remainder of the paper we focus our attention on EP_Γ-reproducing symmetric subdivision schemes, where the set Γ has the symmetric structure specified in Definition 3.3. The next proposition proves two very important properties of EP_Γ-reproducing symmetric subdivision schemes.

Proposition 3.4. *A non-singular, nonstationary subdivision scheme associated with odd-symmetric or even-symmetric symbols $\{a^{(k)}(z),\ k \geq 0\}$ reproduces the pair of exponential polynomials $\{e^{\theta_\ell x},\ e^{-\theta_\ell x}\}$, $\theta_\ell \in \mathbb{R}^+ \cup \mathrm{i}(0,\pi)$, only if $p = 0$ or $p = -\frac{1}{2}$, respectively. Moreover, in case $\theta_\ell = 0$, the subdivision scheme reproduces $\{1, x\}$ only if $p = 0$ or $p = -\frac{1}{2}$, respectively.*

Proof. Let $z_\ell^{(k)} := e^{\frac{-\theta_\ell}{2^{k+1}}}$, $\theta_\ell \in \mathbb{R}^+ \cup \mathrm{i}(0,\pi)$. We know from conditions (2.4) that the reproduction of the pair $\{e^{\theta_\ell x}, e^{-\theta_\ell x}\}$ is equivalent to the existence of a shift parameter p such that $a^{(k)}(z_\ell^{(k)}) = 2(z_\ell^{(k)})^p$ and $a^{(k)}((z_\ell^{(k)})^{-1}) = 2(z_\ell^{(k)})^{-p}$. Thus, if the k-level symbols are odd-symmetric, we can write $2(z_\ell^{(k)})^p = 2(z_\ell^{(k)})^{-p}$, and the latter equation is satisfied only if the shift parameter $p = 0$ is chosen. Otherwise, if the k-level symbols are even-symmetric, we can write $2(z_\ell^{(k)})^{p+1} = 2(z_\ell^{(k)})^{-p}$, and the latter equation is fulfilled only if the shift parameter $p = -\frac{1}{2}$ is fixed.
To conclude the proof we observe that, when $\theta_\ell = 0$, the reproduction of the pair $\{1, x\}$ is obtained by setting $p = 0$ if the k-level symbol is odd-symmetric and $p = -\frac{1}{2}$ if it is even-symmetric, as shown in [15]. $\square$

We continue by analyzing useful algebraic properties fulfilled by symmetric subdivision symbols.

Proposition 3.5. *Let $d \in \mathbb{N}$ and let $\Gamma = \{(\theta_\ell, d), (-\theta_\ell, d)\}$ with $\theta_\ell \in \mathbb{R}^+ \cup \mathrm{i}[0,\pi)$. For $z_\ell^{(k)} = e^{\frac{-\theta_\ell}{2^{k+1}}}$, the even-symmetric subdivision symbols $\{a^{(k)}(z),\ k \geq 0\}$ satisfy*

$$\frac{d^r\, a^{(k)}(z_\ell^{(k)})}{dz^r} = 2\left(-\frac{1}{2}\right)_r \left(z_\ell^{(k)}\right)^{-\frac{1}{2}-r} \qquad r = 0, \ldots, d-1,$$

if and only if the odd-symmetric subdivision symbols $\{b^{(k)}(z),\ k \geq 0\}$ with $b^{(k)}(z) = z\, a^{(k)}(z^2) - 2$ satisfy

$$\frac{d^r b^{(k)}((z_\ell^{(k)})^{\frac{1}{2}})}{dz^r} = 0, \qquad r = 0, \ldots, d-1.$$

Proof. We start showing that the k-level symbol $a^{(k)}(z)$ is even-symmetric if and only if $b^{(k)}(z) = z\, a^{(k)}(z^2) - 2$ is odd-symmetric. Indeed $z\, a^{(k)}(z) = a^{(k)}(z^{-1})$ if and only if $z^2\, a^{(k)}(z^2) = a^{(k)}(z^{-2})$ if and only if $b^{(k)}(z) + 2 = b^{(k)}(z^{-1}) + 2$ if and only if $b^{(k)}(z) = b^{(k)}(z^{-1})$.
The rest of the proof is inductive on r. The case $r = 0$ is easy to check. Therefore we consider the case $r > 0$ and use the Leibniz formula and the induction for $r = 0$ to write the derivatives of $a^{(k)}(z) = z^{-\frac{1}{2}}(b^{(k)}(z^{\frac{1}{2}}) + 2)$ evaluated at $z_\ell^{(k)}$. Recall that $z^{-\frac{1}{2}} = e^{-\frac{1}{2}\log(z)}$ can be defined as a single-valued function, analytic on $\mathbb{C} \setminus (-\infty, 0]$. Thus we have

$$\frac{d^r a^{(k)}(z)}{dz^r}\Big|_{z=z_\ell^{(k)}} = \sum_{s=0}^{r} \binom{r}{s} \frac{d^s(b^{(k)}(z^{\frac{1}{2}}) + 2)}{dz^s} \frac{d^{r-s}(z^{-\frac{1}{2}})}{dz^{r-s}}\Big|_{z=z_\ell^{(k)}} =$$

$$\left(\frac{d^r(z^{-\frac{1}{2}})}{dz^r}(b^{(k)}(z^{\frac{1}{2}}) + 2)\right)\Big|_{z=z_\ell^{(k)}} + \sum_{s=1}^{r} \binom{r}{s} \frac{d^s(b^{(k)}(z^{\frac{1}{2}}) + 2)}{dz^s} \frac{d^{r-s}(z^{-\frac{1}{2}})}{dz^{r-s}}\Big|_{z=z_\ell^{(k)}}$$

$$= 2\left(-\frac{1}{2}\right)_r (z_\ell^{(k)})^{-\frac{1}{2}-r} + \sum_{s=1}^{r} \binom{r}{s} \frac{d^s(b^{(k)}(z^{\frac{1}{2}}))}{dz^s} \frac{d^{r-s}(z^{-\frac{1}{2}})}{dz^{r-s}}\Big|_{z=z_\ell^{(k)}}.$$

We continue by using the Faà di Bruno formula (see [34] or [36]) to write

$$\frac{d^s(b^{(k)}(z^{\frac{1}{2}}))}{dz^s}\Big|_{z=z_\ell^{(k)}} = \sum_{j=1}^{s} \frac{d^j b^{(k)}(y)}{dy^j}\Big|_{y=(z_\ell^{(k)})^{\frac{1}{2}}} A_{j,s}(z)\Big|_{z=z_\ell^{(k)}},$$

with $A_{j,s}(z)$ given functions whose value is important to know only for $j = s$. In particular, we have $A_{s,s}(z) = \left(\frac{1}{2}z^{-\frac{1}{2}}\right)^s$. In fact, for $j = 1, \cdots, s,\ s < r$, using the induction assumption we know that $\frac{d^j b^{(k)}(y)}{dy^j}\Big|_{y=(z_\ell^{(k)})^{\frac{1}{2}}} = 0$ and therefore the above sum reduces to the last term only, that is to

$$\nu_s = \frac{d^s b^{(k)}(y)}{dy^s}\Big|_{y=(z_\ell^{(k)})^{\frac{1}{2}}} \left(\frac{1}{2}(z_\ell^{(k)})^{-\frac{1}{2}}\right)^s, \quad 1 \le s \le r.$$

Hence, using the fact that $\frac{d^r\, a^{(k)}(z)}{dz^r}\Big|_{z=z_\ell^{(k)}} = 2\left(z_\ell^{(k)}\right)^{-\frac{1}{2}-r}\left(-\frac{1}{2}\right)_r$, we arrive at

$$2\left(-\frac{1}{2}\right)_r \left(z_\ell^{(k)}\right)^{-\frac{1}{2}-r} = 2\left(-\frac{1}{2}\right)_r (z_\ell^{(k)})^{-\frac{1}{2}-r} + \sum_{s=1}^{r} \binom{r}{s} \nu_s \left(-\frac{1}{2}\right)_{r-s} (z_\ell^{(k)})^{-\frac{1}{2}-(r-s)}.$$

Now, using again the inductive hypothesis that $\frac{d^s b^{(k)}(y)}{dy^s}\Big|_{y=(z_\ell^{(k)})^{\frac{1}{2}}} = 0$ for all $s = 1, ..., r-1$ we obtain

$$0 = \frac{d^r b^{(k)}(y)}{dy^r}\Big|_{y=(z_\ell^{(k)})^{\frac{1}{2}}} \left(\frac{1}{2}(z_\ell^{(k)})^{-\frac{1}{2}}\right)^r (z_\ell^{(k)})^{-\frac{1}{2}},$$

which is the required value of the r-th derivative of $b^{(k)}(z)$ at $(z_\ell^{(k)})^{\frac{1}{2}}$, i.e.

$$\frac{d^r b^{(k)}(y)}{dy^r}\Big|_{y=(z_\ell^{(k)})^{\frac{1}{2}}} = 0\,.$$

This concludes the induction step and therefore the proof. □

Remark 3.6. *As a by-product of the proof of Proposition 3.5, from* $b^{(k)}(z) = b^{(k)}(z^{-1})$ *we obtain that, for* $z_\ell^{(k)} = 1$*, the condition* $\frac{d^r b^{(k)}((z_\ell^{(k)})^{\frac{1}{2}})}{dz^r} = 0$ *for* $r = 0, \ldots, 2j$*, with* $0 \le 2j \le \tau_\ell - 1$*, implies that* $\frac{d^{2j+1} b^{(k)}((z_\ell^{(k)})^{\frac{1}{2}})}{dz^{2j+1}} = 0$ *and, therefore, the corresponding condition on* $\frac{d^{2j+1} a^{(k)}(z_\ell^{(k)})}{dz^{2j+1}}$.

The next proposition shows that conditions (2.4) are compatible with symmetry properties of subdivision symbols. Indeed we prove that symmetric subdivision symbols are such that, if conditions (2.4) are satisfied at a given $z_\ell^{(k)}$, they are also satisfied at $(z_\ell^{(k)})^{-1}$.

Proposition 3.7. *Let $\{a^{(k)}(z),\ k \geq 0\}$ be the odd-symmetric (even-symmetric) symbols of a non-singular, nonstationary subdivision scheme with shift parameter $p = 0$ ($p = -\frac{1}{2}$). For $z_\ell^{(k)} := e^{\frac{-\theta_\ell}{2^{k+1}}}$ with $\theta_\ell \in \mathbb{R}^+ \cup \mathrm{i}[0, \pi)$ we have that*

$$\frac{d^r\, a^{(k)}(z_\ell^{(k)})}{dz^r} = 2(p)_r \left(z_\ell^{(k)}\right)^{p-r} \qquad r = 0, \ldots, d-1, \quad d \in \mathbb{N}$$

if and only if

$$\frac{d^r\, a^{(k)}((z_\ell^{(k)})^{-1})}{dz^r} = 2(p)_r \left((z_\ell^{(k)})^{-1}\right)^{p-r} \qquad r = 0, \ldots, d-1, \quad d \in \mathbb{N}.$$

Proof. We show the claim by induction on r. The case $r = 0$ has been already considered in Proposition 3.4. For $r > 0$ we first consider the odd-symmetric case and we start proving one of the two implications. Computing the r-th derivative of the equation $a^{(k)}(z) = a^{(k)}(z^{-1})$ via the Faà di Bruno formula (see [34] or [36]) and evaluating it at $z_\ell^{(k)}$, we obtain

$$\frac{d^r a^{(k)}(z)}{dz^r}\Big|_{z=z_\ell^{(k)}} = \sum_{j=1}^{r} \frac{d^j a^{(k)}(y)}{dy^j}\Big|_{y=(z_\ell^{(k)})^{-1}} A_{j,r}(z_\ell^{(k)})$$

with

$$A_{j,r}(z) = \sum_{\mathbf{q}\in\mathbf{M}^j,\, |\mathbf{q}|=r} \frac{r!}{\mathbf{q}!} \frac{(-1)^r z^{-r-j}}{\prod_{i=1}^{r} N(\mathbf{q}, i)!}\,,$$

where

$$\mathbf{M}^j = \{\mathbf{q} = (q_1, q_2, ..., q_j) \in \mathbb{N}^j,\ q_1 \geq q_2 \geq ... \geq q_j \geq 1\}, \qquad |\mathbf{q}| = q_1 + ... + q_j,$$

and $N(\mathbf{q}, i)$ denoting the number of times the positive integer i appears in the j-tuple $\mathbf{q} \in \mathbb{N}^j$. Now, by the inductive hypothesis we know that $\frac{d^r a^{(k)}(z)}{dz^r}\Big|_{z=z_\ell^{(k)}} = 0$ for $r = 1, ..., d-2$ implies $\frac{d^r a^{(k)}(z)}{dz^r}\Big|_{z=(z_\ell^{(k)})^{-1}} = 0$ for $r = 1, ..., d-2$. Hence,

$$\frac{d^{d-1} a^{(k)}(z)}{dz^{d-1}}\Big|_{z=z_\ell^{(k)}} = \frac{d^{d-1} a^{(k)}(y)}{dy^{d-1}}\Big|_{y=(z_\ell^{(k)})^{-1}} A_{d-1,d-1}(z_\ell^{(k)}),$$

and since $A_{d-1,d-1}(z_\ell^{(k)}) = (-1)^{d-1}(z_\ell^{(k)})^{-2(d-1)} \neq 0$, we easily get that

$$\frac{d^{d-1} a^{(k)}(z)}{dz^{d-1}}\Big|_{z=z_\ell^{(k)}} = 0 \quad \Rightarrow \quad \frac{d^{d-1} a^{(k)}(y)}{dy^{d-1}}\Big|_{y=(z_\ell^{(k)})^{-1}} = 0\,,$$

which concludes one direction of the proof in the odd-symmetric case. The proof of the converse implication can be repeated analogously.

For the even-symmetric case, in view of Proposition 3.5, we use the same argument as above for the odd-symmetric k-level symbol $b^{(k)}(z) = z\, a^{(k)}(z^2) - 2$ and for the roots $(z_\ell^{(k)})^{\frac{1}{2}},\ (z_\ell^{(k)})^{-\frac{1}{2}}$, so completing the proof. □

Remark 3.8. *The above proposition proves the equivalence between the conditions for exponential polynomial reproduction given in [17] and in [32, 33] when $p = 0$ or $p = -\frac{1}{2}$.*

4. Symmetric exponential B-spline symbols and their normalization factors

For a symmetric set Γ of cardinality n, defined as in Definition 3.3, we observe that, in case (a), n is even and the value $N := 2\sum_{\ell=1}^{\frac{n}{2}} \tau_\ell$, given by the sum of the multiplicities, is also even. In contrast, in case (b), n is odd as well as the value of the sum of the multiplicities, given by $N := 2\sum_{\ell=1}^{\frac{n-1}{2}} \tau_\ell + 1$.
Now, for any $L \in \mathbb{R}$, let $\lfloor L \rfloor := \max\{M \in \mathbb{Z} : M \leq L\}$ and $\lceil L \rceil := \min\{M \in \mathbb{Z} : M \geq L\}$. For a given $\ell \in \{1, \cdots, \lfloor \frac{n}{2} \rfloor\}$, suppose that $\{(\theta_\ell, \tau_\ell), (-\theta_\ell, \tau_\ell)\} \in \Gamma$ and define $\Gamma_\ell^* := \Gamma \setminus \{(\theta_\ell, \tau_\ell), (-\theta_\ell, \tau_\ell)\}$. Then, if n is even, we introduce the notation

$$\Gamma_{\ell,e} := \begin{cases} \Gamma \setminus \{(\theta_\ell, 1), (-\theta_\ell, 1)\} & \text{if} \quad \tau_\ell = 1, \\ \Gamma_\ell^* \cup \{(\theta_\ell, \tau_\ell - 1), (-\theta_\ell, \tau_\ell - 1)\} & \text{if} \quad \tau_\ell > 1, \end{cases}$$

whereas, if n is odd, we denote by $\Gamma_{\ell,o}$ the set

$$\Gamma_{\ell,o} := \begin{cases} \Gamma \setminus \{(\theta_\ell, 1), (-\theta_\ell, 1), (0,1)\} & \text{if} \quad \tau_\ell = 1, \\ (\Gamma_\ell^* \setminus \{(0,1)\}) \cup \{(\theta_\ell, \tau_\ell - 1), (-\theta_\ell, \tau_\ell - 1)\} & \text{if} \quad \tau_\ell > 1. \end{cases}$$

For a symmetric set Γ as in (3.1) and for any $k \geq 0$ we define a symmetric (not-normalized) exponential B-spline scheme by the sequence of symbols

$$\tilde{B}_{N,\Gamma}^{(k)}(z) := \begin{cases} z^{-\frac{N}{2}} \displaystyle\prod_{\ell=1}^{\frac{n}{2}} \left(e^{\frac{\theta_\ell}{2^{k+1}}} z + 1\right)^{\tau_\ell} \left(e^{\frac{-\theta_\ell}{2^{k+1}}} z + 1\right)^{\tau_\ell}, \text{ if } n \text{ is even} \\ z^{-\frac{N+1}{2}} (z+1) \displaystyle\prod_{\ell=1}^{\frac{n-1}{2}} \left(e^{\frac{\theta_\ell}{2^{k+1}}} z + 1\right)^{\tau_\ell} \left(e^{\frac{-\theta_\ell}{2^{k+1}}} z + 1\right)^{\tau_\ell}, \text{ if } n \text{ is odd.} \end{cases} \tag{4.1}$$

Symbols in (4.1) satisfy the necessary and sufficient conditions for EP_Γ-generation that, accordingly to (2.3), are given by

$$\begin{cases} \left.\begin{aligned} &\tilde{B}_{N,\Gamma}^{(k)}\left(-e^{\frac{\pm\theta_j}{2^{k+1}}}\right) = 0, \\ &\frac{d^r\, \tilde{B}_{N,\Gamma}^{(k)}\left(-e^{\frac{\pm\theta_j}{2^{k+1}}}\right)}{dz^r} = 0, \quad r = 1, \ldots, \tau_j - 1, \end{aligned}\right. & j = 1, \ldots, \frac{n}{2}, \text{ if } n \text{ is even;} \\ \left.\begin{aligned} &\tilde{B}_{N,\Gamma}^{(k)}(-1) = 0, \quad \tilde{B}_{N,\Gamma}^{(k)}\left(-e^{\frac{\pm\theta_j}{2^{k+1}}}\right) = 0, \\ &\frac{d^r\, \tilde{B}_{N,\Gamma}^{(k)}\left(-e^{\frac{\pm\theta_j}{2^{k+1}}}\right)}{dz^r} = 0, \quad r = 1, \ldots, \tau_j - 1, \end{aligned}\right. & j = 1, \ldots, \frac{n-1}{2}, \text{ if } n \text{ is odd.} \end{cases} \tag{4.2}$$

Moreover, when n is even, by definition of $\Gamma_{\ell,e}$ we easily see that for $\ell \in \left\{1, \ldots, \frac{n}{2}\right\}$ $\tilde{B}_{N,\Gamma}^{(k)}(z)$ satisfies the "recursion"

$$\tilde{B}_{N,\Gamma}^{(k)}(z) = z^{-1} \left(e^{\frac{\theta_\ell}{2^{k+1}}} z + 1\right) \left(e^{\frac{-\theta_\ell}{2^{k+1}}} z + 1\right) \tilde{B}_{N-2,\Gamma_{\ell,e}}^{(k)}(z), \tag{4.3}$$

whereas, when n is odd and $\ell \in \left\{1, \ldots, \frac{n-1}{2}\right\}$, we can exploit the definition of $\Gamma_{\ell,o}$ to write the formula

$$\tilde{B}^{(k)}_{N,\Gamma}(z) = z^{-2}(z+1)\left(e^{\frac{\theta_\ell}{2^{k+1}}} z + 1\right)\left(e^{\frac{-\theta_\ell}{2^{k+1}}} z + 1\right)\tilde{B}^{(k)}_{N-3,\Gamma_{\ell,o}}(z)\,. \tag{4.4}$$

For reproduction purposes it may be convenient to consider *normalized exponential B-spline* subdivision schemes. Their symbols are defined by multiplying $\tilde{B}^{(k)}_{N,\Gamma}(z)$ in (4.1) with an extra factor $K^{(k)}_\ell \in \mathbb{R}$, namely by

$$B^{(k)}_{N,\Gamma}(z) := K^{(k)}_\ell\, \tilde{B}^{(k)}_{N,\Gamma}(z) = \begin{cases} K^{(k)}_\ell\, z^{-\frac{N}{2}} \displaystyle\prod_{\ell=1}^{\frac{n}{2}} \left(e^{\frac{\theta_\ell}{2^{k+1}}} z + 1\right)^{\tau_\ell} \left(e^{\frac{-\theta_\ell}{2^{k+1}}} z + 1\right)^{\tau_\ell}, & \text{if } n \text{ is even},\\ K^{(k)}_\ell\, z^{-\frac{N+1}{2}}(z+1) \displaystyle\prod_{\ell=1}^{\frac{n-1}{2}} \left(e^{\frac{\theta_\ell}{2^{k+1}}} z + 1\right)^{\tau_\ell} \left(e^{\frac{-\theta_\ell}{2^{k+1}}} z + 1\right)^{\tau_\ell}, & \text{if } n \text{ is odd}. \end{cases} \tag{4.5}$$

The k-level coefficient $K^{(k)}_\ell$ can be selected in accordance with the parameter p in order to ensure that the normalized exponential B-spline scheme, besides generating EP_Γ, reproduces the pair of exponential polynomials $\{e^{\theta_\ell x},\ e^{-\theta_\ell x}\} \in EP_\Gamma$. This fact is discussed in the next proposition.

Proposition 4.1. *Let Γ be a given symmetric set of the form* (3.1)*, and let $\{(\theta_\ell, \tau_\ell), (-\theta_\ell, \tau_\ell)\} \in \Gamma$. The symbols in (4.5) satisfy the $\{e^{\theta_\ell x},\ e^{-\theta_\ell x}\}$-reproduction condition if*

(i) $p = 0$, $(K^{(k)}_\ell)^{-1} = \left(e^{\frac{-\theta_\ell}{2^{k+1}}} + e^{\frac{\theta_\ell}{2^{k+1}}}\right) \tilde{B}^{(k)}_{N-2,\Gamma_{\ell,e}}\left(e^{\frac{\theta_\ell}{2^{k+1}}}\right)$ *for N even;*

(ii) $p = -\frac{1}{2}$, $(K^{(k)}_\ell)^{-1} = \left(e^{\frac{-\theta_\ell}{2^{k+2}}} + e^{\frac{\theta_\ell}{2^{k+2}}}\right)\left(e^{\frac{-\theta_\ell}{2^{k+1}}} + e^{\frac{\theta_\ell}{2^{k+1}}}\right) \tilde{B}^{(k)}_{N-3,\Gamma_{\ell,o}}\left(e^{\frac{\theta_\ell}{2^{k+1}}}\right)$ *for N odd.*

Proof. Let us start analyzing the case N even. Introducing the abbreviation $z^{(k)}_\ell := e^{\frac{-\theta_\ell}{2^{k+1}}}$, in view of Theorem 2.4 the reproduction of $\{e^{\theta_\ell x},\ e^{-\theta_\ell x}\}$ requires the fulfillment of the conditions

$$B^{(k)}_{N,\Gamma}(z^{(k)}_\ell) = 2(z^{(k)}_\ell)^p, \qquad B^{(k)}_{N,\Gamma}\left((z^{(k)}_\ell)^{-1}\right) = 2(z^{(k)}_\ell)^{-p},$$

that is

$$K^{(k)}_\ell\, \tilde{B}^{(k)}_{N,\Gamma}\left(z^{(k)}_\ell\right) = 2(z^{(k)}_\ell)^p, \qquad K^{(k)}_\ell\, \tilde{B}^{(k)}_{N,\Gamma}\left((z^{(k)}_\ell)^{-1}\right) = 2\left(z^{(k)}_\ell\right)^{-p}.$$

Exploiting the recurrence relation in (4.3) and observing that $\tilde{B}^{(k)}_{N-2,\Gamma_{\ell,e}}\left(z^{(k)}_\ell\right) = \tilde{B}^{(k)}_{N-2,\Gamma_{\ell,e}}\left((z^{(k)}_\ell)^{-1}\right)$, we obtain

$$\begin{aligned} K^{(k)}_\ell\, \left((z^{(k)}_\ell)^2 + 1\right) \tilde{B}^{(k)}_{N-2,\Gamma_{\ell,e}}\left((z^{(k)}_\ell)^{-1}\right) &= (z^{(k)}_\ell)^{p+1},\\ K^{(k)}_\ell\, \left((z^{(k)}_\ell)^{-2} + 1\right) \tilde{B}^{(k)}_{N-2,\Gamma_{\ell,e}}\left((z^{(k)}_\ell)^{-1}\right) &= (z^{(k)}_\ell)^{-p-1}. \end{aligned}$$

The solution of this system in the unknowns p and $K_\ell^{(k)}$ is given by

$$p = 0 \quad \text{and} \quad (K_\ell^{(k)})^{-1} = \left(z_\ell^{(k)} + (z_\ell^{(k)})^{-1}\right) \tilde{B}_{N-2,\Gamma_{\ell,e}}^{(k)} \left((z_\ell^{(k)})^{-1}\right),$$

which concludes the proof of subcase (i).

We continue studying the case N odd. Again, using the recurrence relation (4.4), the two conditions to be satisfied for the reproduction of $\{e^{\theta_\ell x},\ e^{-\theta_\ell x}\}$ can be written as

$$2K_\ell^{(k)} \left(z_\ell^{(k)}\right)^{-2} \left((z_\ell^{(k)})^2 + 1\right) \left(z_\ell^{(k)} + 1\right) \tilde{B}_{N-3,\Gamma_{\ell,o}}^{(k)} \left(z_\ell^{(k)}\right) = 2(z_\ell^{(k)})^p,$$

$$2K_\ell^{(k)} \left(z_\ell^{(k)}\right)^{2} \left((z_\ell^{(k)})^{-2} + 1\right) \left((z_\ell^{(k)})^{-1} + 1\right) \tilde{B}_{N-3,\Gamma_{\ell,o}}^{(k)} \left((z_\ell^{(k)})^{-1}\right) = 2(z_\ell^{(k)})^{-p}.$$

Now, since $N-3$ is even, we have $\tilde{B}_{N-3,\Gamma_{\ell,o}}^{(k)} \left(z_\ell^{(k)}\right) = \tilde{B}_{N-3,\Gamma_{\ell,o}}^{(k)} \left((z_\ell^{(k)})^{-1}\right)$, and thus we can write the simplified expressions

$$K_\ell^{(k)} \left(z_\ell^{(k)} + 1\right) \left((z_\ell^{(k)})^{-1} + z_\ell^{(k)}\right) \tilde{B}_{N-3,\Gamma_{\ell,o}}^{(k)} \left((z_\ell^{(k)})^{-1}\right) = (z_\ell^{(k)})^{p+1},$$

$$K_\ell^{(k)} \left(z_\ell^{(k)} + 1\right) \left((z_\ell^{(k)})^{-1} + z_\ell^{(k)}\right) \tilde{B}_{N-3,\Gamma_{\ell,o}}^{(k)} \left((z_\ell^{(k)})^{-1}\right) = (z_\ell^{(k)})^{-p}.$$

The solution of this system in the unknowns p and $K_\ell^{(k)}$ is given by $p = -\frac{1}{2}$ and

$$(K_\ell^{(k)})^{-1} = \left((z_\ell^{(k)})^{\frac{1}{2}} + (z_\ell^{(k)})^{-\frac{1}{2}}\right) \left(z_\ell^{(k)} + (z_\ell^{(k)})^{-1}\right) \tilde{B}_{N-3,\Gamma_{\ell,o}}^{(k)} \left((z_\ell^{(k)})^{-1}\right),$$

which concludes the proof of subcase (ii). □

Note that similar results concerning the normalization of exponential B-spline symbols are also given in [32]. Two special situations are considered in the next result.

Corollary 4.2. *For $N \geq 2$ the exponential B-spline scheme with symbols $\{B_{N,\Gamma}^{(k)}(z),\ k \geq 0\}$ reproduces $\{1,\ x\}$ if $\{(\theta_\ell, \tau_\ell), (-\theta_\ell, \tau_\ell)\} \in \Gamma$ with $\theta_\ell = 0$, and the k-level coefficient $(K_\ell^{(k)})^{-1}$ and the parameter p are*

$$(K_\ell^{(k)})^{-1} = \begin{cases} 2\tilde{B}_{N-2,\Gamma_{\ell,e}}^{(k)}(1), & \textit{for } N \textit{ even}, \\ 4\tilde{B}_{N-3,\Gamma_{\ell,o}}^{(k)}(1), & \textit{for } N \textit{ odd}, \end{cases} \quad \textit{and } p = \begin{cases} 0, & \textit{for } N \textit{ even}, \\ -\frac{1}{2}, & \textit{for } N \textit{ odd}. \end{cases}$$

Moreover, when $\theta_\ell = 0$ for $\ell = 1, \cdots, \lfloor \frac{n}{2} \rfloor$, the symbol $B_{N,\Gamma}^{(k)}(z)$ does not depend on k any longer and becomes the stationary symbol of the shifted order-N (polynomial) B-spline that we simply denote by $B_N(z) = z^{-\lceil \frac{N}{2} \rceil} \frac{(1+z)^N}{2^{N-1}}$.

5. Deriving the symbols of exponential pseudo-splines and investigating their symmetry properties

For any $p \in \mathbb{R}$ and $N, M \in \mathbb{N}$ the pseudo-spline subdivision scheme is defined to be the stationary scheme with minimal support that generates polynomials of degree $N-1$ and whose symbol, $a_{M,N}(z)$, satisfies the conditions $\frac{d^r a_{M,N}(1)}{dz^r} = 2(p)_r$, $r = 0, \ldots, M-1$ for reproduction of polynomials up to degree $M-1$. Its actual degree of polynomial reproduction is thus $\min\{N-1, M-1\}$ (see [15, 24]).

The main contribution of this paper consists in generalizing the family of (binary) pseudo-spline subdivision schemes to the nonstationary setting. The resulting family is called the family of *(binary) exponential pseudo-spline* subdivision schemes.

For Γ as in (3.1) and $p = 0$ if N is even, while $p = -\frac{1}{2}$ if N is odd, the family of symmetric binary exponential pseudo-splines is defined to be the family of symmetric subdivision schemes with *minimal support* that generates the space of exponential polynomials EP_Γ and whose k-level symbol satisfies the conditions in (2.4) for reproduction of $EP_{\tilde{\Gamma}} \subseteq EP_\Gamma$. Here, $EP_{\tilde{\Gamma}}$ is a subspace of EP_Γ identified by a symmetric set $\tilde{\Gamma} \subseteq \Gamma$, such that $\tilde{\Gamma}$ has cardinality $m \leq n$, with m and n of the same parity. To simplify the notation, we denote by $(\tilde{\theta}_\ell, \tilde{\tau}_\ell)$, $\ell = 1, \cdots, m$, a generic element of $\tilde{\Gamma}$, and we thus define $M := \sum_{j=1}^{m} \tilde{\tau}_j$.

The k-level symbol of an exponential pseudo-spline subdivision scheme is therefore of the form

$$a^{(k)}_{M,N,\Gamma}(z) := B^{(k)}_{N,\Gamma}(z)\, c^{(k)}_{M,\Gamma}(z),$$

where $B^{(k)}_{N,\Gamma}(z)$ is the k-level symbol of the normalized exponential B-spline scheme in (4.5) with $K^{(k)}_\ell$ as in Proposition 4.1, whereas $c^{(k)}_{M,\Gamma}(z)$ is the k-level Laurent polynomial of lowest possible degree such that $a^{(k)}_{M,N,\Gamma}(z)$ satisfies

$$\begin{cases} a^{(k)}_{M,N,\Gamma}(z^{(k)}_\ell) = 2\left(z^{(k)}_\ell\right)^p, \\ \dfrac{d^s\, a^{(k)}_{M,N,\Gamma}(z^{(k)}_\ell)}{dz^s} = 2(p)_s \left(z^{(k)}_\ell\right)^{p-s} \qquad s = 1, \ldots, \tilde{\tau}_\ell - 1, \end{cases} \tag{5.1}$$

for $z^{(k)}_\ell := e^{\frac{-\tilde{\theta}_\ell}{2^{k+1}}}$, $\ell = 1, \ldots, m$, $m \leq n$, and $(\tilde{\theta}_\ell, \tilde{\tau}_\ell) \in \tilde{\Gamma}$. Obviously, (2.3) are satisfied by construction.

Using the Leibniz rule we can write conditions (5.1) in the equivalent form

$$\sum_{i=0}^{s} \binom{s}{i} \frac{d^i\, c^{(k)}_{M,\Gamma}(z^{(k)}_\ell)}{dz^i} \frac{d^{s-i}\, B^{(k)}_{N,\Gamma}(z^{(k)}_\ell)}{dz^{s-i}} = v_{\ell,s}, \quad s = 0, \ldots, \tilde{\tau}_\ell - 1, \tag{5.2}$$

where $v_{\ell,s} := 2(p)_s \left(z^{(k)}_\ell\right)^{p-s}$.

We start by considering the case $m = n$ (which means $M = N$). In the latter case equations (5.2) can be rewritten as the linear system

$$\boldsymbol{\mathcal{A}}\mathbf{w} = \mathbf{v}, \quad \boldsymbol{\mathcal{A}} \in \mathbb{R}^{N\times N}, \quad \mathbf{w} \in \mathbb{R}^N, \quad \mathbf{v} \in \mathbb{R}^N \quad \text{with} \quad N = \sum_{j=1}^{n} \tilde{\tau}_j, \tag{5.3}$$

where $\mathbf{w} := \left(\frac{d^i\, c_{N,\Gamma}^{(k)}(z_\ell^{(k)})}{dz^i},\ i = 0, \cdots, \tilde{\tau}_\ell - 1,\ \ell = 1, \cdots, n\right)^T$, $\mathbf{v}$ is defined as $\mathbf{v} := (v_{1,0}, \ldots, v_{1,\tilde{\tau}_1-1}, v_{2,0}, \ldots, v_{2,\tilde{\tau}_2-1}, \ldots)^T$, and $\boldsymbol{\mathcal{A}}$ is the block diagonal lower triangular matrix given by

$$\boldsymbol{\mathcal{A}} := \begin{bmatrix} \mathcal{A}_1 & & \\ & \ddots & \\ & & \mathcal{A}_n \end{bmatrix}, \quad \mathcal{A}_j \in \mathbb{C}^{\tilde{\tau}_j \times \tilde{\tau}_j},$$

where, for $1 \le j \le n$,

$$\mathcal{A}_j := \begin{bmatrix} B_{N,\Gamma}^{(k)}(z_j^{(k)}) & & & \\ \binom{1}{0}\frac{d^1\, B_{N,\Gamma}^{(k)}(z_j^{(k)})}{dz^1} & B_{N,\Gamma}^{(k)}(z_j^{(k)}) & & \\ \vdots & & \ddots & \\ \binom{\tilde{\tau}_j-1}{0}\frac{d^{\tilde{\tau}_j-1}\, B_{N,\Gamma}^{(k)}(z_j^{(k)})}{dz^{\tilde{\tau}_j-1}} & \binom{\tilde{\tau}_j-1}{1}\frac{d^{\tilde{\tau}_j-2}\, B_{N,\Gamma}^{(k)}(z_j^{(k)})}{dz^{\tilde{\tau}_j-2}} & \ldots & B_{N,\Gamma}^{(k)}(z_j^{(k)}) \end{bmatrix}.$$

The structure of $\mathcal{A}_j^{-1}$ follows from the next result.

Lemma 5.1. *For a given $z_j^{(k)}$ such that $B_{N,\Gamma}^{(k)}(z_j^{(k)}) \neq 0$ the matrix $\mathcal{A}_j$ is invertible and*

$$\mathcal{A}_j^{-1} = \mathcal{G}_j := \begin{bmatrix} G_{N,\Gamma}^{(k)}(z_j^{(k)}) & & & \\ \binom{1}{0}\frac{d^1\, G_{N,\Gamma}^{(k)}(z_j^{(k)})}{dz^1} & G_{N,\Gamma}^{(k)}(z_j^{(k)}) & & \\ \vdots & & \ddots & \\ \binom{\tilde{\tau}_j-1}{0}\frac{d^{\tilde{\tau}_j-1}\, G_{N,\Gamma}^{(k)}(z_j^{(k)})}{dz^{\tilde{\tau}_j-1}} & \binom{\tilde{\tau}_j-1}{1}\frac{d^{\tilde{\tau}_j-2}\, G_{N,\Gamma}^{(k)}(z_j^{(k)})}{dz^{\tilde{\tau}_j-2}} & \ldots & G_{N,\Gamma}^{(k)}(z_j^{(k)}) \end{bmatrix}$$

with $G_{N,\Gamma}^{(k)}(z) := 1/B_{N,\Gamma}^{(k)}(z)$.

Proof. Let $\mathcal{S}_j = \mathcal{A}_j\,\mathcal{G}_j$ be the product matrix. The claim follows from the relation $B_{N,\Gamma}^{(k)}(z)\; G_{N,\Gamma}^{(k)}(z) = 1$. By differentiating and using the Leibniz rule we obtain that, for $s \ge 1$,

$$\sum_{i=0}^{s} \binom{s}{i} \frac{d^i\, G_{N,\Gamma}^{(k)}(z_j^{(k)})}{dz^i} \frac{d^{s-i}\, B_{N,\Gamma}^{(k)}(z_j^{(k)})}{dz^{s-i}} = 0,$$

which means that the subdiagonal entries in the first column of $\mathcal{S}_j$ are zero. For the remaining entries observe that

$$(\mathcal{S}_j)_{\ell,r} = \sum_{i=r}^{\ell} \binom{\ell-1}{i-1}\binom{i-1}{r-1} \frac{d^{\ell-i}\, B^{(k)}_{N,\Gamma}(z^{(k)}_j)}{dz^{\ell-i}} \frac{d^{i-r}\, G^{(k)}_{N,\Gamma}(z^{(k)}_j)}{dz^{i-r}},$$

and, hence, by setting $\tilde{i} := i - r$ and $\tilde{\ell} := \ell - r$

$$(\mathcal{S}_j)_{\ell,r} = \binom{\ell-1}{r-1} \sum_{\tilde{i}=0}^{\tilde{\ell}} \binom{\tilde{\ell}}{\tilde{i}} \frac{d^{\tilde{\ell}-\tilde{i}}\, B^{(k)}_{N,\Gamma}(z^{(k)}_j)}{dz^{\tilde{\ell}-\tilde{i}}} \frac{d^{\tilde{i}}\, G^{(k)}_{N,\Gamma}(z^{(k)}_j)}{dz^{\tilde{i}}} = \delta_{\ell,r},$$

where $\delta_{\ell,r}$ denotes the Kronecker symbol. □

Assuming that Γ is defined as in (3.1), we have that $z^{(k)}_j \neq -z^{(k)}_\ell$, $1 \leq j, \ell \leq N$ and thus Lemma 5.1 yields the following.

Proposition 5.2. *For* $z^{(k)}_\ell := e^{\frac{-\tilde{\theta}_\ell}{2k+1}}$, $\ell = 1, \cdots, n$, *we have*

$$\frac{d^s\, c^{(k)}_{N,\Gamma}(z^{(k)}_\ell)}{dz^s} = \sum_{i=0}^{s} \binom{s}{i} v_{\ell,i} \frac{d^{s-i}\, G^{(k)}_{N,\Gamma}(z^{(k)}_\ell)}{dz^{s-i}}, \quad s = 0, \dots, \tilde{\tau}_\ell - 1, \quad \ell = 1, \dots, n. \tag{5.4}$$

where $v_{\ell,i} := 2\left(z^{(k)}_\ell\right)^{p-i} (p)_i$ *and* $G^{(k)}_{N,\Gamma}(z) := \frac{1}{B^{(k)}_{N,\Gamma}(z)}$.

It is worth pointing out that, once we have specified the support of $c^{(k)}_{N,\Gamma}(z)$, the generalized interpolation conditions (5.4) enable the computation of its coefficients by means of an interpolation process. In particular, the construction of $c^{(k)}_{N,\Gamma}(z)$ can rely upon the following functional approach. Let $\{\beta_i\}_{i=1}^{N}$, $N = \sum_i \tilde{\tau}_i$, denote a finite sequence of nodes generated from the distinct points $z^{(k)}_\ell$, $1 \leq \ell \leq n$, each of them repeated $\tilde{\tau}_\ell$ times. Moreover, for a given $\eta > 0$ let γ_η be the lemniscata defined by $\gamma_\eta = \{z \in \mathbb{C} \colon |\prod_{\ell=1}^{n}(z - z^{(k)}_\ell)^{\tilde{\tau}_\ell}| = \eta^N\}$. It is worth noticing that

$$\prod_{\ell=1}^{n} (z - z^{(k)}_\ell)^{\tilde{\tau}_\ell} = \xi_k (-z)^{\lceil \frac{N}{2} \rceil} B^{(k)}_{N,\Gamma}(-z),$$

for a suitable $\xi_k \in \mathbb{C}$. Let us introduce the infinite sequence $\{\tilde{\beta}_i\}_{i\in\mathbb{N}}$ obtained by cyclically repeating each β_i, i.e., $\tilde{\beta}_i = \beta_{\mathrm{mod}(i,N)}$, $i \geq 1$. For $m_1 \leq m_2$ let $f[\{\tilde{\beta}_i\}_{i=m_1}^{m_2}]$ be the divided difference of the meromorphic function $f(z) := z^\ell a^{(k)}_{N,N,\Gamma}(z)/B^{(k)}_{N,\Gamma}(z)$ on the set of points $\tilde{\beta}_i$, $m_1 \leq i \leq m_2$, where $\ell \in \mathbb{Z}$ is a given fixed integer. Then the relation

$$f(z) = \sum_{i=1}^{+\infty} f[\{\tilde{\beta}_s\}_{s=1}^{i}] \prod_{j=1}^{i-1} (z - \tilde{\beta}_j)$$

holds in the following sense: the partial sums of the Newton series converge uniformly to $f(z)$ in any closed set lying in the interior of γ_η for any $\eta > 0$ such that f is analytic in γ_η. Since

$$\sum_{i=1}^{+\infty} f[\{\tilde{\beta}_s\}_{s=1}^{i}] \prod_{j=1}^{i-1}(z-\tilde{\beta}_j) = \sum_{i=1}^{N} f[\{\tilde{\beta}_s\}_{s=1}^{i}] \prod_{j=1}^{i-1}(z-\tilde{\beta}_j) + B_{N,\Gamma}^{(k)}(-z)R_k(z),$$

one deduces that

$$a_{N,N,\Gamma}^{(k)}(z) = z^{-\ell} B_{N,\Gamma}^{(k)}(z) \sum_{i=1}^{N} f[\{\tilde{\beta}_s\}_{s=1}^{i}] \prod_{j=1}^{i-1}(z-\tilde{\beta}_j) + z^{-\ell} B_{N,\Gamma}^{(k)}(z) B_{N,\Gamma}^{(k)}(-z)R_k(z),$$

and, therefore, in view of (4.2), we can set

$$c_{N,\Gamma}^{(k)}(z) = z^{-\ell} \sum_{i=1}^{N} f[\{\tilde{\beta}_s\}_{s=1}^{i}] \prod_{j=1}^{i-1}(z-\tilde{\beta}_j),$$

which is the shifted Newton form of the Hermite interpolant of $\frac{a_{N,N,\Gamma}^{(k)}(z)}{B_{N,\Gamma}^{(k)}(z)}$ at the points $z_\ell^{(k)},\ \ell = 1, \cdots, n$.

In this way for any value of ℓ we may determine a Laurent polynomial satisfying (5.4) whose support lies in $[-\ell, -\ell+N-1]$. If N is odd, then by choosing $\ell = \frac{N-1}{2}$ we obtain the unique Laurent polynomial supported in $[\frac{1-N}{2}, \frac{N-1}{2}]$. By using the symmetry of both the symbol and the distribution of nodes, and from the uniqueness of the Laurent polynomial, we may conclude that this latter polynomial is symmetric. The case where N is even is a bit more involved. In principle, using the same arguments as above we find that the interpolating Laurent polynomial is supported in $[-\frac{N}{2}, \frac{N}{2}-1]$ or $[-\frac{N}{2}+1, \frac{N}{2}]$. However, by expressing the polynomial in the new variable $t := z + z^{-1}$ we find that there exists a uniquely determined symmetric interpolating Laurent polynomial supported in $[-\frac{N}{2}+1, \frac{N}{2}-1]$. The precise statement is given below.

Proposition 5.3. *The polynomial correction $c_{N,\Gamma}^{(k)}(z)$ is the unique symmetric Laurent polynomial supported in $[-\lceil \frac{N}{2} \rceil + 1, \lceil \frac{N}{2} \rceil - 1]$ which satisfies the conditions* (5.4) *with p, N such that $p = 0$ if N even and $p = -\frac{1}{2}$ if N odd.*

Proof. We are looking for a polynomial of the form

$$c_{N,\Gamma}^{(k)}(z) = c_0 + \sum_{j=1}^{\lceil \frac{N}{2} \rceil - 1} c_j (z^j + z^{-j}),$$

which fulfills the interpolation conditions (5.4). From Proposition 3.7 it follows that the condition in $z_\ell^{(k)}$ implies the same in $(z_\ell^{(k)})^{-1}$ and vice versa. Hence, for N even we have to impose $N/2$ independent conditions, while for N odd we have

$\tilde{\tau}_\ell = 2j+1$ conditions at $z_\ell^{(k)} = 1$ plus $(N-1)/2 - j$ independent conditions. Since from Remark 3.6 the conditions at $z_\ell^{(k)} = 1$ yield $j+1$ independent conditions, we obtain $(N+1)/2 = \lceil \frac{N}{2} \rceil$ conditions also for the odd case. Defining $t := z + z^{-1}$ let us introduce the functions $p_j(t) = z^j + z^{-j}$, $j \geq 0$. Such functions are monic Chebyshev-like polynomials of degree j which, starting from $p_0(t) = 2$, $p_1(t) = t$, satisfy the three-term recurrence relation

$$p_{j+1}(t) = t p_j(t) - p_{j-1}(t), \qquad j \geq 1.$$

Hence, by writing

$$c_{N,\Gamma}^{(k)}(z) = \frac{c_0}{2} p_0(t) + \sum_{j=1}^{\lceil \frac{N}{2} \rceil - 1} c_j p_j(t) = \psi(t),$$

the proof follows from the existence and uniqueness of the interpolating polynomial $\psi(t)$ on the considered set of nodes. □

The results given so far immediately generalize to the case where we consider a subset $\tilde{\Gamma}$ of Γ.

Theorem 5.4. *Let Γ and $\tilde{\Gamma} \subset \Gamma$ be symmetric sets of cardinality n and m, respectively, with m and n of the same parity. Moreover let EP_Γ, $EP_{\tilde{\Gamma}}$ be the corresponding spaces of exponential polynomials, and assume $p = 0$ in case n and m are both even (namely, N and M both even), $p = -\frac{1}{2}$ in case n and m are both odd (namely, N and M both odd). Then there exists a unique symmetric Laurent polynomial $c_{M,\Gamma}^{(k)}(z)$ supported in $[-\lceil \frac{M}{2} \rceil + 1, \lceil \frac{M}{2} \rceil - 1]$ satisfying for $z_\ell^{(k)} := e^{\frac{-\tilde{\theta}_\ell}{2^{k+1}}}$, $\ell = 1, \ldots, m$, $m \leq n$, $(\tilde{\theta}_\ell, \tilde{\tau}_\ell) \in \tilde{\Gamma}$, the generalized interpolation conditions*

$$\frac{d^s\, c_{M,\Gamma}^{(k)}(z_\ell^{(k)})}{dz^s} = \sum_{i=0}^{s} \binom{s}{i} v_{\ell,i} \frac{d^{s-i}\, G_{N,\Gamma}^{(k)}(z_\ell^{(k)})}{dz^{s-i}}, \qquad s = 0, \ldots, \tilde{\tau}_\ell - 1,$$

where $v_{\ell,i} := 2 \left(z_\ell^{(k)}\right)^{p-i} (p)_i$ and $G_{N,\Gamma}^{(k)}(z) := \frac{1}{B_{N,\Gamma}^{(k)}(z)}$.

For the effective construction of the polynomial $c_{M,\Gamma}^{(k)}(z)$ we can proceed as follows. Setting

$$c_{M,\Gamma}^{(k)}(z) = \psi(z + z^{-1}) \tag{5.5}$$

and using the Faà di Bruno Formula we get

$$\frac{d^s\, c_{M,\Gamma}^{(k)}(z_\ell^{(k)})}{dz^s} = \sum_{r=1}^{s} \frac{d^r\, \psi\left(z_\ell^{(k)} + (z_\ell^{(k)})^{-1}\right)}{dz^r} A_{r,s}(z_\ell^{(k)}), \qquad s = 1, \ldots, \tilde{\tau}_\ell - 1. \tag{5.6}$$

Since

$$A_{s,s}(z_\ell^{(k)}) = \left(1 - \frac{1}{{(z_\ell^{(k)})}^2}\right)^s,$$

we obtain that for $z_\ell^{(k)} \neq \pm 1$ the triangular system is invertible and, therefore, it enables the computation of the highest derivative of $\psi(z+z^{-1})$ to be performed. In this way $\psi(z+z^{-1})$ and then $c_{M,\Gamma}^{(k)}(z)$ can be computed by using the customary Hermite interpolation formula. For the case $z_\ell^{(k)} = 1$ it can be shown that only the derivatives of $c_{M,\Gamma}^{(k)}(z)$ of even order give information about the derivatives of $\psi(z+z^{-1})$. The case $z_\ell^{(k)} = -1$ cannot occur due to the definition of $z_\ell^{(k)}$.

Proposition 5.5. *In the case $p = 0$ and $M = N$ even, the subdivision symbol $a_{N,N,\Gamma}^{(k)}(z)$ of the symmetric exponential pseudo-spline scheme, derived from Theorem 5.4, is always interpolatory.*

Proof. The proof exploits the fact that, for $p = 0$ and N even, from a subdivision symbol satisfying (2.3) we are able to construct a subdivision symbol satisfying (2.4), and viceversa. In particular, in the case $p = 0$ and N even, from conditions (5.4) we find that $a_{N,N,\Gamma}^{(k)}(z) - 2 = B_{N,\Gamma}^{(k)}(-z)\, q^{(k)}(z)$ for a certain Laurent polynomial $q^{(k)}(z)$. Hence,

$$a_{N,N,\Gamma}^{(k)}(z) = 2 + B_{N,\Gamma}^{(k)}(-z)\, q^{(k)}(z) = B_{N,\Gamma}^{(k)}(z)\, c_{N,\Gamma}^{(k)}(z).$$

Since the relation holds for any z we also have

$$2 + B_{N,\Gamma}^{(k)}(z)\, q^{(k)}(-z) = B_{N,\Gamma}^{(k)}(-z)\, c_{N,\Gamma}^{(k)}(-z),$$

which gives

$$q^{(k)}(z) = -c_{N,\Gamma}^{(k)}(-z)$$

and, hence,

$$a_{N,N,\Gamma}^{(k)}(z) + a_{N,N,\Gamma}^{(k)}(-z) = 2.$$

□

6. Convergence and regularity of exponential pseudo-spline subdivision schemes

The aim of this section is two-fold. First, we show that the strategy proposed in the previous section allows us to get back polynomial pseudo-splines in the stationary case. Second, we prove convergence and regularity of exponential pseudo-spline subdivision schemes.

To achieve the first goal we introduce a new set of z-functions, that are meant to be shifted B-spline symbols, of the form

$$\bar{B}_N(z) := \frac{(z+1)^N}{2^{N-1}}\, z^{-\frac{N}{2}} = \begin{cases} \frac{(z+z^{-1}+2)^\rho}{2^{2\rho-1}}, & \text{if} \quad N = 2\rho \\ \frac{(z+z^{-1}+2)^{\rho+\frac{1}{2}}}{2^{2\rho}}, & \text{if} \quad N = 2\rho + 1. \end{cases}$$

Obviously, in case N is even, $\bar{B}_N(z) = B_N(z)$, whereas, for N odd, $\bar{B}_N(z) = z^{\frac{1}{2}} B_N(z)$. We also need the following result whose proof is obtained by induction, following the lines of the proof of Proposition 3.5.

Lemma 6.1. *The even-symmetric symbol $a(z)$ satisfies*

$$\frac{d^r\, a(1)}{dz^r} = 2\left(-\frac{1}{2}\right)_r, \quad r \geq 0,$$

if and only if the associated odd-symmetric z-function $b(z) = z^{\frac{1}{2}}\, a(z)$ satisfies

$$\frac{d^r b(1)}{dz^r} = 2\delta_{r,0}, \qquad r \geq 0.$$

By means of these preliminary results, we can prove the following proposition.

Proposition 6.2. *For the order-N B-spline symbol $B_N(z) = \frac{(z+1)^N}{2^{N-1} z^{\lceil \frac{N}{2} \rceil}}$, let $c_M(z)$ be the polynomial correction such that*

$$\sum_{i=0}^{s} \binom{s}{i} \frac{d^i\, c_M(1)}{dz^i} \frac{d^{s-i}\, B_N(1)}{dz^{s-i}} = 2(p)_s, \quad s = 0, \dots, M-1,$$

with $p = 0$ if $N = 2\rho$ and $p = -\frac{1}{2}$ if $N = 2\rho+1$. Then, $a_{M,N}(z) = B_N(z)\, c_M(z)$ is the subdivision symbol of the polynomial pseudo-spline scheme given in [26, Section 6], that is

$$a_{M,N}(z) = \begin{cases} 2\,\sigma^\rho(z) \displaystyle\sum_{s=0}^{\lfloor \frac{M-1}{2} \rfloor} \binom{\rho+s-1}{s} \delta^s(z), & \text{if} \quad N = 2\rho, \\ \frac{z+1}{z}\,\sigma^\rho(z) \displaystyle\sum_{s=0}^{\lfloor \frac{M-1}{2} \rfloor} \binom{\rho+s-\frac{1}{2}}{s} \delta^s(z), & \text{if} \quad N = 2\rho+1, \end{cases}$$

where $\delta(z) := -\frac{(1-z)^2}{4z}$ and $\sigma(z) := \frac{(1+z)^2}{4z}$.

Proof. We start by observing that, since $c_M(z) = \frac{a_{M,N}(z)}{B_N(z)} = \frac{z^{\frac{1}{2}} a_{M,N}(z)}{z^{\frac{1}{2}} B_N(z)}$, in view of Lemma 6.1, the polynomial correction $c_M(z)$ can be equivalently obtained by solving the linear system

$$\sum_{i=0}^{s} \binom{s}{i} \frac{d^i\, c_M(1)}{dz^i} \frac{d^{s-i}\, \bar{B}_N(1)}{dz^{s-i}} = 2v_s, \quad s = 0, \dots, M-1$$

with $\bar{B}_N(z) = 2^{-(N-1)}\, z^{-\frac{N}{2}}\, (z+1)^N$ and $v_s = 2\delta_{s,0}$. We continue by taking

$$\bar{G}_N(z) = \frac{1}{\bar{B}_N(z)} = 2^{2\rho-2p-1}(z+z^{-1}+2)^{-\rho+p}, \quad p \in \left\{0, -\frac{1}{2}\right\}.$$

Thus, using the Faà di Bruno formula we can write

$$\frac{d^r \bar{G}_N(z)}{dz^r} = 2^{2\rho-2p-1} \sum_{j=1}^{r} (-1)^j \binom{\rho-p+j-1}{j} j!(z+z^{-1}+2)^{-\rho+p-j} A_{j,r}(z),$$

where

$$A_{j,r}(z) = \sum_{\mathbf{q}\in\mathbf{M}^j,\,|\mathbf{q}|=r} \frac{r!}{\mathbf{q}!} \frac{\prod_{i=1}^{j}\left(\delta_{q_i,1} + (-1)^{q_i} q_i! z^{-(q_i+1)}\right)}{\prod_{i=1}^{r} N(\mathbf{q},i)!}$$

with

$$\mathbf{M}^j = \{\mathbf{q} = (q_1, q_2, ..., q_j) \in \mathbb{N}^j,\, q_1 \geq q_2 \geq ... \geq q_j \geq 1\}, \qquad |\mathbf{q}| = q_1 + ... + q_j$$

and $N(\mathbf{q}, i)$ denoting the number of times the integer $i \in \mathbb{N}$ appears in the j-tuple $\mathbf{q} \in \mathbb{N}^j$. Evaluating at $z = 1$ we obtain

$$\frac{d^r \bar{G}_N(1)}{dz^r} = \sum_{j=1}^{r} (-1)^j \binom{\rho-p+j-1}{j} j! 2^{-2j-1} A_{j,r}(1)$$

so that, recalling (5.4), for $s = 0, \ldots, M-1$ we find

$$\frac{d^s c_M(1)}{dz^s} = \sum_{i=0}^{s} \binom{s}{i} v_i \sum_{j=1}^{s-i} (-1)^j \binom{\rho-p+j-1}{j} j! 2^{-2j-1} A_{j,s-i}(1).$$

Hence,

$$\begin{aligned} c_M(1) &= 1 \\ \frac{d^s c_M(1)}{dz^s} &= \sum_{j=1}^{s} (-1)^j \binom{\rho-p+j-1}{j} j! 2^{-2j} A_{j,s}(1), \quad s = 1, \ldots, M-1. \end{aligned} \tag{6.1}$$

On the other hand, recalling (5.5)-(5.6) we can write

$$\begin{aligned} c_M(z) &= \psi(z+z^{-1}), \\ \frac{d^s c_M(z)}{dz^s} &= \sum_{j=1}^{s} \frac{d^j \psi(z+z^{-1})}{dz^j} A_{j,s}(z), \quad s = 1, \ldots, M-1, \end{aligned}$$

so that, when evaluating at $z = 1$, we obtain

$$\begin{aligned} c_M(1) &= \psi(2), \\ \frac{d^s c_M(1)}{dz^s} &= \sum_{j=1}^{s} \frac{d^j \psi(2)}{dz^j} A_{j,s}(1), \quad s = 1, \ldots, M-1. \end{aligned} \tag{6.2}$$

In this way, by comparison of (6.1) with (6.2), from all even s we can find the values attained by all derivatives of ψ at 2, *i.e.*

$$\begin{aligned} \psi(2) &= 1, \\ \frac{d^j \psi(2)}{dz^j} &= (-1)^j \binom{\rho-p+j-1}{j} j! 2^{-2j}, \quad j = 1, \ldots, \left\lfloor \frac{M-1}{2} \right\rfloor, \end{aligned}$$

and, exploiting the customary Hermite interpolation formula we can thus get the analytic expression of $c_M(z)$, *i.e.*

$$c_M(z)=\sum_{j=0}^{\lfloor\frac{M-1}{2}\rfloor}\frac{(z+z^{-1}-2)^j}{j!}\,\frac{d^j\psi(2)}{dz^j}=\sum_{j=0}^{\lfloor\frac{M-1}{2}\rfloor}\binom{\rho-p+j-1}{j}\delta^j(z),$$

with $\delta(z)=-\frac{(1-z)^2}{4z}$. Making distinction between N even and N odd, and rewriting $B_N(z)$ in terms of $\sigma(z)=\frac{(1+z)^2}{4z}$, the claim is proven. □

Now, in order to study the asymptotic behaviour of $a^{(k)}_{M,N,\Gamma}(z)$ when k approaches infinity, we recall from [12] the following definition.

Definition 6.3. *The sequence of subdivision masks* $\{\mathbf{a}^{(k)},\ k\geq 0\}$ *and* $\{\mathbf{a}\}$ *are called* asymptotically similar *if*

$$\lim_{k\to+\infty}\|\mathbf{a}^{(k)}-\mathbf{a}\|_\infty=\lim_{k\to+\infty}\max_{i\in\mathrm{supp}(\mathbf{a}^{(k)})\cup\mathrm{supp}(\mathbf{a})}|\mathrm{a}^{(k)}_i-\mathrm{a}_i|=0,$$

or, equivalently in terms of symbols, if $\lim_{k\to+\infty}a^{(k)}(z)=a(z)$*.*

The following proposition proves asymptotical similarity both between $B^{(k)}_{N,\Gamma}(z)$ and $B_N(z)$, and between the polynomial corrections $c^{(k)}_{M,\Gamma}(z)$ and $c_M(z)$.

Proposition 6.4. *As* $k\to+\infty$*, the exponential B-spline symbol* $B^{(k)}_{N,\Gamma}(z)$ *converges to the polynomial B-spline symbol* $B_N(z)$*, and the polynomial correction* $c^{(k)}_{M,\Gamma}(z)$ *approaches the symbol*

$$c_M(z)=\begin{cases}\displaystyle\sum_{s=0}^{\lfloor\frac{M-1}{2}\rfloor}\binom{\rho+s-1}{s}\delta^s(z), & \text{if } p=0\ (\textit{i.e. } N=2\rho),\\ \displaystyle\sum_{s=0}^{\lfloor\frac{M-1}{2}\rfloor}\binom{\rho+s-\frac{1}{2}}{s}\delta^s(z), & \text{if } p=-\frac{1}{2}\ (\textit{i.e. } N=2\rho+1),\end{cases}\tag{6.3}$$

where $\delta(z):=-\frac{(1-z)^2}{4z}$*.*

Proof. We start with the simple observation that $\lim_{k\to+\infty}B^{(k)}_{N,\Gamma}(z)=B_N(z)$. Next, we first consider the case $p=0$ and N even. From equation (5.1) we have that the Hermite interpolant of $f(z):=\frac{a^{(k)}_{M,N,\Gamma}(z)}{B^{(k)}_{N,\Gamma}(z)}$ at the points $z^{(k)}_\ell,\ \ell=1,\cdots,m$ coincides with the Hermite interpolant of $\chi^{(k)}(z):=\frac{2}{B^{(k)}_{N,\Gamma}(z)}$ at the same points. Hence, denoting by $\{\beta^{(k)}_i\}_{i=1}^M$, $M=\sum_{\ell=1}^m\tilde{\tau}_\ell$, a finite sequence of nodes generated from the distinct points $z^{(k)}_\ell$, $1\leq\ell\leq m$, each of them repeated $\tilde{\tau}_\ell$ times, and by $\tilde{\beta}^{(k)}_i=\beta^{(k)}_{\mathrm{mod}(i,M)}$, $i\geq 1$, we get

$$f[\{\tilde{\beta}^{(k)}_s\}_{s=1}^i]=\chi^{(k)}[\{\tilde{\beta}^{(k)}_s\}_{s=1}^i]=\frac{1}{2\pi\mathtt{i}}\int_C\frac{\chi^{(k)}(z)\mathtt{d}z}{\prod_{s=1}^i(z-\tilde{\beta}^{(k)}_i)},$$

where C is a simple closed curve in the complex plane enclosing a simply connected region which contains the points $\{\tilde{\beta}_s^{(k)}\}_{s=1}^{i}$. Therefore,

$$\lim_{k\to+\infty} \chi^{(k)}[\{\tilde{\beta}_s^{(k)}\}_{s=1}^{i}] = \frac{1}{(i-1)!}\frac{\partial^{i-1}}{\partial z^{i-1}}\chi^{(+\infty)}(z)\Big|_{z=1},$$

which means that, as k goes to infinity, $c_{M,\Gamma}^{(k)}(z)$ approaches a shifted Newton form of the Hermite interpolant of $\chi^{(+\infty)}(z) = \frac{2^{-(N-2)}}{z^{-\lceil \frac{N}{2}\rceil}(z+1)^N}$.
The remaining case $p = -1/2$ and N odd reduces to the previous analysis by observing that $c_{M,\Gamma}^{(k)}(z)$ can be computed from

$$za_{M,N,\Gamma}^{(k)}(z^2) = B_{N,\Gamma}^{(k)}(z^2)\, zc_{M,\Gamma}^{(k)}(z^2),$$

by imposing the generalized interpolation conditions (5.1) at the points $(z_\ell^{(k)})^{\frac{1}{2}}$, $(z_\ell^{(k)})^{-\frac{1}{2}}$ for the even symmetric symbol $\bar{a}_{M,N,\Gamma}^{(k)}(z) = za_{M,N,\Gamma}^{(k)}(z^2)$. □

Collecting all previous results we finally arrive at the following important asymptotic result.

Corollary 6.5. *The exponential pseudo-spline subdivision masks are asymptotically similar to the polynomial pseudo-spline subdivision masks, i.e. ,*

$$\lim_{k\to+\infty} a_{M,N,\Gamma}^{(k)}(z) = a_{M,N}(z), \tag{6.4}$$

with $a_{M,N}(z)$ denoting the well-known polynomial pseudo-spline symbol.

Before proceeding, we recall that in [25] and in [23] the authors prove convergence and regularity of the subdivision schemes associated with the symbol $a_{M,N}(z)$ with N even and odd, respectively. In the following we continue analyzing the values attained by $\{a_{M,N,\Gamma}^{(k)}(z),\ k \geq 0\}$ and its derivatives at $z = -1$ and, in turn, we study the convergence and regularity of the associated subdivision schemes.

Proposition 6.6. *Let $M > 1$. The subdivision symbols $\{a_{M,N,\Gamma}^{(k)}(z),\ k \geq 0\}$ are such that*

$$\begin{array}{lcl} |a_{M,N,\Gamma}^{(k)}(1) - 2| & = & O(2^{-kN}), \\ \left|\dfrac{d^s}{dz^s}\, a_{M,N,\Gamma}^{(k)}(-1)\right| & = & O(2^{-k(N-s)}),\quad s = 0,\dots,N-1, \end{array} \qquad k \to +\infty. \tag{6.5}$$

Moreover, the nonstationary subdivision scheme with symbols $\{a_{M,N,\Gamma}^{(k)}(z),\ k \geq 0\}$ converges and has the same regularity as the stationary one with symbol $a_{M,N}(z)$.

Proof. The proof of (6.5) is based on the recent results proven in [18, Theorem 10] and is a direct consequence of the exponential polynomial generation properties of $\{a^{(k)}_{M,N,\Gamma}(z),\ k \geq 0\}$ and the asymptotical similarity of $\{\mathbf{a}^{(k)}_{M,N,\Gamma},\ k \geq 0\}$ to $\{\mathbf{a}_{M,N}\}$, previously shown in Corollary 6.5. Then, for the convergence and regularity result we can rely on [6]. $\square$

7. An example

This section contains an interesting example of a family of nonstationary symmetric exponential pseudo-spline subdivision symbols. As far as we know, this is the first derivation of nonstationary purely exponential pseudo-spline symbols to appear. In fact, the recently published paper [37] merely discusses the interpolatory subcase of a family of exponential pseudo-spline schemes which reproduces function spaces spanned by an arbitrary number of polynomials and just a pair of exponential polynomials.

Let $\theta \in \mathbb{R}^+ \cup \mathrm{i}[0,\pi)$ and, for all $k \geq 0$, define $v^{(k)} = \frac{1}{2}(e^{\frac{\theta}{2^{k+1}}} + e^{-\frac{\theta}{2^{k+1}}})$. We consider the exponential B-spline scheme with k-level symbol

$$B^{(k)}_{N,\Gamma}(z) = \frac{(z+z^{-1}+2v^{(k)})^\rho}{2^{2\rho-1}(v^{(k)})^\rho},$$

with $\rho \in \mathbb{N}$, $\Gamma = \{(\theta,\rho),(-\theta,\rho)\}$ and $N = 2\rho$, which is obtained from the general formulation with $n=2$ and $z_1^{(k)} = e^{\frac{-\theta}{2^{k+1}}}$, $z_2^{(k)} = e^{\frac{\theta}{2^{k+1}}}$. The subdivision scheme with symbol $B^{(k)}_{N,\Gamma}(z)$ generates the space of exponential polynomials

$$\mathrm{span}\{e^{\theta x},\ e^{-\theta x},\ xe^{\theta x},\ xe^{-\theta x},\ \cdots,\ x^{\rho-1}e^{\theta x},\ x^{\rho-1}e^{-\theta x}\}, \tag{7.1}$$

and reproduces $\mathrm{span}\{e^{\theta x},\ e^{-\theta x}\}$ with respect to the parameter shift $p = 0$. In order to apply Theorem 5.4 with $m = n$, namely with $M = N$ (the only possibility we have here), we define

$$G^{(k)}_{N,\Gamma}(z) = 2^{2\rho-1}(v^{(k)})^\rho(z+z^{-1}+2v^{(k)})^{-\rho}$$

and, using the Faà di Bruno formula we compute

$$\frac{d^r G^{(k)}_{N,\Gamma}(z)}{dz^r} = 2^{2\rho-1}(v^{(k)})^\rho \sum_{j=1}^{r}(-1)^j \binom{\rho+j-1}{j} j!(z+z^{-1}+2v^{(k)})^{-\rho-j}\, A_{j,r}(z)$$

where $A_{j,r}(z)$ is the same as the one appearing in the proof of Proposition 6.2. Hence, for $\ell = 1,2$

$$G^{(k)}_{N,\Gamma}(z^{(k)}_\ell) = \frac{1}{2}$$

and

$$\frac{d^r G^{(k)}_{N,\Gamma}(z^{(k)}_\ell)}{dz^r} = \sum_{j=1}^{r}(-1)^j \binom{\rho+j-1}{j} j!2^{-2j-1}(v^{(k)})^{-j}\, A_{j,r}(z^{(k)}_\ell),\ 1 \leq r \leq \rho-1.$$

Therefore, from (5.4) we can write for $\ell = 1, 2$

$$\frac{d^s c_{N,\Gamma}^{(k)}(z_\ell^{(k)})}{dz^s} = \sum_{i=0}^{s} \binom{s}{i} v_{\ell,i} \sum_{j=1}^{s-i} (-1)^j \binom{\rho+j-1}{j} j! 2^{-2j-1} (v^{(k)})^{-j} A_{j,s-i}(z_\ell^{(k)}), \tag{7.2}$$

with $s = 0, ..., \rho-1$. Now, taking into account that, when $p = 0$, then $v_{\ell,i} = 2\delta_{i,0}$, $i = 0, \ldots, s$, equation (7.2) can be rewritten for $\ell = 1, 2$ and $1 \le s \le \rho - 1$ as

$$c_{N,\Gamma}^{(k)}(z_\ell^{(k)}) = 1, \tag{7.3}$$

$$\frac{d^s c_{N,\Gamma}^{(k)}(z_\ell^{(k)})}{dz^s} = \sum_{j=1}^{s} (-1)^j \binom{\rho+j-1}{j} j! 2^{-2j} (v^{(k)})^{-j} A_{j,s}(z_\ell^{(k)}). \tag{7.4}$$

On the other hand, from (5.5)-(5.6) we have that

$$\begin{aligned} c_{N,\Gamma}^{(k)}(z) &= \psi(z+z^{-1}) \\ \frac{d^s c_{N,\Gamma}^{(k)}(z)}{dz^s} &= \sum_{j=1}^{s} \frac{d^j \psi(z+z^{-1})}{dz^j} A_{j,s}(z), \quad s = 1, ..., \rho-1, \end{aligned}$$

which, when evaluated at $z_\ell^{(k)}$, $\ell = 1, 2$, yield

$$c_{N,\Gamma}^{(k)}(z_\ell^{(k)}) = \psi(2v^{(k)}) \tag{7.5}$$

$$\frac{d^s c_{N,\Gamma}^{(k)}(z_\ell^{(k)})}{dz^s} = \sum_{j=1}^{s} \frac{d^j \psi(2v^{(k)})}{dz^j} A_{j,s}(z_\ell^{(k)}), \quad s = 1, ..., \rho-1, \tag{7.6}$$

due to the fact that $z_\ell^{(k)} + (z_\ell^{(k)})^{-1} = 2v^{(k)}$ for $\ell = 1, 2$. At this point, comparing (7.3) with (7.5) and (7.4) with (7.6), we respectively obtain

$$\begin{aligned} \psi(2v^{(k)}) &= 1, \\ \frac{d^j \psi(2v^{(k)})}{dz^j} &= (-1)^j \binom{\rho+j-1}{j} j! 2^{-2j} (v^{(k)})^{-j}, \quad j = 1, ..., \rho-1. \end{aligned}$$

Using the customary Hermite interpolation formula we can thus compute

$$\begin{aligned} \psi(z+z^{-1}) &= \sum_{j=0}^{\rho-1} \frac{(z+z^{-1}-2v^{(k)})^j}{j!} \frac{d^j \psi(2v^{(k)})}{dz^j} \\ &= \sum_{j=0}^{\rho-1} (z+z^{-1}-2v^{(k)})^j (-1)^j \binom{\rho+j-1}{j} 2^{-2j} (v^{(k)})^{-j}, \end{aligned} \tag{7.7}$$

which provides the expression of the polynomial correction $c_{N,\Gamma}^{(k)}(z)$. The obtained symbol $c_{N,\Gamma}^{(k)}(z)$ is the one that allows us to define the symmetric symbol

$a^{(k)}_{N,N,\Gamma}(z) = B^{(k)}_{N,\Gamma}(z) c^{(k)}_{N,\Gamma}(z)$ which reproduces the space of exponential polynomials (7.1) with respect to the parameter shift $p = 0$. Since $p = 0$ and $M = N$ is even, in light of Proposition 5.5, $a^{(k)}_{N,N,\Gamma}(z)$ is an interpolatory symbol. We can thus refer to $c^{(k)}_{N,\Gamma}(z)$ as to the polynomial correction transforming the approximating scheme having symbol $B^{(k)}_{N,\Gamma}(z)$ into the interpolatory scheme with the same generation properties (see [3, 13, 14]).

Remark 7.1. *Since, as previously observed,* $\lim_{k\to+\infty} B^{(k)}_{N,\Gamma} = B_{2\rho}(z)$ *and* $\lim_{k\to+\infty} c^{(k)}_{N,\Gamma}(z) = c_{2\rho}(z)$*, then the derived family of exponential pseudo-spline schemes with k-level symbol* $a^{(k)}_{N,N,\Gamma}(z)$ *can be considered a nonstationary extension of the family of interpolatory* (2ρ)*-point Dubuc-Deslauriers schemes (see [22]). A general construction for families of interpolatory schemes reproducing exponential polynomials was originally proposed in [28]. However, to the best of our knowledge, an algebraic approach for deriving the subdivision symbols of exponential pseudo-spline schemes (including as a special case all nonstationary variants of Dubuc-Deslauriers schemes) was never investigated before.*

For instance, note that, when $\rho = 2$, equation (7.7) yields $c^{(k)}_{N,\Gamma}(z) = -\frac{1}{2v^{(k)}} z + 2 - \frac{1}{2v^{(k)}} z^{-1}$ and the resulting exponential pseudo spline scheme is an interpolatory 4-point scheme with k-level mask

$$\left\{\ldots 0,\, -\frac{1}{16(v^{(k)})^3},\, 0,\, \frac{3(4(v^{(k)})^2-1)}{16(v^{(k)})^3},\, 1,\, \frac{3(4(v^{(k)})^2-1)}{16(v^{(k)})^3},\, 0,\, -\frac{1}{16(v^{(k)})^3},\, 0\ldots\right\} \tag{7.8}$$

while, when $\rho = 3$, from equation (7.7) we obtain $c^{(k)}_{N,\Gamma}(z) = \frac{3}{8(v^{(k)})^2} z^2 - \frac{9}{4v^{(k)}} z + \frac{3+16(v^{(k)})^2}{4(v^{(k)})^2} - \frac{9}{4v^{(k)}} z^{-1} + \frac{3}{8(v^{(k)})^2} z^{-2}$ and thus the resulting exponential pseudo spline scheme is an interpolatory 6-point scheme with k-level mask

$$\begin{aligned}\Big\{0, \cdots, 0,\, &\frac{3}{256(v^{(k)})^5},\, 0,\, -\frac{5(8(v^{(k)})^2-3)}{256(v^{(k)})^5},\, 0,\, \frac{15(8(v^{(k)})^4-4(v^{(k)})^2+1)}{128(v^{(k)})^5},\, 1,\\ &\frac{15(8(v^{(k)})^4-4(v^{(k)})^2+1)}{128(v^{(k)})^5},\, 0,\, -\frac{5(8(v^{(k)})^2-3)}{256(v^{(k)})^5},\, 0,\, \frac{3}{256(v^{(k)})^5},\, 0, \cdots, 0\Big\}.\end{aligned} \tag{7.9}$$

Figure 1 shows how the basic limit function for the interpolatory 4-point and 6-point scheme (with k-level mask in (7.8) and (7.9), respectively) changes when tuning the parameter θ. Such interpolatory 4- and 6-point schemes are new nonstationary variants of the well-known Dubuc-Deslauriers schemes in [22]. They differ from the ones previously proposed in [2, 39] for the space of exponential polynomials they reproduce. Figure 2 illustrates the basic limit function of the extreme members of two different families of exponential pseudo-splines, i.e. exponential splines and interpolatory schemes, pointing out how their shape properties change for a given parameter θ.

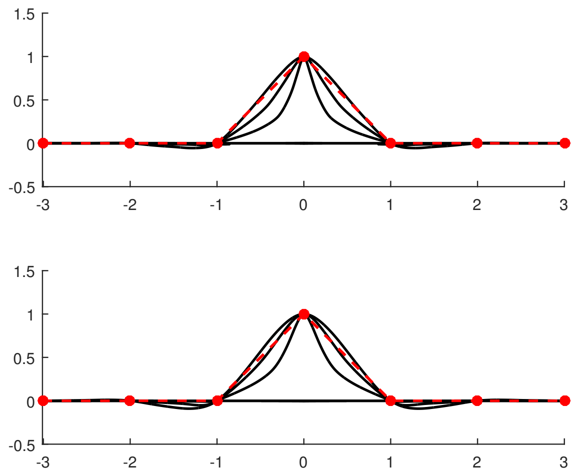

Figure 1: Effect of the parameter θ on the basic limit functions of the interpolatory 4-point scheme (top) and 6-point scheme (bottom) with k-level mask in (7.8) and (7.9), respectively. The graphs are restricted to $[-3, 3]$ and correspond to the real values $\theta \in \{1, 2, 3\}$.

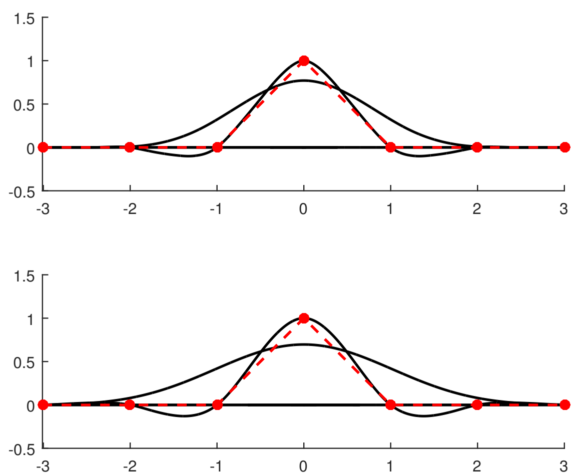

Figure 2: Extreme members of two different families of exponential pseudo-splines with $\theta = \mathrm{i}$: basic limit function of the interpolatory 4-point scheme and order-4 exponential B-spline on $[-3, 3]$ (top); basic limit function of the interpolatory 6-point scheme and order-6 exponential B-spline on $[-3, 3]$ (bottom).

8. Conclusions

In this work we have constructed exponential pseudo-splines, *i.e.* nonstationary counterparts of polynomial pseudo-splines and developments of exponential

B-splines, that show so much generality and flexibility to bode a real practical use in several domains of applications. The construction of exponential pseudo-spline symbols has been obtained by means of an algebraic strategy featuring the following key properties:

- it allows the user to pass from exponential B-spline subdivision schemes generating a space of exponential polynomials to subdivision schemes reproducing the same space, or any desired of its subspaces;
- it provides the subdivision symbols of minimal support that fulfill the conditions ensuring reproduction of the desired space of exponential polynomials;
- it preserves the symmetry properties of the given exponential B-spline symbols;
- it contains the stationary case of polynomial pseudo-spline symbols as a special subcase.

Moreover, we have proved convergence and regularity of the nonstationary subdivision schemes obtained from the repeated application of exponential pseudo-spline symbols, exploiting the property of asymptotical similarity to the stationary symbols of the well-known polynomial pseudo-spline schemes.

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References

[1] Barendrecht, P.J. (2013) Isogeometric Analysis for subdivision surfaces. MSc Thesis, Eindhoven University of Technology.

[2] Beccari, C., Casciola, G. & Romani, L. (2007) A non-stationary uniform tension controlled interpolating 4-point scheme reproducing conics. *Comput. Aided Geom. Design*, **24**, 1–9.

[3] Beccari, C., Casciola, G. & Romani, L. (2010) A unified framework for interpolating and approximating univariate subdivision. *J. Comput. Appl. Math.*, **216**(4), 1169–1180.

[4] Burkhart, D., Hamann, B. & Umlauf, G. (2010) Iso-geometric finite element analysis based on Catmull-Clark subdivision solids. *Computer Graphics Forum*, **29**(5), 1575–1584.

[5] Charina, M. & Conti, C. (2013) Polynomial reproduction of multivariate scalar subdivision schemes. *J. Comput. Appl. Math.*, **240**, 51–61.

[6] Charina, M., Conti, C., Guglielmi, N. & Protasov V. (2014) Regularity of non-stationary multivariate subdivision. *Submitted*, http://arxiv.org/abs/1406.7131

[7] Charina, M., Conti, C. & Romani, L. (2014) Reproduction of exponential polynomials by multivariate non-stationary subdivision schemes with a general dilation matrix. *Numer. Math.*, **127**(2), 223–254.

[8] Christensen, O. & Massopust, P.(2012) Exponential B-splines and the partition of unity property. *Adv. Comput. Math.*, **37**(3), 301–318.

[9] Cirak, F., Ortiz, M. & Schröder, P. (2000) Subdivision surfaces: A new paradigm for thin-shell finite-element analysis. *Int. J. Num. Meth. Eng.*, **47**, 2039-2072.

[10] Cirak, F., Scott, M.J., Antonsson, E.K., Ortiz, M., Schröder, P. (2002) Integrated modeling, finite-element analysis, and engineering design for thin-shell structures using subdivision. *Computer-Aided Design*, **34**, 137-148.

[11] Comtet, L.(1974) *Advanced combinatorics: The art of finite and infinite expansions.* Dordrecht: D. Reidel Publishing Company.

[12] Conti, C., Dyn, N., Manni, C. & Mazure, M.-L. (2015) Convergence of univariate non-stationary subdivision schemes via asymptotical similarity. *Comput. Aided Geom. Design*, **37**, 1–8.

[13] Conti, C., Gemignani, L. & Romani, L. (2009) From symmetric subdivision masks of Hurwitz type to interpolatory subdivision masks. *Linear Algebra Appl.*, **431**, 1971–1987.

[14] Conti, C., Gemignani, L. & Romani, L. (2011) From approximating to interpolatory non-stationary subdivision schemes with the same generation properties. *Adv. Comput. Math.*, **35**(2-4), 217–241.

[15] Conti, C. & Hormann, K. (2011) Polynomial reproduction for univariate subdivision schemes of any arity. *J. Approx. Theory*, **163**, 413–437.

[16] Conti, C. & Romani, L. (2010) Affine combination of B-spline subdivision masks and its non-stationary counterparts. *BIT*, **50**(2), 269–299.

[17] Conti, C. & Romani, L. (2011) Algebraic conditions on non-stationary subdivision symbols for exponential polynomial reproduction. *J. Comput. Applied Math.*, **236**, 543–556.

[18] Conti, C., Romani, L. & Yoon, J. (2015) Approximation order and approximate sum rules in subdivision. *Submitted*, http://arxiv.org/abs/1411.2114v2

[19] Cottrell, J.A., Hughes, Th.J.R., & Bazilevs, Y.(2009) *Isogeometric Analysis: Toward Integration of CAD and FEA*. Chichester, U.K.: John Wiley & Sons.

[20] Delgado-Gonzalo, R., Thevenaz, P., Seelamantula, C.S. & Unser, M. (2012) Snakes with an ellipse reproducing property. *IEEE Transactions on Image Processing*, **21**, 1258–1271.

[21] Delgado-Gonzalo, R. & Unser, M. (2013) Spline-based framework for interactive segmentation in biomedical imaging. *IRBM Ingenierie et Recherche Biomedicale / BioMedical Engineering and Research*, **34**(3), 235–243.

[22] Deslauriers, G., Dubuc, S.(1989) Symmetric iterative interpolation processes. *Constr. Approx.*, **5**, 49–68.

[23] Dong, B., Dyn, N., Hormann, K.(2010) Properties of dual pseudo-splines. *Appl. Comput. Harmon. Anal.*, **29**, 104-110.

[24] Dong, B., Shen, Z.(2006) Linear independence of pseudo-splines. *Proc. Amer. Math. Soc.*, **134**(9), 2685–2694.

[25] Dong, B., Shen, Z.(2007) Pseudo-splines, wavelets and framelets. *Appl. Comput. Harmon. Anal.*, **22**(1), 78-104.

[26] Dyn, N., Hormann, K., Sabin, M.A. & Shen, Z. (2008) Polynomial reproduction by symmetric subdivision schemes. *J. Approx. Theory*, **155**, 28–42.

[27] Dyn, N. & Levin, D. (2002) Subdivision schemes in geometric modelling. *Acta Numer.*, **11**, 73–144.

[28] Dyn, N., Levin, D. & Luzzatto, A. (2003) Exponential reproducing subdivision schemes. *Found. Comput. Math.*, **3**, 187-206.

[29] Dyn, N., Levin, D. & Yoon, J. (2007) Analysis of univariate nonstationary subdivision schemes with application to gaussian-based interpolatory schemes. *SIAM J. Math. Anal.*, **39**(2), 470–488.

[30] Fang, M., Ma, W. & Wang, G. (2010) A generalized curve subdivision scheme of arbitrary order with a tension parameter. *Comput. Aided Geom. Design*, **27**, 720-733.

[31] Hughes, T.J.R., Cottrell, J.A. & Bazilevs, Y. (2005) Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.*, **194**, 4135-4195.

[32] Jeong, B., Lee, Y.J. & Yoon, J. (2013a) Exponential polynomial reproducing property of non-stationary symmetric subdivision schemes and normalized exponential B-splines. *Adv. Comput. Math.*, **38**(3), 647–666.

[33] Jeong, B., Lee, Y.J. & Yoon, J. (2013b) A family of non-stationary subdivision schemes reproducing exponential polynomials. *J. Math. Anal. Appl.*, **402**(1), 207–219.

[34] Johnson, W.P.(2002) The curious history of Faà di Bruno's formula. *Amer. Math. Monthly*, **109**(3).

[35] Li, B.-J., Yu, Z.-L., Yu, B.-W., Su, Z.-X. & Liu, X.-P. (2013) Non-stationary subdivision for exponential polynomials reproduction. *Acta Mathematicae Applicatae Sinica, English Series*, **29**(3), 567–578.

[36] Mortini, R. (2013) The Faà di Bruno's formula revisited. *Elem. Math.*, **68**(1), 33-38.

[37] Novara, P. & Romani, L. (2015) Building blocks for designing arbitrarily smooth subdivision schemes with conic precision. *J. Comput. Appl. Math.*, **279**, 67–79.

[38] Novara, P., Romani, L. & Yoon, J. (2016) Improving smoothness and accuracy of Modified Butterfly subdivision scheme. *Appl. Math. Comput.*, **272**, 64–79.

[39] Romani, L. (2009) From approximating subdivision schemes for exponential splines to high-performance interpolating algorithms. *J. Comput. Applied Math.*, **224**, 383–396.

[40] Romani, L., Hernández Mederos, V. & Estrada Sarlabous, J. (2015) Exact evaluation of a class of nonstationary approximating subdivision algorithms and related applications. *IMA J. Numer. Anal.*, doi:10.1093/imanum/drv008, in press.

[41] Unser, M. & Blu, T. (2005) Cardinal exponential B-splines: part I - theory and filtering algorithms. *IEEE Trans. Signal Process.*, **53**(4), 1425-1438.

[42] Wang, G. & Fang, M. (2008) Unified and extended form of three types of splines. *J. Comput. Applied Math.*, **216**, 498–508.

[43] Warren, J. & Weimer, H. (2002) *Subdivision methods for geometric design - A constructive approach.* San Francisco: Morgan-Kaufmann Publishers.