

# ITERATED HYPER-EXTENSIONS AND AN IDEMPOTENT ULTRAFILTER PROOF OF RADO'S THEOREM

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ABSTRACT. By using nonstandard analysis, and in particular iterated hyper-extensions, we give foundations to a peculiar way of manipulating ultrafilters on the natural numbers and their pseudo-sums. The resulting formalism is suitable for applications in Ramsey theory of numbers. To illustrate the use of our technique, we give a (rather) short proof of Milliken-Taylor's Theorem, and a ultrafilter version of Rado's theorem about partition regularity of diophantine equations.

## INTRODUCTION

The algebraic structure on the space of ultrafilters  $\beta\mathbb{N}$  as given by the pseudo-sum operation  $\mathcal{U} \oplus \mathcal{V}$ , and the related generalizations, have been deeply investigated during the last thirty years, revealing a powerful tool for applications in Ramsey theory and combinatorial number theory (see the monograph [15]). The aim of this paper is to introduce a peculiar formalism grounded on the use of the hyper-natural numbers of nonstandard analysis, that allows to manipulate ultrafilters on  $\mathbb{N}$  and their pseudo-sums in a simplified manner. Especially, we shall be interested in linear combinations  $a_0\mathcal{U} \oplus \dots \oplus a_k\mathcal{U}$  of a given idempotent ultrafilter  $\mathcal{U}$ . To illustrate the use of our technique, we shall give a nonstandard proof of Ramsey Theorem, and a (rather) short proof of Milliken-Taylor's Theorem, a strengthening of the celebrated Hindman's Theorem. Moreover, we shall also prove the following ultrafilter version of Rado's Theorem, that seems to be new.

**Theorem.** *Let  $c_1X_1 + \dots + c_kX_k = 0$  be a diophantine equation with  $c_1 + \dots + c_k = 0$  and  $k > 2$ . Then there exists  $a_0, \dots, a_{k-2} \in \mathbb{N}$  such that for every idempotent ultrafilter  $\mathcal{U}$ , the corresponding linear combination*

$$\mathcal{V} = a_0\mathcal{U} \oplus \dots \oplus a_{k-2}\mathcal{U}$$

*witnesses the injective partition regularity of the given equation, i.e. for every  $A \in \mathcal{V}$  there exist distinct elements  $x_1, \dots, x_k \in A$  with  $c_1x_1 + \dots + c_kx_k = 0$ .*

At the end of the paper, some hints are given for further possible applications and developments of the introduced nonstandard technique.

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We assume the reader to be familiar with the notion of *ultrafilter*, and with the basics of *nonstandard analysis*. In particular, we shall call *star map* or *nonstandard embedding* a function  $A \mapsto *A$  that associates to each mathematical object  $A$  under consideration its *hyper-extension*  $*A$ , and that satisfies the *transfer principle*. Excellent references for the foundations of nonstandard analysis are [6] §4.4, where the classical superstructure approach is presented, and the textbook [12], grounded on the ultrapower construction. The peculiarity of our nonstandard approach is that we shall use iterated hyper-extensions (see the discussion in Section 2).

### 1. $u$ -EQUIVALENT POLYNOMIALS

Before starting to work in a nonstandard setting, in this section we present a general result about linear combinations of a given idempotent ultrafilter (see below for the definition), whose proof will be given in Section 4. As a consequence of its, we prove an ultrafilter version of Rado's theorem.

Throughout the paper,  $\mathbb{N}$  will denote the set of *positive* integers, and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  the set of *non-negative* integers.

Recall the *pseudo-sum* operation between ultrafilters on  $\mathbb{N}_0$ :

$$A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \in \mathbb{N}_0 \mid A - n \in \mathcal{V}\} \in \mathcal{U},$$

where  $A - n = \{m \in \mathbb{N}_0 \mid m + n \in A\}$  is the left-ward *shift* of  $A$  by  $n$ . It can be readily verified that  $\mathcal{U} \oplus \mathcal{V}$  is actually an ultrafilter, and that the pseudo-sum operation is *associative*. If one identifies every *principal ultrafilter*  $\mathfrak{U}_n = \{A \subseteq \mathbb{N}_0 \mid n \in A\}$  with its generator  $n \in \mathbb{N}_0$ , then it is readily seen that the pseudo-sum extends the usual addition, *i.e.*  $\mathfrak{U}_n \oplus \mathfrak{U}_m = \mathfrak{U}_{n+m}$ ; moreover,  $\mathcal{U} \oplus \mathfrak{U}_0 = \mathfrak{U}_0 \oplus \mathcal{U} = \mathcal{U}$  for all  $\mathcal{U}$ . (In fact, it can be proved that the center of  $(\beta\mathbb{N}_0, \oplus)$  is the family  $\{\mathfrak{U}_n \mid n \in \mathbb{N}_0\}$  of the principal ultrafilters. A nonstandard proof of this fact can be found in [11].)

Given an ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$  and a natural number  $h$ , the product  $h\mathcal{U}$  is the ultrafilter defined by putting

$$A \in h\mathcal{U} \iff A/h = \{n \mid nh \in A\} \in \mathcal{U}.$$

Notice that  $0\mathcal{U} = \mathfrak{U}_0$  and  $1\mathcal{U} = \mathcal{U}$  for every  $\mathcal{U}$ .

Particularly relevant for applications are the *idempotent ultrafilters*, namely the *non-principal* ultrafilters  $\mathcal{U}$  such that  $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$ . We remark that their existence is a non-trivial result whose proof requires repeated applications of Zorn's lemma. (To be precise, also the principal ultrafilter  $\mathfrak{U}_0$  has the property  $\mathfrak{U}_0 = \mathfrak{U}_0 \oplus \mathfrak{U}_0$ , but it is not usually considered as an "idempotent" in the literature.)

Let us now introduce an equivalence relation on the *strings* (*i.e.*, finite sequences) of integers.

**Definition 1.1.** The  $u$ -equivalence  $\approx_u$  between strings of integers is the smallest equivalence relation such that:

- The empty string  $\varepsilon \approx_u \langle 0 \rangle$ .
- $\langle a \rangle \approx_u \langle a, a \rangle$  for all  $a \in \mathbb{Z}$ .
- $\approx_u$  is coherent with concatenations<sup>1</sup>, i.e.

$$\sigma \approx_u \sigma' \text{ and } \tau \approx_u \tau' \implies \sigma \frown \tau \approx_u \sigma' \frown \tau'.$$

Two polynomials  $P(X) = \sum_{i=0}^n a_i X^i$  and  $Q(X) = \sum_{j=0}^m b_j X^j$  in  $\mathbb{Z}[X]$  are  $u$ -equivalent when the corresponding strings of coefficients are  $u$ -equivalent:

$$\langle a_0, \dots, a_n \rangle \approx_u \langle b_0, \dots, b_m \rangle$$

So,  $\approx_u$ -equivalence between strings is preserved by inserting or removing zeros, by repeating finitely many times a term or, conversely, by shortening a block of consecutive equal terms. *E.g.*  $\langle 3, 0, 0, -4, 1, 1 \rangle \approx_u \langle 0, 3, -4, -4, 1 \rangle$  and  $\langle 2, 2, 0, 0, 7, 7, 3 \rangle \approx_u \langle 2, 7, 3 \rangle$ , and hence:

- $X^5 + X^4 - 4X^3 + 3 \approx_u X^4 - 4X^3 - 4X^2 + 3X$
- $3X^6 + 7X^5 + 7X^4 + 2X + 2 \approx_u 3X^2 + 7X + 2$ , etc.

As an application of nonstandard analysis, in Section 4 the following will be proved:

**Theorem 4.4** *Let  $a_0, \dots, a_n \in \mathbb{N}_0$ , and assume that there exist [distinct] polynomials  $P_i(X)$  such that*

$$P_1(X) \approx_u \dots \approx_u P_k(X) \approx_u \sum_{i=0}^n a_i X^i \text{ and } c_1 P_1(X) + \dots + c_k P_k(X) = 0.$$

*Then for every idempotent ultrafilter  $\mathcal{U}$  and for every  $A \in a_0\mathcal{U} \oplus \dots \oplus a_n\mathcal{U}$ , there exist [distinct]  $x_i \in A$  such that  $c_1 x_1 + \dots + c_k x_k = 0$ .*

We derive here a straight consequence of the above theorem which is a ultrafilter version of Rado's theorem.

**Theorem 1.2.** *Let  $c_1 X_1 + \dots + c_k X_k = 0$  be a diophantine equation with  $c_1 + \dots + c_k = 0$  and  $k > 2$ . Then there exists  $a_0, \dots, a_{k-2} \in \mathbb{N}$  such that for every idempotent ultrafilter  $\mathcal{U}$ , the corresponding linear combination*

$$\mathcal{V} = a_0\mathcal{U} \oplus \dots \oplus a_{k-2}\mathcal{U}$$

*witnesses the injective partition regularity of the given equation, i.e., for every  $A \in \mathcal{V}$  there exist distinct  $x_i \in A$  such that  $c_1 x_1 + \dots + c_k x_k = 0$ .*

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<sup>1</sup> Recall that if  $\sigma = \langle a_0, \dots, a_k \rangle$  and  $\tau = \langle b_0, \dots, b_h \rangle$ , then their concatenation is defined as  $\sigma \frown \tau = \langle a_0, \dots, a_k, b_0, \dots, b_h \rangle$ .

*Proof.* For arbitrary  $a_0, \dots, a_{k-2}$ , consider the following polynomials:

$$\begin{array}{rcll}
P_1(X) & = & a_0 + a_1X + a_2X^2 + \dots + a_{k-3}X^{k-3} + a_{k-2}X^{k-2} + a_{k-2}X^{k-1} \\
P_2(X) & = & a_0 + a_1X + a_2X^2 + \dots + a_{k-3}X^{k-3} + 0 + a_{k-2}X^{k-1} \\
P_3(X) & = & a_0 + a_1X + a_2X^2 + \dots + 0 + a_{k-3}X^{k-2} + a_{k-2}X^{k-1} \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
P_{k-2}(X) & = & a_0 + a_1X + 0 + a_2X^3 + \dots + a_{k-3}X^{k-2} + a_{k-2}X^{k-1} \\
P_{k-1}(X) & = & a_0 + 0 + a_1X^2 + a_2X^3 + \dots + a_{k-3}X^{k-2} + a_{k-2}X^{k-1} \\
P_k(X) & = & a_0 + a_0X + a_1X^2 + a_2X^3 + \dots + a_{k-3}X^{k-2} + a_{k-2}X^{k-1}
\end{array}$$

Notice that  $P_1(X) \underset{\mathcal{U}}{\approx} \dots \underset{\mathcal{U}}{\approx} P_k(X) \underset{\mathcal{U}}{\approx} \sum_{i=0}^{k-2} a_i X^i$ . In order to apply Theorem 4.4, we need to find suitable coefficients  $a_0, \dots, a_{k-2}$  in such a way that the linear combination  $c_1 P_1(X) + \dots + c_k P_k(X) = 0$ . It is readily verified that this happens if and only if the following conditions are fulfilled:

$$\begin{cases}
(c_1 + \dots + c_k) \cdot a_0 = 0 \\
(c_1 + \dots + c_{k-2}) \cdot a_1 + c_k \cdot a_0 = 0 \\
(c_1 + \dots + c_{k-3}) \cdot a_2 + (c_{k-1} + c_k) \cdot a_1 = 0 \\
\vdots \\
c_1 \cdot a_{k-2} + (c_3 + \dots + c_k) \cdot a_{k-3} = 0 \\
(c_1 + \dots + c_k) \cdot a_{k-2} = 0
\end{cases}$$

The first and the last equations are trivially satisfied because of the hypothesis  $c_1 + \dots + c_k = 0$ . Now assume without loss of generality that the coefficients  $c_1 \geq \dots \geq c_k$  are arranged in non-increasing order. It can be verified in a straightforward manner that the remaining  $k-2$  equations are satisfied by (infinitely many) suitable  $a_0, \dots, a_{k-2} \in \mathbb{N}$ , *e.g.*

$$(*) \quad \begin{cases}
a_0 = c_1 \cdot (c_1 + c_2) \cdot \dots \cdot (c_1 + \dots + c_{k-2}) \\
a_i = b_i \cdot b'_i \text{ for } 0 < i < k-2 \text{ where} \\
\quad b_i = c_1 \cdot (c_1 + c_2) \cdot \dots \cdot (c_1 + \dots + c_{k-2-i}) \text{ and} \\
\quad b'_i = (-1)^i \cdot c_k \cdot (c_k + c_{k-1}) \cdot \dots \cdot (c_k + \dots + c_{k+1-i}) \\
a_{k-2} = (-1)^{k-2} \cdot c_k \cdot (c_k + c_{k-1}) \cdot \dots \cdot (c_k + \dots + c_3)
\end{cases}$$

Remark that by the hypothesis  $c_1 + \dots + c_k = 0$  together with the assumption  $c_1 \geq \dots \geq c_k$ , it follows that all  $a_i > 0$ . Finally, remark also that since all coefficients  $a_0, \dots, a_{k-2} \neq 0$ , the polynomials  $P_i(X)$  are mutually distinct.  $\square$

For instance, consider the diophantine equation

$$3X_1 + X_2 + X_3 - X_4 - 4X_5 = 0$$

where coefficients are arranged in non-increasing order and their sum equals zero. By using (\*) in the above proof, we obtain that  $a_0 = 60$ ,  $a_1 = 48$ ,  $a_2 = 60$ ,  $a_3 = 80$ . So, for every idempotent ultrafilter  $\mathcal{U}$  and for every set

$$A \in 60\mathcal{U} \oplus 48\mathcal{U} \oplus 60\mathcal{U} \oplus 80\mathcal{U}$$

we know that there exist distinct  $x_i \in A$  such that  $3x_1 + x_2 + x_3 - x_4 - 4x_5 = 0$ .

## 2. ITERATED HYPER-EXTENSIONS

For our purposes, we shall need to work in models of nonstandard analysis where hyper-extensions can be iterated, so that one can consider, *e.g.*, the set of hyper-hyper-natural numbers  $^{**}\mathbb{N}$ , the hyper-hyper-hyper-natural numbers  $^{***}\mathbb{N}$ , the hyper-extension  $^*\nu$  of an hyper-natural number  $\nu$ , and so forth. Moreover, we shall use the  $\mathfrak{c}^+$ -*enlargement property*, namely the property that intersections  $\bigcap_{F \in \mathcal{F}} {}^*F$  are nonempty for all families  $|\mathcal{F}| \leq \mathfrak{c}$  of cardinality at most the continuum, and which satisfy the *finite intersection property* (*i.e.*  $A_1 \cap \dots \cap A_n \neq \emptyset$  for every choice of finitely many  $A_i \in \mathcal{F}$ ).

In full generality, all the above requirements are fulfilled by taking a  $\mathfrak{c}^+$ -enlarging nonstandard embedding

$$* : \mathbf{V} \longrightarrow \mathbf{V}$$

which is defined on the *universal class*  $\mathbf{V}$  of all sets. It is now a well-known fact in nonstandard set theory that such “universal” nonstandard embeddings can be constructed within conservative extensions of ZFC where the regularity axiom is replaced by a suitable *anti-foundation axiom*; see *e.g.* [1].<sup>2</sup> (For a comprehensive treatment of nonstandard set theories, we refer the interested reader to the monography [16].)

A suitable axiomatic framework is the nonstandard set theory  $\text{ZFC}[\Omega]$  of [10], which includes all axioms of Zermelo-Fraenkel theory ZFC with choice with the only exception of the regularity axiom, and where for every “ $\in$ -definable” cardinal  $\kappa$ , one has a nonstandard embedding  $J_\kappa : \mathbf{V} \rightarrow \mathbf{V}$  of the universe into itself that satisfies the  $\kappa$ -enlargement property.<sup>3</sup> The resulting theory is conservative over ZFC. (See also the related axiomatics  ${}^*\text{ZFC}$  [7, 8] and Alpha-Theory [9]).

Another suitable setting where iterated hyper-extensions can be considered was introduced by V. Benci in [2]: it consists in a special version of the *superstructure approach*  $* : V(X) \rightarrow V(X)$  where the standard universe and the nonstandard universe coincide. The limitation here is that superstructures  $V(X)$  only satisfies a fragment of ZFC (*e.g.*, replacement fails and there are no infinite ordinals in  $V(X)$ ).

We stress that working with iterated hyper-extensions requires caution. To begin with, recall that in nonstandard analysis one has that  ${}^*n = n$  for all natural numbers  $n \in \mathbb{N}$ ; however, the same property cannot be extended to the hyper-natural numbers  $\xi \in {}^*\mathbb{N}$ . Indeed, by *transfer* one can easily show that  ${}^*\xi > \xi$  for all infinite  $\xi \in {}^*\mathbb{N}$ ; more generally, the following facts hold:

<sup>2</sup> In summary, one takes *transitive Mostowski collapses* of ultrapowers  $\mathbf{V}^I/\mathcal{U}$  of the universe. For any given cardinal  $\kappa$ , the  $\kappa$ -enlargement property is obtained by picking a  $\kappa$ -regular ultrafilter  $\mathcal{U}$ .

<sup>3</sup> Indeed, the separation and replacement schemas hold for all  $\in$ -\*formulas, and  $J_\kappa$  is postulated to satisfy  $\kappa$ -saturation, a stronger property than  $\kappa$ -enlargement.

- ${}^*\mathbb{N} \subsetneq {}^{**}\mathbb{N}$ .
- If  $\xi \in {}^*\mathbb{N} \setminus \mathbb{N}$  then  ${}^*\xi \in {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}$ .
- ${}^*\mathbb{N}$  is an initial segment of  ${}^{**}\mathbb{N}$ , *i.e.*  $\xi < \nu$  for every  $\xi \in {}^*\mathbb{N}$  and for every  $\nu \in {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}$ .

Let  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ . By transferring the fact that  ${}^*f$  and  $f$  agree on  $\mathbb{N}_0$ , one gets that

- $({}^{**}f)(\xi) = {}^*f(\xi)$  for all  $\xi \in {}^*\mathbb{N}_0$ .

Remark that  ${}^*[*f(\xi)] = ({}^{**}f)({}^*\xi)$ , but in general  $({}^{**}f)({}^*\xi) \neq ({}^{**}f)(\xi)$ .

Now denote by “ $k*$ ” the  $k$ -times iterated star map, *i.e.*

$$\begin{cases} {}^{0*}A = A \\ {}^{(k+1)*}A = {}^*(k^*A). \end{cases}$$

As a first application of iterated hyper-extensions, we now present a non-standard proof of Ramsey theorem.

**Lemma 2.1.** *Let  $A \subseteq \mathbb{N}^k$ . If there exists an infinite  $\xi \in {}^*\mathbb{N}$  such that  $(\xi, {}^*\xi, \dots, {}^{(k-1)*}\xi) \in {}^{k*}A$  then there exists an infinite set of natural numbers*

$$H = \{h_1 < h_2 < \dots < h_n < h_{n+1} < \dots\}$$

*such that  $(h_{n_1}, h_{n_2}, \dots, h_{n_k}) \in A$  for all  $n_1 < n_2 < \dots < n_k$ .*

*Proof.* To simplify notation, let us only consider here the particular case  $k = 3$ . The general result is proved exactly in the same fashion, only by using a heavier notation. So, let us assume that  $(\xi, {}^*\xi, {}^{**}\xi) \in {}^{***}A$ . For  $n, n' \in \mathbb{N}$ , let:

- $X = \{n \in \mathbb{N} \mid (n, \xi, {}^*\xi) \in {}^{**}A\}$ ;
- $X_n = \{n' \in \mathbb{N} \mid (n, n', \xi) \in {}^*A\}$ ;
- $X_{nn'} = \{n'' \in \mathbb{N} \mid (n, n', n'') \in A\}$ .

The corresponding hyper-extensions are described as follows:

- ${}^*X = \{\eta \in {}^*\mathbb{N} \mid (\eta, {}^*\xi, {}^{**}\xi) \in {}^{***}A\}$ ;
- ${}^*X_n = \{\eta \in {}^*\mathbb{N} \mid (n, \eta, {}^*\xi) \in {}^{**}A\}$ ;
- ${}^*X_{n,n'} = \{\eta \in {}^*\mathbb{N} \mid (n, n', \eta) \in {}^*A\}$ .

Notice that  $n \in X \Leftrightarrow \xi \in {}^*X_n$ , and  $n' \in X_n \Leftrightarrow \xi \in {}^*X_{nn'}$ . By the hypothesis, we have that  $\xi \in {}^*X$ , so  $X$  is an infinite set and we can pick an element  $h_1 \in X$ . Now,  $\xi \in {}^*X \cap {}^*X_{h_1}$  implies that  $X \cap X_{h_1}$  is infinite, and so we can pick an element  $h_2 > h_1$  in that intersection. But then  $\xi \in {}^*X \cap {}^*X_{h_1} \cap {}^*X_{h_2} \cap {}^*X_{h_1 h_2}$ , and so we can pick an element  $h_3 > h_2$  in the intersection  $X \cap X_{h_1} \cap X_{h_2} \cap X_{h_1 h_2}$ . In particular,  $(h_1, h_2, h_3) \in A$ .

An increasing sequence  $\langle h_n \mid n \in \mathbb{N} \rangle$  that satisfies the desired property is obtained by iterating this procedure, where at each step  $n$  one has

$$\xi \in {}^*X \cap \bigcap_{1 \leq i \leq n} {}^*X_{h_i} \cap \bigcap_{1 \leq i < j \leq n} {}^*X_{h_i h_j},$$

and  $h_{n+1} > h_n$  is picked in the infinite intersection

$$h_{n+1} \in X \cap \bigcap_{1 \leq i \leq n} X_{h_i} \cap \bigcap_{1 \leq i < j \leq n} X_{h_i h_j}.$$

□

As a straight corollary, one obtains:

**Theorem 2.2** (Ramsey). *Let  $[\mathbb{N}]^k = C_1 \sqcup \dots \sqcup C_r$  be a finite partition of the  $k$ -sets of natural numbers.<sup>4</sup> Then there exists an infinite  $H \subseteq \mathbb{N}$  such that all its  $k$ -sets are monochromatic, i.e.  $[H]^k \subseteq C_i$  for some  $i$ .*

*Proof.* Identify the family of  $k$ -sets  $[\mathbb{N}]^k$  with the upper-diagonal in the Cartesian product  $\mathbb{N}^k$ :

$$\left\{ (n_1, n_2, \dots, n_k) \in \mathbb{N}^k \mid n_1 < n_2 < \dots < n_k \right\}.$$

By applying *transfer* to the  $k$ -iterated star map, one gets that  ${}^{k*}([\mathbb{N}]^k) = [{}^{k*}\mathbb{N}]^k$ , and one has the following finite coloring:

$$[{}^{k*}\mathbb{N}]^k = {}^{k*}C_1 \sqcup \dots \sqcup {}^{k*}C_r.$$

Now fix any infinite  $\xi \in {}^*\mathbb{N}$ , and let  $i$  be such that the ordered  $k$ -tuple  $(\xi, {}^*\xi, \dots, ({}^{k-1})*\xi) \in {}^{k*}C_i$  (notice that  $\xi < {}^*\xi < \dots < ({}^{k-1})*\xi$ ). By the Lemma 2.1, we get the existence of an infinite  $H \subseteq \mathbb{N}$  such that  $[H]^k \subseteq [C_i]^k$ . □

### 3. HYPER-NATURAL NUMBERS AS REPRESENTATIVES OF ULTRAFILTERS

Recall that there is a canonical way of associating an ultrafilter to every element  $\alpha \in {}^*\mathbb{N}_0$  (see *e.g.* [5, 19, 11]). Namely, one takes the family of those sets of natural numbers whose hyper-extensions contain  $\alpha$ :

$$\mathfrak{U}_\alpha = \{A \subseteq \mathbb{N}_0 \mid \alpha \in {}^*A\}.$$

It is readily verified from the properties of hyper-extensions that  $\mathfrak{U}_\alpha$  is indeed an ultrafilter, called the *ultrafilter generated* by  $\alpha$ . Notice that  $\mathfrak{U}_\alpha$  is principal if and only if  $\alpha \in \mathbb{N}_0$  is finite. Notice also that if  $\mathcal{U} = \mathfrak{U}_\alpha$  is generated by  $\alpha$ , then  $h\mathcal{U}$  is generated by  $h\alpha$ .

**Definition 3.1.** We say that two elements  $\alpha, \beta \in {}^*\mathbb{N}_0$  are *u-equivalent*, and write  $\alpha \sim_u \beta$ , if they generate the same ultrafilter:  $\mathfrak{U}_\alpha = \mathfrak{U}_\beta$ .

<sup>4</sup> A  $k$ -set is a set with exactly  $k$ -many elements.

So,  $\alpha \underset{u}{\sim} \beta$  when  $\alpha \in {}^*A \Leftrightarrow \beta \in {}^*A$  for all  $A \subseteq \mathbb{N}_0$ . Since every ultrafilter  $\mathcal{U}$  on  $\mathbb{N}_0$  is a family of  $\mathfrak{c}$ -many sets with the finite intersection property, by the hypothesis of  $\mathfrak{c}^+$ -enlargement there exists an element  $\alpha \in \bigcap_{A \in \mathcal{U}} {}^*A$ . This means that every ultrafilter  $\mathcal{U} = \mathfrak{U}_\alpha$  is generated by some number  $\alpha \in {}^*\mathbb{N}_0$ .

The following properties are readily verified:

**Proposition 3.2.** *Let  $\alpha \underset{u}{\sim} \alpha'$  be two  $u$ -equivalent hyper-natural numbers, and let  $n \in \mathbb{N}$  be finite. Then:*

- (1)  $\alpha \pm n \underset{u}{\sim} \alpha' \pm n$ ,
- (2)  $n \cdot \alpha \underset{u}{\sim} n \cdot \alpha'$ ,
- (3)  $\alpha/n \underset{u}{\sim} \alpha'/n$ , provided  $\alpha$  is divisible by  $n$ .

Remark that in general sums in  ${}^*\mathbb{N}_0$  are not coherent with  $u$ -equivalence, *i.e.* it can well be the case that  $\alpha \underset{u}{\sim} \alpha'$  and  $\beta \underset{u}{\sim} \beta'$ , but  $\alpha + \beta \not\underset{u}{\sim} \alpha' + \beta'$ .

We now extend the notion of generated ultrafilter and also consider elements  $\nu \in {}^{k*}\mathbb{N}$  in iterated hyper-extensions of  $\mathbb{N}$ , by putting

$$\mathfrak{U}_\nu = \{A \subseteq \mathbb{N}_0 \mid \nu \in {}^{k*}A\}.$$

The  $u$ -equivalence relation is extended accordingly to all pairs of numbers in the following union

$${}^*\mathbb{N}_0 = \bigcup_{k \in \mathbb{N}} {}^{k*}\mathbb{N}_0.$$

In general, for every  $A \subseteq \mathbb{N}$ , the set  ${}^*A = \bigcup_{k \in \mathbb{N}} {}^{k*}A$  can be seen as the direct limit of the finitely iterated hyper-extensions of  $A$ ; and similarly for functions. In consequence, the map  $\star$  itself is a nonstandard embedding, *i.e.* it satisfies the *transfer principle*. Since  $\star$  is assumed to be  $\mathfrak{c}^+$ -enlarging, it can be easily verified that the same property also holds for  $\star$ .

Remark that the above definitions are coherent. In fact, by starting from the equivalence  $n \in A \Leftrightarrow n \in {}^*A$  which holds for all  $n \in \mathbb{N}_0$ , one can easily show that  $\nu \in {}^{k*}A \Leftrightarrow \nu \in {}^{h*}A$  for all  $\nu \in {}^{k*}\mathbb{N}_0$  and  $h > k$ . In consequence, for all  $\nu \in {}^*\mathbb{N}_0$ , one has  $\nu \underset{u}{\sim} {}^*\nu$ , and hence  ${}^{k*}\nu \underset{u}{\sim} {}^{h*}\nu$  for all  $k, h$ . (A detailed study of  $\underset{u}{\sim}$ -equivalence in  ${}^*\mathbb{N}_0$  can be found in [17].)

We shall use the following characterization of pseudo-sums of ultrafilters.

**Proposition 3.3.** *Let  $\alpha, \beta \in {}^*\mathbb{N}_0$  and  $A \subseteq \mathbb{N}_0$ . Then  $A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta$  if and only if the sum  $\alpha + {}^*\beta \in {}^{**}A$ .*

*Proof.* Consider the set  $\widehat{A} = \{n \in \mathbb{N}_0 \mid A - n \in \mathfrak{U}_\beta\}$ , and notice that its hyper-extension  ${}^*\widehat{A} = \{n \in \mathbb{N}_0 \mid n + \beta \in {}^*A\} = \{\gamma \in {}^*\mathbb{N}_0 \mid \gamma + {}^*\beta \in {}^{**}A\}$ . Then the following equivalences yield the thesis:

$$A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \iff \widehat{A} \in \mathfrak{U}_\alpha \iff \alpha \in {}^*\widehat{A} \iff \alpha + {}^*\beta \in {}^{**}A.$$

□



We already mentioned that when  $\alpha \approx_u \alpha'$  and  $\beta \approx_u \beta'$ , one cannot conclude that  $\alpha + \beta \not\approx_u \alpha' + \beta'$ . However, under the same assumptions, one has that  $\alpha + {}^*\beta \approx_u \alpha + {}^*\beta'$  in  ${}^{**}\mathbb{N}$ , as they generate the same ultrafilter  $\mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta = \mathfrak{U}_{\alpha'} \oplus \mathfrak{U}_{\beta'}$ .

The characterization of pseudo-sums as given above can be extended to linear combinations of ultrafilters in a straightforward manner.

**Corollary 3.4.** *For every  $\xi_0, \dots, \xi_k \in {}^*\mathbb{N}_0$ , and for every  $a_0, \dots, a_k \in \mathbb{N}_0$ , the linear combination  $a_0\mathcal{U} \oplus \dots \oplus a_k\mathcal{U}$  is the ultrafilter generated by the element  $a_0\xi + \dots + a_k{}^{k*}\xi \in (k+1){}^*\mathbb{N}_0$ .*

The class of idempotent ultrafilters was first isolated to provide a simplified proof of Hindman's Theorem, a cornerstone of combinatorics of numbers.

**Theorem (Hindman)** *For every finite coloring  $\mathbb{N} = C_1 \sqcup \dots \sqcup C_r$  there exists an infinite set  $X = \{x_1 < x_2 < \dots < x_n < \dots\}$  such that all its finite sums are monochromatic, i.e. there exists  $i$  such that:*

$$\text{FS}(X) = \left\{ \sum_{i \in F} x_i \mid F \subset \mathbb{N} \text{ nonempty finite} \right\} \subseteq C_i.$$

Starting from the ultrafilter proof of the above theorem, a whole body of new combinatorial results have been then obtained by exploiting the algebraic properties of the space  $(\beta\mathbb{N}_0, \oplus)$  and of its generalizations (see the monograph [15]).

By Proposition 3.3, it directly follows that

**Proposition 3.5.** *Let  $\xi \in {}^*\mathbb{N}$ . The ultrafilter  $\mathfrak{U}_\xi$  is idempotent if and only if  $\xi \approx_u \xi + {}^*\xi$ .*

Next, we show a general result connecting linear combinations of a given idempotent ultrafilter, and  $u$ -equivalence of the corresponding strings of coefficients.

**Theorem 3.6.** *Let  $a_0, a_1, \dots, a_k, b_0, b_1, \dots, b_h \in \mathbb{N}_0$ . Then the following are equivalent:*

(1)  $\langle a_0, a_1, \dots, a_k \rangle \approx_u \langle b_0, b_1, \dots, b_h \rangle$ .

(2) For every idempotent ultrafilter  $\mathcal{U}$ :

$$a_0\mathcal{U} \oplus a_1\mathcal{U} \oplus \dots \oplus a_k\mathcal{U} = b_0\mathcal{U} \oplus b_1\mathcal{U} \oplus \dots \oplus b_h\mathcal{U}.$$

(3) For every  $\xi \in {}^*\mathbb{N}_0$  such that the generated ultrafilter  $\mathfrak{U}_\xi$  is idempotent:

$$a_0\xi + a_1{}^*\xi + \dots + a_k{}^{k*}\xi \approx_u b_0\xi + b_1{}^*\xi + \dots + b_h{}^{h*}\xi.$$

*Proof.* (1)  $\Rightarrow$  (2). For every string  $\sigma = \langle d_0, d_1, \dots, d_n \rangle$  of numbers  $d_i \in \mathbb{N}_0$ , and for every idempotent ultrafilter  $\mathcal{U}$ , denote by

$$\oplus_{\mathcal{U}}(\sigma) = d_0\mathcal{U} \oplus d_1\mathcal{U} \oplus \dots \oplus d_n\mathcal{U}.$$

We have to show that  $\sigma \approx_{\mathcal{U}} \tau \Rightarrow \oplus_{\mathcal{U}}(\sigma) = \oplus_{\mathcal{U}}(\tau)$ . By agreeing that  $\oplus_{\mathcal{U}}(\varepsilon) = \mathfrak{U}_0$ , one trivially has  $\oplus_{\mathcal{U}}(\varepsilon) = \oplus_{\mathcal{U}}(\langle 0 \rangle)$ . Moreover,  $\oplus(\langle a \rangle) = \oplus_{\mathcal{U}}(\langle a, a \rangle)$  because  $a\mathcal{U} \oplus a\mathcal{U} = a(\mathcal{U} \oplus \mathcal{U}) = a\mathcal{U}$ . Now let  $\sigma \approx_{\mathcal{U}} \sigma'$  and  $\tau \approx_{\mathcal{U}} \tau'$ , where we assume by inductive hypothesis that  $\oplus_{\mathcal{U}}(\sigma) = \oplus_{\mathcal{U}}(\sigma')$  and  $\oplus_{\mathcal{U}}(\tau) = \oplus_{\mathcal{U}}(\tau')$ . Then, by associativity of the pseudo-sum, it follows that

$$\oplus_{\mathcal{U}}(\sigma \frown \tau) = [\oplus_{\mathcal{U}}(\sigma)] \oplus [\oplus_{\mathcal{U}}(\tau)] = [\oplus_{\mathcal{U}}(\sigma')] \oplus [\oplus_{\mathcal{U}}(\tau')] = \oplus_{\mathcal{U}}(\sigma' \frown \tau').$$

(2)  $\Rightarrow$  (1). Assume that  $\langle a_0, a_1, \dots, a_k \rangle \not\approx_{\mathcal{U}} \langle b_0, b_1, \dots, b_h \rangle$ . By the previous implication, we can assume without loss of generality that  $a_i \neq a_{i+1}$  for  $i < k$ , and that  $b_j \neq b_{j+1}$  for  $j < h$ . Then we apply the following known result:

- ([18] Theorem 2.19.) *Let  $a_0, \dots, a_k, b_0, \dots, b_h \in \mathbb{N}$  so that  $a_i \neq a_{i+1}$  and  $b_j \neq b_{j+1}$  for any  $i < k$  and  $j < h$ . If  $a_0\mathcal{U} \oplus \dots \oplus a_k\mathcal{U} = b_0\mathcal{U} \oplus \dots \oplus b_h\mathcal{U}$  for some idempotent  $\mathcal{U}$  then  $k = h$  and  $a_i = b_i$  for all  $i$ .*

(2)  $\Leftrightarrow$  (3). By the  $\mathfrak{c}^+$ -enlargement property, every ultrafilter  $\mathcal{U}$  is generated by some element  $\xi \in {}^*\mathbb{N}$ , i.e.  $\mathcal{U} = \mathfrak{U}_{\xi}$ . So, the thesis is a particular case of Corollary 3.4.  $\square$

We shall use the above characterization to justify a neat formalism which is suitable to handle idempotent ultrafilters and their linear combinations. As a first relevant example, let us give a nonstandard ultrafilter proof of Milliken-Taylor's Theorem, a strengthening of Hindman's Theorem.

**Lemma 3.7.** *Let  $\mathcal{U}$  be an idempotent ultrafilter, and let  $a_0, \dots, a_k \in \mathbb{N}$ . For every  $A \in a_0\mathcal{U} \oplus \dots \oplus a_k\mathcal{U}$  there exists an infinite set of natural numbers*

$$X = \{x_1 < x_2 < \dots < x_n < \dots\}$$

*with the property that for every increasing sequence  $I_0 < \dots < I_k$  of nonempty finite sets of natural numbers (i.e.  $\max I_i < \min I_{i+1}$ ), the sum*

$$\sum_{i \in I_0} a_0 x_i + \dots + \sum_{i \in I_k} a_k x_i \in A.$$

*Proof.* By the  $\mathfrak{c}^+$ -enlargement property, we can pick  $\xi \in {}^*\mathbb{N}$  with  $\mathcal{U} = \mathfrak{U}_{\xi}$ . By Corollary 3.4,

$$A \in a_0\mathcal{U} \oplus \dots \oplus a_k\mathcal{U} \iff a_0\xi + a_1{}^*\xi + \dots + a_k{}^{k*}\xi \in {}^{(k+1)*}A,$$

and so we have that

$$\begin{aligned} \xi &\in \left\{ \eta \in {}^*\mathbb{N} \mid a_0\eta + a_1{}^*\xi + \dots + a_k{}^{k*}\xi \in {}^{(k+1)*}A \right\} \\ &= {}^*\left\{ x \in \mathbb{N} \mid a_0x + a_1\xi + \dots + a_k{}^{(k-1)*}\xi \in {}^{k*}A \right\}. \end{aligned}$$

Since  $\langle a_0, a_1, \dots, a_k \rangle \approx_{\mathfrak{u}} \langle a_0, a_0, a_1, \dots, a_k \rangle$ , we also have that  $a_0\xi + a_0^*\xi + a_1^{**}\xi + \dots + a_k^{(k+1)*}\xi \in {}^{(k+2)*}A$ , and hence:

$$\begin{aligned} \xi &\in \left\{ \eta \in {}^*\mathbb{N} \mid a_0\eta + a_0^*\xi + a_1^{**}\xi + \dots + a_k^{(k+1)*}\xi \in {}^{(k+2)*}A \right\} \\ &= {}^* \left\{ x \in \mathbb{N} \mid a_0x + a_0\xi + a_1^*\xi + \dots + a_k^{k*}\xi \in {}^{(k+1)*}A \right\}. \end{aligned}$$

By *transfer*, there exists an element  $x_1$  such that

- $a_0x_1 + a_1\xi + \dots + a_k^{(k-1)*}\xi \in {}^{k*}A$ , and
- $a_0x_1 + a_0\xi + a_1^*\xi + \dots + a_k^{k*}\xi \in {}^{(k+1)*}A$ .

We now proceed by induction on  $n$  and define elements  $x_1 < \dots < x_n$  in such a way that for every increasing sequence of nonempty finite sets  $J_0 < \dots < J_h$  where  $h \leq k$  and  $\max J_h \leq n$ , the following properties are fulfilled:

- (1)  $\sum_{s=0}^h \left( \sum_{i \in J_s} a_s x_i \right) + a_h \xi + a_{h+1}^* \xi + \dots + a_k^{(k-h)*} \xi \in {}^{(k-h+1)*}A$ .
- (2)  $\sum_{s=0}^h \left( \sum_{i \in J_s} a_s x_i \right) + a_{h+1} \xi + a_{h+2}^* \xi + \dots + a_k^{(k-h-1)*} \xi \in {}^{(k-h)*}A$ .

Remark that  $x_1$  actually satisfies the inductive basis  $n = 1$ , because in this case one necessarily has  $h = 0$  and  $J_0 = \{1\}$ . As for the inductive step, notice that

- $\langle a_h, a_{h+1}, \dots, a_k \rangle \approx_{\mathfrak{u}} \langle a_h, a_h, a_{h+1}, \dots, a_k \rangle$ , and
- $\langle a_{h+1}, a_{h+2}, \dots, a_k \rangle \approx_{\mathfrak{u}} \langle a_{h+1}, a_{h+1}, a_{h+2}, \dots, a_k \rangle$ .

So, in consequence of the inductive hypotheses (1) and (2) respectively, one has that

- (3)  $\sum_{s=0}^h \left( \sum_{i \in J_s} a_s x_i \right) + a_h \xi + a_h^* \xi + a_{h+1}^{**} \xi + \dots + a_k^{(k-h+1)*} \xi \in {}^{(k-h+2)*}A$ .
- (4)  $\sum_{s=0}^h \left( \sum_{i \in J_s} a_s x_i \right) + a_{h+1} \xi + a_{h+1}^* \xi + a_{h+2}^{**} \xi + \dots + a_k^{(k-h)*} \xi \in {}^{(k-h+1)*}A$ .

Now, properties (1), (2), (3), (4) say that for every increasing sequence of nonempty finite sets  $J_0 < \dots < J_h$  where  $h \leq k$  and  $\max J_h \leq n$ , the

hyperinteger  $\xi \in {}^*\Gamma(J_0 < \dots < J_h)$  where:

$$\begin{aligned} \Gamma(J_0 < \dots < J_h) = & \\ & \left\{ m \in \mathbb{N} \mid \sum_{s=0}^h \left( \sum_{i \in J_s} a_s x_i \right) + a_h m + a_{h+1} \xi + \dots + a_k^{(k-h-1)*} \xi \in {}^{(k-h)*} A \right. \\ & \& \sum_{s=0}^h \left( \sum_{i \in J_s} a_s x_i \right) + a_{h+1} m + a_{h+2} \xi + \dots + a_k^{(k-h-2)*} \xi \in {}^{(k-h-1)*} A \\ & \& \sum_{s=0}^h \left( \sum_{i \in J_s} a_s x_i \right) + a_h m + a_h \xi + a_{h+1} {}^* \xi \dots + a_k^{(k-h)*} \xi \in {}^{(k-h+1)*} A \\ & \left. \& \sum_{s=0}^h \left( \sum_{i \in J_s} a_s x_i \right) + a_{h+1} m + a_{h+1} \xi + a_{h+2} {}^* \xi \dots + a_k^{(k-h-1)*} \xi \in {}^{(k-h)*} A \right\} \end{aligned}$$

Then  $\xi \in {}^*\Gamma$ , where  $\Gamma$  is the following finite intersection:

$$\Gamma = \bigcap_{\substack{J_0 < \dots < J_h \\ h \leq k, \max J_h \leq n}} \Gamma(J_0 < \dots < J_h).$$

The set  $\Gamma$  is infinite because its hyper-extension contains an infinite hyper-natural number, namely  $\xi$ ; in particular, we can pick an element  $x_{n+1} > x_n$  in  $\Gamma$ . It now only takes a straightforward verification to check that  $x_1 < \dots < x_n < x_{n+1}$  satisfy the desired properties, namely (1) and (2) for every sequence of nonempty finite sets  $J_0 < \dots < J_l$  where  $l \leq k$  and  $\max J_l \leq n+1$ .  $\square$

As a straight corollary, we obtain

**Theorem 3.8** (Milliken-Taylor). *Let a finite coloring  $\mathbb{N} = C_1 \sqcup \dots \sqcup C_r$  be given. For every choice of  $a_0, \dots, a_k \in \mathbb{N}$  there exists an infinite set*

$$X = \{x_1 < x_2 < \dots < x_n < x_{n+1} < \dots\}$$

such that the following sums are monochromatic for every increasing sequence  $I_0 < \dots < I_k$  of nonempty finite sets:

$$\sum_{i \in I_0} a_0 x_i + \dots + \sum_{i \in I_k} a_k x_i.$$

*Proof.* Pick any idempotent ultrafilter  $\mathcal{U}$ , and consider the linear combination  $\mathcal{W} = a_0 \mathcal{U} \oplus \dots \oplus a_k \mathcal{U}$ . Then take  $i$  such that  $C_i \in \mathcal{W}$ , and apply the previous Lemma.  $\square$

## 4. PARTITION REGULARITY AND RADO'S THEOREM

We now aim at showing how the introduced nonstandard approach can be used in partition regularity of linear equations. Let us start with an example. Recall the following known fact.

**Theorem 4.1** ([4] Th. 2.10). *Let  $\mathcal{U}$  be an idempotent ultrafilter. Then every set  $A \in 2\mathcal{U} \oplus \mathcal{U}$  contains a 3-term arithmetic progression.*

A nonstandard proof of the above theorem is obtained by the following simple observation. If the ultrafilter  $\mathcal{U} = \mathfrak{U}_\xi$  is idempotent, then the following three elements of the hyper-hyper-hyper-natural numbers  ${}^{***}\mathbb{N}$  are arranged in arithmetic progression, and they all generate the same ultrafilter  $\mathcal{W} = 2\mathcal{U} \oplus \mathcal{U}$ :

- $\nu = 2\xi + 0 + {}^{**}\xi$
- $\mu = 2\xi + {}^*\xi + {}^{**}\xi$
- $\lambda = 2\xi + 2^*\xi + {}^{**}\xi$

The property that  $\nu \underset{u}{\sim} \mu \underset{u}{\sim} \lambda \underset{u}{\sim} 2\xi + {}^*\xi$  directly follows from Theorem 3.6, since  $\langle 2, 0, 1 \rangle \underset{u}{\sim} \langle 2, 1, 1 \rangle \underset{u}{\sim} \langle 2, 2, 1 \rangle \underset{u}{\sim} \langle 2, 1 \rangle$ . Moreover, by Proposition 3.3, the generated ultrafilter is the following:

$$\mathcal{W} = \mathcal{U}_{2\xi + {}^*\xi} = \mathfrak{U}_{2\xi} \oplus \mathfrak{U}_\xi = 2\mathcal{U} \oplus \mathcal{U}.$$

If  $A \in \mathcal{W}$  then  $\nu, \mu, \lambda \in {}^{***}A$ , and the existence of a 3-term arithmetic progression in  $A$  is proved by applying backward *transfer* to the following property, which holds in  ${}^{***}\mathbb{N}$ :

$$\exists x, y, z \in {}^{***}A \text{ s.t. } y - x = z - y > 0.$$

We now elaborate on this example to prove a general fact which connects partition regularity of equations with  $u$ -equivalence in the direct limit

$${}^*\mathbb{N} = \bigcup_{k \in \mathbb{N}} k^*\mathbb{N}.$$

Recall the following

**Definition 4.2.** An equation  $F(X_1, \dots, X_n) = 0$  is [injectively] *partition regular* on  $\mathbb{N}_0$  if for every finite coloring of  $\mathbb{N}_0 = C_1 \sqcup \dots \sqcup C_r$  there exist [distinct] monochromatic elements  $x_1, \dots, x_n$  which are a solution, *i.e.*  $F(x_1, \dots, x_n) = 0$  and  $x_1, \dots, x_n \in C_i$  for a suitable color  $C_i$ .

It is a well-known fact that partition regularity is intimately connected with ultrafilters. In particular, recall the following:

- $F(X_1, \dots, X_n) = 0$  is [injectively] *partition regular* on  $\mathbb{N}_0$  if and only if there exists an ultrafilter  $\mathcal{V}$  on  $\mathbb{N}_0$  such that in every  $A \in \mathcal{V}$  one finds [distinct] elements  $x_1, \dots, x_n \in A$  with  $F(x_1, \dots, x_n) = 0$ .<sup>5</sup>

<sup>5</sup> A proof of this equivalence can be found *e.g.* in [15] §3.1.

When the above property is satisfied, we say that the ultrafilter  $\mathcal{V}$  is a *witness* of the [injective] partition regularity of  $F(X_1, \dots, X_n) = 0$ .

A useful nonstandard characterization holds.

**Theorem 4.3.** *Let the nonstandard embedding  $*$  satisfy the  $\mathfrak{c}^+$ -enlargement property. Then an ultrafilter  $\mathcal{V}$  on  $\mathbb{N}_0$  witnesses the [injective] partition regularity of the equation  $F(X_1, \dots, X_n) = 0$  if and only if there exists [distinct] hyper-natural numbers  $\xi_1, \dots, \xi_n \in {}^*\mathbb{N}_0$  such that  $\mathcal{V} = \mathfrak{U}_{\xi_1} = \dots = \mathfrak{U}_{\xi_n}$  and  ${}^*F(\xi_1, \dots, \xi_n) = 0$ .*

*Proof.* Assume first that the ultrafilter  $\mathcal{V}$  is a witness. For  $A \in \mathcal{V}$ , let

$$\Gamma(A) = \{(x_1, \dots, x_n) \in A^n \mid [x_i \neq x_j \text{ for } i \neq j] \ \& \ F(x_1, \dots, x_n) = 0\}.$$

Since  $\Gamma(A) \cap \Gamma(B) = \Gamma(A \cap B)$ , by the hypothesis it follows that the family  $\{\Gamma(A) \mid A \in \mathcal{V}\}$  satisfies the finite intersection property and hence, by  $\mathfrak{c}^+$ -enlargement, we can pick  $(\xi_1, \dots, \xi_n) \in \bigcap_{A \in \mathcal{V}} {}^*\Gamma(A)$ . Then it is readily checked from the definitions that the [distinct] components  $\xi_1, \dots, \xi_n$  are such that  $\mathfrak{U}_{\xi_1} = \dots = \mathfrak{U}_{\xi_n} = \mathcal{V}$  and  ${}^*F(\xi_1, \dots, \xi_n) = 0$ .

Conversely, let  $A \in \mathcal{V} = \mathfrak{U}_{\xi_1} = \dots = \mathfrak{U}_{\xi_n}$ . By applying backward *transfer* to the property: “There exist [distinct]  $\xi_1, \dots, \xi_n \in {}^*A$  such that  ${}^*F(\xi_1, \dots, \xi_n) = 0$ ”, one obtains the existence of [distinct]  $x_1, \dots, x_n \in A$  such that  $F(x_1, \dots, x_n) = 0$ , as desired.  $\square$

We can finally prove the result that was used in Section 1 to prove an ultrafilter version of Rado’s Theorem, namely Theorem 1.2.

**Theorem 4.4.** *Let  $a_0, \dots, a_n \in \mathbb{N}_0$ , and assume that there exist [distinct] polynomials  $P_i(X)$  such that*

$$P_1(X) \underset{\mathfrak{u}}{\approx} \dots \underset{\mathfrak{u}}{\approx} P_k(X) \underset{\mathfrak{u}}{\approx} \sum_{i=0}^n a_i X^i \quad \text{and} \quad c_1 P_1(X) + \dots + c_k P_k(X) = 0.$$

*Then for every idempotent ultrafilter  $\mathcal{U}$  and for every  $A \in a_0 \mathcal{U} \oplus \dots \oplus a_n \mathcal{U}$ , there exist [distinct]  $x_i \in A$  such that  $c_1 x_1 + \dots + c_k x_k = 0$ .*

*Proof.* For  $i = 1, \dots, k$ , let the polynomial  $P_i(X) = \sum_{j=0}^{n_i} b_{ij} X^j$  correspond to the string of coefficients  $\langle b_{i0}, b_{i1}, \dots, b_{in_i} \rangle$ . Given the idempotent ultrafilter  $\mathcal{U}$ , pick an hyper-natural number  $\xi \in {}^*\mathbb{N}$  such that  $\mathfrak{U}_\xi = \mathcal{U}$  (this is always possible by  $\mathfrak{c}^+$ -enlargement), and consider the numbers

$$\zeta_i = b_{i0} \xi + b_{i1} {}^*\xi + \dots + b_{in_i} {}^{n_i} \xi \in ({}^{n_i+1})^* \mathbb{N} \subset {}^*\mathbb{N}.$$

Since  $\xi {}^*\mathbb{N}$  is infinite, for every  $d \in \mathbb{N}_0$  one has  $d < \xi$ ,  $d\xi < {}^*\xi$ ,  $d {}^*\xi < {}^{**}\xi$ , and so forth. In consequence, by the hypothesis  $c_1 P_1(X) + \dots + c_k P_k(X) = 0$ , it directly follows that  $c_1 \zeta_1 + \dots + c_k \zeta_k = 0$ . Moreover, by the hypotheses  $P_i(X) \underset{\mathfrak{u}}{\approx} \sum_{i=0}^n a_i X^i$ , Theorem 3.6 guarantees that

$$\mathfrak{U}_{\zeta_1} = \dots = \mathfrak{U}_{\zeta_n} = a_0 \mathcal{U} \oplus \dots \oplus a_n \mathcal{U}.$$

The thesis is finally reached applying the previous Theorem 4.3. (Recall that the nonstandard embedding  $\star$  is  $\mathfrak{c}^+$ -enlarging, because the starting nonstandard embedding  $\ast$  was.)  $\square$

## 5. FINAL REMARKS

A similar characterization of partition regularity as the one given in Theorem 4.3 can also be proved for (possibly infinite) systems of equations. It seems worth investigating the use of such nonstandard characterizations especially for the study of homogeneous non-linear equations (along the lines of [17]) and of infinite systems, which are research areas where very little is known (see [13, 14, 3]). In particular, also the notion of *image partition regularity* would deserve attention. Another possible direction for further research is to consider possible extensions of Theorem 1.2 which are closer to the most general form of Rado's theorem for systems.

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