New Valid Inequalities and Facets for the Simple Plant Location Problem

Laura Galli*   Adam N. Letchford†   Sebastian J. Miller†

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Abstract

The *Simple Plant Location Problem* is a well-known (and $\mathcal{NP}$-hard) combinatorial optimisation problem arising in logistics. We present some new results concerned with the associated family of polyhedra. Our starting point is a family of valid inequalities derived by Karen Aardal. We strengthen that family, using a mixed-integer rounding argument, and then show that the strengthened family can itself be easily generalised. We then show that the original family, the strengthened family, and the generalised family each contain an exponentially large number of facet-defining members.

Keywords: facility location; combinatorial optimisation; branch-and-cut; polyhedral combinatorics

1   Introduction

The *Simple Plant Location Problem* (SPLP), sometimes called the *Uncapacitated Facility Location Problem*, is a classical and much-studied problem in Operational Research. We are given a set I of facilities and a set J of clients. The cost of opening facility $i \in I$ is denoted by f_i, and the cost of assigning client $j \in J$ to facility $i \in I$ is denoted by c_{ij}. The task is to decide where to open facilities, and then to assign each client to an open facility, at minimum cost.

Balinski [2] showed that the set covering problem can be transformed to the SPLP, which implies that the SPLP is $\mathcal{NP}$-hard in the strong sense. Surveys on theory, algorithms and applications of the SPLP include [9, 14, 15, 19, 21]. Here, we are concerned with the *integer programming* approach

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*Dipartimento di Informatica, Università di Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy. E-mail: `Laura.Galli@di.unipi.it`

†Department of Management Science, Lancaster University Management School, Lancaster LA1 4YX, United Kingdom. E-mail: `A.N.Letchford@lancaster.ac.uk`, `sebmiller64@hotmail.co.uk`

to the SPLP, and in particular on valid and facet-defining inequalities for the associated polyhedra. We assume throughout that the reader is familiar with the polyhedral approach to combinatorial optimisation. Readers who are not are referred to standard textbooks, such as [4, 7, 13, 18].

Balinski [2] formulated the SPLP as follows:

$$\min \quad \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i \tag{1}$$

$$\text{s.t.} \quad \sum_{i\in I} x_{ij} = 1 \qquad (j \in J) \tag{2}$$

$$x_{ij} \leq y_i \qquad (i \in I, j \in J) \tag{3}$$

$$x_{ij} \in \{0,1\} \qquad (i \in I, j \in J) \tag{4}$$

$$y_i \in \{0,1\} \qquad (i \in I). \tag{5}$$

Here, y_i is a binary variable, taking the value 1 if and only if a facility i is opened, and x_{ij} is a binary variable, taking the value 1 if and only if client j is assigned to facility i. This formulation is now standard. The constraints (2) are called *assignment constraints*, and the constraints (3) are called *variable upper bounds* (VUBs).

The polyhedra associated with the formulation (1)–(5) have been studied in depth, and several families of valid and facet-defining inequalities are known [1, 3, 5, 6, 8, 10, 12]. We will review these in the next section. We will be particularly interested in a family of inequalities derived by Karen Aardal [1], that we call *(p,q)* inequalities. We will show the following:

- The (p,q) inequalities can be easily strengthened, leading to what we call *aggregated mixed-integer rounding* (AMIR) inequalities.

- The (p,q) and AMIR inequalities in turn can be easily generalised.

- For all p and q such that $p \bmod q = 1$, there exist exponentially many facet-defining (p,q) inequalities, and exponentially many facet-defining generalised (p,q) inequalities that are not (p,q) inequalities.

- For all p and q, there exist exponentially many facet-defining AMIR inequalities, and exponentially many facet-defining generalised AMIR inequalities that are not AMIR inequalities.

In a companion paper [11], we will present *separation algorithms* for these new families and also for some known families.

The paper is structured as follows. The literature is reviewed in Section 2. The new valid inequalities are introduced in Section 3, and the facet proofs are given in Section 4. Finally, some concluding remarks are in Section 5.

The following notation is used throughout the paper. We let m denote $|I|$ and n denote $|J|$. We let $P(m,n)$ denote the SPLP *polytope*, i.e., the convex hull of all pairs $(x,y) \in \mathbb{R}^{(m\times n)+m}$ that satisfy the constraints (2)–(5). To avoid using subscripted subscripts, we sometimes write $x(i,j)$ instead of x_{ij}, and $y(i)$ instead of y_i. Moreover, sometimes we write $x(E)$ for $\sum_{\{i,j\}\in E} x_{ij}$, $x(S:T)$ for $\sum_{i\in S}\sum_{j\in T} x_{ij}$ and $y(S)$ for $\sum_{i\in S} y_i$.

2 Literature Review

Since the SPLP literature is vast, we focus here on papers that present polyhedral results, and refer the reader to [9, 14, 15, 19, 21] for surveys of other approaches.

2.1 Dimension and trivial facets

Cornuéjols & Thizy [10] showed that the dimension of $P(m, n)$ is $mn+m-n$, and that the affine hull is described by the assignment constraints (2). It is also shown that the VUBs (3), the non-negativity constraints $x_{ij} \geq 0$ for all i and j, and the upper bounds $y_i \leq 1$ for all i define trivial facets of $P(m, n)$. Cho *et al.* [5] showed that the assignment constraints and the trivial facets completely describe $P(m, n)$ if and only if $m \leq 2$ or $n \leq 2$ (or both).

2.2 Valid inequalities

Cornuéjols *et al.* [8] showed the following. Let p and q be integers satisfying $2 \leq q < p \leq m$ and $p \leq n$, with p not a multiple of q. Let $s_1, \ldots, s_p$ be distinct facilities, let $t_1, \ldots, t_p$ be distinct clients, and take indices modulo p, so that, for example, s_{p+1} is identified with s_1. Then, the inequality

$$\sum_{i=1}^{p} \sum_{j=i}^{i+q-1} x(s_i, t_j) \leq \sum_{i=1}^{p} y(s_i) + p - \lceil p/q \rceil \tag{6}$$

is valid for $P(m, n)$.

We call the inequalities (6) *circulant* inequalities. Figure 1 represents a circulant inequality with $p = 8$ and $q = 3$. The large and small circles represent facilities and clients, respectively, and the edges represent variables that appear on the left-hand side. Guignard [12] showed that circulant inequalities with $p = q + 1$, which we call *simple*, define facets. Cornuéjols & Thizy [10] showed that non-simple circulant inequalities do not.

The circulant inequalities with $q = 2$ (and therefore p odd) are called *odd cycle* inequalities [5, 10]. Guignard's result implies that the only odd cycle inequalities that define facets are the ones with $p = 3$, which we call *3-cycle* inequalities. Cho *et al.* [5] showed that the addition of the 3-cycle inequalities to the ones present in the LP relaxation of (1)–(5) gives a complete description of $P(m, n)$ when $m = 3$.

Cornuéjols & Thizy [10] presented the following generalisation of the simple circulant inequalities. Let p and t be integers with $2 \leq t < p \leq m$ and $\binom{p}{t} \leq n$. Let $S \subseteq I$ be any facility set with $|S| = p$, and let $T \subseteq J$ be any client set with $|T| = \binom{p}{t}$. Let G be any bipartite graph with node sets S and T, such that, for every set $S' \subset S$ with $|S'| = t$, there exists a $j \in T$ that is connected to each node in S' and no other nodes. (Figure 2
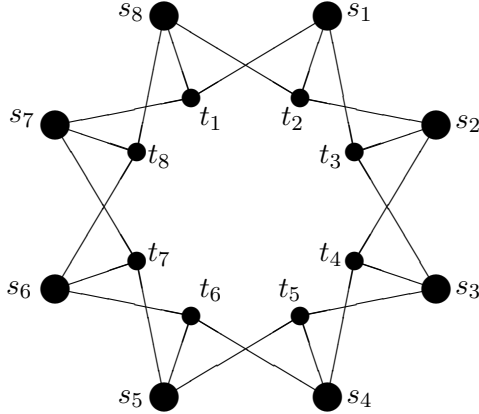
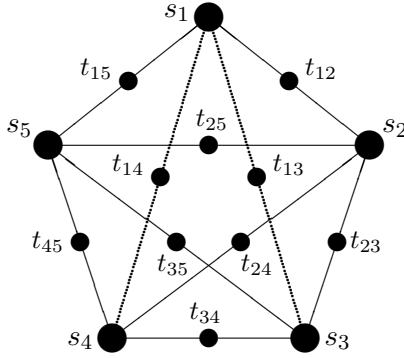Figure 1: Graph corresponding to a circulant inequality with $p = 8$ and $q = 3$.

Figure 2: Graph corresponding to a facility-subset inequality with $p = 5$ and $t = 2$.

shows such a graph G for the case $p = 5$, $t = 2$. The nodes in S are labelled $s_1, \ldots, s_5$ and the nodes in T are labelled $t_{12}, \ldots, t_{45}$.) Finally, let E denote the edge set of G. Then the following inequality defines a facet.

$$x(E) \ \leq \ y(S) + \binom{p}{t} + t - p - 1. \tag{7}$$

We call these inequalities *facility-subset* inequalities. They reduce to simple circulant inequalities when $t = p - 1$.

Another family of facet-defining inequalities was given by Cho *et al.* [6]. Let j_1, j_2 and j_3 be three clients, and let S_{12}, S_{13}, S_{23} and S_{123} be disjoint subsets of I. (We permit S_{123} to be empty.) Then we have:

$$x(S_{12} : \{j_1, j_2\}) \ + \ x(S_{13} : \{j_1, j_3\}) \ + \ x(S_{23} : \{j_2, j_3\}) \ + $$
$$\tfrac{1}{2} x\big(S_{123} : \{j_1, j_2, j_3\}\big) \ \leq \ 1 \ + \ y(S_{12} \cup S_{13} \cup S_{23}) \ + \ \tfrac{1}{2} y(S_{123}).$$
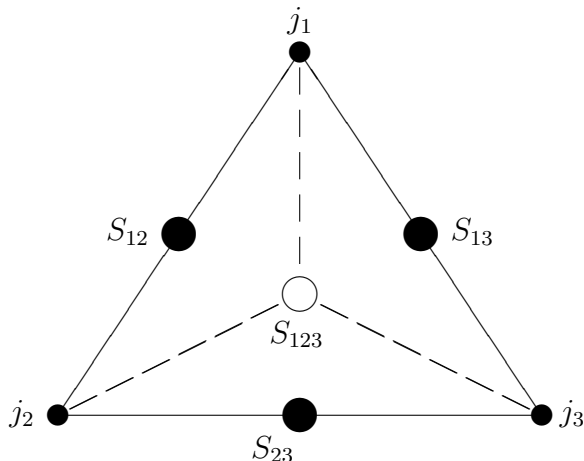
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Figure 3: Graphical representation of 3-client inequalities.

We call these inequalities *3-client* inequalities. (Figure 3 gives a graphical representation of them. The dashed lines represent x variables that have a coefficient of $1/2$ on the left-hand side, and the circle in the centre represents y variables that have a coefficient of $1/2$ on the right-hand side.) They reduce to 3-cycle inequalities when $|S_{12}| = |S_{13}| = |S_{23}| = 1$ and $S_{123} = \emptyset$. It is shown in [6] that the addition of the 3-client inequalities to the constraints present in the LP relaxation of (1)–(5) gives a complete description of $P(m, n)$ when $n = 3$.

Aardal [1] generalised the circulant inequalities as follows. Let p and q be as before, let $S \subseteq I$ be any facility set with $|S| \geq \lceil p/q \rceil$ and let $T \subseteq J$ be any client set with $|T| = p$. Let G be any bipartite graph with node sets S and T, such that each node in S has degree q in G. Finally, let E denote the set of edges of G. Then the inequality

$$x(E) \ \leq \ y(S) + p - \lceil p/q \rceil \tag{8}$$

is valid. We call these inequalities *(p, q)* inequalities. Aardal does not discuss conditions for them to define facets.

Some additional inequalities, called *fan* and *wheel* inequalities, were presented in Cánovas *et al.* [3]. We skip details, for the sake of brevity.

2.3 Lifting and replicating

Recall that non-simple circulant inequalities do not define facets. Cornuéjols & Thizy [10] showed that non-simple circulant inequalities with $p \bmod q = 1$ can be lifted, to make them facet-defining, by increasing the left-hand side coefficients of some x variables from 0 to 1. Cho *et al.* [6] proved that, in fact, *any* valid inequality with binary coefficients on the variables can be made facet-defining by lifting in this way. Unfortunately, lifting itself is a hard (possibly $\mathcal{NP}$-hard) problem. In fact, even lifting the odd cycle inequalities (with $p > 3$) is rather involved [6, 10].

We will also need the following result of Cho *et al.* [6], which enables one to create facets from facets by "replicating" facilities:

Theorem 1 (Cho *et al.* [6]) *Suppose the inequality $\alpha^T x \leq \beta^T y + \gamma$ is valid for $P(m,n)$, where $\alpha \in \{0,1\}^{mn}$ and $\beta \in \{0,1\}^m$. For any fixed $s \in I$, let $\tilde{\alpha} \in \{0,1\}^{(m+1)\times n}$ and $\tilde{\beta} \in \{0,1\}^{m+1}$ be defined by setting $\tilde{\alpha}_{ij} = \alpha_{ij}$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$, $\tilde{\alpha}_{m+1,j} = \alpha_{sj}$ for $j = 1,\ldots,n$, $\tilde{\beta}_i = \beta_i$ for $i = 1,\ldots,m$, and $\tilde{\beta}_{m+1} = \beta_s$. Then the inequality $\tilde{\alpha}^T x \leq \tilde{\beta}^T y + \gamma$ is valid for $P(m+1,n)$. Moreover, it defines a facet of $P(m+1,n)$ if the original inequality defines a facet of $P(m,n)$.*

In what follows, when we speak of "replicating" an inequality, we mean replicating facilities whose y variables have a non-zero coefficient on the right-hand side of the inequality. We remark that replicating a (p,q) inequality yields an inequality of the same type, but this is not so for circulant, simple circulant, odd cycle or facility subset inequalities.

3   New Valid Inequalities

In this section, we present three new families of valid inequalities, and point out some special cases of interest.

3.1   AMIR inequalities

In this subsection, we will present some new inequalities, that dominate the (p,q) inequalities. We will need the following lemma:

Lemma 1 *Let p, q, S and E be as in the definition of the (p,q) inequalities (Subsection 2.2). In any feasible SPLP solution, $x(E) \in \mathbb{Z}_+$, $y(S) \in \mathbb{Z}_+$, $x(E) \leq p$ and $x(E) \leq q\, y(S)$.*

Proof.   Integrality and non-negativity are implied by (4) and (5). The inequality $x(E) \leq p$ is implied by the non-negativity of x and the assignment constraints (2) for the clients in T. The inequality $x(E) \leq q\, y(S)$ is implied by the VUBs (3) for all $\{i,j\} \in E$. $\square$

The new inequalities are presented in the following theorem.

Theorem 2 *Let p, q, S and E be as in the definition of (p,q) inequalities (Subsection 2.2). Let k denote $\lfloor p/q \rfloor$ and r denote $p \bmod q$. (Note that $1 \leq r < q$ by assumption). The following inequality is valid for $P(m,n)$:*

$$x(E) \ \leq \ r\, y(S) + k(q - r). \tag{9}$$

Moreover, it is equivalent to the (p,q) inequality (8) when $r = 1$, but stronger when $r \geq 2$.

Proof. Consider the following two-dimensional mixed-integer set:

$$\{(X,Y) \in \mathbb{R}_+ \times \mathbb{Z}_+ : 0 \le X \le p, \ X \le qY\}.$$

It is well known (see, e.g., [16, 17]) that the only non-trivial facet-defining inequality for the convex hull of this set is the so-called *mixed-integer rounding* (MIR) inequality

$$X \le rY + k(q - r).$$

Replacing X with $x(E)$ and Y with $y(S)$, and using Lemma 1, we obtain the inequality (9). This shows that (9) is valid for $P(m, n)$.

To see that the inequality (9) reduces to a (p, q) inequality when $r = 1$, just note that $kq = p - r$. To see that it dominates the (p, q) inequality when $r > 1$, note that the (p, q) inequality can be derived by dividing the inequality (9) by r, multiplying the trivial inequality $x(E) \le p$ by $(r-1)/r$, and summing together the two resulting inequalities. $\qquad\square$

Figure 4 provides a graphical illustration of Lemma 1 and Theorem 2 for the case $p = 8$ and $q = 3$. The dots represent, for each possible value of $x(E)$, the corresponding lower bound on $y(S)$ given by the inequality $y(S) \ge \lceil x(E)/3 \rceil$. The dashed line represents the inequality $y(S) \ge x(E) - 5$, which is equivalent to the (p, q) inequality (8) in this case. The solid line passing through the points $(6, 2)$ and $(8, 3)$ represents the inequality $2y(S) \ge x(E) - 2$, which is equivalent to the new inequality (9) in this case. It is clear that the (p, q) inequality is redundant.

Since the inequalities (9) were derived by constructing the "aggregated" variables $x(E)$ and $y(S)$, and then applying the MIR procedure, we will call them *aggregated mixed-integer rounding* (AMIR) inequalities. We remark that some apparently similar inequalities, called *residual capacity* inequalities, were derived by Leung & Magnanti [16] for the *capacitated* plant location polytope. However, the residual capacity inequalities become trivial in the uncapacitated case, being implied by the non-negativity condition on x. Moreover, the bipartite graphs associated with residual capacity inequalities are complete, whereas this need not be the case for AMIR inequalities. Therefore the two families of inequalities are distinct.

Note that the coefficients of the y variables in AMIR inequalities are not binary in general. Therefore, Theorem 1 cannot be applied to them. Yet, if one does replicate an AMIR inequality, the result is just another AMIR inequality. Note also that each circulant inequality (6) is equivalent to or dominated by an AMIR inequality of the form:

$$\sum_{i=1}^{p} \sum_{j=i}^{i+q} x(s_i, t_j) \le r \sum_{i=1}^{p} y(s_i) + k(q - r). \tag{10}$$

We call these special AMIR inequalities *strengthened* circulant inequalities. We remark that they are different from the *lifted* circulant inequalities presented in [6, 10].
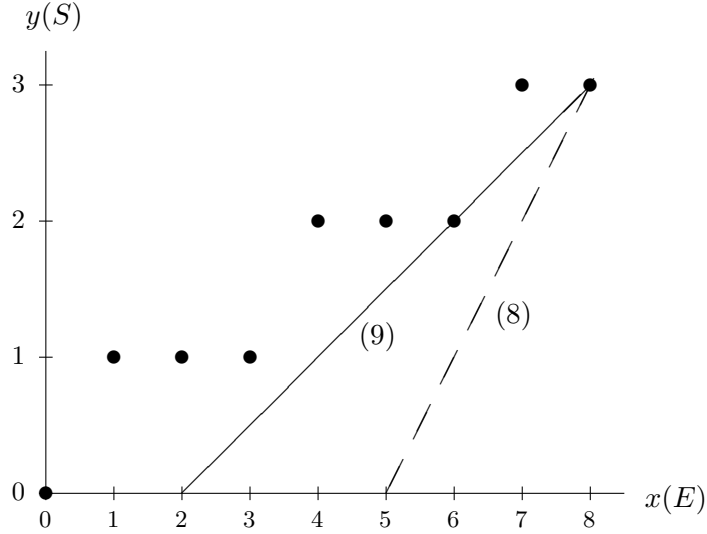
Figure 4: Representation of (p,q) and AMIR inequalities when $p = 8$ and $q = 3$.

3.2 Generalised (p,q) inequalities

Now we present a family of valid inequalities that contains the (p,q) inequalities.

Theorem 3 *Let p and q be positive integers with $2 \le q < p \le m$ and $p \le n$, with p not a multiple of q. Let $T \subseteq J$ be any client set with $|T| = p$, and let $S \subseteq I$ be any facility set such that $|S| \ge \lceil p/q \rceil$. Let G be any bipartite graph with node sets S and T, such that the degree of each node in S is a (positive) multiple of q. For each $i \in S$, let $d(i)$ be the degree of i in G. (Figure 5 shows a suitable graph G for the case $p = 5$ and $q = 2$. Facilities $s_1, \ldots, s_5$ have degree two, but the facility s_6 has degree four.) Finally, let E denote the set of edges of G. Then the following 'generalised (p,q)' inequality*

$$x(E) \ \le \ \sum_{i \in S} \frac{d(i)}{q} y_i + p - \lceil p/q \rceil \tag{11}$$

is valid for $P(m,n)$.

Proof. Let

$$y'(S) = \sum_{i \in S} \frac{d(i)}{q} y_i.$$

Also let k be defined as in Theorem 2.

In a feasible solution, $y'(S)$ must be integer. Moreover, $x(E) \le p$ due to the assignment constraints (2), and $x(E) \le q\,y'(S)$ due to the VUBs (3).
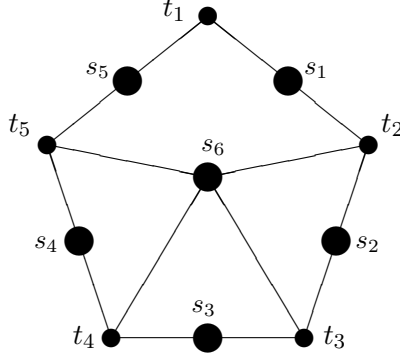
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Figure 5: Graph yielding a generalised $(5, 2)$ inequality.

We now consider two cases. First, suppose $y'(S) \geq k + 1$. Then the right-hand side of (11) is at least p. The inequality (11) is then trivially valid, since $x(E)$ cannot exceed p. Second, suppose that $y'(S) \leq k$. Multiplying the inequality $y(S) \leq k$ by $q - 1$ and re-arranging, we obtain $0 \leq -(q - 1)\, y'(S) + k(q - 1)$. Adding to this the valid inequality $x(E) \leq q\, y'(S)$, we obtain $x(E) \leq y'(S) + k(q - 1)$. This implies the inequality (11), since $k(q - 1) = p - \lfloor p/q \rfloor - (p \bmod q) \leq p - \lceil p/q \rceil$. $\qquad\square$

Note that generalised (p, q) inequalities, like AMIR inequalities, do not have binary coefficients for the y variables. Therefore, Theorem 1 cannot be applied to them either. Yet, if one does replicate a generalised (p, q) inequality, the result is another generalised (p, q) inequality.

3.3   Generalised AMIR inequalities

One can perform the strengthening and the generalisation simultaneously, yielding what we call *generalised AMIR* inequalities. This is shown in the following proposition.

Proposition 1 *Let p, q, S, T, G, E and $d(i)$ be as defined in Theorem 3, and let k and r be defined as in Theorem 2. Then the following 'generalised AMIR' inequality*

$$x(E) \; \leq \; r \sum_{i \in S} \frac{d(i)}{q} y_i \, + k(q - r) \tag{12}$$

is valid for $P(m, n)$.

Proof.   Similar to the proof of Theorem 2. $\qquad\square$

As in the previous two subsections, if one replicates a generalised AMIR inequality, the result is just another generalised AMIR inequality.
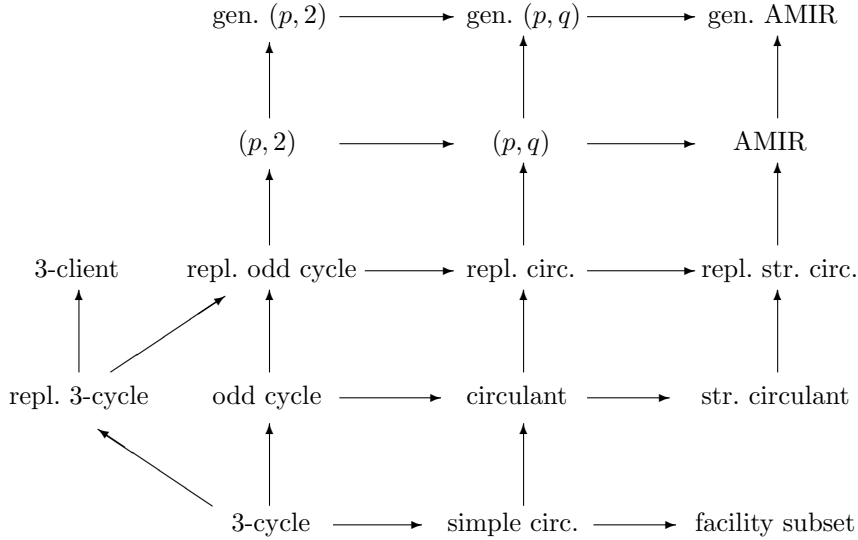
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Figure 6: Hierarchy of inequalities for the SPLP.

To aid the reader, we display in Figure 6 a hierarchy of seventeen key families of valid inequalities. An arrow from one class to another means that the latter is a proper generalisation of, or dominates, the former. The prefixes "repl.", "gen." and "str." stand for "replicated", "generalised" and "strengthened", respectively, and "circ." stands for "circulant". (We do not include the *lifted* odd cycle or circulant inequalities in the figure, partly for clarity, but also because there is no closed-form expression for their left-hand sides. Also, we do not include the *replicated* simple circulant or facility-subset inequalities, for clarity.) One can see that the generalised AMIR inequalities subsume all other inequalities in the figure, apart from the 3-client and facility subset inequalities.

4   New Facets

In this section, we examine conditions under which the (p, q) and AMIR inequalities, and their generalisations, define facets of $P(m, n)$. In Subsection 4.1, we characterise the facet-defining $(p, 2)$ inequalities. In Subsection 4.2, we present some facet-defining (p, q) and AMIR inequalities with $q > 2$. In Subsection 4.3, we present some facet-defining generalised (p, q) and AMIR inequalities.
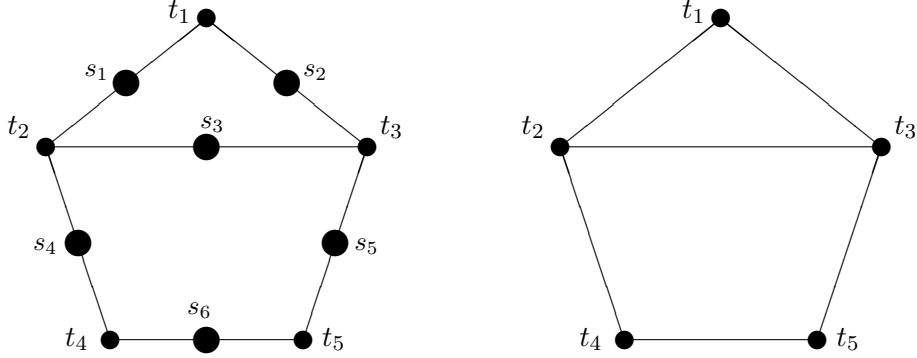
Figure 7: Graph G (left) and associated reduced graph G^- (right).

4.1 Facet-defining $(p, 2)$ inequalities

Recall (from the result of Guignard [12] mentioned in Subsection 2.2) that 3-cycle inequalities define facets. Moreover, from Theorem 1 in Subsection 2.3, replicated 3-cycle inequalities define facets as well. Therefore (see Figure 6) both 3-cycle and replicated 3-cycle inequalities are examples of facet-defining $(p, 2)$ inequalities. A natural question is whether there exist any other facet-defining $(p, 2)$ inequalities. In this subsection, we show that, in fact, there exist exponentially many inequalities of this type.

Recall that, associated with any given (p, q) inequality, there is a bipartite graph G with vertex set $S \cup T$ and edge set E. When $q = 2$, each node in S is adjacent to exactly 2 nodes in T. We will find it helpful to define a *reduced graph*, denoted by G^-. The vertex set of G^- is T, and the edge set, denoted by E^-, is defined as follows. There is an edge $\{u, v\}$ in E^- if and only if there exists some $i \in S$ such that the edges $\{i, u\}$ and $\{i, v\}$ exist in E. This construction is illustrated in Figure 7. The graph G on the left corresponds to a $(p, 2)$ inequality with $S = \{s_1, \ldots, s_6\}$ and $T = \{t_1, \ldots, t_5\}$. The large and small circles represent facilities in S and clients in T, respectively, and the lines represent the edges in E. The inequality is:

$$x(s_1, t_1) + x(s_1, t_2) + \cdots + x(s_6, t_4) + x(s_6, t_5) \ \leq \ 2 + \sum_{i=1}^{6} y(s_i).$$

The corresponding reduced graph G^- is shown on the right. So, for example, the edge $\{t_1, t_2\}$ is present in E^- because the edges $\{s_1, t_1\}$ and $\{s_1, t_2\}$ exist in E.

We now define a certain property that G^- may or may not have:

Definition 1 *The reduced graph $G^- = (T, E^-)$ is said to be* super-matchable *if, for every edge $\{u, v\} \in E^-$ and every node $w \in T \backslash \{u, v\}$, there is a perfect matching of the nodes in $T \setminus \{u, v, w\}$.*
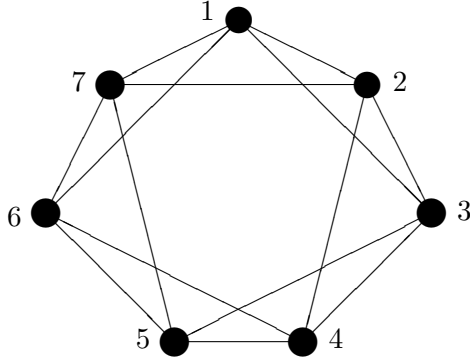
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Figure 8: A super-matchable graph on 7 nodes.

So, for example, the reduced graph G^- shown on the right of Figure 7 is not supermatchable, because, if we take the edge $\{t_1, t_2\}$ and the node t_5, the nodes t_3 and t_4 cannot be matched. On the other hand, the graph shown in Figure 8 is super-matchable. For example, if we take the edge $\{1, 3\}$ and the node 5, the remaining nodes, 2, 4, 6 and 7, can be matched using the edges $\{2, 4\}$ and $\{6, 7\}$.

The property of being super-matchable turns out to be crucial for determining whether a $(p, 2)$ inequality defines a facet.

Theorem 4 *A $(p, 2)$ inequality defines a facet of $P(m, n)$ if and only if the associated reduced graph G^- is super-matchable.*

Proof. From the result in [6] on lifting, mentioned in Subsection 2.3, it suffices to show that a $(p, 2)$ inequality can be lifted if and only if G^- is super-matchable. So, consider an edge $\{u, v\}$ in G^-, and let i be a facility in S such that the edges $\{i, u\}$ and $\{i, v\}$ are in G. (Note that there may be more than one such facility.) Consider the lifted inequality that is obtained by changing the coefficient of x_{iw} from zero to one, for some $w \in T \setminus \{u, v\}$. To see whether this lifted inequality is valid, we attempt to construct a feasible SPLP solution that violates it, in which x_{iw} takes the value 1. Note that, in such a solution, y_i must take the value 1 as well. Then, to maximise the left-hand side of the lifted inequality, it pays to set x_{iu} and x_{iv} to one as well. Now we have a contribution of 3 to the left-hand side and 1 to the right-hand side. Then, the only way the lifted inequality could be violated would be for the remaining $p - 3$ clients in T to be served by only $(p - 3)/2$ of the remaining facilities in S. This can happen if and only if those $p - 3$ clients can be matched in G^-.

To complete the proof, it suffices to show that it is not possible to lift x_{ij} when $i \in I \setminus S$ and/or $j \in J \setminus T$. To see this, just note that any extreme point of $P(m, n)$ that satisfies the $(p, 2)$ inequality at equality and has $x_{ij} = y_i = 1$ violates the lifted inequality. □

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The above theorem explains why the odd cycle inequalities define facets if and only if $p = 3$. Indeed, the reduced graph corresponding to an odd cycle inequality is itself an odd cycle on p nodes. A odd cycle on 3 nodes (i.e., a triangle) is (trivially) super-matchable, but an odd cycle on $p \geq 5$ nodes is not.

The following proposition presents an exponentially large family of super-matchable graphs, which corresponds to an exponentially large family of facet-defining $(p, 2)$ inequalities.

Proposition 2 *Let $p \geq 5$ be an odd integer and let k be an integer with $2 \leq k \leq \lfloor p/2 \rfloor$. Let $G(p, k)$ be the graph with vertex set $\{1, \ldots, p\}$, and an edge between nodes i and j if and only if $|i - j| \bmod p \leq k$. (The graph in Figure 8 is $G(7, 2)$.) Then $G(p, k)$ is super-matchable.*

Proof. One can check that, if one removes *any* three nodes from such a graph, the remaining nodes can be matched. $\square$

We remark that the graphs $G(p, k)$ are a special case of the so-called *antiwebs*, defined by Trotter [20] in the context of set packing. We remark also that replicating a $(p, 2)$ inequality has no effect on the associated reduced graph. Therefore, if a $(p, 2)$ inequality defines a facet, then so does an inequality obtained from it by replicating.

4.2 Facet-defining (p, q) and AMIR inequalities

Now, observe that the simple circulant and replicated simple circulant inequalities, in addition to the facet-defining $(p, 2)$ inequalities described in the previous subsection, are facet-defining AMIR inequalities. A natural question is whether there exist any other facet-defining AMIR inequalities, or indeed (p, q) inequalities. The following theorem and corollary shows that this is indeed the case.

Theorem 5 *Let p and q be positive integers with $2 \leq q < p \leq n$, $m \geq \binom{p}{q}$ and p not a multiple of p. Let S be a set of facilities such that $|S| = \binom{p}{q}$, and let T be a set of clients such that $|T| = p$. Let G be any bipartite graph with node sets S and T, such that, for every set $T' \subset T$ with $|T'| = q$, there exists an $i \in S$ that is connected to each node in T' and no other nodes. (Figure 9 shows a suitable graph G for the case $p = 5$, $q = 2$. The five nodes in T are labelled $t_1, \ldots, t_5$ and the ten nodes in S are labelled $s_{12}, \ldots, s_{45}$.) Let E denote the edge set of G, and let k and r be defined as in Theorem 2. The AMIR inequality (9) defines a facet of $P(m, n)$.*

Proof. Since AMIR inequalities do not have binary coefficients for the y variables in general, we cannot use the same technique that we used to prove Theorem 4. Instead, we use a "dual" proof.

So, let us call an extreme point of $P(m, n)$ a "root" of the AMIR inequality if it satisfies it at equality, and let $\alpha^T x + \beta^T y = \gamma$ be an equation that is satisfied by all roots. One can check (cF. Figure 4) that there are two kinds of roots. One kind has $X = p - r$ and $Y = k$, and the other has $X = p$ and $Y = k + 1$.

Consider any $j \in J \setminus T$ and any root (of either kind) such that client j is assigned to an open facility in S. We can obtain another root (of the same kind) by assigning client j to any other open facility. By symmetry, this shows that α_{ij} takes the same value for all $i \in I$. Due to the assignment constraints (2), we can assume that this value is zero.

Next, consider any $i \in I \setminus S$, and any root (of either kind) such that facility i is closed. By opening that facility, we obtain another root (of the same kind). By symmetry, this shows that $\beta_i = 0$ for all $i \in I \setminus S$.

Next, consider any $j \in T$, and any root of the first kind such that client j is assigned to a facility in $I \setminus S$. By assigning j to any other open facility i such that $\{i, j\} \notin E$, we obtain another root. By symmetry, this shows that α_{ij} takes the same value for all $i \in I$ such that $\{i, j\} \notin E$. Due to the assignment constraints (2), we can assume that this value is zero.

So far, we have shown that the equation $\alpha^T x + \beta^T y = \gamma$ must take the form
$$\sum_{\{i,j\} \in E} \alpha_{ij} x_{ij} = \sum_{i \in S} \beta_i y_i + \gamma.$$

Now consider a root of the second kind. There must exist a facility $i \in S$ that has fewer than q clients in T assigned to it. Accordingly, there must exist a client $j \in T$ that is currently assigned to a facility in $S \setminus \{i\}$, but for which $\{i, j\} \in E$. We can obtain another root of the second kind by assigning that client to i instead. By symmetry, this shows that, for any given client $j \in T$, the coefficient α_{ij} takes the same value for all i such that $\{i, j\} \in E$.

Now, consider any pair $j, j' \in T$, and let $i \in S$ be such that both $\{i, j\}$ and $\{i, j'\}$ belong to E. Consider any root of the first kind such that facility i is open, client j is assigned to facility i, and client j' is assigned to an open facility in $I \setminus S$. We obtain another root by assigning client j' to facility i and assigning client j to the open facility in $I \setminus S$. By symmetry, this shows that α_{ij} takes the same value for all $\{i, j\} \in E$.

We have now shown that the equation $\alpha^T x + \beta^T y = \gamma$ must take the form
$$\alpha_0 \sum_{\{i,j\} \in E} x_{ij} = \sum_{i \in S} \beta_i y_i + \gamma,$$

where α_0 is a scalar. Now, for any $i \in S$, consider a root of the second kind such that facility i is open and exactly r clients are assigned to i. By closing facility i, and assigning each of those r clients to an open facility in $I \setminus \{i\}$, we obtain another root. This shows that $\beta_i = r\alpha_0$. Therefore, the equation
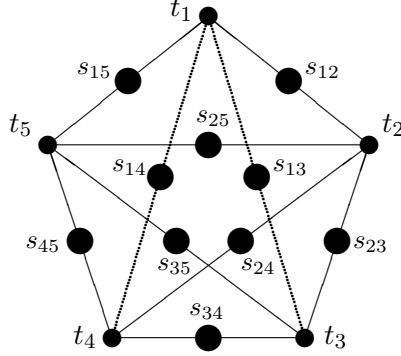
14

Figure 9: Graph G corresponding to a facet-defining AMIR inequality with $p = 5$ and $q = 2$.

$\alpha^T x + \beta^T y = \gamma$ must take the form:

$$\alpha_0 \sum_{\{i,j\} \in E} x_{ij} = r\alpha_0 \sum_{i \in S} y_i + \gamma.$$

Now, if α_0 were equal to zero, γ would have to be zero, and the equation would become vacuous. So $\alpha_0 \neq 0$. Dividing the equation by α_0, we obtain:

$$\sum_{\{i,j\} \in E} x_{ij} = r \sum_{i \in S} y_i + \gamma,$$

and the only possible value for γ is the one given in the AMIR inequality (9). □

Corollary 1 *For all p and q with $2 \leq q < p$ and $p \bmod q = 1$, there exist facet-defining (p, q) inequalities.*

Proof. When $p \bmod q = 1$, the facet-defining AMIR inequalities described in Theorem 5 are (p, q) inequalities. □

Note that the number of facet-defining inequalities described in both Theorem 5 and Corollary 1 grows exponentially with both m and n. Note also that, in these inequalities, there is one facility for each suitable subset of clients. This is to be contrasted with the facility subset inequalities (7), in which there is one client for each suitable subset of facilities.

With a little work, it can be shown that replicating a facet-defining AMIR or (p, q) inequality leads to another facet-defining AMIR or (p, q) inequality, respectively. In this way, one can generate still more facet-defining AMIR and (p, q) inequalities. We omit details for brevity. We also leave open the question of finding a necessary and sufficient condition for an AMIR or (p, q) inequality to define a facet.

Now, recall that the strengthened circulant inequalities (10) are a special case of the AMIR inequalities. For these inequalities, we have the following negative result:

Proposition 3 *Strengthened circulant inequalities do not define facets if either of the following holds: (i) $p > 3q/2$, (ii) $q + 2 \leq p \leq 3q/2$ and $p = m$.*

Proof (sketch). For brevity, we only sketch the proof. Let $x(E)$ denote the left-hand side of the inequality, let S denote $\{s_1, \ldots, s_p\}$ and let r denote $p \bmod q$. Just as in the proof of Theorem 5, every root of a strengthened circulant inequality satisfies one of the following two conditions:

- $x(E) = p$ and $y(S) = \lceil p/q \rceil$,

- $x(E) = p - r$ and $y(S) = \lfloor p/q \rfloor$.

Moreover, the special structure of the graph G associated with a strengthened circulant inequality implies that, for a root of the second kind, the open facilities are "spread out", in the sense that $|(i \bmod p) - (i' \bmod p)| \geq q$ for all pairs i, i' of open facilities.

We now consider four cases:

1. $p > 2q$. One can check that, in this case, all roots (of either kind) satisfy the equation $x(s_i, t_{i+q+r}) = 0$ for $i = 1, \ldots, p$.

2. $3q/2 < p < 2q$ and q is odd. One can check that all roots satisfy the equation $x\big(s_i, t_{i+(q-1)/2}\big) = y(s_i)$ for $i = 1, \ldots, p$.

3. $3q/2 < p < 2q$ and q is even. One can check that all roots satisfy the equations $x(s_i, t_{i+q/2-1}) = x(s_i, t_{i+q/2}) = y(s_i)$ for $i = 1, \ldots, p$.

4. $q + 2 \leq p \leq 3q/2$ and $p = m$. One can check that all roots satisfy the equation $x(s_i, t_{i+q}) = x(s_i, t_{i+q+1})$ for $i = 1, \ldots, p$.

In each case, the inequality cannot define a facet. $\square$

Note that, when $p = q + 1$, the strengthened circulant inequalities reduce to simple circulant inequalities, which are facet-defining. We do not know if there exist any other facet-defining strengthened circulant inequalities.

4.3 Facet-defining generalised (p, q) and AMIR inequalities

Another natural question is whether there exist any facet-defining *generalised* AMIR or (p, q) inequalities that are not just AMIR or (p, q) inequalities. The following theorem shows that this is indeed the case.

Theorem 6 *Let p, q, S, T, G and E be as defined in Theorem 5, and let k and r be defined as in Theorem 2. Now suppose that we enlarge S and E, by adding h new facility nodes to G, along with edges connecting each new facility node to a set of clients in T, in such a way that the degree of each new facility node is a multiple of q. (For example, we could take the graph G shown in Figure 9, which has $p = 5$ and $q = 2$, and add one new facility node, say s_{1234}, which is adjacent to client nodes t_1 to t_4. The resulting graph G' is shown in Figure 10. The labels of the ten original facility nodes have been omitted, for clarity.) The generalised AMIR inequality (12) corresponding to the resulting graph defines a facet of $P(m + h, n)$.*

Proof. Let H denote the set of added facilities. From Theorem 5, the AMIR inequality associated with the original graph G defines a facet of $P(m, n)$. Also, any root of the original AMIR inequality can be converted into a root of the generalised AMIR inequality simply by closing all of the facilities in H. Now, the dimension of $P(m + h, n)$ is $h(n + 1)$ more than that of $P(m, n)$, since we have added h new y variables and hn new x variables. So, it suffices to construct $n + 1$ new affinely-independent roots of the generalised AMIR inequality for each $s \in H$.

Let $s \in H$ be fixed. One of the desired roots can be obtained easily by taking a root of the AMIR inequality, opening facility s, closing the facilities in $H \setminus \{s\}$, and assigning no clients to facility s. To construct the other n roots for the given s, we proceed as follows. Let $T(s)$ be the set of clients that are adjacent to s, and note that $|T(S)|$ equals $d(s)$, the degree of node s in the graph. For each client $t \in T(S)$, we construct a root by opening facility s, assigning the clients in $T(S) \setminus \{t\}$ to facility s, and opening exactly $k + 1 - d(s)/q$ of the facilities in S, chosen in such a way that each of the $p - d(s) + 1$ clients in $(T \setminus T(S)) \cup \{t\}$ can be assigned to one of them. Finally, for each client $j \in J \setminus T(S)$, we construct a root by opening facility s and assigning all of the clients in $T(s) \cup \{j\}$ to it, then opening $k - d(i)/q$ facilities in S and assigning q of the clients in $T \setminus T(s)$ to each of them. $\square$

Corollary 2 *For all p and q with $2 \leq q < p$ and $p \bmod q = 1$, there exist facet-defining generalised (p, q) inequalities that are not (p, q) inequalities.*

Proof. When $p \bmod q = 1$, the facet-defining generalised AMIR inequalities described in Theorem 6 are generalised (p, q) inequalities. $\square$

Again, the number of facet-defining inequalities described in both Theorem 6 and Corollary 2 grows exponentially with both m and n. We leave open the question of finding a necessary and sufficient condition for a generalised (p, q) or AMIR inequality to define a facet.
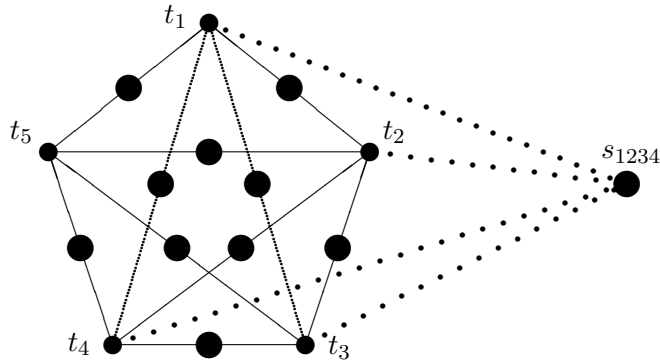
Figure 10: Representation of a facet-defining generalised $(5, 2)$ inequality.

5 Conclusion

Although the SPLP has been studied in depth from a polyhedral point of view, Aardal [1] left open the question of whether there exist any facet-defining (p, q) inequalities. We have shown that, in fact, for any fixed p and q, the number of such inequalities grows exponentially as the number of facilities and/or clients grows. Moreover, we have presented three stronger and/or more general families of valid inequalities, and shown that they too contain exponentially many facet-defining members.

In our companion paper [11], we turn our attention to algorithmic issues. In particular, we present exact and heuristic separation algorithms for several different families of inequalities, and test their practical performance within a branch-and-cut algorithm for the SPLP.

Acknowledgement

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