ON THE REGULARITY UP TO THE BOUNDARY FOR CERTAIN NONLINEAR ELLIPTIC SYSTEMS

Luigi C. Berselli and Carlo R. Grisanti

Dipartimento di Matematica
Via F. Buonarroti 1/c
Pisa, I-56127, Italy

Dedicated to P. Secchi and A. Valli on the occasion of their 60th birthday

Abstract. We consider a class of nonlinear elliptic systems and we prove regularity up to the boundary for second order derivatives. In the proof we trace carefully the dependence on the various parameters of the problem, in order to establish, in a further work, results for more general systems.

1. **Introduction.** We consider a nonlinear elliptic system in $n \geq 2$ space variables, with Dirichlet boundary conditions

$$\begin{aligned} -\nu_0 \Delta u - \nu_1 \nabla \cdot S(\nabla u) = f \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbf{R}^n$ is a smooth and bounded domain, while the unknown is the vector valued field $u : \mathbf{R}^n \to \mathbf{R}^N$. Here, the "extra-stress tensor" is $S(\nabla u)$, where $S : \mathbf{R}^{N\times n} \to \mathbf{R}^{N\times n}$ is a nonlinear function satisfying, for each $A, B \in \mathbf{R}^{N\times n}$ and for some $k_1, k_3 > 0$ and $1 < p < 2$

$$\begin{aligned} & S(A) \cdot A \geq k_1 |A|^p - k_2, \\ & |S(A)| \leq k_3 \left(|A|^{p-1} + 1 \right), \\ & (S(A) - S(B)) \cdot (A - B) \geq 0. \end{aligned} \tag{2}$$

With the exception of an interior regularity result (see Theorem 1.4) which is valid for a wider class of tensors S, the practical example we will treat, which is motivated by a further understanding of the p-Laplace equation and certain shear thinning non-Newtonian fluids is the following

$$S(A) = (\mu + |A|^2)^{\frac{p-2}{2}} A, \qquad 0 \leq \mu, \qquad 1 < p < 2. \tag{3}$$

With the stress tensor defined above, we can estimate that $k_1 = 2^{\frac{p-2}{2}}$, $k_2 \leq 4\mu^{\frac{p}{2}}$ and $k_3 = 1$.

2010 *Mathematics Subject Classification.* Primary: 35J25; Secondary: 35B65, 35J70.

Key words and phrases. Elliptic equations, boundary regularity, degenerate equations.

The authors are supported by MIUR under the project "Nonlinear Hyperbolic Partial Differential Equations, Dispersive and Transport Equations: Theoretical and Applicative Aspects." The authors are members of INDAM/GNAMPA.

For such a stress tensor we will also have the following estimates: There exist $\gamma_1, \gamma_2 > 0$, independent of μ, such that

$$|S(A) - S(B)| \leq \gamma_1(\mu + |A| + |B|)^{p-2}|A - B|, \tag{4}$$

$$(S(A) - S(B)) \cdot (A - B) \geq \gamma_2(\mu + |A| + |B|)^{p-2}|A - B|^2, \tag{5}$$

for a proof see for instance [12]. We highlight that we consider the case $p < 2$, since our work is a starting point for a better understanding of properties of shear-thinning fluids.

The constants $\nu_0 \geq 0$ and $\nu_1 > 0$ play a very relevant role, since when ν_0 is positive the problem is a nonlinear perturbation of the Poisson equation, while when $\nu_0 = 0$ the principal part is non-linear. In particular, the equation is the so-called p-Laplacian, if $\nu_0 = \mu = 0$.

We are mainly interested in obtaining estimates in Lebesgue spaces for the second order derivatives. We use customary Lebesgue spaces $L^p(\Omega)$, with norm $\|\,.\,\|_p$ and Sobolev $W^{k,p}(\Omega)$ spaces with norm $\|\,.\,\|_{k,p}$, see Adams [2] for further properties (when needed we also explicitly write if we are using norms evaluated in open sets different from Ω). For shortness of notation we also write $\partial_j := \frac{\partial}{\partial x_j}$ to denote partial derivatives. In this short paper we consider just the case $\nu_0 > 0$, and we say that $u \in W^{1,2}_0(\Omega)$ is a weak solution to the boundary valued problem (1) if

$$\nu_0 \int_\Omega \nabla u \cdot \nabla \phi \, dx + \nu_1 \int_\Omega S(\nabla u) \cdot \nabla \phi \, dx = \langle f, \phi \rangle \qquad \forall\, \phi \in W^{1,2}_0(\Omega), \tag{6}$$

where $\langle\, ,\, \rangle$ denotes the duality pairing between $W^{1,2}_0(\Omega)$ and its topological dual $W^{-1,2}(\Omega) := (W^{1,2}_0(\Omega))^*$. Due to the Poincaré inequality, we use as equivalent norm $\|\nabla u\|_p \sim \|u\|_{1,p}$. A well-known basic result is the following (see for instance Lions [17, Ch. 2.2])

Theorem 1.1. *Let the tensor S satisfy conditions (2), $\nu_0 > 0$, $\nu_1 \geq 0$, $1 < p \leq 2$, and $f \in W^{-1,2}(\Omega)$, then there exists a unique weak solution u to system (1) and it satisfies*

$$\nu_0\|\nabla u\|_2^2 + \nu_1 k_1 \|\nabla u\|_p^p \leq \frac{1}{\nu_0}\|f\|^2_{W^{-1,2}} + \nu_1 k_2|\Omega|,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

Our principal aim is that of obtaining results of regularity up to the boundary, with a non-zero given f. It is important to note that, differently from most results of partial regularity, we do not assume neither boundedness nor a given sign for the weak solution u. In particular we will use in an essential way that the domain is smooth enough, see Nečas [21, Ch. 1] for the precise definition of a domain of class C^k. The study of this problem seems not complete at all and the main result of this paper is the following

Theorem 1.2. *Let be given $\nu_0 > 0$, $\nu_1 \geq 0$, $\mu > 0$, $1 < p \leq 2$, S defined by means of the relation (3) and $f \in L^2(\Omega)$. Let also Ω be a bounded domain of class C^2, then the unique weak solution to (1) belongs to $W^{2,2}(\Omega)$ and there exists a constant C depending only on Ω and p such that*

$$\|u\|_{2,2} \leq \frac{C}{\nu_0}\|f\|_2.$$

After having established this result, we will use it to study also the degenerate problem.

Theorem 1.3. *Let the same hypotheses as before be satisfied, with the only difference that $\mu = 0$. Then, the same $W^{2,2}(\Omega)$-regularity result holds true.*

In proving the previous results we get also an intermediate interior regularity which holds not only for the specific tensor defined by (3) but for a more general class:

Theorem 1.4. *Let S satisfy conditions (2), (4), (5), $\nu_0 > 0, \nu_1 \geq 0, \mu > 0, 1 < p \leq 2$ and $f \in L^2(\Omega)$, then the weak solution to (1) belongs to $W^{2,2}_{loc}(\Omega)$. Moreover the system (1) is satisfied in a strong form, that is for almost every point (in the sense of the Lebesgue measure) of the open set Ω.*

The proof of the above theorem is developed in subsection 3.3.

Since the problem we study is very classical, we recall the main results about it. The problem with the p-Laplacian $1 < p < 2$ for which concerns interior regularity (especially $C^{1,\alpha}$) has been studied by DiBenedetto [11], Lewis [15] and Tolksdorf [23, 24], independently and more or less in the same period. They extended to the case $p < 2$ earlier well-known results by Uralt'seva [26], Uhlenbeck [25], and Evans [14]. These results concern interior regularity (of bounded solutions) and in many cases the external force must be zero. Further interior results are those by Acerbi and Fusco [1] and Naumann and Wolf [20].

Concerning boundary regularity we can cite Lieberman [16] and more recently Beirão da Veiga and Crispo [6, 7], Crispo [9], Crispo and Maremonti [10]. Anyway, our result differs from all of these since: a) we assume the force to be just in $L^2(\Omega)$; b) we do not have limitations on $p \in]1, 2]$; c) the non flat domain is treated in a slightly different way; d) we also have a Laplacian term which makes the problem slightly simpler; e) the use of (tangential) finite quotients allows calculations which are probably easier to be followed, and which are extremely explicit.

Anyway, we will treat the case without the Laplacian in a forthcoming paper and the motivation for our studies are that equation (1) can be considered as a suitable regularization to study the degenerate problem involving the p-Laplacian.

Moreover, the system we are considering is just a first step to tackle the problem involving incompressible shear-dependent fluids, to extend and improve results (from which we took great inspiration) of Beirão da Veiga [3, 4] and Málek, Nečas, and Růžička [19], mainly to have a very explicit and complete proof of the regularity of second order derivatives of the solution.

2. **Local estimates near the boundary.** In order to treat the regularity up to the boundary, we will employ a technique borrowed from [19] where it is used in the case $p > 2$. The technique is a suitable adaption of the translation method, to obtain estimates for derivatives taken along the boundary of the domain, and then in the normal direction. We use the following notation for $x \in \mathbf{R}^n$, $x = (x', x_n)$, and also we will denote by ∇' the gradient operator acting only on the first $n-1$-variables. By $B(x, R) \subset \mathbf{R}^n$ we denote the open ball centered at x and of radius $R > 0$, while $B'(x', R) \subset \mathbf{R}^{n-1}$ is the $n-1$-dimensional open ball, centered at x' and of radius R.

By the assumption of regularity $\partial\Omega \in C^2$ it follows that for each $\overline{x} \in \partial\Omega$ there exist $0 < \rho_{\overline{x}} < 1$ and a function $a_{\overline{x}}(x')$ of class $C^2(B'(\overline{x}', \rho_{\overline{x}}))$ such that

$$\max_{x' \in B'(\overline{x}', \rho_{\overline{x}})} |\nabla' a_{\overline{x}}(x')| < \frac{1}{6(n-1)} \frac{p-1}{3-p} \quad \text{and} \quad \nabla' a_{\overline{x}}(0) = 0, \tag{7}$$

and also (after a rigid rotation of axis around the point $\overline{x}$, in such a way that the outward normal unit vector at $\overline{x}$ becomes the vector $(0,\dots,0,-1)$) we define

$$
\begin{aligned}
V^+(\overline{x},\rho_{\overline{x}}) &:= \{(x',x_n):\ x'\in B'(\overline{x}',\rho_{\overline{x}}),\ a_{\overline{x}}(x')<x_n<a_{\overline{x}}(x')+\rho_{\overline{x}}\}\subset\Omega,\\
V^=(\overline{x},\rho_{\overline{x}}) &:= \{(x',x_n):\ x'\in B'(\overline{x}',\rho_{\overline{x}}),\ a_{\overline{x}}(x')=x_n\}\subset\partial\Omega,\\
V^-(\overline{x},\rho_{\overline{x}}) &:= \{(x',x_n):\ x'\in B'(\overline{x}',\rho_{\overline{x}}),\ a_{\overline{x}}(x')-\rho_{\overline{x}}<x_n<a_{\overline{x}}(x')\}\cap\overline{\Omega}=\emptyset,
\end{aligned}
$$

for further details, see Nečas [21]. We define

$$
V(\overline{x},\rho_{\overline{x}}) := V^-(\overline{x},\rho_{\overline{x}})\cup V^=(\overline{x},\rho_{\overline{x}})\cup V^+(\overline{x},\rho_{\overline{x}}),
$$

and we consider the following covering of $\partial\Omega$

$$
\partial\Omega\subset\bigcup_{\overline{x}\in\partial\Omega}V(\overline{x},2^{-3}\rho_{\overline{x}}).
$$

Since $\partial\Omega$ is compact, we can select a finite number of points $\overline{x}_m\in\partial\Omega$, with $m=1,\dots,\mathcal{N}$ such that defining $\rho_m:=\rho_{\overline{x}_m}$ the open sets $\{V(\overline{x}_m,2^{-3}\rho_m)\}_{m=1}^{\mathcal{N}}$ still determine a sub-covering of $\partial\Omega$. The corresponding parametrization of $\partial\Omega\cap V(\overline{x}_m,\rho_m)$ is again such that $|\nabla' a_{\overline{x}_m}(x')|$ satisfies (7) in $B'(\overline{x}'_m,\rho_m)$, for $m=1,\dots,\mathcal{N}$. We define $\rho:=\min\{\rho_m\}_{m=1}^{\mathcal{N}}$. Regarding the interior of Ω, there exists an open set $V_0\subset\subset\Omega$ such that

$$
V_0\cup\big(\cup_{m=1}^{\mathcal{N}}V^+(x_m,\rho_m)\big)=\Omega\quad\text{and}\quad d(\Omega_0,\mathbf{R}^n\backslash\Omega)=\rho_0>0.
$$

We fix $m\in\{1,\dots,\mathcal{N}\}$ (and we will show estimates independent of it) denoting for easiness of notation

$$
U:=V(\overline{x}_m,2^{-1}\rho_m),\qquad \Theta:=V(\overline{x}_m,\rho_m),\qquad\text{and}\qquad a:=a_{\overline{x}_m}(x').
$$

2.1. **Tangential derivatives.** We follow now the approach in [19, 3] and instead of flattening the boundary, we perform suitable derivatives in the directions tangential to the boundary and then we use the equations point-wise to determine all the first order derivatives of ∇u. It is important to observe that in this way the equations are not changed (see subsection 3.1), and this allows to make extremely explicit and precise calculations, exploiting in a very clear way the growth of S and the structure of the equations. The role of the smallness assumption (7) will appear only in the estimates for normal derivatives. We are very complete in all this section, since many details will be needed in a future work to handle more challenging problems, related to non-Newtonian fluids. It seems that some of the explicit calculations we are going to do are not easy to find in literature and they are needed to completely justify some hard regularity results. Moreover, we follow a slightly different track, which we believe is more elementary, and easier to be followed by the reader interested in a thorough check of all needed details.

Let be given $h\neq 0$ such that $|h|<\frac{\rho}{2}$ and $s\in\{1,\dots,n-1\}$. As in [19] we define a mapping $T_{s,h}:\ U\to\Theta$ in the following way

$$
T_{s,h}(x):=(x'+h\,\mathrm{e}_s,x_n+a(x'+h\,\mathrm{e}_s)-a(x')),
$$

where e_s is the sth term of the canonical basis of $\mathbf{R}^n$. Observe that

$$
T_{s,h}(U^+)\subseteq\Theta^+,\qquad T_{s,h}(U^=)\subseteq\Theta^=,\qquad T_{s,h}(U^-)\subseteq\Theta^-,
$$

with obvious notation for U^+, Θ^+ and so on. Moreover $T_{s,h}$ is injective, has derivatives up to the second order, and if $[\nabla T_{s,h}(x)]_{ij} := \partial_j [T_{s,h}(x)]_i$, then

$$\nabla T_{s,h}(x) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ [x]_1 & [x]_2 & \dots & [x]_{n-1} & 1 \end{pmatrix},$$

where the symbol $[x]_j$ denotes terms, which are explicitly given by

$$[x]_j := [\nabla T_{s,h}(x)]_{nj} = \partial_j a(x' + h\,\mathrm{e}_s) - \partial_j a(x'), \qquad j = 1, \dots, n-1.$$

The lower-triangular shape of the matrix of first derivatives of the transformation implies that $\det[\nabla T_{s,h}(x)] \equiv 1$, for all $x \in U$.

Let be given $v : \Theta \to \mathbf{R}$ and $s \in \{1, \dots, n-1\}$, we denote by $\partial_{\tau_s} v$ the tangential derivative defined as follows

$$\partial_{\tau_s} v(x) := \lim_{h \to 0} \frac{v(T_{s,h}(x)) - v(x)}{h} = \partial_s v + \partial_n v(x)\,\partial_s a(x'),$$

where the last equality is valid for smooth functions. (If v is vector valued, the same derivative has to be intended component-wise). We have the following results (see for instance Evans [13] and also Málek, Nečas, Růžička [19])

Lemma 2.1. *If $v \in W^{1,p}(\Theta)$ and $0 < |h| < \rho/2$, then*

$$\int_U \left| \frac{v(T_{s,h}(x)) - v(x)}{h} \right|^p dx \le \|\nabla' a\|^p_{L^\infty(\Theta)} \|\nabla v\|^p_{L^p(\Theta)}.$$

On the other hand, given $v \in L^p(\Theta)$, if for some $0 < h_0 < \rho/2$

$$\exists\, C > 0 : \quad \int_U \left| \frac{v(T_{s,h}(x)) - v(x)}{h} \right|^p dx \le C \qquad \forall\, h \in]-h_0, h_0[\setminus\{0\},$$

then the tangential derivative $\partial_{\tau_s} v$ exists in the sense of distributions and also it holds $\|\partial_{\tau_s} v\|^p_{L^p(U)} \le C$.

We observe that the transformation $T_{s,h}$ generalizes the usual translation operator $\tau_{s,h}(x) := (x' + h\,\mathrm{e}_s, x_n)$ which can be used in the case of the half-space $\mathbf{R}^{n-1}$. Moreover in the interior, that is in V_0, one can also make translations in all the directions, see Section 3.3.

2.2. **Commutation terms.** In the case of the translation operator $\tau_{s,h}$ it is immediate to check that it commutes with all space-derivatives, while certain commutation terms appear with the operator $T_{s,h}$, and these terms (even if of lower order, hence negligible) must be carefully estimated.

Without restrictions we can suppose that $s = 1$ (in the other cases we need just to rename variables) and we write $T_h(x) := T_{1,h}(x)$. Observe that in the last row of $\nabla T_h(x)$ we have, by Taylor formula,

$$\big|[\nabla T_h(x)]_{nj}\big| = |\partial_j a(x' + h\,\mathrm{e}_1) - \partial_j a(x')| \le |h| \|D^2 a\|_{L^\infty(\Theta)}, \qquad j = 1, \dots, n-1.$$

We also recall that if $y = (y', y_n) = T_h(x)$, then $x' = y' - h\,\mathrm{e}_1$ and $x_n = y_n + a(y' - h\,\mathrm{e}_1) - a(y') = y_n + a(x') - a(x' + h\,\mathrm{e}_1)$. Hence, the inverse of T_h is the curvilinear translation in the opposite direction, that is $T_{-h} = (T_h)^{-1}$ and

$$\nabla\big((T_h)^{-1}\big)(y) = \nabla T_{-h}(y) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ \{y\}_1 & \{y\}_2 & \dots & \{y\}_{n-1} & 1 \end{pmatrix},$$

where $\{y\}_j$ denote terms, which are given explicitly by

$$\{y\}_j := [\nabla T_{-h}(y)]_{nj} = \partial_j a(y' - h\,\mathrm{e}_1) - \partial_j a(y'), \qquad j = 1, \dots, n-1,$$

We also define the operator Δ_h acting on functions $v : \Theta \to \mathbf{R}$. The function $\Delta_h v : U \to \mathbf{R}$ is defined on U as follows

$$\Delta_h v(x) := v(T_h(x)) - v(x) \qquad \forall\, x \in U,$$

and observe that since $|h| < \rho/2$, then $T_h(U) \subseteq \Theta$.

As in the usual method of translations, the following formula of integration by parts holds true for all couples of function v, w defined on Θ, with the support contained in U. If $x = T_h(y)$ and $y = T_{-h}(x)$, then

$$\begin{aligned} \int_\Theta v\,(\Delta_{-h} w)\,dx &= \int_\Theta v\,(w \circ T_{-h} - w)\,dx \\ &= \int_{U \cap T_h(U)} v(x)(w(T_{-h}(x))\,dx - \int_U v(x)w(x)\,dx \\ &= \int_{T_{-h}(U) \cap U} v(T_h(y))\,w(y)\,dy - \int_U v(x)w(x)\,dx. \end{aligned}$$

Since the support of both functions is contained in U, if $y \notin T_{-h}(U)$, then $v(T_h(y)) = 0$. Hence, the first integral over $T_{-h}(U) \cap U$ is equal to the same computed over the whole U. Then we have

$$\begin{aligned} \int_\Theta v\,(\Delta_{-h} w)\,dx &= \int_U v(T_h(x))\,w(x)\,dx - \int_U v(x)w(x)\,dx \\ &= \int_\Theta v(T_h(x))\,w(x)\,dx - \int_\Theta v(x)w(x)\,dx \\ &= \int_\Theta (v(T_h(x)) - v(x))\,w(x)\,dx \\ &= \int_\Theta (\Delta_h v)\,w\,dx. \end{aligned} \tag{8}$$

This formula is formally the same as in the case of standard Nirenberg translations [22].

Remark 2.2. In the sequel we will use in an extensive way that the latter formula is still valid when *at least one* of the two functions has support contained in U.

We also observe that the composition of the translation operator Δ_h with gradients leads to the following formula

$$\begin{aligned} \nabla(\Delta_h \psi) &= \nabla(\psi \circ T_h) - \nabla\psi = \nabla\psi_{|T_h} \nabla T_h - \nabla\psi \\ &= \nabla\psi_{|T_h} \nabla T_h - \nabla\psi_{|T_h} + \nabla\psi_{|T_h} - \nabla\psi \\ &= \nabla\psi_{|T_h} (\nabla T_h - \mathrm{I}) + \Delta_h(\nabla\psi). \end{aligned} \tag{9}$$

This shows that the commutator $[\nabla, \Delta_h](\psi) = \nabla\psi_{|T_h}(\nabla T_h - \mathrm{I})$ is non-vanishing, but it can be easily estimated. Due to the particular expression of the transformation T_h, and by using Taylor formula we have

$$\|\nabla T_h - \mathrm{I}\|_{L^\infty(U)} \le \|D^2 a\|_{L^\infty(\Theta)}|h|, \qquad \text{for all} \quad |h| < \frac{\rho}{2}. \tag{10}$$

3. **Proof of Theorem 1.2.** In this section we prove the main result of the paper and the proof is split into three-different group of estimates. The first relevant point is that we need to change the reference frame by sending the origin into the point $\overline{x}_m$, for $m = 1, \dots, \mathcal{N}$. Then, we need to apply a rotation in order to have a reference frame (x', x_n) as in the previous section. To do this it is relevant that the system is invariant by these transformations.

3.1. **Rotation invariance.** Let us consider a rotation of the axes $y = Rx$ where $R = (r_{ij})$ is a $n \times n$ matrix such that $R^T R = I$ that is $r_{kj} r_{sj} = \delta_{ks}$. We remark that $\frac{\partial R_k}{\partial x_j}(x) = r_{kj}$ and $|\det(\nabla R)| = 1$.

For a generic weakly differentiable function $v(y)$ we set $u(x) = v(Rx)$ hence we have

$$(\nabla u(x))_{ij} = \frac{\partial u_i}{\partial x_j}(x) = \frac{\partial v_i}{\partial y_k}(Rx)\frac{\partial R_k}{\partial x_j}(x) = (\nabla v(Rx))_{ik} r_{kj}$$

We remark that the transformation preserves the modulus of the gradient, indeed

$$\begin{aligned}|\nabla u(x)|^2 = (\nabla u(x))_{ij}(\nabla u(x))_{ij} &= (\nabla v(Rx))_{ik} r_{kj} (\nabla v(Rx))_{is} r_{sj} \\ &= (\nabla v(Rx))_{ik}(\nabla v(Rx))_{is}\delta_{ks} = |\nabla v(Rx)|^2\end{aligned}$$

For a generic test function $\phi(x) \in W_0^{1,2}(\Omega)$ we set $\psi(y) = \phi(R^T y)$ hence we have $\psi \in W_0^{1,2}(R(\Omega))$,

$$\nabla\psi(y) = \nabla\phi(R^T y) R^T$$

and multiplying on the right by R

$$(\nabla\phi(R^T y))_{ij} = ((\nabla\psi(y)R)_{ij} = (\nabla\psi(y))_{is} r_{sj}.$$

Hence

$$\begin{aligned}
&\int_\Omega \left[\nu_0 + \nu_1(\mu + |\nabla u(x)|^2)^{\frac{p-2}{2}}\right] \nabla u(x)\cdot\nabla\phi(x)\,dx \\
&= \int_\Omega \left[\nu_0 + \nu_1(\mu + |\nabla u(x)|^2)^{\frac{p-2}{2}}\right] (\nabla u(x))_{ij}(\nabla\phi(x))_{ij}\,dx \\
&= \int_\Omega \left[\nu_0 + \nu_1(\mu + |\nabla v(Rx)|^2)^{\frac{p-2}{2}}\right] (\nabla v(Rx))_{ik} r_{kj}(\nabla\phi(x))_{ij}\,dx \\
&= \int_{R(\Omega)} \left[\nu_0 + \nu_1(\mu + |\nabla v(y)|^2)^{\frac{p-2}{2}}\right] (\nabla v(y))_{ik} r_{kj}(\nabla\phi(R^T y))_{ij}|\det(\nabla R)|\,dy \\
&= \int_{R(\Omega)} \left[\nu_0 + \nu_1(\mu + |\nabla v(y)|^2)^{\frac{p-2}{2}}\right] (\nabla v(y))_{ik} r_{kj}(\nabla\psi(y))_{is} r_{sj}\,dy \\
&= \int_{R(\Omega)} \left[\nu_0 + \nu_1(\mu + |\nabla v(y)|^2)^{\frac{p-2}{2}}\right] (\nabla v(y))_{ik}(\nabla\psi(y))_{is}\delta_{ks}\,dy \\
&= \int_{R(\Omega)} \left[\nu_0 + \nu_1(\mu + |\nabla v(y)|^2)^{\frac{p-2}{2}}\right] (\nabla v(y))\cdot(\nabla\psi(y))\,dy
\end{aligned}$$

Considering the right-hand side of the equation we have

$$\int_\Omega f(x)\cdot\phi(x)\,dx = \int_{R(\Omega)} f(R^T y)\cdot\phi(R^T y)|\det(\nabla R)|\,dy = \int_{R(\Omega)} \tilde{f}(y)\cdot\psi(y)\,dy$$

where we have set $\tilde{f}(y) = f(R^T y)$. Hence, if u is a weak solution of the system with force $f \in L^2(\Omega)$ then v is a weak solution of the same type of system with force $\tilde{f} \in L^2(R(\Omega))$. This is what we mean saying that the system is rotation invariant.

Remark 3.1. We want to observe that what we have just described is not an invariance property in the physical sense. Indeed, the correct transformation would be $u(x) = R^T v(Rx)$ but unfortunately the system is not invariant for this kind of change of variables. Roughly speaking we are reading the domain in a reference frame and the function in a different one producing a sort of distortion. This does not affect our system in its mathematical aspects and the main reason is that there is no interaction between the image of the function and the domain. The scene changes dramatically if one deals for instance with Neumann boundary conditions or if in the system there is a transport term, like in Navier-Stokes equations. This is the main reason why our result does not hold without changes to fluid mechanical problems. It has to be remarked that the quantity physically relevant is not ∇u but $Du = \frac{1}{2}(\nabla u + \nabla u^T)$ and making this change in our system makes it rotation invariant in the physical sense.

3.2. **Estimates on tangential derivatives.** In this section we obtain estimates on tangential derivatives. First, with a slight abuse of notation, we denote by u the weak solution of (1) extended by zero for all $x \in \mathbf{R}^n \backslash \Omega$, in such a way that $u \in W^{1,2}(\mathbf{R}^n)$ and formulæ from the previous section using integrals over Θ are well-defined (recall that $\Theta \cap (\mathbf{R}^n \backslash \overline{\Omega}) \neq \emptyset$).

Next, note that we essentially would like to use $\Delta_{-h}(\Delta_h u)$ as test function, but this function does not belong to $W_0^{1,2}(\Omega)$ even if $u \in W_0^{1,2}(\Omega)$, hence we have to localize it with a suitable cut-off function. As in the previous section, we fix $m \in \{1, \dots, \mathcal{N}\}$ and we denote $a(x') := a_{\overline{x}_m}(x')$ as before and we define

$$\Xi := V(\overline{x}_m, 2^{-3}\rho_m), \quad U := V(\overline{x}_m, 2^{-1}\rho_m), \quad \text{and} \quad \Theta := V(\overline{x}_m, \rho_m).$$

We fix a function $0 \leq \xi \leq 1$ of class $C_0^\infty(\Theta)$ such that

$$\xi(x) = \begin{cases} 1 & \forall\, x \in \Xi, \\ 0 & \forall\, x \in \Theta \backslash V(\overline{x}_m, 2^{-2}\rho_m), \end{cases}$$

We then use as test function ϕ in the weak formulation the function

$$\phi(x) := \begin{cases} \Delta_{-h}(\xi^2\, \Delta_h u) & x \in U^+, \\ 0 & x \in \Omega \backslash U^+, \end{cases}$$

with $|h| < \rho_m/8$. Observe that $\phi \in W_0^{1,2}(\Omega)$, but more precisely, due to the choice of the open sets Ξ, U, Θ, and the limitation on h, we have

$$\phi \in W_0^{1,2}(\Theta), \quad \text{with} \quad \operatorname{supp} \phi \subset U^+.$$

Remark 3.2. In the definition of ϕ we have been deliberately somewhat sloppy, since we think that the reader can fill the missing details. Anyway, to be really precise, denoting by a bar the restriction of any function to U^+, we should first consider $\Delta_h(\overline{u})$. Then extend it to 0 for all $x \in \Theta \backslash (U^+ \cup T_h(U^+))$, denoting the extension by $\mathcal{E}_1(\Delta_h(\overline{u}))$. Then $\overline{\xi^2 \mathcal{E}_1(\Delta_h(\overline{u}))} = \xi^2 \mathcal{E}_1(\Delta_h(\overline{u}))_{|U^+}$ is the further restricted function on which we can apply the translation $\Delta_{-h}(\overline{\xi^2 \mathcal{E}_1(\Delta_h(\overline{u}))})$. Finally, we have to extend the latter to 0 for all $x \in \Theta \backslash (T_{-h}(U^+) \cup U^+)$, denoting it $\mathcal{E}_2(\Delta_{-h}(\overline{\xi^2 \mathcal{E}_1(\Delta_h(\overline{u}))}))$.

Observe that since $u_{|\Theta^=} \equiv 0$ (because $u \in W_0^{1,2}(\Omega)$), since $T_h(U^=) \subset \Theta^=$, and due to the choice of the cut-off function, the support of ϕ is contained in U^+.

We now perform the very basic manipulations and integration by parts which will be used several times in the sequel. We employ the function ϕ in the weak formulation (6) and we then obtain

$$\nu_0 \int_\Theta \nabla u \cdot \nabla(\Delta_{-h}(\xi^2\, \Delta_h u))\, dx + \nu_1 \int_\Theta S(\nabla u) \cdot \nabla(\Delta_{-h}(\xi^2\, \Delta_h u))\, dx \\ = \int_\Theta f \Delta_{-h}(\xi^2\, \Delta_h u)\, dx.$$

With suitable manipulations, we will see that, collecting leading terms end commutation terms, coming from the weak formulation, we obtain

$$\nu_0 \int_\Theta \xi^2 |\Delta_h(\nabla u)|^2\, dx + \nu_1 \int_\Theta \xi^2(x) \left(S(\nabla u_{|T_h(x)}) - S(\nabla u_{|x})\right) \cdot (\nabla u_{|T_h(x)} - \nabla u_{|x}) dx \\ = \int_\Theta f \Delta_{-h}(\xi^2\, \Delta_h u)\, dx + \text{“lower-order terms”},$$

where in the lower-order terms there are many integrals due to the presence of the cut-off functions and due to the fact that Δ_h and differentiation do not commute. In the flat case of $\Omega = \mathbf{R}^n_+$ all these lower-order terms vanish, while we need here to check them very carefully, but the reader can understand that the most important terms are the other ones written in the latter equality.

3.2.1. *Estimate for the linear part.* We manipulate the first integral of the weak formulation by using the commutation formula (9) to obtain

$$\int_\Theta \nabla u \cdot \nabla(\Delta_{-h}(\xi^2\, \Delta_h u))\, dx = \int_\Theta \nabla u \cdot \Delta_{-h} \left(2\, \xi\, \nabla\xi\, \Delta_h u + \xi^2 \nabla(\Delta_h u)\right)\, dx \\ + \int_\Theta \nabla u \cdot \left[\nabla(\xi^2 \Delta_h u)_{|T_{-h}(x)}\, [\nabla T_{-h} - \mathrm{I}]\,\right] dx.$$

Next, we observe that in the first integral from the right-hand side of the latter formula (thanks to the cut-off function) we are in the hypotheses of Remark 2.2, hence we can apply (8) to the restriction of ∇u to U^+ and integrate by parts using also (9), to obtain

$$\begin{aligned} &\int_\Theta \nabla u \cdot \Delta_{-h} \left(2\, \xi\, \nabla\xi\, \Delta_h u + \xi^2 \nabla(\Delta_h u)\right)\, dx \\ &= \int_\Theta \Delta_h(\nabla u) \cdot \left(2\xi\, \nabla\xi\, \Delta_h u + \xi^2 \nabla(\Delta_h u)\right)\, dx \\ &= \int_\Theta \Delta_h(\nabla u) \cdot (2\xi\, \nabla\xi\, \Delta_h u)\, dx + \int_\Theta \xi^2 |\Delta_h(\nabla u)|^2\, dx \\ &\qquad + \int_\Theta \xi^2 \Delta_h(\nabla u) \cdot \nabla u_{|T_h(x)}\, [\nabla T_h - \mathrm{I}]\, dx. \end{aligned}$$

Hence, be defining

$$\mathcal{R}_1 := \int_\Theta \nabla u \cdot \left[\nabla(\xi^2 \Delta_h u)_{|T_{-h}(x)}\, [\nabla T_{-h} - \mathrm{I}]\right]\, dx,$$

$$\mathcal{R}_2 := \int_\Theta \xi^2 \Delta_h(\nabla u) \cdot \nabla u_{|T_h(x)} \cdot [\nabla T_h - \mathrm{I}]\, dx,$$

$$\mathcal{R}_3 := \int_\Theta \Delta_h(\nabla u) \cdot (2\xi\,\nabla\xi\,\Delta_h u)\,dx,$$

we have finally

$$\nu_0 \int_\Theta \nabla u \cdot \nabla(\Delta_{-h}(\xi^2\,\Delta_h u))\,dx = \nu_0 \int_\Theta \xi^2 |\Delta_h(\nabla u)|^2\,dx + \nu_0(\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3).$$

We now carefully estimate the three integrals coming from commutation terms as follows, by using also (10) and the bounds

$$\|\xi\|_\infty \le 1 \qquad \text{and} \qquad \|\nabla\xi\|_\infty \le \frac{8}{\rho_m} \le \frac{8}{\rho}.$$

For the first one we have

$$\begin{aligned}
|\mathcal{R}_1| &\le c(a)|h|\,\|\nabla u\|_2 \|\nabla(\xi^2 \Delta_h u)\|_2 \\
&\le c(a)|h|\,\|\nabla u\|_2 \left\{2\|\nabla\xi\Delta_h u\|_2 + \|\xi^2\nabla(\Delta_h u)\|_2\right\} \\
&\le c(a)|h|\,\|\nabla u\|_2 \left\{\frac{16|h|}{\rho}\|\nabla u\|_2 + \|\xi^2\Delta_h(\nabla u)\|_2 + \|\xi^2\nabla u_{|T_h(x)}[\nabla T_h - \mathrm{I}]\|_2\right\} \\
&\le c(a)h^2\|\nabla u\|_2^2 + c(a)|h|\,\|\nabla u\|_2\|\xi\,\Delta_h(\nabla u)\|_2 \\
&\le \frac{1}{4}\|\xi\,\Delta_h(\nabla u)\|_2^2 + c(a)h^2\|\nabla u\|_2^2,
\end{aligned}$$

where all norms are evaluated over Θ and with $c(a)$ we denote a constant depending only on $\|D^2 a\|_\infty$, hence on the smoothness of $\partial\Omega$.

Next, we have

$$|\mathcal{R}_2| \le c(a)|h|\,\|\nabla u\|_2\|\xi\,\Delta_h(\nabla u)\|_2 \le c(a)h^2\|\nabla u\|_2^2 + \frac{1}{4}\|\xi\,\Delta_h(\nabla u)\|_2^2.$$

Concerning the last term, we have

$$|\mathcal{R}_3| \le c(a)|h|\,\|\nabla u\|_2\,\frac{1}{\rho}\|\xi\Delta_h(\nabla u)\|_2 \le c(a)h^2\|\nabla u\|_2^2 + \frac{1}{4}\|\xi\Delta_h(\nabla u)\|_2^2.$$

Hence, by collecting all estimates we get

$$\frac{\nu_0}{4}\|\xi\,\Delta_h(\nabla u)\|_{L^2(\Theta^+)}^2 \le \nu_0 \int_\Theta \nabla u \cdot \nabla(\Delta_{-h}(\xi^2\,\Delta_h u))\,dx + c(a)\nu_0 h^2\|\nabla u\|_{L^2(\Omega)}^2. \tag{11}$$

3.2.2. *Estimate for the monotone term.* We now use the same tools to handle the other term coming from the extra stress tensor. First, in analogy with standard notation used in the context of p-Laplace and p-fluids we define the following quantity

$$\mathcal{I}_h(u) := \int_\Theta \xi^2(x)(\mu + |\nabla u_{|T_h(x)}| + |\nabla u_{|x}|)^{p-2}|\nabla u_{|T_h(x)} - \nabla u_{|x}|^2\,dx,$$

which is the natural one coming out when testing the equations and it is related with the coercivity of the problem. Standard calculations lead us to the following equality

$$\begin{aligned}
\int_\Theta S(\nabla u)\cdot\nabla(\Delta_{-h}(\xi^2\,\Delta_h u))\,dx &= \int_\Theta S(\nabla u)\cdot\Delta_{-h}\left(2\,\xi\,\nabla\xi\,\Delta_h u + \xi^2\nabla(\Delta_h u)\right)\,dx \\
&\quad + \int_\Theta S(\nabla u)\cdot\left(\nabla(\xi^2\Delta_h u)_{|T_{-h}(x)}\,[\nabla T_{-h} - \mathrm{I}]\right)\,dx \\
&=: I_1 + I_2,
\end{aligned}$$

and, as before,

$$
\begin{aligned}
I_1 &= \int_\Theta \Delta_h\left(S(\nabla u)\right)\cdot\left(2\,\xi\,\nabla\xi\,\Delta_h u+\xi^2\nabla(\Delta_h u)\right)\,dx\\
&= \int_\Theta \Delta_h\left(S(\nabla u)\right)\cdot\xi^2\nabla(\Delta_h u)\,dx+\int_\Theta S(\nabla u)\cdot\Delta_{-h}(2\,\xi\,\nabla\xi\,\Delta_h u)\,dx\\
&=: I_3+I_5.
\end{aligned}
$$

The main term is I_3, which is handled as follows:

$$
\begin{aligned}
I_3 &= \int_\Theta \Delta_h\left(S(\nabla u)\right)\cdot\xi^2\Delta_h(\nabla u)\,dx+\int_\Theta \Delta_h\left(S(\nabla u)\right)\cdot\xi^2\nabla u_{|T_h(x)}\left[\nabla T_h-\mathrm{I}\right]\,dx\\
&=: I_4+I_6.
\end{aligned}
$$

Next, by the assumption (5) on the stress-tensor S

$$
\begin{aligned}
I_4 &= \int_\Theta \Delta_h\left(S(\nabla u)\right)\cdot\xi^2\Delta_h(\nabla u)\,dx\\
&= \int_\Theta \xi^2(x)\left(S(\nabla u_{|T_h(x)})-S(\nabla u_{|x})\right)\cdot(\nabla u_{|T_h(x)}-\nabla u_{|x})\,dx\\
&\geq \gamma_2\int_\Theta \xi^2(x)(\mu+|\nabla u_{|T_h(x)}|+|\nabla u_{|x}|)^{p-2}|\nabla u_{|T_h(x)}-\nabla u_{|x}|^2\,dx\\
&= \gamma_2\mathcal{I}_h(u).
\end{aligned}
$$

The other integral is estimated as follows, by using the growth assumption (4) on S, Hölder inequality (for any $\epsilon>0$) and (10)

$$
\begin{aligned}
|I_6| &= \left|\int_\Theta \Delta_h\left(S(\nabla u)\right)\cdot\xi^2\nabla u_{|T_h(x)}\left[\nabla T_h-\mathrm{I}\right]\,dx\right|\\
&\leq \gamma_1 c(a)|h|\int_\Theta \xi^2(\mu+|\nabla u_{|T_h(x)}|+|\nabla u_{|x}|)^{p-2}|\nabla u_{|T_h(x)}-\nabla u_{|x}|\,|\nabla u_{|T_h(x)}|\,dx\\
&\leq \gamma_1 c(a)|h|\int_\Theta \xi^2(\mu+|\nabla u_{|T_h(x)}|+|\nabla u_{|x}|)^{p-1}|\nabla u_{|T_h(x)}-\nabla u_{|x}|\,dx\\
&\leq \gamma_1 c(a)|h|\int_\Theta(\mu+|\nabla u_{|T_h(x)}|+|\nabla u_{|x}|)^{\frac{p-2}{2}}\xi\,|\Delta_h(\nabla u)|(\mu+|\nabla u_{|T_h(x)}|+|\nabla u_{|x}|)^{\frac{p}{2}}\,\xi\,dx\\
&\leq \epsilon\int_\Theta(\mu+|\nabla u_{|T_h(x)}|+|\nabla u_{|x}|)^{p-2}\xi^2|\Delta_h(\nabla u)|^2\,dx+\frac{c(a)\,h^2}{\epsilon}(\mu^p+\|\nabla u\|_p^p)\\
&= \epsilon\,\mathcal{I}_h(u)+\frac{c(a)h^2}{\epsilon}(\mu^p+\|\nabla u\|_p^p).
\end{aligned}
$$

We next estimate I_2 as follows (by using again commutation formulæ, (10) and also Hölder inequality)

$$
\begin{aligned}
|I_2| &\leq c(a)|h|\int_\Theta|S(\nabla u)|\xi^2|\nabla(\Delta_h(u))|\,dx+c(a)|h|\int_\Theta|S(\nabla u)|2\,\xi|\nabla\xi|\,|\Delta_h(u)|\,dx\\
&\leq c(a)k_3|h|\int_\Theta|\nabla u|^{p-1}\xi^2|\Delta_h(\nabla u)|\,dx\\
&\quad+c(a)k_3|h|\int_\Theta\left(|\nabla u|^{p-1}+1\right)\xi^2|\nabla u_{|T_h}\left[\nabla T_h-\mathrm{I}\right]|\,dx\\
&\quad+\frac{k_3c(a)}{\rho}h^2\int_\Theta\left(|\nabla u|^{p-1}+1\right)\left|\frac{\Delta_h(u)}{h}\right|\,dx+c(a)k_3|h|\int_\Theta\xi^2|\Delta_h(\nabla u)|\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le c(a)k_3|h|\int_\Theta (\mu+|\nabla u_{|T_h}|+|\nabla u|)^{\frac{p-1}{2}}\xi|\Delta_h(\nabla u)|\xi\left(\mu+|\nabla u_{|T_h}|+|\nabla u|\right)^{\frac{p}{2}}\,dx\\
&\qquad + c(a)k_3h^2\int_\Theta \left(|\nabla u|^{p-1}+1\right)|\nabla u_{|T_h(x)}|\,dx + c(a)k_3h^2\|\nabla u\|_p^p\\
&+c(a)k_3|h|\int_\Theta \xi|\Delta_h(\nabla u)|\left(\mu+|\nabla u_{|T_h}|+|\nabla u|\right)^{\frac{p-2}{2}}\xi\left(\mu+|\nabla u_{|T_h}|+|\nabla u|\right)^{\frac{2-p}{2}}\,dx\\
&\le \epsilon\,\mathcal{I}_h(u)+c(a)k_3h^2\left(1+\frac{1}{\epsilon}\right)\left(\|\nabla u\|_p^p+\|\nabla u\|_p\right).
\end{aligned}
$$

Concerning the term I_5 we observe that

$$
\begin{aligned}
&\|\xi\Delta_h(\nabla u)\|_p^p =\\
&\int_\Theta |\xi|^p|\Delta_h(\nabla u)|^p(\mu+|\nabla u_{|T_h(x)}|+|\nabla u_{|x}|)^{\frac{(p-2)p}{2}}(\mu+|\nabla u_{|T_h(x)}|+|\nabla u_{|x}|)^{\frac{(2-p)p}{2}}\,dx\\
&\qquad\qquad \le c\,\mathcal{I}_h(u)^{\frac{p}{2}}\left(\mu+\|\nabla u\|_p\right)^{\frac{(2-p)p}{2}},
\end{aligned}
$$

hence

$$
\begin{aligned}
|I_5| &\le \|S(\nabla u)\|_{p'}\|\Delta_{-h}(2\xi\nabla\xi\Delta_h u)\|_p\\
&\le k_3\left(\|\nabla u\|_p^{p-1}+|\Theta|^{\frac{p-1}{p}}\right)|h|\,c(a)\|\nabla(2\xi\nabla\xi\Delta_h u)\|_p\\
&\le c(a)k_3|h|\,\left(\|\nabla u\|_p^{p-1}+1\right)\left[\frac{1}{\rho^2}\|\Delta_h u\|_p+\frac{1}{\rho}\|\xi\nabla(\Delta_h u)\|_p\right]\\
&\le\frac{c(a)k_3|h|}{\rho}\left(\|\nabla u\|_p^{p-1}+1\right)\left[\frac{|h|c(a)\|\nabla u\|_p}{\rho}+\|\xi\Delta_h(\nabla u)\|_p+\|\xi(\nabla T_h-\mathrm{I})\nabla u_{|T_h}\|_p\right]\\
&\le\left(\frac{1}{\rho^2}+\frac{1}{\rho}\right)c(a)k_3\,h^2\left(\|\nabla u\|_p^p+1\right)+\epsilon\,\mathcal{I}_h(u)+\frac{c(a)k_3^2}{\epsilon\rho^2}h^2\left(\|\nabla u\|_p^p+\mu^p+1\right),
\end{aligned}
$$

for all $\epsilon>0$.

By choosing $0<\epsilon$ small enough, say $\epsilon<\gamma_2/8$, we have that

$$
\begin{aligned}
\int_\Theta S(\nabla u)\,\nabla(\Delta_{-h}(\xi^2\,\Delta_h u))\,dx &\ge I_4-|I_6|-|I_5|-|I_2|\\
&\ge\frac{\gamma_2}{2}\mathcal{I}_h(u)-ch^2\left(\|\nabla u\|_p^p+\mu^p+1\right),
\end{aligned}
\tag{12}
$$

where the constant c is independent of u and depends essentially on $\|a\|_{C^2}$.

3.2.3. *The external force.* The external force is estimated as follows

$$
\begin{aligned}
&\left|\int_\Theta f\Delta_{-h}(\xi^2\,\Delta_h u)\,dx\right|\le c(a)|h|\,\|f\|_2\|\nabla(\xi^2\,\Delta_h u)\|_2\\
&\le c(a)|h|\,\|f\|_2\left\{\|2\xi\nabla\xi\,\Delta_h u\|_2+\|\xi^2\,\nabla(\Delta_h u)\|_2\right\}\\
&\le c(a)|h|\,\|f\|_2\left\{\frac{|h|}{\rho}\|\nabla u\|_2+\|\xi^2\,\Delta_h(\nabla u)+\xi^2(\nabla u_{|T_h}[\nabla T_h-\mathrm{I}])\|_2\right\}\\
&\le c(a)|h|\,\|f\|_2\left\{\frac{|h|}{\rho}\|\nabla u\|_2+\|\xi\,\Delta_h(\nabla u)\|_2+|h|\,\|\nabla u\|_2\right\}.
\end{aligned}
$$

Hence, we get

$$
\left|\int_\Theta f\Delta_{-h}(\xi^2\,\Delta_h u)\,dx\right|\le\frac{\nu_0}{8}\|\xi\,\Delta_h(\nabla u)\|_2^2+\frac{c(a)h^2}{\nu_0}\|f\|_2^2+c(a)h^2\|\nabla u\|_2^2.\tag{13}
$$

By collecting estimates (11)-(12)-(13), we obtain

$$\frac{\nu_0}{8}\|\xi\,\Delta_h(\nabla u)\|_2^2 + \nu_1\gamma_2\mathcal{I}_h(u) \leq \frac{h^2}{\nu_0}\|f\|_2^2 + c(a)\nu_1 h^2\left(\|\nabla u\|_2^2 + \mu^p + 1\right). \tag{14}$$

Hence dividing by h^2 we get that

$$\nu_0\left\|\xi\,\frac{\Delta_h(\nabla u)}{h}\right\|_2^2 \leq \frac{c(a)}{\nu_0}\|f\|_2^2 + c(a)\nu_1\left(\|\nabla u\|_2^2 + \mu^p + 1\right) =: C,$$

where C is a bounded function in terms of its arguments $\|f\|_2$, h, ν_1, and $\|a\|_{C^2}$. The constant C depends in a critical way only on ν_0. This implies that on Ξ^+, where the cut-off function satisfies $\xi \equiv 1$, there exists the tangential derivative $\partial_\tau\nabla u \in L^2(\Xi^+)$ and its norm is estimated by C, which depends on the data of the problem. Since estimates on $\Xi = \Xi_m$ are uniform with respect to $m = 1,\dots,\mathcal{N}$ and since $\cup_{m=1}^{\mathcal{N}}\Xi_m = \cup_{m=1}^{\mathcal{N}}V(\overline{x}_m, 2^{-3}\rho_m)$ represents a covering of $\partial\Omega$, these facts imply that in a neighborhood of the boundary all the tangential derivatives of ∇u belong to L^2. We have then proved the following proposition

Proposition 3.3. *Let be given $f \in L^2(\Omega)$, then there exists a constant $\widetilde{C}$, independent of u and of s such that*

$$\|\partial_{\tau_s}\nabla u\|_{L^2(\Xi_m^+)} \leq \frac{\widetilde{C}}{\nu_0}\|f\|_{L^2(\Omega)} \qquad m = 1,\dots,\mathcal{N}, \tag{15}$$

for all tangential derivatives ∂_{τ_s} associated with the transformation $T_{s,h}$, for $s = 1,\dots,n-1$.

3.3. **Interior estimates.** The interior estimates follow now easily from the previous calculations.

We believe that these estimates can be found somewhere in literature, but since we did not find appropriate references, for the sake of completeness, we briefly describe how to get them. We stress that the knowledge of all second derivatives in the interior, will be crucial for the following. Let be given any $\lambda > 0$ and let the open set Ω_λ be defined as follows

$$\Omega_\lambda := \{x \in \Omega : \ d(x, \partial\Omega) > \lambda\}.$$

Then, for any $0 \neq h \in]-\lambda/4, \lambda/4[$, the following transformation is well defined for $s \in \{1,\dots,n\}$, as a mapping $\tau_{s,h}: \ \Omega_\lambda \to \Omega$

$$\tau_{s,h}(x) := (x + h\,\mathrm{e}_s),$$

where e_s is the sth term of the canonical basis of $\mathbf{R}^n$. These are the usual translations in all directions and we also define

$$(\delta_h f)(x) := f(\tau_h(x)) - f(x) \qquad \forall\, x \in \Omega_\lambda.$$

We observe that now the operator δ_h commutes with spatial derivatives, hence relations in the previous section become much easier. Take then a smooth function $\xi \in C_0^\infty(\Omega)$ such that

$$\xi(x) = \begin{cases} 1 & \forall\, x \in \Omega_\lambda, \\ 0 & \forall\, x \in \Omega\backslash\Omega_{\lambda/2}. \end{cases}$$

By using as test function

$$\phi(x) = \begin{cases} \delta_{-h}(\xi^2\,\delta_h u) & x \in \Omega_{\lambda/2}, \\ 0 & x \in \Omega\backslash\Omega_{\lambda/2}, \end{cases}$$

for $0 \neq h \in]-\lambda/8, \lambda/8[$, we can easily deduce, re-doing previous calculations, that

$$\nu_0 \| \delta_h(\nabla u)\|^2_{L^2(\Omega_\lambda)} \leq \frac{h^2}{\nu_0}\|f\|_2^2 + c\nu_1 h^2 \left(\|\nabla u\|_2^2 + \mu^p + 1\right).$$

This proves that for any compact set $\Omega' \subset\subset \Omega$

$$\partial_{ij} u \in L^2(\Omega') \qquad \forall\, i, j = 1, \dots, n.$$

Observe that the norm of the second order derivatives is not bounded uniformly (at this stage) since the constants in the above estimate depend on the choice of the cut-off function ξ which gradient increases when Ω_λ gets closer and closer to $\partial\Omega$.

Anyway this is enough to prove that $u \in W^{2,2}_{loc}(\Omega)$. Moreover for any $0 < h < \lambda$ and $1 \leq s \leq n$, by (4) we have the estimate

$$\begin{aligned}
&\int_{\Omega_{2\lambda}} |S(Du(x + he_s)) - S(Du(x)|^2 \, dx \\
&\leq \gamma_1^2 \int_{\Omega_{2\lambda}} (\mu + |Du(x + he_s)| + |Du(x)|)^{2(p-2)} \, |Du(x + he_s) - Du(x)|^2 \, dx \\
&\qquad\qquad\qquad\qquad \leq \gamma_1^2 \mu^{2(p-2)} h^2 \|D^2 u\|^2_{2,\Omega_\lambda}.
\end{aligned}$$

It follows that $\partial_s S(Du) \in W^{1,2}(\Omega_{2\lambda})$ for any $s = 1, \dots, n$ and for any $\lambda > 0$. In particular $\nabla \cdot S(Du)$ is a function il $L^2_{loc}(\Omega)$ and we can write system (1) almost everywhere in Ω. This concludes the proof of Theorem 1.4, which will be of paramount importance in the following subsection.

3.4. **Estimates for the second order "normal derivative".**

In this subsection we will consider only the case

$$S(A) = (\mu + |A|^2)^{\frac{p-2}{2}} A, \quad \mu > 0.$$

A very relevant point is that we are considering a non-degenerate stress tensor, that is we are treating the case $\mu > 0$. With this assumption, Theorem 1.4 allows us to write point-wisely the equations, in order to recover regularity of the normal derivatives and also of all other second order partial derivatives, near to the boundary. In this context near to the boundary means that we will prove not only that $\partial_{ij} u(x) \in L^2_{loc}(\Omega)$, but also that $\partial_{ij} u(x) \in L^2(\Omega)$, and this will be obtained by showing that the estimates are uniform in $\Omega' \subset\subset \Omega$. So the knowledge of the interior regularity plays a fundamental role in the proof.

We estimate now the quantity $\partial_{nn} u$ (which we call "normal derivative" since it corresponds to the normal derivative at $x = 0$) close to the boundary and in the reference frame we use in Θ. We use the name "normal derivative" since in the flat case this operation corresponds to the normal derivative.

We recall that in Ξ^+ we proved existence of $\partial_\tau u$ as function of $L^2(\Xi^+)$. We now prove, in the reference frame corresponding to the definition of Ξ^+ that $\partial_{nn} u \in L^2(\Xi^+)$). First observe that, since we know the interior regularity, we can write the equations point-wise for a.e. $x \in \Theta^+ \subset \Omega$ and we obtain, for any $i = 1, \dots, N$

$$\sum_{j,k=1}^{n} \left[\nu_0 + \nu_1 (\mu + |\nabla u|^2)^{\frac{p-2}{2}}\right] \partial_{jj} u_i + (p-2)\nu_1 (\mu + |\nabla u|^2)^{\frac{p-4}{2}} \partial_{jk} u_i \, \partial_k u_i \, \partial_j u_i = -f_i.$$

We can explicitly write the normal derivative as follows

$$\begin{aligned}&\Big[\nu_0+\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}+(p-2)\nu_1(\mu+|\nabla u|^2)^{\frac{p-4}{2}}(\partial_n u_i)^2\Big]\,\partial_{nn}u_i\\&\qquad=-\Big[\nu_0+\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big]\sum_{j\neq n}\partial_{jj}u_i\\&\qquad\quad+(2-p)\nu_1(\mu+|\nabla u|^2)^{\frac{p-4}{2}}\left[\sum_{(j,k)\neq(n,n)}\partial_{jk}u_i\partial_k u_i\,\partial_j u_i\right]-f_i.\end{aligned}$$

We observe now that, since $1<p<2$, then both $p-2<0$ and $p-1>0$, hence

$$\begin{aligned}&\nu_0+\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}+(p-2)\nu_1(\mu+|\nabla u|^2)^{\frac{p-4}{2}}(\partial_n u_i)^2\\&\quad\geq\nu_0+\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}+(p-2)\nu_1(\mu+|\nabla u|^2)^{\frac{p-4}{2}}|\nabla u|^2\\&\quad\geq\nu_0+(p-1)\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\\&\quad\geq\nu_0>0.\end{aligned}$$

We then obtain

$$\begin{aligned}&\Big[\nu_0+(p-1)\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big]\;|\partial_{nn}u_i|\\&\quad\leq\Big[\nu_0+\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big]\sum_{j\neq n}|\partial_{jj}u_i|\\&\qquad\quad+(2-p)\nu_1(\mu+|\nabla u|^2)^{\frac{p-4}{2}}|\nabla u|^2\Big(\sum_{(j,k)\neq(n,n)}|\partial_{jk}u_i|\Big)+|f|\\&\quad\leq\Big[\nu_0+\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big]\sum_{j\neq n}|\partial_{jj}u_i|\\&\qquad\quad+(2-p)\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big(\sum_{(j,k)\neq(n,n)}|\partial_{jk}u_i|\Big)+|f|,\end{aligned}$$

so finally

$$\begin{aligned}&\Big[\nu_0+(p-1)\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big]\;|\partial_{nn}u_i|\\&\quad\leq\Big[\nu_0+(3-p)\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big]\Big(\sum_{(j,k)\neq(n,n)}|\partial_{jk}u_i|\Big)+|f|.\end{aligned}\tag{16}$$

We set, for a fixed $i\in\{1,\dots,N\}$

$$g_{rs}:=\partial_{rs}u_i+(\partial_r a)\partial_{ns}u_i\qquad r=1,\dots,n-1,\;\;s=1,\dots,n,\tag{17}$$

and observe that $g_{rs}:\Xi^+\to\mathbf{R}$ is exactly $\partial_{\tau_r}\partial_s u_i$, that is the tangential derivative of $\partial_s u_i$ calculated by means of $T_{r,h}$ and –due to the results from Proposition 3.3– we know that $g_{rs}\in L^2(\Xi^+)$. By (17) we obtain, for any $1\leq k\leq n-1$

$$\partial_{nk}u_i=\partial_{kn}u_i=g_{kn}-(\partial_k a)(\partial_{nn}u_i).\tag{18}$$

Using (17) and (18) we obtain, for any $1\leq j,k\leq n-1$

$$\begin{aligned}\partial_{jk}u_i&=g_{jk}-(\partial_j a)(\partial_{nk}u_i)=\\&=g_{jk}-(\partial_j a)g_{kn}+(\partial_j a)(\partial_k a)(\partial_{nn}u_i).\end{aligned}\tag{19}$$

By (18) and (19) we get

$$
\begin{aligned}
\sum_{(j,k)\neq(n,n)} |\partial_{jk}u_i| &= \sum_{j,k=1}^{n-1} |\partial_{jk}u_i| + 2\sum_{k=1}^{n-1} |\partial_{nk}u_i| \\
&\leq 2\Big(\sum_{j=1}^{n-1}\sum_{k=1}^{n} |g_{jk}|\Big) + \Big(\sum_{j,k=1}^{n-1} |\partial_j a||g_{kn}|\Big) \\
&\quad + \left[\|\nabla' a\|_\infty^2 (n-1)^2 + 2\|\nabla' a\|_\infty (n-1)\right] |\partial_{nn}u_i|
\end{aligned}
\tag{20}
$$

and we can substitute in the right hand side of (16) to get

$$
\begin{aligned}
&\Big[\nu_0 + (p-1)\nu_1(\mu + |\nabla u|^2)^{\frac{p-2}{2}} \\
&- \left[\|\nabla' a\|_\infty^2 (n-1)^2 + 2\|\nabla' a\|_\infty (n-1)\right] \Big(\nu_0 + (3-p)\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big)\Big] |\partial_{nn}u_i| \\
&\leq \Big[\nu_0 + (3-p)\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big] \Bigg[2\Big(\sum_{j=1}^{n-1}\sum_{k=1}^{n} |g_{jk}|\Big) + \Big(\sum_{j,k=1}^{n-1} |\partial_j a||g_{kn}|\Big)\Bigg] + |f|.
\end{aligned}
$$

Hence, if (7) holds, since $\frac{p-1}{3-p} < 1$ we have that

$$
\|\nabla' a\|_\infty^2 (n-1)^2 + 2\|\nabla' a\|_\infty (n-1) < \frac{p-1}{2(3-p)},
$$

hence

$$
\begin{aligned}
&\nu_0 + (p-1)\nu_1(\mu + |\nabla u|^2)^{\frac{p-2}{2}} \\
&- \left[\|\nabla' a\|_\infty^2 (n-1)^2 + 2\|\nabla' a\|_\infty (n-1)\right] \Big(\nu_0 + (3-p)\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big) \\
&\qquad\qquad \geq \frac{p-1}{2(3-p)} \Big(\nu_0 + (3-p)\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}\Big),
\end{aligned}
$$

then, for a.e. $x \in \Xi^+$,

$$
\begin{aligned}
|\partial_{nn}u_i| &\leq 2\frac{3-p}{p-1}\Bigg[2\Big(\sum_{j=1}^{n-1}\sum_{k=1}^{n} |g_{jk}|\Big) + \Big(\sum_{j,k=1}^{n-1} |\partial_j a||g_{kn}|\Big)\Bigg] \\
&\qquad + \frac{3-p}{p-1}\frac{2|f|}{\nu_0 + (3-p)\nu_1(\mu+|\nabla u|^2)^{\frac{p-2}{2}}} \\
&\leq 2\,\frac{3-p}{p-1}\Bigg[2\Big(\sum_{j=1}^{n-1}\sum_{k=1}^{n} |g_{jk}|\Big) + \Big(\sum_{j,k=1}^{n-1} |\partial_j a||g_{kn}|\Big)\Bigg] + \frac{2}{\nu_0}|f|.
\end{aligned}
$$

By using Proposition 3.3 we finally obtain that there exists $C(p,n) > 0$, depending only on p and on n, such that

$$
\|\partial_{nn}u_i\|_{L^2(\Xi^+)} \leq \frac{C(p,n)}{\nu_0}\|f\|_{L^2(\Omega)} \qquad i = 1,\dots,N. \tag{21}
$$

3.5. **Estimates for all non-tangential derivatives.** We now use the results of the previous section to end the proof of the main result. We observe that by using (18) and (19), we obtain by comparison

$$
\|\partial_{jk}u_i\|_{L^2(\Xi^+)} \leq \frac{3C(p,n)}{\nu_0}\|f\|_{L^2(\Omega)} \qquad j = 1,\dots,n-1,\ k = 1,\dots,n,\ i = 1,\dots,N \tag{22}
$$

since all terms from the right-hand sides belong to $L^2(\Xi^+)$.

Finally collecting the estimates (15)-(21)-(22), and by observing that estimates in $\Xi_m \subset V_m$ are uniform in $m = 1, \dots, \mathcal{N}$, we end the proof of Theorem 1.2.

4. **The degenerate case.** In this section we consider the degenerate case $\mu = 0$, that is we consider the problem, for $\nu_0, \nu_1 > 0$

$$\begin{aligned} -\nu_0 \Delta u - \nu_1 \nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{23}$$

which corresponds to a Laplacian plus a p-Laplacian.

The difficulty stems in the fact that is it not possible to use the calculations of the previous section in a direct manner, since many terms can have a non-controlled growth near the points where $\nabla u = 0$. To this end we follow path similar to Barrett and Liu [18] and [8] and we use problem (1) as a suitable approximation to (23). To this end we explicitly denote by u^μ the solution to (1), where $\mu > 0$ is given.

The results of Theorem 1.2 show that, there exists C, independent of $\mu \in]0, 1[$ (the upper bound is unessential, since we are interested in the behavior near zero) such that

$$\|u^\mu\|_{W^{2,2}(\Omega)} \leq \frac{C}{\nu_0} \|f\|_{L^2(\Omega)} \qquad \text{for each given } \mu > 0.$$

Hence, by using weak compactness and Rellich theorem, we can select a sequence $\{\mu_r\}_{r \in \mathbf{N}}$ such that $\mu_r \to 0^+$ and a function $u \in W^{2,2}(\Omega)$ such that

$$\begin{cases} u^{\mu_r} \rightharpoonup u & \text{in } W^{2,2}(\Omega), \\ u^{\mu_r} \to u & \text{in } W^{1,q}(\Omega) \quad \begin{cases} \text{for all } q < 2^* = \dfrac{2n}{n-2}, & \text{if } n \geq 3, \\ \text{for all } q \geq 1, & \text{if } n = 2, \end{cases} \\ \nabla u^{\mu_r} \to \nabla u & \text{a.e. in } \Omega. \end{cases}$$

We next observe that

$$\left| (\mu_r + |\nabla u^{\mu_r}(x)|^2)^{\frac{p-2}{2}} \nabla u^{\mu_r}(x) \right| \leq |\nabla u^{\mu_r}(x)|^{p-1} \qquad \text{a.e. } x \in \Omega,$$

hence, recalling that u^{μ_r} is a weak solution and since $p < 2$ implies $p' > 2$, there exists $C > 0$ independent of $r \in \mathbf{N}$ such that

$$\begin{aligned} \left\| (\mu_r + |\nabla u^{\mu_r}(x)|^2)^{\frac{p-2}{2}} \nabla u^{\mu_r}(x) \right\|_{L^2} &\leq |\Omega|^{\frac{2-p}{2p}} \left\| (\mu_r + |\nabla u^{\mu_r}(x)|^2)^{\frac{p-2}{2}} \nabla u^{\mu_r}(x) \right\|_{L^{p'}} \\ &\leq |\Omega|^{\frac{2-p}{2p}} \|\nabla u^{\mu_r}(x)\|_{L^p}^{p-1} \leq C. \end{aligned}$$

Next, by recalling that Ω is bounded and $\psi^{\mu_r} := (\mu_r + |\nabla u^{\mu_r}(x)|^2)^{\frac{p-2}{2}} \nabla u^{\mu_r}(x)$ is such that

$$\begin{aligned} &\|\psi^{\mu_r}\|_{L^2} \leq C \\ &\psi^{\mu_r} \to \psi^0 := |\nabla u(x)|^{p-2} \nabla u(x) \qquad \text{a.e. } x \in \Omega, \end{aligned}$$

by Lions [17, Lemma I.1.3] it follows that $\psi^{\mu_r} \rightharpoonup \psi^0$ in $L^2(\Omega)$, that is

$$(\mu_r + |\nabla u^{\mu_r}(x)|^2)^{\frac{p-2}{2}} \nabla u^{\mu_r}(x) \rightharpoonup |\nabla u(x)|^{p-2} \nabla u(x) \qquad \text{in } L^2(\Omega).$$

Hence, we can pass to the limit in the weak formulation in both terms on the left-hand side of (6) to get

$$\lim_{r\to+\infty}\nu_0\int_\Omega \nabla u^{\mu_r}\cdot\nabla\phi\,dx+\nu_1\int_\Omega(\mu_r+|\nabla u^{\mu_r}|^2)^{\frac{p-2}{2}}\nabla u^{\mu_r}\cdot\nabla\phi\,dx$$
$$=\nu_0\int_\Omega\nabla u\cdot\nabla\phi\,dx+\nu_1\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\phi=\int_\Omega f\cdot\phi\,dx\quad\forall\,\phi\in W_0^{1,2}(\Omega).$$

This shows that u is a weak solution to (23) and by uniqueness it is also the unique strong-$W^{2,2}(\Omega)$ solution to (23).

REFERENCES

[1] E. Acerbi and N. Fusco, Regularity for minimizers of nonquadratic functionals: The case $1 < p < 2$, *J. Math. Anal. Appl.*, **140** (1989), 115–135.

[2] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[3] H. Beirão da Veiga, On the Ladyzhenskaya-Smagorinsky turbulence model of the Navier-Stokes equations in smooth domains. The regularity problem, *J. Eur. Math. Soc. (JEMS)*, **11** (2009), 127–167.

[4] H. Beirão da Veiga, On the global regularity of shear thinning flows in smooth domains, *J. Math. Anal. Appl.*, **349** (2009), 335–360.

[5] H. Beirão da Veiga, Navier-Stokes equations with shear-thickening viscosity. Regularity up to the boundary, *J. Math. Fluid Mech.*, **11** (2009), 233–257.

[6] H. Beirão da Veiga and F. Crispo, On the global $W^{2,q}$ regularity for nonlinear N-systems of the p-Laplacian type in n space variables, *Nonlinear Anal.*, **75** (2012), 4346–4354.

[7] H. Beirão da Veiga and F. Crispo, On the global regularity for nonlinear systems of the p-Laplacian type, *Discrete Contin. Dyn. Syst. Ser. S*, **6** (2013), 1173–1191.

[8] L. C. Berselli, L. Diening and M. Růžička, Existence of strong solutions for incompressible fluids with shear dependent viscosities, *J. Math. Fluid Mech.*, **12** (2010), 101–132.

[9] F. Crispo, A note on the global regularity of steady flows of generalized Newtonian fluids, *Port. Math.*, **66** (2009), 211–223.

[10] F. Crispo and P. Maremonti, On the higher regularity of solutions to the p-Laplacean system in the subquadratic case, *Riv. Math. Univ. Parma (N.S.)*, **5** (2014), 39–63.

[11] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.*, **7** (1983), 827–850.

[12] L. Diening, C. Ebmeyer and M. Růžička, Optimal convergence for the implicit space-time discretization of parabolic systems with p-structure, *SIAM J. Numer. Anal.*, **45** (2007), 457–472 (electronic).

[13] L. C. Evans, *Partial Differential Equations*, 2nd edition, Graduate Studies in Mathematics, 19, American Mathematical Society, Providence, RI, 2010.

[14] L. C. Evans, A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e, *J. Differential Equations*, **45** (1982), 356–373.

[15] J. L. Lewis, Regularity of the derivatives of solutions to certain degenerate elliptic equations, *Indiana Univ. Math. J.*, **32** (1983), 849–858.

[16] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.*, **12** (1988), 1203–1219.

[17] J.-L. Lions, *Quelques Méthodes de Résolution Des Problèmes Aux Limites Non Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[18] W. B. Liu and J. W. Barrett, A remark on the regularity of the solutions of the p-Laplacian and its application to their finite element approximation, *J. Math. Anal. Appl.*, **178** (1993), 470–487.

[19] J. Málek, J. Nečas and M. Růžička, On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: the case $p \geq 2$, *Adv. Differential Equations*, **6** (2001), 257–302. Available from: `http://projecteuclid.org/euclid.ade/1357141212`.

[20] J. Naumann and J. Wolf, On the interior regularity of weak solutions of degenerate elliptic systems (the case $1 < p < 2$), *Rend. Sem. Mat. Univ. Padova*, **88** (1992), 55–81. Available from: `http://www.numdam.org/item?id=RSMUP_1992__88__55_0`.

[21] J. Nečas, *Les Méthodes Directes en Théorie des Équations Elliptiques*, Masson et Cie, Éditeurs, Paris, 1967.

[22] L. Nirenberg, Remarks on strongly elliptic partial differential equations, *Comm. Pure Appl. Math.*, **8** (1955), 649–675.

[23] P. Tolksdorf, Everywhere-regularity for some quasilinear systems with a lack of ellipticity, *Ann. Mat. Pura Appl. (4)*, **134** (1983), 241–266.

[24] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations*, **51** (1984), 126–150.

[25] K. Uhlenbeck, Regularity for a class of non-linear elliptic systems, *Acta Math.*, **138** (1977), 219–240.

[26] N. N. Ural′ceva, Degenerate quasilinear elliptic systems, *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **7** (1968), 184–222.

Received September 2014; revised February 2015.

E-mail address: luigi.carlo.berselli@unipi.it
E-mail address: carlo.romano.grisanti@unipi.it