Variations on a theme: Clifford's parallelism in elliptic space

Contents

Introduction

Euclidean parallelism in ordinary 3–space can be defined in a number of equivalent ways. According to Zacharias,[1] possible definitions can be grouped into three classes; straight lines can be considered to be parallel when i) they lie in the same plane and they do not intersect; ii) they have the same direction; iii) they are equidistant.

A notion of parallelism can be attributed to non–Euclidean geometries too but, depending on the geometrical setting, namely hyperbolic or elliptic, to use a denomination introduced by Klein, one has to single out a particular characterization among the three just mentioned. Indeed, it is only in the Euclidean case that all of the three groups of definitions turn out to be equivalent.

For example, in order to introduce Lobachevsky's parallelism in (real) hyperbolic space, one can adopt the first characterization and define two parallels to a given line as those limit lines that separate concurrent lines from those that are not. By employing Cayley–Klein projective model,[2] one can define Lobachevsky's parallels to a given line as those two lines that intersect the given line at a point of the absolute (ruled) quadric. In other words, one may define two lines to be parallel in the sense of Lobachevsky if they meet the same couple of generators of the absolute quadric (i.e. the canonical quadric of hyperbolic space) that belong to *different* systems (of generators). It might be helpful to recall that every point of a ruled quadric is obtained as the intersection of two straight lines that lie upon the quadric. All these straight lines can be grouped into two systems; lines that belong to the same system are *skew* while lines belonging to different systems always intersect in a point of the quadric itself.[3] In view of this definition, it is clear for example that property iii) must be surrendered.

[1] [Zacharias 1913, p. 870-871]; see also [Vaney 1929, p. 1-2].

[2] On Cayley–Klein projective models of non–Euclidean geometries, one can consult [Giering 1982].

[3] In this connection see, for example, [Castelnuovo 1904, p. 633] or the appendix at the end of the present paper.

It is a discovery due to Clifford that a kind of parallelism could also be introduced in elliptic 3–space. The notion can be most easily understood using the Cayley–Klein projective model of elliptic space. Contrary to the definition of parallelism adopted in the hyperbolic case, we cannot require that two real lines meet at a point of the fundamental quadric for the simple reason that there are no *distinct real* lines in the elliptic case that meet at a point of the absolute quadric.[4] Nonetheless, since in this case too the quadric admits a double system of (imaginary) generators, we can define two straight lines to be parallel if they meet two generators that belong to the *same* system of generators. This requires us to consider complex extensions of straight lines to the complex extension of $\mathbb{P}^3(\mathbb{R})$.[5] It turns out that such parallels are skew lines (i.e. they are not coplanar) that are also equidistant. Furthermore, there exist motions (i.e. isometries of the elliptic space) that have the property of leaving congruences of parallel lines invariant,[6] precisely as certain translations in Euclidean space conserve all the lines which are parallel to a given line.[7] It is interesting to observe that none of these properties is shared by Lobachevsky's parallels, while, in turn, Clifford's parallels are never coplanar and they never intersect (*a fortiori* in a point of the fundamental quadric). As is remarked in [Bonola 1912, Appendix II], equidistant lines can be coplanar only in Euclidean space , while skew parallels can exist only in elliptic space .

It was observed by [Bortolotti 1935, p. 288] that Lobachevsky's and Clifford's parallelism are, in a certain sense, complementary phenomena. Usually properties of Euclidean parallels seem to split into two classes whose intersection is empty: one class of properties characterizes Lobaschevsky's parallels, the other Clifford's parallels. In many respects, Clifford's parallelism seems acquire a privileged position with respect to Lobachevsky's since it retains almost all the essential characteristics of ordinary Euclidean parallelism.

The present article gives a historical account of the emergence and further developments of this curious and fascinating phenomenon. The history of Clifford's parallelism is worth telling, not only for its intrinsic interest in the wider field of researches in the history of non–Euclidean geometry but also because it assumed, over the decades following its discovery (in 1873), a paradigmatic role. Indeed, new developments in geometrical theories, such as Cayley–Klein projective models of non–Euclidean spaces, the systematization of Riemannian geometry, the theory of connections, and finally the search for generalizations of the notion of Riemannian manifolds often tried to assimilate Clifford's parallelism within new theoretical frameworks. As will be seen, the general aim of mathematicians seems to have been twofold, depending on the context: on one hand, to investigate more deeply its essence and its theoretical grounds; on the other, either to employ it at a heuristic level to test new conceptual tools or to provide relatively simple examples of more general phenomena.

Section 1 discusses the genesis of the notion in W. K. Clifford's work by indicating its original motivation in the search for an algebraic and geometrical treatment of the kinematics and the dynamics of rigid bodies in elliptic 3–space. Section 2 offers a detailed account of Klein's interpretation of Clifford's parallelism in the context of projective models of non–Euclidean geometries. Section 3 deals with the Italian school of differential geometry, namely with L. Bianchi and G. Fubini's fruitful attempts at providing an analysis of elliptic space and Clifford's parallelism in the framework of Riemannian geometry.

[4]See in this respect [Klein 1928, p. 236] and also [Klein 1928, p. 49].

[5]See appendix. In the following, as was usual before a rigorous formalization of these concepts was attained, a straight line in $\mathbb{P}^3(\mathbb{R})$ will be tacitly identified with its complex extension, i.e. a real straight line of $\mathbb{P}^3(\mathbb{C})$.

[6]A congruence is a set of lines which depend on two parameters.

[7]For a list of some properties of Clifford's parallels, see [Bonola 1912, Appendix II]

Section 4 treats in detail the problem of understanding Clifford's parallelism in the light of Levi–Civita's theory of parallel displacement. In this context, [Bortolotti 1925] is discussed in which the Italian mathematician Bortolotti succeeded in characterizing Clifford's parallelism as a parallel displacement of an absolute kind (i.e. independent of the path of displacement). Finally, an appendix is attached at the end of the paper that recalls some elementary facts concerning the theory of polarity with respect to a given quadric surface.

The general aim of this paper is twofold: both to provide a historical analysis of Clifford's parallelism *per se* and to emphasize its role, in particular within the context of the Italian school of differential geometry, in the historical development of the dawning theory of connections. Indeed, as will be seen, Clifford's parallelism represented a highly non–trivial example of absolute parallelism which fostered the search for extension of the classical framework of Levi–Civita's connection. The historical analysis is here restricted to the Italian milieu with special emphasis on the work by Bortolotti. However, in a forthcoming paper, we will provide a discussion of Élie Cartan's response to the problem posed by Clifford's parallelism. In particular, the important role played by Clifford's parallelism in guiding Cartan's researches on absolute parallelisms and symmetric spaces will be analyzed.

1 A new kind of parallelism in elliptic space

What nowadays is known as Clifford's parallelism in 3–dimensional elliptic space was first sketched by Clifford in a short memoir [Clifford 1873] that was presented at the London Mathematical Society in June 1873. Its title *Preliminary Sketch of Biquaternions* seems to have little to do with the geometry of elliptic space, however, the main focus of Clifford was to provide an algebraic and geometrical analysis of the kinematics and dynamics of rigid bodies.

This subject situated Clifford's investigations in a fertile branch of research that, since the works of Plücker and Möbius, had pursued a geometrization programme for mechanics based upon projective geometry.[8] Already in 1866, for example, Plücker had tried to "connect, in mechanics, translatory and rotatory movements with each other by a principle in geometry, analogous to that of reciprocity [Plücker 1866, p. 361]". In 1871, Klein had extended this connection to non–Euclidean geometries by observing:

> Let us introduce such a metric [Cayley's metric] and let us replace, at the same time, the ∞^6 movements of our space with as many linear transformations that leave invariant the fundamental quadric surface; thus, we can speak in the same way of forces acting along a straight line or about a straight line and of motions along a straight line or about it. Both kinds or forces and motions would be equivalent. A rotation about a straight line is thus equivalent to a translation along its polar with respect to the fundamental quadric surface. Similarly, a force acting along a straight line is tantamount to a force which produces a rotation about its conjugate polar [Klein 1871a, p. 412].

In Great Britain as well, in the early 1870s, the geometrical representation of mechanical entities such as the motion of, and action upon, rigid bodies was at the center of the investigations of the astronomer and mathematician Robert Stawell Ball, who developed a peculiar approach that welded rotational and linear quantities into a single geometrical element, the *screw*. According to Ball's definition,[9] a screw is a straight line with which

[8] On the historical development of geometrical mechanics, see [Ziegler 1985].

[9] See e.g. [Ball 1876, 1-2 §].

a definite magnitude, termed the pitch, is associated. The fecundity of this definition derived from the two fundamental theorems of Chasles and Poinsot according to which: any rigid–body motion could be described as a rotation about a unique axis and a translation parallel to it (Chasles' theorem); any system of forces upon a rigid body can be replaced by a force along a unique axis and a couple in a plane perpendicular to it (Poinsot's theorem). Indeed, a general rigid body motion could interpreted geometrically as a screw whose axis coincides with the rotation axis and whose associated magnitude (the *pitch* of the screw) is given by the ratio of the moduli of the translational and rotational velocity. Analogously, in the case of forces acting on a rigid body, dynamics is described in term of a screw whose axis is the line of action of the single resultant force and the pitch is given by the ratio of the magnitude of the couple to the magnitude of the force.

Apparently, Clifford's main aim in his [Clifford 1873] was to provide an algebra for this extended range of mechanical entities, such as screws. Precisely in the same way as quaternions were introduced by Hamilton in order to extend the set of 3–dimensional vectors to a non–commutative algebra which is closed not only under addition but also under multiplication and division, biquaternions were introduced by Clifford in order to extend the set of screws (Clifford called them *motors*) to an algebra which is closed under addition, multiplication and division. As Hamilton interpreted a quaternion as the ratio of two 3–dimensional vectors, for Clifford too, a biquaternion could regarded as a ratio of two screws. The term "biquaternion" was not a new one; Hamilton had already employed it to denote a quaternion $ai + bj + ck + d$ whose coefficients are complex numbers; however, since, as Clifford explicitly remarked, "it is convenient to suppose from the beginning that all scalars may be complex", he decided to employ it in a new way (thus, denoting something essentially different from a complex quaternion).

From a purely algebraic point of view, a biquaternion is an expression of the form $q + \omega r$, where q, r are Hamilton's quaternions while ω is an operator which converts every motor into a vector parallel to the axis of the motor and of magnitude equal to the magnitude of the rotating part of the motor (Clifford named it *rotor*). As a consequence of this definition, since the rotating part of a vector is null, one has: $\omega^2 = 0$. In actual fact, however, the value to be attributed to ω^2, remained somehow ambiguous in Clifford's treatment: depending upon the geometrical setting under consideration (parabolic, elliptic or hyperbolic) in which the calculus of biquaternions is carried out, ω^2 is forced to assume different values ($0, 1$ or -1, respectively). Indeed, as is clear from the analysis which Clifford carried out in sections III, IV and V of his paper, in the case of elliptic geometry the value of ω^2 is $\omega^2 = 1$.

Clifford's parallels made their appearance in section III, where Clifford aimed at developing an algebra of motors in the special case of elliptic geometry. The main reason for such a specialization seems to lie in a technical difficulty which prevented him from interpreting a biquaternion as a ratio of motors in a fully general case. The problem is presented by Clifford as follows. Consider two motors which are denoted by the expressions $\alpha + \omega\beta$ and $\gamma + \omega\delta$ ($\alpha, \beta, \gamma, \delta$ are rotors; expressions like $\alpha + \omega\beta$ consists of a translational and a rotational part, in accordance with the aforementioned definition of motor); one wants to provide a meaning to the ratio $\dfrac{\alpha + \omega\beta}{\gamma + \omega\delta}$ and to write it as a biquaternion $q + \omega r$ in such a way that the latter could be interpreted as the operator transforming $\gamma + \omega\delta$ into $\alpha + \omega\beta$, $\gamma + \omega\delta = (q + \omega r)(\alpha + \omega\beta)$. However, Clifford observed, this equation is not susceptible of interpretation as in the case of a single quaternion q which transforms a vector into another. Indeed, he wrote, "the expression $q + \omega r$ does not denote the sum of geometrical operations, which can be applied to the motor as a whole; and the ratio of two motors is only expressed by a symbol as the sum of two parts, each of which separately has a definite meaning in certain other cases, but not in the case in point. [...] this difficulty will be partly overcome by showing that the system here sketched is the limit of another in which it does not occur". Likely, Clifford was here referring to the possibility of regarding the Euclidean geometry as a limit case of the elliptic geometry where, he claimed, such a difficulty could be evaded.

The reason for this was that, in this particular geometrical setting, Clifford was able to provide a decomposition of a general motion (a *motor*) into what he called left and right vectors, which, as will be seen, are a special kind of transformations of elliptic space, similar in some respects to translations of ordinary space. Such a decomposition allowed him to provide a precise geometrical meaning to both of the addends in $q + \omega r$.

The geometrical setting is that of the so–called Cayley–Klein geometries. Clifford explicitly referred to [Cayley 1859] and [Klein 1871b]. Thus, he shared Cayley's and Klein's view according to which metric geometry could be regarded as a part of descriptive (i.e. projective) geometry. This interpretation of metric geometry could be attained by fixing a quadric surface (the absolute) in (real) projective space (or complex extension thereof) which, in the case of elliptic geometry, is characterized by the fact that all of its points are imaginary.

After recalling the basic formulas for distance and angle in terms of cross–ratio, Clifford considered two straight lines a, b (not lying in the same plane) and observed that in general two lines that are polars of one another can be drawn so that each meets a, b *at right angles*. These polar lines, Clifford claimed without proof, can be determined as the lines which meet the two lines a, b and their polars a', b'. Indeed, any line that cuts both a and a' is perpendicular to both (this is a consequence of the trivial fact that any point of a is conjugate to any point of a'). Thus, any line which meets the four lines a, a', b, b' cuts them all at right angles. Now, the three lines a, a', b determine a ruled quadric surface[10] since they are (pairwise) skew[11] and the fourth line b' meets this quadric in two points P and Q. The generators p, q of the opposite system (opposite with respect to the system of generators to which a, a', b belong) passing through P and Q are common transversals of the four lines a, a', b, b' and thus cut them all at right angles. The fact that p and q are polar one of the other follows by assuming that there are no more than two common perpendiculars.

However, there is an exceptional case that is of the utmost importance to the analysis. This is the case in which the line b' belongs to the same system of generators as the lines a, a', b. Then, there exists an infinite number of common perpendiculars to the lines a, a', b, b' that coincide precisely with the second system of generators of the quadric. It can be proved in this case that the skew lines a and b are equidistant and, furthermore, that they cut the same two generators of one system of the absolute quadric. Depending upon the type of system of generators cut by the two lines a and b, Clifford spoke of *right* or *left* parallels. This denomination appeared to him most appropriate since he observed, again without proof, that "there are many points of analogy between the *parallels* here defined and those of parabolic geometry." Clifford cited a few of them, namely: i) the isogonality property according to which a line cuts two parallels at equal angles, ii) the existence of a ruled quadric of zero curvature generated by all the parallel lines which meet a given line (this is the well known *Clifford's quadric*).

Let us now turn to the aforementioned decomposition of a general movement (motor) of elliptic space into the sum of two vector motions which, in a way, can be assimilated to translations. In this respect, Clifford remarked:

[10] This is an auxiliary quadric that is not to be confused with the absolute quadric of elliptic geometry.

[11] See [Castelnuovo 1904, §361-365].

> A twist-velocity of a rigid body must be regarded as having *two* axes. For a motion of translation along any axis is the same thing as a rotation about the polar axis, and *vice versa*. Hence a twist-velocity is compounded of rotation–velocities about two polar axes; say these are θ, ϕ. Then the motion may be regarded either as a twist–velocity about a screw whose pitch is $\dfrac{\phi}{\theta}$ and whose axis is the first axis, or about a screw whose pitch is $\dfrac{\theta}{\phi}$ and whose axis is the polar axis. In general, then, a motor has two axes and is expressible in one way only as the sum of two polar rotors. There is, however, one case of exception which the axes of a motor are indeterminate; that, namely, in which the magnitudes of the two polar rotors are equal. [...] Such [...] a motor of pitch unity, or which is its own polar, may [...] be regarded as having the nature of a *vector* [...]. For we may define a vector as a motor whose axes are indeterminate; and the case we are now considering is the only case of such indetermination which occurs in elliptic geometry. Vectors will be called *right* or *left* according as the twist of them is right– or left–handed [Clifford 1873, p. 390].

On the basis of these observations, Clifford stated the proposition according to which every motor in elliptic space is the sum of a right and a left vector. Indeed, Clifford observed, if A indicates a motor and A' is its polar motor then $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ and, by construction, $A + A'$ and $A - A'$ are right– and left–vectors.

Clifford's treatment suffers from a certain lack of technical details but we can try to make sense of it by making recourse to more familiar tools as follows. A twist–velocity can be interpreted as motion of the elliptic space, i.e. as a real homography leaving the absolute quadric invariant. In general, such a motion leaves invariant four generators g_1, g_2, g_1', g_2' of the absolute quadric (they are two couples belonging to different systems of generators). This set of four lines determines a skew quadrilateral whose diagonals are straight lines that are polar one of the other and are left invariant by the motion itself. These are the *axes* of the motor to which Clifford referred. Furthermore, the correspondence between translation along a line and rotation about its polar is due to an elementary property of polarity according to which, while a point moves along a line l, its polar plane rotates about l', the polar line of l. Furthermore, the case of exception of a motor whose pitch is equal to 1 corresponds to what are nowadays called *Clifford's translations.* In view of Clifford's observation according to which a motor is compounded of rotation–velocities (θ, ϕ) about polar axes we recognize that the notion of motor with pitch equal to 1 can be interpreted in the light of the characterization of Clifford's translation provided by Coxeter in [Coxeter 1998, p. 135]: "a Clifford's translation may be defined as the product of rotations through equal angles θ about two absolute polar lines". Finally, the asserted decomposition of any motor into right– and left–vectors corresponds to the possibility of writing a general motion Ψ of the elliptic space into the product of two Clifford's translations.

On the basis of these premises, Clifford was able to obviate the above-mentioned difficulty concerning the definition of a biquaternion as the ratio of two motors. To this end, he introduced the operator ω such that $\omega^2 = 1$; it is defined by means of the following prescription: if α is a rotor, then "$\omega\alpha$ will denote the rotor polar to α and equal to it in magnitude". Then he observed that a general motor could be expressed in the form $\alpha + \omega\beta$, where α and β denote two rotors; he then defined two operators, namely $\xi = \dfrac{1+\omega}{2}$ and $\eta = \dfrac{1-\omega}{2}$. In view of the decomposition described above, Clifford could finally write a generic motor as $\xi\gamma + \eta\delta$ and thus provide the summands of a biquaternion with a geometrical meaning. Indeed, the operator which converts a motor $\xi\gamma + \eta\delta$ into a motor $\xi\alpha + \eta\beta$ could be chosen to be $\xi q + \eta r$ where q, r are quaternions equal to the ratios of rotors $\alpha/\gamma, \beta/\delta$, respectively. As a consequence of $\eta\xi = \xi\eta = 0$, each of the two summands ξq and ηr acquired a definite meaning.

Far from being a fully satisfying treatment of the algebra of motions, [Clifford 1873] remained, in many respects, an exposition of audacious and fertile ideas which required a patient work of systematization. Attempts at it were undertaken by R. Ball himself, with whom the theory of screws had originated, A. Buchheim in the early 1880s and, some years later, by E. Study. In particular, Ball devoted to the topic a memoir which he presented to the Royal Irish Academy in November 1881. Interestingly, Ball was able to characterize Clifford's vector motions of elliptic 3–space in terms of invariance properties of the generators of the fundamental quadric, a characterization which had remained implicit in Clifford's treatment; thus preparing the ground for Klein's subsequent analysis. In this respect, Ball observed:

> [The] name [vector] was applied by Clifford to a particular description of displacement which a body can receive in elliptic space. In the most general displacement of a rigid system two right generators and two left generators of the absolute remained unaltered. In the movement which we call a *vector*, two of the generators of one system and all of the generators of the other remained unaltered. Clifford had shown that when a body is displaced by a vector each point of the body moves through equal distances along parallel lines. [...] He distinguishes between a right vector and a left vector, according to which system of generators remains unaltered [Ball 1881, pp. 159-160].

Systematic investigations on biquaternions were carried out by Arthur Buchheim (1859-1888). A most brilliant student of the Savilian Professor of Geometry in Oxford, Henry Smith, who "spoke of him as the most promising young mathematician that had appeared in the University of Oxford for a long series of years", he devoted a great deal of attention to providing a comprehensible treatment of Clifford's pioneering views by seeking for an algebraic foundations of the new concept and also by exploring their geometrical meaning. Before his premature death in September 1888, Buchheim, who also studied with Klein in Leipzig, devoted three papers to this project of systematization. Of these, [Buchheim 1883] provided a reinterpretation of Clifford's calculus of screws in the light of Grassmann's *Ausdehnungslehre*. In particular, Buchheim was able to identify the notion of screw with that of linear complex and Clifford's operator ω with that of Grassmann's *conjugation* K (*Ergänzung*)[12] By means of these tools, extraneous to Clifford's original treatment, in [Buchheim 1883, p. 91] Buchheim deduced conditions for Clifford's parallelism in a very easy way; if a and b denote two lines, then a, b are Clifford parallel lines if $a \mp Ka = \lambda(b \mp Kb)$, λ being an arbitrary constant. Only a year later, Buchheim came back to the same topic in [Buchheim 1884] by providing an extension of Grassmann's calculus to the three kinds of uniform space (i.e. parabolic, hyperbolic and elliptic). The idea was simply that of interpreting the operator K (ω in Clifford's notation) no longer as the *Ergänzung* but as the polar of a given geometric figure with respect to the absolute (in the sense of Cayley and Klein). Eventually, this insight was fully developed in a long memoir [Buchheim 1885] where he not only furnished an algebraic discussion of biquaternions but also conveyed a very interesting treatment of their useful geometrical applications. In particular, the algebra of the operator ω was explicitly linked to the curvature of the space

[12]In this respect, see [Buchheim 1883, p. 85].

under consideration; indeed, Buchheim shew how a consistent definition of the operator ω could be attained by posing $\omega^2 = k$, where k indicates the curvature of the space which is equal to $-1, 0, 1$ in the hyperbolic, parabolic, and elliptic cases respectively.

A few years later, namely in [Study 1891], Eduard Study also had recourse to the notion of biquaternion in his researches upon the representation of motions in space, and much later in [Study 1913] he provided a general outline of the application of biquaternions in theoretical kinematics.

2 Klein's *Zur Nicht–Euklidischen Geometrie*

After the introduction by Clifford of the notion of parallelism in 3–dimensional elliptic space, the matter was thoroughly discussed by Klein in the first part of [Klein 1890], which appeared in the *Mathematische Annalen.* Klein had met Clifford in September 1873, on the occasion of a session of the British Association for the Advancement of Science in Bradford (U.K.) where Clifford had delivered a lecture with the title "On a surface of zero curvature and finite extent". Since the text of Clifford's conference was not extant, Klein thought it useful to tackle the topic anew and to discuss in more analytical detail some results that Clifford had sketchily with dealt in various memoirs of his, such as [Clifford 1873].[13]

Undoubtedly, Klein must have welcomed the publication of [Clifford 1873] with great favor, as Clifford took great profit from Klein's unitary approach to (non–Euclidean) geometry through the general framework offered by projective geometry that he (Klein) had been developing since the early 1870s.[14] In particular, Klein committed himself to providing an analytical expressions of those transformations of the elliptic space that Clifford had called "vector motions". This special type of displacement of elliptic space plays a prominent role in the theory of Clifford parallels.

Klein started his analysis by first recalling the essential notions of the projective interpretation of non–Euclidean geometry: he considered real 3–dimensional projective space $\mathbb{P}^3(\mathbb{R})$ and its extension to the complex projective space $\mathbb{P}^3(\mathbb{C})$; then he fixed a quadric surface $\mathcal{Q} \subset \mathbb{P}^3(\mathbb{C})$ which he called the *absolute*. Thus, a metrical geometry can be interpreted as the theory of the projective relations of certain geometrical forms that are fixed by the introduction of the quadric surface with all other geometrical forms. In the case of elliptic geometry the *absolute* is a quadric $\mathcal{Q}$ with real coefficients but with no real points (*nulltheilege Fläche*),

$$\mathcal{Q} = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3(\mathbb{C}) \quad | \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\right\}.$$

This is a ruled quadric which admits two systems of (complex) generators (*Erzeugenden*). They can easily be obtained starting from the following decomposition of the quadric equation:

$$(x_0 + ix_1)(x_0 - ix_1) + (x_2 + ix_3)(x_2 - ix_3) = 0.$$

The first system of generators parameterized by the complex number λ is given by[15]:

$$\{R_\lambda\}: \qquad \lambda = \frac{x_0 + ix_1}{x_2 - ix_3} = -\frac{x_2 + ix_3}{x_0 - ix_1}. \tag{1}$$

[13]On the relationship between Clifford and Klein, see [Ziegler 1985, p. 175].

[14]On Klein's early geometrical works, see [Rowe 1989].

[15]To uniform notation throughout the paper, the purely conventional choice of naming one system of generators either "first" or "second" system adopted by Klein has been modified.

In turn, the second system can be written as follows ($\mu \in \mathbb{C}$):

$$\{L_\mu\}: \qquad \mu = -\frac{x_0 + ix_1}{x_2 + ix_3} = \frac{x_2 - ix_3}{x_0 - ix_1}. \tag{2}$$

At this point, Klein considered all the *real* collineations of $\mathbb{P}^3(\mathbb{C})$ which fulfill the following requirement:[16] i) they leave the absolute quadric invariant; ii) they transform generators of one system into generators of the same system. As a consequence of the fact that a collineation transforms straight lines into straight lines in a bijective way, these collineations induce, in a natural way, projective transformations of the parameters $\lambda, \mu \in \mathbb{C}$:

$$\lambda' = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}, \qquad \mu' = \frac{\alpha'\mu + \beta'}{\gamma'\mu + \delta'}.$$

Among such collineations, there exist special ones, which Klein called scrolls (*Schiebungen*) of the first and the second type which are characterized by the property of transforming generators of one system while leaving invariant the generators of the other. Analytically, that means for a *Schiebung* of the first type:

$$\Phi : (\lambda, \mu) \mapsto (\lambda', \mu') \qquad \text{with} \qquad \lambda' = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}, \qquad \mu' = \mu,$$

and analogously for scrolls of the second type.

Let us now focus upon scrolls of the first type only. Besides leaving invariant the generators of the second system $\{L_\mu\}$, the maps Φ leave two generators R_{λ_1}, R_{λ_2} of the first system pointwise–invariant. The parameters λ_1, λ_2 are the solutions of the quadratic equation $\lambda = \dfrac{\alpha\lambda + \beta}{\gamma\lambda + \delta}$. It should be observed that if R_{λ_1} is left invariant by a scroll, then $R_{\lambda_2} = \bar{R}_{\lambda_1}$, i.e. the two generators of the first system that are left invariant are complex conjugate one of the other.

Furthermore, under the action of a scroll of the first type, every point of the space is displaced along straights line which intersects the generators $R_{\lambda_1}, R_{\lambda_2}$.

In order to obtain an analytical expression for the scrolls of the first type, Klein deduced the following:

Lemma 1 *Let $\bar{R}_\lambda$ indicate the straight line which is conjugate to a generator R_λ of the first system. Then $\bar{R}_\lambda$ is a generator of the first system too, i.e., there exists $\lambda^* \in \mathbb{C}$ such that $\bar{R}_\lambda = R_{\lambda^*}$. Furthermore, if λ has polar expression $\lambda = re^{i\phi}$, then $\lambda^* = -\frac{1}{r}e^{i\phi}$.*

On the basis of this result, Klein's well-known correspondence between homographies of the complex plane and rotations on the sphere could be applied. Let us see why and how.

Klein recalled a standard technique which represents the extended complex plane $\mathbb{C} \cup \infty$ on the unit sphere

$$S^2 = \left\{(\xi, \eta, \zeta) \in \mathbb{R}^3 | \xi^2 + \eta^2 + \zeta^2 = 1\right\}$$

by means of the stereographic projection $\pi : S^2 \to \mathbb{C} \cup \infty$ on the equatorial plane that is given, for $\zeta \neq 1$, by:

$$x + iy = \pi(\xi, \eta, \zeta) = \frac{\xi + i\eta}{1 - \zeta}.$$

Points of the complex plane of type $re^{i\phi}$ and $-\frac{1}{r}e^{i\phi}$ (just as λ and λ^* in the preceding lemma) are called diametral points since they correspond, through the stereographic

[16]Such collineations are sometimes called proper or direct collineations; see e.g. [Klein 1928, p. 112].

projection, to antipodal points on the sphere. As a consequence of this, the projective transformation $\lambda' = \Phi(\lambda) = \dfrac{\alpha\lambda + \beta}{\gamma\lambda + \delta}$, corresponding to a scroll of the first kind, was characterized by Klein as a Möbius transformation that is conjugate to a rotation on the sphere; more explicitly, Φ can be written as $\pi R \pi^{-1}$, where R is a rotation. Already in his *Vorlesungen über das Ikosaeder*, Klein had conveyed an explicit formula for transformations of this kind, which can be written as follows:

$$\lambda' = \frac{(d+ic)\lambda - (b-ia)}{(b+ia)\lambda + (d-ic)}, \qquad a, b, c, d \in \mathbb{R}, \quad a^2 + b^2 + c^2 + d^2 = 1. \tag{3}$$

In view of this result, the analytic expression for scrolls of the first type is achieved by observing that from (1) and (2) one obtains:

$$\begin{cases} \rho x_0 = \lambda\mu + 1, \\ \rho x_1 = i(-\lambda\mu + 1), \\ \rho x_2 = \mu - \lambda, \\ \rho x_3 = i(\lambda + \mu). \end{cases} \tag{4}$$

where ρ is an arbitrary proportionality factor. Upon substitution of (λ, μ) with (λ', μ), where λ' is given by (3), Klein attained the desired expression:

$$\begin{cases} x_0' = +dx_0 - cx_1 + bx_2 - ax_3, \\ x_1' = +cx_0 + dx_1 - ax_2 - bx_3, \\ x_2' = -bx_0 + ax_1 + dx_2 + cx_3, \\ x_3' = +ax_0 + bx_1 - cx_2 + dx_3, \end{cases} \tag{5}$$

for a general scroll of the first kind. In a completely similar way, upon consideration of the second system of generators of the absolute quadric Klein deduced the expression for a general scroll of the second kind which reads as follows:

$$\begin{cases} x_0' = +\delta x_0 - \gamma x_1 + \beta x_2 + \alpha x_3, \\ x_1' = +\gamma x_0 + \delta x_1 - \alpha x_2 + \beta x_3, \\ x_2' = -\beta x_0 + \alpha x_1 + \delta x_2 + \gamma x_3, \\ x_3' = -\alpha x_0 - \beta x_1 - \gamma x_2 + \delta x_3, \end{cases} \tag{6}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are such that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$. Thus, we can reformulate Klein's result in the following

Theorem 1 *A real collineation of the complex projective space $\mathbb{P}^3(\mathbb{C})$, $\Psi : x \mapsto x'$ that leaves the absolute quadric $\mathcal{Q}$ invariant and projectively transforms the generators $\{R_\lambda\}$ of the first system while leaving invariant the generators $\{L_\mu\}$ of the second system can be written in the form (5). In a completely analogous way, an analytical representation for the scrolls of the second kind can be obtained. Furthermore, by the action of a scroll of the first kind, a point of the space moves along the straight line that meets the generators left invariant by the scroll itself. In other words, to any scroll there is an associated congruence of straight lines, i.e. a 2–parameter set of straight lines which meet on two generators of the quadric.*[17]

Klein also pointed out that an isometry of the elliptic space could be represented by means of unitary quaternions in a way that renders evident the decomposition of a general

[17] For a proof that this set of straight lines is a 2–parameter family, see for example [Bianchi 1902, pp. 446-447].

motion into scrolls of first and second kind. By indicating by q the unitary quaternion $q = d+ai+bi+ck$ and by identifying a point with homogeneous coordinates (x_1, x_2, x_3, x_4) with the quaternion $x_3 + ix_2 + jx_1 + kx_0$, a scroll of the first kind could be represented by

$$x'_3 + ix'_2 + jx'_1 + kx'_0 = (x_3 + ix_2 + jx_1 + kx_0) \cdot q,$$

where it is intended that the product follows the multiplication rules for i, j, k. If a scroll of the second type is represented analogously by left quaternion multiplication by $q' = \delta + \alpha i + \beta j + \gamma k$, then a general motion of the elliptic space, Klein remarked, could be written as:

$$x'_3 + ix'_2 + jx'_1 + kx'_0 = q' \cdot (x_3 + ix_2 + jx_1 + kx_0) \cdot q.$$

At this point, Klein turned to investigating the geometrical significance of the scrolls. To this end, he recalled the expression of the non–Euclidean distance (the so–called Cayley–Klein metric) between two points $x, x' \in \mathbb{P}^3(\mathbb{R})$ which takes on the following form:

$$\mathrm{d}(x, x') = \arccos \frac{\sum_{j=0}^{3} x_j x'_j}{\sqrt{\sum_{j=0}^{3} x_j^2}\sqrt{\sum_{j=0}^{3} x'^2_j}}. \tag{7}$$

Klein observed, that if x' is the image under the scroll Ψ of an arbitrary point x then $\mathrm{d}(x, x')$ is constant. Furthermore, he explicitly pointed out an outstanding analogy with the group of translations in ordinary, Euclidean space that in a way admitted a generalization of the notion of Euclidean parallelism. Just as in ordinary space parallel straight lines are preserved under a translation that is parallel to a given line, so in elliptic geometry a given scroll preserves the line congruence corresponding to it via theorem (1). Consequently, it was a natural choice to define two straight lines to be parallel in the sense of Clifford if they belong to the same line congruence or, equivalently, if they meet the same conjugate pair of (imaginary) generators of the absolute. Depending on the kind of congruence or of generators under consideration, Klein spoke of *right* (*rechtsgewundene*) parallel straight lines or of *left* (*linksgewundene*) parallel straight lines.

Interestingly, in [Klein 1890, p. 552] Klein referred to the introduction by Clifford of parallels in elliptic space as a new theory of parallels (*neue Parallelentheorie*) and explicitly emphasized the distinctiveness of the new parallelism with respect to the ordinary (i.e. Lobachevsky's) parallelism of non–Euclidean geometry. The passage in question is worth quoting:

> In the light of the theorem just given [Klein referred to Theorem 1], the ground for Clifford's new theory of parallels is attained. What parallel lines in parabolic (Euclidean) space are, is a well established notion. As for the definition of parallel lines in non–Euclidean space, we simply have to pay attention to the fact that the definition must coincide with the ordinary Euclidean one as soon as the non–Euclidean space reduces to the Euclidean one. This condition is fulfilled through the ordinary position according to which lines are called parallel that intersect in a point at infinity (i.e. in a point of the fundamental surface of second degree). However, as Clifford has emphasized, the parallel lines thus defined lose almost all the elegant properties that one encounters in Euclidean space. These properties, according to Clifford, rely essentially upon this: by the action of certain motions of the space, Euclidean parallel lines can be displaced along themselves. But the lines that belong to a congruence (of either type) just referred to have precisely this property in non–Euclidean space.

Thus, a bundle of Euclidean parallel lines can be regarded as the limiting case (*Ausartung*) of such a congruence. Hence the following suggestion (*Vorschlag*): to treat as parallel lines in non–Euclidean geometry those (skew) lines which belong to the same congruence (either of first or second type) or, which is the same, that intersect the same couple of imaginary generators (either of first or of second type) of our fundamental surface (*Fundamentalflǐche*).

3 An approach based upon differential geometry: Bianchi and Fubini

Soon after Klein's reexamination of Clifford's parallelism in the framework of projective geometry, Luigi Bianchi devoted a great deal of attention to it. As early as his degree thesis (1877) Bianchi explicitly referred to Klein's geometrical work, namely [Klein 1871b], as a valuable source of inspiration for his early investigations on the applicability of surfaces in spaces of constant curvature.

After his graduation at the Scuola Normale Superiore in Pisa, where he was student of Enrico Betti and Ulisse Dini, Bianchi spent a year (Autumn 1879 to Autumn 1880) of study under Klein's guidance in Munich, during which, besides dealing with the theory of elliptic curves, he surely had the possibility of better assimilating Klein's viewpoint on non–Euclidean geometries.

The first couple of papers[18] by Bianchi on the subject of Clifford's parallelism appeared in 1895 and 1896 when Bianchi tackled the subject of Clifford's zero–curvature surface that was mentioned before. In 1899, Bianchi included some of his results on this subject in the German edition of his lectures on differential geometry. But it was in the greatly augmented second Italian edition of his *Lezioni di Geometria Differenziale* that Bianchi presented a complete account of Clifford's parallels in a chapter devoted to spaces of constant curvature.

Bianchi's approach was based upon a reinterpretation of Clifford's and Klein's results in the light of metric Riemannian geometry to which he had been introduced by the works of Eugenio Beltrami and Dini himself. After discussing the typical form of the metric for spaces of constant curvature, Bianchi derived the metric of the elliptic space from the Cayley–Klein distance as follows.

He first introduced homogeneous coordinates $[x_0, \ldots, x_n]$ for the (complex) projective space and fixed the quadric surface $\Omega(x, x) = \sum_{i=0}^{n} x_i^2 = 0$, in order to obtain the so–called geodetic representation of elliptic space in Weierstrass's coordinates. This was a particular representation of spaces with constant curvature that Beltrami had already used in his classical memoirs [Beltrami 1868a] and [Beltrami 1868b]. The denomination "geodetic representation" means that geodesics of the non–Euclidean space and those of the representative Euclidean space are paired off by the representation map.[19] Furthermore, Weierstrass's coordinates are deduced from Beltrami's geodetic representation by passing to homogeneous coordinates. By indicating with $\mathrm{d}(x, x')$ the Cayley–Klein distance between two points x, x', one has, in accordance with (7):

$$\cos\left(\frac{\mathrm{d}}{R}\right) = \frac{\Omega(x, x')}{\sqrt{\Omega(x, x) \cdot \Omega(x', x')}}, \tag{8}$$

[18] [Bianchi 1895] and [Bianchi 1896].

[19] In [Beltrami1865] Beltrami has shown that surfaces of constant curvature can be represented upon a plane in such a way that geodetic lines are mapped into straight lines and also that this property is shared by no other surfaces.

where R is a real constant.

Interestingly, Bianchi did not deduce the Cayley–Klein metric from projective considerations; on the contrary, he observed that (8) essentially coincides with the expression for the length of a geodetic arc in Beltrami's (geodetic) representation as given for example in [Beltrami 1868b]. In this respect, it should be borne in mind that Klein himself had been explicit in recognizing the essential equivalence of Beltrami's formulas with those of the so–called Cayley–Klein metric for non–Euclidean geometries when he remarked that "[...] there is barely a step to be taken to pass from Beltrami's formulas to those of Cayley".[20]

In order to derive the expression for the Riemannian metric corresponding to (8), Bianchi considered two infinitesimally near points x_i and $x_i + dx_i$ upon an arbitrary curve and indicated with ϵ the infinitesimal increment of the arc–length in passing from x_i to $x_i + dx_i$. Then he posed

$$\mathrm{d} = \epsilon \qquad x'_i = x_i + \frac{dx_i}{ds}\epsilon + \frac{1}{2}\frac{d^2x_i}{ds^2}\epsilon^2 + o(\epsilon^2)$$

and, upon replacement of these expressions into (8), he obtained:

$$1 - \frac{\epsilon^2}{2R^2} + o(\epsilon^2) = \Omega(x,x) + \epsilon\sum_{i=0}^{n} x_i\frac{dx_i}{ds} + \frac{1}{2}\sum_{i=0}^{n} x_i\frac{d^2x_i}{ds^2}\epsilon^2 + o(\epsilon^2). \tag{9}$$

By equating the coefficients of the terms in ϵ^2, Bianchi deduced $\sum x_i \dfrac{d^2x_i}{ds^2} = \dfrac{1}{R^2}$ and thus, as a consequence of the fact that one can suppose $\Omega(x,x) = \sum_{i=0}^{n} x_i^2 = 1$, he could conclude that $\sum_{i=0}^{n}\left(\dfrac{dx_i}{ds}\right)^2 = \dfrac{1}{R^2}$ or, equivalently, that $ds^2 = R^2\sum dx_i^2$, where it is understood that the x_i's are not independent variables but they are linked by $\Omega(x,x) = 1$.

In this framework, as was said, geodetic lines are mapped into straight lines of the representative Euclidean space. Thus, a geodesic is defined by the giving of a point x_i, $i = 0,\ldots,n$ and of the direction parameters ξ_i, $i = 0,\ldots,n$ of the normal plane to it[21]. The coordinates of any other point x'_i belonging to the geodesic is given by the equations

$$x'_i = \lambda x_i + \mu\xi_i, \qquad i = 0,\ldots,n$$

where $\lambda^2 + \mu^2 = 1$. Now let d indicate the distance between x_i and x'_i. Then, as a consequence of (8), Bianchi obtained $\lambda = \cos\dfrac{\mathrm{d}}{R}$ and $\mu = \sin\dfrac{\mathrm{d}}{R}$ and thus:

$$x'_i = \cos\left(\frac{\mathrm{d}}{R}\right)x_i + \sin\left(\frac{\mathrm{d}}{R}\right)\xi_i, \qquad i = 0,\ldots,n.$$

It is then evident that straight lines are closed and of a finite length that is equal to πR.

On the basis of these results, which were specialized to the case $n = 3$, Bianchi set out to provide a discussion of Clifford's parallelism that, contrary to Clifford's and Klein's presentations, started from a definition of the elliptic scrolls Ψ (*scorrimenti*) as those motions of the elliptic space which are characterized by the property that $\mathrm{d}(x,\Psi x) = \mathrm{cost}$ for every x. Their characterization as biaxial homographies of the underlying projective space, whose axes are identified with a couple of generators of the absolute quadric, was obtained as a consequence of the aforementioned metric definition.

[20] See [Klein 1871b, p. 578].

[21] ξ_i are such that they satisfy $\sum_i \xi_i x_i = 0$.

In 1900, Guido Fubini, one of Bianchi's most brilliant students at the Scuola Normale Superiore in Pisa, dealt with Clifford's parallelism in his degree thesis. Once again, the approach was that of metric geometry. Fubini's main achievement consisted of an important technical innovation that obviated the need for extremely long and tedious calculations in the study of curves and surface in elliptic space. This was the introduction of new line coordinates, which Fubini called scroll parameters (*parametri di scorrimento*). The idea was simple but at the same time very ingenious. As was seen, a straight line (i.e. a geodetic line of the elliptic space) could be identified by means of the coordinates (x_0, x_1, x_2, x_3) of one of its points and the coordinates $(\xi_0, \xi_1, \xi_2, \xi_3)$ of a plane normal to it in (x_0, x_1, x_2, x_3). In other words, a straight line r is determined by specification of two conjugate points upon it. Now, there exist two scrolls (one right–handed and one left–handed) which take the point (x) to the point (ξ). Their parameters are (A, B, C, D) and $(\alpha, \beta, \gamma, \delta)$. As a consequence of $\sum_{i=0}^{3} x_i \xi_i = 0$ and of the explicit expressions for scrolls (5)-(6), one gets $A = \alpha = 0$ and $B^2 + C^2 + D^2 = \beta^2 + \gamma^2 + \delta^2 = 1$. These quantities are what Fubini termed the scroll parameters of the line. It is easy to see that they can serve as line coordinates since they are nothing other than a linear combination of the usual Plücker coordinates. Indeed, by indicating with $p_{ik} = x_i \xi_k - x_k \xi_i$ $(i, k = 0, \ldots, 3)$ the Plücker coordinates of the line, the scroll parameters take on the following form:

$$\begin{cases} B = p_{01} + p_{23}, & C = p_{02} + p_{31}, \quad D = p_{03} + p_{12} \\ \beta = p_{01} - p_{23}, & \gamma = p_{02} - p_{31}, \quad \delta = p_{03} - p_{12}. \end{cases} \tag{10}$$

An elementary application of this new notion led to a straightforward procedure for the determination of the two parallels through a point $y = (y_0, y_1, y_2, y_3)$ to a given line. Indeed, the scroll parameters realize the following property which Fubini termed *invariance under parallelism*. This means that if two lines r, r' have equal (in module) parameters of the same triple, B, C, D or β, γ, δ, then the two lines are Clifford parallel; in particular, if $(B, C, D) = \pm(B', C', D')$, r and r' are right Clifford's parallels, while if $(\beta, \gamma, \delta) = \pm(\beta', \gamma', \delta)$, r, r' are left Clifford's parallel. As a consequence of this, we get the following expressions for the cosine directions $(\bar{\xi})$ of the two parallels:

$$\begin{cases} \bar{\xi}_0 = -By_1 - Cy_2 - Dy_3 \\ \bar{\xi}_1 = By_0 - Dy_2 + Cy_3 \\ \bar{\xi}_3 = Cy_0 + Dy_1 - By_3 \\ \bar{\xi}_4 = Dy_0 - Cy_1 + By_2, \end{cases} \tag{11}$$

for the right Clifford parallel to r;

$$\begin{cases} \bar{\xi}_0 = -\beta y_1 - \gamma y_2 - \delta y_3 \\ \bar{\xi}_1 = \beta y_0 + \delta y_2 - \gamma y_3 \\ \bar{\xi}_3 = \gamma y_0 - \delta y_1 + \beta y_3 \\ \bar{\xi}_4 = \delta y_0 + \gamma y_1 - \beta y_2. \end{cases} \tag{12}$$

for the left Clifford parallel. Another application of scroll parameters, which is useful for our purposes,[22] was offered by easy determination of the angle of parallelism (*angolo di parallelismo*) between a pair of Clifford parallels through a point to a given straight line r. The scope of Fubini's thesis went far beyond the mere introduction of more efficient techniques of calculus; indeed, it made good use of the notion of scroll parameter by applying it to a wide class of problems, already tackled by Bianchi in [Bianchi 1896], in the realm of the theory of curves and surfaces in elliptic 3–space.

[22] See next section.

4 Absolute parallelism *vs.* Levi–Civita's parallelism: Enea Bortolotti

Clifford's parallelism had been considered for decades as an isolated phenomenon which was not susceptible of being interpreted within a general theoretical framework. Still in 1917, when Levi–Civita first proposed a geometrical interpretation of Christoffel 3–index symbols by introducing the notion of metric connection in Riemannian geometry,[23] it was remarked that Clifford's parallelism (together with Lobachevsky parallelism of hyperbolic geometry) represented, in a way, an anomaly that could not be framed within the new theoretical scheme. Indeed, as Levi–Civita explicitly observed, although elliptic space (regarded as a Riemannian manifold of constant curvature) could be equipped with Levi–Civita's parallelism, nonetheless Clifford's parallelism constituted a distinct phenomenon.

The following fact can be singled out as the most outstanding difference between the two: while Levi–Civita's prescription for parallel displacement depended essentially upon the chosen path, the notion of Clifford's parallelism was completely independent of the path, thus qualifying it as a kind of absolute parallel displacement.

Interestingly, Levi–Civita chose a rather roundabout strategy to prove the irreducibility of one kind of parallelism to the other. The matter was tackled by him in [Levi-Civita 1917, §6]. He first specialized his notion of parallelism to the case of manifolds with constant positive Riemannian curvature and observed that, when such manifolds are interpreted as submanifolds of Euclidean–space, there is a close link between the ordinary parallelism of the ambient space and Levi–Civita's parallelism of the immersed manifold. Indeed, let the submanifold V_n be represented by the the equation:

$$\sum_{i=0}^{n} x_i^2 = 1. \tag{13}$$

A vector stemming from V_n is identified with a direction of the surrounding Euclidean space. Its components, denoted by α_i, satisfy the following condition which translates the fact that the direction belongs to the hyperplane tangent to V_n:

$$\sum_{i=0}^{n} \alpha_i x_i = 0. \tag{14}$$

On the basis of these premises, Levi–Civita was able to characterize the notion of parallelism upon V_n in terms of the ordinary parallelism in Euclidean space.

Theorem 2 (Levi–Civita 1917) *Let C be a geodetic curve in V_n. Let $\vec{\alpha} = (\alpha_0, \ldots, \alpha_n)$ be a direction orthogonal to C at some point $x \in C$. Then, the conditions which guarantee that $\vec{\alpha}(s)$ (now consider as a function of the arc–length s) is parallel displaced along C are $\alpha_i'(s)$=0, $i = 0, \ldots, n$. In other words, as far as orthogonal directions are concerned, Levi–Civita's parallelism in V_n coincides with ordinary parallelism in the surrounding Euclidean space.* [Levi-Civita 1917, p. 18]

If this result is specialized to the case $n = 3$, Levi–Civita observed, it is clear that the notion of parallelism just introduced has no relation at all with the so–called Clifford's parallelism. Indeed, if one considers a geodetic line C and considers through each point of it a line belonging to the same Clifford's congruence, the lines so obtained, though orthogonal to C, are not parallels (in the ordinary sense) of the surrounding space.

[23] On the history of Levi–Civita's parallelism and its early reception, see [Reich 1992].

As Levi–Civita proved in full detail,[24] it is indeed impossible to find a geodetic curve C through each point of which lines belonging to the same Clifford's congruence are drawn with equal inclination (as would be required by ordinary parallelism).

In his lectures on "infinitesimal Geometry" which he delivered at the University of Pisa in 1922, Bianchi provided a treatment of Levi–Civita's innovative views "by presenting them in analytical form without recourse to infinitesimal considerations which had represented Levi–Civita's starting point [Bianchi 1922, p. 150]". On this occasion, Bianchi criticized Levi–Civita's employment of the denomination "parallelismo" to denote the notion that Levi–Civita had introduced in 1917. A partial justification of such usage, Bianchi admitted, was provided by the fact that Levi-Civita's parallelism coincides with the ordinary parallelism in the case of Euclidean space. Nonetheless, he wrote:

> [...] in every other curved space, including all spaces with non–vanishing constant Riemannian curvature, Levi–Civita's parallelism is essentially bound to the path along which the displacement is carried out. This circumstance, together with the not irrelevant fact that in the geometry of spaces with constant, non–vanishing curvature the name parallelism is employed with a distinct and absolute meaning (Lobachevsky's parallelism and Clifford's parallelism) render it doubtful whether the usage of this word should be regarded as appropriate. Instead, I believe it more convenient to speak of *bound* parallelism of Levi–Civita's type.

Obviously, Bianchi's criticism was not directed towards the notion of Levi–Civita's parallelism in itself which, on the contrary, he regarded very highly as a herald of fertile advances. Nonetheless, his perplexity upon an apparently lexical issue hid a much more fundamental question pertaining the relationship between Levi–Civita's parallelism and those types of parallelism (*in primis* Clifford's parallelism) which were termed by him absolute, in view of their independence from the path of displacement.

Some years later, Enea Bortolotti[25] (1896-1942), a former student of Bianchi at the *Scuola Normale Superiore* in Pisa, took up these observations and produced a detailed treatment of Clifford's (and Lobachevsky's) parallelism that aimed at reinterpreting it in the light of the absolute differential calculus of Ricci and at providing an intrinsic characterization thereof.

Clearly influenced by Bianchi's views, in the introduction to [Bortolotti 1925] where Bortolotti explained the main motivation at the basis of his investigations, he wrote:

> Already in 1917, in his classical memoir on *parallelism in any variety*, Levi–Civita remarked that the notion introduced by him was totally distinct from that of Clifford in spaces S_3 with constant curvature. Also, the parallelism of Lobachevsky, which can be defined in any S_3 with negative constant curvature, as is well-known, never coincides with that of Levi–Civita. The only exception is represented by Euclidean space, in which case all three types of parallelism coincide. This circumstance induces us to think that, despite the fact that the notions of *absolute parallelism* according to Clifford and Lobaschevsky are distinct from that of *bound* (*vincolato*) parallelism of Levi–Civita and that it is not possible to reduce one to the other, there should exist between them some noteworthy relationship [Bortolotti 1925, p. 821].

[24] See in particular footnote 13 in [Levi-Civita 1917, p. 18].

[25] For a biographical sketch of E. Bortolotti, see [Bompiani 1942]. Bortolotti's interest in the theory of connection, namely in the notion of Levi–Civita's parallelism can be traced back to January 1923, when he wrote to Levi–Civita to ask him for some reading advice on the topic. See Archivio Levi–Civita, Biblioteca della Accademia dei Lincei, E. Bortolotti, 1923.

Bortolotti's primary object was thus to draw a comparison between Levi–Civita's parallelism on the one side and the parallelisms of Clifford and of Lobaschevsky on the other. The result was unexpected since he was able to obtain an invariant characterization of the latter in terms of the former. This achievement could be attained using the notion of *associated directions* that Bianchi had recently introduced in [Bianchi 1922] while providing his own reformulation of Levi–Civita's parallelism. This innovation grew out of Bianchi's investigation of what he termed *subordinate parallelism*, that is into the notion of parallelism on Riemannian submanifolds V_m that is naturally induced by Levi–Civita's parallelism in the ambient manifold V_n $(1 < m < n)$.

In modern terms, Bianchi's associated directions are, modulo an invariant factor, no other than the values of the covariant derivative of the vector field ξ along the path γ. Bianchi posed:

$$\Omega_i = \frac{d\xi_i}{dt} + \sum_{\lambda,\mu=1}^{n} \left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\} \xi_\lambda \frac{dx_\mu}{dt}, \tag{15}$$

and defined the associated directions η_i, $i = 1, \ldots, n$ by means of the following formulas:

$$\eta_i = C\Omega_i, \qquad i = 1, \ldots, n, \tag{16}$$

where R is a scalar quantity defined by

$$\frac{1}{C} = \sqrt{\sum_{ik} a_{ik}\Omega_i\Omega_k}$$

Since, as was proved by Bianchi in full detail, the quantities Ω_i transform under a change of coordinates in a controvariant manner, $\frac{1}{C}$ is an invariant. Bianchi termed it *associated curvature* (*curvatura associata*). Its geometrical meaning can be described as a measure of the rapidity of deviation from Levi–Civita's parallelism of the directions $\xi(t)$ along the curve γ.

In view of its intuitive geometrical interpretation, it must have been a natural choice for Bortolotti to employ Bianchi's notion of associated directions in order to provide a characterization of Clifford's parallelism in terms of its deviation from Levi–Civita's. Let us see in some detail how this characterization was attained in [Bortolotti 1925].

In following Bianchi's treatment as described in the preceding section, Bortolotti employed Weierstrass' coordinates (x_0, x_1, x_2, x_3) for the elliptic 3–space; the corresponding metric is thus given by $ds^2 = R^2 \sum_{i=0}^{3} dx_i^2$, where, as usual, it is intended that x_0, x_1, x_2, x_3 are linked by the relation $\sum_{i=0}^{3} x_i^2 = 1$. The first step of Bortolotti's analysis consisted in deriving the analytical expression for finite Clifford's displacement (*trasporto finito per parallelismo di Clifford*), i.e. the analytical prescription according to which a direction ξ at a given point x is Clifford parallelly displaced at another point $\bar{x}$ a finite distance from x. To this end, it was crucial to use Fubini's scroll parameters, which we referred to in the previous section. Let r be a geodesic in S^3, i.e. a straight line of the associated projective space; let x_i and ξ_i indicate the coordinates of one of its point P and the coordinates of the normal plane in P, respectively. Then, following Fubini, the scroll parameters of r are given by

$$\left\{ \begin{array}{l} B = (\xi_1 x_0 - \xi_0 x_1) \pm (\xi_3 x_2 - \xi_2 x_3) \\ C = (\xi_2 x_0 - \xi_0 x_2) \pm (\xi_1 x_3 - \xi_3 x_1) \\ D = (\xi_3 x_0 - \xi_0 x_3) \pm (\xi_2 x_1 - \xi_1 x_2), \end{array} \right. \tag{17}$$

where it is intended that the upper and the lower signs refer to right–handed and left–handed scrolls, respectively. After that the six scroll parameters of r have been calculated, the directions $\bar{\xi}$ of the right and left Clifford's parallels drawn through a point $\bar{x}$ are given by the following equations:

$$\begin{cases} \bar{\xi}_0 = -B\bar{x}_1 - C\bar{x}_2 - D\bar{x}_3 \\ \bar{\xi}_1 = B\bar{x}_0 \mp D\bar{x}_2 \pm C\bar{x}_3 \\ \bar{\xi}_2 = C\bar{x}_0 \pm D\bar{x}_1 \mp B\bar{x}_3 \\ \bar{\xi}_3 = D\bar{x}_0 \mp C\bar{x}_1 \pm B\bar{x}_2. \end{cases} \tag{18}$$

Upon substitution of the expressions for B, C, D as given by (17), Bortolotti finally obtained the equation for finite Clifford's displacement (there are actually two of them):

$$\bar{\xi}_i = \mp(-1)^i \begin{vmatrix} \bar{x}_k & \bar{x}_r & \bar{x}_s \\ x_k & x_r & x_s \\ \xi_k & \xi_r & \xi_s \end{vmatrix} - x_i \sum_{\lambda=0}^{3} \bar{x}_\lambda \xi_\lambda + \xi_i \sum_{\lambda=0}^{3} \bar{x}_\lambda x_\lambda, \tag{19}$$

where $(ikrs)$ is any even permutation of (0123).

Interestingly, Bortolotti provided a very intuitive description of the geometrical significance of Clifford's displacement as compared to Levi–Civita's displacement *along geodetic lines*. He obtained the following[26]

Theorem 3 (Bortolotti 1925) *If a given direction ξ is (Levi–Civita) parallelly displaced along an orthogonal geodesic, then at every point of the geodesic the Levi–Civita's parallel is the bisector of the two Clifford's parallels.*

As for Levi-Civita's parallelism in S_3, Levi-Civita himself had explicitly written down[27] the differential equations for parallel displacement of a given vector ξ_i, $(i = 0, \ldots, 3)$ (to be regarded as function of the arc length s of a curve γ) along a curve $x_i(s) = \gamma(s)$. By indicating with $\lambda_i = R\dfrac{dx_i}{ds}$, $(i = 0, \ldots, 3)$ the direction cosines of the tangent vector $\dfrac{d\vec{x}}{ds}$ and with $\cos\phi$ the cosine of the angle between the vectors $\vec{\lambda}$ and $\vec{\xi}$ (i.e. $\cos\phi = \sum_i \lambda_i \xi_i$); these equations read as follows:

$$\frac{d\xi_i}{ds} = -\frac{\cos\phi}{R} x_i, \qquad i = 0, \ldots, 3. \tag{20}$$

At this point, Bortolotti recalled that equations (20) could be most easily integrated along a geodesic to give:

$$\bar{\xi}_i^L = \xi_i - \cos\phi \frac{\cos\dfrac{s}{R} - 1}{\sin\dfrac{s}{R}} \left(x_i + \bar{x}_i\right), \tag{21}$$

where it is intended that $\bar{\xi}^L$ indicates the direction parallel to ξ in the point $\bar{x}$ that is obtained by Levi–Civita's displacement along the geodetic line joining x to $\bar{x}$; s is the arc–length of the geodesic joining x to $\bar{x}$. Now, by defining ψ to be the angle between the two Clifford's parallels $\bar{\xi}^C$ and the Levi–Civita's parallel $\bar{\xi}^L$, i.e. $\cos\psi = \sum_{i=0}^{3} \bar{\xi}_i^C \bar{\xi}_i^L$, it immediately follows from equations (19) and (21) that, *independently* from the kind of Clifford's parallelism, one has:

$$\cos\psi = \cos\phi \left(\cos\frac{s}{R} - 1\right) + \cos\frac{s}{R}. \tag{22}$$

[26]The same result had been obtained by Cartan in [Cartan 1924].

[27]See [Levi-Civita 1917, p. 18].

The assertion of the theorem is readily deduced by setting $\cos\phi = 0$. Indeed, as Bortolotti observed, one has $\cos\psi = \cos\frac{s}{R}$, i.e., $\psi = \frac{s}{R}$ which is exactly one half of the angle between the two Clifford's parallels.[28]

Theorem (3) provided a vivid illustration of the geometrical relationship between the two parallelisms; nonetheless, Bortolotti aimed at characterization of this relationship also at an infinitesimal level, that is by means of the differential equations presiding over parallel displacement. To this end, he first deduced the infinitesimal counterpart of equations (19); by setting $\lambda_i = \frac{1}{R}\frac{dx_i}{ds}$ ($x_i(s), i = 0, \ldots, 3$ is now a generic curve, not necessarily a geodesic), he obtained:

$$\frac{d\xi_i^C}{ds} = \mp\frac{(-1)^i}{R}\begin{vmatrix}\lambda_k & \lambda_r & \lambda_s\\ x_k & x_r & x_s\\ \xi_k & \xi_r & \xi_s\end{vmatrix} - x_i\frac{\cos\phi}{R}, \tag{23}$$

where $(ikrs)$ is an even permutation of (0123). Now, since equations (20) could be interpreted by saying that the covariant derivative of ξ along the curve $x = x(s)$, which is the same as Bianchi's associated direction Ω_i, is vanishing, equations (23) could be rewritten as[29]:

$$\Omega_i = \mp\frac{(-1)^i}{R}\begin{vmatrix}\lambda_k & \lambda_r & \lambda_s\\ x_k & x_r & x_s\\ \xi_k & \xi_r & \xi_s\end{vmatrix} \tag{24}$$

Correspondingly, Bortolotti determined Bianchi's associated curvature C that in this case is easily seen to be equal to: $C = \frac{1}{R}|\sin\phi|$, ϕ being $\widehat{\lambda\xi}$.

This achievement constituted the starting point of an extensive research programme that Bortolotti developed in a series of papers that appeared in late 1920s. His aim was to generalize Clifford's (and Lobachevsky's) parallelism to general Riemannian manifolds other than those of constant Riemannian curvature. A first result in this direction, which was limited to the case of 3–dimensional Riemannian manifolds, was attained already in [Bortolotti 1925] where Bortolotti indicated a generalization of the parallel displacement defined by equations (23). The idea was to replace the curvature $K = \frac{1}{R^2}$ of the space by the sectional curvature of the 2–plane generated by ξ and λ. Further investigations into absolute parallelism were carried out by him in a brief memoir [Bortolotti 1927] presented by Levi–Civita to the *Reale Istituto Veneto* in January 1927, where he provided a generalization of a type of absolute parallelism in Riemannian manifolds introduced by Schouten and Cartan in 1926 in the realm of the geometry of Lie groups.[30] Over the following years, Bortolotti developed a peculiar approach to the theory of connections that took great profit of Giuseppe Vitali's researches on absolute differential calculus. Before his premature death in 1942, Bortolotti became the undisputed Italian leader in the field of connection theory. As Fabio Conforto put it, Bortolotti's "main scientific achievements are concerned with the study of spaces with connection, a theory which he brought back to Italy where it was born [undoubtedly, here Conforto referred to Levi–Civita's 1917 paper]; Bortolotti enriched it with new concepts and results by constantly striving to grasp the geometrical significance of the analytical developments beyond the intricacy of the formalism".

[28] As is clear from its expression, the angle between the two Clifford's parallels depends upon the distance of the point x_i and $\bar{x}_i$. For an easy computation of this angle (first calculated by Fubini in [Fubini 1900]), see e.g. [Bianchi 1902, p. 453].

[29] An equivalent form for Clifford's displacement which does not require any recourse to Weierstrass' coordinates was provided by Bortolotti in [Bortolotti 1930, p. 243].

[30] See [Cartan–Schouten 1926a].

5 Conclusions

Starting from the early 1920s, Clifford's parallelism, which until then had represented an isolated phenomenon, could finally be understood within the theoretical framework of the dawning theory of connections. In this respect the contribution of Bortolotti constituted one of the first reflections of this general tendency. Almost at the same time, in a joint work [Cartan–Schouten 1926a] with Jan Arnoldus Schouten, Cartan succeeded in providing group-theoretic grounds for the explanation of this phenomenon.[31] By using techniques and notions stemming from his theory of generalized (non–holonomic) spaces he was able to interpret Clifford's parallelism as particular case of a property shared by all Lie groups. More explicitly, Cartan and Schouten's main achievement consisted in the discovery of three different types of connections with which a given Lie group manifold can be endowed. Two of them define an absolute parallelism (precisely as Clifford's parallelism is) and the other, insofar as far as the Lie group can be provided with a Riemannian structure, defines a parallelism of Levi–Civita's type. As a consequence of the fact that the elliptic $3-$space can be regarded as a simple compact Lie group, a higher standpoint was thus obtained in virtue of which a clear comprehension of the relation between Levi–Civita's and Clifford's parallelisms was finally attained.

[Cartan–Schouten 1926a] marked somehow the beginning of a new and most interesting story which eventually led Cartan (and Schouten) to build up a new discipline of research, the geometry of Lie groups. According to this innovative standpoint, which implied at the same time major technical innovation with respect to classical differential geometry, Lie groups came to be regarded as proper abstract manifolds deserving of geometrical investigation in their own respect.

6 Appendix

This section provides some elementary material concerning projective geometry and the theory of polarity with respect to a given quadric. Since the discussion of sections (1) and (2) heavily relies upon these notions, it appears useful to gather some relevant information in the present appendix.[32]

A second order surface, i.e. a *quadric*, is the locus of points (either complex or real) in the 3–dimensional projective space that satisfies a second-order equation; by employing homogeneous coordinates (x_0, x_1, x_2, x_3), the equation of a quadric $\mathcal{Q}$ can be written as follows:

$$f(x) = \sum_{i,j=0}^{3} a_{ij} x_i x_j = 0, \tag{25}$$

where a_{ij} are symmetric constant coefficients. Consider now two distinct points of the space $P = (x_i), P' = (y_i)$; let PP' be the straight line connecting them consisting of the points of type $(kx_i + y_i)$, $i = 0, \ldots, 3$. The intersections of PP' with $\mathcal{Q}$ are easily obtained by replacing in (25) x_i with $kx_i + y_i$, $i = 0, \ldots, 3$. By doing so and by setting $f\binom{x}{y} = \sum_{i,j=0}^{3} a_{ij} x_i y_j$, one obtains:

$$k^2 \cdot f(x) + 2k \cdot f\begin{pmatrix} x \\ y \end{pmatrix} + f(y) = 0. \tag{26}$$

[31] We will deal with the collaboration between Cartan and Schouten upon this topic in a forthcoming paper.

[32] For a more extensive treatment, see e.g. [Bianchi 1904], [Castelnuovo 1904] and [Seidenberg 1962] upon which this appendix relies.

Now, by denoting by k_1, k_2 the solutions of (26), the two intersections Q, Q' are given by $Q = (x_i + k_1 y_i)$ and $Q' = (x_i + k_2 y_i)$. In the case in which $f\binom{x}{y} = 0$ (i.e. when k_1, k_2 are equal in modulus but with opposite sign), the cross–ratio of the four point P, P', Q, Q' is harmonic and the points P, P' are said to be *conjugate* with respect to the quadric. The set of points (x_0, x_1, x_2, x_3) which are conjugate to a fixed point (x_0', x_1', x_2', x_3') is given by the equation

$$(a_{00}x_0' + \ldots + a_{03}x_3')x_0 + \ldots + (a_{30}x_0' + \ldots + a_{33}x_3')x_3 = 0.$$

This set of points represents a plane which is called the *polar plane* with respect to the point (x_0', x_1', x_2', x_3') (the *pole* of the plane). In the case of an irreducible quadric, i.e. when the determinant of the matrix $[a_{ij}]$, $i, j = 0, \ldots, 3$ is not equal to zero, the quadric determines a bijective correspondence α between points and planes; to the point (y_0, y_1, y_2, y_3) there corresponds the plane $\sum_{k=0}^{3} \xi_k x_k = 0$, where $\xi_k = \sum_{i=0}^{3} a_{ki} y_i$. Furthermore, if a point P runs along a straight line $P'P''$ so that its coordinates can be written as $x_i = x_i' + kx_i''$, then the coordinates of the corresponding polar plane are given by

$$\xi_i = \xi_i' + k\xi_i'', \qquad i = 0, \ldots, 3.^{33} \tag{27}$$

Any two planes of the form (27) intersect in a straight line that is called the *polar line* with respect to the line $P'P''$. In other terms, while the pole P runs along a straight line r, the corresponding polar planes rotate around a fixed axis which coincides with the polar line $r' = \alpha(r)$. As a consequence of this, any point of a given straight line r is conjugate to *any* point lying on the polar line (with respect to r) r'.

It seems helpful to recall some basic properties of ruled quadric. To this end, we appeal to the synthetic definition of quadric in terms of *reguli* of straight lines. By a *regulus* one means the set of lines meeting three mutually skew lines of the projective space. In addition to the analytical definition given above, a quadric surface can be defined as the set of points (either real or complex) lying on the lines of a regulus (such regulus will be said to be the first regulus of the quadric or the first system of generators of the quadric). It can be proved that any two lines of a regulus are mutually skew. Furthermore, if a line meets three lines of the first regulus of the quadric then it meets all the lines of the regulus. As a consequence of this, the lines that meet all the lines of the first regulus form a new regulus that is called the second regulus of the quadric (or alternatively, the second system of generators of the quadric). In a completely symmetrical way, the first regulus consists of all the lines that meet all the lines of the second regulus. Finally, through any point of the quadric there are exactly two lines belonging to distinct reguli.

In the case of the imaginary quadric surface which is relevant in the context of the Cayley–Klein projective model of elliptic space,

$$\mathcal{Q} = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3(\mathbb{C}) \quad | \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\right\},$$

through every (imaginary) point we have two imaginary lines belonging to distinct reguli whose analytic expression is provided by formulae (1) and (2).

Let us consider the so–called elliptic polarity α induced by the absolute quadric $\mathcal{Q}$ in the real projective space $\mathbb{P}^3(\mathbb{R})$. It can be defined by the following equivalence:

$$(x_0, x_1, x_2, x_3) \in \alpha((y_0, y_1, y_2, y_3)) \Leftrightarrow x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0, \quad x, y \in \mathbb{P}^3(\mathbb{R})$$

Accordingly, we have an induced polarity involving straight lines which is denoted by α, as well. In the light of section (1), Clifford's original definition of parallels can be

[33] It is intended that (x_i') and (x_i'') are homogeneous coordinates of the points P', P'' respectively.

reformulated in the following way: two distinct straight lines a, b in $\mathbb{P}^3(\mathbb{R})$ are said to be Clifford's parallels if the four straight lines $a, b, \alpha(a), \alpha(b)$ belong to one regulus of $\mathbb{P}^3(\mathbb{R})$.[34]

This definition does not require any recourse to a complex extension of the real projective space, which is instead required if one wants to define Clifford's parallels in terms of the generators of the absolute quadric. In order to do so, one has to embed the real projective space into its complex extension $\mathbb{P}^3(\mathbb{C})$. A point $x \in \mathbb{P}^3(\mathbb{C})$ is said to be real if $x = (kx_0, kx_1, kx_2, kx_3)$ for $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $k \in \mathbb{C}$. Points in $\mathbb{P}^3(\mathbb{C})$ which are not real, are said to be imaginary. Accordingly, a straight line l is said to be real if there exist two real points x, y such that l consists of all points of type $\lambda x + \mu y$ for $\lambda, \mu \in \mathbb{C}$.[35] As a consequence of this, a straight line in $\mathbb{P}^3(\mathbb{C})$ is real if $l = \bar{l}$, where the bar denotes complex conjugation.[36] It is easy to see that a real line in $\mathbb{P}^3(\mathbb{C})$ is completely determined by one imaginary point. In this context, it is important to note that a line in $\mathbb{P}^3(\mathbb{R})$ was usually identified (for example, by Klein) with its complex extension in $\mathbb{P}^3(\mathbb{C})$.

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[34] This is the formalization of Clifford's original definition provided in [Betten and Riesinger 2012, pp. 32-33].

[35] In in this way a line in $\mathbb{P}^3(\mathbb{R})$ can be extended to a real line in $\mathbb{P}^3(\mathbb{C})$.

[36] It should be noticed that a real line l actually contains imaginary points. However, if $z \in l$ then also $\bar{z} \in l$.

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