NODAL SOLUTIONS FOR THE CHOQUARD EQUATION

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Abstract. We consider the general Choquard equations

$$-\Delta u + u = \bigl(I_\alpha * |u|^p\bigr)|u|^{p-2}u$$

where I_α is a Riesz potential. We construct minimal action odd solutions for $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and minimal action nodal solutions for $p \in (2, \frac{N+\alpha}{N-2})$. We introduce a new minimax principle for least action nodal solutions and we develop new concentration-compactness lemmas for sign-changing Palais–Smale sequences. The nonlinear Schrödinger equation, which is the nonlocal counterpart of the Choquard equation, does not have such solutions.

Contents

1. Introduction

We study the general Choquard equation

$$(\mathcal{C}) \qquad - \Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 1$, $\alpha \in (0, N)$ and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined at each point $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \qquad \text{where} \qquad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}.$$

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When $N = 3$, $\alpha = 2$ and $p = 2$, the equation ($\mathcal{C}$) has appeared in several contexts of quantum physics and is known as the *Choquard–Pekar equation* [15, 24], the *Schrödinger–Newton equation* [13, 14, 20] and the *stationary Hartree equation*.

The action functional $\mathcal{A}$ associated to the Choquard equation ($\mathcal{C}$) is defined for each function u in the Sobolev space $H^1(\mathbb{R}^N)$ by

$$\mathcal{A}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p.$$

In view of the Hardy–Littlewood–Sobolev inequality, which states that if $s \in (1, \frac{N}{\alpha})$ then for every $v \in L^s(\mathbb{R}^N)$, $I_\alpha * v \in L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |I_\alpha * v|^{\frac{Ns}{N-\alpha s}} \le C\Big(\int_{\mathbb{R}^N} |v|^s\Big)^{\frac{N}{N-\alpha s}}, \tag{1.1}$$

(see for example [16, theorem 4.3]), and of the classical Sobolev embedding, the action functional $\mathcal{A}$ is well-defined and continuously differentiable whenever

$$\frac{N-2}{N+\alpha} \le \frac{1}{p} \le \frac{N}{N+\alpha}.$$

A natural constraint for the equation is the *Nehari constraint* $\langle \mathcal{A}'(u), u\rangle = 0$ which leads to search for solutions by minimizing the action functional on the *Nehari manifold*

$$\mathcal{N}_0 = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \;:\; \langle \mathcal{A}'(u), u\rangle = 0\}.$$

The existence of such a solution has been proved when

$$\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha};$$

these assumptions are optimal [15, 17, 21].

We are interested in the construction of *nodal solutions* to ($\mathcal{C}$), that is, solutions to ($\mathcal{C}$) that change sign. The easiest way to construct such solutions is to impose an odd symmetry constraint. More precisely we consider the Sobolev space of odd functions

$$H^1_{\mathrm{odd}}(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) \;:\; \text{for almost every } (x', x_N) \in \mathbb{R}^N,\; u(x', -x_N) = -u(x', x_N)\},$$

we define the odd Nehari manifold

$$\mathcal{N}_{\mathrm{odd}} = \mathcal{N}_0 \cap H^1_{\mathrm{odd}}(\mathbb{R}^N)$$

and the corresponding level

$$c_{\mathrm{odd}} = \inf_{\mathcal{N}_{\mathrm{odd}}} \mathcal{A}.$$

Our first result is that this level c_{odd} is achieved.

Theorem 1. *If $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$, then there exists a weak solution $u \in H^1_{\mathrm{odd}}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ to the Choquard equation ($\mathcal{C}$) such that $\mathcal{A}(u) = c_{\mathrm{odd}}$. Moreover, u has constant sign on each of the half-spaces $\mathbb{R}^N_+$ and $\mathbb{R}^N_-$ and u is axially symmetric with respect to an axis perpendicular to $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \{0\}$.*

Nodal solutions with higher level of symmetries and thus larger action have already been constructed [8–10].

The proof of theorem 1 relies on two ingredients: a compactness property up to translation under the strict inequality $c_{\text{odd}} < 2c_0$ obtained by a concentration–compactness argument (proposition 2.3) and the proof of the latter strict inequality (proposition 2.4).

Another notion of solution is that of *least action nodal solution*, which has been well studied for local problems [5–7]. As for these local problems, we define the constrained Nehari nodal set (as in the local case, in contrast with $\mathcal{N}_0$ and $\mathcal{N}_{\text{odd}}$, the set $\mathcal{N}_{\text{nod}}$ *is not a manifold*),

$$\mathcal{N}_{\text{nod}} = \{u \in H^1(\mathbb{R}^N) \ : \ u^+ \neq 0 \neq u^-, \\ \langle \mathcal{A}'(u), u^+ \rangle = 0 \text{ and } \langle \mathcal{A}'(u), u^- \rangle = 0\},$$

where $u^+ = \max(u, 0) \geq 0$ and $u^- = \min(u, 0) \leq 0$. (In contrast with the local case, we have for every $u \in \mathcal{N}_{\text{nod}}$, $\langle \mathcal{A}'(u), u^+ \rangle < \langle \mathcal{A}'(u^+), u^+ \rangle$, and thus $u^+ \notin \mathcal{N}_0$ and $u^- \notin \mathcal{N}_0$.) We prove that when $p > 2$, the associated level

$$c_{\text{nod}} = \inf_{\mathcal{N}_{\text{nod}}} \mathcal{A}$$

is achieved.

Theorem 2. *If* $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{1}{2}$, *then there exists a weak solution* $u \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ *to the Choquard equation* ($\mathcal{C}$) *such that* $\mathcal{A}(u) = c_{\text{nod}}$, *and* u *changes sign.*

The restriction on the exponent p can only be satisfied when $\alpha > N - 4$. We understand that u changes sign if the sets $\{x \in \mathbb{R}^N \ : \ u(x) > 0\}$ and $\{x \in \mathbb{R}^N \ : \ u(x) < 0\}$ have both positive measure.

We do not know whether the solutions constructed in theorem 2 are odd and coincide thus with those of theorem 1 or even whether the solutions of theorem 2 have axial symmetry as those of theorem 1. We leave these questions as open problems.

The proof of theorem 2 is based on a new reformulation of the minimization problem as a minimax problem that allows to apply a minimax principle with location information (proposition 3.2) and a new compactness property up to translations under the condition $c_{\text{nod}} < 2c_0$ proved by concentration–compactness (proposition 3.5), in the proof of which we introduce suitable methods and estimates (see lemma 3.6). The latter strict inequality is deduced from the inequality $c_{\text{nod}} \leq c_{\text{odd}}$.

Compared to theorem 1, theorem 2 introduces the additional restriction $p > 2$. This assumption is almost optimal: in the locally sublinear case $p < 2$, the level c_{nod} is not achieved.

Theorem 3. *If* $\max(\frac{N-2}{N+\alpha}, \frac{1}{2}) < \frac{1}{p} < \frac{N}{N+\alpha}$, *then* $c_{\text{nod}} = c_0$ *is not achieved in* $\mathcal{N}_{\text{nod}}$.

Theorem 3 shows that minimizing the action on the Nehari nodal set does not provide a nodal solution; there might however exist a minimal action nodal solution that would be constructed in another fashion.

We do not answer in the present work whether c_{nod} is achieved when $p = 2$ and $\alpha > N - 4$. In a forthcoming manuscript in collaboration with V. Moroz, we extend theorem 3 to the case $p = 2$ by taking the limit $p \searrow 2$ [12].

If we compare the results in the present paper to well-established features of the *stationary nonlinear Schödinger equation*

$$-\Delta u + u = |u|^{2p-2} u, \tag{1.2}$$

which is the local counterpart of the Choquard equation ($\mathcal{C}$), theorems 1 and 2 are quite surprising. The action functional associated to (1.2) is defined by

$$\mathcal{A}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} |u|^{2p},$$

which is well-defined and continuously differentiable when $\frac{1}{2} - \frac{1}{N} < \frac{1}{p} < \frac{1}{2}$. Since in this case $\mathcal{A}(u) = \mathcal{A}(u^+) + \mathcal{A}(u^-)$, it can be easily proved by a density argument that

$$c_{\text{odd}} = c_{\text{nod}} = 2c_0.$$

Therefore if one of the infimums c_{odd} or c_{nod} is achieved at u, then both u^+ and u^- should achieve c_0 in $\mathcal{N}_0$. This is impossible, since by the strong maximum principle $u^+ > 0$ and $u^- > 0$ almost everywhere on the space $\mathbb{R}^N$. This nonexistence of minimal action nodal solutions also contrasts with theorem 3: for the nonlocal problem c_{nod} is too small to be achieved whereas for the local one this level is too large.

2. Minimal action odd solution

In this section we prove theorem 1 about the existence of solutions under an oddness constraint.

2.1. **Variational principle.**

We first observe that the corresponding level c_{odd} is positive.

Proposition 2.1 (Nondegeneracy of the level). *If* $\frac{N-2}{N+\alpha} \le \frac{1}{p} \le \frac{N}{N+\alpha}$, *then*

$$c_{\text{odd}} > 0.$$

Proof. Since $\mathcal{N}_{\text{odd}} = \mathcal{N}_0 \cap H^1_{\text{odd}}(\mathbb{R}^N) \subset \mathcal{N}_0$ we have $c_{\text{odd}} \ge c_0$. The conclusion follows then from the fact that $c_0 > 0$ [21]. □

A first step in the construction of our solution is the existence a Palais–Smale sequence.

Proposition 2.2 (Existence of a Palais-Smale sequence). *If* $\frac{N-2}{N+\alpha} \le \frac{1}{p} \le \frac{N}{N+\alpha}$, *then there exists a sequence* $(u_n)_{n\in\mathbb{N}}$ *in* $H^1_{\text{odd}}(\mathbb{R}^N)$ *such that, as* $n \to \infty$,

$$\mathcal{A}(u_n) \to c_{\text{odd}} \qquad \textit{and} \qquad \mathcal{A}'(u_n) \to 0 \quad \textit{in } H^1_{\text{odd}}(\mathbb{R}^N)'.$$

Proof. We first recall that the level c_{odd} can be rewritten as a mountain pass minimax level:

$$c_{\text{odd}} = \inf_{\gamma\in\Gamma} \sup_{[0,1]} A \circ \gamma,$$

where the class of paths Γ is defined by

$$\Gamma = \{\gamma \in C\big([0,1], H^1_{\mathrm{odd}}(\mathbb{R}^N)\big) \ :\ \gamma(0) = 0 \text{ and } \mathcal{A}(\gamma(1)) < 0\}$$

(see for example [30, theorem 4.2]). By the general minimax principle [30, theorem 2.8], there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1_{\mathrm{odd}}(\mathbb{R}^N)$ such that the sequence $\big(\mathcal{A}(u_n)\big)_{n\in\mathbb{N}}$ converges to c_{odd} and the sequence $\big(\mathcal{A}'(u_n)\big)_{n\in\mathbb{N}}$ converges strongly to 0 in the dual space $H^1_{\mathrm{odd}}(\mathbb{R}^N)'$. □

2.2. **Palais–Smale condition.** We would now like to construct out of the Palais–Smale sequence of proposition 2.2 a solution to our problem. We shall prove that the functional $\mathcal{A}|_{H^1_{\mathrm{odd}}(\mathbb{R}^N)}$ satisfies the Palais–Smale condition up to translations at the level c_{odd} if the strict inequality $c_{\mathrm{odd}} < 2c_0$ holds.

Proposition 2.3 (Palais–Smale condition)**.** *Assume that* $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$. *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $H^1_{\mathrm{odd}}(\mathbb{R}^N)$ *such that, as* $n \to \infty$,

$$\mathcal{A}(u_n) \to c_{\mathrm{odd}} \qquad \text{and} \qquad \mathcal{A}'(u_n) \to 0 \quad \text{in } H^1_{\mathrm{odd}}(\mathbb{R}^N)'.$$

If

$$c_{\mathrm{odd}} < 2c_0,$$

then there exists a sequence of points $(a_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}^{N-1}\times\{0\} \subset \mathbb{R}^N$ *such that the subsequence* $\big(u_{n_k}(\cdot - a_{n_k})\big)_{k\in\mathbb{N}}$ *converges strongly in* $H^1(\mathbb{R}^N)$ *to* $u \in H^1_{\mathrm{odd}}(\mathbb{R}^N)$. *Moreover*

$$\mathcal{A}(u) = c_{\mathrm{odd}} \qquad \text{and} \qquad \mathcal{A}'(u) = 0 \quad \text{in } H^1(\mathbb{R}^N)'.$$

Proof. First, we observe that, as $n \to \infty$,

$$\begin{aligned}\Big(\frac{1}{2} - \frac{1}{2p}\Big)\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2 &= \mathcal{A}(u_n) - \frac{1}{2p}\langle \mathcal{A}'(u_n), u_n\rangle \\ &= \mathcal{A}(u_n) + o\Big(\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2\Big)^{\frac{1}{2}}\Big) \\ &= c_{\mathrm{odd}} + o\Big(\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2\Big)^{\frac{1}{2}}\Big).\end{aligned} \tag{2.1}$$

In particular, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in the space $H^1(\mathbb{R}^N)$.

We now claim that there exists $R > 0$ such that

$$\liminf_{n\to\infty} \int_{D_R} |u_n|^{\frac{2Np}{N+\alpha}} > 0, \tag{2.2}$$

where the set $D_R \subset \mathbb{R}^N$ is the infinite slab

$$D_R = \mathbb{R}^{N-1} \times [-R, R].$$

We assume by contradiction that for each $R > 0$,

$$\liminf_{n\to\infty} \int_{D_R} |u_n|^{\frac{2Np}{N+\alpha}} = 0.$$

We define for each $n \in \mathbb{N}$ the functions $v_n = \chi_{\mathbb{R}^{N-1}\times(0,\infty)} u_n$ and $\tilde{v}_n = \chi_{\mathbb{R}^{N-1}\times(-\infty,0)} u_n$. Since $u_n \in H^1_{\mathrm{odd}}(\mathbb{R}^N)$, we have $v_n \in H^1_0(\mathbb{R}^{N-1}\times(0,\infty)) \subset$

$H^1(\mathbb{R}^N)$ and $\tilde{v}_n \in H^1_0(\mathbb{R}^{N-1} \times (-\infty, 0)) \subset H^1(\mathbb{R}^N)$. We now compute

$$\int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|\tilde{v}_n|^p \le 2 \int_{\mathbb{R}^N} \int_{D_R} I_\alpha(x-y)|v_n(y)|^p|\tilde{v}_n(x)|^p \,\mathrm{d}y\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus D_R} \int_{\mathbb{R}^N\setminus D_R} I_\alpha(x-y)|v_n(y)|^p|\tilde{v}_n(x)|^p \,\mathrm{d}y\,\mathrm{d}x$$

By definition of the region D_R we have, if $\beta \in (\alpha, N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|\tilde{v}_n|^p &\le 2\int_{D_R} (I_\alpha * |u_n|^p)|u_n|^p + \int_{\mathbb{R}^N} ((\chi_{\mathbb{R}^N\setminus B_{2R}} I_\alpha) * |u_n|^p)|u_n|^p \\ &\le 2\int_{D_R} (I_\alpha * |u_n|^p)|u_n|^p + \frac{C}{R^{\beta-\alpha}} \int_{\mathbb{R}^N} ((\chi_{\mathbb{R}^N\setminus B_{2R}} I_\beta) * |u_n|^p)|u_n|^p\end{aligned}$$

Since by assumption $p > \frac{N+\alpha}{N}$, we can take β such that moreover $\beta < (p-1)N$, and then by the Hardy–Littlewood–Sobolev inequality (1.1) and the classical Sobolev inequality, we obtain that

$$\int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|\tilde{v}_n|^p \le C'\Bigl(\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2\Bigr)^{\frac{p}{2}}\Bigl(\int_{D_R} |u_n|^{\frac{2Np}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}} + \frac{C''}{R^{\beta-\alpha}}\Bigl(\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2\Bigr)^p,$$

from which we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|\tilde{v}_n|^p = 0. \tag{2.3}$$

For each $n \in \mathbb{N}$, we fix $t_n \in (0, \infty)$ so that $t_n v_n \in \mathcal{N}_0$ or, equivalently,

$$t_n^{2p-2} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla v_n|^2 + |v_n|^2}{\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2}{\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p - 2\int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|\tilde{v}_n|^p}. \tag{2.4}$$

For every $n \in \mathbb{N}$, we have

$$\mathcal{A}(t_n u_n) = 2\mathcal{A}(t_n v_n) - \frac{t_n^{2p}}{p} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|\tilde{v}_n|^p$$

By (2.1), (2.3) and (2.4), in view of proposition 2.1, we note that $\lim_{n\to\infty} t_n = 1$ and thus in view of (2.3) again we conclude that

$$c_{\mathrm{odd}} = \lim_{n\to\infty} \mathcal{A}(u_n) = \lim_{n\to\infty} \mathcal{A}(t_n u_n) = 2 \lim_{n\to\infty} \mathcal{A}(t_n v_n) \ge 2c_0,$$

in contradiction with the assumption $c_{\mathrm{odd}} < 2c_0$ of the proposition.

We can now fix $R > 0$ such that (2.2) holds. We take a function $\eta \in C^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} \eta \subset D_{3R/2}$, $\eta = 1$ on D_R, $\eta \le 1$ on $\mathbb{R}^N$ and

$\nabla \eta \in L^\infty(\mathbb{R}^N)$. We have the inequality [18, lemma I.1; 21, lemma 2.3; 27, (2.4); 30, lemma 1.21]

$$\begin{aligned}\int_{D_R} |u_n|^{\frac{2Np}{N+\alpha}} &\le \int_{\mathbb{R}^N} |\eta u_n|^{\frac{2Np}{N+\alpha}}\\ &\le C\Bigl(\sup_{a\in\mathbb{R}^N} \int_{B_{R/2}(a)} |\eta u_n|^{\frac{2Np}{N+\alpha}}\Bigr)^{1-\frac{N+\alpha}{Np}} \int_{\mathbb{R}^N} |\nabla(\eta u_n)|^2 + |\eta u_n|^2\\ &\le C'\Bigl(\sup_{a\in\mathbb{R}^{N-1}\times\{0\}} \int_{B_{2R}(a)} |u_n|^{\frac{2Np}{N+\alpha}}\Bigr)^{1-\frac{N+\alpha}{Np}} \int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2.\end{aligned}$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in the space $H^1(\mathbb{R}^N)$ we deduce from (2.2) that there exists a sequence of points $(a_n)_{n\in\mathbb{N}}$ in the hyperplane $\mathbb{R}^{N-1}\times\{0\}$ such that

$$\liminf_{n\to\infty} \int_{B_{2R}(a_n)} |u_n|^{\frac{2Np}{N+\alpha}} > 0.$$

Up to translations and a subsequence, we can assume that the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly in $H^1(\mathbb{R}^N)$ to a function $u \in H^1(\mathbb{R}^N)$.

Since the action functional $\mathcal{A}$ is invariant under odd reflections, we note that for every $n \in \mathbb{N}$, $\mathcal{A}(u_n) = 0$ on $H^1_{\mathrm{odd}}(\mathbb{R}^N)^\perp$ by the symmetric criticality principle [23] (see also [30, theorem 1.28]). This allows to deduce from the strong convergence of the sequence $(\mathcal{A}'(u_n))_{n\in\mathbb{N}}$ to 0 in $H^1_{\mathrm{odd}}(\mathbb{R}^N)'$ the strong convergence to 0 of the sequence $(\mathcal{A}'(u_n))_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^N)'$.

For any test function $\varphi \in C^1_c(\mathbb{R}^N)$, by the weak convergence of the sequence $(u_n)_{n\in\mathbb{N}}$, we first have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla\varphi + u_n\varphi = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi + u\varphi.$$

By the classical Rellich–Kondrashov compactness theorem, the sequence $(|u_n|^p)_{n\in\mathbb{N}}$ converges locally in measure to $|u|^p$ and by the Sobolev inequality, this sequence is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Therefore, the sequence $(|u_n|^p)_{n\in\mathbb{N}}$ converges weakly to $|u|^p$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ (see for example [2, proposition 4.7.12; 31, proposition 5.4.7]), and, by the Hardy–Littlewood–Sobolev inequality (1.1), the sequence $(I_\alpha * |u_n|^p)_{n\in\mathbb{N}}$ converges weakly in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ to $I_\alpha * |u|^p$. By the Rellich–Kondrashov theorem again, the sequence $((I_\alpha * |u_n|^p)|u_n|^{p-2}u_n)_{n\in\mathbb{N}}$ converges weakly in $L^{\frac{2N}{N+2}}(K)$ for every compact set $K \subset \mathbb{R}^N$. Therefore we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n\varphi = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}u\varphi.$$

We have thus proved that

$$\mathcal{A}'(u) = 0 = \lim_{n\to\infty} \mathcal{A}'(u_n).$$

Finally, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{A}(u_n) &= \lim_{n\to\infty} \mathcal{A}(u_n) - \frac{1}{2p}\langle \mathcal{A}'(u_n), u_n\rangle\\ &= \lim_{n\to\infty} \Bigl(\frac{1}{2} - \frac{1}{2p}\Bigr) \int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2\\ &\geq \Bigl(\frac{1}{2} - \frac{1}{2p}\Bigr) \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\\ &= \mathcal{A}(u) - \frac{1}{2p}\langle \mathcal{A}'(u), u\rangle = \mathcal{A}(u),\end{aligned}$$

from which we conclude that $\mathcal{A}(u) = c_{\mathrm{odd}}$ and that the sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly to u in $H^1(\mathbb{R}^N)$. □

2.3. **Strict inequality.** It remains now to establish the strict inequality $c_{\mathrm{odd}} < 2c_0$.

Proposition 2.4. *If $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$, then*

$$c_{\mathrm{odd}} < 2c_0.$$

Proof. It is known that the Choquard equation has a least action solution [21]. More precisely, there exists $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\mathcal{A}'(v) = 0$ and

$$\mathcal{A}(v) = \inf_{\mathcal{N}_0} \mathcal{A}.$$

We take a function $\eta \in C^2_c(\mathbb{R}^N)$ such that $\eta = 1$ on B_1, $0 \leq \eta \leq 1$ on $\mathbb{R}^N$ and $\operatorname{supp}\eta \subset B_2$ and we define for each $R > 0$ the function $\eta_R \in C^2_c(\mathbb{R}^N)$ for every $x \in \mathbb{R}^N$ by $\eta_R(x) = \eta(x/R)$. We define now the function $u_R : \mathbb{R}^N \to \mathbb{R}$ for each $x = (x', x_N) \in \mathbb{R}^N$ by

$$u_R(x) = (\eta_R v)(x', x_N - 2R) - (\eta_R v)(x', -x_N - 2R).$$

It is clear that $u_R \in H^1_{\mathrm{odd}}(\mathbb{R}^N)$.

We observe that $\langle \mathcal{A}'(t_R u_R), t_R u_R\rangle = 0$ if and only if $t_R \in (0, \infty)$ satisfies

$$t_R^{2p-2} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_R|^2 + |u_R|^2}{\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u_R|^p)|u_R|^p}.$$

Such a t_R always exists and

$$\mathcal{A}(t_R u_R) = \Bigl(\frac{1}{2} - \frac{1}{p}\Bigr) \frac{\Bigl(\displaystyle\int_{\mathbb{R}^N} |\nabla u_R|^2 + |u_R|^2\Bigr)^{\frac{p}{p-1}}}{\Bigl(\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u_R|^p)|u_R|^p\Bigr)^{\frac{1}{p-1}}}.$$

The proposition will follow once we have established that for some $R > 0$

$$\frac{\Bigl(\displaystyle\int_{\mathbb{R}^N} |\nabla u_R|^2 + |u_R|^2\Bigr)^{\frac{p}{p-1}}}{\Bigl(\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u_R|^p)|u_R|^p\Bigr)^{\frac{1}{p-1}}} < 2\frac{\Bigl(\displaystyle\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2\Bigr)^{\frac{p}{p-1}}}{\Bigl(\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p\Bigr)^{\frac{1}{p-1}}}. \tag{2.5}$$

We begin by estimating the denominator in the left-hand side of (2.5). We first observe that, by construction of the function u_R

$$\int_{\mathbb{R}^N} (I_\alpha * |u_R|^p)|u_R|^p \geq 2\int_{\mathbb{R}^N} (I_\alpha * |\eta_R v|^p)|\eta_R v|^p + 2\frac{A_\alpha}{(4R)^{N-\alpha}}\Bigl(\int_{\mathbb{R}^N} |\eta_R v|^p\Bigr)^2.$$

For the first term, we have

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |\eta_R v|^p)|\eta_R v|^p &= \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p - 2\int_{\mathbb{R}^N} (I_\alpha * |v|^p)(1-\eta_R^p)|v|^p\\ &\quad + \int_{\mathbb{R}^N} (I_\alpha * (1-\eta_R^p)|v|^p)(1-\eta_R^p)|v|^p\\ &\geq \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p - 2\int_{\mathbb{R}^N} (I_\alpha * |v|^p)(1-\eta_R^p)|v|^p.\end{aligned}$$

By the asymptotic properties of $I_\alpha * |v|^p$ [21, theorem 4], we have

$$\lim_{|x|\to\infty} \frac{(I_\alpha * |v|^p)}{I_\alpha(x)} = \int_{\mathbb{R}^N} |v|^p,$$

so that

$$2\int_{\mathbb{R}^N} (I_\alpha * |v|^p)(1-\eta_R^p)|v|^p \leq C\int_{\mathbb{R}^N\setminus B_R} \frac{|v(x)|^p}{|x|^{N-\alpha}}\,\mathrm{d}x.$$

We have thus

$$\begin{aligned}&\int_{\mathbb{R}^N} (I_\alpha * |u_R|^p)|u_R|^p\\ &\quad\geq 2\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p + \frac{2A_\alpha}{(4R)^{N-\alpha}}\Bigl(\int_{B_R} |v|^p\Bigr)^2 - C\int_{\mathbb{R}^N\setminus B_R} \frac{|v(x)|^p}{|x|^{N-\alpha}}\,\mathrm{d}x.\end{aligned}$$

We now use the information that we have on the decay of the least action solution v [21]. If $p < 2$, then $v(x) = O(|x|^{-(N-\alpha)/(2-p)})$ as $|x| \to \infty$ and

$$\int_{\mathbb{R}^N\setminus B_R} \frac{|v(x)|^p}{|x|^{N-\alpha}}\,\mathrm{d}x = O\Bigl(\frac{1}{R^{\frac{Np-2\alpha}{2-p}}}\Bigr) = o\Bigl(\frac{1}{R^{N-\alpha}}\Bigr),$$

since $p > \frac{N+\alpha}{N} > \frac{2N}{2N-\alpha}$. If $p \geq 2$, then v decays exponentially at infinity. We have thus the asymptotic lower bound

$$\begin{aligned}&\int_{\mathbb{R}^N} (I_\alpha * |u_R|^p)|u_R|^p\\ &\qquad\geq 2\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p + \frac{2A_\alpha}{(4R)^{N-\alpha}}\Bigl(\int_{\mathbb{R}^N} |v|^p\Bigr)^2 + o\Bigl(\frac{1}{R^{N-\alpha}}\Bigr).\end{aligned}\tag{2.6}$$

For the numerator in (2.5), we compute by integration by parts

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_R|^2 + |u_R|^2 &= 2\int_{\mathbb{R}^N} |\nabla(\eta_R v)|^2 + |\eta_R v|^2\\ &= 2\int_{\mathbb{R}^N} \eta_R^2(|\nabla v|^2 + |v|^2) - 2\int_{\mathbb{R}^N} \eta_R(\Delta\eta_R)|v|^2\\ &\leq 2\int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2) + \frac{C}{R^2}\int_{B_{2R}\setminus B_R} |v|^2.\end{aligned}$$

If $p < 2$, we have by the decay of the solution v

$$\frac{1}{R^2}\int_{B_{2R}\setminus B_R} |v|^2 = O\Bigl(\frac{1}{R^{\frac{Np-2\alpha}{2-p}+2}}\Bigr) = o\Bigl(\frac{1}{R^{N-\alpha}}\Bigr).$$

In the case where $p \geq 2$, the solution v decays exponentially. We conclude thus that

$$\int_{\mathbb{R}^N} |\nabla u_R|^2 + |u_R|^2 = 2 \int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2 + o\Bigl(\frac{1}{R^{N-\alpha}}\Bigr). \tag{2.7}$$

We derive from the asymptotic bounds (2.6) and (2.7), an asymptotic bound on the quotient:

$$\begin{aligned}
&\frac{\Bigl(\displaystyle\int_{\mathbb{R}^N} |\nabla u_R|^2 + |u_R|^2\Bigr)^{\frac{p}{p-1}}}{\Bigl(\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u_R|^p)|u_R|^p\Bigr)^{\frac{1}{p-1}}}\\
&\qquad\le 2\frac{\Bigl(\displaystyle\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2\Bigr)^{\frac{p}{p-1}}}{\Bigl(\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p\Bigr)^{\frac{1}{p-1}}}\left(1 - \frac{pA_\alpha\Bigl(\displaystyle\int_{B_R} |v|^p\Bigr)^2}{(p-1)(4R)^{N-\alpha}\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p}\right.\\
&\qquad\qquad\left. + o\Bigl(\frac{1}{R^{N-\alpha}}\Bigr)\right).
\end{aligned}$$

The inequality (2.5) holds thus when R is large enough, and the conclusion follows. □

2.4. **Existence of a minimal action odd solution.** We have now developped all the tools to prove the existence of a least action odd solution to the Choquard equation, corresponding to the existence part of theorem 1.

Proposition 2.5. *If $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$, then there exists solution $u \in H^1_{\mathrm{odd}}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ to the Choquard equation $(\mathcal{C})$ such that $\mathcal{A}(u) = c_{\mathrm{odd}}$.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence given by proposition 2.2. In view of proposition 2.4, proposition 2.3 is applicable and gives the required weak solution $u \in H^1(\mathbb{R}^N)$. By the regularity theory for the Choquard equation [21, proposition 4.1] (see also [8, lemma A.10]), $u \in C^2(\mathbb{R}^N)$. □

2.5. **Sign and symmetry properties.** We complete the proof of theorem 1 by showing that such solutions have a simple sign and symmetry structure.

Proposition 2.6. *If $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{1}{2}$ and if $u \in H^1_{\mathrm{odd}}(\mathbb{R}^N)$ is a solution to the Choquard equation $(\mathcal{C})$ such that $\mathcal{A}(u) = c_{\mathrm{odd}}$, then u has constant sign on $\mathbb{R}^N_+$ and u is axially symmetric with respect to an axis perpendicular to $\partial\mathbb{R}^N_+$.*

The proof takes profit of the structure of the problem to rewrite it as a groundstate of a problem on the halfspace where quite fortunately the strategy for proving similar properties of groundstates of the Choquard equation still works [21, Propositions 5.1 and 5.2] (see also [22, propositions 5.2 and 5.3].

Proof of proposition 2.6. We first rewrite the problem of finding odd solutions on $\mathbb{R}^N$ as a groundstate problem on $\mathbb{R}^N_+$ whose nonlocal term has a more intricate structure.

Claim 1. *For every* $v \in H^1_{\mathrm{odd}}(\mathbb{R}^N)$,

$$\mathcal{A}(v) = \tilde{\mathcal{A}}(v|_{\mathbb{R}^N_+}),$$

where $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0, \infty)$ *and the functional* $\tilde{\mathcal{A}} : H^1_0(\mathbb{R}^N_+) \mapsto \mathbb{R}$ *is defined for* $w \in H^1_0(\mathbb{R}^N_+)$ *by*

$$\tilde{\mathcal{A}}(w) = \int_{\mathbb{R}^N_+} |\nabla w|^2 + |w|^2 - \frac{1}{p} \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} K(|x'-y'|, x_N, y_N) |u(x)|^p |u(y)|^p \,\mathrm{d}x \,\mathrm{d}y,$$

with $x = (x', x_N)$, $y = (y', y_N)$ *and the kernel* $K : (0, \infty)^3 \to \mathbb{R}$ *defined for each* $(r, s, t) \in (0, \infty)^3$ *by*

$$K(r, s, t) = \frac{A_\alpha}{\bigl(r^2 + (s-t)^2\bigr)^{\frac{N-\alpha}{2}}} + \frac{A_\alpha}{\bigl(r^2 + (s+t)^2\bigr)^{\frac{N-\alpha}{2}}}.$$

In particular, $u \in \tilde{\mathcal{N}}_{\mathrm{nod}}$, *where*

$$\tilde{\mathcal{N}}_{\mathrm{nod}} = \{ w \in H^1_0(\mathbb{R}^N_+) \ : \ \langle \tilde{\mathcal{A}}'(w), w \rangle = 0 \}$$

and

$$\tilde{\mathcal{A}}(u) = \inf_{\tilde{\mathcal{N}}_{\mathrm{nod}}} \tilde{\mathcal{A}}.$$

Proof of the claim. This follows from the fact that if $u \in H^1_{\mathrm{odd}}(\mathbb{R}^N)$, then $u|_{\mathbb{R}^N_+} \in H^1_0(\mathbb{R}^N_+)$ and by direct computation of the integrals. ◇

Claim 2. *One has either* $u > 0$ *almost everywhere on* $\mathbb{R}^N_+$ *or* $u < 0$ *almost everywhere on* $\mathbb{R}^N_+$.

Proof of the claim. Let $w = u|_{\mathbb{R}^N_+}$. We observe that $|w| \in H^1_0(\mathbb{R}^N_+)$,

$$\tilde{\mathcal{A}}(|w|) = \tilde{\mathcal{A}}(w) = c_{\mathrm{odd}} \qquad \text{and} \qquad \langle \tilde{\mathcal{A}}'(|w|), |w| \rangle = \langle \tilde{\mathcal{A}}'(w), w \rangle.$$

Therefore, if we define $\bar{u} \in H^1_{\mathrm{odd}}(\mathbb{R}^N)$ for almost every $x = (x', x_N) \in \mathbb{R}^N$ by

$$\bar{u}(x) = \begin{cases} |w|(x', x_N) & \text{if } x_N > 0, \\ -|w|(x', x_N) & \text{if } x_N < 0, \end{cases}$$

the function $\bar{u}$ is a weak solution to the Choquard equation $(\mathcal{C})$. This function $\bar{u}$ is thus of class C^2 [21, proposition 4.1] (see also [8, lemma A.10]) and, in the classical sense, it satisfies

$$-\Delta \bar{u} + \bar{u} \geq 0 \qquad \text{in } \mathbb{R}^N_+.$$

By the usual strong maximum principle for classical supersolutions, we conclude that $|u| = \bar{u} > 0$ in $\mathbb{R}^N_+$. Since the function u was also a solution to the Choquard equation $(\mathcal{C})$, it is also a continuous function, and we have thus either $u = |u| > 0$ in $\mathbb{R}^N_+$ or $u = -|u| < 0$ in $\mathbb{R}^N_+$. ◇

Claim 3. *The solution* u *is axially symmetric with respect to an axis parallel to* $\{0\} \times \mathbb{R} \subset \mathbb{R}^N$.

Proof of the claim. Let H be a closed affine half-space perpendicular to $\partial \mathbb{R}^N_+$ and let $\sigma_H : \mathbb{R}^N \to \mathbb{R}^N$ be the reflection with respect to ∂H. We recall that the polarization or two-point rearrangement with respect to H of w is the function $w^H : \mathbb{R}^N \to \mathbb{R}$ defined for each $x \in \mathbb{R}^N$ by [1, 4]

$$w^H(x) = \begin{cases} \max(w(x), w(\sigma_H(x))) & \text{if } x \in H, \\ \min(w(x), w(\sigma_H(x))) & \text{if } x \in \mathbb{R}^N \setminus H. \end{cases}$$

Since ∂H is perpendicular to $\mathbb{R}^{N-1} \times \{0\}$, we have $\sigma_H(\mathbb{R}^N_+) = \mathbb{R}^N_+$ so that $w^H \in H^1_0(\mathbb{R}^N)$ and [4, lemma 5.3]

$$\int_{\mathbb{R}^N_+} |\nabla w^H|^2 + |w^H|^2 = \int_{\mathbb{R}^N_+} |\nabla w|^2 + |w|^2.$$

Moreover, we also have

$$\begin{aligned}(2.8)\quad &\int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} K(|x' - y'|, x_N, y_N) |w^H(x)|^p |w^H(y)|^p \,\mathrm{d}x \,\mathrm{d}y \\ &= \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * (|u|^p)^H)(|u|^p)^H \ge \frac{1}{2} \int_{\mathbb{R}^N_+} (I_\alpha * |u|^p)\, |u|^p \\ &= \int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} K(|x' - y'|, x_N, y_N)\, |w(x)|^p \,|w(y)|^p \,\mathrm{d}x \,\mathrm{d}y,\end{aligned}$$

with equality if and only if either $|u|^H = |u|$ almost everywhere on $\mathbb{R}^N_+$ or $|u|^H = |u| \circ \sigma_H$ almost everywhere on $\mathbb{R}^N_+$ [21, lemma 5.3] (see also [1, corollary 4; 28, proposition 8]), or equivalently, since by claim 2 w has constant sign on $\mathbb{R}^N$, either $w^H = w$ almost everywhere on $\mathbb{R}^N_+$ or $w^H = w \circ \sigma_H$ almost everywhere on $\mathbb{R}^N_+$.

If the inequality (2.8) was strict, then, since $p > 1$ there would exist $\tau \in (0, 1)$ such that $\tau w^H \in \tilde{\mathcal{N}}$ and we would have

$$\mathcal{A}(\tau u^H) < \tilde{\mathcal{A}}(w) = c_{\mathrm{odd}},$$

in contradiction with claim 1.

We have thus proved that for every affine half-space $H \subset \mathbb{R}^N$ whose boundary ∂H is perpendicular to $\partial \mathbb{R}^N_+$, either $w^H = w$ almost everywhere on $\mathbb{R}^N_+$ or $w^H = w \circ \sigma_H$ almost everywhere on $\mathbb{R}^N_+$. This implies that w is axially symmetric with respect to an axis perpendicular to $\partial \mathbb{R}^N_+$ [21, lemma 5.3; 29, proposition 3.15], which is equivalent to the claim. ◇

The proposition follows directly from claims 2 and 3. □

3. Minimal action nodal solution

This section is devoted to the proof of theorem 2 on the existence of a least action nodal solution.

3.1. **Minimax principle.**

We begin by observing that the counterpart of proposition 2.1 holds.

Proposition 3.1 (Nondegeneracy of the level)**.** *If* $\frac{N-2}{N+\alpha} \le \frac{1}{p} \le \frac{N}{N+\alpha}$, *then*

$$c_{\mathrm{nod}} > 0.$$

Proof. In view of the inequality $c_0 > 0$ [21], it suffices to note that since $\mathcal{N}_{\mathrm{nod}} \subset \mathcal{N}_0$ we have $c_{\mathrm{nod}} \ge c_0$. □

We first reformulate the minimization problem as a minimax problem.

Proposition 3.2 (Minimax principle). *If $\frac{N-2}{N+\alpha} \le \frac{1}{p} < \frac{1}{2}$, then for every $\varepsilon > 0$,*

$$c_{\text{nod}} = \inf_{\gamma \in \Gamma} \sup_{\mathbb{B}^2} \mathcal{A} \circ \gamma,$$

where

$$\Gamma = \Big\{\gamma \in C(\mathbb{B}^2; H^1_0(\mathbb{R}^N)) : \xi(\gamma(\partial \mathbb{B}^2)) \not\ni 0,\ \deg(\xi \circ \gamma) = 1$$
$$\text{and } (\mathcal{A} \circ \gamma)^{\frac{p-1}{p-2}} \le c_{\text{nod}}^{\frac{p-1}{p-2}} + \varepsilon - c_0^{\frac{p-1}{p-2}} \text{ on } \partial\mathbb{B}^2\Big\},$$

where the map $\xi = (\xi_+, \xi_-) \in C(H^1(\mathbb{R}^N); \mathbb{R}^2)$ is defined for each $u \in H^1(\mathbb{R}^N)$ by

$$\xi_\pm(u) = \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u_\pm|^p}{\displaystyle\int_{\mathbb{R}^N} |\nabla u_\pm|^2 + |u_\pm|^2} - 1 & \text{if } u_\pm \ne 0,\\ -1 & \text{if } u_\pm = 0. \end{cases}$$

Moreover, for every $\gamma \in \Gamma$, $\mathcal{N}_{\text{nod}} \cap \gamma(\mathbb{B}^2) \ne 0$.

In this statement $\mathbb{B}^2$ denotes the closed unit disc in the plane $\mathbb{R}^2$ and deg is the classical topological degree of Brouwer, or equivalently, the winding number (see for example [19, §5.3; 25, chapter 6]).

The continuity of the map ξ on the subset of constant-sign functions in $H^1(\mathbb{R}^N)$ follows from the Hardy–Littlewood–Sobolev inequality (1.1) and the classical Sobolev inequality, and requires the assumption $p > 2$.

The map ξ is the nonlocal counterpart of a map appearing in the variational characterization of least action nodal solutions by Cerami, Solimini and Struwe for local Schrödinger type problems [7], which is done in the framework of critical point theory in ordered spaces whereas our minimax principle works in the more classical framework of Banach spaces.

Proof of proposition 3.2. We denote the right-hand side in the equality to be proven as $\tilde{c}$ and we first prove that that $\tilde{c} \ge c_{\text{nod}}$. Let $\gamma \in \Gamma$. Since $\deg(\xi \circ \gamma) = 1$, by the existence property of the degree, there exists $t^* \in \mathbb{B}^2$ such that $(\xi \circ \gamma)(t^*) = 0$. It follows then that $\gamma(t_*) \in \mathcal{N}_{\text{nod}} = \xi^{-1}(0)$ and thus

$$\sup_{\mathbb{B}^2} \mathcal{A} \circ \gamma \ge \gamma(t_*) \ge c_{\text{nod}},$$

so that $\tilde{c} \ge c_{\text{nod}}$.

We now prove that $\tilde{c} \le c_{\text{nod}}$. For a given $u \in \mathcal{N}_{\text{nod}}$, we define the map $\tilde{\gamma} : [0, \infty)^2 \to H^1(\mathbb{R}^N)$ for every $(t_+, t_-) \in [0, \infty)^2$ by

$$\tilde{\gamma}(t_+, t_-) = t_+^{\frac{1}{p}} u^+ + t_-^{\frac{1}{p}} u^-.$$

We compute for each $(t_+, t_-) \in [0, \infty)^2$

$$\mathcal{A}(\tilde{\gamma}(t_+, t_-)) = \frac{t_+^{2/p}}{2} \int_{\mathbb{R}^N} |\nabla u^+|^2 + |u^+|^2 + \frac{t_-^{2/p}}{2} \int_{\mathbb{R}^N} |\nabla u^-|^2 + |u^-|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} \big|I_{\alpha/2} * (t_+|u^+|^p + t_-|u^-|^p)\big|^2. \tag{3.1}$$

The function $\mathcal{A} \circ \tilde{\gamma}$ is thus strictly concave and $(\mathcal{A} \circ \tilde{\gamma})'(1,1) = 0$. Hence, $(1,1)$ is the unique maximum point of the function $\mathcal{A} \circ \tilde{\gamma}$.

We also have in particular

$$\mathcal{A}(\tilde{\gamma}(t_+, 0)) = \frac{t_+^{2/p}}{2} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u^+|^p - \frac{t_+^2}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u^+|^p)|u^+|^p,$$

and therefore

$$\mathcal{A}(\tilde{\gamma}(t_+, 0)) \le \Bigl(\frac{1}{2} - \frac{1}{2p}\Bigr) \frac{\Bigl(\displaystyle\int_{\mathbb{R}^n} (I_\alpha * |u|^p)|u^+|^p\Bigr)^{\frac{p}{p-1}}}{\Bigl(\displaystyle\int_{\mathbb{R}^n} (I_\alpha * |u^+|^p)|u^+|^p\Bigr)^{\frac{1}{p-1}}}. \tag{3.2}$$

By the semigroup property of the Riesz potential $I_\alpha = I_{\alpha/2} * I_{\alpha/2}$ (see for example [16, theorem 5.9]) and by the Cauchy–Schwarz inequality,

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u^+|^p &= \int_{\mathbb{R}^N} (I_{\alpha/2} * |u^+|^p)(I_{\alpha/2} * |u|^p) \\ &\le \Bigl(\int_{\mathbb{R}^N} |I_{\alpha/2} * |u|^p|^2\Bigr)^{\frac{1}{2}} \Bigl(\int_{\mathbb{R}^N} |I_{\alpha/2} * |u^+|^p|^2\Bigr)^{\frac{1}{2}} \\ &= \Bigl(\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p)\Bigr)^{\frac{1}{2}} \Bigl(\int_{\mathbb{R}^N} (I_\alpha * |u^+|^p)|u^+|^p)\Bigr)^{\frac{1}{2}}, \end{aligned} \tag{3.3}$$

and therefore by (3.2) and (3.3)

$$\mathcal{A}(\tilde{\gamma}(t_+, 0)) \le \Bigl(\frac{1}{2} - \frac{1}{2p}\Bigr) \Bigl(\int_{\mathbb{R}^n} (I_\alpha * |u|^p)|u^+|^p\Bigr)^{\frac{p-2}{p-1}} \Bigl(\int_{\mathbb{R}^n} (I_\alpha * |u|^p)|u|^p\Bigr)^{\frac{1}{p-1}}$$

We deduce therefrom that for every $(t_+, t_-) \in [0, \infty)^2$,

$$\mathcal{A}(t_+^{1/p} u^+)^{\frac{p-1}{p-2}} + \mathcal{A}(t_-^{1/p} u^-)^{\frac{p-1}{p-2}} \le \mathcal{A}(u)^{\frac{p-1}{p-2}}. \tag{3.4}$$

Since $u_\pm \ne 0$, we have

$$\sup_{t_\pm \in [0,\infty)} \mathcal{A}(t_\pm^{1/p} u^\pm) \ge c_0,$$

we conclude that

$$\sup_{t \in \partial([0,\infty)^2)} (\mathcal{A} \circ \tilde{\gamma})^{\frac{p-1}{p-2}} \le \mathcal{A}(u)^{\frac{p-1}{p-2}} - c_0^{\frac{p-1}{p-2}}.$$

Moreover, we have by (3.1)

$$\lim_{|t| \to \infty} \mathcal{A}(\tilde{\gamma}(t)) = -\infty.$$

It remains to compute the degree of the map $\xi \circ \tilde{\gamma}$ on a suitable set homeomorphic to $\mathbb{B}^2$. We compute for each $(t_+, t_-) \in [0, \infty)^2$, since $u \in \mathcal{N}_{\mathrm{nod}}$

$$\begin{aligned} &(t_+, t_-) \cdot \xi(\tilde{\gamma}(t_+, t_-)) \\ &\quad = t_+^{3-\frac{2}{p}} \frac{\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u^+|^p)|u^+|^p}{\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u^+|^p} + t_-^{3-\frac{2}{p}} \frac{\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u^-|^p)|u^-|^p}{\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u^-|^p} - t_+ - t_- \end{aligned}$$

Since $p > 2$, we can now take $R > \sqrt{2}$ large enough so that if $t \in [0, \infty)^2 \cap \partial B_R$, then

$$t \cdot \xi(\tilde{\gamma}(t)) > 0 \qquad \text{and} \qquad (\mathcal{A} \circ \tilde{\gamma}(t))^{\frac{p-1}{p-2}} \le \mathcal{A}(u)^{\frac{p-1}{p-2}} - c_0^{\frac{p-1}{p-2}}.$$

We now define the homotopy $H : [0, 1] \times [0, \infty)^2$ for each $(\tau, t_+, t_-) \in [0, 1] \times [0, \infty)^2$ by

$$H(\tau, t) = \tau(\xi \circ \tilde{\gamma})(t) + (1 - \tau)(t_+ - 1, t_- - 1).$$

By the choice of R, for every $(\tau, t) \in [0, 1] \times \partial([0, \infty)^2 \cap B_R)$, $H(\tau, t) \neq 0$, and thus by the homotopy invariance property of the degree, $\deg(\xi \circ \tilde{\gamma}|_{(0,\infty)^2 \cap B_R}) = 1$. If we set $\gamma = \tilde{\gamma} \circ \Phi$, where $\Phi : \mathbb{B}^2 \to [0, \infty) \cap \bar{B}_R$ is an orientation preserving homeomorphism, we have $\gamma \in \Gamma$ and $\sup_{\mathbb{B}^2} \mathcal{A} \circ \gamma = \mathcal{A}(u)$.

We have thus proved that if $u \in \mathcal{N}_{\mathrm{nod}}$ and if $\mathcal{A}(u)^{\frac{p-1}{p-2}} \le c_{\mathrm{nod}}^{\frac{p-1}{p-2}} + \varepsilon$, then

$$\mathcal{A}(u) \ge \tilde{c},$$

from which we deduce that $c_{\mathrm{nod}} \ge \tilde{c}$. □

We would like to point out that the inequality (3.4) in the proof of proposition 3.2 gives a *lower bound* on the level c_{nod} that refines the degeneracy given for $p > 2$ by theorem 3.

Corollary 3.3. *If* $\frac{N-2}{N+\alpha} \le \frac{1}{p} \le \frac{1}{2}$, *then*

$$c_{\mathrm{nod}} \ge 2^{\frac{p-2}{p-1}} c_0.$$

In particular, corollary (3.3) allows theorem 3 to hold when $p = 2$.

Proposition 3.4 (Existence of a Palais-Smale sequence)**.** *If* $\frac{N-2}{N+\alpha} \le \frac{1}{p} < \frac{1}{2}$, *then there exists a sequence* $(u_n)_{n\in\mathbb{N}}$ *in* $H^1(\mathbb{R}^N)$ *such that*

$$\mathcal{A}(u_n) \to c_{\mathrm{nod}}, \quad \operatorname{dist}(u_n, \mathcal{N}_{\mathrm{nod}}) \to 0 \quad \textit{and} \quad \mathcal{A}'(u_n) \to 0 \quad \textit{in } H^1(\mathbb{R}^N)',$$

as $n \to \infty$.

Proof. We take Γ given by proposition 3.2 with $\varepsilon < c_0^{\frac{p-1}{p-2}}$. The location theorem [30, theorem 2.20] (see also [3, theorem 2][26, theorem 2.12]) is applicable and gives the conclusion. □

3.2. **Convergence of the Palais–Smale sequence.** We prove that Palais–Smale sequences at the level c_{nod} and localized near the Nehari nodal set $\mathcal{N}_{\mathrm{nod}}$ converge strongly up to a subsequence and up to translations.

Proposition 3.5. *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $H^1_{\mathrm{odd}}(\mathbb{R}^N)$ *such that, as* $n \to \infty$,

$$\mathcal{A}(u_n) \to c_{\mathrm{nod}}, \quad \operatorname{dist}(u_n, \mathcal{N}_{\mathrm{nod}}) \to 0 \quad \textit{and} \quad \mathcal{A}'(u_n) \to 0 \quad \textit{in } H^1(\mathbb{R}^N)'.$$

If $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{1}{2}$ *and if*

$$c_{\mathrm{nod}} < 2c_0,$$

then there exists a sequence of points $(a_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}^N$ *such that the subsequence* $\big(u_{n_k}(\cdot - a_{n_k})\big)_{k\in\mathbb{N}}$ *converges strongly in* $H^1(\mathbb{R}^N)$ *to* $u \in H^1(\mathbb{R}^N)$. *Moreover*

$$\mathcal{A}(u) = c_{\mathrm{nod}}, \qquad u \in \mathcal{N}_{\mathrm{nod}} \qquad \textit{and} \qquad \mathcal{A}'(u) = 0 \quad \textit{in } H^1(\mathbb{R}^N)'.$$

Palais–Smale conditions have been already proved by concentration–compactness arguments for local semilinear elliptic problems [7, 11].

Proof of proposition 3.5. We shall proceed through several claims on the sequence $(u_n)_{n\in\mathbb{N}}$.

Claim 1. *The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in the space $H^1(\mathbb{R}^N)$.*

Proof of the claim. We write, as $n \to \infty$,

$$\Bigl(\frac{1}{2} - \frac{1}{2p}\Bigr) \int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2 = \mathcal{A}(u_n) - \frac{1}{2p} \langle \mathcal{A}'(u_n), u_n\rangle$$
$$= \mathcal{A}(u_n) + o\Bigl(\Bigl(\int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2\Bigr)^{\frac{1}{2}}\Bigr).$$

from which the claim follows. ◇

We now show that neither positive nor the negative parts of the sequence $(u_n)_{n\in\mathbb{N}}$ tend to 0.

Claim 2. *The functional $u \in H^1(\mathbb{R}^N) \mapsto \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u^\pm|^p$ is uniformly continuous on bounded subsets of the space $H^1(\mathbb{R}^N)$.*

We bring to the attention of the reader that the related map $u \in H^1(\mathbb{R}^N) \mapsto \int_{\mathbb{R}^N} |\nabla u^\pm|^2 + |u^\pm|^2$ *is not uniformly continuous* on bounded sets.

Proof of the claim. For every $u, v \in H^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u^\pm|^p - \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v^\pm|^p$$
$$= \frac{1}{2} \int_{\mathbb{R}^N} \bigl(I_\alpha * (|u|^p + |v|^p)\bigr)\bigl(|u^\pm|^p - |v^\pm|^p\bigr)$$
$$+ \frac{1}{2} \int_{\mathbb{R}^N} \bigl(I_\alpha * (|u|^p - |v|^p)\bigr)\bigl(|u^\pm|^p + |v^\pm|^p\bigr).$$

By the classical Hardy–Littlewood–Sobolev inequality (1.1), we obtain

$$\Bigl|\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u^\pm|^p - \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v^\pm|^p\Bigr|$$
$$\le C\Bigl(\int_{\mathbb{R}^N} (|u|^p + |v|^p)^{\frac{2N}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}} \Bigl(\int_{\mathbb{R}^N} \bigl||u^\pm|^p - |v^\pm|^p\bigr|^{\frac{2N}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}}$$
$$+ C\Bigl(\int_{\mathbb{R}^N} (|u^\pm|^p + |v^\pm|^p)^{\frac{2N}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}} \Bigl(\int_{\mathbb{R}^N} \bigl||u|^p - |v|^p\bigr|^{\frac{2N}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}}.$$

Since for every $s, t \in \mathbb{R}$, one has $\bigl||s^\pm|^p - |t^\pm|^p\bigr| \le \bigl||s|^p - |t|^p\bigr|$ and $|s^\pm|^p + |t^\pm|^p \le |s|^p + |t|^p$, the latter estimate can be simplified to

$$\Bigl|\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u^\pm|^p - \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v^\pm|^p\Bigr|$$
$$\le 2C\Bigl(\int_{\mathbb{R}^N} (|u|^p + |v|^p)^{\frac{2N}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}} \Bigl(\int_{\mathbb{R}^N} \bigl||u|^p - |v|^p\bigr|^{\frac{2N}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}}.$$

Next, for each $s, t \in \mathbb{R}$, we have, since $p \geq 2$,

$$
\begin{aligned}
\big||s|^p - |t|^p\big| &\le p|s-t| \int_0^1 p|\tau s + (1-\tau)t|^{p-1}\,\mathrm{d}\tau \\
&\le p|s-t| \int_0^1 \tau|s|^{p-1} + (1-\tau)|t|^{p-1}\,\mathrm{d}\tau \\
&= \frac{p}{2}|s-t|\bigl(|s|^{p-1} + |t|^{p-1}\bigr).
\end{aligned}
$$

Hence, we have,

$$
\begin{aligned}
\Bigl|\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u^\pm|^p - \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v^\pm|^p\Bigr| \\
\le 2C\Bigl(\frac{p}{2}\Bigr)^{\frac{N+\alpha}{2N}} \Bigl(\int_{\mathbb{R}^N} (|u|^p + |v|^p)^{\frac{2N}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}} \\
\times \Bigl(\int_{\mathbb{R}^N} \bigl(|u|^{p-1} + |v|^{p-1}\bigr)^{\frac{2N}{N+\alpha}} |u-v|^{\frac{2N}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}}.
\end{aligned}
$$

Therefore, by the Hölder inequality,

$$
\begin{aligned}
\Bigl|\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u^\pm|^p - \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v^\pm|^p\Bigr| \\
\le C'\Bigl(\int_{\mathbb{R}^N} |u|^{\frac{2Np}{N+\alpha}} + |v|^{\frac{2Np}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{N}(1-\frac{1}{2p})} \Bigl(\int_{\mathbb{R}^N} |u-v|^{\frac{2Np}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2Np}}.
\end{aligned}
$$

This shows that the map is uniformly continuous on $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. Since by assumption, $\frac{N-2}{N+\alpha} \le \frac{1}{p} \le \frac{N}{N+\alpha}$, in view of the classical Sobolev embedding theorem, the embedding $H^1(\mathbb{R}^N) \subset L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ is uniformly continuous, and the claim follows. ◇

Claim 3. *We have*

$$
\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n^\pm|^2 + |u_n^\pm|^2 = \liminf_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n^\pm|^p > 0.
$$

Proof of the claim. First we observe that if $v \in \mathcal{N}_{\mathrm{nod}}$, then by the Hardy–Littewood–Sobolev inequality (1.1), the Sobolev inequality and the definition of the nodal Nehari set $\mathcal{N}_{\mathrm{nod}}$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v^\pm|^p &\le C\Bigl(\int_{\mathbb{R}^N} |v|^{\frac{2Np}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}} \Bigl(\int_{\mathbb{R}^N} |v^\pm|^{\frac{2Np}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}} \\
&\le C'\Bigl(\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2\Bigr)^{\frac{p}{2}} \Bigl(\int_{\mathbb{R}^N} |\nabla v^\pm|^2 + |v^\pm|^2\Bigr)^{\frac{p}{2}} \\
&= C'\Bigl(\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2\Bigr)^{\frac{p}{2}} \Bigl(\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v^\pm|^p\Bigr)^{\frac{p}{2}}.
\end{aligned}
$$

Since $v^\pm \ne 0$, we deduce that

$$
\inf_{v\in\mathcal{N}_{\mathrm{nod}}} \Bigl(\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2\Bigr)^{\frac{p}{2}} \Bigl(\int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v^\pm|^p\Bigr)^{\frac{p-2}{2}} > 0.
$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in the space $H^1(\mathbb{R}^N)$ and since $\lim_{n\to\infty} \mathrm{dist}(u_n, \mathcal{N}_{\mathrm{odd}}) = 0$, we deduce from the lower bound above and from the uniform continuity property of claim 2 that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n^\pm|^p > 0.$$

Since $\lim_{n\to\infty}\langle \mathcal{A}'(u_n), u_n^\pm\rangle = 0$, the conclusion follows. ◇

Claim 4. *There exists $R > 0$ such that*

$$\limsup_{n\to\infty} \sup_{a\in\mathbb{R}^N} \int_{B_R(a)} |u_n^+|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |u_n^-|^{\frac{2Np}{N+\alpha}} > 0.$$

Proof of the claim. We assume by contradiction that for every $R > 0$,

$$\lim_{n\to\infty} \sup_{a\in\mathbb{R}^N} \int_{B_R(a)} |u_n^+|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |u_n^-|^{\frac{2Np}{N+\alpha}} = 0.$$

Then by lemma 3.6 below, since the sequences $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ are bounded in the space $H^1(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n^+|^p)|u_n^-|^p = 0.$$

We now take $t_{n,\pm} \in (0,\infty)$ such that $t_{n,\pm}u_n^\pm \in \mathcal{N}_0$. Since

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n^\pm|^2 + |u_n^\pm|^2 &= \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n^\pm|^p + \langle \mathcal{A}'(u_n), u_n^\pm\rangle\\ &= \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n^\pm|^p + o(1)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |u_n^\pm|^p)|u_n^\pm|^p &= \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n^\pm|^p - \int_{\mathbb{R}^N} (I_\alpha * |u_n^+|^p)|u_n^-|^p\\ &= \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n^\pm|^p + o(1),\end{aligned}$$

it follows, in view of claim 3, that $\lim_{n\to\infty} t_{n,\pm} = 1$. We compute

$$\begin{aligned}&\mathcal{A}(t_{n,+}u_n^+ + t_{n,-}u_n^-)\\ &\qquad = \mathcal{A}(t_{n,+}u_n^+) + \mathcal{A}(t_{n,-}u_n^-) - \frac{t_{n,+}^p t_{n,-}^p}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_n^+|^p)|u_n^-|^p.\end{aligned}$$

and we deduce that

$$\begin{aligned}c_{\text{nod}} &= \lim_{n\to\infty} \mathcal{A}(t_{n,+}u_n^+ + t_{n,-}u_n^-)\\ &\ge \liminf_{n\to\infty} \mathcal{A}(t_{n,+}u_n^+) + \liminf_{n\to\infty} \mathcal{A}(t_{n,-}u_n^-) \ge 2c_0,\end{aligned}$$

in contradiction with the assumption of the proposition. ◇

We now conclude the proof of the proposition. Up to a translation and a subsequence, we can assume that

$$\liminf_{n\to\infty} \int_{B_R(a)} |u_n^\pm|^{\frac{2Np}{N+\alpha}} \ge 0$$

and that the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly to some $u \in H^1(\mathbb{R}^N)$. As in the proof of proposition 2.3, by the weak convergence and by the classical Rellich–Kondrashov compactness theorem, $\mathcal{A}'(u) = 0$ and $u^\pm \neq 0$, whence

$u \in \mathcal{N}_{\text{nod}}$. We also have by lower semicontinuity of the Sobolev norm under weak convergence

$$\begin{aligned}\lim_{n\to\infty} \mathcal{A}(u_n) &= \lim_{n\to\infty} \mathcal{A}(u_n) - \frac{1}{2p}\langle \mathcal{A}'(u_n), u_n\rangle \\ &= \lim_{n\to\infty} \Bigl(\frac{1}{2} - \frac{1}{2p}\Bigr) \int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2 \\ &\geq \Bigl(\frac{1}{2} - \frac{1}{2p}\Bigr) \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \\ &= \mathcal{A}(u) - \frac{1}{2p}\langle \mathcal{A}'(u), u\rangle = \mathcal{A}(u),\end{aligned}$$

from which we deduce that $\mathcal{A}(u) = c_{\text{nod}}$ and the strong convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ in the space $H^1(\mathbb{R}^N)$. □

Lemma 3.6. *If $\frac{N-2}{N+\alpha} \leq \frac{1}{p} < \frac{N}{N+\alpha}$, then for every $\beta \in \bigl(\alpha, \min(1, p-1)N\bigr)$, there exists $C > 0$ such that for every $u, v \in H^1(\mathbb{R}^N)$,*

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha |u|^p)|v|^p \leq C\Bigl(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2\Bigr)^{\frac{1}{2}} \\ \Bigl(\sup_{a\in\mathbb{R}^N} \int_{B_R(a)} |u|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |v|^{\frac{2Np}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}(1-\frac{1}{p})} \\ + \frac{C}{R^{\beta-\alpha}}\Bigl(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2\Bigr)^{\frac{p}{2}}\end{aligned}$$

When $p = \frac{N+\alpha}{N}$, then $(p-1)N = \alpha$ and there is no β that would satisfy the assumptions.

Proof of lemma 3.6. We first decompose the integral as

$$\int_{\mathbb{R}^N} (I_\alpha |u|^p)|v|^p = \int_{\mathbb{R}^N} ((\chi_{B_{R/2}} I_\alpha) * |u|^p)|v|^p + \int_{\mathbb{R}^N} ((\chi_{\mathbb{R}^N\setminus B_{R/2}} I_\alpha) * |u|^p)|v|^p.$$

We then observe that if $\beta \in (\alpha, N)$, then

$$\int_{\mathbb{R}^N} ((\chi_{\mathbb{R}^N\setminus B_{R/2}} * I_\alpha)|u|^p)|v|^p \leq \frac{C}{R^{\beta-\alpha}} \int_{\mathbb{R}^N} (I_\beta * |u|^p)|v|^p.$$

If $\beta < (p-1)N$, then by the Hardy–Littewood–Sobolev inequality and by the Sobolev inequality, we have

$$\int_{\mathbb{R}^N} ((\chi_{\mathbb{R}^N\setminus B_{R/2}} I_\alpha) * |u|^p)|v|^p \leq \frac{C'}{R^{\beta-\alpha}}\Bigl(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\Bigr)^{\frac{p}{2}}\Bigl(\int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2\Bigr)^{\frac{p}{2}}.$$

Next, we have

$$\begin{aligned}\int_{\mathbb{R}^N} ((\chi_{B_{R/2}} I_\alpha) * |u|^p)|v|^p \\ \leq \frac{C'}{R^N} \int_{\mathbb{R}^N} \int_{B_R(a)} \int_{B_R(a)} I_\alpha(x-y)|u(x)|^p |v(y)|^p \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}a.\end{aligned}$$

For every $a \in \mathbb{R}^N$, we have, by the Hardy–Littewood–Sobolev inequality (1.1) and the classical Sobolev inequality on the ball $B_R(a)$,

$$\int_{B_R(a)} \int_{B_R(a)} I_\alpha(x-y)|u(x)|^p |v(y)|^p \,\mathrm{d}x \,\mathrm{d}y$$

$$
\begin{aligned}
&\le C''\Bigl(\int_{B_R(a)} |u|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |v|^{\frac{2Np}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}}\\
&\qquad\le C'''\Bigl(\int_{B_R(a)} |u|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |v|^{\frac{2Np}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}(1-\frac{1}{p})}\\
&\qquad\qquad\times\Bigl(\int_{B_R(a)} |\nabla u|^2+|u|^2 \int_{B_R(a)} |\nabla v|^2+|v|^2\Bigr)^{\frac{1}{2}}.
\end{aligned}
$$

We now integrate this estimate with respect to $a \in \mathbb{R}^N$ and apply the Cauchy–Schwarz inequality to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \bigl((\chi_{B_{R/2}} I_\alpha)|u|^p\bigr)|v|^p\\
&\le \frac{C'''}{R^N}\Bigl(\int_{\mathbb{R}^N}\Bigl(\int_{B_R(a)} |\nabla u|^2+|u|^2\Bigr)\,\mathrm{d}a\Bigr)^{\frac{1}{2}}\Bigl(\int_{\mathbb{R}^N}\Bigl(\int_{B_R(a)} |\nabla v|^2+|v|^2\Bigr)\,\mathrm{d}a\Bigr)^{\frac{1}{2}}\\
&\qquad\qquad\times\Bigl(\sup_{a\in\mathbb{R}^N}\int_{B_R(a)} |u|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |v|^{\frac{2Np}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}(1-\frac{1}{p})}\\
&\qquad= C''''\Bigl(\int_{\mathbb{R}^N} |\nabla u|^2+|u|^2 \int_{\mathbb{R}^N} |\nabla v|^2+|v|^2\Bigr)^{\frac{1}{2}}\\
&\qquad\qquad\times\Bigl(\sup_{a\in\mathbb{R}^N}\int_{B_R(a)} |u|^{\frac{2Np}{N+\alpha}} \int_{B_R(a)} |v|^{\frac{2Np}{N+\alpha}}\Bigr)^{\frac{N+\alpha}{2N}(1-\frac{1}{p})}.
\end{aligned}
$$

□

3.3. **Existence of a minimal action nodal solution.** In order to prove theorem 2, we finally establish the strict inequality.

Proposition 3.7. *If* $\frac{N-2}{N+\alpha} \le \frac{1}{p} \le \frac{N}{N+\alpha}$*, then*

$$\mathcal{N}_{\text{odd}} \subset \mathcal{N}_{\text{nod}}.$$

In particular,

$$c_{\text{nod}} \le c_{\text{odd}}.$$

Proof. If $u \in \mathcal{N}_{\text{odd}}$, then since $u \in \mathcal{N}_0$,

$$\langle \mathcal{A}'(u), u^+\rangle + \langle \mathcal{A}'(u), u^-\rangle = \langle \mathcal{A}'(u), u\rangle = 0.$$

On the other hand, since $u \in H^1_{\text{odd}}(\mathbb{R}^N)$, by the invariance of u under odd reflections,

$$\langle \mathcal{A}'(u), u^+\rangle = \langle \mathcal{A}'(u), u^-\rangle,$$

and the conclusion follows. □

We can now prove theorem 2 about the existence of minimal action nodal solutions.

Proof of theorem 2. Proposition 3.4 gives the existence of a localized Palais–Smale sequence $(u_n)_{n\in\mathbb{N}}$. By propositions 2.4 and 3.7, the strict inequality $c_{\text{nod}} < 2c_0$ holds. Hence we can apply proposition 3.5 to reach the conclusion. The solution u is twice continuously differentiable by the Choquard equation's regularity theory [21, proposition 4.1] (see also [8, lemma A.10]). □

3.4. **Degeneracy in the locally sublinear case.** We conclude this paper by proving that $c_{\mathrm{nod}} = c_0$ if $p < 2$.

Proof of theorem 3. We observe that if $u \in \mathcal{N}_0$, then $|u| \in \mathcal{N}_0$. Together with a density argument, this shows that

$$c_0 = \inf\{\mathcal{A}(u) \;:\; u \in \mathcal{N}_0 \cap C^1_c(\mathbb{R}^N) \text{ and } u \geq 0 \text{ on } \mathbb{R}^N\}.$$

Let now $u \in \mathcal{N}_0 \cap C^1_c(\mathbb{R}^N)$ such that $u \geq 0$ on $\mathbb{R}^N$. We choose a point $a \notin \operatorname{supp} u$ and a function $\varphi \in C^1_c(\mathbb{R}^N) \setminus \{0\}$ such that $\varphi \geq 0$ and we define for each $\delta > 0$ the function $u_\delta : \mathbb{R}^N \to \mathbb{R}$ for every $x \in \mathbb{R}^N$ by

$$u_\delta(x) = u(x) - \delta^{\frac{2}{2-p}} \varphi\Big(\frac{x-a}{\delta}\Big).$$

We observe that, if $\delta > 0$ is small enough, $u_\delta^+ = u^+$. By a direct computation, we have $t_+ u_\delta^+ + t_- u_\delta^- \in \mathcal{N}_{\mathrm{nod}}$ if and only if

(3.5)
$$\begin{cases} (t_+^{2-p} - t_+^p) \displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p = t_-^p \delta^{N + \frac{2p}{2-p}} J_\delta, \\ \qquad t_-^{2-p} \displaystyle\int_{\mathbb{R}^N} |\nabla \varphi|^2 + \delta^2 |\varphi|^2 = t_+^p J_\delta + t_-^p \delta^{\alpha + \frac{2p}{2-p}} \displaystyle\int_{\mathbb{R}^N} (I_\alpha * |\varphi|^p)|\varphi|^p, \end{cases}$$

where

$$J_\delta = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)(\delta z + a)(\varphi(z))^p \,\mathrm{d}z.$$

We observe that when $\delta = 0$, the system reduces to

$$\begin{cases} (t_+^{2-p} - t_+^p) \displaystyle\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p = 0, \\ \qquad t_-^{2-p} \displaystyle\int_{\mathbb{R}^N} |\nabla \varphi|^2 = t_+^p (I_\alpha * |u|^p)(a) \displaystyle\int_{\mathbb{R}^N} |\varphi|^p, \end{cases}$$

which has a unique solution. By the implicit function theorem, for $\delta > 0$ small enough there exists $(t_{+,\delta}, t_{-,\delta}) \in (0, \infty)^2$ that solves the system (3.5) and such that

$$\lim_{\delta \to 0} (t_{+,\delta}, t_{-,\delta}) = \Big(1, \Big((I_\alpha * |u|^p)(a) \int_{\mathbb{R}^N} |\varphi|^p \Big/ \int_{\mathbb{R}^N} |\nabla \varphi|^2\Big)^{\frac{1}{2-p}}\Big).$$

Since $(N-2)(2-p) > -4$, we have $t_+ u_\delta^+ + t_- u_\delta^- \to u$ strongly in $H^1(\mathbb{R}^N)$ as $\delta \to 0$, and it follows that

$$\inf_{\mathcal{N}_{\mathrm{nod}}} \mathcal{A} \leq \mathcal{A}(u),$$

and thus $c_{\mathrm{nod}} \leq c_0$.

We assume now that the function $u \in \mathcal{N}_{\mathrm{nod}}$ minimizes the action functional $\mathcal{A}$ on the nodal Nehari set $\mathcal{N}_{\mathrm{nod}}$. Since $c_0 = c_{\mathrm{nod}}$, we deduce that u also minimizes $\mathcal{A}$ over the Nehari manifold $\mathcal{N}_0$. By the properties of such groundstates [21, proposition 5.1], either $u^+ = 0$ or $u^- = 0$, in contradiction with the assumption $u \in \mathcal{N}_{\mathrm{nod}}$ and the definition of the Nehari nodal set $\mathcal{N}_{\mathrm{nod}}$. ∎

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