Transbounday Pollution Externalities: Think Globally, Act Locally?*

Davide La Torre[†] Danilo Liuzzi[‡] Simone Marsiglio[§]

Forthcoming in Journal of Mathematical Economics

Abstract

We analyze the implications of transboundary pollution externalities on environmental policymaking in a spatial setting, in which pollution diffuses across the global spatial economy independently of the specific location in which it is originally generated. This framework gives rise to a simple regional optimal pollution control problem allowing us to compare the global and local solutions in which, respectively, the transboundary externality is and is not taken into account in the determination of the optimal policy by individual local policymakers. We show that it is not obvious that transboundary externalities are a source of inefficiency per se since this is strictly related to the spatial features of the initial distribution of pollution. If the initial pollution distribution is spatially homogeneous then the local and global solutions will coincide and thus no efficiency loss will arise from transboundary externalities, but if it is spatially heterogeneous the local solution will be suboptimal and thus a global approach to environmental problems will be needed to achieve efficiency. From a normative perspective, in this latter (and most realistic) case we also quantify the amount of policy intervention needed at local level in order to achieve the globally desirable goal of pollution eradication in the long run. Our conclusions hold true in a number of different settings, including situations in which the spatial domain is either bounded or unbounded, and situations in which macroeconomic-environmental feedback effects are taken into account.

Keywords: Local and Global Solutions, Regional Optimal Control, Spatial Dynamics, Transboundary Pollution Externalities
JEL Classification: C60, H23, Q28

1 Introduction

After decades of debates it has finally grown a shared consensus on the fact that anthropogenetic activities, and in particular economic activities, are an important determinant of environmental problems and climate change (Oreskes, 2004, IPCC, 2014). Policymakers need thus to critically intervene to reduce the accumulation of polluting emissions in the atmosphere in order to ensure that economic development is

*We are grateful to Davide Fiaschi, Mauro Sodini, Anastasios Xepapadeas and the participants in the Workshop on Space and Growth: Theoretical and Empirical Models (Pisa, Italy) and in the SURED 2020 Conference (Ascona, Switzerland) for insightful discussions. We wish to thank the Associate Editor, Raouf Boucekkine, and three anonymous referees for their constructive comments on earlier drafts of the paper. All remaining errors and omissions are our own sole responsibility. Simone Marsiglio has been supported by the Italian Ministry of University and Research as a part of the PRIN 2017 program the Italian Ministry of University and Research as a part of the PRIN 2017 program (grant protocol number 2017FKHBA8), and this work is supported by the Universitá di Pisa under the "PRA – Progetti di Ricerca di Ateneo"(Institutional Research Grants) - Project no. PRA_2020_79 "Sustainable development: economic, environmental and social issues".

[†]SKEMA Business School - Université Côte d'Azur, Sophia Antipolis Campus, France. Contact: davide.latorre@skema.edu.

[‡]University of Milan, Department of Economics, Management and Quantitative Methods, Milan, Italy. Contact: danilo.liuzzi@unimi.it.

[§]University of Pisa, Department of Economics and Management, via Cosimo Ridolfi 10, Pisa 56124, Italy. Contact: simone.marsiglio@unipi.it.

effectively addressed towards a sustainable path. However, understanding how to determine the optimal size of these policy interventions is not simple at all, especially because the stock of pollution in specific locations is strongly affected by the level of emissions generated in other locations as well, a phenomenon referred to as transboundary externality (Ansuategi and Perrings, 2000; Ansuategi, 2003). In order to ensure that such externalities are effectively accounted for in the design of environmental policy it has often been suggested that local policymakers should adopt a global perspective and collaborate with one another. This argument has been summarized in the popular motto *"think globally, act locally"*[1] (or *"think global, act local"*). Despite the fact that the benefits of such a collaborative approach to policymaking are quite clear, understanding how to effectively implement collaboration is not simple at all, since quantifying the size of such transboundary externalities and what individual policymakers should do is all but trivial. The goal of this paper is to partly contribute to this debate by formally analyzing the desirability of a think globally, act locally approach and the type of intervention that local policymakers should opt for in order to effectively implement it.

Specifically, we analyze a pollution control problem in the presence of transboundary externalities. On the one hand, several works have introduced transboundary externalities in the form of spatial spillovers in the contexts of capital accumulation (Brito, 2004; Camacho and Zou, 2004; Camacho et al., 2008; Boucekkine et al., 2009, 2013a, 2013b, 2019) and environmental problems (Anita et al., 2013, 2015; Brock and Xepapadeas, 2008, 2010, 2017; Camacho and Pérez–Barahona, 2015; La Torre et al., 2015, 2019a, 2019b), but none in a pollution control framework. On the other hand, several works have discussed different types of pollution control problems under uncertainty and irreversibility (Bawa, 1975; Forster, 1972, 1975; Keeler et al., 1973; van der Ploeg and Withagen, 1991; Athanassoglou and Xepapadeas, 2012; Saltari and Travaglini, 2014; La Torre et al., 2017), but to the best of our knowledge, none has ever considered the effects of transboundary externalities as spatial spillovers. We therefore try to bridge these two branches of the economics literature by analyzing a finite time horizon framework in which the social planner tries to minimize the social costs of pollution (La Torre et al., 2017) by accounting for the fact that polluting emissions, independently of where they are originated, naturally spread in the atmosphere affecting the pollution stock of every location (La Torre et al., 2015). Such a setting allows us to compare the "global" and "local" solutions in which, respectively, the transboundary externality is and is not taken into account in the determination of the optimal environmental policy by single local policymakers. This approach gives rise to a simple "regional optimal control problem", which is a specific type of spatially-structured optimal control problem aiming to understand whether the local solution is effectively optimal also globally (Lions, 1973, 1988). This class of problem was born as an application of the think globally, act locally motto in mathematics, thus our paper partly relates to this literature as well (see Anita and Capasso, 2018, for a recent survey of applications in epidemiology). To the best of our knowledge, ours is the first paper bringing a regional optimal control approach in economics and applying it to a transboundary pollution control problem.

Different from extant works on transboundary pollution externalities, which compare the spatial and a-spatial outcomes (La Torre et al., 2015, 2019a; de Frutos and Martin-Herran, 2019b), our analysis is based on the comparison of local and global solutions, characterizing the decentralized and centralized outcomes, respectively (Ogawa and Wildasin, 2009).[2] Specifically, the global solution represents a situation in which the global policymaker determines the optimal level of intervention in each single local economy by internalizing the effects of transboundary pollution externalities at local levels, while the local solution represents a situation in which single local policymakers determine their optimal level of local intervention by completing neglecting the existence of such transboundary pollution externalities. However, the evolution

[1]The motto "think globally, act locally" seems to have origins in the pioneering work of the Scottish urban planner Geedes (1915) which, even if not explicitly containing the phrase, clearly discusses the underlying argument.

[2]Even if in completely different settings and with goals substantially different from ours, also La Torre et al. (2019b) compare the decentralized and centralized outcomes in spatial frameworks. Specifically, they analyze the effects of (non-transboundary) pollution on mortality and population dynamics in a framework with spatial migration.

of the pollution stock, both in a local and global solutions, depends on the level of intervention in each single location and on its transboundary effects, which are internalized in the determination of the global intervention level while they are not considered when determining the local one. Therefore, in our framework we always consider the existence of spatial spillovers (i.e., the economy is spatially structured and pollution spreads across space), thus our local solution does not represent the a-spatial equivalent of our global solution. Moreover, in the local solution single local policymakers do not interact strategically exploiting free-riding opportunities (i.e., there is no game structure), thus our local solution is not the non-cooperative counterpart of a global cooperative game equilibrium. Our setting thus significantly departs from some recent works which have analyzed the implications of transboundary pollution in differential game contexts to analyze the spatial implications of strategic interactions, comparing spatial vs a-spatial and cooperative vs non-cooperative equilibrium outcomes (de Frutos et al., 2019; de Frutos and Martin-Herran, 2019a, 2019b; Boucekkine et al., 2020). Our different perspective allows us to investigate whether despite the presence of transboundary pollution externalities in a spatial setting a local approach to policymaking may be enough to achieve globally desirable outcomes, and eventually how such desirable outcomes may be implemented through coordination at the local level.

The analysis of the local and global solutions in our simple regional optimal control problem, due to its peculiar quadratic-linear formulation, allows us to analytically derive two interesting sets of conclusions. First, we show that transboundary externalities are not a source of inefficiency per se since the spatial structure of the initial distribution of pollution plays a critical role in determining their efficiency implications. Indeed, if the initial pollution distribution is homogeneous the local and global solutions will perfectly coincide and thus no efficiency loss will arise from transboundary externalities. It is the existence of some heterogeneity in the initial pollution distribution that leads the presence of transboundary externalities to place a wedge between the local and the global solutions, requiring thus a global approach to environmental problems to restore efficiency, consistently with the think globally, act locally argument. To some extent these results are consistent with Ogawa and Wildasin's (2009), who show, in a completely different and purely static setting, that despite the presence of transboundary externalities there exist situations in which local policymaking may give rise to efficient outcomes globally; different from them, we can clearly determine the situations in which this conclusion may or may not hold true, identifying the spatial features of the initial pollution distribution as the driver of this eventual lack of efficiency losses. Second, in the case of heterogeneity in the initial pollution distribution, which clearly represents the most realistic scenario from a real world perspective, we quantify how to implement coordination at local level in order to achieve globally desirable goals. Specifically, we show that if every local economy implements an environmental policy stringent enough (and we determine what stringent enough exactly means), then the global total pollution level will fall. If this is the case, then over the long run the entire global economy will be able to achieve a completely pollution-free status. These two results jointly suggest that from a normative perspective it may well be possible to determine whether and how global collaboration needs to be implemented locally in order to deal with global pollution problems. To the best of our knowledge such a neat and clear characterization of these issues from an economic point of view has never been provided before. We also show that our main conclusions do not depend on the peculiarities of our model's formulation but they rather extend to more general and complicated frameworks.

This paper proceeds as follows. Section 2 discusses environmental policy in an a-spatial setting, where there are no transboundary externalities and there is no spatial dimension. This represents a benchmark for our following analysis, and it basically consists of a pollution control problem over finite horizon, in which the stock of pollution is entirely determined by local economic and environmental conditions. We derive the optimal policy which characterizes the environmental tax in a setting abstracting from a spatial dimension and thus from spatial spillovers. Section 3 extends our baseline model in order to allow for a spatial dimension in which transboundary externalities occur as a result of spatial pollution diffusion over a bounded spatial domain. We distinguish between the local optimal policy, in which the environmental tax is determined without considering the effects of transboundary pollution externalities, and the global optimal policy, in which the environmental tax is determined by internalizing such transboundary pollution externalities. We show that, only if there exists some spatial heterogeneity in the initial pollution distribution, the localized approach to environmental policy will give rise to suboptimal solutions, suggesting that coordination across local policymakers will be essential to deal with environmental problems only in such a setting. We also quantify the minimal level of intervention needed at local level for the global total level of pollution to decrease over time, and we show that if such a minimal level of intervention is implemented in every local economy then over the long run the entire global economy will be able to achieve a completely pollution-free status. The specific linear-quadratic formulation of our model allows us to derive the explicit local and global solutions for the spatio-temporal dynamic path of the environmental tax and the pollution level, which we illustrate through a numerical example to clearly visualize the extent to which the two solutions may differ. Section 4 presents some extensions of our baseline model showing that our main results straightforwardly apply also in more general frameworks, including traditional macroeconomic-environmental settings with macroeconomic and environmental feedback effects. Section 5 presents a further extension of our baseline model in which the spatial domain is unbounded showing that also in such a setting our main conclusions apply. Also in such a framework we derive the explicit solutions for the spatio-temporal dynamic path of the environmental tax and the pollution level and we graphically visualize through a numerical example the extent to which the local and global solutions may differ. Section 6 as usual concludes and suggests directions for future research. The proofs of most of our results are presented in appendix A.

2 The A-Spatial Model

We consider a deterministic version of the finite time horizon pollution control problem recently presented in La Torre et al. (2017). The macroeconomic framework is very simple but as we shall show in section 4 our main conclusions will hold true also in more complicated settings. Agents consume entirely their disposable income, $c_t = (1 - \tau_t)y_t$, where c_t is consumption, y_t income and $\tau_t \in (0, 1)$ the tax rate. The final consumption good, y_t, is produced through a linear production function, $y_t = ak_t$, where $a > 0$ is a technological parameter and k_t capital. Capital grows at a constant exogenous rate $g > 0$, and productive activities generate pollution, p_t. The tax revenue is entirely allocated to environmental policy aiming to reduce pollution accumulation, and one unit of output devoted to environmental preservation reduces one unit of pollution. Pollution dynamics is summarized by the following equation $\dot{p}_t = [\eta(1 - \tau_t) - \delta]\, p_t$, where $\eta > 0$ is the rate at which output growth (in excess of the environmental absorption capacity) generates emissions[3] and $\delta > 0$ the natural pollution decay rate. We assume that $\eta > \delta$ such that environmental regulation is effectively needed to avoid explosive pollution dynamics. In this setting, the policy instrument τ_t represents the environmental tax aiming to manage the economic-environmental trade off. The social planner wishes to minimize the social cost of pollution by choosing the optimal level of the tax rate. The social cost function, $\mathcal{C}$, is the weighted sum of two terms: the expected discounted ($\rho > 0$ is the discount factor) sum of instantaneous social losses depending on both environmental and economic factors, and the discounted environmental damage associated with the level of pollution remaining at the end of the planning horizon, T. Both the loss function $c(p_t, \tau_t)$ and the damage function $d(p_T)$ are assumed to take a quadratic form as follows: $c(p_t, \tau_t) = \frac{p_t^2(1+\tau_t^2)}{2}$ and $d(p_T) = \frac{p_T^2}{2}$, respectively. Note that the loss function increases both with the pollution stock and the tax rate, penalizing thus deviations from the no-pollution scenarios and the no-environmental-regulation scenario. The weight of the social losses and the environmental damage are

[3]We may think that $\eta = \eta(1+g_t)$, where g_t is the time-varying growth of output which increases the growth rate of pollution due to emissions in excess of the environmental absorption capacity, such that even in the absence of output growth pollution growth is constant. The assumption of a constant output growth at the rate g is just a mere simplifying assumption. For an extension of the baseline model to the case of time-varying output growth due to capital accumulation, see section 5 in La Torre et al. (2017).

given by $\theta \in [0,1]$ and $1-\theta$, respectively, such that $\frac{1-\theta}{\theta}$ represents the relative weight of the environmental damage in terms of the social losses.

Therefore, the social planner needs to choose the level of the environmental tax in order to minimize the social cost, given the evolution of pollution and its initial condition. The planner's optimal control problem[4] reads as follows:

$$\min_{\tau_t} \quad \mathcal{C} = \int_0^T \frac{p_t^2(1+\tau_t^2)}{2} e^{-\rho t} dt + \frac{1-\theta}{\theta}\frac{p_T^2}{2} e^{-\rho T} \tag{1}$$

$$s.t. \quad \dot{p}_t = [\eta(1-\tau_t) - \delta]\, p_t \tag{2}$$

$$p_0 > 0 \text{ given}, \tag{3}$$

The objective function in the above problem reflects sustainability considerations related to intertemporal equity (Chichilnisky et al., 1995; Chichilnisky, 1997; Colapinto et al., 2017). In particular, it is consistent with the so-called Chichilnisky's criterion which proposes to consider a weighted average between the discounted sum of instantaneous costs and the long run cost associated with pollution in order to make sure that future generations' wellbeing is effectively taken into account in the determination of the current optimal policy (Chichilnisky, 1997). The larger $1-\theta$ (i.e., the lower θ) the larger the weight attached to future generations, suggesting, as we shall clarify later, that environmental policy will tend to be stricter in order to allow future generations to live in a cleaner environment; thus $1-\theta$ represents the degree of sustainability concern (La Torre et al., 2017). Note that encompassing sustainability issues in the analysis requires us to add a second term in the objective function quantifying the level of pollution that future generations will need to bear. In order for this term to effectively matter in the planning problem, the time horizon needs to be finite to represent future generations not too far away in the future. A more detailed discussion of the peculiarities of the problem under investigation can be found in La Torre et al. (2017).

In order to simplify the above problem, we can define $u_t = p_t\tau_t$ which allows us to obtain a linear-quadratic model's formulation which can be explicitly solved in closed form. Through this variable change, the above problem can be rewritten as follows:

$$\min_{u_t} \quad \mathcal{C} = \int_0^T \frac{p_t^2 + u_t^2}{2} e^{-\rho t} dt + \frac{1-\theta}{\theta}\frac{p_T^2}{2} e^{-\rho T} \tag{4}$$

$$s.t. \quad \dot{p}_t = (\eta - \delta)\, p_t - \eta u_t \tag{5}$$

$$p_0 > 0 \text{ given}, \tag{6}$$

The current value Hamiltonian function, $\mathcal{H}\big(p_t, u_t, \lambda_t\big)$, read as:

$$\mathcal{H} \quad = \quad \frac{p_t^2 + u_t^2}{2} + \lambda_t \left[(\eta-\delta)p_t - \eta u_t\right],$$

where λ_t is the costate variable. The FOCs for a minimum are given by the following expressions:

$$\dot{\lambda}_t \quad = \quad \rho\lambda_t - p_t - (\eta-\delta)\lambda_t \tag{7}$$

$$u_t \quad = \quad \eta\lambda_t. \tag{8}$$

Substituting the latter expression in the former allows us to obtain the following system of differential equations for the state and control variables:

$$\dot{u}_t \quad = \quad (\rho - \eta + \delta)u_t - \eta p_t \tag{9}$$

$$\dot{p}_t \quad = \quad (\eta-\delta)\, p_t - \eta u_t, \tag{10}$$

[4] It is possible to show that such a social cost minimization problem is equivalent to a social welfare maximization problem in which the utility function takes some specific form (see section 4).

which, jointly with the terminal condition $u_T = \eta \frac{1-\theta}{\theta} p_T$, completely characterize the optimal solution of our a-spatial control problem. By defining the following vector z_t and matrix Θ:

$$z_t = \begin{bmatrix} p_t \\ u_t \end{bmatrix}, \qquad \Theta = \begin{bmatrix} \eta - \delta & -\eta \\ -\eta & \rho - \eta + \delta \end{bmatrix}$$

the system (9) and (10) can be rewritten as as follows:

$$\dot{z}_t = \Theta z_t$$

whose solution is given by:

$$z_t = C e^{\Theta t},$$

where $C = [C_1, C_2]^T$. Since the two eigenvalues are both real and distinct, it is not complicated to calculate the exponential term $e^{\Theta t}$ as follows:

$$e^{\Theta t} = \begin{bmatrix} e^{\frac{1}{2}\rho t}\left(\cosh\left(\frac{1}{2}\xi t\right) - \frac{(2(\delta-\eta)+\rho)\sinh\left(\frac{1}{2}\xi t\right)}{\xi}\right) & -2\frac{e^{\frac{1}{2}\rho t}\sinh\left(\frac{1}{2}\xi t\right)\eta}{\xi} \\ -2\frac{e^{\frac{1}{2}\rho t}\sinh\left(\frac{1}{2}\xi t\right)\eta}{\xi} & e^{\frac{1}{2}\rho t}\left(\cosh\left(\frac{1}{2}\xi t\right) + \frac{(2(\delta-\eta)+\rho)\sinh\left(\frac{1}{2}\xi t\right)}{\xi}\right) \end{bmatrix} \tag{11}$$

where $\xi = \sqrt{[2(\delta - \eta) + \rho]^2 + 4\eta^2}$. The explicit solution of the system above is therefore given by the following expressions:

$$u_t^* = \frac{\eta\left((1-\theta)\,\xi - \{(1-\theta)\,[2(\delta-\eta)+\rho] - 2\,\theta\}\tanh\left[\frac{1}{2}\xi\,(T-t)\right]\right)p_0}{e^{\frac{1}{2}\rho T}\left\{\theta\,\xi + \theta\,[2(\delta-\eta)+\rho] + 2\,\eta^2\,(1-\theta)\tanh\left(\frac{1}{2}\xi T\right)\right\}} \tag{12}$$

$$p_t^* = \frac{\left\{\theta\,[2(\delta-\eta)+\rho] + 2\,\eta^2\,(1-\theta)\tanh\left[\frac{1}{2}\xi\,(T-t)\right] + \theta\,\xi\right\}p_0}{e^{\frac{1}{2}\rho T}\left\{\theta\,\xi + \theta\,[2(\delta-\eta)+\rho] + 2\,\eta^2\,(1-\theta)\tanh\left(\frac{1}{2}\xi T\right)\right\}}, \tag{13}$$

from which it follows that the optimal environmental tax rate, $\tau_t = \frac{u_t}{p_t}$, read as follows:

$$\tau_t^* = \frac{\eta\left((1-\theta)\,\xi - \{(1-\theta)\,[2(\delta-\eta)+\rho] - 2\,\theta\}\tanh\left[\frac{1}{2}\xi\,(T-t)\right]\right)}{\theta\,[2(\delta-\eta)+\rho] + 2\,\eta^2\,(1-\theta)\tanh\left[\frac{1}{2}\xi\,(T-t)\right] + \theta\,\xi} \tag{14}$$

This expression can be rewritten as in La Torre et al. (2017) as follows:

$$\tau_t = \frac{1}{2\eta}\left\{2(\eta-\delta) - \rho + \sqrt{[2(\eta-\delta)-\rho]^2 + 4\eta^2}\tanh\left[\frac{\sqrt{[2(\eta-\delta)-\rho]^2+4\eta^2}(T-t)}{2} + \right.\right.$$
$$\left.\left. + \operatorname{arctanh}\left(\frac{2(1-\theta)\eta^2 - 2(\eta-\delta)\theta + \rho\theta}{\theta\sqrt{[2(\eta-\delta)-\rho]^2+4\eta^2}}\right)\right]\right\} \tag{15}$$

where $\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$ and $\operatorname{arctanh}(z) = \frac{\log(1+z) - log(1-z)}{2}$, with $-1 < z < 1$, are the hyperbolic tangent function and its inverse, respectively. From the (14) we can note that the optimal environmental tax is time-varying and intuitively increases with the degree of sustainability concern, and similarly also the optimal pollution level changes over time to reflect the environmental tax dynamics. Moreover, the optimal environmental tax is completely independent of the initial pollution level p_0, suggesting that the only determinants of the optimal policy are environmental (η and δ) and economic (ρ, θ, T) parameters. Note that in the determination of the above optimal policy, since the model abstracts completely from a spatial dimension, there is no room from transboundary externalities and thus the policymaker cannot account for them.

3 The Spatial Model: Bounded Spatial Domain

We now introduce a spatial dimension in the above problem to allow for transboundary pollution externalities. There exist two alternative forms of transboundary pollution externalities, with static and dynamic effects respectively (La Torre et al., 2015). On the one hand, pollution may be transboundary because at any moment in time the stock of pollution in each location is affected not only by the emissions generated in that single location but also by those generated in other locations as well; this characterizes a static externality with simultaneous effects between economies, and the implications of this type of externality have been extensively discussed in the pollution control literature, for example in the context of dynamic games (Long, 1992). On the other hand, pollution may be transboundary because of the effects of spatial diffusion, which lead the stock of pollution in each location to spread out across space affecting thus the stock of pollution in other locations as well; this characterizes a dynamic externality with delayed effects between economies and such delays are associated with the amount of time required for pollution to spatially propagate; to the best of our knowledge, the implications of this type of externality have not been analyzed in a pollution control setting yet. Given the lack of studies on this latter interpretation of transboundary pollution externalities and given the physical characteristics of pollution which is generally transported by air currents and deposited with precipitation, along with the fact that such natural phenomena take time (OECD, 1977), it seems convenient to model the transboundary characteristic of pollution as a dynamic externality.

We assume that the economy develops along a linear city (Hotelling, 1929), where all activities take place, and in particular pollution, even if generated in a specific location, diffuses across the whole economy (La Torre et al., 2015). We denote with $\tau_{x,t}$ and $p_{x,t}$ the tax rate and the pollution stock in the position x at date t, in a closed and compact set $\Omega \subset \mathbb{R}$. We also assume that the economy is isolated, in the sense that there is no flow of pollution that can cross the boundary in the normal direction and the component on the boundary is purely tangential. Mathematically this is described by the so-called Neumann's conditions, which state that the normal derivatives at the borders of Ω are null: $\frac{\partial \tau_{x,t}}{\partial x} = \frac{\partial p_{x,t}}{\partial x} = 0$. Other constraints, such us Dirichlet or Robin boundary conditions, might be imposed as well, however these conditions will somehow force the pollution level to assume a specific value at the boundary and thus to follow a specific path over time, which do not seem to fit the purpose of this paper which rather aims at exploring the pollution dynamics in a closed economy without external impositions. In this framework, a set of locations x may be interpreted as a specific local economy while the entire spatial domain as the global economy; such a possibility to distinguish between local and global economies allows us to compare the local and global solutions of the pollution control problem. Modeling the spatial domain as a compact interval implies that we are considering an isolated global economy (i.e., an island country located far away from other countries) in which pollution is entirely determined by the behavior of the different local economies which compose it (i.e., the subnational unities within the country). Despite this may seem a very restrictive assumption, as we shall see in section 5, our main results will not depend on the boundedness of the spatial domain.

In our spatial framework, the social planner's problem can be summarized as follows:

$$\min_{\tau_{x,t}} \quad \mathcal{C} = \int_0^T \int_\Omega \frac{p_{x,t}^2(1+\tau_{x,t}^2)}{2} e^{-\rho t} dx dt + \frac{1-\theta}{\theta} \int_\Omega \frac{p_{x,T}^2}{2} e^{-\rho T} dx \tag{16}$$

$$s.t. \quad \frac{\partial p_{x,t}}{\partial t} = d \frac{\partial^2 p_{x,t}}{\partial x^2} + \left[\eta(1-\tau_{x,t}) - \delta\right] p_{x,t} \tag{17}$$

$$\frac{\partial \tau_{x,t}}{\partial x} = \frac{\partial p_{x,t}}{\partial x} = 0, x \in \{x_a, x_b\}, \tag{18}$$

$$p_{x,0} > 0 \text{ given} \tag{19}$$

With respect to the a-spatial control problem discussed in the previous section, two important differences arise. (i) The social planner wishes to minimize the social costs of pollution within the spatial economy, thus the level of pollution and environmental tax in each location within Ω jointly contribute to determine the choice of the optimal level of policy intervention. (ii) The dynamics of pollution is described by a partial differential equation (PDE) which characterizes its spatio-temporal evolution, and the parameter $d > 0$ represents the diffusion coefficient which measures the speed at which the pollution stock spreads across space. Therefore, the term $d\frac{\partial^2 p_{x,t}}{\partial x^2}$ in (17) introduces some transboundary pollution externalities since the increase in the stock of pollution in each location x is affected by the stock of pollution in other locations $x' \neq x$ as well, implying that all locations mutually affect one another. Note that such a type of transboundary externality implies that individual local economies are all interrelated in a dynamic sense, since it will take time for the effects of pollution diffusion to manifest themselves.[5] These two substantive differences with respect to the a-spatial model imply that the environmental outcomes in different local economies are all interrelated and thus they need to be fully accounted for in the determination of the optimal policy by a global social planner, while a local social planner may neglect to consider them; this might suggests that a global approach to environmental policy may be superior to a local approach. Despite such an intuitive conclusion may appear obvious, as we shall formally prove later, it will not always hold true and only in specific circumstances local policymaking will fail to achieve global efficiency.

We assume without loss of generality that the spatial economy develops along the interval $[x_a, x_b]$, and is composed by M (non-overlapping, except for sets of zero Lebesgue measure) local economies, such that the interval $[x_a, x_b]$ is split in M different subintervals $[x_j, x_{j+1}]$, $j = 1, ..., M$, with $x_1 = x_a$ and $x_{M+1} = x_b$, each of which represents a single individual local economy. In a local control problem each local economy j is ruled by a local social planner who solves the problem (16) - (19) over the interval $[x_j, x_{j+1}]$ neglecting the transboundary nature of pollution generated in other local economies, considering thus the economy j to be completely isolated from other economies $j' \neq j$ (and thus Neumann's conditions apply to the borders x_j nd x_{j+1}). The optimal policy determined in this context represents what an individual policymaker would independently choose without considering what other individual policymakers might do. In a global control problem there exists a global social planner who solves the problem (16) - (19) over the entire spatial domain $[x_a, x_b]$, accounting for transboundary externalities arising within the whole spatial economy (and thus Neumann's conditions apply only to the borders x_a and x_b). The optimal policy determined in this framework characterizes what policymakers collaborating with one another would agreed upon, or what a global intergovernmental policymaker would determine and impose to local policymakers. Therefore, the optimization problems faced by a local and a global social planner present the same mathematical structure, and the only difference between them is related to the size of the interval over which their pollution control decisions are made. We now describe more specifically the characteristics of the local and global control problems deriving our local and global solutions to then move to their comparison in order to assess the efficiency implications of transboundary pollution externalities.

3.1 The Local Solution

The local solution corresponds to the decentralized outcome in the local control problem in which single local social planners behave by completely neglecting the transboundary pollution externalities. Thus, the environmental tax is determined without internalizing at local level the consequences of pollution diffusion within the entire spatial domain. After simplifying the problem by defining a new control variable $u^j_{x,t} = p^j_{x,t}\tau^j_{x,t}$ as in our previous a-spatial analysis, the problem faced by a local social planner in the local economy

[5] It may be possible to include also a static trasboundary externality by including an integral term to account for the extent to which the level of pollution in each single location instantaneously propagates across the whole economy (see La Torre et al., 2015, for a discussion of the different effects of static and dynamic externalities). This will make the pollution dynamic equation become an integro-partial differential equation, substantially complicating the analysis without qualitatively modifying our main results.

j reads as follows:

$$\min_{u^j_{x,t}} \quad \mathcal{C}^j = \int_0^T \int_{x_j}^{x_{j+1}} \frac{(p^j_{x,t})^2 + (u^j_{x,t})^2}{2} e^{-\rho t} dx dt + \frac{1-\theta}{\theta} \int_{x_j}^{x_{j+1}} \frac{(p^j_{x,T})^2}{2} e^{-\rho T} dx \tag{20}$$

$$s.t. \quad \frac{\partial p^j_{x,t}}{\partial t} = d \frac{\partial^2 p^j_{x,t}}{\partial x^2} + (\eta - \delta)\, p^j_{x,t} - \eta u^j_{x,t} \tag{21}$$

$$\frac{\partial u^j_{x,t}}{\partial x} = \frac{\partial p^j_{x,t}}{\partial x} = 0, x \in \{x_j, x_{j+1}\}, \tag{22}$$

$$p^j_{x,0} > 0 \text{ given}, \tag{23}$$

where the local policymaker determines the optimal level of intervention by completely ignoring the surrounding local economies and thus without taking into account the transboundary pollution externalities generated in other local economies. Since the local social planner neglects to account for what happens in other local economies, his optimization problem is constrained by Neumann's boundary conditions.

We approach the spatial optimal control problem (20) – (23) by following a variational method (Troltzsch, 2010; Boucekkine et al., 2013a). The generalized current value Hamiltonian function, $\mathcal{H}^j(p_{x,t}, u_{x,t}, \lambda_{x,t})$, reads as follows:

$$\mathcal{H}^j \quad = \quad \frac{(p^j_{x,t})^2 + (u^j_{x,t})^2}{2} + \lambda^j_{x,t} \left[d \frac{\partial^2 p^j_{x,t}}{\partial x^2} + (\eta - \delta) p^j_{x,t} - \eta u^j_{x,t} \right]$$

where $\lambda^j_{x,t}$ is the costate variable. The FOCs for a minimum are given by the following expressions:

$$\frac{\partial \lambda^j_{x,t}}{\partial t} \quad = \quad \rho \lambda^j_{x,t} - d \frac{\partial^2 \lambda^j_{x,t}}{\partial x^2} - p^j_{x,t} - (\eta - \delta) \lambda^j_{x,t} \tag{24}$$

$$u^j_{x,t} \quad = \quad \eta \lambda^j_{x,t} \tag{25}$$

Rearranging this last expression in terms of $\lambda^j_{x,t}$ and substituting this into (21) and (24) we obtain the following system of PDEs:

$$\frac{\partial p^j_{x,t}}{\partial t} \quad = \quad d \frac{\partial^2 p^j_{x,t}}{\partial x^2} + (\eta - \delta)\, p^j_{x,t} - \eta u^j_{x,t} \overline{u}^j_{x,t} \tag{26}$$

$$\frac{\partial u^j_{x,t}}{\partial t} \quad = \quad \rho u^j_{x,t} - d \frac{\partial^2 u^j_{x,t}}{\partial x^2} - \eta p^j_{x,t} - (\eta - \delta) u^j_{x,t}, \tag{27}$$

which, jointly with the following boundary conditions:

$$\overline{p}^j_{x,0} = p^j_0(x) \tag{28}$$

$$\overline{u}^j_{x,T} = \eta \, \frac{1-\theta}{\theta} \overline{p}^j_{x,T} \tag{29}$$

$$\frac{\partial p^j_{x_j,t}}{\partial x} = \frac{\partial p^j_{x_{j+1},t}}{\partial x} = 0 \quad \forall t \in [0, T] \tag{30}$$

$$\frac{\partial u^j_{x_j,t}}{\partial x} = \frac{\partial u^j_{x_{j+1},t}}{\partial x} = 0 \quad \forall t \in [0, T], \tag{31}$$

completely characterize the optimal level of intervention and pollution in the local economy j, along with the optimal environmental tax at local level, $\tau^j_{x,t} = \frac{u^j_{x,t}}{p^j_{x,t}}$. Since in a decentralized setting the policymaker neglects to account for the transboundary nature of pollution generated in other local economies, we need to determine the true level of pollution arising in the single local economy by considering the fact that the interrelations between local economies implies that the environmental tax in each single local economy impacts the stock of pollution not only locally but also globally. Indeed, the collection of all local environmental taxes within the global economy, given by: $\overline{\tau}_{x,t} = \tau^j_{x,t}$ for $x \in [x_j, x_{j+1}]$, determines the evolution of the pollution stock at global level. Thus, by substituting $\overline{\tau}_{x,t}$ into (17), we obtain the following PDE:

$$\frac{\partial \overline{p}_{x,t}}{\partial t} = d\frac{\partial^2 \overline{p}_{x,t}}{\partial x^2} + (\eta - \delta)\,\overline{p}_{x,t} - \eta \overline{\tau}_{x,t}\overline{p}_{x,t} \tag{32}$$

subject to the boundary conditions:

$$\frac{\partial \overline{p}_{x_a,t}}{\partial x} = \frac{\partial \overline{p}_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T], \tag{33}$$

Therefore our local solution is represented by the pair $(\overline{\tau}_{x,t},\, \overline{p}_{x,t})$ satisfying the above FOCs, given by (26) and (27) subject to (28) - (31) for all M local economies, along with (32) and (33).

3.2 The Global Solution

The global solution corresponds to the centralized outcome in the global control problem in which the global social planner quantifies and accounts for the transboundary pollution externalities between local economies, and thus he determines the environmental tax in each local economy in order to internalize the externality. Thus, the global control problem reads as follows:

$$\begin{aligned}
\min_{u_{x,t}} \quad & \mathcal{C} = \int_0^T \int_{x_a}^{x_b} \frac{p^2_{x,t} + u^2_{x,t}}{2} e^{-\rho t} dx dt + \frac{1-\theta}{\theta} \int_{x_a}^{x_b} \frac{p^2_{x,T}}{2} e^{-\rho T} dx & (34)\\
s.t. \quad & \frac{\partial p_{x,t}}{\partial t} = d\frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta - \delta)\, p_{x,t} - \eta u_{x,t} & (35)\\
& \frac{\partial u_{x,t}}{\partial x} = \frac{\partial p_{x,t}}{\partial x} = 0, x \in \{x_a, x_b\}, & (36)\\
& p_{x,0} > 0 \text{ given}, & (37)
\end{aligned}$$

where, different from what happens in the local control problem, Neumann's boundary conditions apply only to the borders of the entire spatial economy. It is straightforward to see that the mathematical structure of the local and global control problems is identical, and in particular the global control problem is a large-scale version of the local problem.

In the global control problem, by following the same approach we have earlier employed, the FOCs lead to the following system of PDEs:

$$\begin{aligned}
\frac{\partial p_{x,t}}{\partial t} &= d\frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta - \delta)\, p_{x,t} - \eta u_{x,t} & (38)\\
\frac{\partial u_{x,t}}{\partial t} &= \rho u_{x,t} - d\frac{\partial^2 u_{x,t}}{\partial x^2} - \eta p_{x,t} - (\eta - \delta) u_{x,t}, & (39)
\end{aligned}$$

which, jointly with the following boundary conditions:

$$\begin{aligned}
& p_{x,0} = p_0(x) & (40)\\
& u_{x,T} = \eta\, \frac{1-\theta}{\theta} p_{x,T} & (41)\\
& \frac{\partial p_{x_a,t}}{\partial x} = \frac{\partial p_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T] & (42)\\
& \frac{\partial u_{x_a,t}}{\partial x} = \frac{\partial u_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T], & (43)
\end{aligned}$$

completely characterize the optimal level of intervention and pollution in the global economy along with the optimal global environmental tax, given by $\tau_{x,t} = \frac{u_{x,t}}{p_{x,t}}$.. Therefore the global solution is represented by the pair $(\tau_{x,t},\, p_{x,t})$ satisfying the above FOCs, given by (38) and (39) subject to (40) - (43). Comparing the global solution with the local one earlier derived, we can notice that the pollution dynamic equations are exactly the same (by substituting $u_{x,t} = \tau_{x,t} p_{x,t}$ in (38)), while the evolution of the environmental tax presents some dissimilarity. Indeed, even if the expressions are similar, the presence in the local FOCs of the Neumann's boundary conditions at the borders between local economies (which is not present in the global FOCs) eventually introduces a discepancy between the optimal level of intervention at local and global level. This difference is entirely driven by the different treatment of the transboundary pollution externality, which is totally accounted for in the optimization process in the global solution while it is not considered at all in the local solution.

3.3 Local vs Global Policymaking

Having characterized the local and global solutions, we now wish to compare them in order to understand whether and how they eventually differ in order to understand whether a global approach to policymaking is effectively superior to a local approach. The analysis of the optimality conditions earlier derived in a centralized and a decentralized setting allows us to derive some interesting results about the eventual differences between the local and global solutions. These are summarized in the following Propositions 1 and 2.

Proposition 1. *Assume that $p_0(x) = p_0$ along with $\tau_{x,t} = \overline{\tau}_{x,t}$ and $p_{x,t} = \overline{p}_{x,t}$ satisfying (26) - (31) along with (32) - (33). Then the pair $(\overline{\tau}_{x,t}, \overline{p}_{x,t})$ represents the optimal solution of the control problem (34) – (37).*

Proposition 2. *Assume that $p_0(x) \neq p_0$ along with $\tau_{x,t} = \overline{\tau}_{x,t}$ and $p_{x,t} = \overline{p}_{x,t}$ satisfying (26) - (31) along with (32) - (33). Then the pair $(\overline{\tau}_{x,t}, \overline{p}_{x,t})$ does not represent the optimal solution of the control problem (34) – (37).*

Proposition 1 states that in a framework in which there is no spatial heterogeneity in the initial pollution distribution, then the local approach to policymaking is optimal also from a global perspective, since the global solution coincides with the local solution. Proposition 2 states that, whenever there exists some spatial heterogeneity in the initial pollution distribution, the local approach to policymaking is suboptimal from a global perspective, since the local solution does not solve the global control problem. These results are quite intuitive: in the absence of heterogeneity transboundary pollution externalities do not play any role and so they are not a source of efficiency loss with respect to the global social optimum, thus local policymakers can safely determine their optimal intervention levels without taking into account what is happening in the surrounding local economies. In the presence of heterogeneity transboundary pollution externalities become critical since they place a wedge between the global social optimum and the locally determined outcome, therefore designing local policies without accounting for such externalities will lead to suboptimal results globally. Since a large number of studies document the existence of sizeable environmental inefficiencies due local policymaking (Helland and Withford, 2003; Sigman, 2005; Monogan III et al., 2017), from a real world perspective, our results suggest that the existence of initial spatial heterogeneity is the most interesting and realistic situation, as confirmed also by the fact that different local economies are characterized by idiosyncratic economic and environmental disturbances which have determined their economic history impacting on their specific initial pollution levels.[6] In such a scenario Proposition 2 provides strong support for the think globally, act locally argument. In a completely different static setting with heterogeneous jurisdictions subject to transboundary (static) pollution externalities, Ogawa and Wildasin (2009) show that local policymaking generally leads to globally efficient outcomes; different from them, our results are derived in a spatio-temporal dynamic framework and are more general since we show that the global efficiency of local policymaking is only one possibility whose occurrence strictly depends on the spatial features of the initial pollution distribution.

[6]This is consistent with the path-dependency argument frequently discussed in the economic geography literature resulting from agglomeration and external economies (Krugman, 1991; Fujita et al., 1999; Fujita and Thisse, 2002).

From our above discussion, it is clear that in the most realistic situations policy coordination across local economies is essential in order to deal with environmental problems. However, understanding how to implement coordination is not simple since it is not straightforward to quantify what the intervention level of single local economies should be. We will now try to shed some light on this by analyzing the patterns of the pollution and the environmental tax dynamics in a framework with heterogeneous initial spatial pollution distribution. It is then possible to prove the following results, summarized by Propositions 3, 4 and 5

Proposition 3. *Let $(\tau_{x,t}, p_{x,t}) \geq 0$ be the globally optimal solution of the spatial optimal control problem (34) – (37) with initial pollution level given by $p_0(x)$, along with $\tau_{min} = \min_{(x,t)\in[x_a,x_b]\times[0,T]} \tau_{x,t}$. Define the global total pollution at the time $t \in [0,T]$ as follows:*

$$p_t^{tot} = \int_{x_a}^{x_b} p_{x,t} dx. \tag{44}$$

If $\tau_{min} > \frac{\eta-\delta}{\eta}$, then p_t^{tot} will be non-increasing over time.

Proposition 4. *Let $(\tau_{x,t}, p_{x,t}) \geq 0$ be the globally optimal solution of the spatial optimal control problem (34) – (37) with initial pollution level given by $p_0(x)$. Let $\tau_{min} = \min_{(x,t)\in[x_a,x_b]\times[0,T]} \tau_{x,t}$; then $p_{x,t} \leq e^{(\eta-\delta-\eta\tau_{min})t} h_{x,t}$ where $h_{x,t}$ is the solution of the following problem:*

$$\begin{aligned}
\frac{\partial h_{x,t}}{\partial t} &= d\frac{\partial^2 h_{x,t}}{\partial x^2} \quad on\ (x_a, x_b) \times (0,T) && (45)\\
\frac{\partial h_{x_a,t}}{\partial x} &= \frac{\partial h_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0,T] && (46)\\
h_{x,0} &= p_0(x) \quad in\ (x_a, x_b) && (47)
\end{aligned}$$

Proposition 5. *If $\tau_{min} > \frac{\eta-\delta}{\eta}$ then $\lim_{T\to+\infty} p_{x,T} = 0$ for any $x \in [x_a, x_b]$.*

Proposition 3 identifies a lower bound for the environmental tax in any local economy allowing to achieve a reduction in the global total pollution level. If each local economy implements at any moment in time an environmental policy stringent enough ($\tau_{x,t}^j > \frac{\eta-\delta}{\eta}, \forall x, t$) then it will be possible to effectively observe a pollution reduction in the global economy. This result can be interpreted from a normative perspective to determine the minimal policy that needs to be implemented locally. Proposition 4 states that, independently of what single local economies do, the minimum of the tax rate in different local economies at different moments in time, $\tau_{min}^j = \min_{(x,t)\in[x_a,x_b]\times[0,T]} \tau_{x,t}^j$, allows to determine an upper bound for the pollution stock at global level. Therefore, the lowest tax rate implemented by single local economies provides us with important information about the maximal pollution level that the global economy will need to bear. Proposition 5 states that if the minimum of the tax is large enough (i.e., $\tau_{min}^j > \frac{\eta-\delta}{\eta}$) then it will be possible for the global economy to achieve a completely pollution-free status in the long run. The lowest tax rate implemented by single local economies can thus be informative also of whether the pollution problem can be effectively eliminated in the long run. These three propositions jointly allow us not only to clearly understand that some collaboration across local economies is needed, but also to quantify from a normative perspective the minimal level of policy intervention required to achieve the globally desirable goal of pollution elimination.

3.4 An Analytical Solution

Our results thus far have been derived by focusing on the FOCs for our global and local control problems. However, because the model (16) – (19) has a linear-quadratic structure, it is possible to solve it in closed form to gain some further understanding on the difference between the local and the global solutions. This result is summarized in the next two propositions.

Proposition 6. *Let us define:*

$$A_0 = \frac{1}{x_b - x_a} \int_{x_a}^{x_b} p_0(x) dx \qquad A_n = \frac{2}{x_b - x_a} \int_{x_a}^{x_b} p_0(x) \cos\left(n\pi \left[\frac{x - x_a}{x_b - x_a}\right]\right) dx$$

$$B_0 = \left[\frac{\theta e_{21}^{\Theta T} - \eta(1-\theta) e_{11}^{\Theta T}}{\eta(1-\theta) e_{12}^{\Theta T} - \theta e_{22}^{\Theta T}}\right] A_0 \qquad B_n = \left[\frac{\theta e_{21}^{\Theta T} - \eta(1-\theta) e_{11}^{\Theta T}}{\eta(1-\theta) e_{12}^{\Theta T} - \theta e_{22}^{\Theta T}}\right] A_n e^{-d\left(\frac{n\pi}{x_b - x_a}\right)^2 T}$$

and $e^{\Theta t}$ *is the exponential matrix given in (11). Then, the globally global optimal tax* $\tau_{x,t}$ *is given by:*

$$\tau_{x,t} = \frac{e_{21}^{\Theta t} A_0 + e_{22}^{\Theta t} B_0 + \sum_{n\geq 1} \left(e_{21}^{\Theta t} A_n + e_{22}^{\Theta t} B_n\right) e^{-d\left(\frac{n\pi}{x_b - x_a}\right)^2 t} \cos\left(n\pi \left[\frac{x - x_a}{x_b - x_a}\right]\right)}{e_{11}^{\Theta t} A_0 + e_{12}^{\Theta t} B_0 + \sum_{n\geq 1} \left(e_{11}^{\Theta t} A_n + e_{12}^{\Theta t} B_n\right) e^{-d\left(\frac{n\pi}{x_b - x_a}\right)^2 t} \cos\left(n\pi \left[\frac{x - x_a}{x_b - x_a}\right]\right)},$$

while the globally optimal pollution level, $p_{x,t}$*, solves the PDE:*

$$\frac{\partial p_{x,t}}{\partial t} = d \frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta - \delta) p_{x,t} - \eta \tau_{x,t} p_{x,t}$$

jointly with the following boundary conditions:

$$\begin{aligned} p_{x,0} &= p_0(x) \quad \forall x \in [x_a, x_b] \\ \frac{\partial p_{x_a,t}}{\partial x} &= \frac{\partial p_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T] \end{aligned}$$

and it is provided by the following expression:

$$p_{x,t} = e_{11}^{\Theta t} A_0 + e_{12}^{\Theta t} B_0 + \sum_{n\geq 1} \left(e_{11}^{\Theta t} A_n + e_{12}^{\Theta t} B_n\right) e^{-d\left(\frac{n\pi}{x_b - x_a}\right)^2 t} \cos\left(n\pi \left[\frac{x - x_a}{x_b - x_a}\right]\right)$$

Proposition 7. *Let us define:*

$$A_0^j = \frac{1}{x_{j+1} - x_j} \int_{x_j}^{x_{j+1}} p_0(x) dx \qquad A_n^j = \frac{2}{x_{j+1} - x_j} \int_{x_j}^{x_{j+1}} p_0(x) \cos\left(n\pi \left[\frac{x - x_j}{x_{j+1} - x_j}\right]\right) dx$$

$$B_0^j = \left[\frac{\theta e_{21}^{\Theta T} - \eta(1-\theta) e_{11}^{\Theta T}}{\eta(1-\theta) e_{12}^{\Theta T} - \theta e_{22}^{\Theta T}}\right] A_0^j \qquad B_n^j = \left[\frac{\theta e_{21}^{\Theta T} - \eta(1-\theta) e_{11}^{\Theta T}}{\eta(1-\theta) e_{12}^{\Theta T} - \theta e_{22}^{\Theta T}}\right] A_n^j e^{-d\left(\frac{n\pi}{x_{j+1} - x_j}\right)^2 T}$$

and $e^{\Theta t}$ *is the exponential matrix given in (11). Then, the locally optimal tax* $\bar{\tau}_{x,t}$ *is given by:*

$$\bar{\tau}_{x,t} = \tau_{x,t}^j \quad x \in [x_j, x_{j+1}]$$

where

$$\tau_{x,t}^j = \frac{e_{21}^{\Theta t} A_0 + e_{22}^{\Theta t} B_0 + \sum_{n\geq 1} \left(e_{21}^{\Theta t} A_n + e_{22}^{\Theta t} B_n\right) e^{-d\left(\frac{n\pi}{x_{j+1} - x_j}\right)^2 t} \cos\left(n\pi \left[\frac{x - x_j}{x_{j+1} - x_j}\right]\right)}{e_{11}^{\Theta t} A_0 + e_{12}^{\Theta t} B_0 + \sum_{n\geq 1} \left(e_{11}^{\Theta t} A_n + e_{12}^{\Theta t} B_n\right) e^{-d\left(\frac{n\pi}{x_{j+1} - x_j}\right)^2 t} \cos\left(n\pi \left[\frac{x - x_j}{x_{j+1} - x_j}\right]\right)},$$

while the locally optimal pollution level, $\bar{p}_{x,t}$*, solves the following PDE:*

$$\frac{\partial \bar{p}_{x,t}}{\partial t} = d \frac{\partial^2 \bar{p}_{x,t}}{\partial x^2} + (\eta - \delta) \bar{p}_{x,t} - \eta \bar{\tau}_{x,t} \bar{p}_{x,t}$$

jointly with the following boundary conditions:

$$\begin{aligned} p_{x,0} &= p_0(x) \quad \forall x \in [x_a, x_b] \\ \frac{\partial p_{x_a,t}}{\partial x} &= \frac{\partial p_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T] \end{aligned}$$

Propositions 6 and 7 determine explicitly the spatio-temporal dynamic path of the the environmental tax in global and local settings from which it is possible to determine the evolution of pollution, which in turn is explicitly determined in the global solution but not in the local one. We can notice that the two solutions differ for the presence of different intervals at which the boundary conditions are associated, which yields some difference in the environmental tax and thus in the pollution stock between the local and global solutions. Despite the presence of analytical expressions for the global and local solutions, they turn out to be particularly cumbersome and thus it is not possible to perform any comparative statics exercises in order to understand how they depend on the different economic and environmental parameters. Nevertheless, they allow to explicitly verify Proposition 1. Indeed, as shown in the following proposition, in the absence of spatial heterogeneity in the initial distribution of pollution, the global solution of our transboundary pollution control problem boils down to the local solution.

Proposition 8. *Suppose $p_0(x) = p_0$ for all $x \in [x_a, x_b]$. Then the globally optimal pair $(\tau_{x,t}, p_{x,t})$ is given by*

$$\tau_{x,t} = \frac{e_{21}^{\Theta t} + e_{22}^{\Theta t}\left(\frac{\theta e_{21}^{\Theta T} - \eta(1-\theta)e_{11}^{\Theta T}}{\eta(1-\theta)e_{12}^{\Theta T} - \theta e_{22}^{\Theta T}}\right)}{e_{11}^{\Theta t} + e_{12}^{\Theta t}\left(\frac{\theta e_{21}^{\Theta T} - \eta(1-\theta)e_{11}^{\Theta T}}{\eta(1-\theta)e_{12}^{\Theta T} - \theta e_{22}^{\Theta T}}\right)}$$

and

$$p_{x,t} = e_{11}^{\Theta t} + e_{12}^{\Theta t}\left(\frac{\theta e_{21}^{\Theta T} - \eta(1-\theta)e_{11}^{\Theta T}}{\eta(1-\theta)e_{12}^{\Theta T} - \theta e_{22}^{\Theta T}}\right)$$

where $\mathrm{e}^{\Theta t}$ is given by (11). The globally optimal pair $(p_{x,t}, \tau_{x,t})$ perfectly coincides with the locally optimal pair $(\overline{p}_{x,t}, \overline{\tau}_{x,t})$.

In order to better understand the extent to which the local and global solutions may differ, we can exploit our closed-form solutions from Propositions 6 and 7 to visualize such an eventual difference. Therefore, we now present some numerical example in which the parameters take the same values employed by La Torre et al. (2017) in their calibration based on global CO_2 data under an intermediate degree of sustainability concern. Specifically, we set parameters as follows: $\eta = 0.051$, $\delta = 0.05$, $\rho = 0.04$, $\theta = 0.5$, $T = 30$ and $d = 0.01$.

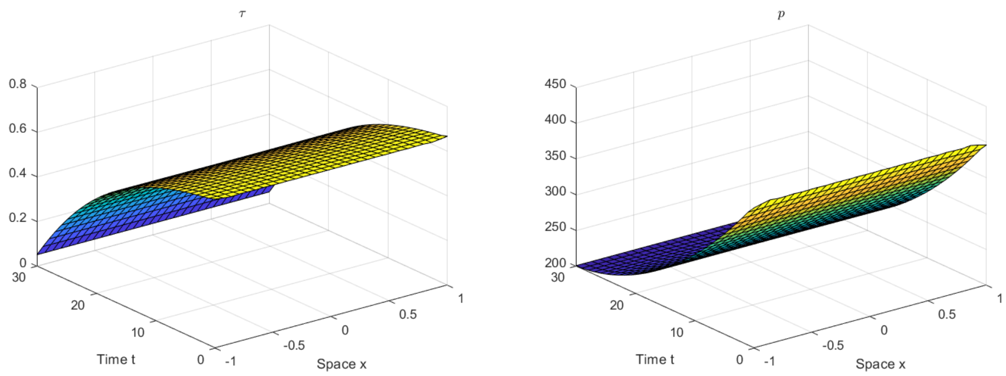

Figure 1: Spatio-temporal dynamics of the environmental tax (left) and pollution (right) with no spatial heterogeneity in the initial pollution distribution. Local and global solutions perfectly coinciding.

We start by illustrating the pollution and environmental tax dynamics in a framework with no spatial heterogeneity in the initial pollution distribution. Without loss of generality, we assume that $p_0(x) = p_0 = 400.23$, representing today's initial concentration of CO_2 (400.23 parts per million in year 2015), and that the global economy, which develops along the $[-1, 1]$ interval, is composed by three local economies developing on the [-1, -1/3], [-1/3,1/3] and [1/3, 1] intervals, respectively. From Proposition 1 (and Proposition 8) we know that the local and global solutions will coincide, and these identical solutions are shown in Figure 1 where we plot the spatio-temporal evolution of the environmental tax (left panel) and pollution (right panel). Since in this setting transboundary pollution externalities do not play any role every local economy behaves exactly in the same way, and optimality implies that the tax rate initially exceeds its final level in order to effectively achieve a reduction in the stock of pollution.

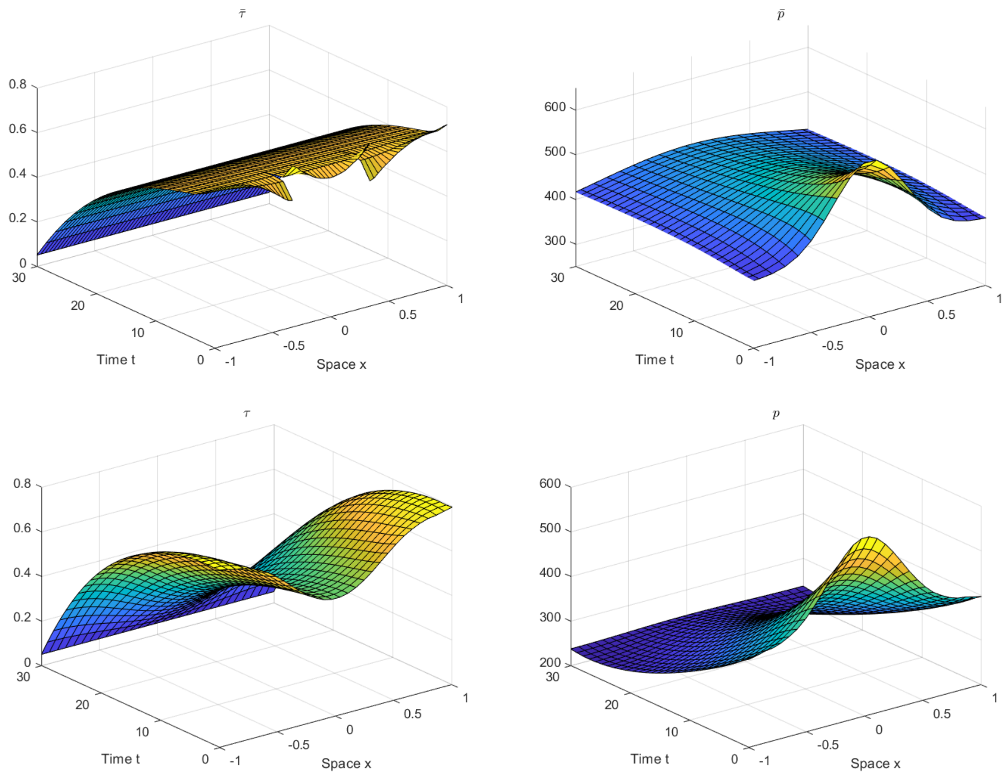

Figure 2: Spatio-temporal dynamics of the environmental tax (left) and pollution (right) with spatial heterogeneity in the initial pollution distribution. Local solution (top) and global solution (bottom) not coinciding.

We now move to the most interesting framework in which there is some spatial heterogeneity in the initial pollution distribution. In order to characterize such heterogeneity we assume that $p_0(x) = p_0 + \frac{1}{2}p_0 e^{-5x^2}$, suggesting that on the average in the entire global economy the concentration of CO_2 is roughly 400.23 parts per million, with a peak in pollution concentration in central locations and lower concentrations in the later ones. Specifically, the central local economy is characterized by higher levels of pollution at the center and lower levels towards its borders, while the lateral local economies by higher levels towards the border with the central local economy and lower levels towards their external borders. From Proposition 2 we know that global and local solutions do not coincide and thus we will need to analyze them separately. This is shown in Figure 2 where we represent the local solution (top panels) and the global solutions (bottom panels).In both the local and global solutions the tax rate starts high to then monotonically decrease over time and such initially high effort allows to obtain a substantial reduction in the stock of pollution within the spatial economy. In both the local and global solutions, the spatial heterogeneity in the optimal tax is such that the tax is initially lower (higher) in the locations where pollution concentrations are higher (lower). This kind of counterintuitive result is due to the effect of transboundary pollution externalities which consists of homogenizing the spatial distribution of pollution: provided that the tax is high enough to ensure a reduction of pollution, in the locations with a high initial stock pollution will tend to fall more rapidly than in the locations with a lower pollution stock as a natural result of diffusion, and as such environmental policy can be less stringent in such locations. In other words, from the social planner's point of view it is more effective to let diffusion do its work in rather than imposing a higher tax rate: in a sense, the social planner internalizes the physical mechanism of diffusion allowing it to optimally interact with the transboundary nature of pollution in order to minimize the social costs. From a qualitative perspective the local and global solutions show quite a different pattern, which gives rise to substantially different quantitative outcomes: in the global solution the environmental tax is on average higher than in the local solution yielding to a lower pollution stock at the end of the planning horizon. In the central local economy the local environmental tax is initially higher than in the global one and thus its pollution level is lower, while in the lateral local economies the local environmental tax is lower than the global one and thus their pollution level is higher; this can be explained exactly as before by the referring to the physical mechanism of diffusion. Note also that such differences between the local and global taxes fade away over time while the differences between the local and global pollution persist also at the end of the planning horizon. As a final remark, note that both in the local and the global solutions, at the end of the planning horizon pollution and as a result the environmental tax are almost spatially homogenous: this is again due to the fact that diffusion acts as a convergence mechanism which tends to smooth spatial differences out (Boucekkine et al., 2009; La Torre et al., 2015).

4 Extensions

We now consider some extensions of our baseline model in order to show that our main results related to the suboptimality of the local approach to the pollution problem still hold true even in more general settings.

4.1 Welfare Maximization

We first show that our social cost minimization formulation gives rise to an equivalent social welfare maximization problem in which the utility function takes a specific functional form. Consider a framework in which the social planner wishes to maximize social welfare $\mathcal{W}$ by determining the optimal environmental tax rate. Social welfare is the weighted sum of two different terms: the expected discounted sum of instantaneous utilities, and the discounted damage associated with the remaining level of pollution at the end of the planning horizon, T. The instantaneous utility function, $U(c_t, p_t)$, is increasing in consumption and decreasing in pollution, and consumption is given by the following expression: $c_t = (1-\tau_t)y_t$ since households consume completely their disposable income. The damage function takes the same form earlier discussed: $d(p_T) = \frac{p_T^2}{2}$. Therefore, by focusing only on the a-spatial model, we may consider the following problem, with p_0 given:

$$\max_{\tau_t} \quad \mathcal{W} = \int_0^T U\left((1-\tau_t)y_t, p_t\right) e^{-\rho t} dt - \frac{1-\theta}{\theta}\frac{p_T^2}{2}e^{-\rho T} \tag{48}$$

$$s.t. \quad \dot{p}_t = [\eta(1-\tau_t) - \delta]p_t \tag{49}$$

Since income is completely exogenous (and growing at the constant rate $g > 0$), the above optimization model is equivalent to a problem in which the second dynamic constraint is disregarded and $U\left(c_t, p_t\right)$ is replaced by $U(x_t, p_t)$ where $x_t = \frac{c_t}{y_t}$ represents the propensity to consume out of income and thus $0 < x_t < 1$. Therefore, given p_0, we can equivalently analyze the problem below:

$$\max_{\tau_t} \quad \mathcal{W} = \int_0^T U(1-\tau_t, p_t) e^{-\rho t} dt - \frac{1-\theta}{\theta}\frac{p_T^2}{2}e^{-\rho T} \tag{50}$$

$$s.t. \quad \dot{p}_t = [\eta(1-\tau_t) - \delta]p_t \tag{51}$$

By focusing on a specific functional form of the utility function, adapted to the peculiarity of the problem above in which the utility argument x_t takes values between zero and one, it is possible to show that the model is completely equivalent to the cost minimization setup that we have earlier discussed. Specifically, let us consider the following utility function: $U(x_t, p_t) = -\frac{p_t^2}{2}\left[1 + (1 - x_t)^2\right]$, where $U'_x > 0$, $U''_x < 0$, $U'_p < 0$, and $U''_p < 0$. Since $x_t = 1 - \tau_t$, the model boils down to the following:

$$\max_{\tau_t} \quad \mathcal{W} = \int_0^T -p_t^2 \left(\frac{1+\tau_t^2}{2}\right) e^{-\rho t} dt - \frac{1-\theta}{\theta}\frac{p_T^2}{2} e^{-\rho T} \tag{52}$$

$$s.t. \quad \dot{p}_t = [\eta(1-\tau_t) - \delta] p_t \tag{53}$$

which with simple algebra can be straightforwardly shown to be totally equivalent to the model below:

$$\min_{\tau_t} \quad \mathcal{C} = \int_0^T p_t^2 \left(\frac{1+\tau_t^2}{2}\right) e^{-\rho t} dt + \frac{1-\theta}{\theta}\frac{p_T^2}{2} e^{-\rho T} \tag{54}$$

$$s.t. \quad \dot{p}_t = [\eta(1-\tau_t) - \delta] p_t, \tag{55}$$

which perfectly coincides with our original problem (1 - (2). Therefore, we can conclude that the results that we have derived from a cost minimization problem apply also in a standard welfare maximization setup.

4.2 Convex Functions

We now go back to our social cost minimization formulation in a spatial context and consider a setup in which the instantaneous loss function, the end-of-planning damage function and the pollution accumulation functions are convex. Specifically, we assume that the instantaneous loss function $c(p_{x,t}, \tau_{x,t})$ is increasing and convex in both its arguments, that is $c_p > 0$, $c_\tau > 0$, $c_{pp} > 0$ and $c_{\tau\tau} > 0$, and that the end-of-planning damage function $D(p_{x,T})$ is similarly increasing and convex in pollution, that is $D > 0$ and $D_{pp} > 0$. We also assume that pollution accumulation is determined by a function $f(p_{x,t}, \tau_{x,t})$ which is increasing in pollution and decreasing in the tax rate, $f_p > 0$ and $f_\tau < 0$, but convex in both its arguments, $f_{pp} \geq 0$ and $f_{\tau\tau} \geq 0$.

In a global control framework, the social planner faces the following optimization problem:

$$\min_{\tau(x,t)} \quad \mathcal{C} = \int_0^T \int_{x_a}^{x_b} c(p_{x,t}, \tau_{x,t}) e^{-\rho t} dx dt + \frac{1-\theta}{\theta} \int_{x_a}^{x_b} D(p_{x,T}) e^{-\rho T} dx \tag{56}$$

$$s.t. \quad \frac{\partial p_{x,t}}{\partial t} = d\frac{\partial^2 p_{x,t}}{\partial x^2} + f(p_{x,t}, \tau_{x,t}) \tag{57}$$

$$\frac{\partial \tau_{x,t}}{\partial x} = \frac{\partial p_{x,t}}{\partial x} = 0, x \in \{x_a, x_b\}, \tag{58}$$

$$p_{x,0} > 0 \text{ given}, \tag{59}$$

from which the FOCs read as follows:

$$c_\tau + \lambda f_\tau = 0 \tag{60}$$

$$\frac{\partial \lambda}{\partial t} = \rho\lambda - c_p - \lambda f_p - d\frac{\partial^2 \lambda}{\partial x^2} \tag{61}$$

jointly with the following boundary conditions:

$$p_{x,0} = p_0(x) \tag{62}$$

$$\lambda = \frac{1-\theta}{\theta} D_p \tag{63}$$

$$\frac{\partial p_{x_a,t}}{\partial x} = \frac{\partial p_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T] \tag{64}$$

$$\frac{\partial \tau_{x_a,t}}{\partial x} = \frac{\partial \tau_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T], \tag{65}$$

In a local control context, instead, the problem faced by a local social planner in the local economy j reads as follows:

$$\min_{\tau^j_{x,t}} \quad \mathcal{C}^j = \int_0^T \int_{x_j}^{x_{j+1}} c(p^j_{x,t}, \tau^j_{x,t}) e^{-\rho t} dx dt + \frac{1-\theta}{\theta} \int_{x_j}^{x_{j+1}} D(p^j_{x,T}) e^{-\rho T} dx \tag{66}$$

$$s.t. \quad \frac{\partial p^j_{x,t}}{\partial t} = d \frac{\partial^2 p^j_{x,t}}{\partial x^2} + f(p^j_{x,t}, \tau^j_{x,t}) \tag{67}$$

$$\frac{\partial \tau^j_{x,t}}{\partial x} = \frac{\partial p^j_{x,t}}{\partial x} = 0, x \in \{x_j, x_{j+1}\}, \tag{68}$$

$$p^j_{x,0} > 0 \text{ given,} \tag{69}$$

from which we can derive the following FOCs:

$$c_{\tau^j} + \lambda f_{\tau^j} = 0 \tag{70}$$

$$\frac{\partial \lambda^j}{\partial t} = \rho \lambda^j - c_{p^j} - \lambda f_{p^j} - d \frac{\partial^2 \lambda^j}{\partial x^2} \tag{71}$$

jointly with the following boundary conditions:

$$p^j_{x,0} = p^j_0(x) \tag{72}$$

$$\lambda^j = \frac{1-\theta}{\theta} D_{p^j} \tag{73}$$

$$\frac{\partial p^j_{x_j,t}}{\partial x} = \frac{\partial p^j_{x_{j+1},t}}{\partial x} = 0 \quad \forall t \in [0, T] \tag{74}$$

$$\frac{\partial \tau^j_{x_j,t}}{\partial x} = \frac{\partial \tau^j_{x_{j+1},t}}{\partial x} = 0 \quad \forall t \in [0, T], \tag{75}$$

By comparing (60) - (65) with (70)- (75), it is straightforward to conclude that the presence in the local FOCs of the Neumann's boundary conditions at the borders between local economies introduces a difference between the local and the global solutions, but such a difference disappears when the initial pollution distribution is homogeneous. This suggests that Propositions 1 and 2 still hold true independently of the specific functional forms assumed in the analysis.

4.3 Capital Accumulation

We now consider a setting in which there is capital accumulation and so optimal saving. This framework brings our analysis into a traditional macroeconomic context in which the objective function represents social welfare. Social welfare is the sum of two terms: the infinite discounted sum of utilities and the end-of-planning horizon utility. The finite-time utility function $u(c_{x,t}, k_{x,t}, \tau_{x,t}, p_{x,t})$ where $c_{x,t}$ denotes consumption and $k_{x,t}$ capital, is assumed to be increasing in consumption and capital but decreasing in the tax rate and pollution, and to be concave in all of its arguments. The end-of-planning horizon utility $v(k_{x,T}, p_{x,T})$ is increasing in capital and decreasing in pollution and concave in both the arguments. The accumulation of capital is determined by a function $g(c_{x,t}, k_{x,t}, \tau_{x,t}, p_{x,t})$ which is increasing in capital, decreasing in the remaining arguments, and concave in all of them. The accumulation of pollution function $f(k_{x,t}, \tau_{x,t}, p_{x,t})$, other than the properties earlier discussed, is increasing and concave in capital. Consistent with economic growth and environment literature (see Xepapadeas, 2005, for a survey), such a framework allows us to consider mutual economic-environmental feedbacks: the stock of capital determines the evolution of the pollution stock, which in turn affects the evolution of capital.

By assuming for the sake of simplicity that the diffusion parameter is the same for both capital and pollution, the social planner's problem in a global control framework reads as follows:

$$\min_{\tau(x,t)} \quad \mathcal{W} = \int_0^T \int_{x_a}^{x_b} u(c_{x,t}, k_{x,t}, \tau_{x,t}, p_{x,t}) e^{-\rho t} dx dt + \frac{1-\theta}{\theta} \int_{x_a}^{x_b} v(k_{x,T}, p_{x,T}) e^{-\rho T} dx \tag{76}$$

$$s.t. \quad \frac{\partial p_{x,t}}{\partial t} = d\frac{\partial^2 p_{x,t}}{\partial x^2} + f(c_{x,t}, k_{x,t}, \tau_{x,t}, p_{x,t}) \tag{77}$$

$$\frac{\partial k_{x,t}}{\partial t} = d\frac{\partial^2 k_{x,t}}{\partial x^2} + g(c_{x,t}, k_{x,t}, \tau_{x,t}, p_{x,t}) \tag{78}$$

$$\frac{\partial \tau_{x,t}}{\partial x} = \frac{\partial p_{x,t}}{\partial x} = \frac{\partial c_{x,t}}{\partial x} = \frac{\partial k_{x,t}}{\partial x} = 0, \tag{79}$$

$$p_{x,0} > 0, \ k_{x,0} > 0 \text{ given}, \tag{80}$$

where $d\frac{\partial^2 k_{x,t}}{\partial x^2}$ denotes the capital migration externalities (Boucekkine et al., 2009). From the above problem, by denoting with ξ the costate variable for capital, the following FOCs follow:

$$u_c + \lambda g_c = 0 \tag{81}$$

$$u_\tau + \lambda g_\tau + \xi f_\tau = 0 \tag{82}$$

$$\frac{\partial \lambda}{\partial t} = \rho\lambda - u_k - \lambda g_k - \xi f_k - d\frac{\partial^2 \lambda}{\partial x^2} \tag{83}$$

$$\frac{\partial \xi}{\partial t} = \rho\xi - u_p - \lambda g_p - \xi f_p - d\frac{\partial^2 \xi}{\partial x^2} \tag{84}$$

jointly with the following boundary conditions:

$$p_{x,0} = p_0(x) \tag{85}$$

$$\lambda = \frac{1-\theta}{\theta} v_p \tag{86}$$

$$\xi = \frac{1-\theta}{\theta} v_k \tag{87}$$

$$\frac{\partial p_{x_a,t}}{\partial x} = \frac{\partial p_{x_b,t}}{\partial x} = 0 \tag{88}$$

$$\frac{\partial \tau_{x_a,t}}{\partial x} = \frac{\partial \tau_{x_b,t}}{\partial x} = 0 \tag{89}$$

In a local control context, instead, the problem faced by a local social planner in the local economy j reads as follows:

$$\min_{\tau^j_{x,t}, c^j_{x,t}} \quad \mathcal{W}^j = \int_0^T \int_{x_j}^{x_{j+1}} u(c^j_{x,t}, k^j_{x,t}, \tau^j_{x,t}, p^j_{x,t}) e^{-\rho t} dx dt + \frac{1-\theta}{\theta} \int_{x_j}^{x_{j+1}} v(k^j_{x,T}, p^j_{x,T}) e^{-\rho T} dx \tag{90}$$

$$s.t. \quad \frac{\partial p^j_{x,t}}{\partial t} = d\frac{\partial^2 p^j_{x,t}}{\partial x^2} + f(c^j_{x,t}, k^j_{x,t}, \tau^j_{x,t}, p^j_{x,t}) \tag{91}$$

$$\frac{\partial k^j_{x,t}}{\partial t} = d\frac{\partial^2 k^j_{x,t}}{\partial x^2} + g(c^j_{x,t}, k^j_{x,t}, \tau^j_{x,t}, p^j_{x,t}) \tag{92}$$

$$\frac{\partial \tau^j_{x,t}}{\partial x} = \frac{\partial p^j_{x,t}}{\partial x} = \frac{\partial c^j_{x,t}}{\partial x} = \frac{\partial k^j_{x,t}}{\partial x} = 0, \quad x \in \{x_j, x_{j+1}\}, \tag{93}$$

$$p^j_{x,0} > 0, \ k^j_{x,0} > 0 \text{ given} \tag{94}$$

From the above problem the following FOCs follow:

$$u_{c^j} + \lambda^j g_{c^j} = 0 \tag{95}$$

$$u_{\tau^j} + \lambda^j g_{\tau^j} + \xi^j f_{\tau^j} = 0 \tag{96}$$

$$\frac{\partial \lambda^j}{\partial t} = \rho\lambda^j - u_{k^j} - \lambda g_k - \xi^j f_{k^j} - d\frac{\partial^2 \lambda^j}{\partial x^2} \tag{97}$$

$$\frac{\partial \xi^j}{\partial t} = \rho\xi^j - u_{p^j} - \lambda g_p - \xi^j f_{p^j} - d\frac{\partial^2 \xi^j}{\partial x^2} \tag{98}$$

jointly with the following boundary conditions:

$$p^j_{x,0} = p^j_0(x) \tag{99}$$

$$\lambda^j a = \frac{1-\theta}{\theta} v_{p^j} \tag{100}$$

$$\xi^j = \frac{1-\theta}{\theta} v_{k^j} \tag{101}$$

$$\frac{\partial p^j_{x_j,t}}{\partial x} = \frac{\partial p^j_{x_{j+1},t}}{\partial x} = 0 \quad \forall t \in [0,T] \tag{102}$$

$$\frac{\partial \tau^j_{x_j,t}}{\partial x} = \frac{\partial \tau^j_{x_{j+1},t}}{\partial x} = 0 \quad \forall t \in [0,T], \tag{103}$$

By comparing (81) - (89) with (95) - (103), it is straightforward to conclude that the presence in the local FOCs of the Neumann's boundary conditions at the borders between local economies introduces a difference between the local and the global solutions, but such a difference disappears when the initial pollution distribution is homogeneous. This suggests that Propositions 1 and 2 still hold true independently of the fact that we introduce a richer macroeconomic framework with mutual economic-environmental feedback effects in the analysis.

5 The Spatial Model: Unbounded Spatial Domain

We now consider a further extension of our baseline model, in which the spatial domain is no longer bounded but rather unbounded. Different from what assumed earlier, now we assume that $x \in \mathbb{R}$ and thus there are no natural borders for the global economy and that the initial pollution level is uniformly bounded over $\mathbb{R}$, that is there exists a positive constant $\kappa > 0$ such that $0 \leq p_0(x) \leq \kappa$ for all $x \in \mathbb{R}$. In order to distinguish between global and local solutions, we assume that the global economy is composed by an infinite number of local economies such that the real line $(-\infty, +\infty)$ is split into a sequence of infinite intervals $[x_j, x_{j+1}]$, $j = 1, ..., \infty$, such that $(-\infty, +\infty) = \bigcup_{j=-\infty}^{+\infty} [x_j, x_{j+1}]$. Modeling the spatial domain as the whole real line implies that we are considering an integrated global economy (i.e., the world economy, in which countries all interconnected one another) in which pollution, even if generated in locations very apart one another, depends on the behavior of all the different locations which compose it (i.e., the national countries within the world economy). In our baseline setup with a bounded spatial domain the behavior of pollution at the borders plays a critical role, and in particular the Neumann conditions guarantee that the endogenous patterns emerging from the optimizing choices of the social planner are not induced by setting the pollution level at some arbitrary value at the borders. Also in our extension to a unbounded spatial domain in which natural borders do not exist, Neumann conditions play a similar role but are intended in the limit sense when the spatial variables tend either to $+\infty$ or $-\infty$. In this setting the global control problem reads as follows:

$$\min_{u_{x,t}} \quad \mathcal{C} = \int_0^T \int_{-\infty}^{+\infty} \frac{p^2_{x,t} + u^2_{x,t}}{2} e^{-\rho t} dx dt + \frac{1-\theta}{\theta} \int_{-\infty}^{+\infty} \frac{p^2_{x,T}}{2} e^{-\rho T} dx \tag{104}$$

$$s.t. \quad \frac{\partial p_{x,t}}{\partial t} = d \frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta - \delta)\, p_{x,t} - \eta u_{x,t} \quad on\ (-\infty, +\infty) \times (0, T) \tag{105}$$

$$\lim_{x \to \pm\infty} \frac{\partial p_{x,t}}{\partial x} = \frac{\partial u_{x,t}}{\partial x} = 0 \quad \forall t \in [0,T] \tag{106}$$

$$p_{x,0} > 0 \quad in\ (-\infty, +\infty) \tag{107}$$

Most of the calculations presented in section 3 apply in this case as well, and it is possible to show that the following system of PDEs:

$$\frac{\partial p_{x,t}}{\partial t} = d\frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta - \delta)\, p_{x,t} - \eta u_{x,t} \tag{108}$$

$$\frac{\partial u_{x,t}}{\partial t} = \rho u_{x,t} - d\frac{\partial^2 u_{x,t}}{\partial x^2} - \eta p_{x,t} - (\eta - \delta) u_{x,t}, \tag{109}$$

jointly with the following boundary conditions:

$$p_{x,0} = p_0(x) \tag{110}$$

$$\lim_{x\to\pm\infty} \frac{\partial p_{x,t}}{\partial x} = \frac{\partial u_{x,t}}{\partial x} = 0 \quad \forall t \in [0, T] \tag{111}$$

$$u_{x,T} = \eta\, \frac{1-\theta}{\theta} p_{x,T} \tag{112}$$

characterize our global solution given by the pair $(\tau_{x,t}, p_{x,t})$ where $\tau_{x,t} = \frac{u_{x,t}}{p_{x,t}}$. The problem faced by the social planner in the local economy j reads instead as follows:

$$\min_{u^j_{x,t}} \quad \mathcal{C}^j = \int_0^T \int_{x_j}^{x_{j+1}} \frac{(p^j_{x,t})^2 + (u^j_{x,t})^2}{2} e^{-\rho t} dx dt + \frac{1-\theta}{\theta} \int_{x_j}^{x_{j+1}} \frac{(p^j_{x,T})^2}{2} e^{-\rho T} dx \tag{113}$$

$$s.t. \quad \frac{\partial p^j_{x,t}}{\partial t} = d\frac{\partial^2 p^j_{x,t}}{\partial x^2} + (\eta - \delta)\, p^j_{x,t} - \eta u^j_{x,t} \tag{114}$$

$$\frac{\partial u^j_{x,t}}{\partial x} = \frac{\partial p^j_{x,t}}{\partial x} = 0, x \in \{x_j, x_{j+1}\}, \tag{115}$$

$$p^j_{x,0} > 0 \text{ given,} \tag{116}$$

It is straightforward to observe that the local control problem in the case of an unbounded spatial domain perfectly coincide with that earlier presented for the bounded spatial domain case, since all local economies extend over a finite spatial interval. Therefore, the above problem leads exactly to the same FOCs and the same local environmental tax $\tau^j_{x,t}$ earlier derived in our baseline model. Since the local policymaker does not account for the transboundary nature of pollution generated in other local economies, we determine the true level of pollution arising in the single local economy by substituting the collection of all local environmental taxes within the global economy, $\overline{\tau}_{x,t} = \tau^j_{x,t}$ for $x \in [x_j, x_{j+1}]$, in the evolution of the pollution stock at global level, which yields:

$$\frac{\partial \overline{p}_{x,t}}{\partial t} = d\frac{\partial^2 \overline{p}_{x,t}}{\partial x^2} + (\eta - \delta)\, \overline{p}_{x,t} - \eta \overline{\tau}_{x,t} \overline{p}_{x,t} \tag{117}$$

subject to the boundary conditions:

$$\lim_{x\to\pm\infty} \frac{\partial \overline{p}_{x,t}}{\partial x} = 0 \quad \forall t \in [0, T] \tag{118}$$

The collection of the local environmental taxes and the global pollution stock $(\overline{\tau}_{x,t},\ \overline{p}_{x,t})$ characterize our local solution. Exactly as in our baseline model the constraint imposed by the presence in the local FOCs of the Neumann's boundary conditions at the borders between local economies places eventually a wedge between the local and the global solutions. Therefore, exactly the same results as in Propositions 1 and 2 hold: in the case in which the initial pollution distribution is homogeneous the local solution coincides with the global one, while in the case in which the initial pollution distribution is heterogeneous the local and the global solutions differ with the local solution being suboptimal. This suggests that our conclusions regarding the desirability of a global approach to environmental problems, consistently with the think globally, act locally motto, are independent of the specific assumptions on the structure of the spatial domain.

Moreover, by applying the same arguments employed in Propositions 4 and 5, it is possible to prove the following result.

Proposition 9. *Let $(\tau_{x,t}, p_{x,t}) \geq 0$ be the globally optimal solution of the spatial optimal control problem (104) – (107) with initial pollution level given by $p_0(x)$. Let $\tau_{min} = \min_{(x,t)\in(-\infty,+\infty)\times[0,T]} \tau_{x,t}$; then $p_{x,t} \leq e^{(\eta-\delta-\eta\tau_{min})t} h_{x,t}$ where $h_{x,t}$ is the solution of the following problem:*

$$\frac{\partial h_{x,t}}{\partial t} = d\frac{\partial^2 h_{x,t}}{\partial x^2} \quad \text{on } (-\infty,+\infty)\times(0,T) \tag{119}$$

$$h_{x,0} = p_0(x) \quad \text{in } (-\infty,+\infty) \tag{120}$$

where $h_{x,t}$ is then the well-known classical solution of the heat equation over an unbounded domain provided by means of the Green functions as follows:

$$h_{x,t} = \frac{1}{2\sqrt{\pi d t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4dt}} p_0(y)dy.$$

Moreover, if $\tau_{min} > \frac{\eta-\delta}{\eta}$ then $\lim_{T\to+\infty} p_{x,T} = 0$ for any $x \in \mathbb{R}$.

Consistent with what discussed in our baseline model, Proposition 9 states that the minimum of the tax rate in different local economies at different moments in time, τ^j_{min}, determines an upper bound for the pollution stock in each locations, and if such a minimum of the tax is large enough (i.e., $\tau^j_{min} > \frac{\eta-\delta}{\eta}$) then each location within the global economy will achieve a completely pollution-free status in the long run. This allows us to quantify the minimal level of policy intervention required to achieve the desirable goal of pollution elimination in the global economy and such a minimal level of collaboration across local economies perfectly coincide with that determined in our baseline model, suggesting that also our conclusion regarding the amount of collaboration needed to reduce global pollution is independent of the specific assumptions on the structure of the spatial domain.

5.1 An Analytical Solution

Even in the presence of an unbounded spatial domain, given the specific linear-quadratic structure of our model, it is possible to solve it in closed form to gain some further understanding on the difference between the local and the global solutions. This result is summarized in the next proposition.

Proposition 10. *Define:*

$$\tilde{z}^1_{x,T} = \frac{1}{2\sqrt{\pi d T}}\left[\frac{\theta e^{\Theta T}_{21} - \eta(1-\theta)e^{\Theta T}_{11}}{\eta(1-\theta)e^{\Theta T}_{12} - \theta e^{\Theta T}_{22}}\right]\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4dT}} p_0(y)dy,$$

and $e^{\Theta t}$ is the exponential matrix given in (11). Then, the globally optimal tax, $\tau_{x,t}$, is given by:

$$\tau_{x,t} = \frac{e^{\Theta t}_{21}\left(\frac{1}{2\sqrt{\pi d t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4dt}} p_0(y)dy\right) + e^{\Theta t}_{22}\left(\frac{1}{2\sqrt{\pi d(T-t)}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4d(T-t)}} \tilde{z}^1_{y,T}dy\right)}{e^{\Theta t}_{11}\left(\frac{1}{2\sqrt{\pi d t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4dt}} p_0(y)dy\right) + e^{\Theta t}_{12}\left(\frac{1}{2\sqrt{\pi d(T-t)}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4d(T-t)}} \tilde{z}^1_{y,T}dy\right)},$$

while the globally optimal pollution level, $p_{x,t}$, solves the following PDE:

$$\frac{\partial p_{x,t}}{\partial t} = d\frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta-\delta)\,p_{x,t} - \eta\tau_{x,t}p_{x,t}$$

jointly with the following boundary conditions:

$$p_{x,0} = p_0(x)$$

$$\lim_{x\to\pm\infty}\frac{\partial p_{x,t}}{\partial x} = 0 \quad \forall t \in [0,T],$$

and it is provided in closed-form by the following expression:

$$p_{x,t} = e^{\Theta t}_{11}\left(\frac{1}{2\sqrt{\pi d t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4dt}} p_0(y)dy\right) + e^{\Theta t}_{12}\left(\frac{1}{2\sqrt{\pi d(T-t)}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4d(T-t)}} \tilde{z}^1_{y,T}dy\right)$$

Proposition 10 determines explicitly the spatio-temporal dynamic path of the pollution level, $p_{x,t}$, and the environmental tax, $\tau_{x,t}$, in the global setting. The local solution is determined by solving an infinite family of problems, each of them being defined on a bounded domain, and thus it coincides with the local solution earlier derived in Proposition 7. Therefore, in each interval $[x_j, x_{j+1}]$, the local tax is determined exactly as in the bounded domain case, and so is the collection of locally optimal tax and the evolution of pollution, with the extremes in the boundary conditions x_a and x_b replaced by $-\infty$ and $+\infty$, respectively. Exactly as in the bounded domain case, the presence of different intervals at which the boundary conditions are associated introduces some difference in the environmental tax and thus in the pollution stock between the local and global solutions. The same comments presented for the closed-form solution of our baseline model apply: the analytical expressions in Proposition 10 are so cumbersome to prevent the possibility to understand how they depend on the different economic and environmental parameters.

Note that the closed-form expression for the dynamic path of the environmental rate and the pollution level in Proposition 10 can be expressed in a simpler form by interpreting it as the expected value of a random variable. More precisely, recall that if Z is a standard normal distribution, its density is given by:

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

and the expected value of the composition $g(Z)$, where $g : \mathbb{R} \to \mathbb{R}$ is a real function, is given by

$$\mathbb{E}[g(Z)] = \int_{-\infty}^{+\infty} g(s) f_Z(s) ds$$

By introducing the change of variable $z = \frac{y-x}{\sqrt{2dt}}$ and replacing $dy = \sqrt{2dt}dz$, the expressions for the global solution in Proposition 10 can be rewritten as follows:

$$\tau_{x,t} = \frac{e^{\Theta t}_{21} \mathbb{E} p_0(x + \sqrt{2dt}Z)) + e^{\Theta t}_{22} \mathbb{E} \tilde{z}^1(x + \sqrt{2d(T-t)}Z, T))}{e^{\Theta t}_{11} \mathbb{E} p_0(x + \sqrt{2dt}Z)) + e^{\Theta t}_{12} \mathbb{E} \tilde{z}^1(x + \sqrt{2d(T-t)}Z, T))}$$

$$p_{x,t} = e^{\Theta t}_{11} \mathbb{E} p_0(x + \sqrt{2dt}Z)) + e^{\Theta t}_{12} \mathbb{E} \tilde{z}^1(x + \sqrt{2d(T-t)}Z, T))$$

The above stochastic formulation in terms of the expected value of a rescaled normal distribution allows to express the closed-form solutions in global setting in a more compact form and to use a Monte-Carlo simulation to approximate the optimal paths in our following numerical example. From an economic perspective, the pollution expression states that, as expected, in the long-run the level of pollution in a single location x will get more and more affected by pollution externalities over time. Despite the difficulties in interpreting the spatio-temporal dynamic path of the pollution level and the environmental tax, Proposition 10 allows us to explicitly verify Proposition 1. Indeed, as shown in the following proposition, in the absence of spatial heterogeneity in the initial distribution of pollution, the globally optimal dynamic path of our transboundary pollution control problem boils down to the locally optimal one.

Proposition 11. *Suppose that $p_0(x) = p_0$ for all $x \in (-\infty, +\infty)$. The globally optimal pair $(\tau_{x,t}, p_{x,t})$ is given by:*

$$\tau_{x,t} = \frac{e^{\Theta t}_{21} + e^{\Theta T}_{22} \left(\frac{\theta e^{\Theta T}_{21} - \eta(1-\theta) e^{\Theta T}_{11}}{\eta(1-\theta) e^{\Theta T}_{12} - \theta e^{\Theta T}_{22}} \right)}{e^{\Theta t}_{11} + e^{\Theta T}_{12} \left(\frac{\theta e^{\Theta T}_{21} - \eta(1-\theta) e^{\Theta T}_{11}}{\eta(1-\theta) e^{\Theta T}_{12} - \theta e^{\Theta T}_{22}} \right)}$$

and

$$p_{x,t} = e^{\Theta t}_{11} + e^{\Theta T}_{12} \left(\frac{\theta e^{\Theta T}_{21} - \eta(1-\theta) e^{\Theta T}_{11}}{\eta(1-\theta) e^{\Theta T}_{12} - \theta e^{\Theta T}_{22}} \right)$$

where $e^{\Theta t}$ is given by (11). The globally optimal pair $(p_{x,t}, \tau_{x,t})$ perfectly coincides with the locally optimal pair $(\overline{p}_{x,t}, \overline{\tau}_{x,t})$.

In order to better understand the extent to which the local and global solutions may differ, we can exploit our closed-form solutions from Proposition 10 to visualize such an eventual difference. Therefore, we now present some numerical example based on the same parameter values employed in our baseline model, by focusing only on the most interesting case in which the initial pollution distribution is heterogeneous in $x \in \mathbb{R}$ (while in our baseline model $x \in [-1, 1]$). For the sake of simplicity, we assume that out of the $[-1, 1]$ region, the spatial economy is uninhabited and thus the initial pollution level is null and thus also the environmental tax is null, while in the inhabited region $[-1, 1]$ the initial pollution distribution is exactly as in our bounded domain case, that is $p_0(x) = p_0 + \frac{1}{2} p_0 e^{-5x^2}$. Our numerical example is illustrated in Figure 3

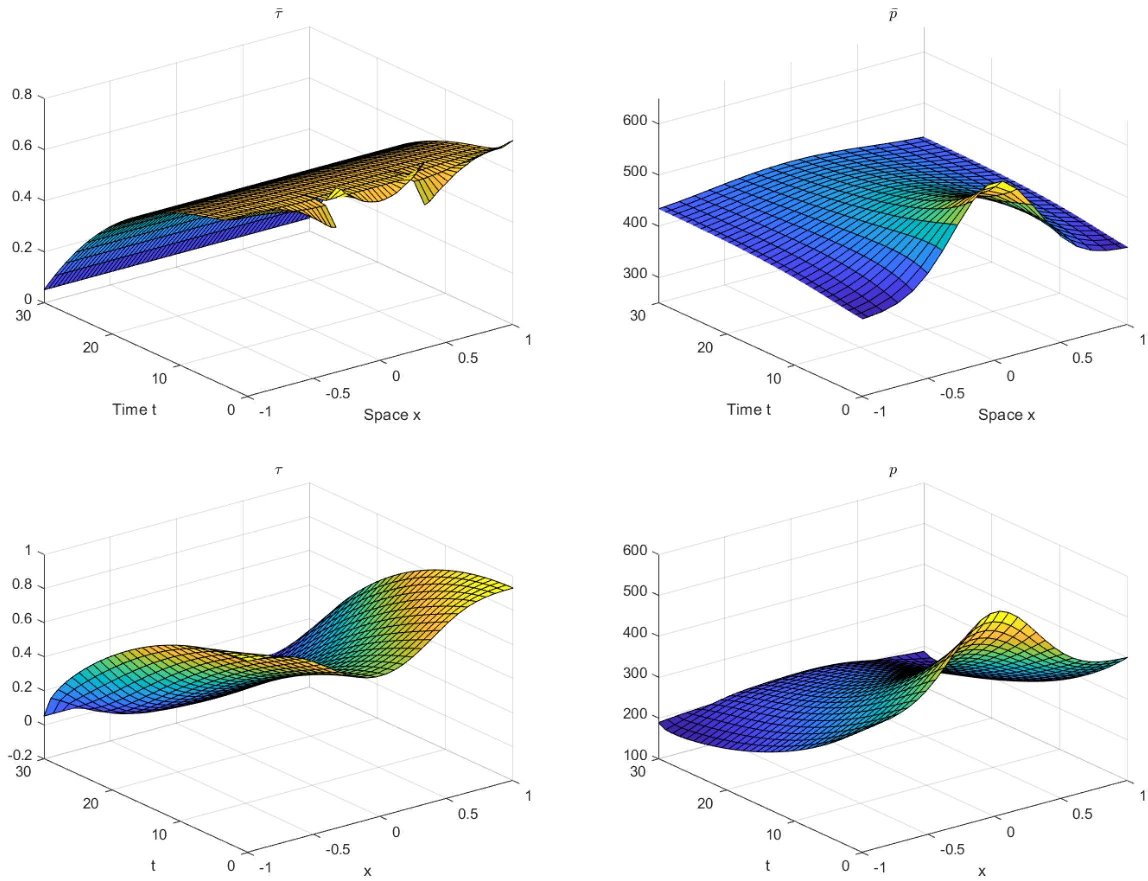

Figure 3: Spatio-temporal dynamics of the environmental tax (left) and pollution (right) with spatial heterogeneity in the initial pollution distribution. Local solution (top) and global solution (bottom) not coinciding.

where we represent both the local solution (top panel) and the global solution (bottom panel) in the spatial interval [-1, 1], that is a small subset of the entire spatial domain. From a qualitative point of view, the local and global solutions behave as in our baseline model in which the spatial domain is bounded: the tax rate starts high to then decrease over time in order to allow for a monotonic reduction in the pollution stock, and the physical mechanisms of pollution diffusion lead to a spatially heterogeneous pattern in both the environmental tax and the pollution level. Also in the unbounded domain case, there exist important qualitative and quantitative differences between the local and global solutions, and at the end of the planning horizon the global solution leads to a lower level of pollution than the local solution. By comparing the pollution stock in the global solution between the bounded and unbounded domain cases, we can observe that the final pollution stock is lower in the former case, and this is simply due to the effects of diffusion: since pollution diffuses across a larger spatial domain it tends to fall more rapidly in the [-1,1] region where it is initially higher.

5.2 Extensions

By following exactly the same arguments presented earlier in the bounded spatial domain case, it is straightforward to show that the results of Propositions 1 and 2 apply also in more general settings than the specific one just discussed. In particular, even in an unbounded spatial domain framework, by allowing the instantaneous loss function, the end-of-planning damage function and the pollution accumulation functions to be convex, the FOCs in the local and global cases will be given by (70)- (75) and (60) - (65), respectively (with the extremes in the boundary conditions x_a and x_b replaced by $-\infty$ and $+\infty$). By introducing capital accumulation and optimal consumption, along the lines discussed in section 4, the FOCs in the local and global cases will be given by (95) - (103) and (81) - (89), respectively (again, with the extremes in the boundary conditions x_a and x_b replaced by $-\infty$ and $+\infty$). Therefore, exactly the same comments we highlighted earlier hold true, confirming that the suboptimality of the local approach to the pollution problem (in the case of heterogeneous initial spatial pollution distribution) still applies in general settings and independently of the specific structure of the spatial domain.

6 Conclusion

Environmental policy is essential in order to achieve sustainable development but understanding the optimal level of policy intervention is not simple, especially because of the presence of transboundary externalities. The popular motto think globally, act locally summarizes the view that policy coordination across individual policymakers is necessary to preserve our common environment. In this paper we try to formally analyze whether this is really the case by focusing on a pollution control problem over a finite horizon and in a spatial framework. This setting gives rise to a regional optimal control problem which allows us to contrast the local and global solutions in which, respectively, the transboundary externality is and is not taken into account in the determination of the optimal policy by individual local policymakers. We show that whenever the initial spatial distribution of pollution is homogeneous then the local and the global solutions coincide, while whenever this is heterogeneous the two solutions differ meaning that the local solution is suboptimal. Therefore, the transboundary externalities per se are not a source of inefficiency in locally determined outcomes, but the inefficiency arises from the interaction between transboundary externalities and spatial heterogeneity in the initial pollution distribution. In the context with heterogeneous initial pollution distribution, which represents the most realistic case from a real world perspective, coordination across local policymakers is the best approach consistently with the think globally, act locally argument. We also quantify the minimal level of policy intervention that needs to be implemented locally in order to achieve the globally desirable goal of pollution elimination. Indeed, we show that whenever every local economy implements an environmental policy stringent enough, then the global pollution level will fall and over the long run the entire global economy will be able to achieve a completely pollution-free status. We also show that our main conclusions do not depend on the peculiarities of our model's formulation but they rather extend to more general and complicated frameworks.

To the best of our knowledge, no other paper has explicitly analyzed a regional optimal control problem in economics nor the potential differences between local and global solutions arising from an economic problem in spatial settings in a way comparable to ours. The approach has therefore been a bit simplistic and the analysis could be extended along multiple directions. Some further heterogeneity in the characteristics or the objectives of local policymakers can be introduced in order to represent the realistic situation in which some local economies (due for example to binding budget constraints) within the global economy cannot optimally determine their level of intervention. A certain degree od strategic interaction between local policymakers can be introduced in order to account for the possibility of free-riding on the environmental policies implemented in neighbor locations and understand the efficiency implications of such strategic interactions for local policymaking. In our analysis we have considered only dynamic transboundary pollution externalities while it may be interesting to understand whether the introduction of static transboundary externalities may affect our qualitative results. These further issues are left for future research.

A Technical Appendix

In this section we present all the proofs of the results discussed in the paper.

Proof of Proposition 1

The proof is straightforward and it easily follows by noticing that the objective function is convex and the pair $(\overline{p}, \overline{\tau})$ solves the first-order optimality conditions over each interval (x_j, x_{j+1}) and, because both $\overline{p}$ and $\overline{\tau}$ are homogeneous over x, they can be extended to the whole interval $[x_a, x_b]$ and, therefore, they satisfy (38) and (39) over $[x_a, x_b]$.

Proof of Proposition 2

The proof is straightforward and it easily follows by noticing that ($\tau^j_{x,t}$ solves the first-order optimality conditions over each interval (x_j, x_{j+1}) along with the optimal local level of pollution $p^j_{x,t}$), but the local solution $\bar{\tau}_{x,t}$ and $\bar{p}_{x,t}$ do not satisfy the optimality conditions described by equations (38) and (39) over the whole interval $[x_a, x_b]$ (for instance over the points x_j) and the boundary conditions (40) - (43).

Proof of Proposition 3

As (p, τ) is optimal, it satisfies the first-order optimality conditions. In particular, by defining $u = \tau p$, it is true that:

$$\frac{\partial p_{x,t}}{\partial t} = d\frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta - \delta)\, p_{x,t} - \eta u_{x,t} \tag{121}$$

Then, by using the fact that $u_{x,t} = \tau_{x,t} p_{x,t}$ and the definition of τ_{min}, we can prove that the following inequality is true:

$$\frac{\partial p_{x,t}}{\partial t} \leq \frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta - \delta - \eta\tau_{min}) p_{x,t} \tag{122}$$

By integrating from x_a to x_b we obtain:

$$\frac{d}{dt}\int_{x_a}^{x_b} p_{x,t}dx \leq \int_{x_a}^{x^b} \frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta - \delta - \eta\tau_{min}) \int_{x_a}^{x_b} p_{x,t}dx, \tag{123}$$

which implies:

$$\frac{d}{dt} p_t^{tot} \leq (\eta - \delta - \eta\tau_{min}) p_t^{tot} \leq 0, \tag{124}$$

and then the thesis follows.

Proof of Proposition 4

The following proposition will be instrumental to proving Proposition 4.

Proposition 12. *(Friedman, 2008) Let ρ be smooth and suppose that:*

$$\begin{aligned} \frac{\partial \rho}{\partial t} - d\frac{\partial^2 \rho}{\partial x^2} &\geq -c\rho \quad in\ (x_a, x_b) \times (0, T) && (125)\\ \frac{\partial \rho}{\partial n} &\geq 0, \quad on\ \{x_a, x_b\} \times (0, T) && (126)\\ \rho(0, x) &\geq 0 \quad in\ (x_a, x_b) && (127) \end{aligned}$$

where d is a positive real number and $c \in\ R$. Then $\rho \geq 0$ in $(x_a, x_b) \times (0, T)$.

From the above proposition we can now prove our thesis. If $(p_{x,t}, \tau_{x,t}) \geq 0$ is the optimal solution of the spatial optimal control problem (16) – (19) and, by defining $u_{x,t} = \tau_{x,t}p_{x,t}$, then it has been already proved that

$$\frac{\partial p_{x,t}}{\partial t} \leq \frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta - \delta - \eta\tau_{min})p_{x,t} \tag{128}$$

Let us denote by $\tilde{p}_{x,t}$ the solution to the following problem:

$$\begin{aligned} \frac{\partial p_{x,t}}{\partial t} &= d\frac{\partial^2 p_{x,t}}{\partial x^2} + (\eta - \delta - \eta\tau_{min})p_{x,t} && (129) \\ p_{x,0} &= p_0(x) && (130) \\ \frac{\partial p_{x_a,t}}{\partial x} &= \frac{\partial p_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T] && (131) \end{aligned}$$

Let us define $\rho_{x,t} = \tilde{p}_{x,t} - p_{x,t}$. By computing the equation for ρ, we get:

$$\begin{aligned} \frac{\partial \rho_{x,t}}{\partial t} &\geq d\frac{\partial^2 \rho_{x,t}}{\partial x^2} + (\eta - \delta - \eta\tau_{min})\rho_{x,t} && (132) \\ \rho_{x,t} &= 0 && (133) \\ \frac{\partial \rho_{x_a,t}}{\partial x} &= \frac{\partial \rho_{x_b,t}}{\partial x} = 0 \quad \forall t \in [0, T] && (134) \end{aligned}$$

From the above result Proposition 12, we can deduce that $\rho \geq 0$ and then $p_{x,t} \leq \tilde{p}_{x,t}$. On the other hand, by using the following change of variable

$$\tilde{p}_{x,t} = e^{(\eta-\delta-\eta\tau_{min})t}h_{x,t} \tag{135}$$

it is easy to show that if $\bar{p}$ solves the above problem given by (129) - (131), then $h_{x,t}$ will be the well-known classical solution of the heat equation with Neumann boundary conditions, given by the following expression:

$$h_{x,t} = \sum_{n\geq 0} B_n e^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2 t} \cos\left[\frac{n\pi(x - x_a)}{x_b - x_a}\right] \tag{136}$$

where:

$$B_0 = \frac{1}{x_b - x_a}\int_{x_a}^{x_b} p_0(x)dx \tag{137}$$

and:

$$B_n = \frac{2}{x_b - x_a}\int_{x_a}^{x_b} p_0(x)\cos\left[\frac{n\pi(x - x_a)}{x_b - x_a}\right]dx \tag{138}$$

Proof of Proposition 5

The proposition follows as a corollary of Proposition 4 which states that $p_{x,t} \leq e^{(\eta-\delta-\eta\tau_{min})t}h_{x,t}$. If $\tau_{min} > \frac{\eta-\delta}{\eta}$, for any $x \in [x_a, x_b]$ it is straightforward to conclude the following:

$$\lim_{T\to\infty} p_{x,T} \leq \lim_{T\to\infty} e^{(\eta-\delta-\eta\tau_{min})T}h_{x,T} = 0. \tag{139}$$

Proof of Proposition 6

By defining the variable $u_{x,t} = \tau_{x,t}p_{x,t}$, the vector-value variable $z_{x,t}$, and the matrix Θ as for the one dimensional case,

$$z_{x,t} = \begin{bmatrix} p_{x,t} \\ u_{x,t} \end{bmatrix}, \qquad \Theta = \begin{bmatrix} \eta - \delta & -\eta \\ -\eta & \rho - \eta + \delta \end{bmatrix},$$

and the diffusion matrix D as follows:

$$D = \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix},$$

we can write the first-order optimality conditions in a more compact form as follows;

$$\frac{\partial z_{x,t}}{\partial t} = D\frac{\partial^2 z_{x,t}}{\partial x^2} + \Theta z_{x,t},$$

This is a system of two heat equations. By using the change of variable $\tilde{z}_{x,t} = e^{-\Theta t} z_{x,t}$ it is easy to show that the solution to this equation takes the following form:

$$z_{x,t} = e^{\Theta t}\tilde{z}_{x,t}$$

where $\tilde{z}_{x,t}$ is the solution to the classical heat equation given by

$$\tilde{z}_{x,t} = \begin{bmatrix} A_0 + \sum_{n\geq 1} A_n e^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2 t} \cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right) \\ B_0 + \sum_{n\geq 1} B_n e^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2 (T-t)} \cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right) \end{bmatrix}$$

and

$$e^{\Theta t} = \begin{bmatrix} e^{\frac{\rho t}{2}}\left(\cosh\left(\frac{\xi t}{2}\right) - \frac{(2(\delta-\eta)+\rho)\sinh\left(\frac{\xi t}{2}\right)}{\xi}\right) & -2\frac{e^{\frac{\rho t}{2}}\sinh\left(\frac{\xi t}{2}\right)\eta}{\xi} \\ -2\frac{e^{\frac{\rho t}{2}}\sinh\left(\frac{\xi t}{2}\right)\eta}{\xi} & e^{\frac{\rho t}{2}}\left(\cosh\left(\frac{\xi t}{2}\right) + \frac{(2(\delta-\eta)+\rho)\sinh\left(\frac{\xi t}{2}\right)}{\xi}\right) \end{bmatrix} \tag{140}$$

If we plug $t = 0$ we get that $e^{\Theta 0} = I$ and then the first component of z boils down to

$$\tilde{z}^0_{x,0} = p_0(x) = A_0 + \sum_{n\geq 1} A_n \cos\left(2n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right)$$

which implies that A_0 and A_n are the Fourier coefficients of p_0, that is

$$A_0 = \frac{1}{x_b - x_a}\int_{x_a}^{x_b} p_0(x)dx$$

and

$$A_n = \frac{2}{x_b - x_a}\int_{x_a}^{x_b} p_0(x)\cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right)dx$$

For the terminal condition, instead, let us plug $t = T$ into the expression of $z_{x,t}$. We get

$$z_{x,T} = e^{\Theta T}\begin{bmatrix} \tilde{z}^0_{x,T} \\ \tilde{z}^1_{x,T} \end{bmatrix} = e^{\Theta T}\begin{bmatrix} A_0 + \sum_{n\geq 1} A_n e^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2 T} \cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right) \\ B_0 + \sum_{n\geq 1} B_n \cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right) \end{bmatrix}$$

and by using the terminal condition

$$u_{x,T} = \eta\,\frac{1-\theta}{\theta} p_{x,T}$$

we get the system

$$\frac{\theta}{\eta(1-\theta)} = \frac{e^{\Theta T}_{11}(A_0 + \sum_{n\geq 1} A_n e^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2 T}\cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right)) + e^{\Theta T}_{12}(B_0 + \sum_{n\geq 1} B_n\cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right))}{e^{\Theta T}_{21}(A_0 + \sum_{n\geq 1} A_n e^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2 T}\cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right)) + e^{\Theta T}_{22}(B_0 + \sum_{n\geq 1} B_n\cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right))}$$

which can be transformed into

$$\frac{\theta}{\eta(1-\theta)} = \frac{e^{\Theta T}_{11}A_0 + e^{\Theta T}_{12}B_0 + \sum_{n\geq 1}\left(e^{\Theta T}_{11}A_n e^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2 T} + e^{\Theta T}_{12}B_n\right)\cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right)}{e^{\Theta T}_{21}A_0 + e^{\Theta T}_{22}B_0 + \sum_{n\geq 1}\left(e^{\Theta T}_{21}A_n e^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2 T} + e^{\Theta T}_{22}B_n\right)\cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right)}$$

and then

$$\theta\left\{e_{21}^{\Theta T}A_0+e_{22}^{\Theta T}B_0+\sum_{n\geq1}\left(e_{21}^{\Theta T}A_ne^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2T}+e_{22}^{\Theta T}B_n\right)\cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right)\right\}=$$

$$\eta(1-\theta)\left\{e_{11}^{\Theta T}A_0+e_{12}^{\Theta T}B_0+\sum_{n\geq1}\left(e_{11}^{\Theta T}A_ne^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2T}+e_{12}^{\Theta T}B_n\right)\cos\left(n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right)\right\}$$

and finally we can get the expressions of B_n and B_0 in terms of A_n and A_0 by noticing that

$$\theta\left(e_{21}^{\Theta T}A_0+e_{22}^{\Theta T}B_0\right)=\eta(1-\theta)\left(e_{11}^{\Theta T}A_0+e_{12}^{\Theta T}B_0\right)$$

$$\left(\theta e_{21}^{\Theta T}-\eta(1-\theta)e_{11}^{\Theta T}\right)A_0=\left(\eta(1-\theta)e_{12}^{\Theta T}-\theta e_{22}^{\Theta T}\right)B_0$$

which implies that

$$B_0=\left[\frac{\theta e_{21}^{\Theta T}-\eta(1-\theta)e_{11}^{\Theta T}}{\eta(1-\theta)e_{12}^{\Theta T}-\theta e_{22}^{\Theta T}}\right]A_0 \tag{141}$$

and

$$\left(\theta e_{21}^{\Theta T}-\eta(1-\theta)e_{11}^{\Theta T}\right)A_ne^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2T}=\eta(1-\theta)e_{12}^{\Theta T}B_n-\theta e_{22}^{\Theta T}B_n$$

$$\left(\theta e_{21}^{\Theta T}-\eta(1-\theta)e_{11}^{\Theta T}\right)A_ne^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2T}=\left(\eta(1-\theta)e_{12}^{\Theta T}-\theta e_{22}^{\Theta T}\right)B_n$$

which allows to determine B_n as

$$B_n=\left[\frac{\theta e_{21}^{\Theta T}-\eta(1-\theta)e_{11}^{\Theta T}}{\eta(1-\theta)e_{12}^{\Theta T}-\theta e_{22}^{\Theta T}}\right]A_ne^{-d\left(\frac{n\pi}{x_b-x_a}\right)^2T} \tag{142}$$

This completes the proof and the calculations of the Fourier coefficients.

Proof of Proposition 7

The proof of this result is equal to the one presented in Proposition 6 where the interval $[x_a, x_b]$ is replaced by $[x_j, x_{j+1}]$.

Proof of Proposition 8

The proof follows by noticing that if $p_0(x)=p_0$ for all $x\in[x_a,x_b]$ then the Fourier coefficients are equal to:

$$A_0=p_0$$

$$A_n=\frac{2}{x_b-x_a}\int_{x_a}^{x_b}p_0(x)\cos\left(2n\pi\left[\frac{x-x_a}{x_b-x_a}\right]\right)dx=0$$

$$B_0=\left[\frac{\theta e_{21}^{\Theta T}-\eta(1-\theta)e_{11}^{\Theta T}}{\eta(1-\theta)e_{12}^{\Theta T}-\theta e_{22}^{\Theta T}}\right]p_0$$

$$B_n=\left[\frac{\theta e_{21}^{\Theta T}-\eta(1-\theta)e_{11}^{\Theta T}}{\eta(1-\theta)e_{12}^{\Theta T}-\theta e_{22}^{\Theta T}}\right]A_ne^{-d\left(\frac{2n\pi}{x_b-x_a}\right)^2T}=0$$

and this implies the thesis.

Proof of Proposition 9

The proof of this result is equal to the one of Proposition 4 in which the bounded interval is replaced with the unbounded one.

Proof of Proposition 10

By repeating the same steps as in the baseline model, the first-order conditions can be written in a more compact form as

$$\frac{\partial z_{x,t}}{\partial t} = D\frac{\partial^2 z_{x,t}}{\partial x^2} + \Theta z_{x,t}$$

and by introducing the change of variable $\tilde{z} = e^{-\Theta t}z$ it is easy to show that the solution to this equation takes the following form:

$$z_{x,t} = e^{\Theta t}\tilde{z}_{x,t}.$$

Now the first component of $\tilde{z}$, $\tilde{z}^0_{x,t}$, solves the following heat equation

$$\frac{\partial \tilde{z}^0_{x,t}}{\partial t} = d\frac{\partial^2 \tilde{z}^0_{x,t}}{\partial x^2}$$

with initial condition $z^0_{x,0} = p_0(x)$. The solution is then provided by means of the Green functions as follows:

$$\tilde{z}^0_{x,t} = \frac{1}{2\sqrt{\pi dt}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4dt}} p_0(y)dy$$

The second component of $\tilde{z}$, $\tilde{z}^1_{x,t}$, satisfies the following backward heat equation

$$\frac{\partial \tilde{z}^1_{x,t}}{\partial t} = -d\frac{\partial^2 \tilde{z}^1_{x,t}}{\partial x^2}$$

and the solution is then provided by means of the Green functions as follows:

$$\tilde{z}^1_{x,t} = \frac{1}{2\sqrt{\pi d(T-t)}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4d(T-t)}} \tilde{z}^1_{y,T}dy$$

Using the terminal condition for u and p, we get

$$\frac{\theta}{\eta(1-\theta)} = \frac{p_{x,T}}{u_{x,T}} = \frac{e^{\Theta T}_{11}\tilde{z}^0_{x,T} + e^{\Theta T}_{12}\tilde{z}^1_{x,T}}{e^{\Theta T}_{21}\tilde{z}^0_{x,T} + e^{\Theta T}_{22}\tilde{z}^1_{x,T}}$$

which implies that

$$\theta e^{\Theta T}_{21}\tilde{z}^0_{x,T} + \theta e^{\Theta T}_{22}\tilde{z}^1_{x,T} = \eta(1-\theta)e^{\Theta T}_{11}\tilde{z}^0_{x,T} + \eta(1-\theta)e^{\Theta T}_{12}\tilde{z}^1_{x,T}$$
$$\left[\theta e^{\Theta T}_{21} - \eta(1-\theta)e^{\Theta T}_{11}\right]\tilde{z}^0_{x,T} = \left[\eta(1-\theta)e^{\Theta T}_{12} - \theta e^{\Theta T}_{22}\right]\tilde{z}^1_{x,T}$$

and, finally,

$$\tilde{z}^1_{x,T} = \left[\frac{\theta e^{\Theta T}_{21} - \eta(1-\theta)e^{\Theta T}_{11}}{\eta(1-\theta)e^{\Theta T}_{12} - \theta e^{\Theta T}_{22}}\right]\tilde{z}^0_{x,T}$$

This completes the proof as it allows to determine the closed-form expression of both components of the vector z.

Proof of Proposition 11

The proof follows by noticing that when $p_0(x) = p_0$, then

$$\tilde{z}^1_{x,T} = \left[\frac{\theta e^{\Theta T}_{21} - \eta(1-\theta)e^{\Theta T}_{11}}{\eta(1-\theta)e^{\Theta T}_{12} - \theta e^{\Theta T}_{22}}\right]p_0$$

by using the fact that

$$\frac{1}{2\sqrt{\pi dT}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4dT}}dy = 1$$

Then the globally optimal pair $(p_{x,t}, u_{x,t})$ and the locally optimal pair $(\overline{p}_{x,t}, \overline{u}_{x,t})$ are given by

$$\begin{bmatrix} p_{x,t} \\ u_{x,t} \end{bmatrix} = \begin{bmatrix} \overline{p}_{x,t} \\ \overline{u}_{x,t} \end{bmatrix} = p_0 e^{\Theta t} \begin{bmatrix} 1 \\ \left[\frac{\theta e_{21}^{\Theta T} - \eta(1-\theta) e_{11}^{\Theta T}}{\eta(1-\theta) e_{12}^{\Theta T} - \theta e_{22}^{\Theta T}}\right] \end{bmatrix} \tag{143}$$

which implies that

$$\bar{\tau}_{x,t} = \tau_{x,t} = \frac{e_{21}^{\Theta t} + e_{22}^{\Theta t}\left(\frac{\theta e_{21}^{\Theta T} - \eta(1-\theta) e_{11}^{\Theta T}}{\eta(1-\theta) e_{12}^{\Theta T} - \theta e_{22}^{\Theta T}}\right)}{e_{11}^{\Theta t} + e_{12}^{\Theta t}\left(\frac{\theta e_{21}^{\Theta T} - \eta(1-\theta) e_{11}^{\Theta T}}{\eta(1-\theta) e_{12}^{\Theta T} - \theta e_{22}^{\Theta T}}\right)}$$

References

1. Anita, S., Capasso, V. (2018). The interplay between models and public health policies: regional control for a class of spatially structured epidemics (think globally, act locally), Mathematical Biosciences and Engineering 15, 1–20
2. Anita, S., Capasso, V., Kunze, H., La Torre, D. (2013). Optimal control and long-run dynamics for a spatial economic growth model with physical capital accumulation and pollution diffusion, Applied Mathematics Letters 26, 908–912
3. Anita, S., Capasso, V., Kunze, H., La Torre, D. (2015). Dynamics and optimal control in a spatially structured economic growth model with pollution diffusion and environmental taxation, Applied Mathematics Letters 42, 36-40
4. Ansuategi, A. (2003). Economic growth and transboundary pollution in Europe: an empirical analysis, Environmental and Resource Economics 26, 305–328
5. Ansuategi, A., Perrings , C.A. (2000). Transboundary externalities in the environmental transition hypothesis, Environmental and Resource Economics 17, 353-–373
6. Athanassoglou, S., Xepapadeas, A. (2012). Pollution control with uncertain stock dynamics: when, and how, to be precautious, Journal of Environmental Economics and Management 63, 304-–320
7. Bawa, V.S. (1975). On optimal pollution control policies, Management Science 21, 1397–1404
8. Boucekkine, R., Camacho, C., Zou, B. (2009). Bridging the gap between growth theory and economic geography: the spatial Ramsey model, Macroeconomic Dynamics 13, 20–45
9. Boucekkine, R., Camacho, C., Fabbri, G. (2013a). On the optimal control of some parabolic differential equations arising in economics, Serdica Mathematical Journal 39, 331–354
10. Boucekkine, R., Camacho, C., Fabbri, G. (2013b). Spatial dynamics and convergence: the spatial AK model, Journal of Economic Theory 148, 2719–2736
11. Boucekkine, R., Fabbri, G., Federico, S., Gozzi, F. (2019). Growth and agglomeration in the heterogeneous space: a generalized AK approach, Journal of Economic Geography 19, 1287–1318
12. Boucekkine, R., Fabbri, G., Federico, S., Gozzi, F. (2020). A dynamic theory of spatial externalities, UCLouvain Discussion Paper 2020/17
13. Brito, P. (2004). The dynamics of growth and distribution in a spatially heterogeneous world, UECE-ISEG, Technical University of Lisbon
14. Brock, W.A., Xepapadeas, A. (2004). Spatial analysis: development of descriptive and normative methods with applications to economic–ecological modelling, Working Papers 2004.159, Fondazione Eni Enrico Mattei
15. Brock, W.A., Xepapadeas, A. (2008). Diffusion-induced instability and pattern formation in infinite horizon recursive optimal control, Journal of Economic Dynamics & Control 32, 2745–2787
16. Brock, W.A., Xepapadeas, A. (2010). Pattern formations, spatial externalities and regulation in a coupled economic–ecological system, Journal of Environmental Economics and Management 59, 149–164

17. Brock, W.A., Xepapadeas, A. (2017). Climate change policy under polar amplification, European Economic Review 94, 263–282
18. Camacho, C., Pérez–Barahona, A. (2015). Land use dynamics and the environment, Journal of Economic Dynamics & Control 52, 96–118
19. Camacho, C., Zou, B. (2004). The spatial Solow model, Economics Bulletin 18, 1–11
20. Camacho, C., Zou, B., Briani, M. (2008). On the dynamics of capital accumulation across space, European Journal of Operational Research 186 2, 451–465
21. Chichilnisky, G., Heal, G., Beltratti, A. (1995). The green golden rule, Economics Letters 49, 174–179
22. Chichilnisky, G. (1997). What is sustainable development?, Land Economics 73, 476–491
23. Colapinto, C., Liuzzi, D., Marsiglio, S. (2017). Sustainability and intertemporal equity: a multicriteria approach, Annals of Operations Research 251, 271–284
24. Forster, B.A. (1972). A note on the optimal control of pollution, Journal of Economic Theory 5, 537–539
25. Forster, B.A. (1975).Optimal pollution control with a nonconstant exponential rate of decay, Journal of Environmental Economics and Management 2, 1–6
26. Friedman, A. (2008). Partial differential equations of parabolic type (Dover Ed.)
27. de Frutos J., Martin-Herran G. (2019a). Spatial effects and strategic behavior in a multiregional transboundary pollution dynamic game, Journal of Environmental Economics and Management 97, 182–207
28. de Frutos F., Martin-Herran G. (2019b). Spatial vs. non-spatial transboundary pollution control in a class of cooperative and non-cooperative dynamic games, European Journal of Operational Research 276, 379–394
29. de Frutos J., Lopez P., Martin-Herran G. (2019). Equilibrium strategies in a multiregional transboundary pollution differential game with spatially distributed controls, arXiv:1912.00481
30. Fujita, M., Krugman, P., Venables, A. (1999). The spatial economy. Cities, regions and international trade (MIT Press)
31. Fujita, M., Thisse, J.F. (2002). Economics of agglomeration (Cambridge University Press)
32. Geddes, P. (1915). Cities in evolution (London: Williams)
33. Hotelling, H. (1929). Stability in competition, Economic Journal 39, 41–57
34. Helland, E., Withford, A. (2003). Pollution incidence and political jurisdiction: evidence from the TRI, Journal of Environmental Economics and Management 46, 403–424
35. Intergovernmental Panel on Climate Change, IPCC (2014). Climate change 2014: synthesis report summary for policymakers (IPCC, Geneva), available at: https://www.ipcc.ch/pdf/assessment-report/ar5/syr/AR5_SYR_FINAL_SPM.pdf
36. Keeler, E., Spencer, M., Zeckhauser, R. (1973). The optimal control of pollution, Journal of Economic Theory 4, 19–34
37. Krugman, P. (1991). Increasing returns and economic geography, Journal of Political Economy 99, 483–499
38. La Torre, D., Liuzzi, D., Marsiglio, S. (2015). Pollution diffusion and abatement activities across space and over time, Mathematical Social Sciences 78, 48–63
39. La Torre, D., Liuzzi, D., Marsiglio, S. (2017). Pollution control under uncertainty and sustainability concern, Environmental and Resource Economics 67, 885—903
40. La Torre, D., Liuzzi, D., Marsiglio, S. (2019a). Population and geography do matter for sustainable development, Environment and Development Economics 24, 201-–223

41. La Torre, D., Liuzzi, D., Marsiglio, S. (2019b). The optimal population size under pollution and migration externalities: a spatial control approach, Mathematical Modelling of Natural Phenomena 14, 104
42. Lions, J.L. (1973). The optimal control of distributed systems, Russian Mathematical Surveys 28, 13–46
43. Lions, J.L. (1988). Controlabilité exacte, stabilisation et perturbation de systémes distribués (Masson: Paris)
44. Long, N.V. (1992). Pollution control: a differential game approach, Annals of Operations Research 37, 283–296
45. McAsey, M., Mou, L., Han, W. (2012). Convergence of the forward–backward sweep method in optimal control, Computational Optimization and Applications 53, 207–226
46. Monogan III, J., Konisky, D., Woods, N. (2017). Gone with the wind: federalism and the strategic location of air polluters, American Journal of Political Science 61, 257–27
47. Ogawa, H., Wildasin, D.E. (2009). Think locally, act locally: spillovers, spillbacks, and efficient decentralized policymaking, American Economic Review 99, 1206–1217
48. Organisation for Economic Cooperation and Development (1977). Long-range transport of air pollutants. Measurements and findings (OECD: Paris)
49. Oreskes, N. (2004). The scientific consensus on climate change, Science 306, 1686
50. Saltari, E., Travaglini, G. (2014). Pollution control under emission constraints: switching between regimes, Energy Economics, forthcoming
51. Sigman, H. (2005). Transboundary spillovers and decentralization of environmental policies, Journal of Environmental Economics and Management 50, 82–10
52. Troltzsch, F. (2010). Optimal control of partial differential equations: theory, methods and applications (American Mathematical Society)
53. van der Ploeg, F., Withagen, C. (1991). Pollution control and the Ramsey problem, Environmental and Resource Economics 1, 215–236
54. Xepapadeas, A. (2005). Economic growth and the environment, in (Mäler, K.G., Vincent, J., Eds.), "Handbook of Environmental Economics", vol. 3., ch. 23, 1219–1271 (Elsevier: Amsterdam, Netherlands)