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Flat connections in three-manifolds and classical Chern–Simons invariant

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Abstract

A general method for the construction of smooth flat connections on 3-manifolds is introduced. The procedure is strictly connected with the deduction of the fundamental group of a manifold M by means of a Heegaard splitting presentation of M. For any given matrix representation of the fundamental group of M, a corresponding flat connection A on M is specified. It is shown that the associated classical Chern–Simons invariant assumes then a canonical form which is given by the sum of two contributions: the first term is determined by the intersections of the curves in the Heegaard diagram, and the second term is the volume of a region in the representation group which is determined by the representation of $\pi_1(M)$ and by the Heegaard gluing homeomorphism. Examples of flat connections in topologically nontrivial manifolds are presented and the computations of the associated classical Chern–Simons invariants are illustrated.
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1. Introduction

Each $SU(N)$-connection, with $N \geq 2$, in a closed and oriented 3-manifold M can be represented by a 1-form $A = A_\mu dx^\mu$ which takes values in the Lie algebra of $SU(N)$. The Chern–Simons function $S[A]$,

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$$
\begin{aligned}
S[A] &= \int_M \mathcal{L}_{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + i\tfrac{2}{3} A \wedge A \wedge A\right) \\
&= \frac{1}{8\pi^2} \int_M d^3x \, \epsilon^{\mu\nu\lambda} \, \mathrm{Tr}\left(A_\mu(x)\partial_\nu A_\lambda(x) + i\tfrac{2}{3} A_\mu(x) A_\nu(x) A_\lambda(x)\right) \,, \qquad (1.1)
\end{aligned}
$$

can be understood as the Morse function of an infinite dimensional Morse theory, on which the instanton Floer homology [1] and the gauge theory interpretation [2] of the Casson invariant [3] are based. Under a local gauge transformation

$$A_\mu(x) \longrightarrow A^\Omega_\mu(x) = \Omega^{-1}(x) A_\mu(x) \Omega(x) - i\Omega^{-1}(x)\partial_\mu \Omega(x) \,, \qquad (1.2)$$

where Ω is a map from M into $SU(N)$, the function $S[A]$ transforms as

$$S[A^\Omega] = S[A] + I_\Omega \,, \qquad (1.3)$$

where the integer $I_\Omega \in \mathbb{Z}$,

$$I_\Omega = \frac{1}{24\pi^2} \int_M \mathrm{Tr}\left(\Omega^{-1} d\Omega \wedge \Omega^{-1} d\Omega \wedge \Omega^{-1} d\Omega\right) \,, \qquad (1.4)$$

can be used to label the homotopy class of Ω. The stationary points of the function (1.1) correspond to flat connections, *i.e.* connections with vanishing curvature $F(A) = 2dA + i[A, A] = 0$. We shall now concentrate on flat connections exclusively. Let A be a flat connection in M, and let $\gamma \subset M$ be an oriented path connecting the starting point x_0 to the final point x_1. The associated holonomy $\gamma \to h_\gamma[A] \in SU(N)$ is given by the path-ordered integral

$$h_\gamma[A] = \mathrm{P}\, e^{i\int_\gamma A} \,, \qquad (1.5)$$

which is computed along γ. Under a gauge transformation $A \to A^\Omega$, one finds

$$h_\gamma[A^\Omega] = \Omega^{-1}(x_0)\, h_\gamma[A]\, \Omega(x_1) \,. \qquad (1.6)$$

Let us consider the set of holonomies which are associated with the closed oriented paths such that $x_0 = x_1 = x_b$, for a given base point x_b. Since the element $h_\gamma[A] \in SU(N)$ is invariant under homotopy transformations acting on γ, this set of holonomies specifies a matrix representation of the fundamental group $\pi_1(M)$ in the group $SU(N)$. Because of equation (1.3), the classical Chern–Simons invariant $cs[A]$,

$$cs[A] = S[A] \mod \mathbb{Z} \,, \qquad (1.7)$$

is well defined for the gauge orbits of flat $SU(N)$-connections on M, and it is well defined [4] for the $SU(N)$ representations of $\pi_1(M)$ modulo the action of group conjugation. If the orientation of M is modified, one gets $cs[A] \to -cs[A]$.

In the case of the structure group $SU(2)$, methods for the computation of $cs[A]$ have been presented in References [5–10], where a few non-unitary gauge groups have also been considered. In all the examples that have been examined, $cs[A]$ turns out to be a rational number. In the case of three dimensional hyperbolic geometry, the associated $PSL(2, \mathbb{C})$ classical invariant [7,11–13] combines the real volume and imaginary Chern–Simons parts in a complex geometric invariant. The Baseilhac–Benedetti invariant [14] with group $PSL(2, \mathbb{C})$ represents some kind of corresponding quantum invariant.

Precisely because flat connections represent stationary points of the function (1.1), flat connections and the corresponding value of $cs[A]$ play an important role in the quantum Chern–Simons gauge field theory [15]. For instance, the path-integral solution of the abelian Chern–Simons theory has recently been produced [16,17]. In this case, flat connections dominate the functional integration and the value of the partition function is given by the sum over the gauge orbits of flat connections of the exponential of the classical Chern–Simons invariant. The classical abelian Chern–Simons invariant is strictly related [16,17] with the intersection quadratic form on the torsion group of M, which also enters the abelian Reshetikhin–Turaev [18,19] surgery invariant.

In general, the precise expression of the flat connections is an essential ingredient for the computation of the observables of the quantum Chern–Simons theory by means of the path-integral method. In this article we shall mainly be interested in nonabelian flat connections. We will show that, given a representation ρ of $\pi_1(M)$ and a Heegaard splitting presentation [20] of M (with the related Heegaard diagram), by means of a general construction one can define a corresponding smooth flat connection A on M. The method that we describe is related with the deduction [21] of a presentation of the fundamental group of a manifold M by means of a Heegaard splitting of M. Then the associated invariant $cs[A]$ assumes a canonical form, which can be written as the sum of two contributions. The first term is determined by the intersections of the curves in the Heegaard diagram and can be interpreted as a sort of "coloured intersection form". Whereas the second term is the Wess–Zumino volume of a region in the structure group $SU(N)$ which is determined by the representation of $\pi_1(M)$ and by the Heegaard gluing homeomorphism.

The procedure that we present for the determination of the flat connections can find possible applications also in the description of the topological states of matter [22,23]. A discussion on the importance of topological configurations and of the holonomy operators in gauge theories can be found for instance in Ref. [24].

Our article is organised as follows. Section 2 contains a brief description of the main results of the present article. The general construction of flat connections in a generic 3-manifold M by means of a Heegaard splitting presentation of M is discussed in Section 3. The canonical form of the corresponding classical Chern–Simons invariant is derived in Section 4, where a two dimensional formula of the Wess–Zumino group volume is also produced. In the remaining sections, our method is illustrated by a few examples. Flat connections in lens spaces are discussed in Section 5 and a non-abelian representation of the fundamental group of a particular 3-manifold is considered in Section 6; computations of the corresponding classical Chern–Simons invariants are presented. The case of the Poincaré sphere is discussed in Section 7. One example of a general formula of the classic Chern–Simons invariant for a particular class of Seifert manifolds is given in Section 8. Finally, Section 9 contains the conclusions.

2. Outlook

The main steps of our construction can be summarised as follows. For any given $SU(N)$ representation ρ of $\pi_1(M)$,

$$\rho : \pi_1(M) \to SU(N) \, , \tag{2.1}$$

one can find a corresponding flat connection A on M whose structure is determined by a Heegaard splitting presentation $M = H_L \cup_f H_R$ of M. In this presentation, the manifold M is interpreted as the union of two handlebodies H_L and H_R which are glued by means of the homeomorphism $f : \partial H_L \to \partial H_R$ of their boundaries, as sketched in Fig. 1.

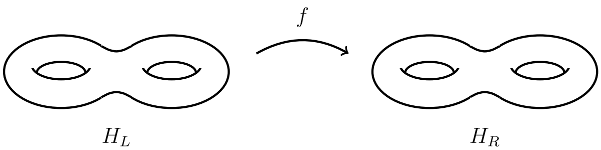


Fig. 1. Attaching homemorphism $f : \partial H_L \to \partial H_R$.

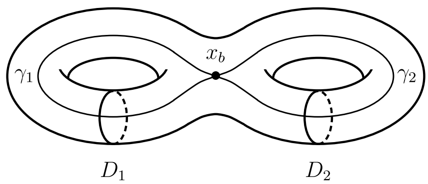


Fig. 2. Generators $\{\gamma_1, \gamma_2\}$ and meridinal discs $\{D_1, D_2\}$ in a handlebody of genus 2.

Let the fundamental group of M be defined with respect to a base point x_b which belongs to the boundaries of the two handlebodies. Then the representation ρ of $\pi_1(M)$ canonically defines a representation of the fundamental group of each of the two handlebodies H_L and H_R. As shown in Fig. 2, in each handlebody the generators of its fundamental group can be related with a set of corresponding disjoint meridinal discs. To each meridinal disc is associated a matrix which is specified by the representation ρ; this matrix can be interpreted as a "colour" which is attached to each meridinal disc. With the help of these coloured meridinal discs, one can construct a smooth flat connection A^0_L in H_L—and similarly a smooth flat connection A^0_R in H_R—whose holonomies correspond to the elements of the representation ρ in the handlebody H_L (or H_R). The precise definition of A^0_L and A^0_R is given in Section 3.

In general, A^0_L and A^0_R do not coincide with the restrictions in H_L and H_R of a single connection A in M, because the images—under f—of the boundaries of the meridinal discs of H_L are not the boundaries of meridinal discs of H_R. So, in order to define a connection A which is globally defined in M, one needs to combine A^0_L with A^0_R in a suitable way. In facts, the exact matching of the gauge fields A^0_L and A^0_R in M is specified by the homeomorphism f through the Heegaard diagram, which shows precisely how the boundaries of the meridinal discs of H_L are pasted onto the surface ∂H_R, in which the boundaries of the meridinal discs of H_R are also placed. Let us denote by $f * A^0_L$ the image of A^0_L under f. The crucial point now is that, on the surface ∂H_R, the connections A^0_R and $f * A^0_L$ are gauge related

$$f * A^0_L = U_0^{-1} A^0_R U_0 - i U_0^{-1} dU_0, \quad \text{on } \partial H_R \,, \tag{2.2}$$

because their holonomies define the same representation of $\pi_1(\partial H_R)$. The value of the map U_0 from the surface ∂H_R on the group $SU(N)$ is uniquely determined by equation (2.2) and by the condition $U_0(x_b) = 1$. In facts, we will demonstrate that

$$U_0(x) = \Phi_R^{-1}(x)\Phi_{f*L}(x), \quad \text{for } x \in \partial H_R \,, \tag{2.3}$$

where Φ_R and Φ_{f*L} denote the developing maps associated respectively with A^0_R and $f * A^0_L$ from the universal covering of ∂H_R into the group $SU(N)$. The definition of the developing map will be briefly recalled in Section 3.3. Then the map U_0 can smoothly be extended to the whole handlebody H_R; this extension will be denoted by U. The values of $U : H_R \to SU(N)$ inside H_R are not constrained and can be chosen without restrictions apart from smoothness. As far as the computation of the classical Chern–Simons invariant is concerned, the particular choice of the extension U of U_0 turns out to be irrelevant. To sum up, the connection A—which is well defined in M and whose holonomies determine the representation ρ—takes the form

$$A = \begin{cases} A_L^0 & \text{in } H_L \, ; \\ \\ U^{-1} A_R^0 U - i U^{-1} dU & \text{in } H_R \, ; \end{cases} \tag{2.4}$$

the correct matching of these two components is ensured by equation (2.2). The expression (2.4) of the connection implies

Proposition 1. *The classical Chern–Simons invariant (1.7), evaluated for the $SU(N)$ flat connection (2.4), takes the form*

$$cs[A] = \mathcal{X}[A] + \Gamma[U] \quad \text{mod } \mathbb{Z} \, , \tag{2.5}$$

where

$$\mathcal{X}[A] = \frac{1}{8\pi^2} \int_{\partial H_R} \text{Tr} \Big[U_0^{-1} A_R^0 U_0 \wedge f * A_L^0 \Big] \, , \tag{2.6}$$

and

$$\Gamma[U] = \frac{1}{24\pi^2} \int_{H_R} \text{Tr} \Big[U^{-1} dU \wedge U^{-1} dU \wedge U^{-1} dU \Big] \, . \tag{2.7}$$

The function $\mathcal{X}[A]$ is defined on the surface ∂H_R, and similarly the value of the Wess–Zumino volume $\Gamma[U]$ mod $\mathbb{Z}$ only depends [25–27] on the values of U in ∂H_R (*i.e.*, it only depends on U_0). A canonical dependence of Γ on U_0 will be produced in Section 4.4. Therefore both terms in expression (2.5) are determined by the data on the two-dimensional surface ∂H_R of the Heegaard splitting presentation $M = H_L \cup_f H_R$ exclusively. This is why the particular choice of the extension of U_0 inside H_R is irrelevant. The remaining part of this article contains the proof of Proposition 1 and a detailed description of the construction of the flat connection A. Examples will also be given, which elucidate the general procedure and illustrate the computation of $cs[A]$.

3. Flat connections

Given a matrix representation ρ of $\pi_1(M)$, we would like to determine a corresponding flat connection A on M whose holonomies agree with ρ; then we shall compute $S[A]$.

In order to present a canonical construction which is not necessarily related with the properties of the representation space, we shall use a Heegaard splitting presentation $M = H_L \cup_f H_R$ of M. The construction of A is made of two steps. First, in each of the two handlebodies H_L and H_R we define a flat connection, A_L^0 and A_R^0 respectively, whose holonomies coincide with the elements of the matrix representation of the fundamental group of the handlebody which is induced by ρ. Second, the components A_L^0 and A_R^0 are combined according to the Heegaard diagram to define A on M.

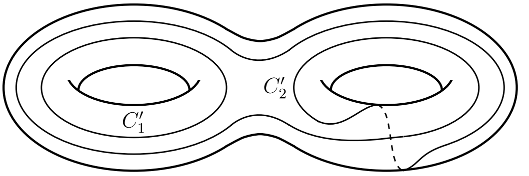


Fig. 3. Example of a genus 2 Heegaard diagram.

3.1. Heegaard splitting

Let us recall [4,20] that the fundamental group of a three-dimensional oriented handlebody H of genus g is a free group with g generators $\{\gamma_1, \gamma_2, ..., \gamma_g\}$. A disc D in H is called a meridinal disc if the boundary of D belongs to the boundary of H, $\partial D \subset \partial H$, and ∂D is homotopically trivial in H. Let $\{D_1, D_2, ..., D_g\}$ be a set of disjoint meridinal discs in H such that $H - \{D_1, D_2, ..., D_g\}$ is homeomorphic with a 3-ball with $2g$ removed disjoint discs in its boundary. These meridinal discs $\{D_1, D_2, ..., D_g\}$ can be put in a one-to-one correspondence with the g handles of the handlebody H or, equivalently, with the generators of $\pi_1(H)$, and can be oriented in such a way that the intersection of γ_j with D_k is δ_{jk}. For instance, in the case of a handlebody of genus 2, a possible choice of the generators $\{\gamma_1, \gamma_2\}$ and of the discs $\{D_1, D_2\}$ is illustrated in Fig. 2, where the base point x_b is also shown.

By means of a Heegaard presentation $M = H_L \cup_f H_R$ of the 3-manifold M, which is specified by the homeomorphism

$$f \; : \; \partial H_L \; \to \; \partial H_R \; , \tag{3.1}$$

one can find a presentation of the fundamental group $\pi_1(M)$. Suppose that the two handlebodies H_L and H_R have genus g. Let $\{D_1, D_2, ..., D_g\}$ be a set of disjoint meridinal discs in H_L which are associated with the g handles of H_L. The homeomorphism $f : \partial H_L \to \partial H_R$ is specified—up to ambient isotopy—by the images $C'_j = f(C_j)$ in ∂H_R of the boundaries $C_j = \partial D_j$, for $j = 1, 2, .., g$. Thus each Heegaard splitting can be described by a diagram which shows the set of the characteristic curves $\{C'_j\}$ on the surface ∂H_R. One example of Heegaard diagram is shown in Fig. 3.

Let $\{\gamma_1, \gamma_2, ..., \gamma_g\}$ be a complete set of generators for $\pi_1(H_R)$ which are associated to a complete set of meridinal discs of H_R. The fundamental group of M is specified by adding to the generators $\{\gamma_1, \gamma_2, ..., \gamma_g\}$ the constraints which implement the homotopy triviality condition of the curves $\{C'_j\}$. Indeed, since each curve C_j is homotopically trivial in M, the fundamental group of M admits [20,21] the presentation

$$\pi_1(M) = \langle \, \gamma_1, \gamma_2, ..., \gamma_g \, | \, [C'_1] = 1, ..., [C'_g] = 1 \, \rangle \; , \tag{3.2}$$

where $[C'_j]$ denotes the $\pi_1(H_R)$ homotopy class of C'_j expressed in terms of the generators $\{\gamma_1, \gamma_2, ..., \gamma_g\}$. The classes $[C'_j]$ are determined by the intersections of the boundaries of the meridinal discs of H_L and H_R, which can be inferred from the Heegaard diagram.

3.2. Flat connection in a handlebody

Let us consider the handlebody H_L of the Heegard splitting $M = H_L \cup_f H_R$ of genus g and a corresponding set $\{D_1, D_2, ..., D_g\}$ of disjoint meridinal discs in H_L. For each $j = 1, 2, ..., g$,

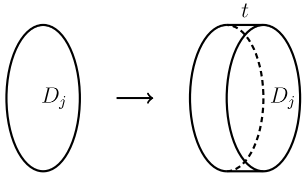


Fig. 4. Disc D_j and the neighbourhood N_j of D_j.

consider a collared neighbourhood N_j of D_j in H_L. As shown in Fig. 4, N_j is homeomorphic with a cylinder $D_j \times [0, \epsilon]$ parametrised as $(z \in \mathbb{C}, |z| \leq 1) \times (0 \leq t \leq \epsilon)$.

The strip $(|z| = 1) \times (0 \leq t \leq \epsilon)$ belongs to the surface ∂H_L. The flat $SU(N)$-connection on H_L we are interested in will be denoted by A_L^0; A_L^0 is vanishing in $H_L - \{N_1, N_2, ..., N_g\}$ and, inside each region N_j, A_L^0 is determined by $\rho(\gamma_j)$. More precisely, suppose that

$$\rho(\gamma_j) = e^{ib_j} \ , \tag{3.3}$$

where the hermitian traceless matrix b_j belongs to the Lie algebra of $SU(N)$. Let $\theta(t)$ be a $\mathcal{C}^\infty$ real function, with $\theta'(t) = d\theta(t)/dt > 0$, satisfying $\theta(0) = 0$ and $\theta(\epsilon) = 1$. Then the value of A_L^0 in the region N_j is given by

$$A_L^0\Big|_{N_j} = b_j \theta'(t) dt \ . \tag{3.4}$$

The orientation of the parametrisation (or the sign in equation (3.4)) is fixed so that the holonomy of the connection (3.4) coincides with expression (3.3). As a consequence of equation (3.4) one has $dA_L^0 = 0$ and also, since $N_j \cap N_k = \emptyset$ for $j \neq k$, one finds $A_L^0 \wedge A_L^0 = 0$.

By construction, the smooth 1-form A_L^0 represents a flat connection on H_L whose holonomies coincide with the matrices that represent the elements of the fundamental group of H_L. The restriction of A_L^0 on the boundary ∂H_L has support on g ribbons and its values are determined by equation (3.4); the j-th ribbon represents a collared neighbourhood of the curve $C_j = \partial D_j$ in ∂H_L. The same construction can be applied to define a flat connection A_R^0 on H_R.

3.3. Flat connection in a 3-manifold

Let us now construct a flat connection A in $M = H_L \cup_f H_R$ which is associated with the representation ρ of $\pi_1(M)$. As far as the value of A on H_L is concerned, one can put

$$A\Big|_{H_L} = A_L^0 \ . \tag{3.5}$$

The image $f * A_L^0$ of A_L^0 under the homeomorphism $f : \partial H_L \to \partial H_R$ does not coincide in general with A_R^0 in ∂H_R. But since $f * A_L^0$ and A_R^0 are associated with the same matrix representation of $\pi_1(\partial H_R)$, the values of $f * A_L^0$ and A_R^0 on ∂H_R are related by a gauge transformation, $f * A_L^0 = U_0^{-1} A_R^0 U_0 - i U_0^{-1} dU_0$, as shown in equation (2.2), in which U_0 must assume the unit value at the base point x_b. Then the map U_0 can smoothly be extended in H_R, let U denote this extension. The value of A on H_R is taken to be

$$A\Big|_{H_R} = U^{-1} A_R^0 U - i U^{-1} dU \ . \tag{3.6}$$

The value of U_0 on the surface ∂H_R represents a fundamental ingredient of our construction, so we now describe how it can be determined. To this end, we need to introduce the concept of developing map.

Let us recall that any flat $SU(N)$-connection A defined in a space X can be locally trivialized because, inside a simply connected neighbourhood of any given point of X, A can be written as $A = -i\Phi^{-1}d\Phi$. The value of Φ coincides with the holonomy of A. When the representation of $\pi_1(X)$ determined by A is not trivial, Φ cannot be extended to the whole space X. A global trivialisation of A can be found in the universal covering $\widehat{X}$ of X; in this case, the map $\Phi : \widehat{X} \to SU(N)$ represents the developing map. For any element γ of $\pi_1(X)$ acting on $\widehat{X}$ by covering transformations, the developing map satisfies

$$\Phi(\gamma \cdot x) = h_\gamma[A] \cdot \Phi(x) \, , \tag{3.7}$$

in agreement with equations (1.6). Now, on the surface ∂H_R we have the two flat connections $f * A_L^0$ and A_R^0 which are related by a gauge transformation, equation (2.2). Thus, for each oriented path $\gamma \subset \partial H_R$ connecting the starting point x_0 with the final point x, the corresponding holonomies are related according to equation (1.6) which takes the form

$$U_0^{-1}(x_0)\, h_\gamma[A_R^0]\, U_0(x) = h_\gamma[f * A_L^0] \, . \tag{3.8}$$

From this equation one obtains $U_0(x) = h_\gamma^{-1}[A_R^0]\, U_0(x_0)\, h_\gamma[f * A_L^0]$. When the starting point x_0 coincides with the base point x_b of the fundamental group, one has $U(x_b) = 1$, and then

$$U_0(x) = h_\gamma^{-1}[A_R^0]\, h_\gamma[f * A_L^0] \quad , \quad \text{for} \quad x \in \partial H_R \, . \tag{3.9}$$

This equation is equivalent to the relation (2.3). Indeed, because of the transformation property (3.7), the combination $\Phi_R^{-1}\Phi_{f*L}$ is invariant under covering translations acting on the universal covering of ∂H_R (and then $\Phi_R^{-1}\Phi_{f*L}$ is really a map from ∂H_R into $SU(N)$), and locally coincides with the product $h_\gamma^{-1}[A_R^0]\, h_\gamma[f * A_L^0]$ appearing in equation (3.9).

4. The invariant

4.1. Proof of Proposition 1

The Chern–Simons function $S[A]$ of the connection (2.4)—whose components in H_L and H_R are shown in equations (3.5) and (3.6)—is given by

$$S[A] = \int_M \mathcal{L}_{CS}(A) = \int_{H_L} \mathcal{L}_{CS}(A) + \int_{H_R} \mathcal{L}_{CS}(A) \, . \tag{4.1}$$

Since $dA_L^0 = 0$ and $A_L^0 \wedge A_L^0 = 0$, one has

$$\int_{H_L} \mathcal{L}_{CS}(A) = \int_{H_L} \mathcal{L}_{CS}(A_L^0) = 0 \, . \tag{4.2}$$

Moreover, a direct computation shows that

$$\int_{H_R} \mathcal{L}_{CS}(A) = \int_{H_R} \mathcal{L}_{CS}(A_R^0) - \frac{i}{8\pi^2} \int_{H_R} d\,\mathrm{Tr}\left[A_R^0 \wedge dU U^{-1}\right] + \frac{1}{24\pi^2} \int_{H_R} \mathrm{Tr}\left[U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU\right] . \tag{4.3}$$

As before, the first term on the r.h.s of equation (4.3) is vanishing

$$\int_{H_R} \mathcal{L}_{CS}(A_R^0) = 0 \, . \tag{4.4}$$

By using equation (2.2), the second term can be written as the surface integral

$$\mathcal{X}[A] = \frac{1}{8\pi^2} \int_{\partial H_R} \mathrm{Tr}\left[U_0^{-1} A_R^0 U_0 \wedge f * A_L^0 \right] . \tag{4.5}$$

By combining equations (4.1)–(4.5) one finally gets

$$\begin{aligned} S[A] = & \frac{1}{8\pi^2} \int_{\partial H_R} \mathrm{Tr}\left[U_0^{-1} A_R^0 U_0 \wedge f * A_L^0 \right] \\ & + \frac{1}{24\pi^2} \int_{H_R} \mathrm{Tr}\left[U^{-1} dU \wedge U^{-1} dU \wedge U^{-1} dU \right] , \end{aligned} \tag{4.6}$$

which implies equation (2.5). This concludes the proof of Proposition 1.

The term $\mathcal{X}[A]$ can be understood as a sort of coloured intersection form, because its value is determined by the trace of the representation matrices—belonging to the Lie algebra of the group—which are associated with the boundaries of the meridinal discs of the two handlebodies which intersect each other in the Heegaard diagram. Indeed, on the surface ∂H_R, A_R^0 is different from zero inside collar neighbourhoods of the boundaries of the meridinal discs of H_R, whereas $f * A_L^0$ is different from zero inside collar neighbourhoods of the images—under f—of the boundaries of the meridinal discs of H_L. Thus, in the computation of $\mathcal{X}[A]$, only the intersection regions of the curves of the Heegaard diagram give nonvanishing contributions. But since the intersections of the boundaries of the meridinal discs of H_L and H_R determine the relations entering the presentation (3.2) of $\pi_1(M)$, an important part of the input, which is involved in the computation of $\mathcal{X}[A]$, is given by the fundamental group presentation (3.2). It turns out that the computation of $\mathcal{X}[A]$ can also be accomplished by means of intersection theory techniques by colouring the de Rham–Federer currents [28,29] of the disks $\{D_j\}$.

When the representation ρ is abelian, $\Gamma[U]$ vanishes and the classical Chern–Simons invariant is completely specified by $\mathcal{X}[A]$ which assumes the simplified form

$$cs[A]\Big|_{abelian} = \mathcal{X}[A]\Big|_{abelian} = \frac{1}{8\pi^2} \int_{\partial H_R} \mathrm{Tr}\left[A_R^0 \wedge f * A_L^0 \right] \quad \mathrm{mod}\ \mathbb{Z} \, . \tag{4.7}$$

4.2. Group volume

The term $\Gamma[U]$ can be interpreted as the 3-volume of the region of the structure group which is bounded by the image of the surface ∂H_R under the map $\Phi_R^{-1} \Phi_{f*L} : \partial H_R \to SU(N)$. In this case also, the combination $\Phi_R^{-1} \Phi_{f*L}$ of the two developing maps, which are associated with $f * A_L^0$ and A_R^0, is characterized by the homeomorphism $f : \partial H_L \to \partial H_R$ which topologically identifies M.

In general, the direct computation of $\Gamma[U]$ is not trivial, and the following properties of $\Gamma[U]$ turn out to be useful. When $U(x)$ can be written as

$$U(x) = W(x)\, Z(x) \, , \tag{4.8}$$

where $W(x) \in SU(N)$ and $Z(x) \in SU(N)$, one obtains

$$\Gamma[U = WZ] = \Gamma[W] + \Gamma[Z] + \frac{1}{8\pi^2} \int_{\partial H_R} \mathrm{Tr}\left[dZZ^{-1} \wedge W^{-1} dW \right] . \tag{4.9}$$

By means of equation (4.9) one can easily derive the relation

$$\Gamma[U = VHV^{-1}] = \Gamma[H] - \frac{1}{8\pi^2} \int_{\partial H_R} \mathrm{Tr}\left[V^{-1} dV \wedge \left(H^{-1} dH + dHH^{-1} \right) \right] + \frac{1}{8\pi^2} \int_{\partial H_R} \mathrm{Tr}\left[V^{-1} dV\, H \wedge V^{-1} dV H^{-1} \right] . \tag{4.10}$$

With a clever choice of the matrices $V(x)$ and $H(x)$, equation (4.10) assumes a simplified form. Indeed any generic map $U(x) \in SU(N)$ can locally be written in the form $U(x) = V(x)H(x)V^{-1}(x)$ where

$$H(x) = \exp(iC(x)) , \tag{4.11}$$

and $C(x)$ belongs to the $(N-1)$-dimensional abelian Cartan subalgebra of the Lie algebra of $SU(N)$. In this case, one has $\Gamma[H] = 0$ and

$$H^{-1}(x) dH(x) = dH(x)\, H^{-1}(x) = i\, dC(x) . \tag{4.12}$$

Therefore relation (4.10) becomes

$$\Gamma[VHV^{-1}] = \frac{1}{8\pi^2} \int_{\partial H_R} \left\{ 2i\, \mathrm{Tr}\left[dC \wedge V^{-1} dV \right] + \mathrm{Tr}\left[e^{-iC}\, V^{-1} dV\, e^{iC} \wedge V^{-1} dV \right] \right\} , \tag{4.13}$$

where it is understood that one possibly needs to decompose the integral into a sum of integrals computed in different regions of ∂H_R where $V(x)$ and $H(x)$ are well defined [30]. Expression (4.13) explicitly shows that the value of $\Gamma[U]$ (modulo integers) is completely specified by the value of U on the surface ∂H_R.

In the case of the structure group $SU(2) \sim S^3$, the computation of $\Gamma[U]$ can be reduced to the computation of the volume of a given polyhedron in a space of constant curvature. Discussions on this last problem can be found, for instance, in the articles [31–38].

4.3. Canonical extension

The reduction of the Wess–Zumino volume $\Gamma[U]$ into a surface integral on ∂H_R can be done in several inequivalent ways, which also depend on the choice of the extension of U_0 from the surface ∂H_R in H_R. Let us now describe the result which is obtained by means of a canonical extension of U_0. We shall concentrate on the structure group $SU(2)$, the generalisation to a generic group $SU(N)$ is quite simple.

Suppose that the value of U_0 on the surface ∂H_R can be written as

$$U_0(x, y) = e^{i\boldsymbol{n}(x,y)\boldsymbol{\sigma}} = e^{i\sum_{a=1}^{3} n^a(x,y)\,\sigma^a} = \cos n(x, y) + i\, \widehat{\boldsymbol{n}}(x, y)\boldsymbol{\sigma}\, \sin n(x, y) , \tag{4.14}$$

where (x, y) designate coordinates of ∂H_R, $n = \left[\sum_{b=1}^{3} n^b n^b\right]^{1/2}$, the components of the unit vector $\widehat{\boldsymbol{n}}$ are given by $\widehat{\boldsymbol{n}}^a = n^a/n$, and $\{\sigma^a\}$ (with $a = 1, 2, 3$) denote the Pauli sigma matrices. The canonical extension of U_0 is defined by

$$U(\tau, x, y) = e^{i\,\tau\, \boldsymbol{n}(x,y)\boldsymbol{\sigma}} \, , \tag{4.15}$$

where the homotopy parameter τ takes values in the range $0 \le \tau \le 1$. When $\tau = 1$ one recovers the expression (4.14), whereas in the $\tau \to 0$ limit one finds $U = 1$. A direct computation gives

$$\mathrm{Tr}\left(U^{-1}\partial_\tau U \left[U^{-1}\partial_x U, U^{-1}\partial_y U\right]\right) = \frac{2i}{n^2} \sin^2(\tau n)\, \mathrm{Tr}\left(\Sigma \left[\partial_y \Sigma\, , \partial_x \Sigma\right]\right) \, , \tag{4.16}$$

in which $\Sigma(x, y) = \sum_{a=1}^{3} n^a(x, y)\sigma^a$. Therefore, by using the identity

$$\int_0^1 d\tau \, \sin^2(\tau n) = \frac{1}{2}\left[1 - \frac{\sin(2n)}{2n}\right] , \tag{4.17}$$

one gets

$$\Gamma[U] = \frac{-i}{8\pi^2} \int_{\partial H_R} \frac{1}{n^2}\left[1 - \frac{\sin(2n)}{2n}\right] \mathrm{Tr}\,(\Sigma\, d\Sigma \wedge d\Sigma) \ . \tag{4.18}$$

This equation will be used in Section 6, Section 7 and Section 8.

4.4. Rationality

As it has already been mentioned, in all the considered examples the value of the $SU(N)$ classical Chern–Simons invariant is given by a rational number. Let us now present a proof of this property for a particular class of 3-manifolds. Suppose that the universal covering $\widetilde{M}$ of the three-manifold M is homeomorphic with S^3, so that M can be identified with the orbit space [39] which is obtained by means of covering translations (acting on S^3) which correspond to the elements of the fundamental group $\pi_1(M)$. Given a flat connection A on M, let us denote by $\widetilde{A}$ the flat connection on $\widetilde{M} \sim S^3$ which is the upstairs preimage of A. By construction, one has

$$S[A]\Big|_M = \frac{1}{|\pi_1(M)|}\, S[\widetilde{A}]\Big|_{S^3} \, , \tag{4.19}$$

where $|\pi_1(M)|$ denotes the order of $\pi_1(M)$. On the other hand, since S^3 is simply connected, one can find a map $\Omega : S^3 \to SU(N)$ such that

$$\widetilde{A} = -i\Omega^{-1} d\Omega \, , \tag{4.20}$$

and then

$$S[\widetilde{A}]\Big|_{S^3} = \frac{1}{24\pi^2} \int_{S^3} \mathrm{Tr}\left(\Omega^{-1}d\Omega \wedge \Omega^{-1}d\Omega \wedge \Omega^{-1}d\Omega\right) = n \, , \tag{4.21}$$

where n is an integer. Equations (4.19) and (4.21) imply

$$cs[A]\Big|_M = \frac{n}{|\pi_1(M)|} \quad \mod \mathbb{Z} \, , \tag{4.22}$$

which shows that, for this type of manifolds, the value of $cs[A]$ is indeed a rational number.

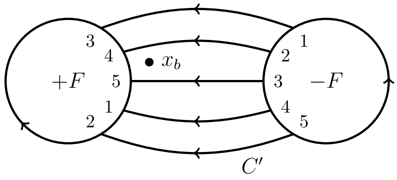


Fig. 5. Heegaard diagram for the lens space $L(5,2)$, with base point x_b displayed.

Let us now present a few examples of computations of $cs[A]$; in the first instance, the representation of the fundamental group of the 3-manifold is abelian, whereas nonabelian representations are considered in the remaining examples.

5. First example

In order to illustrate how to compute $\mathcal{X}[A]$, let us consider the lens spaces $L(p,q)$, where the coprime integers p and q verify $p > 1$ and $1 \leq q < p$. The manifolds $L(p,q)$ admit [4,20] a genus 1 Heegaard splitting presentation, $L(p,q) = H_L \cup_f H_R$ where H_L and H_R are solid tori. The fundamental group of $L(p,q)$ is the abelian group $\pi_1(L(p,q)) = \mathbb{Z}_p$.

5.1. The representation

Let us concentrate, for example, on $L(5,2)$ whose Heegaard diagram is shown in Fig. 5, where the image C' of a meridian C of the solid torus H_L is displayed on the surface ∂H_R. The torus ∂H_R is represented by the surface of a 2-sphere with two removed discs $+F$ and $-F$. The boundaries of $+F$ and $-F$ must be identified (the points with the same label coincide). A possible choice of the base point x_b of the fundamental group is also depicted.

In the solid torus H_L, let the meridian C be the boundary of the meridinal disc $D_L \subset H_L$, which is oriented so that the intersection of D_L with the generator $\gamma_L \subset H_L$ of $\pi_1(H_L)$ is $+1$. Suppose that the representation $\rho : \pi_1(L(5,2)) = \mathbb{Z}_5 \to SU(4)$ is specified by

$$\rho(\gamma_L) = \exp\left[i \frac{2\pi}{5} Y \right] , \tag{5.1}$$

where Y is given by

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} . \tag{5.2}$$

Let $N_L \subset H_L$ be a collared neighbourhood of D_L parametrised by $(z \in \mathbb{C}, |z| \leq 1) \times (0 \leq t \leq \epsilon)$. The flat connection A_L^0 on H_L is vanishing in $H_L - N_L$, whereas the value of A_L^0 in N_L is given by

$$A_L^0 \Big|_{N_L} = \frac{2\pi}{5} Y \theta'(t) dt . \tag{5.3}$$

The restriction of A_L^0 on the boundary ∂H_L is nonvanishing inside a strip which is a collared neighbourhood of C. Therefore the image $f * A_L^0$ of A_L^0 on ∂H_R is different from zero in a collared neighbourhood of C'.

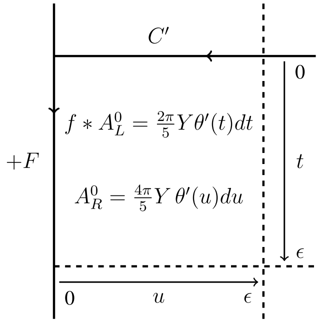


Fig. 6. Values of the connections inside one intersection region.

Let us now consider H_R. The meridinal disc $D_R \subset H_R$ can be chosen in such a way that the boundary of D_R coincides with the boundaries of $+F$ (and $-F$) of Fig. 5. The image on ∂H_R of the corresponding generator γ_R of $\pi_1(H_R)$ is associated to $+F$, and it can be represented by an arrow intersecting the boundary of the disc $+F$ and oriented in the outward direction. As in the previous case, we introduce a collared neighbourhood $N_R \subset H_R$ of D_R parametrised by $(z' \in \mathbb{C}, |z'| \leq 1) \times (0 \leq u \leq \epsilon)$. The flat connection A_R^0 is vanishing in $H_R - N_R$ and, inside N_R, one has

$$A_R^0 \Big|_{N_R} = \widetilde{Y} \, \theta'(u) du \; , \tag{5.4}$$

where $\widetilde{Y}$ represents an element of the Lie algebra of $SU(N)$. The restriction of A_R^0 on the boundary ∂H_R is nonvanishing inside a collared neighbourhood of $\partial(+F)$. The value taken by A_R^0 must be consistent with the given representation $\rho : \pi_1(L(5,2)) \to SU(4)$ which is specified by equation (5.1). In order to determine A_R^0, one can consider a closed path $\gamma \subset \partial H_R$ with base point x_b. One needs to impose that the holonomy of A_R^0 along γ must coincide with the holonomy of $f * A_L^0$ along γ. One then finds $\widetilde{Y} = (4\pi/5)Y$, and consequently

$$A_R^0 \Big|_{N_R} = \frac{4\pi}{5} Y \, \theta'(u) du \; . \tag{5.5}$$

As shown in the Heegaard diagram of Fig. 5, the collar neighbourhood of C' and the collar neighbourhood of $\partial(+F)$—where the connections $f * A_L^0$ and A_R^0 are nonvanishing—intersect in five (rectangular) regions of ∂H_R. Only inside these rectangular regions is the 2-form $A_R^0 \wedge f * A_L^0$ different from zero. As far as the computation of the Chern–Simons invariant is concerned, these five regions are equivalent and give the same contribution to $\mathcal{X}[A]$. The values of the connections inside one of the five rectangular intersection regions are shown in Fig. 6.

In the intersection region shown in Fig. 6, one then finds

$$\int\limits_{\text{one region}} \mathrm{Tr}\left[A_R^0 \wedge f * A_L^0 \right] = -\frac{8\pi^2}{25} \int\limits_0^\epsilon dt \, \theta'(t) \int\limits_0^\epsilon du \, \theta'(u) \, \mathrm{Tr}\left[Y^2 \right] = -\frac{96\pi^2}{25} \; . \tag{5.6}$$

Therefore the value of the classical Chern–Simons invariant which, in this abelian case, takes the form

$$cs[A] = \frac{1}{8\pi^2} \int\limits_{\partial H_R} \mathrm{Tr}\left[A_R^0 \wedge f * A_L^0 \right] \quad \mathrm{mod} \, \mathbb{Z} \, , \tag{5.7}$$

is given by

$$cs[A] = \frac{5 \times \left(-96\pi^2/25\right)}{8\pi^2} \quad \mathrm{mod}\ \mathbb{Z} \;= \frac{3}{5} \quad \mathrm{mod}\ \mathbb{Z}\,. \tag{5.8}$$

5.2. Lens spaces in general

For a generic lens space $L(p,q)$, the corresponding Heegaard diagram has the same structure of the diagram shown in Fig. 5. The curve C' on ∂H_R and the boundary of the disc $(+F)$ give rise to p intersection regions. Since the group $\pi_1(L(p,q))$ is abelian, the analogues of equations (5.3) and (5.5) take the form

$$A_L^0\Big|_{N_L} = \frac{2\pi}{p} M\theta'(t)dt\,, \tag{5.9}$$

and

$$A_R^0\Big|_{N_R} = \frac{2\pi q}{p} M\,\theta'(u)du\,, \tag{5.10}$$

where the matrix M belongs to the Lie algebra of $SU(N)$ and satisfies

$$e^{i2\pi M} = 1\,. \tag{5.11}$$

Therefore the expression of the classical Chern–Simons invariant (5.7) is given by

$$cs[A] = -\frac{1}{8\pi^2}\left\{\frac{(2\pi)^2 q}{p^2}\,\mathrm{Tr}\left(M^2\right) \times p\right\} = -\frac{q}{p}\left[\tfrac{1}{2}\,\mathrm{Tr}\left(M^2\right)\right] \quad \mathrm{mod}\ \mathbb{Z}\,. \tag{5.12}$$

Expression (5.12) is in agreement with the results [16,17] obtained in the case of the abelian Chern–Simons theory, where it has been shown that the value of the Chern–Simons action is specified by the quadratic intersection form on the torsion component of the homology group of the manifold.

6. Second example

Let us consider the 3-manifold Σ_3 which is homeomorphic with the cyclic 3-fold branched covering of S^3 which is branched over the trefoil [20]. Σ_3 admits a Heegaard splitting presentation of genus 2 and the corresponding Heegaard diagram is shown in Fig. 7. The surface ∂H_R is represented by the surface of a 2-sphere with four removed discs: the boundaries of $+F$ and $-F$ (and similarly the boundaries of $+G$ and $-G$) must be identified. In Fig. 7, the two characteristic curves C'_1 and C'_2 are represented by the continuous and the dashed curve respectively, and the base point x_b is also shown.

The two meridinal discs D_{1R} and D_{2R} of H_R are chosen so that their boundaries coincide with the boundaries of the discs $+F$ and $+G$ respectively. The corresponding generators γ_1 and γ_2 of $\pi_1(H_R)$ can be represented by two arrows which are based on the boundaries of $+F$ and $+G$ and oriented in the outward direction. By taking into account the constraints coming from the requirement of homotopy triviality of the curves C'_1 and C'_2, one finds a presentation of the fundamental group of Σ_3,

$$\pi(\Sigma_3) = \langle \gamma_1\,\gamma_2 \,|\, \gamma_1^2 = \gamma_2^2 = (\gamma_1\gamma_2)^2\,\rangle\,. \tag{6.1}$$

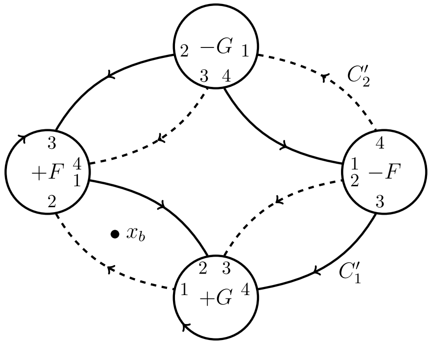


Fig. 7. Heegaard diagram for Σ_3, with base point x_b displayed.

The group $\pi(\Sigma_3)$ is usually called [20] the quaternionic group; it has eight elements which can be denoted by $\{\pm 1, \pm i, \pm j, \pm k\}$, in which $ij = k$, $ki = j$ and $jk = i$.

Let the representation $\rho : \pi_1(\Sigma_3) \to SU(2)$ be given by

$$\gamma_1 \to g_1 = \exp\left[i(\pi/2)\sigma^1\right] = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma^1 \quad ,$$
$$\gamma_2 \to g_2 = \exp\left[i(\pi/2)\sigma^2\right] = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma^2 \quad . \tag{6.2}$$

The corresponding flat connection A_R^0 on H_R vanishes in $H_R - \{N_{1R}, N_{2R}\}$, where N_{1R} and N_{2R} are collared neighbourhoods of the two meridinal discs $\{D_{1R}, D_{2R}\}$ of H_R, and

$$A_R^0 = \begin{cases} A_R^0\Big|_{N_{1R}} = \frac{\pi}{2}\sigma^1\theta'(u)du \quad ; \\ \\ A_R^0\Big|_{N_{2R}} = \frac{\pi}{2}\sigma^2\theta'(v)dv \quad . \end{cases} \tag{6.3}$$

With the choice of the base point x_b shown in Fig. 7, the flat connection A_L^0 on H_L turns out to be

$$A_L^0 = \begin{cases} A_L^0\Big|_{N_{1L}} = \frac{\pi}{2}\sigma^1\theta'(t)dt \quad ; \\ \\ A_L^0\Big|_{N_{2L}} = \frac{\pi}{2}\sigma^2\theta'(s)ds \quad ; \end{cases} \tag{6.4}$$

where N_{1L} and N_{2L} are collared neighbourhoods of the two meridinal discs $\{D_{1L}, D_{2L}\}$ of H_L, and A_L^0 vanishes on $H_L - \{N_{1L}, N_{2L}\}$. Note that, on the surface ∂H_R, $f * A_L^0$ is nonvanishing inside the two ribbons which constitute collared neighbourhoods of the curve C_1' and C_2', whereas A_R^0 is nonvanishing inside the collared neighbourhoods of ∂D_{1R} and ∂D_{2R}. In the region of the surface ∂H_R where both $f * A_L^0$ and A_R^0 are vanishing, the values taken by the map U_0 entering equation (2.2) are shown in Fig. 8.

We now need to specify the values of $U_0 = \Phi_R^{-1}\Phi_{f*L}$ in the eight intersections regions of ∂H_R where both $f * A_L^0$ and A_R^0 are not vanishing. The value of U_0 is defined in equation (3.9). In each region, we shall introduce the variables X and Y according to a correspondence of the type

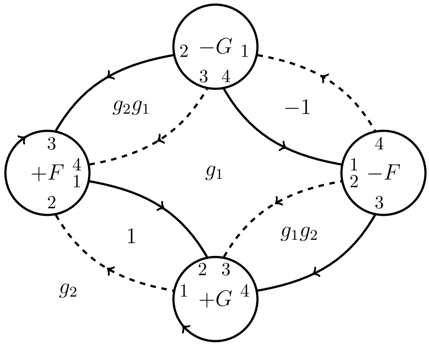
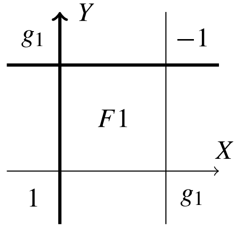


Fig. 8. Values of the map U_0 in the region where $f * A_L^0$ and A_R^0 are vanishing.

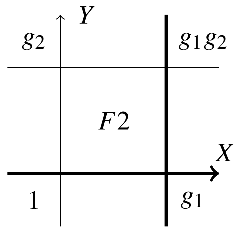
$$
\begin{aligned}
dX &= \theta'(t)dt \quad , \quad 0 \le X \le 1 \\
dY &= \theta'(u)du \quad , \quad 0 \le Y \le 1 \, .
\end{aligned}
\tag{6.5}
$$

The intersection regions are denoted as $\{F1, F2, F3, F4, G1, G2, G3, G4\}$ with the convention that, for instance, the region $F3$ (or $G3$) is a rectangle in which one of the vertices is the point denoted by the number 3 of the boundary of the disk $+F$ (or $+G$). The values of U_0 in these eight regions are in order; in each of the corresponding pictures, the values of U_0 at the vertices of the rectangle are also reported.

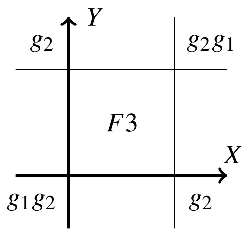
$$[F1]: \qquad U_0 = e^{i\frac{\pi}{2}(X+Y)\sigma^1}$$



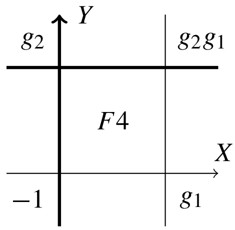
$$[F2]: \qquad U_0 = e^{i\frac{\pi}{2}X\sigma^1} e^{i\frac{\pi}{2}Y\sigma^2}$$



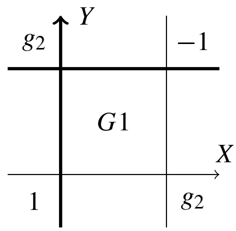
$$[F3]: \qquad U_0 = e^{i\frac{\pi}{2}(1-X-Y)\sigma^1} e^{i\frac{\pi}{2}\sigma^2}$$



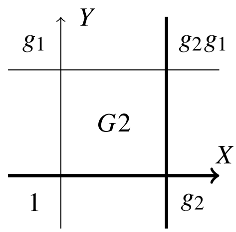
$$[F4]: \qquad U_0 = e^{-i\frac{\pi}{2}X\sigma^1} e^{i\frac{\pi}{2}(2-Y)\sigma^2}$$



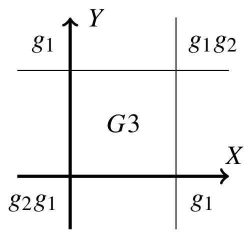
$$[G1]: \qquad U_0 = e^{i\frac{\pi}{2}(X+Y)\sigma^2}$$



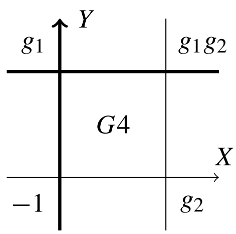
$$[G2]: \qquad U_0 = e^{i\frac{\pi}{2}X\sigma^2} e^{i\frac{\pi}{2}Y\sigma^1}$$



$$[G3]: \qquad U_0 = e^{i\frac{\pi}{2}(1-X-Y)\sigma^2} e^{i\frac{\pi}{2}\sigma^1}$$



$$[G4]: \qquad U_0 = e^{i\frac{\pi}{2}(2-X)\sigma^2} e^{-i\frac{\pi}{2}Y\sigma^1}$$



By using the value of U_0 in the eight intersections regions $\{F1, F2, F3, F4, G1, G2, G3, G4\}$, the contribution $\mathcal{X}[A]$, defined in equation (4.5), of the Chern–Simons invariant can easily be determined. One finds

$$\mathcal{X}[A] = \frac{1}{8\pi^2}\,\mathrm{Tr}\Big\{-\tfrac{\pi}{4}\sigma^1\sigma^1 + \tfrac{\pi}{4}\sigma^1\sigma^2 + \tfrac{\pi}{4}\sigma^1\sigma^1 + \tfrac{\pi}{4}\sigma^1\sigma^2 \\ -\tfrac{\pi}{4}\sigma^2\sigma^2 + \tfrac{\pi}{4}\sigma^2\sigma^1 + \tfrac{\pi}{4}\sigma^2\sigma^2 + \tfrac{\pi}{4}\sigma^2\sigma^1\Big\} = 0\,. \qquad (6.6)$$

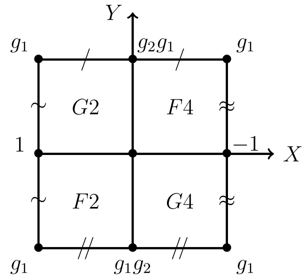


Fig. 9. Images of the regions $\{F2, F4, G2, G4\}$ parametrised in equation (6.7).

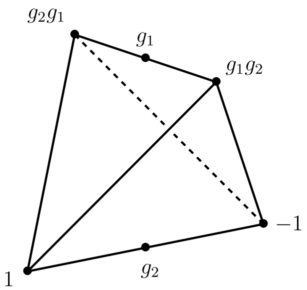


Fig. 10. Closed surface specified by $\Phi_R^{-1}\Phi_{f*L} : \partial H_R \to SU(2)$.

Let us now consider the computation of the contribution $\Gamma[A]$ of equation (2.7). Under the map $U_0 = \Phi_R^{-1}\Phi_{f*L} : \partial H_R \to SU(2)$, the images of the rectangles $\{F1, F3, G1, G3\}$ are degenerate (they have codimension two). Whereas the images of the remaining four rectangles $\{F2, F4, G2, G4\}$ constitute a closed surface of genus zero in $SU(2) \sim S^3$.

As sketched in Fig. 9, the set of the images of $\{F2, F4, G2, G4\}$ can be globally parametrised by new variables $-1 \le X \le 1$ and $-1 \le Y \le 1$ according to the relations

$$
\begin{aligned}
&[G2]: \quad && U_0 = e^{i\frac{\pi}{2}(X+1)\sigma^2} e^{i\frac{\pi}{2}Y\sigma^1} = e^{i\frac{\pi}{2}X\sigma^2} e^{-i\frac{\pi}{2}Y\sigma^1}\, i\sigma^2 = \widetilde{U}_0\, i\sigma^2 \, , \\
&[F4]: \quad && U_0 = e^{-i\frac{\pi}{2}Y\sigma^1} e^{i\frac{\pi}{2}(1+X)\sigma^2} = e^{-i\frac{\pi}{2}Y\sigma^1} e^{i\frac{\pi}{2}X\sigma^2}\, i\sigma^2 = \widetilde{U}_0\, i\sigma^2 \, , \\
&[F2]: \quad && U_0 = e^{-i\frac{\pi}{2}Y\sigma^1} e^{i\frac{\pi}{2}(1+X)\sigma^2} = e^{-i\frac{\pi}{2}Y\sigma^1} e^{i\frac{\pi}{2}X\sigma^2}\, i\sigma^2 = \widetilde{U}_0\, i\sigma^2 \, , \\
&[G4]: \quad && U_0 = e^{i\frac{\pi}{2}(X+1)\sigma^2} e^{i\frac{\pi}{2}Y\sigma^1} = e^{i\frac{\pi}{2}X\sigma^2} e^{-i\frac{\pi}{2}Y\sigma^1}\, i\sigma^2 = \widetilde{U}_0\, i\sigma^2 \, .
\end{aligned}
\tag{6.7}
$$

The images of $\{F2, F4, G2, G4\}$ are glued as shown in Fig. 9; the edges which are labelled by the same symbol must be identified. Therefore, the closed surface which is specified by $\Phi_R^{-1}\Phi_{f*L} : \partial H_R \to SU(2)$ is topologically equivalent to the tetrahedron shown in Fig. 10. Relations (6.7) show that $U_0(X, Y)$ can globally be written as $U_0(X, Y) = \widetilde{U}_0(X, Y)\, i\sigma^2$, therefore if $\widetilde{U}$ denotes the extension of $\widetilde{U}_0$ in H_R, one has

$$
\Gamma[U] = \Gamma[\widetilde{U}] \, . \tag{6.8}
$$

In order to determine the value of $\Gamma[\widetilde{U}]$ one can use symmetry arguments.

The manifold $SU(2) \sim S^3$ can be represented as the union of two equivalent (with the same volume) balls in $\mathbb{R}^3$ of radius $\pi/2$ with identified boundaries, $SU(2) \sim \mathcal{B}_1 \cup \mathcal{B}_2$. Indeed each element of $SU(2)$ can be written as

$$
e^{i\boldsymbol{\theta\sigma}} = \cos(|\boldsymbol{\theta}|) + i\widehat{\boldsymbol{\theta}}\boldsymbol{\sigma}\, \sin(|\boldsymbol{\theta}|) \, ,
$$

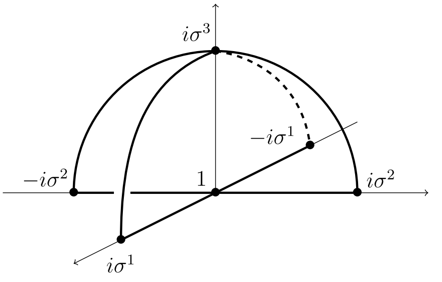


Fig. 11. $\widetilde{U}_0$ images in $\mathcal{B}_1$ of the boundaries of the regions $\{F2, F4, G2, G4\}$.

where $|\boldsymbol{\theta}| = [\boldsymbol{\theta}\boldsymbol{\theta}]^{1/2}$ and $\widehat{\boldsymbol{\theta}} = (\boldsymbol{\theta}/|\boldsymbol{\theta}|)$. The ball $\mathcal{B}_1$ contains the elements with $0 \leq |\boldsymbol{\theta}| \leq \pi/2$, and $\mathcal{B}_2$ contains the elements with $(\pi/2) \leq |\boldsymbol{\theta}| \leq \pi$.

The application $\widetilde{U}_0 : \partial H_R \to SU(2)$ maps the boundaries of the rectangles $\{F2, F4\}$ and $\{G2, G4\}$ into the eight edges in $\mathcal{B}_1$ shown in Fig. 11. Equation (6.7) and the picture of Fig. 11 demonstrate that the surface $\widetilde{U}_0 : \partial H_R \to SU(2)$ is symmetric under rotations of $\pi/2$ around the σ^3 axis and bounds a region $\mathcal{R}$ of $SU(2)$ which is contained in half of the ball $\mathcal{B}_1$. According to the reasoning of Section 4.4, the volume of this region $\mathcal{R}$ must take the value $n/8$, where n is an integer. This integer n is less than 4 because $\mathcal{R}$ is contained inside $\mathcal{B}_1$ and satisfies $n \leq 2$ because $\mathcal{R}$ is contained inside half of $\mathcal{B}_1$. Finally, the value $n = 2$ is excluded because a direct inspection shows that $\mathcal{R}$ does not cover the upper half-part of $\mathcal{B}_1$ completely. Therefore one finally obtains

$$\Gamma[U] = \Gamma[\widetilde{U}] = \frac{1}{8} \, . \tag{6.9}$$

In Section 8 it will be shown that equation (6.9) is also in agreement with a direct computation of $\Gamma[U]$ that we have performed by means of the canonical expression (4.18). Finally, the validity of the result (6.9) has also been verified by means of a numerical evaluation of the integral (4.18). To sum up, in the case of the manifold Σ_3 with the specified representation (6.2) of its fundamental group, the value of the classical Chern–Simons invariant is given by

$$cs[A] = \frac{1}{8} \quad \text{mod}\ \mathbb{Z} \, . \tag{6.10}$$

7. Poincaré sphere

The Poincaré sphere $\mathcal{P}$ admits a genus 2 Heegaard splitting presentation. The corresponding Heegaard diagram [20] is shown in Fig. 12. One of the two characteristic curves, $C_1' = f(C_1)$, is described by the continuous line, whereas the second curve $C_2' = f(C_2)$ is represented by the dashed path; x_b designates the base point for the fundamental group.

Let the generators $\{\gamma_1, \gamma_2\}$ of $\pi_1(H_R)$ be associated with $+F$ and $+G$ respectively and oriented in the outward direction. According to the Heegaard diagram of Fig. 12, the homotopy class of C_1' is given by $\gamma_1^{-4}\gamma_2\gamma_1\gamma_2$, whereas the class of C_2' is equal to $\gamma_2^{-2}\gamma_1\gamma_2\gamma_1$. Therefore the fundamental group of $\mathcal{P}$ admits the presentation

$$\pi_1(\mathcal{P}) = \langle\, \gamma_1, \gamma_2 \,|\, \gamma_1^5 = \gamma_2^3 = (\gamma_1\gamma_2)^2 \,\rangle \, , \tag{7.1}$$

which corresponds to the binary icosahedral (or dodecahedral) group of order 120. Since the abelianization of $\pi_1(\mathcal{P})$ is trivial, $\mathcal{P}$ is a homology sphere. A nontrivial representation $\rho : \pi_1(\mathcal{P}) \to SU(2)$ is given [40,41] by

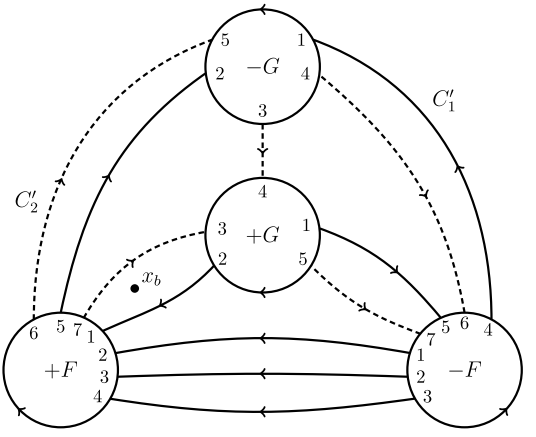


Fig. 12. Heegaard diagram for the Poincaré sphere.

$$
\begin{aligned}
\rho(\gamma_1) &= g_1 = e^{ib_1} = \exp\left[i \frac{\pi}{5} \sigma \right] , \\
\rho(\gamma_2) &= g_2 = e^{ib_2} = \exp\left[i \frac{\pi}{3} \widetilde{\sigma} \right] ,
\end{aligned} \tag{7.2}
$$

where

$$
\begin{aligned}
\sigma &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \\
\widetilde{\sigma} &= r \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sqrt{1 - r^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \\
r &= \frac{\cos(\pi/3)\,\cos(\pi/5)}{\sin(\pi/3)\,\sin(\pi/5)} .
\end{aligned} \tag{7.3}
$$

Equation (7.2) specifies the values of A_R^0,

$$
A_R^0 = \begin{cases} b_1\, \theta'(t_1) dt_1 & \text{inside a neighbourhood of } +F \,; \\ b_2\, \theta'(t_2) dt_2 & \text{inside a neighbourhood of } +G \,; \\ 0 & \text{otherwise} \;. \end{cases} \tag{7.4}
$$

The values of A_L^0 are determined by equation (7.2) and by the choice of the base point. Indeed, let the generators $\{\lambda_1, \lambda_2\}$ of $\pi_1(H_L)$ be associated with C_1 and C_2 respectively. Then, from the Heegaard diagram and the position for the base point, one finds

$$
\begin{aligned}
\rho(\lambda_1) &= g_1 = e^{ib_1} = \exp\left[i \frac{\pi}{5} \sigma \right] , \\
\rho(\lambda_2) &= g_2 = e^{ib_2} = \exp\left[i \frac{\pi}{3} \widetilde{\sigma} \right] .
\end{aligned} \tag{7.5}
$$

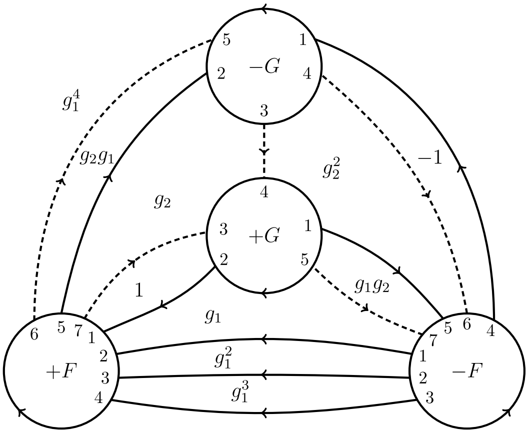


Fig. 13. Values of U_0 in the region where $f * A_L^0$ and A_R^0 vanish.

Consequently, the image of A_L^0 under the gluing homeomorphism f takes values

$$f * A_L^0 = \begin{cases} b_1\,\theta'(u_1)du_1 & \text{inside a neighbourhood of } C_1' \,; \\ b_2\,\theta'(u_2)du_2 & \text{inside a neighbourhood of } C_2' \,; \\ 0 & \text{otherwise} \;. \end{cases} \tag{7.6}$$

One can now determine the map $U_0 = \Phi_R^{-1}\Phi_{f*L} : \partial H_R \to SU(2)$. In the region of the surface ∂H_R where both $f * A_L^0$ and A_R^0 are vanishing, the values of U_0 are shown in Fig. 13. By using the method illustrated in the previous examples, one can compute the classical Chern–Simons invariant. The intersection component is given by

$$\begin{aligned} \mathcal{X}[A] = \frac{1}{8\pi^2}\Big\{ & -4\,\mathrm{Tr}\,(b_1b_1) - 2\,\mathrm{Tr}\,(b_2b_2) + 4\,\mathrm{Tr}\,(b_1b_2) \\ & + \mathrm{Tr}\left(b_1g_2b_1g_2^{-1}\right) + \mathrm{Tr}\left(b_2g_1b_2g_1^{-1}\right)\Big\} \\ = -\frac{2}{15} + \frac{1}{2}&\left[\frac{1}{5}\frac{\cos(\pi/3)}{\sin(\pi/5)} + \frac{1}{3}\frac{\cos(\pi/5)}{\sin(\pi/3)}\right]^2 \;. \end{aligned} \tag{7.7}$$

The image of the map $\Phi_R^{-1}\Phi_{f*L} : \partial H_R \to SU(2)$ is a genus 0 surface in the group $SU(2)$. We skip the details, which anyway can be obtained from the Heegaard diagram and equations (7.2)–(7.6). Numerical computations of the integral (4.18) give the following value of the Wess–Zumino volume (with 10^{-10} precision)

$$\Gamma[A] = 0.0090687883\cdots \;. \tag{7.8}$$

Therefore, the value of the classical Chern–Simons invariant associated with the representation (7.2) of $\pi_1(\mathcal{P})$ turns out to be

$$cs[A] = -0.0083333333\cdots = -\frac{1}{120} \quad \mathrm{mod}\,\mathbb{Z}\,, \tag{7.9}$$

where the last identity is a consequence of the fact that $|\pi_1(\mathcal{P})| = 120$. The result (7.9) has also been obtained by means of a complete computation of the integral (4.18); this issue is elaborated in Section 8.

8. Computations of the Wess–Zumino volume

The computation of $\Gamma[U]$ by means of the canonical expression (4.18) presents general features that are consequences of our construction of the flat connection A by means of a Heegaard splitting presentation of M. This allows the derivation of universal formulae of the classical Chern–Simons invariant for quite wide classes of manifolds. We present here one example; details will be produced in a forthcoming article.

Let us consider the set of Seifert spaces $\Sigma(m, n, -2)$ of genus zero with three singular fibres which are characterised by the integer surgery coefficients $(m, 1)$, $(n, 1)$ and $(2, -1)$. The manifolds $\Sigma(m, n, -2)$ admit [4,40] a genus two Heegaard splitting $M = H_L \cup_f H_R$ and their fundamental group can be presented as

$$\pi_1(M) = \langle \gamma_1, \gamma_2 \,|\, \gamma_1^m = \gamma_2^n = (\gamma_1\gamma_2)^2 \rangle \, , \tag{8.1}$$

for nontrivial positive integers m and n. The manifold Σ_3 discussed in Section 6 and the Poincaré manifold $\mathcal{P}$ considered in Section 7 are examples belonging to this class of manifolds. Let us introduce the representation of $\pi_1(M)$ in the group $SU(2)$ given by

$$\begin{aligned} \gamma_1 &\to g_1 = \exp[i\theta_1\sigma] \quad , \\ \gamma_2 &\to g_2 = \exp[i\theta_2\widetilde{\sigma}] \quad , \end{aligned} \tag{8.2}$$

where σ and $\widetilde{\sigma}$ are combinations of the sigma matrices satisfying $\sigma^2 = 1 = \widetilde{\sigma}^2$, and

$$g_1^m = g_2^n = (g_1g_2)^2 = -1 \, . \tag{8.3}$$

In this case, the value of the surface integral (4.5) is given by

$$\mathcal{X}[A] = -\frac{1}{4}\left\{ m\left[\frac{\theta_1}{\pi}\right]^2 + n\left[\frac{\theta_2}{\pi}\right]^2 - 2\left[\left(\frac{\theta_2}{\pi}\right)\frac{\cos\theta_1}{\sin\theta_2} + \left(\frac{\theta_1}{\pi}\right)\frac{\cos\theta_2}{\sin\theta_1}\right]^2 \right\} . \tag{8.4}$$

As it has been shown in the previous examples, the image of the map $\Phi_R^{-1}\Phi_{f*L} : \partial H_R \to SU(2)$ is a genus 0 surface in the group $SU(2)$. The corresponding Wess–Zumino volume turns out to be

$$\Gamma[U] = \frac{1}{4}\left\{ \frac{1}{2} - 2\left[\left(\frac{\theta_2}{\pi}\right)\frac{\cos\theta_1}{\sin\theta_2} + \left(\frac{\theta_1}{\pi}\right)\frac{\cos\theta_2}{\sin\theta_1}\right]^2 \right\} . \tag{8.5}$$

So that the value of the classical Chern–Simons invariant for the manifolds $\Sigma(m, n, -2)$ reads

$$cs[A] = -\frac{1}{4}\left\{ m\left[\frac{\theta_1}{\pi}\right]^2 + n\left[\frac{\theta_2}{\pi}\right]^2 - \frac{1}{2}\right\} \quad \text{mod } \mathbb{Z} \, . \tag{8.6}$$

When $m = n = 2$, expression (8.6) gives the value of the classical Chern–Simons invariant appearing in equation (6.9); and for $m = 5, n = 3$, expression (8.6) coincides with equation (7.9). Equation (8.6) is valid for generic values of m and n; for those particular values of m and n such that $\Sigma(m, n, -2)$ is a Seifert homology sphere, our equation (8.6) is in agreement with the results of Fintushel and Stern [5] and Kirk and Klassen [6] for Seifert spheres.

9. Conclusions

Given a $SU(N)$ representation ρ of the fundamental group of a 3-manifold M, we have shown how to define a corresponding flat connection A on M such that the holonomy of A coincides with ρ. Our construction is based on a Heegaard splitting presentation of M, so that the relationship between A and the topology of M is displayed. The relative classical Chern–Simons invariant $cs[A]$ is naturally decomposed into the sum of two contributions: a sort of coloured intersection form, which is specified by the Heegaard diagram, and a Wess–Zumino volume of a region of $SU(N)$ which is determined by the non-commutative structure of the ρ representation of $\pi_1(M)$. A canonical expression for the Wess–Zumino volume, as function of the boundary data exclusively, has been produced. A few illustrative examples of flat connections and of classical Chern–Simons invariant computations have been presented.

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