

Three-dimensional ferromagnetic CP^{N-1} models

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We investigate the critical behavior of three-dimensional ferromagnetic CP^{N-1} models, which are characterized by a global $U(N)$ and a local $U(1)$ symmetry. We perform numerical simulations of a lattice model for $N = 2, 3$, and 4. For $N = 2$ we find a critical transition in the Heisenberg $O(3)$ universality class, while for $N = 3$ and 4 the system undergoes a first-order transition. For $N = 3$ the transition is very weak and a clear signature of its discontinuous nature is only observed for sizes $L \gtrsim 50$. We also determine the critical behavior for a large class of lattice Hamiltonians in the large- N limit. The results confirm the existence of a stable large- N CP^{N-1} fixed point. However, this evidence contradicts the standard picture obtained in the Landau-Ginzburg-Wilson (LGW) framework using a gauge-invariant order parameter: the presence of a cubic term in the effective LGW field theory for any $N \geq 3$ would usually be taken as an indication that these models generically undergo first-order transitions.

I. INTRODUCTION

CP^{N-1} models are a class of quantum field theories in which the fundamental field is a complex N -component unit vector $\mathbf{z}(\mathbf{x})$, associated with an element of the complex projective manifold CP^{N-1} . They are characterized by a global $U(N)$ symmetry

$$\mathbf{z}(\mathbf{x}) \rightarrow U\mathbf{z}(\mathbf{x}) \quad U \in U(N), \quad (1)$$

and a local $U(1)$ gauge symmetry

$$\mathbf{z}(\mathbf{x}) \rightarrow e^{i\Lambda(\mathbf{x})}\mathbf{z}(\mathbf{x}). \quad (2)$$

The corresponding statistical field theory is defined by

$$\mathcal{Z} = \int [d\mathbf{z}] \exp \left[- \int d\mathbf{x} \mathcal{L}(\mathbf{z}) \right],$$

$$\mathcal{L} = \frac{1}{2g} \overline{D_\mu \mathbf{z}} \cdot D_\mu \mathbf{z}, \quad D_\mu = \partial_\mu + iA_\mu, \quad (3)$$

where $A_\mu = i\bar{\mathbf{z}} \cdot \partial_\mu \mathbf{z}$ is a composite gauge field, which transforms as $A_\mu(\mathbf{x}) \rightarrow A_\mu(\mathbf{x}) - \partial_\mu \Lambda(\mathbf{x})$ under the gauge transformations (2). For $N = 2$ the CP^1 field theory is locally isomorphic to the $O(3)$ non-linear σ model with the identification $s^a = \sum_{ij} \bar{z}^i \sigma_{ij}^a z^j$, where $a = 1, 2, 3$ and σ^a are the Pauli matrices.

Three-dimensional (3D) CP^{N-1} models emerge as effective theories of $SU(N)$ quantum antiferromagnets [1–7] and of scalar electrodynamics with a compact $U(1)$ gauge group. The two-dimensional (2D) CP^{N-1} model is instead an interesting theoretical laboratory to study quantum field theories of fundamental interactions as it shares several features with quantum chromodynamics (QCD), the theory that describes the hadronic strong interactions [8, 9].

The simplest lattice formulation of CP^{N-1} model is obtained by a straightforward discretization of the continuum theory (3). One considers N -component complex

unit vectors $\mathbf{z}_\mathbf{x}$ defined on the sites of a lattice and the nearest-neighbor Hamiltonian

$$H_s = -J \sum_{\mathbf{x}\mu} |\bar{\mathbf{z}}_\mathbf{x} \cdot \mathbf{z}_{\mathbf{x}+\hat{\mu}}|^2, \quad (4)$$

where the sum is over the lattice sites \mathbf{x} and the lattice directions μ ($\mu = 1, \dots, d$), and $\hat{\mu} = \hat{1}, \hat{2}, \dots$ are unit vectors along the lattice directions.

In three dimensions CP^{N-1} models are expected to undergo a finite-temperature transition between a high- and a low-temperature phase. Its nature may be investigated by resorting to the Landau-Ginzburg-Wilson (LGW) field-theoretical approach [10–14]. In this framework the critical features are uniquely specified by the nature of the order parameter associated with the critical modes, by the symmetries of the model, and by the symmetries of the phases coexisting at the transition, the so-called symmetry-breaking pattern. In the presence of gauge symmetries, the traditional LGW approach starts by considering a gauge-invariant order parameter, effectively integrating out the gauge degrees of freedom, and by constructing a LGW field theory that is invariant under the global symmetries of the original model.

The order parameter of the transition in ferromagnetic CP^{N-1} models is usually identified with the gauge-invariant site variable

$$Q_\mathbf{x}^{ab} = \bar{z}_\mathbf{x}^a z_\mathbf{x}^b - \frac{1}{N} \delta^{ab}, \quad (5)$$

which is a hermitian and traceless $N \times N$ matrix. It transforms as

$$Q_\mathbf{x} \rightarrow U^\dagger Q_\mathbf{x} U, \quad (6)$$

under the global $U(N)$ transformations (1). The order-parameter field in the corresponding LGW theory is therefore a traceless hermitian matrix field $\Phi^{ab}(\mathbf{x})$, which can be formally defined as the average of $Q_\mathbf{x}^{ab}$ over a large

but finite lattice domain. The LGW field theory is obtained by considering the most general fourth-order polynomial in Φ consistent with the $U(N)$ symmetry (6):

$$\begin{aligned} \mathcal{H}_{\text{LGW}} = & \text{Tr}(\partial_\mu \Phi)^2 + r \text{Tr} \Phi^2 \\ & + w \text{tr} \Phi^3 + u (\text{Tr} \Phi^2)^2 + v \text{Tr} \Phi^4. \end{aligned} \quad (7)$$

A continuous transition is possible if the renormalization-group (RG) flow computed in the LGW theory has a stable fixed point.

For $N = 2$, the cubic term in Eq. (7) vanishes and the two quartic terms are equivalent. Therefore, one recovers the $O(3)$ -symmetric LGW theory, consistently with the equivalence between the CP^1 and the Heisenberg model. For $N \geq 3$, the cubic term is generically expected to be present. The presence of a cubic term in the LGW Hamiltonian is usually taken as an indication that phase transitions occurring in this class of systems are generically of first order. Indeed, a straightforward mean-field analysis shows that the transition is of first order in four dimensions. The nature of the transition should not change sufficiently close to four dimensions, as long as statistical fluctuations are small. In particular, it is usually assumed that the four-dimensional mean-field result also applies in three dimensions. In this scenario, continuous transitions may still occur, but they require a fine tuning of the microscopic parameters leading to the effective cancellation of the cubic term. If this occurs, the critical behavior is controlled by the stable fixed point of the RG flow of the LGW theory (7) with $w = 0$. Such a LGW model was studied in detail in Ref. [15]. For $N = 3$ it is equivalent to the $O(8)$ vector ϕ^4 theory, therefore there is a stable fixed point, while no fixed point was identified for $N \geq 4$ by using field-theoretical methods.

The prediction of a first-order transition is apparently contradicted by recent numerical studies [5, 16, 17]. Refs. [16, 17] report evidence of a continuous transition in a loop model supposed to belong to the same universality class as that of the 3D CP^2 model. Their results hint at a new CP^2 universality class, [17] characterized by the correlation-length critical exponent $\nu = 0.536(12)$. This result is not consistent with the standard LGW picture we reported above. In the standard scenario in which generic CP^2 models undergo first-order transitions due to the cubic term in the corresponding LGW theory, a continuous transition may only emerge from a tuning of the parameters achieving its cancellation. Therefore, the continuous transition associated with the effective cancellation of the cubic term is described by the stable fixed point of the LGW theory without cubic term, which belongs to the $O(8)$ vector universality class, whose critical exponent [15] $\nu = 0.85(2)$ definitely differs from that reported in Ref. [17]. Therefore, the emergence of a new CP^2 universality class is in apparent contradiction with the predictions of the standard LGW picture, in which the presence of a cubic term in the effective LGW Hamiltonian implies that first-order phase transitions. To make the existence of the CP^2 universality class plausible even in the presence of a cubic term, Ref. [17] proposed a

double expansion around $N = 2$ (where the cubic term vanishes) and $\epsilon = 4 - d$, arguing that a continuous transition may be possible for values of N sufficiently close to $N = 2$.

An additional apparent contradiction emerges when considering CP^{N-1} models in the large- N limit. In this limit one may argue that 3D CP^{N-1} models may undergo a continuous transition, because the corresponding continuum field theory is expected to share the same critical behavior as the abelian Higgs model for an N -component complex scalar field coupled to a dynamical $U(1)$ gauge field [9]. This equivalence is conjectured to extend to finite N at the critical point [9]. The RG flow of the abelian Higgs model presents a stable fixed point for a sufficiently large number of components [18]. More precisely, in the perturbative ϵ expansion ($\epsilon = 4 - d$) one finds a stable fixed point for $N > N_c$, with [9, 18] $N_c = 90 + 24\sqrt{15} + O(\epsilon)$. Thus, for large values of N , 3D CP^{N-1} models may undergo a continuous transition, in the same universality class as that occurring in the abelian Higgs model.

The above results may suggest that the critical modes at the transition are not exclusively associated with the gauge-invariant order parameter Q defined in Eq. (5). Other features, for instance the gauge degrees of freedom, may become relevant, requiring an effective description different from that of the LGW theory (7).

We mention that the LGW description apparently fails also in other systems in which a gauge symmetry is present, for instance in antiferromagnetic 3D CP^{N-1} and RP^{N-1} models [19, 20], although the discrepancies are not related to the presence of a cubic term in the corresponding LGW theory.

In this paper we discuss the critical behavior of ferromagnetic 3D CP^{N-1} models, presenting additional numerical and analytical results, which may help clarify the above apparent contradictions between the actual behavior of lattice CP^{N-1} models and the LGW approach. We consider an alternative lattice formulation of 3D CP^{N-1} models, which contains an additional $U(1)$ gauge variable and is linear with respect to all lattice variables, and perform numerical simulations for $N = 2$, $N = 3$, and $N = 4$. Moreover, we reconsider the large- N limit, presenting analytical results for the phase diagram. Our results for $N = 2, 3, 4$ are consistent with the standard LGW picture. Indeed, while the transition is continuous for $N = 2$, for $N = 3$ and 4 the transition is of first order (for $N = 4$ we confirm the results of Ref. [21]), although very weak for $N = 3$.

The paper is organized as follows. In Sec. II we define the 3D CP^{N-1} model we consider and the observables that are determined in the Monte Carlo simulations. In Sec. III we summarize the main features of the finite-size scaling (FSS) theory at continuous and first-order transitions, which allow us to distinguish the nature of the transitions observed in lattice 3D CP^{N-1} models. Then, in Secs. IV, V, and VI, we report a numerical study of the 3D CP^{N-1} models for $N = 2$, $N = 4$, and finally

for $N = 3$ (the most controversial case), respectively. For $N = 2$ the transition is continuous and belongs to the $O(3)$ vector universality class, while it is of first order for $N = 4$. For $N = 3$, the transition is also of first order, but it is so weak that its first-order nature only emerges when relatively large systems are considered. In Sec. VII we discuss the large- N limit of lattice CP^{N-1} models. Finally, in Sec. VIII we summarize our main results and draw our conclusions. In App. A we discuss the large- N behavior of a general class of CP^{N-1} models.

II. LATTICE THREE-DIMENSIONAL CP^{N-1} MODELS

In our study we consider a lattice formulation, in which gauge invariance is guaranteed by introducing a $U(1)$ link variable $\lambda_{\mathbf{x},\mu} \equiv e^{i\theta_{\mathbf{x},\mu}}$ [22–24]. The Hamiltonian is given by

$$H_\lambda = -tN \sum_{\mathbf{x},\mu} (\bar{\mathbf{z}}_{\mathbf{x}} \cdot \lambda_{\mathbf{x},\mu} \mathbf{z}_{\mathbf{x}+\hat{\mu}} + \text{c.c.}), \quad (8)$$

where the sum is over all lattice points \mathbf{x} and lattice directions μ . In the simulations we set $t = 1$. The factor N is introduced for convenience; with this definition, the large- N limit is defined by taking $N \rightarrow \infty$ keeping β fixed [25].

One can easily check that Hamiltonian (8) is invariant under the global and local $U(1)$ transformations (1) and (2). Moreover, by integrating out the gauge field, one can rewrite the partition function as

$$Z = \sum_{\{z,\lambda\}} e^{-\beta H_\lambda} = \sum_{\{z\}} \prod_{\mathbf{x},\mu} I_0(2\beta N t |\bar{\mathbf{z}}_{\mathbf{x}} \cdot \mathbf{z}_{\mathbf{x}+\hat{\mu}}|), \quad (9)$$

where $I_0(x)$ is a modified Bessel function. The corresponding effective Hamiltonian is

$$H_{\lambda,\text{eff}} = -\beta^{-1} \sum_{\mathbf{x},\mu} \ln I_0(2\beta N t |\bar{\mathbf{z}}_{\mathbf{x}} \cdot \mathbf{z}_{\mathbf{x}+\hat{\mu}}|). \quad (10)$$

We recall that $I_0(x) = I_0(-x)$, $I_0(x) = 1 + x^2/4 + O(x^4)$, and $I_0(x) \approx e^x/\sqrt{2\pi x}$ for large x . The above formulas imply that, independently of the sign of t , the model defined by the Hamiltonian (10), or equivalently Hamiltonian (8), is ferromagnetic, as is the formulation (4) with $J > 0$. For $N = 2$ Hamiltonian (10) is a variant of the standard $O(3)$ -symmetric (Heisenberg) spin model, which is equivalent to formulation (4).

The lattice formulation (8) is numerically more convenient than the formulation (4). The main reason is that Hamiltonian (8) is linear with respect to all variables $\mathbf{z}_{\mathbf{x}}$ and $\lambda_{\mathbf{x},\mu}$, unlike the standard Hamiltonian (4). This leads to notable advantages for Monte Carlo (MC) simulations. For linear Hamiltonians one can use overrelaxed algorithms [26–28], which are more efficient than the standard Metropolis algorithm, the only one that

can be straightforwardly used for the nonlinear Hamiltonian (4). We employ overrelaxed updates obtained by a stochastic mixing of microcanonical and standard Metropolis updates of the lattice variables [29].

In our numerical study we consider cubic lattices of linear size L with periodic boundary conditions. We compute the energy density and the specific heat, defined respectively as

$$E = \frac{1}{NV} \langle H_\lambda \rangle, \quad C = \frac{1}{N^2 V} (\langle H_\lambda^2 \rangle - \langle H_\lambda \rangle^2), \quad (11)$$

where $V = L^3$. We consider correlations of the gauge invariant operator $Q_{\mathbf{x}}^{ab}$ defined in Eq. (5). Its two-point correlation function is defined as

$$G(\mathbf{x} - \mathbf{y}) = \langle \text{Tr} Q_{\mathbf{x}} Q_{\mathbf{y}} \rangle, \quad (12)$$

where the translation invariance of the system has been taken into account. The susceptibility and the correlation length are defined as

$$\chi = \sum_{\mathbf{x}} G(\mathbf{x}) = \tilde{G}(\mathbf{0}), \quad (13)$$

$$\xi^2 \equiv \frac{1}{4 \sin^2(\pi/L)} \frac{\tilde{G}(\mathbf{0}) - \tilde{G}(\mathbf{p}_m)}{\tilde{G}(\mathbf{p}_m)}, \quad (14)$$

where $\tilde{G}(\mathbf{p}) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} G(\mathbf{x})$ is the Fourier transform of $G(\mathbf{x})$, and $\mathbf{p}_m = (2\pi/L, 0, 0)$ is the minimum nonzero lattice momentum. We also consider the Binder parameter

$$U = \frac{\langle \mu_2^2 \rangle}{\langle \mu_2 \rangle^2}, \quad \mu_2 = \frac{1}{V^2} \sum_{\mathbf{x},\mathbf{y}} \text{Tr} Q_{\mathbf{x}} Q_{\mathbf{y}}. \quad (15)$$

III. SUMMARY OF FINITE-SIZE SCALING THEORY

In this section we summarize the finite-size scaling (FSS) theory at continuous and first-order transitions. It will be useful later to distinguish the nature of the transition in 3D CP^{N-1} models.

A. Continuous transitions

At continuous transitions the FSS limit is obtained by taking $\beta \rightarrow \beta_c$ and $L \rightarrow \infty$ keeping

$$X \equiv (\beta - \beta_c) L^{1/\nu} \quad (16)$$

fixed, where β_c is the inverse critical temperature and ν is the correlation-length exponent. Any RG invariant quantity R , such as $R_\xi \equiv \xi/L$ and U , is expected to asymptotically behave as

$$R(\beta, L) = f_R(X) + O(L^{-\omega}), \quad (17)$$

where $f_R(X)$ is a function, which is universal apart from a trivial normalization of its argument and which only

depends on the shape of the lattice and on the boundary conditions. In particular, the quantity $R^* \equiv f_R(0)$ is universal. The approach to the asymptotic behavior is controlled by the universal exponent $\omega > 0$, which is associated with the leading irrelevant RG operator.

Assuming that the scaling function of a RG invariant quantity R_1 is monotonic—this is generally the case for R_ξ and therefore we will usually set $R_1 = R_\xi$ —we may also write

$$R_2(\beta, L) = F_R(R_1) + O(L^{-\omega}), \quad (18)$$

for any $R_2 \neq R_1$, where $F_R(x)$ is a universal scaling function as well. Eq. (18) is particularly convenient, as it allows a direct check of universality, without the need of tuning any parameter.

In order to estimate the exponent η , one may analyze the FSS behavior of the susceptibility. It scales as

$$\chi(\beta, L) \sim L^{2-\eta} [f_\chi(X) + O(L^{-\omega})], \quad (19)$$

or, equivalently, as

$$\chi(\beta, L) \sim L^{2-\eta} [F_\chi(R_\xi) + O(L^{-\omega})]. \quad (20)$$

The behavior of the specific heat at the transition is [14]

$$C(\beta, L) \approx C_{\text{reg}}(\beta) + L^{\alpha/\nu} f_C(X), \quad (21)$$

where $C_{\text{reg}}(\beta)$ denotes the regular background, which is an analytic function of β . This contribution is the dominant one for $\alpha < 0$, or, correspondingly, for $\nu > 2/3$. When $\alpha > 0$, we may also write the above equation as

$$C(\beta, L) \approx L^{\alpha/\nu} F_C(R_\xi) + O(L^{-\alpha/\nu}, L^{-\omega}). \quad (22)$$

B. First-order transitions

At first-order transitions the probability distributions of the energy and of the magnetization are expected to show a double peak for large values of L . Therefore, two peaks in the distributions are often taken as an indication of a first-order transition. However, as discussed, e.g., in Refs. [30–33] and references therein, the observation of two maxima in the distribution of the energy is not sufficient to conclude that the transition is a first-order one. For instance, in the two-dimensional Potts model with $q = 3$ and $q = 4$ [31, 32], double-peak distributions are observed, even if the transition is known to be continuous. Analogously, in the 3D Ising model the distribution of the magnetization has two maxima [34]. In order to identify definitely a first-order transition, it is necessary to perform a more careful analysis of the large- L scaling behavior of the distributions or, equivalently, of the specific heat and the of Binder cumulants [35–39].

If E_+ and E_- are the values of the energy corresponding to the two maxima of the energy-density distribution, the latent heat Δ_h is given by $\Delta_h = E_+ - E_-$. An alternative estimate of the latent heat can be obtained from

the FSS of the specific heat C . According to the standard phenomenological theory [35] of first-order transitions, for a lattice of size L there exists a value $\beta_{\text{max},C}(L)$ of β where C takes its maximum value $C_{\text{max}}(L)$. For large volumes, we have

$$C_{\text{max}}(L) = V \left[\frac{1}{4} \Delta_h^2 + O(1/V) \right], \quad (23)$$

$$\beta_{\text{max},C}(L) - \beta_c \approx c V^{-1}, \quad (24)$$

where $V = L^d$.

The Binder parameter U can also be used to characterize a first-order transition. As discussed in Ref. [36] (see also Ref. [38] for the extension to models with continuous symmetries), the distribution of the order parameter is also expected to show two peaks at M_+ and M_- , $M_- < M_+$, with $M_- \rightarrow 0$ as $L \rightarrow \infty$ since there is no spontaneous magnetization in the high-temperature phase. As a consequence, the behavior of the Binder parameter $U(\beta, L)$ at fixed L must show a maximum $U_{\text{max}}(L)$ at fixed L (for sufficiently large L) at $\beta = \beta_{\text{max},U}(L) < \beta_c$ with

$$U_{\text{max}} = a V + O(1), \quad (25)$$

$$\beta_{\text{max},U}(L) - \beta_c \approx b V^{-1}. \quad (26)$$

Note that FSS also holds at first-order transitions [38, 40–42], although it is more sensitive to the geometry and and to the nature of the boundary conditions [39]; for instance, FSS differs for boundary conditions that favor or do not favor the different phases coexisting at the transition [43, 44]. In the case of cubic systems with periodic boundary conditions, FSS behavior is typically characterized by an effective exponent $\nu = 1/d = 1/3$ (thus $\alpha/\nu = d = 3$). An exponent ν larger than $1/d = 1/3$ indicates a continuous transition.

IV. NUMERICAL ANALYSIS OF THE 3D CP¹ MODEL

To begin with, we discuss the 3D CP¹ lattice model (8), corresponding to $N = 2$. As already mentioned in the introduction, the transition in 3D CP¹ models should belong to the 3D O(3) vector (Heisenberg) universality class, whose universal features are known with high accuracy [14]. Accurate estimates of the critical exponents have been obtained by various methods; we quote [45, 46]

$$\nu_h = 0.7117(5), \quad \eta_h = 0.0378(3), \quad (27)$$

so that $\alpha_h = 2 - 3\nu_h = -0.1351(15)$. Moreover, accurate results have also been obtained for the RG invariant quantities R_ξ and U . For cubic systems with periodic boundary conditions, we have [45]

$$R_\xi^* = 0.5639(2), \quad U^* = 1.1394(3), \quad (28)$$

at the critical point.

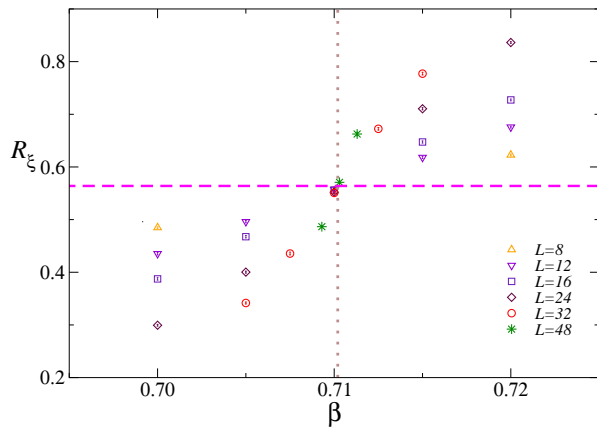


FIG. 1: Estimates of $R_\xi \equiv \xi/L$ for the CP^1 lattice model and several lattice sizes L up to $L = 48$. The horizontal dashed line corresponds to the universal value $R_\xi^* = 0.5639(2)$ of R_ξ obtained for the 3D Heisenberg universality class. The vertical dotted line indicates our best estimate $\beta_c = 0.7102(1)$ of the critical point.

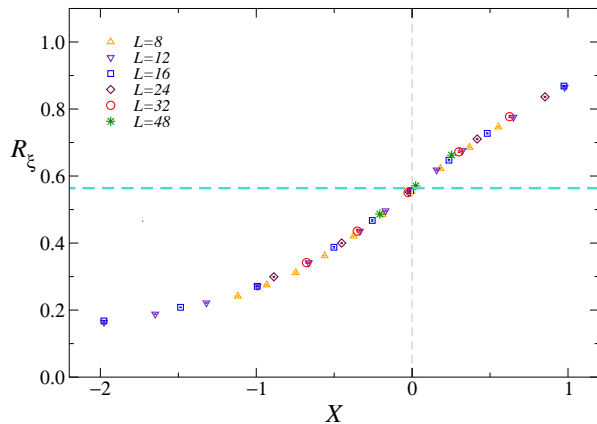


FIG. 2: Scaling plot of R_ξ for the CP^1 model, versus $X \equiv (\beta - \beta_c)L^{1/\nu_h}$, using $\nu_h = 0.7117$ and $\beta_c = 0.7102$. The horizontal dashed line corresponds to the universal value $R_\xi^* = 0.5639(2)$ at the critical point ($X = 0$) for the 3D Heisenberg universality class.

In order to verify that the critical behavior of the lattice CP^1 model (8) belongs to the 3D Heisenberg universality class, we show that the FSS behavior in the CP^1 model has the same universal features (same critical exponents and same values for R_ξ^* and U^*) as in the 3D Heisenberg model.

Figure 1 shows our results for the ratio $R_\xi = \xi/L$. The data corresponding to different value of L have a crossing point, which provides us a first estimate of the critical temperature, $\beta_c \approx 0.710$. The slopes of the curves are related to the critical exponent ν . Their behavior is nicely consistent with the Heisenberg value. An accurate estimate of the critical point is obtained by assuming the Heisenberg critical exponents and fitting the data to

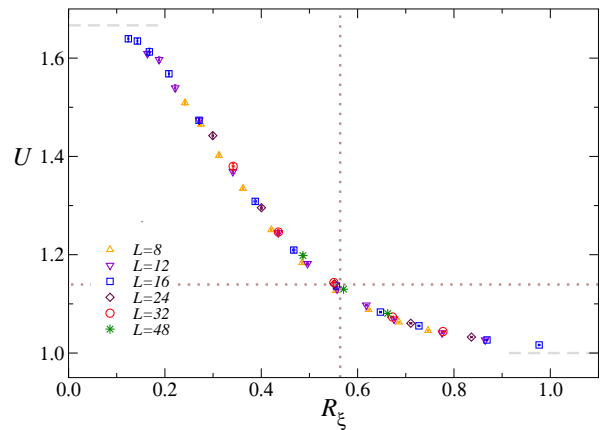


FIG. 3: Plot of U versus R_ξ for the CP^1 model. The horizontal and vertical dotted lines correspond to the universal values $U^* = 1.1394(3)$ and $R_\xi^* = 0.5639(2)$, which are the universal values of R_ξ and U at the critical point for the 3D Heisenberg universality class, respectively.

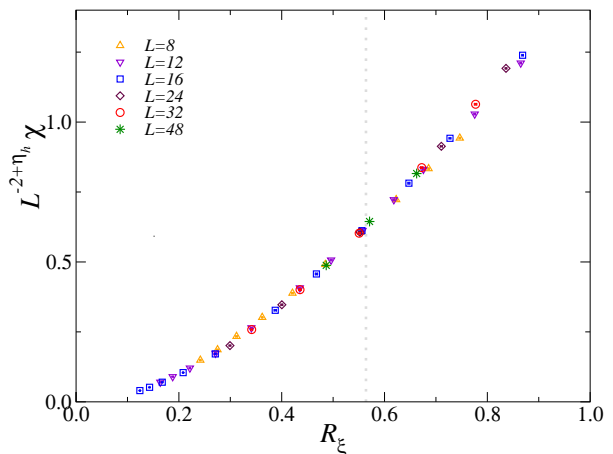


FIG. 4: Scaling behavior of the susceptibility χ for the CP^1 model: plot of $L^{-2+\eta_h}\chi$ versus R_ξ , using the 3D Heisenberg value $\eta_h = 0.0378$.

Eq. (17). In particular, we fit R_ξ to the simple ansatz

$$R_\xi = R_\xi^* + c_1 X, \quad X = (\beta - \beta_c)L^{1/\nu_h}, \quad (29)$$

using the known estimates of R_ξ^* and ν_h . In the fit we only consider the data belonging to a small interval around β_c , where the behavior of R_ξ looks linear. We obtain $\beta_c = 0.7102(1)$. Fig. 2 shows R_ξ versus $X \equiv (\beta - \beta_c)L^{1/\nu_h}$. The quality of the collapse of the data onto a unique scaling curve provides a striking evidence that the exponent ν for the lattice CP^1 model (8) is that of the Heisenberg universality class. Scaling corrections are very small; they are expected to decay as $L^{-\omega_h}$ with [14, 45, 47] $\omega_h = 0.78(1)$. We also note the agreement of the $X = 0$ value of the curves with the estimate (28) of R_ξ^* .

The Heisenberg nature of the transition is also supported by the data for the Binder parameter U . In Fig. 3

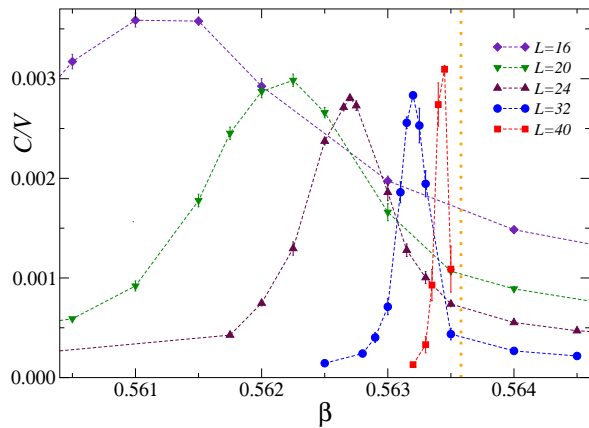


FIG. 5: Plot of the ratio C/L^3 versus β for several lattice sizes L up to $L = 40$, where C is the specific heat. Results for the CP^3 lattice model. The vertical dotted line corresponds to the estimate $\beta_c \approx 0.5636$, obtained by extrapolating the position of the maximum of the specific heat, using Eq. (24). The dotted lines connecting the data for the same size L are only meant to guide the eye.

we plot U versus R_ξ . The data scale on a single curve, in agreement with Eq. (18), and U takes the value U^* reported in Eq. (28) when $R_\xi = R_\xi^*$.

The scaling behavior of the susceptibility is also in agreement with Heisenberg behavior. If we fix η_h to the Heisenberg value, we observe very good scaling, see Fig. 4. Since $\alpha_h < 0$, the leading asymptotic behavior of the specific heat is given by the nonuniversal regular part, see Eq. (21). Thus, C does not show any particular feature that can be used to identify the universality class.

We may safely conclude that the above numerical FSS analysis confirms that the critical behavior of the lattice formulation (8) for $N = 2$ belongs to the 3D $O(3)$ vector universality class, as expected.

V. NUMERICAL ANALYSIS OF THE 3D CP^3 MODEL

In this section we discuss the CP^3 lattice model, i.e. model (4) for $N = 4$. This model was already considered in Ref. [21], where the transition was identified as being of first order. Here, we present additional simulations on larger lattices, which confirm the first-order nature of the finite-temperature transition, in agreement with the predictions of the corresponding LGW field theory.

In Fig. 5 we report the specific heat up to $L = 40$. For each L , it shows a maximum which increases approximately as the volume, thus approaching the asymptotic behavior predicted by Eq. (23) for a first-order transition. We extrapolate the position $\beta_{\max,C}$ of the maximum using Eq. (24), obtaining $\beta_c = 0.5636(1)$ for the transition point. Moreover, using Eq. (23) we can also estimate the latent heat, obtaining $\Delta_h = 0.11(1)$.

The first-order nature of the transition is also sup-

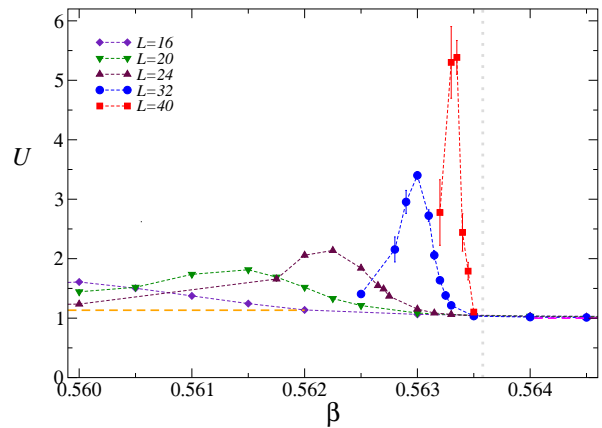


FIG. 6: Estimates of the Binder parameter U for the CP^3 lattice model, for several lattice sizes L up to $L = 40$. The vertical dotted line corresponds to the estimate $\beta_c \approx 0.5636$, while the horizontal dashed line corresponds to $U_h = 17/15$, the high-temperature value of U . The dotted lines connecting the data for the same size L are only meant to guide the eye.

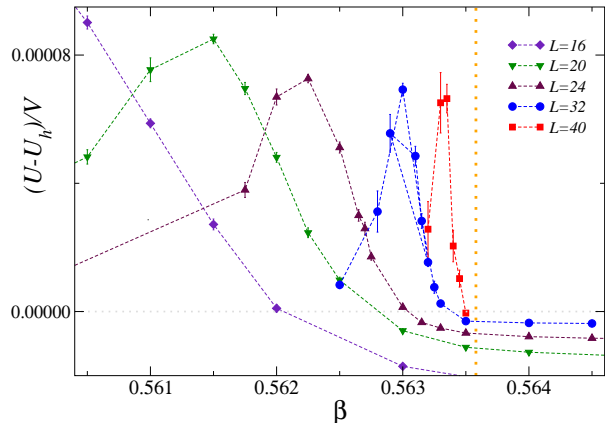


FIG. 7: Estimates of $(U - U_h)/L^3$ in the CP^3 lattice model for several lattice sizes L up to $L = 40$. Here U is the Binder parameter and $U_h = 17/15$ its high-temperature value. The vertical dotted line corresponds to the estimate $\beta_c \approx 0.5636$. The dotted lines connecting the data for the same size L are only meant to guide the eye.

ported by the behavior of the Binder parameter U . For each L , data show a peak at a value $\beta_{\max,U}(L)$, where U is significantly larger than the high-temperature and low-temperature values given by

$$U_h \equiv \lim_{\beta \rightarrow 0} U = \frac{N^2 + 1}{N^2 - 1} = \frac{17}{15}, \quad (30)$$

$$U_l \equiv \lim_{\beta \rightarrow \infty} U = 1.$$

Data are in substantial agreement with the predictions (25) and (26). This is clearly shown in Fig. 7, where we plot $(U - U_h)/L^3$ ($U_h = 17/15$ is the high-temperature value). As L increases the maximum of such quantity apparently approaches a finite nonzero value, in agreement with Eq. (25). Note that the subtraction of the constant

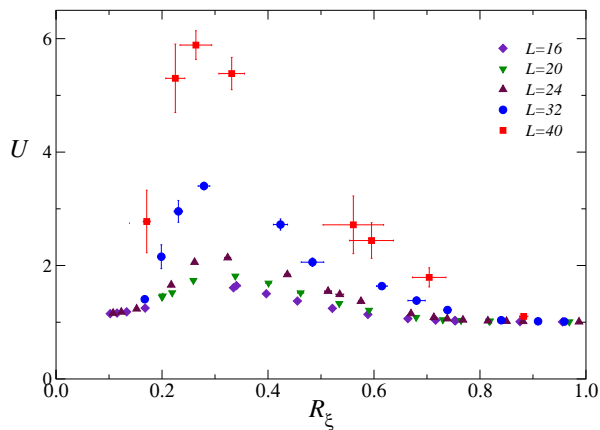


FIG. 8: Plot of the Binder parameter U versus R_ξ for the CP^3 lattice model, for several lattice sizes L up to $L = 40$. Data do not scale, at variance with what happens in the CP^1 model, characterized by a continuous transition.

U_h does not change the asymptotic L^3 behavior. It is natural to expect that it provides a reasonable approximation of the $O(1)$ contribution in Eq. (25) (we subtracted $U_h = 17/15$ instead of $U_l = 1$, because the maximum is located in the high-temperature phase). Moreover, the large- L extrapolation of the position of the maximum using Eq. (26) is consistent with the value obtained from the analysis of the specific heat.

Finally, in Fig. 8 we plot U versus R_ξ . The data do not converge to an asymptotic curve, at variance with what happens in the CP^1 model where the transition is continuous, see Fig. 3.

In conclusion, numerical results provide a robust evidence that the finite-temperature phase transition is of first order in the CP^3 model (4). This confirms the results of Ref. [21], and, in particular, the standard LGW predictions. We recall that a first-order transition was also found in Refs. [16, 17] for an alternative lattice loop formulation corresponding to the 3D CP^3 model.

VI. NUMERICAL ANALYSIS OF THE 3D CP^2 MODEL

We now study the behavior of the 3D CP^2 model, which is more controversial. On the one hand, the usual LGW scenario predicts a first-order transition, as observed in the CP^3 model; on the other hand, the finite-size scaling analysis [16, 17] of a loop model that is expected to be in the same universality class as that of the lattice CP^2 model favors a continuous transition with critical exponents $\nu_n = 0.536(13)$, $\alpha_n = 0.39(4)$, and $\eta_n = 0.23(2)$. As we shall see, our FSS analysis of the lattice CP^2 model up to $L = 96$ favors a weak first-order transition.

To identify the transition temperature, we consider the ratio R_ξ as a function of β . As shown in Figure 9, the data for different values of L clearly show a cross-

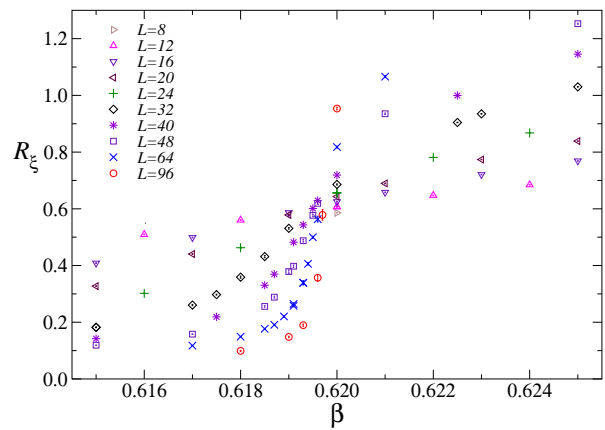


FIG. 9: Estimates of $R_\xi \equiv \xi/L$ for the CP^2 lattice model and several lattice sizes L up to $L = 96$.

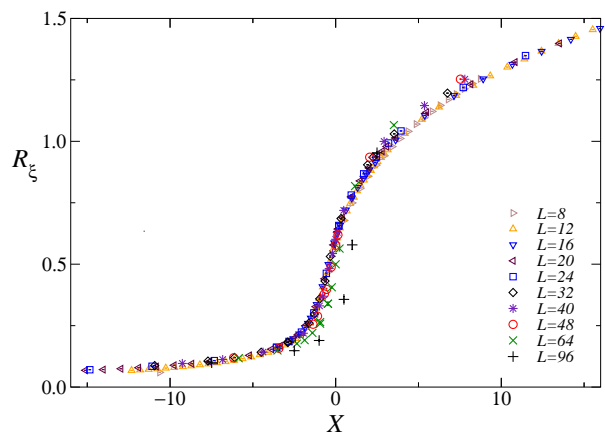


FIG. 10: Scaling plot of R_ξ versus $X \equiv (\beta - \beta_c)L^{1/\nu_n}$, using $\nu_n = 0.536$, which is the exponent obtained in Ref. [17], and $\beta_c = 0.6195$, which is obtained by fitting the data around β_c (up to $L = 48$) to Eq. (29), keeping ν_n fixed. Results for the CP^2 model. Data up to $L = 48$ apparently collapse onto a single curve, but discrepancies appear when considering $L = 64$ and $L = 96$ data; no significant improvement is observed by only shifting the value of β_c .

ing point, indicating a finite-temperature transition for $\beta \approx 0.620$. If the CP^2 model has a continuous transition, $R_\xi(\beta, L)$ should scale as predicted by Eq. (17). In this case, we expect the transition to be in the same universality class as that discussed in Ref. [17], and hence exponents should be the same for the two models. Therefore, we have analyzed our data setting $\nu = 0.536$, the estimate of Ref. [17], and keeping β_c as a free parameter. We obtain a reasonable collapse of the data corresponding to sizes $L \leq 48$. Systematic deviations are instead observed for the data with $L = 96$ and, to a lesser extent, with $L = 64$, suggesting the presence of a crossover for $L \approx 100$. To provide a more convincing evidence that the behavior for $L \leq 48$ is only a transient small-size behavior, we have repeated the determination of β_c , using only the data corresponding to $L \leq 48$. We obtain

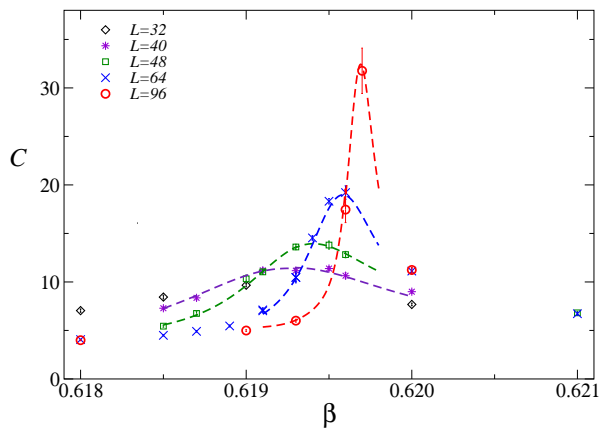


FIG. 11: Specific heat C for the CP^2 lattice model as a function of β , for several lattice sizes L , up to $L = 96$. The dashed lines are interpolations obtained using reweighting techniques [48].

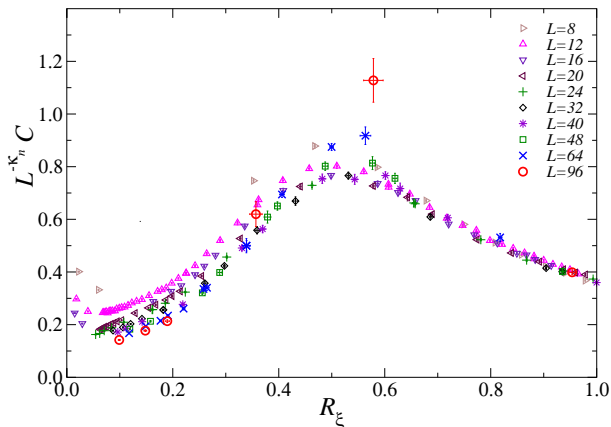


FIG. 12: Plot of $L^{-\kappa_n} C$ versus R_ξ for the CP^2 lattice model. We set $\kappa_n = 0.73$, obtained using the relation $\kappa_n \equiv \alpha_n/\nu_n = (2/\nu_n - 3)$ and the estimate $\nu_n = 0.536(13)$ of Ref. [17]. For a continuous transition with $\alpha > 0$, data should asymptotically collapse onto a single curve, see Eq. (22).

$\beta_c \approx 0.6195$. The corresponding scaling plot is reported in Fig. 10. Up to $L = 48$, the scaling is good, but the largest-size data obtained on lattices with $L = 64, 96$ are clearly not consistent with a continuous transition with $\nu = 0.536$. We have also tested if the data for the susceptibility χ defined in Eq. (13) satisfy the scaling behavior (20), using the estimate $\eta_n = 0.23(2)$ of Ref. [17] (η is the only free parameter). Scaling is quite poor, even if we only consider data with $L \leq 48$. For this set of small-size results, a reasonable scaling is observed if we take a significantly smaller value of η , $\eta \lesssim 0.1$. However, the results for $L = 96$ appear to be off the curve even for this value of η .

Similar conclusions are reached from the analysis of the specific heat C , reported in Fig. 11. In Fig. 12, we plot $L^{-\kappa_n} C$ as a function of R_ξ , where $\kappa_n = \alpha_n/\nu_n = (3/\nu_n - 1)$. Using [17] $\nu_n = 0.536$, we fix $\kappa_n \approx 0.73$. For a

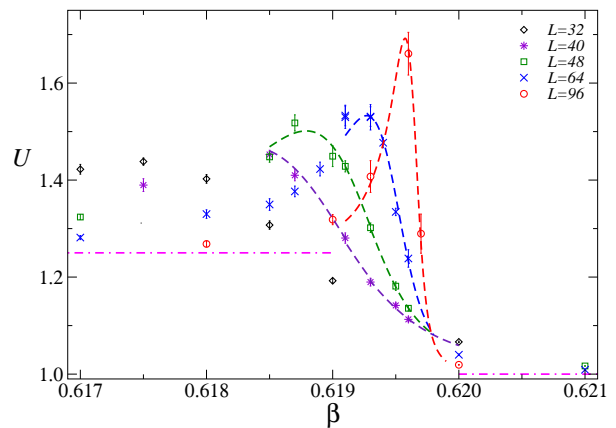


FIG. 13: Estimates of the Binder parameter U for the CP^2 lattice model and several lattice sizes L up to $L = 96$. The horizontal dashed lines correspond to $U = 5/4$ and $U = 1$, which are the asymptotic values of U for $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, respectively. The interpolating dashed lines have been obtained using reweighting techniques [48].

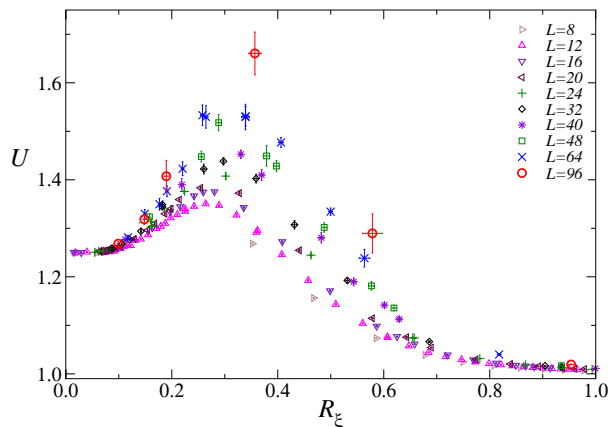


FIG. 14: The Binder parameter U versus R_ξ for the CP^2 model and several lattice sizes L up to $L = 96$.

continuous transition, all data should asymptotically fall onto a single curve, see Eq. (22). The data up to $L = 48$ are in reasonable agreement with the predicted scaling behavior, but this is not the case for those corresponding to $L = 64$ and 96 . The maximum $C_{\max}(L)$, which can be computed quite precisely using the reweighting method of Ref. [48], increases indeed faster than $L^{0.73}$, although we are still very far from observing the asymptotic behavior $C_{\max}(L) \sim L^3$ appropriate for first-order transitions.

The strongest indication for a first-order transition comes from the anomalous behavior of the Binder parameter U . The numerical results, reported in Fig. 13, show that the maximum $U_{\max}(L)$ does not converge to a finite value as L increases, at variance with what is expected for a continuous transition. It is also useful to plot U versus R_ξ . For a continuous transition data should collapse onto a single curve (note that there are no free parameters), and indeed they do in the CP^1 model, see Fig. 3.

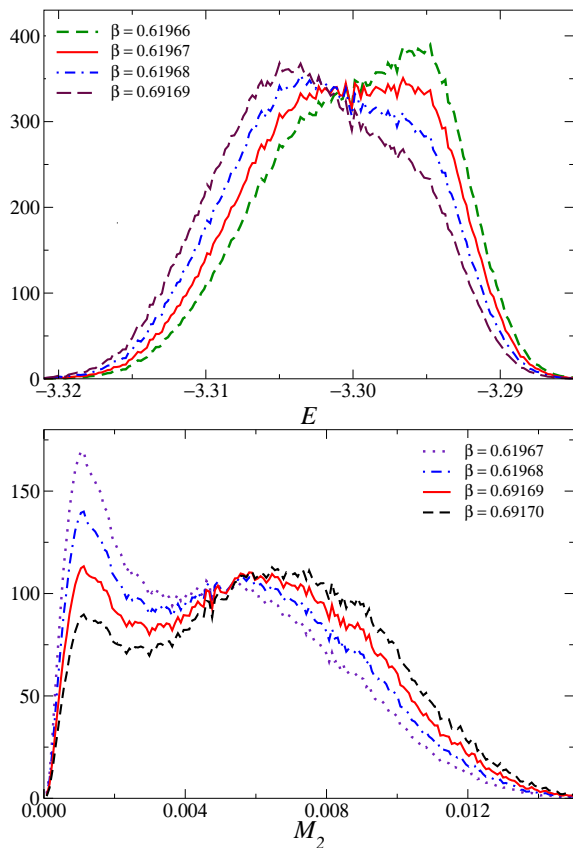


FIG. 15: Distributions of the energy density (top) and of μ_2 defined in Eq. (15) (bottom); see Eq. (31). We report results for several values of β close to the transition point and $L = 96$. The distributions have been estimated using reweighting techniques [48].

Instead, here no evidence of scaling emerges, see Fig. 14. The plot is similar to that obtained for the CP^3 model, see Fig. 8, where the transition is of first order. Note that the plot of U versus R_ξ is apparently the most appropriate one to identify the nature of the transition. Indeed, at variance with the previous analyses, here there are no transient deceiving effects: even if we only consider data with, say, $L \leq 32$, we would still observe a poor scaling behavior in the interval $0.3 \lesssim R_\xi \lesssim 0.6$.

Finally, we consider the distributions of the energy density and of the quantity μ_2 , defined in Eq. (15):

$$\begin{aligned} P_E(E) &= \langle \delta[E - H_\lambda / (NV)] \rangle, \\ P_M(M_2) &= \langle \delta(M_2 - \mu_2) \rangle. \end{aligned} \quad (31)$$

Up to $L = 64$, the distribution $P_M(M_2)$ does not show a double-peak structure, although, for $L = 48$ and $L = 64$, one can identify a somewhat flat top region, which is analogous to that observed for the energy when $L = 96$ (see below). Two distinct peaks are only identified for $L = 96$, see Fig. 15. In the case of the energy density instead, even for $L = 96$ we do not observe a double-peak structure. However, the distribution corresponding to $\beta = 0.61967$ shows an intermediate flat top region,

suggesting that two peaks will appear for largest values of L . The width of the flat region (we make the reasonable assumption that such width is larger than the distance between the positions of the two maxima that will appear for large L) provides an upper bound for the latent heat. We obtain $\Delta_h < 0.01$, which is very small, much smaller than $\Delta_h \approx 0.11$ obtained for the CP^3 model.

In conclusion, the FSS analysis of the data up to $L \approx 100$ supports the presence of a weak first-order transition. For $L \lesssim 50$, most of the data show an intermediate regime where a behavior compatible with a continuous transition is observed. In this regime, results are roughly consistent with those obtained from the analysis of the CP^2 loop model [16, 17] with system sizes up to $L \approx 100$. However, our data corresponding to $L = 64, 96$ are not consistent with the conclusions of Refs. [16, 17]. It is interesting to note that the most robust evidence against a continuous transition is provided by the Binder parameter, which, however, was not considered in Refs. [16, 17].

From our results we might conclude that no CP^2 universality class exists. In this case, the scaling behavior observed in Ref. [17] is not asymptotic, but simply an intermediate transient regime, as it occurs in the lattice formulation we consider. In this case, also in the loop model one would observe first-order behavior for large values of L . However, we would like to stress that we cannot completely exclude that a CP^2 fixed point really exists. Indeed, it is *a priori* possible that the CP^2 lattice model is outside the attraction domain of the fixed point. With increasing L , the renormalization-group flow for the lattice model gets initially closer to the CP^2 fixed point, giving somehow rise to an intermediate scaling regime, and then it moves away to infinity, as expected for a first-order transition.

VII. THE LATTICE CP^{N-1} MODELS IN THE LARGE- N LIMIT

To obtain further insight on the critical behavior of CP^{N-1} models, we consider the large- N limit. We study a very general class of Hamiltonians given by

$$H = -N \sum_{\mathbf{x}\mu} W(|\bar{\mathbf{z}}_{\mathbf{x}+\hat{\mu}} \cdot \mathbf{z}_{\mathbf{x}}|^2), \quad (32)$$

where $W(x)$ is an increasing function of x in $[0, 1]$. Hamiltonian (4) corresponds to $W(x) = x$. The gauge Hamiltonian (7) can also be studied in this framework. Indeed, in the large- N limit, Hamiltonian (7) is equivalent to

$$H = -2N \sum_{\mathbf{x}\mu} |\bar{\mathbf{z}}_{\mathbf{x}+\hat{\mu}} \cdot \mathbf{z}_{\mathbf{x}}|. \quad (33)$$

It therefore corresponds to the choice $W(x) = 2\sqrt{x}$.

The general analysis for $N = \infty$ is presented in the Appendix. We find that a very general class of models undergoes a finite-temperature first-order transition. This class includes all Hamiltonians for which $W'(x)$ is

bounded in $[0, 1]$. In particular, it includes Hamiltonian (4), but not the gauge Hamiltonian (7) (since the latter corresponds to $W(x) = 2\sqrt{x}$, $W'(x) = 1/\sqrt{x}$ is not bounded in $[0, 1]$). This result generalizes Ref. [49], that showed that model (4) has a first-order transition in two dimensions. If we additionally assume that $W''(x) \geq 0$, we can completely characterize the coexisting phases. In the high-temperature phase, the correlation length vanishes identically and the system is disordered. In the low-temperature phase, instead, the system is ordered and the global symmetry is broken. The behavior we find for large N in this class of systems is very similar to what we observe in the gauge model for $N = 3$ and 4.

It is also possible to observe continuous transitions. This is the case of the gauge model (7) or of linear combinations of the Hamiltonians (7) and (4). It is important to stress that the analysis shows that continuous transitions can be obtained without needing an exact tuning of a Hamiltonian parameter. It therefore confirms the existence of a stable CP^{N-1} fixed point for large N [9]. However, this does not guarantee that all CP^{N-1} models undergo a continuous transition. There is indeed a large class of theories that undergo a first-order transition. In the renormalization-group language, these models do not belong to the attraction domain of the fixed point, so that, in the renormalization-group evolution, they flow to infinity, giving rise to a discontinuous nonuniversal transition.

VIII. SUMMARY AND CONCLUSIONS

In the last fifty years the LGW approach has played a fundamental role as it has provided both qualitative predictions and accurate quantitative estimates for the universal properties of critical transitions [12–14]. The approach has also been applied to systems with gauge symmetries, the most notable case being QCD. Considering an appropriate gauge-invariant observable as order parameter, the LGW approach was used [50–52] to predict the nature of the chiral transition in the massless limit of quarks. that is presently looked for in heavy-ion collisions. However, we have recently noted [19, 20] that, if a gauge invariant order parameter is used, for some systems that enjoy a gauge invariance the standard LGW scenario is not consistent with the numerical and/or theoretical results obtained with other methods. This is the case of ferromagnetic and antiferromagnetic CP^{N-1} models and of antiferromagnetic RP^{N-1} systems [19, 20], which are invariant under $\text{U}(1)$ and \mathbb{Z}_2 gauge transformations, respectively.

Here we consider the ferromagnetic CP^{N-1} models that are invariant under $\text{U}(1)$ transformations. If one adopts the usual LGW strategy, one defines a gauge-invariant order parameter and considers the most general LGW Φ^4 theory compatible with the symmetries, cf. Eq. (7). In the case of CP^{N-1} models with $N \geq 3$, a cubic term is present in the LGW Hamiltonian. In the

mean-field approximation this implies that the transition is of first-order. Such prediction is expected to hold close to four dimensions, where statistical fluctuations are not expected to be sufficiently strong to change the nature of the transition, and, in particular, to be valid in three dimensions. In this scenario, one would predict the absence of continuous transitions for all values of N satisfying $N \geq 3$ in generic CP^{N-1} models. If the model undergoes a finite-temperature transition, as it usually does, such transition should generically be of first order.

These conclusions disagree with the general arguments of Ref. [9], that discusses CP^{N-1} models in the large- N limit, and the numerical results of Refs. [16, 17] for $N = 3$. In particular, for large values of N , it is possible to show [9] that the continuum CP^{N-1} field theory is equivalent to an effective abelian Higgs model, for which a large- N stable fixed point can be identified using the standard perturbative ϵ expansion [9, 18]. Therefore, in this limit CP^{N-1} models may undergo a continuous finite-temperature transition.

In this work we reconsider the problem for ferromagnetic CP^{N-1} models. We perform numerical simulations of the lattice model (7), in which gauge invariance is guaranteed by the presence of an explicit gauge $\text{U}(1)$ link variable [22–24]. Such a formulation is particularly convenient from a numerical point of view and allows us to obtain accurate results for systems of size $L \leq 96$. For $N = 2$ we confirm the equivalence of the CP^1 model with the $\text{O}(3)$ Heisenberg vector model, while, for $N = 3$ and 4, we find that the transition is of first order. The results for $N = 4$ agree with those of Ref. [21], obtained on smaller lattices. For $N = 3$ the first-order singularity is very weak and indeed, a clear signature of the nature of the transition is only obtained for sizes $L \gtrsim 50$. For smaller sizes, most of the numerical data are consistent with a continuous transition characterized by critical exponents roughly equal to those obtained in Ref. [17]. Therefore, our numerical results for $N = 2, 3$, and 4 are consistent with the standard LGW picture, in which one predicts a first-order transition because of the presence of a cubic term in the LGW Hamiltonian.

Although we have no evidence of the existence of a CP^{N-1} fixed point for $N = 3$, as claimed in Refs. [16, 17], our results do not exclude it either. Indeed, it is *a priori* possible that the CP^2 lattice model we consider is outside the attraction domain of the stable fixed point that controls the critical behavior of loop models [16, 17]. Clearly, simulations of the loop model on larger lattices with a more careful choice of the observables are needed to confirm that the loop model has indeed a continuous transition and to exclude that the observed behavior is a small-size apparent scaling behavior analogous to that we have observed for $N \lesssim 50$.

We have also investigated the behavior of CP^{N-1} models in the large- N limit. In this case, depending on the specific Hamiltonian, it is possible to observe both first-order and continuous transitions. The latter ones can be obtained without any particular tuning of the Hamilto-

nian parameters, therefore confirming the existence of a large- N stable fixed point.

Summarizing, we can conclude that the usual LGW picture is not always able to provide the correct qualitative description of the finite-temperature transitions occurring in 3D CP^{N-1} lattice models. This conclusion is quite robust in the large- N limit, as it is obtained by analytic theoretical arguments. In this regime the gauge degrees of freedom must play a role, as it should be also expected on the basis of the equivalence of the CP^{N-1} field theory with the abelian Higgs model that describes a dynamical $\text{U}(1)$ gauge field interacting with a $\text{U}(N)$ matter scalar field. For small values of N , the existence of a 3D CP^{N-1} universality class is less robust, as it only relies on the numerical result of Refs. [16, 17] for $N = 3$. For the lattice model we consider, there is no evidence of a critical transition. Clearly additional numerical investigations are needed to settle the question.

In view of the equivalence of the abelian Higgs model with the CP^{N-1} model for large values of N [9], it is also worthwhile to discuss the critical behavior of the corresponding lattice model. Its simplest version is given by the Hamiltonian

$$H = H_{\text{gauge}} + H_{\lambda}, \quad (34)$$

where H_{λ} is the CP^{N-1} Hamiltonian (7) and

$$H_{\text{gauge}} = c_g \sum_{\mathbf{x}\mu\nu} (\lambda_{\mathbf{x},\mu} \lambda_{\mathbf{x}+\hat{\mu},\nu} \bar{\lambda}_{\mathbf{x}+\hat{\nu},\mu} \bar{\lambda}_{\mathbf{x},\nu} + \text{c.c.}), \quad (35)$$

is the usual Wilson Hamiltonian for the gauge field. This theory has been extensively studied for $N = 2$ [2, 3] and $N = 4$ [21]. In particular, for $N = 4$, the numerical results of Ref. [21] show that the first-order transition becomes weaker by increasing the coupling c_g , and turns apparently into a continuous one. A similar scenario may emerge for $N = 3$. It is however, not obvious if these results are relevant for CP^{N-1} models. Indeed, the equivalence of the abelian Higgs model (34) with the CP^{N-1} has only been proved in the large- N limit. Therefore, we cannot take the results on the existence of a continuous transition in the Abelian Higgs model as an indication of the existence of a stable CP^{N-1} fixed point. Additional work is clearly needed.

Appendix A: Large- N solution

In this Appendix we will solve the most general CP^{N-1} theory in the large- N limit, generalizing to CP^{N-1} the results of Refs. [53, 54]. We start from the general Hamiltonian

$$H = -N \sum_{\langle xy \rangle} W(|\bar{\mathbf{z}}_y \cdot \mathbf{z}_x|^2), \quad (A1)$$

where $W(x)$ is an increasing function of x for $0 \leq x \leq 1$ to guarantee ferromagnetism (therefore $W'(x) > 0$), and

the sum is over all links $\langle xy \rangle$ of a lattice. For definiteness, we will consider a d dimensional (hyper)-cubic lattice of size L with periodic boundary conditions.

To linearize the dependence of the Hamiltonian on the spin variables \mathbf{z}_x , we introduce a set of auxiliary fields [53]. To each lattice link we associate the real fields ρ_{xy} and λ_{xy} , and a complex field σ_{xy} , while to each lattice site we associate a real field μ_x [55]. Using the identities (note that ρ_{xy} is integrated in the interval $[0, 1]$)

$$\begin{aligned} e^{N\beta W(|\bar{\mathbf{z}}_y \cdot \mathbf{z}_x|^2)} &\propto \int d\rho_{xy} d\lambda_{xy} \\ &\exp [N\beta \lambda_{xy} (|\bar{\mathbf{z}}_y \cdot \mathbf{z}_x|^2 - \rho_{xy}) + N\beta W(\rho_{xy})], \\ e^{N\beta \lambda_{xy} |\bar{\mathbf{z}}_y \cdot \mathbf{z}_x|^2} &\propto \int d\sigma_{xy} d\bar{\sigma}_{xy} \\ &\exp [-N\beta \lambda_{xy} (|\sigma_{xy}|^2 - \sigma_{xy} \bar{\mathbf{z}}_y \cdot \mathbf{z}_x - \bar{\sigma}_{xy} \mathbf{z}_y \cdot \bar{\mathbf{z}}_x)], \\ \delta(|z_x|^2 - 1) &\propto \int d\mu_x \exp[-N\beta (|z_x|^2 - 1)\mu_x], \end{aligned} \quad (A2)$$

we can rewrite the partition function as

$$Z = \int \prod_{\langle xy \rangle} d\rho_{xy} d\lambda_{xy} d\sigma_{xy} d\bar{\sigma}_{xy} \prod_x d\mu_x d\mathbf{z}_x d\bar{\mathbf{z}}_x e^{N\beta A}, \quad (A3)$$

where

$$\begin{aligned} A = & - \sum_{\langle xy \rangle} [\lambda_{xy} (|\sigma_{xy}|^2 - \sigma_{xy} \bar{\mathbf{z}}_y \cdot \mathbf{z}_x - \bar{\sigma}_{xy} \mathbf{z}_y \cdot \bar{\mathbf{z}}_x) \\ & + \lambda_{xy} \rho_{xy} - W(\rho_{xy})] - \sum_x (|z_x|^2 - 1)\mu_x. \end{aligned} \quad (A4)$$

We perform a saddle-point expansion writing

$$\begin{aligned} \lambda_{xy} &= \alpha + \hat{\lambda}_{xy}, \\ \rho_{xy} &= \tau + \hat{\rho}_{xy}, \\ \mu_{xy} &= \gamma + \hat{\mu}_{xy}, \\ \sigma_{xy} &= \delta + \hat{\sigma}_{xy}. \end{aligned} \quad (A5)$$

We first consider the case $\alpha\delta = 0$. In this case, we have

$$\langle \bar{\mathbf{z}}_x \cdot \mathbf{z}_y \rangle = \frac{1}{\gamma\beta} \delta_{xy}. \quad (A6)$$

For $\alpha = 0$, the saddle-point equations give $W'(\tau) = 0$, which is in contrast with our assumption that $W(x)$ is an increasing function of x . We thus assume that $\alpha \neq 0$ and $\delta = 0$. The saddle-point equations give $\tau = 0$, $\alpha = W'(0)$, and $\gamma\beta = 1$. At the saddle point we have simply

$$\frac{\beta A}{L^d} = \mathcal{A}_1 = \beta dW(0) + 1. \quad (A7)$$

We will call this case the white-noise solution, because of the absence of correlations, see Eq. (A6).

If $\alpha, \delta \neq 0$, we can assume, using gauge invariance, that δ is real. If we define $m_0^2 = (\gamma - 2d\delta\alpha)/\delta\alpha$ we obtain

$$\langle \bar{\mathbf{z}}_x \cdot \mathbf{z}_y \rangle = \frac{1}{\beta\delta\alpha} L^{-d} \sum_p \frac{e^{ip(x-y)}}{\hat{p}^2 + m_0^2}, \quad (A8)$$

where $\hat{p}_\mu = 2 \sin p_\mu/2$ and $\hat{p}^2 = \sum_\mu \hat{p}_\mu^2$. We define

$$\begin{aligned} I(m_0^2, L) &= \frac{1}{L^d} \sum_p \frac{1}{\hat{p}^2 + m_0^2}, \\ J(m_0^2, L) &= \frac{1}{L^d} \sum_p \frac{\cos p_x}{\hat{p}^2 + m_0^2} \\ &= \frac{1}{2d} [(2d + m_0^2)I(m_0^2, L) - 1]. \end{aligned} \quad (\text{A9})$$

The saddle-point equations become

$$\begin{aligned} 1 &= \frac{1}{\beta\delta\alpha} I(m_0^2, L), \\ \alpha - W'(\tau) &= 0, \\ \tau + \delta^2 - \frac{2}{\beta\alpha} J(m_0^2, L) &= 0, \\ \delta - \frac{1}{\delta\beta\alpha} J(m_0^2, L) &= 0, \end{aligned} \quad (\text{A10})$$

which give

$$\begin{aligned} \delta &= \sqrt{\tau}, \\ \alpha &= W'(\tau), \\ \tau &= \left(\frac{J(m_0^2, L)}{I(m_0^2, L)} \right)^2, \\ \beta &= \frac{I(m_0^2, L)}{\sqrt{\tau} W'(\tau)}. \end{aligned} \quad (\text{A11})$$

The quantity βA is given by

$$\frac{\beta A}{L^d} = -\beta d[\alpha\delta^2 + \alpha\tau - W(\tau)] + \beta\gamma - g(m_0^2, L) - \ln(\beta\delta\alpha), \quad (\text{A12})$$

where

$$g(m_0^2, L) = L^{-d} \sum_p \ln(\hat{p}^2 + m_0^2). \quad (\text{A13})$$

Using the gap equation, we can rewrite it as

$$\frac{\beta A}{L^d} = \mathcal{A}_2(m_0^2, L) = \beta dW(\tau) + F(m_0^2, L), \quad (\text{A14})$$

where

$$\begin{aligned} F(m_0^2, L) &= (m_0^2 + 2d)I(m_0^2, L) - 2dJ(m_0^2, L) \\ &\quad - g(m_0^2, L) - \ln I(m_0^2, L). \end{aligned} \quad (\text{A15})$$

To identify the correct phase, for each β , we should determine the saddle point that maximizes A . Note that $F(m_0^2, L)$ is always an increasing function of m_0^{-2} that takes the value 1 for $m_0 = \infty$. Given that $W(\tau) > W(0)$, since $W(x)$ is an increasing function of x , we obtain $\mathcal{A}_2(m_0^2, L) > \mathcal{A}_1$ for any finite value of m_0^2 . Therefore, the solution of Eq. (A11), if it exists, is always more stable than the white-noise one.

We will now focus on the three-dimensional case. As discussed in Ref. [54], one should take the infinite-volume

limit of $I(m_0^2, L)$ carefully. For $m_0^2 \neq 0$, the limit $L \rightarrow \infty$ is simply

$$I_\infty(m_0^2) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\hat{p}^2 + m_0^2}, \quad (\text{A16})$$

which varies between 0 (for $m_0^2 = +\infty$) and a maximum value [56]

$$\begin{aligned} I^* &= I_\infty(0) \\ &= \frac{\sqrt{6}}{192\pi^3} \Gamma\left(\frac{1}{24}\right) \Gamma\left(\frac{5}{24}\right) \Gamma\left(\frac{7}{24}\right) \Gamma\left(\frac{11}{24}\right). \end{aligned} \quad (\text{A17})$$

Numerically, $I^* \approx 0.252731$. For $m_0^2 = 0$ we should take into account the presence of a diverging zero mode. Therefore, for $L \rightarrow \infty$ and $m_0^2 \rightarrow 0$, we have

$$I(m_0^2, L) = L^{-d} \sum_{p \neq 0} \frac{1}{\hat{p}^2 + m_0^2} + \frac{1}{L^3 m_0^2} \approx I^* + \frac{1}{L^3 m_0^2}. \quad (\text{A18})$$

Thus, for any $K > I^*$, there is always a solution $m_0(L)$ of the equation $I(m_0(L)^2, L) = K$, with $m_0(L)^2 \sim L^{-3}$. Thus, in three dimensions there is a condensate phase that we can access by considering $m_0^2 \rightarrow 0$ and $L \rightarrow \infty$ at fixed $\Delta^{-1} = m_0(L)^2 L^3$. In this limit $I(m_0^2, L)$ converges to $I^* + \Delta$. Note [54] that the zero-mode is irrelevant for $g(m_0^2, L)$. If $m_0^2 \rightarrow 0$ and $L \rightarrow \infty$ at fixed $\Delta^{-1} = m_0(L)^2 L^3$ we simply obtain

$$g(m_0^2, L) \rightarrow g_\infty(0) = \int \frac{d^3p}{(2\pi)^3} \ln \hat{p}^2. \quad (\text{A19})$$

The presence of the zero mode implies that, for $L \rightarrow \infty$, all quantities become functions of m_0^2 or Δ , depending on how the limit is taken. With an abuse of notation, we will replace the L dependence with Δ in the following, to underscore that all quantities are either functions of m_0^2 (and in this case $\Delta = 0$), or of Δ (in this case $m_0^2 = 0$).

We will now show that for β small enough, there is no solution of the saddle-point equations (A11), provided that $W'(\tau)$ is finite for $0 \leq \tau \leq 1$. Indeed, as m_0^2 and the condensate value Δ vary, β , computed from Eq. (A11), is always larger than a specific value β_{\min} . Therefore, for $\beta < \beta_{\min}$ the white-noise solution is the relevant one. To prove this point consider the saddle-point equations (A11). The equation for β can be rewritten as

$$\beta = \frac{H(m_0^2, \Delta)}{W'(\tau)} \quad H(m_0^2, \Delta) = \frac{I(m_0^2, \Delta)^2}{J(m_0^2, \Delta)}. \quad (\text{A20})$$

Expressing $J(m_0^2, \Delta)$ in terms of $I(m_0^2, \Delta)$, one can easily verify that:

- i) for $m_0^2 \rightarrow \infty$, the function $H(m_0^2, \Delta = 0)$ converges to 1;
- ii) as m_0^2 decreases, also $H(m_0^2, \Delta = 0)$ decreases; the function keeps decreasing also in the condensate phase up to $\Delta = \Delta_{\min} = 1/3 - I^*$, where it assumes the value $H_{\min} = H(0, \Delta_{\min}) = 2/3$;

iii) for $\Delta > \Delta_{\min}$, $H(0, \Delta)$ increases, going to infinity as $\Delta \rightarrow \infty$.

As for τ , it increases from zero ($m_0^2 \rightarrow \infty$) to 1 ($m_0^2 = 0$, $\Delta \rightarrow \infty$). If M is the maximum value that $W'(\tau)$ takes in $0 \leq \tau \leq 1$, we immediately verify that β , computed from (A20) can never become smaller than $2/(3M)$. In the interval $0 \leq \beta < \beta_{\min}$ the relevant solution is the white-noise one, corresponding to $m_0^2 = \infty$. Instead, for $\beta > \beta_{\min}$, the stable phase corresponds to a solution of Eq. (A11), since the free energy densities satisfy $\mathcal{A}_2(m_0^2, L) > \mathcal{A}_1$. Therefore, the point $\beta = \beta_{\min}$ is a first-order transition point, where m_0 changes discontinuously. For $\beta > \beta_{\min}$ there are may be several solutions of Eq. (A11). If, furthermore, we assume that also $W'(\tau)$ is an increasing function of τ , we can easily conclude that the stable solution for $\beta \geq \beta_{\min}$ always corresponds to $m_0^2 = 0$ and $\Delta > 0$. At the first-order transition there is coexistence between a disordered phase with $1/m_0 = 0$ (the correlation length vanishes) and an ordered phase with $m_0 = 0$ and $\Delta > 0$.

It is interesting to note that the previous argument does not apply to the gauge CP^{N-1} Hamiltonian for which

$$H = -\frac{1}{\beta} \sum_{\langle xy \rangle} \ln I_0(2N\beta |\bar{\mathbf{z}}_y \cdot \mathbf{z}_x|). \quad (\text{A21})$$

In the large- N limit it is equivalent to

$$H \approx -2N \sum_{\langle xy \rangle} |\bar{\mathbf{z}}_y \cdot \mathbf{z}_x|, \quad (\text{A22})$$

so that $W(\tau) = 2\sqrt{\tau}$ and $W'(\tau) = 1/\sqrt{\tau}$. The derivative $W'(\tau)$ is clearly not bounded in $[0, 1]$ and therefore the previous argument does not apply. For this model the gap equation (A11) becomes simply

$$\beta = I(m_0^2, L). \quad (\text{A23})$$

For this Hamiltonian there is only a continuous transition for $m_0 = 0$.

We can consider Hamiltonians that interpolate between Hamiltonians (4) and (7). For instance, we can consider

$$W(\tau) = a\tau + 2(1-a)\sqrt{\tau}. \quad (\text{A24})$$

In this case we observe a first-order transition for $0.7582 \approx a \leq 1$. In the opposite case there is a transition for $m_0 = 0$. The first-order transition is however different from the one we discussed before, as here the high-temperature coexisting phase corresponds to a finite nonzero value of m_0^2 .

To conclude, let us note that, in the derivation of the large- N solution, one explicitly breaks gauge invariance, which is forbidden by Elitzur's theorem [57]. It is usually assumed this to be only a technical problem. Results for gauge-invariant quantities are expected to be correct. This has been extensively checked in two dimensions [25].

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