ERRATUM TO "ORLIK-SOLOMON-TYPE PRESENTATIONS FOR THE COHOMOLOGY ALGEBRA OF TORIC ARRANGEMENTS"

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Abstract. In this short note we correct the statement of the main result of [1]. That paper presented the rational cohomology ring of a toric arrangement by generators and relations. One of the series of relations given in [1] is indexed over the set circuits in the arrangement's arithmetic matroid. That series of relations should however be indexed over all sets X with $|X| = \mathrm{rk}(X) + 1$. Below we give the complete and correct presentation of the rational cohomology ring.

We state the correct version of [1, Theorem 6.13]:

Theorem 1. *Let $\mathcal{A}$ be an essential toric arrangement. The rational cohomology algebra of the complement $H^*(M(\mathcal{A}), \mathbb{Q})$ is isomorphic to the algebra $\mathcal{E}$ with*

- *Set of generators $e_{W,A;B}$, where W ranges over all layers of $\mathcal{A}$, A is a set generating W and B is disjoint from A and such that $A \sqcup B$ is an independent set; the degree of the generator $e_{W,A;B}$ is $|A \sqcup B|$.*
- *The following types of relations:*
  - *For any two generators $e_{W,A;B}$, $e_{W',A';B'}$,*

    $$e_{W,A;B}e_{W',A';B'} = 0$$

    if $A \sqcup B \sqcup A' \sqcup B'$ is a dependent set, and otherwise

    $$e_{W,A;B}e_{W',A';B'} = (-1)^{\ell(A\cup B, A'\cup B')} \sum_{L\in\pi_0(W\cap W')} e_{L,A\cup A';B\cup B'}. \tag{1}$$

  - *For every linear dependency $\sum_{i\in E} n_i\chi_i = 0$ with $n_i \in \mathbb{Z}$, a relation*

    $$\sum_{i\in E} n_i e_{T,\emptyset;\{i\}} = 0. \tag{2}$$

  - *For every subset $X \subseteq E$ where $\mathrm{rk}(X) = |X| - 1$ write $X = C \sqcup F$ with C the unique circuit in X. Consider the associated (unique) linear dependency $\sum_{i\in C} n_i\chi_i = 0$ with $n_i \in \mathbb{Z}$, and for every connected*

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component L of $\cap_{i\in X} H_i$ a relation

$$\sum_{j\in C} \sum_{\substack{A,B\subset X\\ A\supseteq F\\ X=A\sqcup B\sqcup\{j\}\\ |B| \text{ even}\\ W\supseteq L}} (-1)^{|A_{\le j}|} c_B \frac{m(A)}{m(A\cup B)} e_{W,A;B} = 0 \tag{3}$$

where, for all $i \in C$, $c_i := \operatorname{sgn} n_i$, $c_B = \prod_{i\in B} c_i$.

The only difference between Theorem 1 and [1, Theorem 6.13] consists in eq. (3) that hold not only for every circuit C but also for all X with $|X| = \operatorname{rk}(X)+1$. This difference is important as shown in the following example.

Example 2. Consider the central toric arrangement in $T = (\mathbb{C}^*)^3$ given by the four hypertori $H_1 = \{x = 1\}$, $H_2 = \{y = 1\}$, $H_3 = \{xy = 1\}$, and $H_4 = \{xy^{-1}z^3 = 1\}$. The zero-dimensional layers are the points $p = (1, 1, 1)$, $q = (1, 1, \zeta_3)$, and $r = (1, 1, \zeta_3^2)$. Let W be the layer $H_1 \cap H_2 \cap H_3$. The relations given by (3) are the following:

$$\overline{\omega}_{W,\{1,2\}} - \overline{\omega}_{W,\{1,3\}} + \overline{\omega}_{W,\{2,3\}} + \psi_1\psi_2 - \psi_1\psi_3 - \psi_2\psi_3 = 0$$

$$\overline{\omega}_{s,\{1,2,4\}} - \overline{\omega}_{s,\{1,3,4\}} + \overline{\omega}_{s,\{2,3,4\}} + \frac{1}{3}\psi_1\psi_2\overline{\omega}_4 - \frac{1}{3}\psi_1\psi_3\overline{\omega}_4 - \frac{1}{3}\psi_2\psi_3\overline{\omega}_4 = 0$$

for all $s = p, q, r$. Notice that the relation in degree two does not imply the relations in degree three.

The proof of Theorem 1 is the same as in [1] with the following corrections (see also the preprint arXiv:1806.02195v3).

Let $X \subseteq E$ with $|X| = \operatorname{rk}(X)+1$, then X can be written uniquely as $C \sqcup F$ where C is a circuit and $F = X \setminus C$. Theorem 6.12 of [1] holds in a wider generality: the set C does not need to be a circuit but can be any $X \subseteq E$ with $|X| = \operatorname{rk}(X) + 1$. It can be proven by choosing a suitable separating cover of X: for $i \in C$ define $a_i = m(X) \prod_{j\in C\setminus\{i\}} m(C \setminus \{j\})$ and $a_i = m(X)$ for $i \in F$. Let $\Lambda(X)$ be the lattice generated by $\frac{\chi_i}{a_i}$ for all $i \in X$, it defines a covering $\pi_U : U \to T$ of tori (cf. [1, Definition 6.6]).

Lemmas 6.3, 6.4, 6.5, 6.7, 6.8, and 6.10 of [1] should be corrected by changing C with X and with minor changes in their proofs. The proof of [1, Theorem 6.12] needs an extra step: let L be a connected component of $\bigcap_{i\in X} H_i$ and p a point in L, we use the relation (12) of [1] in the torus U to obtain:

$$\sum_{j\in C} \sum_{\substack{A,B\subset C\\ C=A\sqcup B\sqcup\{j\}\\ |B| \text{ even}}} (-1)^{|A_{\le j}|} \overline{\eta}^U_{A,B}(q) c_B = 0.$$

for all $q \in \pi_U^{-1}(p)$. We multiply this equation by $\overline{\eta}^U_{F,\emptyset}(q)$, and using the equality $\overline{\eta}^U_{A,B}(q)\overline{\eta}^U_{F,\emptyset}(q) = (-1)^{|F_{\le j}|+\ell(C,F)}\overline{\eta}^U_{A\sqcup F,B}(q)$ we obtain:

$$\sum_{j\in C} \sum_{\substack{A,B\subset X\\ A\supseteq F\\ X=A\sqcup B\sqcup\{j\}\\ |B| \text{ even}}} (-1)^{|A_{\le j}|} \overline{\eta}^U_{A,B}(q) c_B = 0.$$

This corrects the proof of [1, Theorem 6.12].

In the proof of Theorem 6.13 it was claimed that the old relations allow one to write each $e_{W,A;B}$ in term of generators with A a no-broken-circuit set. In the following we prove the claim by using the new relations.

Indeed if A contains a broken-circuit $A_1 \subseteq A$, i.e. $A_1 = C \setminus \min(C)$ for a circuit C, consider the relation (3) for $X = C \cup A$: it expresses the element $e_{W,A;B}$ as a linear combination of some $e_{W',A';B'}$ with $|A'| < |A|$ or with $|A'| = |A|$ and A' lexicographically smaller than A. We have inductively proved the claim.

References

[1] Filippo Callegaro, Michele D'Adderio, Emanuele Delucchi, Luca Migliorini, and Pagaria Roberto, *Orlik-Solomon type presentations for the cohomology algebra of toric arrangements*, Trans. Amer. Math. Soc. **373** (2019), no. 3, 1909–1940.

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