Population Growth: A Pure Welfarist Approach

by Thomas Renström* and Luca Spataro**

Abstract

In this work we propose a framework based on welfarist principles to deal with several issues concerned with population economics models, such as the Repugnant Conclusion, both in Absolute (see Parfit D, (1984). Reasons and Persons*, Oxford/New York: Oxford University Press) and Relative sense (see Michel P, Pestieau P, (1998).* Optimal Population Without Repugnant Aspects*, Genus, 54 (3/4), 25-34), the shape of childbearing costs and population dynamics, under both normative and positive perspectives. We show that the basic formulation can avoid both the assumption of high childbearing costs and the Absolute Repugnant Conclusion (ARC) but cannot avoid the Relative Repugnant Conclusion (RRC). Moreover, optimal fertility is increased by technological shocks and displays cycles. Both ARC and RRC can be avoided by extending the model to a decentralized economy with consumption externalities; in the latter model, a technological shock reduces long run fertility and can generate cycles along the transitional path.*

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* Durham University Business School, Durham (UK). Email: t.i.renstrom@durham.ac.uk.

**Corresponding author. Dipartimento di Economia e Management, University of Pisa (Italy). Email: luca.spataro@unipi.it, tel: +39 050 2216217

1. Introduction

The study of population growth has a long tradition in economics[1]. Typically, welfarist principles are adopted[2] such as Classical Utilitarianism (CU, or Benthamite approach), where the objective is the sum of the utilities over the population, or average utilitarianism (or Millian approach), where the objective is the welfare of a representative individual, or some mixture of both, such as Number Dampened Utilitarianism (NDU) proposed by Ng (1986)[3]. However, these approaches may pose several problems. For example, in the absence of convex childbearing costs, CU cannot avoid the Repugnant Conclusion (RC henceforth; see Parfit 1976, 1984, Blackorby et al. 2002), whereby any state in which each member of the population enjoys a life above neutrality is declared inferior to a state in which each member of a larger population lives a life with lower utility (Blackorby et al. 1995, 2002)[4]. In such a case, one should at any point in time drive individual consumption levels to neutrality, and typically (if neutrality consumption is zero) obtain the largest possible (infinite) population size. If there are physical constraints on reproductive capacity, at each point in time one should set the reproductive rate to its physical maximum. It is this counter-intuitive result that has been coined 'repugnant' in the literature, simply because there is no trade off between consumption per person and the number of individuals (a higher number of individuals is always preferred).

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[1] Optimal population growth has been firstly analysed by Samuelson (1975, 1976), Deardorff (1976) and, more recently, by Jaeger and Kuhle (2009), de la Croix et al. (2012), Pestieau and Ponthiere (2014). See Doepke (2008) for an in-depth review on this topic.

[2] Among works using non utilitarian principles (i.e. extensions of Pareto-dominance), see, for example, Golosov et al. (2007), who work with discrete fertility levels, and the discussion in Broome (1996). Michel and Wigniolle (2009) and Conde-Ruiz et al. (2010) use the Representative-Consumer and *Millian* efficiency criteria; respectively, for OLG models with children choices selected from a continuum. In these works, parents derive directly their utility from the number of children they bear, not from the utility of their descendants. In fact, as noted by Conde-Ruiz et al. (2010), such a criterion "might be regarded as a form of utilitarianism, called average utilitarianism" (p. 155). For a reconciliation between the benefit-cost analysis and the welfare-based approach, see Mertens and Rubinchik (2017).

[3] More recently, NDU has been adopted in AK-growth models by Boucekkine and Fabbri (2013) and Marsiglio (2014).

[4] In fact, the debate can be dated back to the so-called Edgeworth's conjecture (1925), according to which classical utilitarianism leads to a bigger population size and lower standard of living.

In models of economic growth à la Ramsey-Cass-Koopmans with endogenous population such a conclusion takes the form of an *upper corner solution* for the population growth rate (society reproduces at its physical maximum rate). The literature has tried to avoid RC (the corner property), by either modifying the social welfare criterion or introducing convex costs associated with the number of children each household has, which however might seem unsatisfactory from several points of view. NDU (as in the case of Becker-Barro, 1988)[5], can avoid the RC, but as pointed out, among others, by Michel and Pestieau (1998), at the cost of two undesirable outcomes. The first is the requirement of high child-raising cost for an interior solution to exist (p. 25). In the words of Becker and Barro (1988) (p. 9): "consumption is positive only when children are a financial burden; that is, when the cost of rearing a child exceeds the present value of his *lifetime* earnings"[6]. The second is the so called *relative* RC (RRC). In the words of Michel and Perstieau (1998) (p. 27): "The relative repugnant solution occurs when any increase in the resources brought by each additional individual leads to a drop in equilibrium consumption." (p. 27), that is an inverse relationship between equilibrium consumption and personal income. Although the authors do not provide further comments, we believe that the repugnancy of such an outcome consists in the fact that a growing economy would be associated with decreasing individual consumption (and thus decreasing individual utility), which does not seem a desirable outcome for a society.

A possible way out of this problem is represented by Critical Level Utilitarianism (CLU), in which the objective function is derived from axioms (see Blackorby et al. 1995). The critical level is defined as a social utility value (α) of an extra person that, if added to the (unaffected) population, would make

[5] Razin and Sadka (1995) interpret Barro-Becker's approach as a compromise between Millian average utility and Benthamite sum of utilities.

[6] In NDU à la Becker-Barro, (1988) and Barro-Becker (1989) childbearing costs, although necessary to avoid RC, do not affect directly the steady state level of the population growth rate. Finally, such preferences do not satisfy the axiom of "Existence Independence of Unconcerned Individuals", so that rankings of alternatives may depend on the utilities of unaffected people such as the long dead. See Blackorby et al. (2001). For other approaches, see, for example, Lagerlöf (2015).

society as well off as without that person[7]. CLU can avoid the RC, in the sense it produces a trade off between individual consumption and population size, and producing interior solutions. Of course, in dynamic models, it can very well be the case that (if population growth is positive) population size grows infinite (in infinite time), but there is no desire to drive it to infinite in an instant or a desire to at each point in time have the highest physically possible population growth rate. Likewise, if population growth is negative, population would asymptotically disappear (in infinite time). However, again, there is no desire to make it disappear in an instant (which is the case for Average Utilitarianism), thus avoiding the corner property. In principle, there is little guidance on what the critical level should be.

The population literature has to a great extent focused on choice between population sizes often in a static framework (see, for example, Nerlove et al. 1982, 1985 and Shiell 2008). However, dynamic models have the possibility of addressing population over time, and thus analysing the distribution of population sizes over time (i.e. the population growth rate).

One of the first attempts to characterize the problem in a dynamic setting with CLU dates back to Michel and Pestieau (1998), who show that both the RC (or, in their notation, absolute RC - ARC) and the relative RC (RRC) can only be avoided if α is above individual neutrality level (typically) and is not constant. However, they work in a simplified framework of a small open economy without economic growth. Moreover, they suggest a solution (setting the critical level as a function of the wage) that is "at odds" with that of the welfarist approach (Michel and Pestieau footnote 3, p. 30).

Hence, we build on the intuition of Michel and Pestieau (1998) and we make a step further by introducing general equilibrium economic growth in a closed economy.

Our research question is then the following: is there an axiomatically founded social welfare criterion that allows to avoid the RC without "unwanted side-results" (Michel and Pestieau 1998, p. 25)

[7] Other possible solutions have been, as already mentioned, to incorporate the number of children as an argument of the utility function (see, for example, Borck 2011, Galor 2011; Prettner and Strulik 2016, Ehrlich and Kim 2015). In the spirit of the welfarist tradition we prefer to abstract from this assumption, by allowing for intergenerational altruism.

such as the RRC or too high childbearing costs, in a Cass-Koopmans-Ramsey economy[8]?

On one hand, in the present paper we follow previous literature by defining the RC as an (upper) corner solution for the rate of growth of population[9]. We avoid endogenizing technological progress because we want to keep our analysis as close as possible to the Cass-Koopmans-Ramsey approach and to focus not only on steady states, but also on intertemporal redistribution issues through richer transitional dynamics.

More in detail, we wish to contribute to this debate by proposing a new population criterion, "Relative Critical Level Utilitarianism" (RCLU), where the criterion is axiomatically founded and is the critical level allowed to depend on past utility levels. It allows for non-trivial intergenerational comparisons. If an economy is at a low consumption level, we argue that ethically individuals should not be prevented from being born because of a fixed critical level being too high, or if the economy is at a high consumption level, the constant critical level would induce a large increase in population size. A more flexible criterion is when the critical level depends positively on past consumption. If past consumption was low, there would still be a desire to reproduce, since the critical level is also lower. We develop the Relative Critical Level Utility criterion and study its properties in terms of population growth rates (and thereby the distribution of population over time), consumption, and capital accumulation. Since our analysis is more general than the previous literature we can also compare our results with the special cases of constant critical level (CLU) and classical utilitarianism (when the critical level is zero). Through this approach, which is *normative* in nature, we will show that while RC and high childbearing costs can be avoided, RRC still persists and *optimal* population growth is positively affected by technological shocks, although displaying nontrivial dynamics (i.e. cycles).

RRC has not received the same attention as RC in population economics literature, and one might even question whether the former is really repugnant. Although the discussion of the philosophical

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[8] As already mentioned, several authors (i.e. Palivos and Yip 1993; Razin and Yuen 1995; Boucekkine and Fabbri 2013; and Marsiglio 2014) have analysed the relationship between social preferences, economic growth and endogenous fertility in endogenous growth models. An example of endogenous growth model with endogenous population growth and RCLU is contained in Renström and Spataro (2015).

[9] On this issue see Renström and Spataro (2011) and Spataro and Renström (2012).

implications of RRC is beyond the scope of the present paper, we argue that both the RRC and the issue of the assumption of high childbearing costs are relevant also for theoretical and positive arguments. In fact, as already noticed, most existing models with endogenous population need to assume high costs for childbearing and, moreover, would predict a negative relation between individual's income and consumption. However, both the assumption and the conclusion are unsatisfactory, being at odds with the consolidated empirical evidence.

Hence, in the second part of the paper we extend the model to the case in which externalities in consumption exist (somewhat in line with the "Catching-up with the Jones" literature) and, following most existing literature (including Becker-Barro 1988 and Barro-Becker 1989) we here adopt a positive approach. In this scenario (in which the decentralized equilibrium will be suboptimal due to the presence of consumption externalities), we show that both RC and RRC can be avoided under reasonable assumptions on preferences and childbearing costs. Moreover, we will show that cycles may appear around the steady state and that, differently from previous model, an increase of total factor productivity reduces fertility.

Some final comments are in order**.** First, we depart from previous recent literature on optimal population size, mostly concerned with OLG economies or with endogenous growth models, in that we focus on infinitely lived dynasties (each generation lives for one period), with zero per capita growth (the only source of growth being population), for the sake of comparability with the original contributions of Becker-Barro (1988), Barro-Becker (1989) and Michel and Pestieau (1998). Second, for the same reason and without loss of generality, we run our analysis in discrete time. Third, although we restrict our analysis on symmetric equilibria, we show that the RCLU criterion can be also used when the source of heterogeneity is not between generations, but also within generations. The analysis of the latter case, allowed for in the formal derivation of the RCLU criterion in the Appendix, is left for future research.

The paper is organized as follows: in section 2 we present the model under the normative perspective, in subsection 2.3 we provide the solution and in subsection 2.4 we discuss the existence and the stability of the long run equilibrium and discuss the effects of technological shocks on optimal capital intensity, per capita consumption and fertility. In subsection 2.5 we comment our results with a focus on the role of our critical level function. In section 3 we adopt a positive approach and extend the model to the case of a decentralized economy with atomistic dynasties where each dynasty does not recognize its influence on critical level consumption. Finally, an Appendix contains the axiomatic foundation of RCLU and the Proofs of Propositions.

2. Normative perspective: the centralized economy

We start our analysis under a normative perspective. For doing this, in the present section we build up the model structure of a centralized economy and present the results. The extension to a decentralized economy that will be used for the positive analysis will be presented in section 3.

2.1 Preferences

We concentrate on a single dynasty (household) or a policymaker choosing consumption and population growth over time, so as to maximize:

$$W(u_{t-1}, N_t, u_t, N_{t+1} \dots) = \sum_{s=0}^{\infty} \beta^s N_{t+s}[u_{t+s} - \hat{\alpha}(u_{t-1+s})] \tag{1}$$

where N is the population (family) size, u the instantaneous utility function, $\beta \in (0,1)$ the intergenerational discount factor and $\hat{\alpha}(u_{t-1+s})$ is the critical level utility. In Appendix A we show that this is the only formulation satisfying: Independence of the Long Dead, Parental Dependence, Stationarity, Independence of Distant Future Generations, Independence of Utilities of Unconcerned Individuals, Anonymity, Strong Pareto and Relative Critical Level Dependence.

Note that, differently from previous literature, we allow such a critical value to be a function of previous generation's utility (only if $\hat{\alpha}(u_{t-1+s})$ is a constant this social ordering would coincide with CLU). We call our population criterion "Relative CLU" (RCLU). More precisely, it seems plausible to assume $\hat{\alpha}' > 0$ (that is, the higher utility/consumption of parents, the higher CLU).

Hence, we propose a population criterion in the spirit of CLU, but where the judgment (the critical level of utility for life worth living) is relative to the existing generation's level of wellbeing. According to such a criterion, a society at low level of utility sets a lower threshold of utility for the next generation, and a society with high living standard sets a higher level. So if parents had a good life, they require their children to have a good life as well, and vice versa. We call such a criterion Relative Critical Level Utilitarianism (RCLU). It is reasonable that societies 10000 years ago had an entirely different target level of utility for life worth living than societies today. Also, societies following the more flexible population criterion will find it easier to adapt to fluctuations in the external environment. For example, if food resources become scarce for a period and utility falls below an absolute critical level, society does not stop reproducing under RCLU, but may still view life worth living if the children have a utility related to parent's utility. Societies following (absolute) CLU may become extinct as they stop reproducing when utility for a number of generations fall below the absolute threshold level.

As for population dynamics, we denote the population growth rate as n_t, i.e.

$$N_{t+1} = (1 + n_t)N_t. \tag{2}$$

Furthermore, we assume that there are exogenous lower and upper bounds on the population growth rate: $n_t \in [\underline{n}, \bar{n}]$. Realistically, there is a physical constraint at each period of time on how many children a parent can have. There is also a constraint on how low the population growth can be. The reason for the latter assumption is twofold: first, we do not allow individuals to be eliminated from the population (in that there is no axiomatic foundation for that); moreover, even if nobody wants to reproduce there will always be accidental births (we will also assume that $\underline{n} > -1$: population cannot disappear in one period).

Equation (1) can also be written as[10]:

$$W = \sum_{t=0}^{\infty} \beta^t \,[N_t u(c_t) - N_{t-1}\alpha(c_{t-1})] = \sum_{t=0}^{\infty} \beta^t\, N_t[u(c_t) - \alpha(c_t)\beta(1 + n_t)] - \alpha(c_{-1})N_0 \tag{3}$$

where $u(c_t)$ is the intratemporal utility level, with $u' > 0, u'' < 0$, and $\alpha(c_t)$ the critical level function and $\alpha(c_{-1})$ and N_0 given at period 0. We follow the standard convention in population ethics and normalize lifetime utilities so that a lifetime-utility level of zero represents neutrality, hence $u(0) = 0$ (see Blackorby et at. 2002 and 2005, chap. 2 and the references therein). Note that the sum in eq. (3) is finite only if $\beta(1 + \bar{n}) < 1$, which we assume throughout the paper, as it is standard in growth models (see for example Barro and Becker 1989, eq. 24). Even though N_t can go to infinity in the long run, we can still use (3) as the objective function if $\lim_{T\to\infty} \beta^T N_T = 0$. In fact, this is the transversality condition, which we will show is satisfied (see footnote 12). This is also why we choose to present the optimization problem with N as a state and n as a control (and also to detect possible corners for n).

[10] Without loss of generality we have redefined the critical level function as $\hat{\alpha}(u_{t-1+s}) \equiv \alpha(c_{t-1+s})$.

2.2. Technology

Assuming a constant returns-to-scale (CRS) production technology, $F(K_t, L_t) = A_t G(K_t, L_t)$, with A_t the parameter representing total factor productivity, and a zero capital depreciation, the capital accumulation equation is:

$$K_{t+1} = F(K_t, N_t) + K_t - c_t N_t - \theta(n_t) N_t \tag{4}$$

where $\theta(n_t)$ is the unit cost for raising children. For the sake of tractability and in the spirit of Barro-Becker (1988) and Michel-Pestieau (1998), $\theta(n_t)$ is assumed to be a linear function. Moreover, it is normalised in such a way that when n=0, the per-adult cost is exactly equal to θ, because in this case population is constant, i.e. each adult gives birth to one child. Hence, $\theta(n_t) = \theta \cdot (1 + n_t)$.

2.3 Solution

The Lagrangean function associated with the household's problem is the following:

$$L_t = \sum_{t=0}^{\infty} \{\beta^t N_t [u(c_t) - \alpha(c_t)\beta(1 + n_t)] + q_t[F(K_t, N_t) + K_t - K_{t+1} - c_t N_t - \theta(1 + n_t)N_t] + \lambda_t[(1 + n_t)N_t - N_{t+1}] + \nu_t(\bar{n} - n_t) + \vartheta(n_t - \underline{n})\} \tag{5}$$

where we have omitted the term $-\alpha(c_{-1})N_0$ from eq. (3) as it is a constant at date 0. The term $\lambda_t[(1 + n_t)N_t - N_{t+1}]$ in the Lagrangean function associated with eq. (2) captures the fact that at each time period the population size is given (and thus is a state variable) and can only be controlled by the choice of n (which is a control variable). The law of motion for the population size is provided by (2). Hence, λ_t can be interpreted as the shadow value of population.

The first order conditions of the problem imply:

$$\frac{\partial L}{\partial c_t} = \beta^t N_t [u'(c_t) - \alpha'(c_t)\beta(1 + n_t)] - N_t q_t = 0 \Rightarrow \beta^t [u'(c_t) - \alpha'(c_t)\beta(1 + n_t)] = q_t \tag{6}$$

$$\frac{\partial L}{\partial n_t} = -\beta^{t+1} N_t \alpha(c_t) - q_t \theta N_t + \lambda_t N_t = 0 \Rightarrow \lambda_t = \beta^{t+1}\alpha(c_t) + q_t \theta \tag{7}$$

and from (6), eq. (7) becomes

$$\lambda_t = \beta^{t+1}\alpha(c_t) + \beta^t [u'(c_t) - \alpha'(c_t)\beta(1 + n_t)]\theta\ . \tag{8}$$

The other FOCs yield:

$$\frac{\partial L}{\partial N_t} = \beta^t [u(c_t) - \alpha(c_t)\beta(1 + n_t)] + q_t [F_{N_t} - c_t - \theta(1 + n_t)] + \lambda_t(1 + n_t) - \lambda_{t-1} = 0 \tag{9}$$

$$\frac{\partial L}{\partial K_{t+1}} = -q_t + q_{t+1}(1 + F_{K_{t+1}}) = 0 \tag{10}$$

eqs. (2) and (4). Let us define the capital intensity $k \equiv K/N$, such that, by exploiting CRS in the production function and assuming that L=N we can write: $F(K, N) = Nf(k), F_N = f(k) - f'k$. Moreover, using (6), and adapting q_{t+1} from (10) into (6) we get:

$$\frac{u'_{t+1} - \alpha'_{t+1}\beta(1+n_{t+1})}{u'_t - \alpha'_t\beta(1+n_t)} = \frac{1}{\beta(1+F_{K_{t+1}})} \tag{11}$$

Furthermore, backward shifting (8) and (10) and plugging into (9) and using (8) and (6) we have:

$$u_t - \alpha_{t-1} = [u'_t - \alpha'_t\beta(1+n_t)]\{c_t - F_{N_t} + \theta(1+F_{K_t})\} \tag{12}$$

Hence, eqs. (11), (12), eq. (4) in per capita terms, that is:

$$(1+n_t)k_{t+1} = f_t + k_t - c_t - \theta(1+n_t) \tag{13}$$

fully characterize our dynamic system. Equation (12) states that, at the optimum, both consumption and fertility should be chosen in such a way that the addition to social welfare of increasing the population at the margin, $u_t - \alpha_{t-1}$, should equal the marginal value (in utility units) of what a new-born net takes out of society, $[u'_t - \alpha'_t\beta(1+n_t)]\{c_t - F_{N_t} + \theta(1+F_{K_t})\}$. The term in square brackets is the marginal utility value of capital, and the term in curly brackets is the difference between what she consumes, $c_t + \theta(1+F_{K_t})$ and what she brings, F_{N_t} (the marginal value of labor). The intertemporal price enters because the child cost occurs in the previous period. We should notice that if the per capita capital stock is larger, then the individual brings more in terms of the labour endowment, and the intertemporal price is also lower. Furthermore, the marginal value of capital $\beta^{-t}q_t$ (from eq. (6)) is also lower, implying the right hand side of (12) is lower. Thus, for larger per capita capital stock, the individual takes out less from society. This means a lower level of per capita consumption is required for being indifferent bringing more individuals into being (left hand side of 12).

2.3.1. The steady-state equilibrium

From eqs. (11), (13) and (12) respectively, the steady-state solutions for capital intensity (k^*), population growth rate (n^*) and per capita consumption (c^*) follow:

$$f'^* = \frac{1}{\beta} - 1 \tag{14}$$

$$1 + n^* = \frac{f^* + k^* - c^*}{k^* + \theta} \tag{15}$$

$$c^*: \frac{u^*-\alpha^*}{k^*+\theta} = \frac{[u\prime^*-\alpha\prime^*\beta(1+n^*)]}{\beta}[1-\beta(1+n^*)] \quad (16)$$ [11]

(where the equality in (16) stems from (12) and (14)). It should be noticed that the transversality conditions will hold[12].

Some comments on the solution are worth noting. Eq. (14) pins down a unique positive value of steady-state capital intensity k^*, as is standard in a Cass-Koopmans-Ramsey models. Eq. (15) states that at the optimum both fertility, n^* and consumption, c^* must satisfy the resources available for the economy.

2.4. Results

2.4.1. Existence of the interior equilibrium

In this subsection we will provide the conditions for the existence of a steady state solution with interior n (i.e. avoidance of the RC). Moreover, we will pinpoint the conditions for avoiding the assumption of a too high childbearing cost (common in the previous literature). Preliminarily, from eqs. (14)-(15) and using CRS (so that $f^* = f^* - k^*f'^* + k^*f'^* = F_N^* + k^*f'^*$), eq. (15) becomes:

$$c = F_N^* + \frac{k^*}{\beta} - (1+n)\cdot(k^*+\theta) \quad (15')$$

Let us define c^{max} and c^{min} as follows:

$$c^{max} \equiv F_N^* + \frac{k^*}{\beta} \quad (17)$$

$$c^{min} \equiv F_N^* - \frac{\theta}{\beta} \quad (18)$$

where c^{max} is the solution for c of (15) if $1+n=0$ (lower bound for $n=-1$, population disappears in one period) and c^{min} is the solution for c of (15) if $(1+n)\beta = 1$ (upper bound for $n = \frac{1-\beta}{\beta}$, the objective

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[11] Notice, that, by eq. (6), $[u' - \alpha'\beta(1+n)] > 0$, that is, along the optimal path, marginal utility of consumption in the Social Welfare function must be positive. Hence, by (16), assumption $\beta(1+\bar{n}) < 1$, at the optimum $u - \alpha > 0$ (the social marginal benefit of a new born must be positive).

[12] Transversality conditions read as $\lim_{T\to\infty} q_T K_T = 0$ and $\lim_{T\to\infty} \lambda_T N_T = 0$. From (6) $q_T K_T = [u' - \alpha'\beta(1+n)]\beta^T K_T = [u' - \alpha'\beta(1+n)]k\beta^T N_T$. If c, n, k go to a steady state, the first transversality condition becomes: $\lim_{T\to\infty} \beta^T N_T = 0$, which is also the second transversality condition (following from eq. (8)). Finally, $\lim_{T\to\infty} \beta^T N_T = \lim_{T\to\infty} \prod_{t=0}^{T-1} \beta(1+n_t) N_0 = 0$, last equality follows from $\beta(1+n_t) \leq \beta(1+\bar{n}) < 1$.

function (3) explodes). Both c^{min} and c^{max} are parametrically given numbers (for example, under Cobb-Douglas production function $f = Ak^{\gamma}$ and, normalizing A such that steady state per capita GDP equals unity, we have $c^{max} = 1 - \gamma + \frac{\gamma}{1-\beta}$ and $c^{min} = 1 - \gamma - \frac{\theta}{\beta}$).

Hence, it follows that parameter restrictions for an interior solution for n must be such that the equilibrium per capita consumption lies in the interval: $c^* \in \left(c^{min}, c^{max}\right)$. Recognizing that (15') can be written as $1 + n = \frac{c^{max} - c}{k+\theta}$ and substituting for $(1 + n)$ into (16) and exploiting $c^{max} - c^{min} = \frac{k+\theta}{\beta} > 0$, eq. (16) reads as:

$$\frac{u^* - \alpha^*}{u'^* - \alpha'^*} = (c^* - c^{min}) + \frac{\alpha'^*}{u'^* - \alpha'^*} \frac{\left(c^* - c^{min}\right)^2}{c^{max} - c^{min}} \tag{19}$$

Now we can provide the following Proposition concerning existence and uniqueness of an interior solution at the steady state (we omit stars superscripts for the sake of readability).

Proposition 1: *Necessary and sufficient for existence of a unique interior steady state equilibrium is*:

$$u(c^{max}) - u'(c^{max}) \cdot (c^{max} - c^{min}) > \alpha(c^{max})$$

and

I) If $c^{min} < 0, \frac{u(0) - \alpha(0)}{u'(0) - \alpha'(0)} \leq 0$;

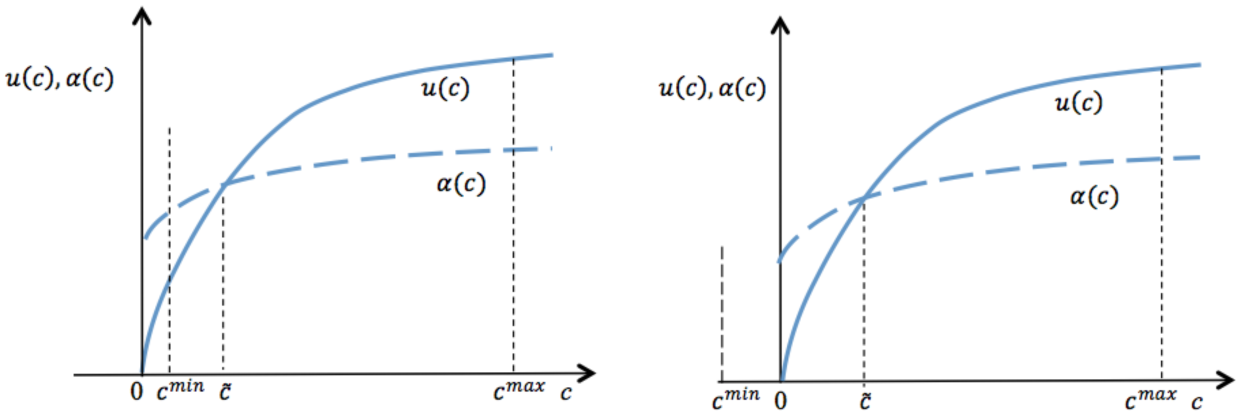
II) If $c^{min} > 0, u\left(c^{min}\right) < \alpha\left(c^{min}\right)$.

Proof: Proof: See Appendix C. □

The conditions contained in the above Proposition are concerned with the shape of individual's preferences and the critical level function on the relevant domain. The first condition tells that the critical level function cannot be too large in relation to utility for large consumption levels ($u(c^{max}) - u'(c^{max}) \cdot (c^{max} - c^{min}) > \alpha(c^{max})$). Part I) and part II) are related to the critical-level-consumption level $\tilde{c}$, for which $u(\tilde{c}) = \alpha(\tilde{c})$, which is required lying in the relevant consumption range $\left(c^{min}, c^{max}\right)$. If $c^{min} < 0$, one needs $u(0) < \alpha(0)$, so that $\tilde{c}$ lies above zero; If $c^{min} > 0$, one needs $u\left(c^{min}\right) < \alpha\left(c^{min}\right)$, so that $\tilde{c}$ lies above c^{min}. Therefore both $u(c)$ and $\alpha(c)$ must cross within the relevant consumption range, with $u(c)$ crossing from below. Figure 1 summarizes the content of Proposition 1 through a graphical representation.

Notice hat condition II) of Proposition 1 allows for sufficiently low childbearing costs, given that, when parameters as set in such a way that $c^{min} > 0$, then it follows that $F_N > (1 + F_K)\theta$. In the remainder of the paper we will focus on the case $c^{min} > 0$, for which the conditions provided in Propositon 1 are satisfied, for example, with $\alpha(c) = \alpha_0 + \tilde{\alpha} \cdot u(c)$, $\alpha_0 > 0$ properly chosen and $\tilde{\alpha} \in (0,1)$.

Figure 1. Graphical representation of the conditions for existence and uniqueness of an interior solution (for $c^{min} > 0$ and $c^{min} < 0$ respectively)



For parametric utility functions we can translate the conditions in Proposition 1 to restrictions on parameters. In our computations we will use normalized negative exponential utility, $u = 1 - e^{-\delta c}$, with $\delta > 0$. For this utility function, under Cobb-Douglas production function $f = Ak^{\gamma}$ and normalizing A such that steady state per capita GDP equals unity, the necessary and sufficient conditions for existence and uniqueness of interior equilibrium provided in Proposition 1 read as:

$$e^{\delta\left(1-\gamma-\frac{\theta}{\beta}\right)} < \frac{1-\tilde{\alpha}}{1-\tilde{\alpha}-\alpha_0} \text{ and } \delta\frac{\frac{\gamma}{1-\beta}+\frac{\theta}{\beta}}{1-\tilde{\alpha}-\alpha_0} + \frac{1-\tilde{\alpha}}{1-\tilde{\alpha}-\alpha_0} < e^{\delta\left(1-\gamma+\frac{\gamma}{1-\beta}\right)}$$

The first condition is related to $c^{min} < \tilde{c}$ and if childbearing costs are high, so that $c^{min} < 0$, this condition always holds, otherwise it constitutes a parametric restriction. The second inequality is related to $\tilde{c} < c^{max}$. We see that with low childbearing costs ($c^{min} > 0$) the first inequality requires $\tilde{\alpha}$ and α_0 to be sufficiently high (to guarantee the crossing point in Figure 1). The second inequality requires $\tilde{\alpha}$ and α_0 not to be too large to avoid the α function being too close to u for high consumption levels (see Figure 1).

2.4.2. Characteristics of the equilibrium: stability and cycles

In this subsection we analyse the stability of the equilibrium and characterize its dynamic properties, that is the shape of the transition path of per capita consumption, capital intensity and fertility. We can summarize our findings on the equilibrium with interior solution for fertility through the following Proposition:

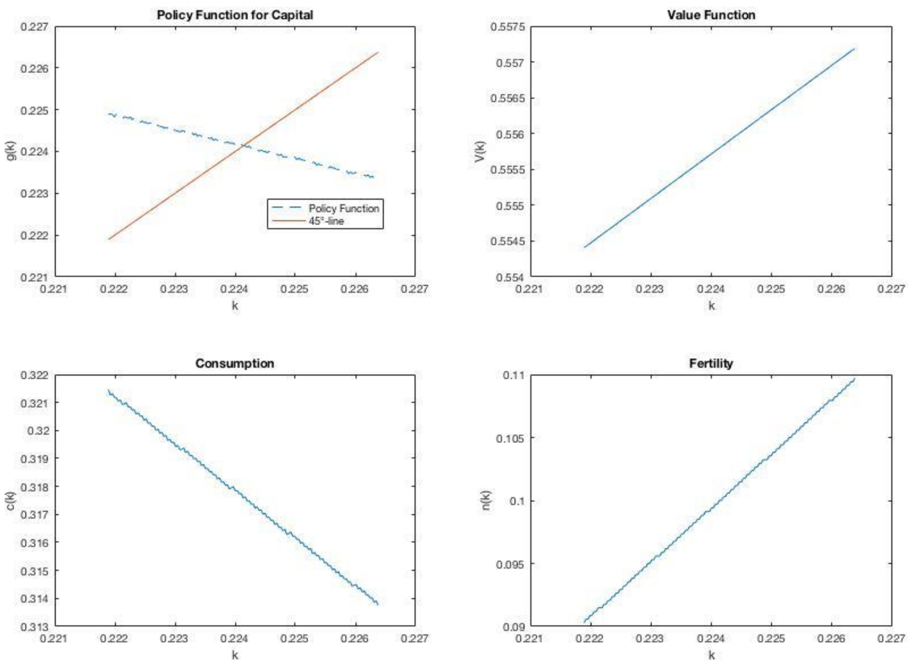
Proposition 2: *Necessary and sufficient for stability of an interior steady state equilibrium is* $(k+\theta)Tf''\beta v' > 2(\alpha')^2$, *with* $v' = \frac{u'-\beta(1+n)\alpha'}{\beta} > 0$ *and* $T \equiv [u'' - \beta(1+n)\alpha''(c)](k+\theta) + 2\beta\alpha' < 0$.

Under stability, the equilibrium displays converging cycles.

Proof: See Appendix B. □

Although the latter conditions do not provide clear restrictions on the parameters, we have used them as a check in the numerical simulations that we present below.

Figure 2: Policy functions for capital, consumption and fertility



Parameters: TFP (A)=1.17, capital share=0.1, $u = 1 - e^{-0.6c}$, $\alpha(c) = 0.01 + 0.2 \cdot u$, $\theta = 0.6$, $\beta = 0.68$. $k^* = 0.224$, $n^* = 0.1$, $c^* = 0.31$, $c^{min} = 0.012$.

For illustrative purposes in Figure 2 we present the policy functions for *k*, *c* and *n* for the case with normalized negative exponential utility function and Cobb-Douglas production function.

For doing this, we have computed our solution through value function iteration over a discrete space in *k* and *n*. This method would allow us to pick up any corner solution if it were present. Notice that the policy function for capital crosses the 45°-line from above, pointing to the existence of cycles in the neighbourhood of the steady state (as in Jones and Schoonbroodt 2016).

Intuitively, at a larger per capita capital stock, a new individual brings more to society on the resource side, and therefore requires less in utility for social welfare (from eq. (12)). Consequently, a larger per capita capital stock will imply lower per capita consumption, everything else equal. Therefore one cannot have a transition towards a steady state where both capital and consumption grow. If society starts off with a low level of capital and moves towards a higher steady state capital stock, consumption cannot continuously decline, as it would then have to be too high from the outset, preventing capital from growing. Therefore, both capital and consumption will cycle during a transition towards a steady state. This is the reason for the policy functions for consumption and capital in Figure 2 being downward sloping.

2.4.3. The effects of a technological change

In this subsection we carry out comparative statics in order to assess the effects of changes in total factor productivity, A, on the equilibrium values of consumption, capital and fertility. We can summarize our findings through the following Proposition:

Proposition 3: *A positive technological shock increases the long run optimal capital intensity and the population growth rate*.

Proof: Define $f(k) = Ag(k)$, so that $f' = Ag'$. From eq. (14) one gets $\frac{f'}{A}dA + f''dk = 0$ and:

$$\frac{dk}{dA} = -\frac{f'}{Af''} > 0 \qquad \text{(P3.1)}$$

Next, by total differentiation of eqs. (15)-(16) it follows:

$$\frac{u'-\alpha'}{k+\theta}dc - \frac{u-\alpha}{(k+\theta)^2}dk = \frac{u''-\alpha''\beta(1+n)}{\beta}[1-\beta(1+n)]dc - [u' + \alpha' - 2\alpha'\beta(1+n)]dn \qquad \text{(P3.2)}$$

and, also using (14),

$$dn = \frac{f}{A(k+\theta)}dA + \frac{1-\beta(1+n)}{\beta(k+\theta)}dk - \frac{1}{(k+\theta)}dc \qquad \text{(P3.3)}$$

Finally, exploiting eqs. (P3.1)-(P3.3) yields:

$$\frac{dn}{dA} = \left[\frac{1-\beta(1+n)}{\beta(k+\theta)}\right]\left(\frac{T-\beta\alpha'}{T}\right)\frac{dk}{dA} + \frac{f}{A}\frac{(T-2\beta\alpha') - \beta\frac{(u'-\alpha')}{[1-\beta(1+n)]}}{T(k+\theta)} \qquad \text{(P3.4)}$$

Given that under concavity T<0 (see Appendix B), then also $T - \beta\alpha' < 0$ and $T - 2\beta\alpha' < 0$. Given that $u' - \alpha' > 0$ (from eqs. (6), (16), (19) and (P1.4)), then $\frac{dn}{dA} > 0$. □

In the light of the analysis carried out so far, we can interpret our findings as follows. A technological improvement renders capital more productive, making it convenient for the economy to invest higher amounts of resources in capital accumulation, in production of goods and in giving birth to more children. Notice that the Repugnant conclusion (i.e. upper-corner solution for population rate of growth) is avoided, even with low costs for raising children. Finally, notice that, at the optimal steady sate, it must be that $u' - \alpha' > 0$: in words, marginal utility of consumption must be positive not only for society (see footnote 11) but also for each single individual, that is, an increase in consumption gives rise to a greater increase in utility than the increase in the critical level function[13].

We now turn to the issue of whether the Relative Repugnant Conclusion (RRC), whereby an increase in personal income (i.e. wage) produces a decrease of long run per capita consumption, can be avoided. We still focus on the effects of a positive change in total factor productivity A, given that, at the steady state, this change increases the marginal productivity of labour (F_N) without affecting the interest rate (which remains constant according to eq. (14)). We can summarise our findings as follows:

Proposition 4: *A positive technological shock decreases the long run optimal per capita consumption*.

Proof: From eqs. (P3.2) and (P3.3) we obtain:

$$\frac{dc}{dA} = \left[\frac{1-\beta(1+n)}{T}\right] \alpha' \frac{dk}{dA} + \frac{f\beta}{A} \frac{2\alpha' + \frac{(u'-\alpha')}{[1-\beta(1+n)]}}{T} \tag{20}$$

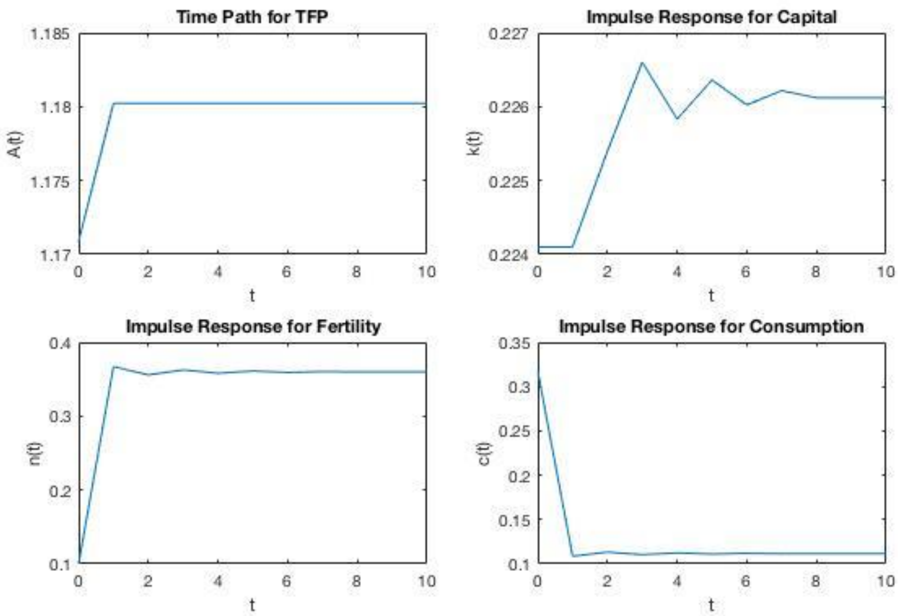
The sign of the first object at the RHS of eq. (20) $\left(\frac{1-\beta(1+n)}{T}\right)$ is non-positive (by (P3.1) and under stability $T < 0$; see Appendix B.3). As for the second object, given that, by restrictions stemming from eq. (6), $\beta(1 + \bar{n}) < 1$ and $u' - \alpha' > 0$, it follows that its sign is negative, so that $\frac{dc}{dA} < 0$. □

[13] For example, if the critical level function is a parameter times utility, $\alpha(c) = \tilde{\alpha} \cdot u(c)$, the condition above would be satisfied for $\tilde{\alpha} < 1$.

Hence, we can conclude that the adoption of a RCLU, while avoiding the assumption of too high childbearing costs, cannot avoid the Relative Repugnant Conclusion (that is, a decrease of consumption when income increases).

In fact, this means that, when per capita income increases due to an increase in total factor productivity, society finds it optimal to increase population size, since the value of the labour endowment an individual will bring increases, and to increase capital accumulation too. However, resource constraint imposes to do this at the cost of lower per capita consumption, while society as a whole will be eventually better off (due to bigger population).

Figure 3: Impulse functions for capital, fertility consumption after an increase of TFP



Parameters: same as Fig. 1 with a change in A of +0.8%.

In Figure 3 we provide the impulse functions for k, n and c after an increase of Total factor productivity (TFP), which confirms the results provided in Proposition 4. Moreover, the Figure confirms the presence of cyclical convergence as was shown in Proposition 2.

2.5. Comments on the results

We can now discuss briefly the results presented so far by comparing them with those stemming from previous related literature.

First, notice that, when $\alpha(c)$ =0 and $\alpha'(c) = 0$, we are in the CU case (or in the special case of Becker-Barro (1988) with the altruism parameter (1-α)=1, in their notation). and (12') reads as

$$u - cu' + u'[F_N - \theta(1 + r)] = 0. \qquad (12'')$$

Under concavity of $u(c)$ and normalization $u(0)$=0, then $u(c) - cu'(c) > 0$, so that an interior solution for n can only arise if $F_N - \theta(1 + r) < 0$, that is, under very high childbearing costs (the cost for rearing children is higher than their lifetime income in present value). This is the undesirable outcome (or assumption) unveiled, among others, by Michel and Pestieau (1998). Moreover, in this case it can be shown that the RRC emerges, that is an increase of wage reduces equilibrium per capita consumption (see Michel and Pestieau 1998); in our paper this can be seen by observation of (20) for $\alpha(c)$ =0 and $\alpha'(c) = 0$. As we have shown in Proposition 1, under our model the assumption of high childbearing costs is not necessary, provided that $\alpha(c)$ takes on a positive value.

With $\alpha(c) > 0$ and $\alpha'(c) = 0$ (fixed critical level), eq. (12) at the steady state reads as:

$$\frac{u-\alpha}{u'} - c = \{\theta(1 + F_K) - F_N\} \qquad (12''')$$

which resembles eq. (13) in Renström and Spataro (2011). In that model it is shown that the RC can be avoided at the steady state (even in the absence of childbearing costs) but not the RRC (see here eq. 20 with $\alpha(c) > 0$ and $\alpha'(c) = 0$). Moreover, as in Renström and Spataro (2011), when $\alpha'(c) = 0$ the model has a corner solution for n outside the steady state. Under RCLU, we obtain interior solutions for n also outside the steady state, so our upper and lower limit for n are not binding; if we were to set $\alpha'(c) = 0$ and ignoring any binding limits for n, the model would jump immediately to steady state k through a discrete jump in N. This manifests itself in our Appendix B, eq. 23, $\alpha'(c) = 0$).

On the contrary, $\alpha(c) > 0$ and $\alpha'(c) > 0$ allow to avoid RC, high childbearing costs and provide nontrivial transitional dynamics, although do not avoid the RRC.

A final remark is worth doing. Up to now we have not investigated the long-run level of population (apart from verifying that transversality conditions are satisfied). By eq. (2), we know that population will asymptotically grow to infinite $n > 0$, will be constant if $n = 0$, will shrink asymptotically to zero if $n < 0$. All these outcomes are well compatible with the axioms that we posed to build the RCLU social ordering. However, there is no axiom preventing population from possible extinction and the latter result would, indeed, be a paradox[14]. That the population size asymptotically

[14] We thank an anonymous referee for pointing us to this issue.

approaches zero is an artefact of the infinitely divisible population size. However we can think that there is a smallest unit of population size, say one person. If this is the case, N becomes smaller and smaller and eventually becomes close to its smallest value. At that point in time there is a discrete choice between its smallest value and zero.

The Bellman equation of the problem can be written as (see eq. (B.1) in Appendix B):

$V(K_t, N_t, c_{t-1}) = max\{N_t[u(c_t) - \alpha(c_{t-1})] + \beta V(K_{t+1}, N_{t+1}, c_t)\}$

Suppose that, at the steady state $n < 0$, so that population is shrinking towards a certain level $\underline{N}$, the latter being the level at which population is no further divisible ($\underline{N}$ can represent one individual or one household, etc.). Hence, at a certain point in time t the choice will be between two possible options, either $N_{t+1} = 0$ or $N_{t+1} = \underline{N}$. If $N_{t+1} = 0$ is chosen, then $V(K_{t+1}, 0, c_t) = 0$. Hence, at period t we have:

$$V(K_t, N_t, c_{t-1}) = max\begin{cases} 0 \\ max_{c_t}\underline{N}[u(c_t) - \alpha(c_{t-1})] + \beta V\big(K_{t+1}, \underline{N}, c_t\big) \end{cases}$$

Since along the optimal path $u(c_t) > \alpha(c_{t-1})$, thus society will not choose to vanish.

Finally, we examine the role of the critical level function α in our model. We wish to see how the level of the critical level function and its slope affect steady state population growth and consumption. To simplify our analysis, we assume $\alpha(c) = \alpha_0 + \tilde{\alpha} \cdot u(c)$, with $\alpha_0 > 0$ and $\tilde{\alpha} \in (0,1)$. Then, by total differentiation of (14), (15) and (16) w.r.t. α we get (recall that k^* is independent of α, and omitting superscript for steady state values)

$$dn = \frac{dc}{k+\theta} \tag{21}$$

and

$$\frac{u\prime-\alpha\prime}{k+\theta}dc - \frac{d\alpha}{k+\theta} = \frac{u''-\alpha''\beta(1+n)}{\beta}[1-\beta(1+n)]dc - \alpha'[1-\beta(1+n)]dn - [u' - \alpha'\beta(1+n)]dn - (1+n)[1-\beta(1+n)]d\alpha' \tag{22}$$

with $d\alpha = \frac{\partial\alpha}{\partial\alpha_i}d\alpha_i$ and $d\alpha' = \frac{\partial\alpha\prime}{\partial\alpha_i}d\alpha_i$, $\alpha_i = \alpha_0, \tilde{\alpha}$. Combining the two equations above and after some manipulation we get:

$$-[1-\beta(1+n)]\frac{T}{\beta}dn = [1-\beta(1+n)](1+n)d\alpha' - \frac{d\alpha}{k+\theta} \tag{23}$$

with $T < 0$, If $\alpha_i = \alpha_0$, then $d\alpha' = 0$, so that (23) can be written as:

$$\frac{dn}{d\alpha_0} = (k+\theta)[1-\beta(1+n)]\frac{T}{\beta} < 0 \tag{24}$$

If $\alpha_i = \tilde{\alpha}$, then (23) becomes:

$$-[1-\beta(1+n)]\frac{T}{\beta}dn = \left\{[1-\beta(1+n)](1+n)u' - \frac{u}{k+\theta}\right\}d\tilde{\alpha}$$

Rewriting $\frac{u}{k+\theta}$ as $\frac{u-\alpha+\alpha_0}{(1-\tilde{\alpha})(k+\theta)}$, exploiting (16) and rearranging terms, we get:

$$\frac{dn}{d\tilde{\alpha}} = -\frac{\alpha_0}{(1-\tilde{\alpha})(k+\theta)} < 0 \qquad (25)$$

Finally, from (21), (24) and (25) we get:

$$\frac{dc}{d\tilde{\alpha}} > 0, \frac{dc}{d\alpha_0} > 0$$

Hence, we can conclude that, every else equal, societies endowed the higher critical level function are more likely the shrink asymptotically, while those with lower critical level function are more likely to experience population growth.

3. Positive perspective: decentralized economy with atomistic dynasties

Up to now we have shown that the basic framework can avoid the emergence of RC and getting an interior solution without too high childbearing costs, but cannot avoid the RRC. Moreover, our analysis was mainly normative, although we performed some comparative statics exercises on the optimal steady state values for consumption, capital and fertility. Indeed, to the extent to which social preferences with RCLU, besides stemming from socially plausible axioms, also obtain from aggregation of individual preferences, then our analysis may also be interpreted as being positive in nature and the results as describing the behaviour of households and of the dynamics of population growth in response to technological shocks.

In fact, one can recognize that preferences describing the aggregate economy can be obtained, for example, by aggregating over individuals who in turn have preferences as those adopted in the relative-income literature, put forward by Duesenberry (1949) and Easterlin (1974, 1995) or in the "catching up with the Jones" (as in Alonso-Carrera et al. 2005) and "habit formation" literature as in de la Croix and Michel (1999)[15]. However, under the *positive* interpretation of the model, the negative effect on

[15] For a review of the above mentioned literature see, among others, Macunovich (1998). In fact, if one assumes that individuals are entailed with both intergenerational altruism and relative-income (or relative-welfare) preferences, with

per capita consumption and the negative effect on fertility of total productivity may appear unsatisfactory.

Hence, we now extend the model to the case in which atomistic dynasties do not recognize their influence on critical level consumption. In particular, we assume that the critical level utility depends on previous period consumption, that is $\alpha = \alpha(\bar{c}_{t-1})$. As a consequence, the decentralized solution, differently from the basic framework, will be suboptimal from a social welfare point of view (in fact, if the policymaker aimed at restoring optimality by correcting for the externality caused by consumption in the critical level, then we would get back to the model analysed in section 2).

3.1. Model setup and results

From now on we will adopt an explicitly positive approach and, in particular, we will tackle the issue of whether the RC and RRC are avoided or not under reasonable childbearing costs and we will also analyse the changes on fertility produced by technological shocks. Besides providing the results for a decentralized competitive economy, we will focus on identical dynasties. CRS apply to production technology also in this case. Identical dynasties take the time paths of α_t, the interest rate (r_t) and the wage rate (w_t) as given. In equilibrium the refererence consumption coincides with the individual dynasties' consumption.

The dynastic welfare function now reads as:

$$W = \sum_{t=0}^{\infty} N_t [u(c_t) - \beta(1+n_t)\alpha(\bar{c}_t)] \quad (26)$$

under the individual (dynasty) budget constraint:

$$k_{t+1} + \theta = \frac{(1+r_t)k_t + w_t - c_t}{1+n_t}. \quad (27)$$

The economy-wide resource constraint, taken as given at individual level, is:

$$\bar{k}_{t+1} + \theta = \frac{\bar{k}_t + f(\bar{k}_t) - \bar{c}_t}{1+\bar{n}_t}. \quad (28)$$

reference group being previous generation's income (or welfare), then an individual's preferences could be written as: $U_t = (u_t - \alpha u_{t-1}) + \beta \frac{N_{t+1}}{N_t} U_{t+1}$ such that, aggregating over individuals, we obtain: $W_t = N_t U_t = N_t(u_t - \alpha u_{t-1}) + \beta N_{t+1} U_{t+1}$, which coincides with eq. (1) in the text.

In equilibrium $w_t = f(\bar{k}_t) - f'(\bar{k}_t)\bar{k}_t$ and $r_t = f'(\bar{k}_t)$ and $k_t = \bar{k}_t$, $c_t = \bar{c}_t$, $n_t = \bar{n}_t$, although individuals will observe these relations only ex-post, after making their own private choices. Maximization of (26) w.r.t. k_{t+1}, c_t and n_t yields (see Appendix D):

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1 + r_{t+1}) \tag{29}$$

$$\frac{u(c_t) - \alpha(\bar{c}_{t-1})}{u'(c_t)} = c_t - w_t + \frac{\theta}{\beta} \tag{30}$$

and eq. (28). Finally, at the steady state, the symmetric equilibrium yields

$$1 = \beta(1 + r) \tag{31}$$

$$\frac{u(c) - \alpha(c)}{u'(c)} = c - w + \theta(1 + r) \tag{32}$$

$$1 + n = \frac{(1+r)k + w - c}{k + \theta} \tag{33}$$

together with market clearing prices: $w = F_N$ and $r = F_K$. In Appendix D we provide conditions for existence, uniqueness and stability of the steady state equilibrium. In this case cycles may or may not arise. Moreover, childbearing costs may well be assumed to be sufficiently low (see Appendix D.2 and section 3.2). As for the effects of a change in the TFP, we can write the following:

Proposition 5: *A positive technological shock increases the long run capital intensity and per capita consumption. The effect on the population growth rate is ambiguous. Sufficient for* $\frac{dn}{dA} < 0$ *(necessary and sufficient for* $\frac{dn}{dA} \leq 0$ *if* $\theta = 0$*) is* $\left(1 - \frac{\boldsymbol{c^{min}}}{\boldsymbol{c}}\right)\sigma(c) + \frac{\boldsymbol{c^{min}}}{\boldsymbol{c}} > \frac{\boldsymbol{\alpha'}}{\boldsymbol{u'}}$*, where* $\sigma(c) \equiv -\frac{u''(c)c}{u'(c)}$*.*

Proof: Total differentiation of (31) and (32) with respect to *A*, exploiting market clearing prices and CRS implies, respectively:

$$\frac{dk}{dA} = -\frac{f'}{Af''} > 0. \tag{P5.1}$$

$$\frac{dc}{dA} = \frac{-f/A}{\sigma(c)\left[1 - \frac{\alpha'}{u'\sigma(c)} - \frac{w - \frac{\theta}{\beta}}{c}\right]} \tag{P5.2}$$

Given that under stability the denominator of (P5.2) is negative (see D.3), it follows that $\frac{dc}{dA} > 0$. As for fertility, total differentiation of (33) provides:

$$\frac{dn}{dA} = \left[\frac{1 - \beta(1+n)}{\beta}\right]\frac{dk}{dA} + \frac{f}{A} - \frac{dc}{dA} \tag{P5.3}$$

and its sign is ambiguous. However, by defining $z \equiv 1-\beta(1+n)$, from (27) and (28) we get $u(c)-\alpha(c)=u'(c)(c-c^{min})$ and $c=zc^{max}+(1-z)c^{min}$. Total differentiation of both equations yields:

$$u'\left(c^{min}+\frac{\theta}{\beta}\right)-\left[\alpha'+u''(c-c^{min})\right]\left[c+(1-z)\frac{\theta}{\beta}\right]=\left[\alpha'+u''(c-c^{min})\right]\left(c^{max}-c^{min}\right)\frac{dz}{dk}k$$

First, notice that LHS of the latter equation is increasing in $\frac{\theta}{\beta}$. Hence, for $\theta \geq 0$ sufficient for $\frac{dz}{dA}=\frac{dz}{dk}\frac{dk}{dA}\geq 0$ (i.e. $\frac{dn}{dA}\leq 0$) is $\left[\alpha'+u''(c-c^{min})\right]<0$, that is $\left(1-\frac{c^{min}}{c}\right)\sigma(c)+\frac{c^{min}}{c}>\frac{\alpha'}{u'}$.□

Hence, we can conclude that under reasonable restrictions on the parameters and given sufficiently low childbearing costs, both RC and RRC are avoided and population growth is decreased by a positive technological shock.

Intuitively, our results show that, when per capita income increases due to an increase in total factor productivity, individuals find it optimal to increase per capita consumption and capital accumulation. This happens because individuals do not take into account the (negative) externality that higher consumption produces in terms of higher critical level. However, resource constraint imposes to do this at the cost of a lower population size.

3.2. An example

In this subsection we provide an example for a CES utility of the form $u(c)=\frac{c^{1-\sigma}}{1-\sigma}$ and $\alpha(c)=\tilde{\alpha}\cdot u(c)$, with both σ and $\tilde{\alpha}\in(0,1)$. In this case sufficient for existence of the steady state equilibrium is (See Appendix D for details)

$$\tilde{\alpha}>\sigma. \quad \text{(E.1)}$$

The latter condition is also sufficient for uniqueness and stability of the steady state. Moreover, cycles may arise or not depending on the parameters. More precisely, under Cobb-Douglas production function $f=Ak^{\gamma}$, from eqs. (26)-(28) steady state solutions for k, c, and n are:

$$k^*=\left(\frac{1-\beta}{\beta A\gamma}\right)^{-\frac{1}{1-\gamma}} \quad \text{(E.2)}$$

$$c^*=\frac{1-\sigma}{\tilde{\alpha}-\sigma}c^{min} \quad \text{(E.3)}$$

$$\beta(1+n^*)=\frac{c^{max}-c^*}{c^{max}-c^{min}} \quad \text{(E.4)}$$

Notice that, from eq. (E.3), a necessary condition for obtaining a steady state solution is $c^{min} > 0$, that is $F_N > (1 + F_k) \cdot \theta$. The common assumption of very high child costs does not even apply. We show in Appendix D that condition for stability of the equilibrium is:

$$\left|\frac{d\bar{k}_{t+1}}{d\bar{k}_t}\right| = \left|\frac{1 - \frac{\beta\alpha'}{-u''(\bar{k}+\theta)}}{\beta(1+n)}\right| < 1 \qquad \text{(E.5)}$$

So that we can two possible stable equilibria may arise:

a) $1 > \frac{\beta\alpha'}{-u''(\bar{k}+\theta)} > 1 - \beta(1+n)$

b) $1 < \frac{\beta\alpha'}{-u''(\bar{k}+\theta)} < 1 + \beta(1+n)$

Under our assumptions on the shape of u and α and given (E.1) the latter inequalities read as:

a) $\frac{c^{max} - c^{min}}{c^{min}} > \frac{\tilde{\alpha}}{\sigma}\frac{1-\sigma}{\tilde{\alpha}-\sigma}$

b) $\frac{\sigma(1-\tilde{\alpha})+\tilde{\alpha}(1-\sigma)}{2\sigma(\tilde{\alpha}-\sigma)} < \frac{c^{max} - c^{min}}{c^{min}} < \frac{\tilde{\alpha}}{\sigma}\frac{1-\sigma}{\tilde{\alpha}-\sigma}$.

Hence, in case a) the equilibrium in the neighbourhood of the steady state will display no cycles, while in case b) it will.

Finally, using eq. (E.3) and the definition of c^{min} provided in subsection 2.4, we have:

$$\frac{dc^*}{dA} = \frac{1-\sigma}{\tilde{\alpha}-\sigma}\frac{dF_N}{dA} > 0 \qquad \text{(E.6)}$$

and, given that $1 - \beta(1+n) = \frac{c^* - c^{min}}{c^{max} - c^{min}} = \frac{1-\tilde{\alpha}}{\tilde{\alpha}-\sigma}\frac{\beta F_N - \theta}{k+\theta} = \frac{1-\tilde{\alpha}}{\tilde{\alpha}-\sigma}\frac{\beta\frac{F_N}{k} - \frac{\theta}{k}}{1+\frac{\theta}{k}}$, where the ratio $\frac{F_N}{k}$ is constant in the Cobb-Douglas case, we end up with the following derivative:

$$\frac{dn^*}{dA} = -\frac{1-\sigma}{\tilde{\alpha}-\sigma}\frac{1}{\beta}\frac{\theta}{k^2}\left[1 + \beta\frac{\frac{c^{min}}{k}}{\left(1+\frac{\theta}{k}\right)^2}\right]\frac{dk}{dA} < 0 \qquad \text{(E.7)}$$

with $\frac{dn^*}{dA} = 0$ if $\theta = 0$.

4. Conclusions

Building on the intuition provided by Michel and Pestieau (1998), we propose an extended version of critical level utilitarianism (CLU) which we name Relative CLU. It consists in an axiomatically founded and variable critical level utility obtained by conditioning the critical level used by parents or society for deciding to give birth to an extra child on their own standard of living. We use the basic formulation for a normative analysis and an extended version for a positive analysis concerning: a) the Absolute Repugnant Conclusion (that is, maximum population growth rate); b) Relative Repugnant Conclusion (that is, consumption decreases after increases in personal income); c) too high childbearing costs; d) the dynamics of fertility after technological shocks.

We show that the basic formulation can avoid both the assumption of high childbearing costs and the Absolute Repugnant Conclusion but cannot avoid the Relative Repugnant Conclusion. Moreover, optimal fertility is increased by technological shocks and displays cycles.

If we take the view that our social welfare criterion can also be applied at individual dynastic level, we can aggregate the dynasties and obtain the equilibrium to the decentralized economy. Then our analysis could also be interpreted as descriptive (positive) in nature. However, given that under a positive perspective, both the persistence of the Relative Repugnant Conclusion and the positive effect of technology on fertility may appear unsatisfactory, we then present an extension of the model, a decentralized economy where each dynasty does not recognize that it affects the aggregate consumption level (consumption externalities). In this scenario, show that both Absolute and Relative Repugnant Conclusions can be avoided without resorting to unrealistically high childbearing costs. Moreover, a technological shock reduces long run fertility and can generate cycles along the transitional path.

References

Alonso-Carrera, J., Caballé, J., Raurich, X., (2005): "Growth, habit formation, and catching-up with the Joneses", *European Economic Review* 49:6, 1665-91;

Barro R.J. and Becker (1989): "Fertility Choice in a Model of Economic Growth", *Econometrica*, 57: 481-501;

Becker, G.S. and Barro, R.J., (1988): "A Reformulation of the Economic Theory of Fertility", *Quarterly Journal of Economics*, 103: 1-25;

Blackorby, C., Bossert, W., and Donaldson, D. (1995): "Intertemporal population ethics: critical-level utilitarian principles", *Econometrica,* 63(6): 1303-20;

Blackorby, C., Bossert, W., and Donaldson, D. (1997): "Birth-date dependent population ethics: critical-level principles", *Journal of Economic Theory,* 77(2): 260-84;

Blackorby, C., Bossert, W., and Donaldson, D. (2001): "The Axiomatic Approach to Population Ethics", *Discussion Paper 01-03*, University of British Columbia, Department of Economics;

Blackorby, C., Bossert, W., and Donaldson, D. (2002): "Critical –level population principles and the repugnant conclusion", *Department of Economics, University of British Columbia Discussion paper*, 02-16;

Blackorby, C., Bossert, W., and Donaldson, D. (2005): "*Population issues in social choice theory, welfare economics, and ethics*", Cambridge University Press;

Borck, R. (2011): "Federalism, fertility, and growth", *The Scandinavian Journal of Economics*, *113*(1), 30-54;

Bouchekkine, R., Fabbri, G. (2013): "Assessing Parfit's *Repugnant Conclusion* with a canonical endogenous set-up", *Journal of Population economics*, 26, 751-767;

Broome, J. (1996): "The Welfare Economics of Population", *Oxford Economic Papers*, 48(2): 177-193;

Cass, D. (1965): "Optimum Growth in an Aggregative Model of Capital Accumulation", *The Review of Economic Studies*, 32(3): 233-40;

Conde-Ruiz, J.I., Giménez, Pérez-Nievas, E.L. (2010): "Millian efficiency with endogenous fertility", *The Review of Economic Studies*, 77 (1), 154-187;

Cremer, H., Gahvari, F., and Pestieau, P. (2006): "Pensions with endogenous and stochastic fertility", *Journal of Public Economics*, 90: 2303–21;

Deardorff, A.V. (1976): "The Optimum Growth Rate for Population: Comment, *International Economic Review*, vol. 17(2): 510-15;

de la Croix, D., Michel, P. (1999): "Optimal growth when tastes are inherited", *Journal of Economic Dynamics and Control*, 23: 519-537;

de la Croix, D., Pestieau, P., and Ponthiere, G. (2012): "How powerful is demography? The Serendipity Theorem revisited", *Journal of Population Economics*, 25(3): 899-922;

Doepke, M. (2008): "Growth takeoffs", In: Steven N., Durlauf, Lawrence E., Blume (Eds.), *The New* Palgrave *Dictionary of Economics*. Second ed., Palgrave MacMillan, London;

Duesenberry, J.S., (1949): "*Income, Saving and the Theory of Consumer Behavior*", Harvard University

Press, Cambridge, M.A;

Easterlin, R.A. (1974; updated in 1995): "*Does economic growth improve the human lot? Some empirical evidence*", in David, P.A. and Reder M.W. (eds), *Nations and Households in Economic Growth*, *Essays in Honor of Moses Abramowitz*, Academic Press, NY: 89-125;

Easterlin, R.A. (1995): "Will raising the incomes of happiness of all?" *Journal of Economic Behavior and Organization*, 27(1), 35-47;

Edgeworth, F.Y. (1925): *Papers relating to political economy*, Vol III. MacMillan, London;

Ehrlich, I. and Kim, J. (2015): "Immigration, Human Capital Formation, and Endogenous Economic Growth", *Journal of Human Capital*, *9*(4), 518-563;

Galor, O. (2011): *Unified Growth Theory*. Princeton, NJ: Princeton University Press;

Golosov, M., Jones, L.E., and Tertilt M., (2007): "Efficiency with Endogenous Population Growth", *Econometrica*, 75: 1039-71;

Jaeger K. and Kuhle, W. (2009): "The optimum growth rate for population reconsidered", *Journal of Population Economics*, 22(1): 23-41;

Jones, L. E., and Schoonbroodt, A. (2016): "Baby busts and baby booms: the fertility response to shocks in dynastic models", *Review of Economic Dynamics*, *22*, 157-178;

Lagerlöf, N. P. (2015): "Malthus in Sweden", *The Scandinavian Journal of Economics*, *117*(4), 1091-1133;

Koopmans, T.C. (1960): "Stationary Ordinal Utility and Impatience"**,** *Econometrica*, 28(2): 287-309;

Koopmans, T.C. (1965): "On the Concept of Optimal Economic Growth", *Pontificiae Academiae Scientiarum Scripta Varia*, 28: 225-287. Reprinted as Cowles Foundation Paper 238 (1966);

Macunovich, D.J. (1998): "Fertility and the Easterlin hypothesis: An assessment of the literature", *Journal of Population Economics*, 11: 53-111;

Marsiglio, S. (2014): "Reassessing Edgeworth's conjecture when population dynamics is stochastic", *Journal of Macroeconomics* 42, 130–140;

Mertens, J. F., Rubinchik, A. (2017): "Discounting and welfare evaluation of policies". *Journal of Public Economic Theory*, *19*(5), 903-920;

Michel, P., Pestieau, P. (1998): "Optimal Population Without Repugnant Aspects", *Genus,* 54 (3/4): pp. 25-34;

Michel, P., Wigniolle, B. (2009): "Pareto-Efficiency and Endogenous Fertility: A Simple Model", *Mathematical Population Studies,*16(1), 36-59*;*

Nerlove, M., Razin, A. and Sadka, E. (1982): "Population size and the social welfare functions of Bentham and Mill", *Economics Letters*, *10*(1-2), 61-64;

Nerlove, M., Razin, A. and Sadka, E. (1985): "Population size: individual choice and social optima", *The Quarterly Journal of Economics*, *100*(2), 321-334;

Ng, Y.K. (1986): "Social Criteria for Evaluating Population Change: an Alternative to the Blackorby-Donaldson Criterion", *Journal of Public Economics* 29: 375-81;

Palivos, T., Yip, C.K. (1993): "Optimal population size and endogenous growth", *Economics Letters* 41, 107–110;

Parfit, D. (1976): "On Doing the Best for Our Children", in *Ethics and Populations*, ed. By M. Bayles. Cambridge: Schenkman;

Parfit, D. (1984): "*Reasons and Persons*", Oxford/New York: Oxford University Press;

Pestieau, P. and Ponthiere, G. (2014): "Optimal fertility along the life cycle", *Economic Theory*, 55: 185-224;

Prettner, K., and Strulik, H. (2016): "Technology, trade, and growth: The role of education", *Macroeconomic Dynamics*, *20*(05), 1381-1394;

Ramsey, F.P. (1928): "A Mathematical Theory of Saving", *Economic Journal*, 38: 543-59;

Razin, A. and Sadka, E. (1995): *Population Economics,* The MIT Press, Cambridge, MA;

Razin, A. and Yuen, C.W. (1995): "Utilitarian tradeoff between population growth and income growth", *Journal of Population Economics* 8, 81–87;

Renström, T.I. and Spataro, L (2011): "The Optimum Growth Rate for Population under Critical-Level Utilitarianism"; *Journal of Population Economics*, 24(3): 1181-201;

Renström, T.I. and Spataro, L (2015): "Population Growth and human capital: a Welfarist Approach", *The Manchester School* 83 (S2): 110-141.

Samuelson, P.A. (1975): "The Optimum Growth Rate for Population" *International Economic Review*, 16(3): 531-38;

Samuelson, P.A. (1976): "The Optimum Growth Rate for Population: Agreement and Evaluations", *International Economic Review*, 17(2), 516-525;

Shiell, L. (2008): "The Repugnant Conclusion and Utilitarianism under Domain Restrictions", *Journal of Public Economic Theory,* 10(6): 1011-31;

Spataro, L. and Renström, T.I. (2012): "Optimal taxation, critical-level utilitarianism and economic growth", *Journal of Public Economics*, 96: 727-738.

Technical Appendix

Appendix A. **Axiomatization of Relative Critical Level Utilitarianism (RCLU).**

Following the approach by Koopmans (1960), we begin by assuming a general welfare function defined over population and utility alternatives. We then introduce one by one the postulates (axioms) to arrive at the final welfare representation. The approach has the advantage that the exposition is simpler and far more accessible, and one can clearly see how the welfare function is developed.

For any time t we define, purely for notational convenience, the alternatives, X_t, as the utility vector $u_t=\{u_t^1,\ldots,u_t^N\}$ of generation t and the size of the next generation, N_{t+1}, i.e. $X_t=\{u_t, N_{t+1}\}$.[16] We assume the population criterion is represented by a general welfare function (assumed twice differentiable).

$$W_t = W_t(\ldots, X_{t-2}, X_{t-1}, X_t, X_{t+1}, \ldots) \quad \text{(A.1)}$$

We begin with a number of independence and dependence postulates (axioms). It is reasonable to assume that the choice of utility vectors and population sizes in the sufficiently distant future is independent of the outcomes of the earlier generations. That is, the preferences over u_t and N_{t+1} for large s should be independent of u_{t-1-s} and N_{t-s}. This is the axiom of Independence of the Long Dead as in Blackorby et al. (1995). To allow the possibility that population judgments depend on the utility level of the parental generation, we must allow for X_{t+1} to depend on X_t. Denote a sequence of all future alternatives as $X_t^f=\{X_{t+1}, X_{t+2},\ldots\}$ and past as $X_t^p=\{X_{t-1}, X_{t-2},\ldots\}$.

Axiom 1: *Independence of the Long Dead*

For all $X_{t-1}^p, \hat{X}_{t-1}^p, X_{t-1}, X_t, \hat{X}_t, X_t^f, \hat{X}_t^f$

$$W_t\left(X_{t-1}^p, X_{t-1}, X_t, X_t^f\right) \geq W_t\left(X_{t-1}^p, X_{t-1}, \hat{X}_t, \hat{X}_t^f\right)$$

implies

[16] We could have defined X_t as $=\{u_t, N_t\}$ or not used the notation X_t at all, without changing the results.

$$W_t(\hat{X}_{t-1}^p, X_{t-1}, X_t, X_t^f) \geq W_t(\hat{X}_{t-1}^p, X_{t-1}, \hat{X}_t, \hat{X}_t^f). \qquad \square$$

Axiom 2: *Parental Dependence*[17]

Preferences over X_t depend on X_{t-1}. □

First, independence of the long dead implies that any indifference relation between u_{t+1} and N_{t+2} (at time t+1) must be independent of the value of X_{t-1}. That is the preferences over X_{t+1} must be represented by some $\widetilde{W}_{t+1}$:

$$\widetilde{W}_{t+1} = \widetilde{W}_{t+1}(X_t, X_{t+1}, X_{t+2} \dots) \tag{A.2}$$

and any preferences over current and future X must be represented by some V_t, i.e. the aggregator function in Koopmans (1960)[18]:

$$W_t = V_t\left(\phi(X_{t-1}, X_t), \widetilde{W}_{t+1}(X_t, X_{t+1}, X_{t+2} \dots)\right). \tag{A.3}$$

Consequently, Independence of the Long Dead and Parental Welfare Dependence imply a weakly separable welfare function of the form (A.3).

Notice that $\widetilde{W}_{t+1}$ is potentially different from W_t, however, if the function is different, then the preferences over X_{t+1} as of time t+1 (given a particular history of X_t), would be different from the preferences over X_{t+1} as of time *t-s, s*≥1, which would generate time-inconsistency. We shall therefore assume time-independence (or time consistency):

Axiom 3: *Stationarity (Time Independence/Time Consistency)*

For all $X_{t-1}, X_t, X_t^f \hat{X}_t^f$

$$W_t(X_{t-1}, X_t, X_t^f) \geq W_t(X_{t-1}, \hat{X}_t, \hat{X}_t^f) \text{ iff } \quad W_t(X_t, X_t^f) \geq W_t(\hat{X}_t, \hat{X}_t^f) \qquad \square$$

Notice that the first inequality gives

$$V\left(\phi(X_{t-1}, X_t), \widetilde{W}_{t+1}(X_t, X_t^f)\right) \geq V\left(\phi(X_{t-1}, X_t), \widetilde{W}_{t+1}(X_t, \hat{X}_t^f)\right)$$

By the latter inequality in the definition of stationarity it follows that $\widetilde{W}_{t+1}(X_t^f)$ is equal to $G(W_{t+1}(X_t^f))$ where G is a monotone transformation. Now, with

[17] If generations do not overlap this is Dependence of the Recently Dead.

[18] Note that we cannot in general have $W_t = V_t(\phi(X_{t-1}), \widetilde{W}_{t+1}(X_t, X_{t+1}, X_{t+2} \dots))$ as then the preferences over X_t would be independent of X_{t-1}, and thus violate Parental Welfare Dependence. However, we could allow for this formulation as a special case and would then obtain Critical Level Utilitarianism.

$$W_t = V_t(\phi(X_{t-1}, X_t), G(W_{t+1}(X_t, X_t^f)) \quad \text{(A.5)}$$

and the further monotone transformation G^{-1}, we obtain, as in Koopmans (1960)

$$W(X_{t-1}, X_t, X_{t+1} \dots) = V\big(\phi(X_{t-1}, X_t), W(X_t, X_{t+1}, X_{t+2} \dots)\big) \quad \text{(A.6)}$$

that is a recursively separable social welfare function.

This welfare function, however, implies that in general the preferences over, say, u_t and N_{t+1}, depend on the values of u and N of *all* future generations. We limit such dependence as follows.

Axiom 4: *Independence of Distant Future Generations*

Any indifference relation between u_t^i and N_{t+1} is constant with respect to $W(X_t^f)$.[19] □

One more recursion of (A.6) gives

$$W(X_{t-1}, X_t, X_{t+1} \dots) = V\left(\phi(X_{t-1}, X_t), V\big(\phi(X_t, X_{t+1}), W(X_t^f)\big)\right) \quad \text{(A.7)}$$

To save on notation, denote

$$\phi^t = \phi(X_{t-1}, X_t) \quad \text{(A.8)}$$

$$\phi^{t+1} = \phi(X_t, X_{t+1}) \quad \text{(A.8')}$$

Any indifference relation between N_{t+1} and u_t^i is given by

$$\left.\frac{\partial N_{t+1}}{\partial u_t^i}\right|_{dW=0} = \frac{\frac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial \phi^t}\frac{\partial \phi^t}{\partial u_t^i} + \frac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial V(\phi^{t+1}, W)}\frac{\partial V(\phi^{t+1}, W)}{\partial \phi^{t+1}}\frac{\partial \phi^{t+1}}{\partial u_t^i}}{\frac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial \phi^t}\frac{\partial \phi^t}{\partial N_{t+1}} + \frac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial V(\phi^{t+1}, W)}\frac{\partial V(\phi^{t+1}, W)}{\partial \phi^{t+1}}\frac{\partial \phi^{t+1}}{\partial N_{t+1}}} \quad \text{(A.9)}$$

For this indifference relation to be constant with respect to $W(X_t^f)$, the ratio of the partial derivatives must be constant:

$$\frac{\frac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial V(\phi^{t+1}, W)}\frac{\partial V(\phi^{t+1}, W)}{\partial \phi^{t+1}}}{\frac{\partial V(\phi^t, V(\phi^{t+1}, W))}{\partial \phi^t}} = \beta \quad \text{(A.10)}$$

Note that β could in fact a discount factor if it is smaller than 1. Although one might argue that applying discounting to intergenerational allocation of resources could be ethically unappealing, we aim at following the lines of Blackorby et al. (1997). (A.10) implies that the function V must be linear:

$$W(X_{t-1}, X_t, X_{t+1}, \dots) = \phi(X_{t-1}, X_t) + \beta[\phi(X_t, X_{t+1}) + \beta W((X_{t+1}, X_{t+2}, \dots)] \quad \text{(A.11)}$$

Or by recursion

[19] Preferences over X_t are independent of X_{t+s}, for all $s \geq 2$. We note that we cannot have independence of X_{t+1}, without sacrificing dependence on X_{t-1}. Thus, a necessary consequence of Parental Dependence and Independence of the Long Dead, is dependence of at least the next (immediate) generation.

$$W(X_{t-1}, X_t, X_{t+1}, \dots) = \sum_{s=0}^{\infty} \beta^s \phi(X_{t-1+s}, X_{t+s}) \quad \text{(A.12)}$$

that is, a discounted sum of generational welfare functions.

Having established the consequences of Independence of the Long Dead, Independence of Distant Future Generations, and Parental Dependence, we now turn to the specifics of the population choice. We now write the pairs $X_{t-1} = \{u_{t-1}, N_t\}$ and $X_t = \{u_t, N_{t+1}\}$ explicitly. Recall that u_t is a vector of individual utilities $\{u_t^1, \dots, u_t^N\}$, i.e. $\phi_t = \phi(u_{t-1}, N_t, u_t, N_{t+1})$. Note that N_{t+1} can be dropped at time t as we are only considering welfarist criteria (population only matters to the extent it brings individual utilities), then

$$\phi_t = \phi(u_{t-1}, N_t, u_t) \quad \text{(A.13)}$$

We will now define our population criterion for each generation.

Relative Critical Level

A society is indifferent adding individuals to an existing population, everything else equal, if the utility of those added equals a critical level function $\tilde{\alpha}(u_{t-1})$, depending on the utility of consumption of someone in the previous generation. If individuals differ within a generation, this level of consumption must be identified. We label this as critical-level consumption, c_{t-1}^r.

The critical level cannot depend on the population size of the previous generation (only the critical-consumption level), as can be seen in (A.13) (it is ruled out by Independence of the Long Dead, as N_{t-1} was a decision taken by the N_{t-2} generation). The critical level must also satisfy anonymity (to be introduced later), implying it must be invariant with respect to any permutation (renaming the indexes).

A permissible relative critical level is the utility of a fraction, $0<\delta<1$, of the consumption level of the top *r*th individual of cohort *t*-1:

$$\tilde{\alpha}(u_{t-1}^r) = u(\delta c_{t-1}^r) \quad \text{(A.14)}$$

This level is invariant to adding individuals of critical level consumption, if delta is not too large. To see this, order consumption (by renaming individuals) as $c_{t-1}^1 \geq c_{t-1}^2 \geq \cdots \geq c_{t-1}^r \geq \cdots \geq c_{t-1}^{N_{t-1}}$.

Adding individuals of the critical level (applied to their generation), will not change the level of c_{t-1}^{r} as long as $c_{t-1}^{r} \geq \delta c_{t-2}^{r}$. Furthermore, the critical level is invariant with respect to renaming individuals (as the ordering of consumption is independent of the 'names').

It should be noted that average consumption as critical-level consumption does not have the property of being independent of adding individuals of the critical level (as it would be declining). The same is also true for median consumption. However, if $c_{t-1}^{i} = c_{t-1}$ for all I (as under a first-best allocation) then the average, the median, and the top *r*th levels are the same. Consequently, (A.11) is:

$$W(X_{t-1}, X_t) = \phi(u_{t-1}^{r}, N_t, u_t) + \beta\phi(u_t^{r}, N_{t+1}, u_{t+1}) + \beta^2 W(X_{t+1}, X_{t+2}) \quad \text{(A.15)}$$

For given population size, the social preferences over individual utilities may depend on population size and past utilities. We shall impose that the social preferences over utilities within a cohort should be independent of utilities of unconcerned individuals (but not necessarily their existence), that is if the utility vector $\{u_t\}$ is preferred to another utility vector $\{u_t'\}$, this should be the case regardless utilities of other individuals. We require

Axiom 5: *Independence of Utilities of Unconcerned Individuals*

Preferences over $\{u_t^i\}$ and $\{u_t^j\}$ are independent of $\{u_t^h\}$ for $h \neq i,j$, and of u_{t-1}^{r}.

□

We also require[20]:

Axiom 6: *Anonymity*

Preferences over $\{u_t^i\}$ are independent of identity *i*. □

Axiom 7: *Strong Pareto Principle*

Welfare is increasing in each u_t^i. □

Axiom 5 first requires (A.15) to be weakly separable in $\{u_t\}$, i.e.

$$W(X_{t-1}, \dots) = \phi\big(\Pi(u_{t-1}^{r}, N_t), N_t, \Psi(u_t, N_t)\big), +\beta\phi\big(\Pi(u_t^{r}, N_{t+1}), N_{t+1}, \Psi(u_{t+1}, N_{t+1})\big) + \beta^2 W(X_{t+1}, \dots)$$

for some functions Π and Ψ, that is the vector u_t enters as a function $\Psi(u_t)$, such that $\phi(u_{t-1}, N_t, \Psi(u_t), N_{t+1})$. Second, it requires $\Psi(u_t)$ to be an additive function. To see this, differentiate the welfare function with respect to u_t^j and u_t^i to obtain:

[20] These are birth-date dependent statements, as in Blackorby et al. (1997).

$$\frac{\frac{\partial W(X_{t-1},\dots)}{\partial u_t^j}}{\frac{\partial W(X_{t-1},\dots)}{\partial u_t^i}} = \frac{\frac{\partial \Psi}{\partial u_t^j}}{\frac{\partial \Psi}{\partial u_t^i}}$$

which can be independent of uh, $h \neq i,j$, only if it is additive: $\Psi = \sum_{i=1}^{N_t} \psi(u_t^i, N_t)$. Anonymity requires the function ψ to be independent of i, and must be a strictly increasing function due to the Strong Pareto Principle. Differentiating with respect to $u^r{}_t$ and $u^i{}_t$ gives:

$$\frac{\frac{\partial W(X_{t-1},\dots)}{\partial u_t^r}}{\frac{\partial W(X_{t-1},\dots)}{\partial u_t^i}} = \frac{\frac{\partial \phi^t}{\partial \Psi}\frac{\partial \psi(u_t^r, N_t)}{\partial u_t^r} + \beta \frac{\partial \phi^{t+1}}{\partial \Pi}\frac{\partial \Pi(u_t^r, N_{t+1})}{\partial u_t^r}}{\frac{\partial \phi^t}{\partial \Psi}\frac{\partial \psi(u_t^i, N_t)}{\partial u_t^i}}$$

which can be independent of uh, $h \neq i,r$, only if it is ϕ is linear. Consequently

$$W(X_{t-1},\dots) = \Pi(u_{t-1}^r, N_t) + \sum_{i=1}^{N_t} \psi\big(u_t^i, N_t\big) + \beta[\Pi(u_t^r, N_{t+1}) + \sum_{i=1}^{N_t} \psi\big(u_{t+1}^i, N_{t+1}\big)] + \beta^2 W(X_{t+1},\dots)$$

(A.16)

All the reasoning up until now has been for fixed populations. We now turn to the population criterion itself.

A society is indifferent adding individuals to an existing population, everything else equal, if the utility of those added equals a critical level function $\tilde{\alpha}(u_{t-1}^r)$.

Axiom 8: *Relative Critical Level Utilitarianism*

Adding m number of individuals, with utilities at the relative-critical level, $\tilde{\alpha}(u_{t-1}^r)$ does not change the value of the social welfare function.

□

Axiom 8 implies that

$$W(X_{t-1},\dots) = \Pi(u_{t-1}^r, N_t + m) + \sum_{i=1}^{N_t} \psi\big(u_t^i, N_t + m\big) + m\psi(\tilde{\alpha}(u_{t-1}^r), N_t + m) + \beta(\Pi + \Psi) + \beta^2 W(X_{t+1},\dots) \quad \text{(A.17)}$$

is invariant with respect to m.

Notice that Π and Ψ in period t+1 are unaffected by the population size N_t (a consequence of our definition of critical-level utility). Then (A.17), by definition of critical level utility, must be constant in m, i.e.

$$\frac{dW}{dm}=\frac{\partial\Pi(u_{t-1}^{r},N_t+m)}{\partial N_t}+\sum_{i=1}^{N_t}\frac{\partial\psi\left(u_t^i,N_t+m\right)}{\partial N_t}+\psi(\tilde{\alpha}(u_{t-1}^{r}),N_t+m)+m\frac{\partial\psi(\tilde{\alpha}(u_{t-1}^{r}),N_t+m)}{\partial N_t}=0$$

Since $\tilde{\alpha}(u_{t-1}^{r})$ cannot depend on the utilities of the other individuals, the derivative of ψ w.r.t. N_t must be zero, consequently

$$\psi(\tilde{\alpha}(u_{t-1}^{r}))=-\frac{\partial\Pi(u_{t-1}^{r},N_t+m)}{\partial N_t}$$

Then, $\Pi(u_{t-1}^{r},N_t)=-\psi(\tilde{\alpha}(u_{t-1}^{r}))N_t$, which substituted into (A.17) gives

Proposition A.1 (Relative Critical Level Utilitarianism)

A social welfare function (representing a social welfare ordering over population and consumption choice) that satisfies Independence of the Long Dead, Stationarity, Independence of Distant Future Generations, Independence of Utilities of Unconcerned Individuals, Anonymity, Pareto Principle and Relative Critical Level, must take the form:

$$W(u_{t-1},N_t,u_t,N_{t+1}\dots)=\sum_{s=0}^{\infty}\beta^s\sum_{i=1}^{N_{t+s}}\left[\psi\left(u_{t+s}^i\right)-\psi(\tilde{\alpha}(u_{t-1+s}^{r}))\right] \tag{A.18}$$

where $\psi'\left(u_{t+s}^i\right)-\psi'(\tilde{\alpha}(u_{t-1+s}^{r}))\tilde{\alpha}'(u_{t-1+s}^{r})>0$ for any s.

Note that if the Relative Critical Level does not depend on past utility, then α is constant, and the population principle reduces to Generalised Critical Level Utilitarianism as in Blackorby et al. (1995). If the critical level is zero, it reduces to Generalised Classical Utilitarianism.

When considering a first best allocation within each generation, i.e. $u_t^1=u_t^2=\dots=u_t^N$ the function ψ is redundant, and we have

$$W(u_{t-1},N_t,u_t,N_{t+1}\dots)=\sum_{s=0}^{\infty}\beta^s N_{t+s}[u_{t+s}-\hat{\alpha}(u_{t-1+s}^{r})] \tag{A.19}$$

Without loss of generality we can redefine the critical level function as $\hat{\alpha}(u_{t-1+s})\equiv\alpha(c_{t-1+s})$.

APPENDIX B. Characterization of the basic model

1. The value Function in the basic framework

The Bellman equation of the problem can be written as:

$$V(K_t,N_t,c_{t-1})=max\{N_t[u(c_t)-\alpha(c_{t-1})]+\beta V(K_{t+1},N_{t+1},c_t)\} \tag{B.1}$$

The envelope condition with respect to c_{t-1} gives $\frac{\partial V}{\partial c_{t-1}}=-\alpha'(c_{t-1})N_t$, which shows that the value function is additively separable in c_{t-1}. Hence, we can define:

$$\tilde{V}(K_t, N_t) \equiv V(K_t, N_t, c_{t-1}) + N_t \alpha(c_{t-1}) \quad \text{(B.2)}$$

so that , substituting into B.1 and exploiting $N_{t+1} = (1+n_t)N_t$ we get:

$$\tilde{V}(K_t, N_t) = max\{N_t[u(c_t) - \beta(1+n_t)\alpha(c_t)] + \beta\tilde{V}(K_{t+1}, N_{t+1})\} \quad \text{(B.3)}$$

The envelope conditions with respect to N_t and K_t of (B.3) and using resource constraint we get:

$$\frac{\partial \tilde{V}_t}{\partial K_t} = \beta \frac{\partial \tilde{V}}{\partial K_{t+1}}\left(1 + F_{K_t}\right) \quad \text{(B.4)}$$

$$\frac{\partial \tilde{V}_t}{\partial N_t} = [u(c_t) - \beta(1+n_t)\alpha(c_t)] + \beta \frac{\partial \tilde{V}_{t+1}}{\partial K_{t+1}}\left[F_{N_t} - c_t - \theta(1+n_t)\right] + \beta \frac{\partial \tilde{V}_{t+1}}{\partial N_{t+1}}(1+n_t) \quad \text{(B.5)}$$

so that, exploiting again resource constraint we get

$$\frac{\partial \tilde{V}_t}{\partial K_t} K_t + \frac{\partial \tilde{V}_t}{\partial N_t} N_t = max\left\{N_t[u(c_t) - \beta(1+n_t)\alpha(c_t)] + \beta\left[\frac{\partial \tilde{V}_{t+1}}{\partial K_{t+1}} K_{t+1} + \frac{\partial \tilde{V}_{t+1}}{\partial N_{t+1}} N_{t+1}\right]\right\} \quad \text{(B.6)}$$

which proves that $\tilde{V}$ is homogenous of degree one. Hence, we can define

$$v(k) = \frac{\tilde{V}(K,N)}{N} = \tilde{v}(k,1) \quad \text{(B.7)}$$

so that (B.3) can we written as:

$$v(k_t) = max\{[u(c_t) - \beta(1+n_t)\alpha(c_t)] + \beta(1+n_t)v(k_{t+1})\} \quad \text{(B.8)}$$

under the constraint

$$(1+n_t)k_{t+1} = f_t + k_t - c_t - \theta(1+n_t) \quad \text{(B.9)}$$

2. Dynamics of the system, concavity and stability of the steady state

FOCs w.r.t. c_t and n_t give

$$H_c \equiv u'(c_t) - \beta(1+n_t)\alpha'(c_t) - \beta v'(k_{t+1}) = 0 \quad \text{(B.10)}$$

$$H_n \equiv -\beta\alpha(c_t) + \beta v(k_{t+1}) - \beta v'(k_{t+1})(k_{t+1} + \theta) = 0 \quad \text{(B.11)}$$

and SOCs

$$H_{cc} = u''(c_t) - \beta(1+n_t)\alpha''(c_t) + \frac{\beta}{(1+n_t)} v''(k_{t+1}) \quad \text{(B.12)}$$

$$H_{cn} = -\beta\alpha'(c_t) + \beta\frac{k_{t+1}+\theta}{1+n_t} v''(k_{t+1}) \quad \text{(B.13)}$$

$$H_{nn} = \beta\frac{(k_{t+1}+\theta)^2}{1+n_t} v''(k_{t+1}) \quad \text{(B.14)}$$

with H the Hessian matrix in *c* and *n* and

$$|H| = \beta v''(k_{t+1})\frac{k_{t+1}+\theta}{1+n_t}\{[(u''(c_t) - \beta(1+n_t)\alpha''(c_t)](k_{t+1}+\theta) + 2\beta\alpha'(c_t)\} - [\beta a'(c_t)]^2 \quad \text{(B.15)}$$

the determinant of the Hessian matrix (which needs to be positive for concavity). By total differentiation of (B.10) and (B.11) it descends that:

$$H_{cc}\frac{dc_t}{dk_t}+H_{cn}\frac{dn_t}{dk_t}-\beta v''(k_{t+1})\frac{\partial k_{t+1}}{\partial k_t}=0 \tag{B.16}$$

$$H_{cn}\frac{dc_t}{dk_t}+H_{nn}\frac{dn_t}{dk_t}-\beta(k_{t+1}+\theta)v''(k_{t+1})\frac{\partial k_{t+1}}{\partial k_t}=0 \tag{B.17}$$

By recognizing that, from resource constraint $\frac{\partial k_{t+1}}{\partial k_t}=\frac{1+f_t'}{1+n_t}$ and using Cramer's rule we obtain:

$$\frac{dc_t}{dk_t}=\beta(1+f'_t)\frac{v''(k_t)}{1+n_t}\frac{H_{nn}-(k_{t+1}+\theta)H_{cn}}{|H|}=\beta(1+f'_t)\alpha'(c_t)\frac{v''(k_t)}{1+n_t}\frac{\beta(k_{t+1}+\theta)}{|H|} \tag{B.18}$$

$$\frac{dn_t}{dk_t}=\beta(1+f'_t)\frac{v''(k_t)}{1+n_t}\frac{H_{cc}(k_{t+1}+\theta)-H_{cn}}{|H|}=\beta(1+f'_t)\frac{v''(k_t)}{1+n_t}\frac{[(u''(c_t)-\beta(1+n_t)\alpha''(c_t)](k_{t+1}+\theta)+\beta\alpha'(c_t)}{|H|}$$

(B.19)

Finally, by totally differentiating the resource constraint one gets

$$\frac{dk_{t+1}}{dk_t}=\frac{1+f'_t}{1+n_t}-\frac{1}{1+n_t}\frac{dc_t}{dk_t}-\frac{k_{t+1}+\theta}{1+n_t}\frac{dn_t}{dk_t} \tag{B.20}$$

and exploiting (B.18) and (B.19), (B.20) becomes

$$\frac{dk_{t+1}}{dk_t}=-\frac{(1+f'_t)}{1+n_t}\frac{(\beta\alpha'(c_t))^2}{|H|} \tag{B.21}$$

The envelope condition with respect to k_t is:

$$v'(k_t)=\beta v'(k_{t+1})(1+f_t') \tag{B.22}$$

Eqs. (B.18), (B.19) and (B.21) fully characterize the dynamics of the system. The steady state conditions follow from the FOCs.

3. Stability

In order to have stability of the steady state equilibrium, by recognizing that $(1+f')\beta=1$, it must be:

$$\left|\frac{dk_{t+1}}{dk_t}\right|=\frac{(\alpha')^2}{1+n}\frac{\beta}{|H|}<1 \tag{B.23}$$

(B.15) in steady state can be written as:

$$|H|+(\beta\alpha')^2=\beta v''\frac{k+\theta}{1+n}T \tag{B.24}$$

where

$$T\equiv[u''-\beta(1+n)\alpha''(c)](k+\theta)+2\beta\alpha'. \tag{B.25}$$

Moreover, by differentiating (B.22) we get that at the steady state

$$v''\left(1-\frac{dk_{t+1}}{dk_t}\right)=f''\beta v' \quad \text{(B.26)}$$

Using (B.21) as well as the steady state condition $(1+f')\beta=1$ in (B.26) and combining with (B.24) gives a quadratic equation in $|H|$:

$$[|H|+(\beta\alpha')^2]\left[|H|+\frac{1}{\beta(1+n)}(\beta\alpha')^2\right]=|H|f''\beta^2 v'\frac{k+\theta}{1+n}T \quad \text{(B.27)}$$

Let us define $x\equiv\ \beta\frac{(\alpha')^2}{1+n}$, $s\equiv f''\beta^2 v'\frac{k+\theta}{1+n}T$, $\Delta\equiv|H|-\frac{\beta}{(1+n)}(\alpha')^2$, $y\equiv\ 1+\beta(1+n)$, so that eq. (B.27) can be written as the quadratic form in Δ:

$$\Delta^2+(2x+xy-s)\Delta+2x^2y-xs=0 \quad \text{(B.28)}$$

the roots are

$$\Delta_{1,2}=\frac{s-x(2+y)}{2}\pm\frac{\sqrt{[s-x(2+y)]^2+4x[s-x(2+y)]+4x^2y}}{2}$$

Recall that, for stability to hold

$$\Delta\equiv\ |H|-\frac{\beta}{(1+n)}(\alpha')^2>0 \quad \text{(B.29)}$$

In order to get a positive root, it must be that $4x[s-x(2+y)]+4x^2y>0$,[21] which is true iff $s>2x$, i.e. iff

$$(k+\theta)Tf''\beta v'>2(\alpha')^2 \quad \text{(B.30)}$$

with $v'=\frac{u'-\beta(1+n)\alpha'}{\beta}$.

Notice that, by eq. (B.29), $\Delta>0$ implies $|H|$>0. To conclude:

1) stability of the equilibrium implies $|H|$>0 (by B.29), which provides concavity of the objective function w.r.t. to *c* and *n*;
2) $T<0$ (from B.27, since $f''<0$) and, by eq. (B.15), $v''<0$, that is, concavity of the value function w.r.t. *k* is guaranteed;
3) From eq. (B.21) the system displays cycles in *k* in the neighbourhood of the steady state;
4) From eq. (B.19), fertility has the same qualitative dynamics as capital (i.e. they move in the same direction in each period);
5) From eq. (B.18), consumption moves in the opposite direction of *k*.

[21] Of course, a stricter condition is $s>2x+xy$ (sufficient), but we look for the weakest condition.

Appendix C: Proof of Proposition 1

The first derivatives of both sides of (19) w.r.t. c are:

$$\frac{\partial LHS}{\partial c} = 1 - \frac{u-\alpha}{u'-\alpha'}\frac{u''-\alpha''}{u'-\alpha'} \tag{P1.1}$$

$$\frac{\partial RHS}{\partial c} = 1 + 2\frac{\alpha'}{u'-\alpha'}\frac{(c-c^{min})}{(c^{max}-c^{min})} + \left(\frac{\alpha''}{u'-\alpha'} - \frac{\alpha'}{u'-\alpha'}\frac{u''-\alpha''}{u'-\alpha'}\right)\frac{(c-c^{min})^2}{(c^{max}-c^{min})} \tag{P1.2}$$

In Appendix B we show that, necessary for concavity of the objective function is

$$T \equiv \beta[2\alpha' + u''(c^{max} - c^{min}) - \alpha''(c^{max} - c)] < 0\ . \tag{P1.3}$$

Combining (P1.1)-(P1.3) it follows that concavity holds iff $\frac{\partial LHS}{\partial c} > \frac{\partial RHS}{\partial c}$ at the steady state. Let us also notice that $RHS(c^{min}) = 0, RHS(c^{max}) > 0$, and $\frac{\partial LHS}{\partial c} > 0$. Finally, the roots to (19) are:

$$c - c^{min} = -\frac{1-\alpha}{2\alpha}(c^{max} - c^{min}) \pm \sqrt{\left[\frac{1-\alpha}{2\alpha}(c^{max} - c^{min})\right]^2 + \frac{u-\alpha}{u\prime-\alpha\prime}} \tag{P1.4}$$

and only one is positive (because $LHS = \frac{u-\alpha}{u'-\alpha'} > 0$ at the crossing point). Hence, the system has either zero or one equilibrium. In either cases we can conclude that necessary for a crossing point is $LHS(c^{max}) > RHS(c^{max})$ (that is, LHS and RHS cross at some $c^* \in (c^{min}, c^{max})$), which reads as:

$$u(c^{max}) - u'(c^{max})(c^{max} - c^{min}) > \alpha'(c^{max}) \tag{P1.5}$$

Sufficient for one crossing point is then either $LHS(c^{min}) < RHS(c^{min})$ (if $c^{min} > 0$) or $LHS(0) < RHS(0)$ (if $c^{min} < 0$), giving the conditions $u(c^{min}) < \alpha(c^{min})$ and $\frac{u(0)-\alpha(0)}{u'(0)-\alpha\prime(0)} \leq 0$, respectively. □

Appendix D. Decentralized model with atomistic dynasties

1. The Value function

Each dynasty chooses C_t, N_t and K_{t+1}, taking as given the time paths of α_t, the interest rate (r_t) and the wage rate (w_t). Those latter quantities are functions of the economy-wide capital stock. Consequently the transition equation for the economy-wide capital stock is needed when formulating the Bellman equation (in order to 'compute' α_t, r_t and w_t). The value function, therefore, depends on own K_t, N_t and economy-wide $\overline{K}_t, \overline{N}_t$. By following the same lines as in Appendix B (invoking the envelope conditions w.r.t. $K_t, N_t, \overline{K}_t, \overline{N}_t$) we can write the Bellman equation in per capita terms as:

$$v(k_t;\bar{k}_t) = max\{[u(c_t) - \beta(1+n_t)\alpha(\bar{c}_t)] + \beta(1+n_t)v(k_{t+1};\bar{k}_{t+1})\} \quad \text{(D.1)}$$

subject to: $k_{t+1} + \theta = \frac{(1+r_t)k_t + w_t - c_t}{1+n_t}$ (D.2)

taking as given: $\bar{k}_{t+1} + \theta = \frac{\bar{k}_t + f(\bar{k}_t) - \bar{c}_t}{1+\bar{n}_t}$. (D.3)

and $w_t = f(\bar{k}_t) - f'(\bar{k}_t)\bar{k}_t$ and $r_t = f'(\bar{k}_t)$. FOCs w.r.t. c and n and envelope condition w.r.t. k yield:

$$v'(k_t) = \beta v'(k_{t+1})(1+r_t) \quad \text{(D.4)}$$

$$u'(c_t) - \beta v'(k_{t+1}) = 0 \quad \text{(D.5)}$$

$$-\beta\alpha(\bar{c}_t) + \beta v(k_{t+1}) - \beta v'(k_{t+1})(k_{t+1}+\theta) = 0 \quad \text{(D.6)}$$

where the latter condition holds under interiority of solution for n. Differentiation of (D.6) w.r.t. k_{t+1} gives:

$$-\beta v''(k_{t+1})(k_{t+1}+\theta)dk_{t+1} = 0 \quad \text{(D.7)}$$

which implies that $v''(k_{t+1}) = 0$, that is, the value function is linear in k_{t+1}, so that it can be written as

$$v(k) = \Phi(\bar{k}) + k \cdot \Psi(\bar{k}) \quad \text{(D.8)}$$

for some function Φ, Ψ, so that, eqs. (D.4)-(D.6) can be written as:

$$\Psi(\bar{k}_t) = \beta(1+r_t)\Psi(\bar{k}_{t+1}) \quad \text{(D.9)}$$

$$u'(c_t) - \beta\Psi(\bar{k}_{t+1}) = 0 \quad \text{(D.10)}$$

$$\alpha(\bar{c}_t) = \Phi(\bar{k}_{t+1}) - \theta\Psi(\bar{k}_{t+1}). \quad \text{(D.11)}$$

Bellman equation (D.1) now reads as:

$$v(k_t;\bar{k}_t) = max\{u(c_t) + \beta(1+n_t)(k_{t+1}+\theta)\Psi(\bar{k}_{t+1})\} \quad \text{(D.12)}$$

and, exploiting (D.8), (D.9) and (D.2):

$$\Phi(\bar{k}_t) = \left\{u(c_t) + \frac{w_t - c_t}{1+r_t}\Psi(\bar{k}_t)\right\} \quad \text{(D.13)}$$

Note that differentiating (D.13) w.r.t. c_t reproduces (D.10), so that, when taking derivatives w.r.t. $\bar{k}_t$ the effect on c_t can be ignored.

Differentiating (D.13) w.r.t. $\bar{k}_t$, noting that $w_t = f(\bar{k}_t) - f'(\bar{k}_t)\bar{k}_t$ and $r_t = f'(\bar{k}_t)$ yields

$$\Phi'(\bar{k}_t) = \left[-\frac{f''(\bar{k}_t)\bar{k}_t}{1+r_t} - f''(\bar{k}_t)\frac{w_t - c_t}{(1+r_t)^2}\right]\Psi(\bar{k}_t) + \frac{w_t - c_t}{1+r_t}\Psi(\bar{k}_t) \quad \text{(D.14)}$$

so that, exploiting (D.2) and (D.3) (need be positive)

$$\Phi'(\bar{k}_t) - \theta\Psi'(\bar{k}_t) = -\frac{f''(\bar{k}_t)}{(1+r_t)^2}(1+\bar{n}_t)(\bar{k}_{t+1}+\theta)\Psi(\bar{k}_t) + \left[\frac{(1+\bar{n}_t)(\bar{k}_{t+1}+\theta)}{(1+r_t)} - (k_t+\theta)\right]\Psi'(\bar{k}_t). \quad \text{(D.15)}$$

At the steady state we get

$$\Phi'(\bar{k}) - \theta\Psi'(\bar{k}) = -(\bar{k}+\theta)\{\beta^2 f''(\bar{k})(1+\bar{n})\Psi(\bar{k}) + [1-\beta(1+\bar{n})]\Psi'(k)\}. \quad \text{(D.16)}$$

This equation will be used later. Next, differentiation of (D.9) w.r.t. $\bar{k}_t$ and equilibrium prices provide:

$$\Psi'(\bar{k}_t) = \beta f''(\bar{k}_t)\Psi(\bar{k}_{t+1}) + \beta[1+f'(\bar{k}_t)]\Psi'(\bar{k}_t)\frac{d\bar{k}_{t+1}}{d\bar{k}_t}. \quad \text{(D.17)}$$

In steady state (D.17) reads:

$$\frac{d\bar{k}_{t+1}}{d\bar{k}_t} = 1 - \beta f''(\bar{k})\frac{\Psi(\bar{k})}{\Psi'(\bar{k})}. \quad \text{(D.18)}$$

Next, substituting for (D.9) into (D.10) and differentiating w.r.t. $\bar{k}_t$ we get:

$$u''(c_t)\frac{dc_t}{d\bar{k}_t} = -\frac{f''(\bar{k}_t)}{(1+r_t)^2}\Psi(\bar{k}_t) + \frac{\Psi'(\bar{k}_t)}{(1+r_t)} \quad \text{(D.19)}$$

Differentiating (D.11)

$$\alpha'(\bar{c}_t)\frac{dc_t}{d\bar{k}_t} = [\Phi'(\bar{k}_{t+1}) - \theta\Psi'(\bar{k}_{t+1})]\frac{d\bar{k}_{t+1}}{d\bar{k}_t} \quad \text{(D.20)}$$

Noticing that in equilibrium $c_t = \bar{c}_t$, then combining (D.19) and (D.20) gives:

$$\frac{\alpha'(\bar{c}_t)}{u''(\bar{c}_t)}\left[-\frac{f''(\bar{k}_t)}{(1+r_t)^2}\Psi(\bar{k}_t) + \frac{\Psi'(\bar{k}_t)}{(1+r_t)}\right] = [\Phi'(\bar{k}_t) - \theta\Psi'(\bar{k}_t)]\frac{d\bar{k}_{t+1}}{d\bar{k}_t} \quad \text{(D.21)}$$

and (D.21) in steady state becomes:

$$\beta\frac{\alpha'(\bar{c}_t)}{u''(\bar{c}_t)}\Psi'(\bar{k})\left[1 - \beta f''\frac{\Psi(\bar{k})}{\Psi'(\bar{k})}\right] = [\Phi'(\bar{k}) - \theta\Psi'(\bar{k})]\frac{d\bar{k}_{t+1}}{d\bar{k}_t} \quad \text{(D.22)}$$

which, using (D.18) gives

$$\beta\frac{\alpha'(\bar{c}_t)}{u''(\bar{c}_t)}\Psi'(\bar{k}) = [\Phi'(\bar{k}) - \theta\Psi'(\bar{k})]. \quad \text{(D.23)}$$

Exploiting (D.23) and (D.16) we get that

$$\frac{1 - \frac{\beta\alpha'(\bar{c})}{-u''(\bar{c})(\bar{k}+\theta)}}{\beta(1+n)} = \left[1 - \beta f''\frac{\Psi(\bar{k})}{\Psi'(\bar{k})}\right] = \frac{d\bar{k}_{t+1}}{d\bar{k}_t} \quad \text{(D.24)}$$

where the last equality comes from (D.18).

2. Existence of the steady state equilibrium

From eqs. (D.10), (D.11) and (D.13), computed at the steady state, we get:

$$\frac{u(c) - \alpha(c)}{u'(c)} = c - w + \frac{\theta}{\beta} \quad \text{(D.25)}$$

The latter equation can be written as:

$$u(c) - cu'(c) = \alpha(c) - \left(w - \frac{\theta}{\beta}\right) u'(c) \qquad \text{(D.26)}$$

Notice that, under concavity of $u(c)$ and normalization $u(0) = 0$, the LHS of (D.26) has a positive sign, so that, if the solution is interior, also the RHS. In previous works on endogenous population, given that $\alpha(c) = 0$, interior solution needed the assumption of very high costs for childbearing, that is $w - \frac{\theta}{\beta} < 0$. Let us assume that $w \geq \frac{\theta}{\beta}$ (that is, $F_N \geq \theta(1 + r)$). Eq. (D.26) can be written as:

$$\eta(c)\left[1 - \frac{\alpha(c)}{u(c)}\right] - 1 = -\left(w - \frac{\theta}{\beta}\right)\frac{u\prime(c)}{c} \qquad \text{(D.27)}$$

with $\eta(c) \equiv \frac{u(c)}{cu'(c)} > 1$ under concavity of $u(c)$. Given that the RHS of (D.27) is negative under our assumptions, existence of the equilibrium implies the restriction

$$\left[1 - \frac{\alpha(c)}{u(c)}\right] < \frac{1}{\eta(c)} \qquad \text{(D.28)}$$

For example, under CES utility and $\alpha(c) = \tilde{\alpha}u(c)$ the latter condition reads $\tilde{\alpha} \geq \sigma$.

3. Uniqueness and stability of the steady state equilibrium

As for uniqueness and stability of the steady state equilibrium, let us assume $\lim_{c \to 0} cu'(c) = 0$, $\alpha'(c) > 0 \ and \ \alpha(0) = 0$.

Thus, as for (D.26), we have LHS(c=0)=0, and RHS(c=0) <0 (possibly $-\infty$). Differentiating (D.26) yields:

$$\frac{\partial LHS}{\partial c} = -cu''(c) > 0 \qquad \text{(D.29)}$$

$$\frac{\partial RHS}{\partial c} = \alpha'(c) - \left(w - \frac{\theta}{\beta}\right) u''(c) > 0 \qquad \text{(D.30)}$$

Necessary for existence and uniqueness of an interior solution is that $\frac{\frac{\partial LHS}{\partial c}}{\frac{\partial RHS}{\partial c}} < 1$, that is

$$\frac{\alpha'(c)}{u'(c)}\frac{1}{\sigma(c)} + \frac{w - \frac{\theta}{\beta}}{c} > 1 \qquad \text{(D.31)}$$

where $\sigma(c) \equiv -u''(c)c/u'(c)$. Sufficient for inequality (D.31) to hold is that

$$\frac{\alpha'}{u'}\frac{1}{\sigma} \geq 1 \qquad \text{(D.32)}$$

For example, under CES utility and $\alpha(c) = \tilde{\alpha}u(c)$, (D.32) reads again $\tilde{\alpha} \geq \sigma$. Moreover, exploiting steady state expression of (D.3), $w = f(\bar{k}\,) - f'(\bar{k}\,)\bar{k}$ and $(1 + f')\beta = 1$, (D.31) reads as

$$1 - \beta(1+n) < \frac{\beta\alpha'(\bar{c})}{-u''(\bar{c})(\bar{k}+\theta)} \tag{D.33}$$

which is also the condition for stability of the equilibrium. Hence, if an interior equilibrium exists, it is also stable.