(p, d)-ELLIPTIC CURVES OF GENUS TWO

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Abstract. We study stable curves of arithmetic genus 2 which admit two morphisms of finite degree p, resp. d, onto smooth elliptic curves, with particular attention to the case p prime.

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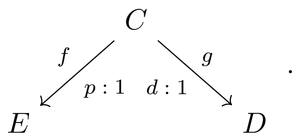
1. Introduction

In this paper we consider stable curves of arithmetic genus 2 which admit a (p, d)-elliptic configurations, namely two morphisms of finite degree p, resp. d, onto smooth elliptic curves D and E:



A curve of genus 2 is called (p, d)-elliptic curve if it admits a (p, d)-elliptic configuration. If $p = 2, d = 3$ we use the terminology bi-tri-elliptic curve.

Frey and Kani in [FK91] studied genus 2 coverings of elliptic curves, i.e., genus 2 d-elliptic curves. A fundamental tool in their research was the observation that given a degree d map $f : C \to E$ onto an elliptic curve then there exists a complementary map $f' : C \to E'$ of the same degree d onto a second elliptic curve (see §2). Later Kani in [Kan97] studied in details the arithmetic properties of such (d, d)-elliptic configurations, also providing existence results.

In this paper we study (p, d)-elliptic configurations following the construction described by Frey and Kani in [FK91].

It will turn out that stable (p, d)-elliptic curves of arithmetic genus 2 are automatically of compact type, i.e., they have compact Jacobian; thus in the first section we recall the Frey-Kani construction, noting that it extends to curves of compact type.

In Section 3 we study (p, d)-elliptic curves, with particular attention to the case p prime. In Theorem 3.12 we give a classification of such curves and in Section 3.C

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we show that for every pair of integers $p, d > 1$ there exists a smooth (p, d)-elliptic curve of genus 2.

The original motivation for this article was the study of bi-tri-elliptic configurations, which parametrise certain strata in the boundary of the moduli space of stable Godeaux surfaces (see [FPR16]). Thus we describe the geometry of bi-tri-elliptic configurations in a little more detail in the last section.

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2. d-elliptic curves of genus 2

Here we recall and slightly refine some results from [FK91, §1], where the focus is on smooth curves and on the case d odd (see below).

2.A. Set-up and preliminaries.

We work over an algebraically closed field $\mathbb{K}$ whose characteristic does not divide the degree d of the finite morphisms that we consider. Throughout all this section C is a stable curve of genus 2 and $J = J(C)$ is the Jacobian of C.

Definition 2.1 — Let $d \geq 2$ be an integer. We say that C is *d-elliptic* if there exists a finite degree d morphism $f\colon C \to E$ such that E is a smooth curve of genus 1 and f does not factor through an étale cover of E; we call f a *d-elliptic map*. Sometimes, a d-elliptic map is called an "elliptic subcover" and the curve C is said to have an "elliptic differential" (cf. [Kan97]); our choice of terminology is due to the fact that we wish to emphasize the degree d of the map. For $d = 2, 3$, the curve C is also called *bi-elliptic*, resp. *tri-elliptic*.

An isomorphism of d-elliptic curves $f_i\colon C_i \to E_i$, $i = 1, 2$, is a pair of isomorphisms $\varphi\colon C_1 \to C_2$ and $\bar\varphi\colon E_1 \to E_2$ such that $f_2 \circ \varphi = \bar\varphi \circ f_1$.

For an abelian variety A we denote by $A[d]$ its subgroup of d-torsion points. If A is principally polarised then there is a non degenerate alternating pairing $e_d\colon A[d] \times A[d] \to \mu_d$ (where μ_d denotes the d-th roots of unity) called Weil pairing (or Riemann form [Mum74, Ch.IV, §20]).

If A' is an abelian variety, then we call a group homomorphism $\alpha\colon A[d] \to A'[d]$ *anti-symplectic* if for every $P, Q \in A[d]$ one has:

$$e_d(\alpha(P), \alpha(Q)) = e_d(P, Q)^{-1},$$

or, equivalently, if the graph of α is an isotropic subgroup of $(A \times A')[d]$.

2.B. The Frey-Kani construction.

Now assume that C is a stable genus 2 curve of compact type, i.e., it is either smooth or the union of two elliptic curves intersecting in one point. Notice that the Jacobian $J = J(C)$ is a principally polarised abelian surface.

Let $f\colon C \to E$ be a d-elliptic map on C. The pull back map $f^*\colon E \to J$ is injective, hence the norm map $f_*\colon J \to E$ has connected kernel E'. Since the composition $f_*f^*\colon E \to E$ is multiplication by d, the abelian subvarieties E' and f^*E of J intersect in $E[d]$ and we have a tower of isogenies

$$E \times E' \xrightarrow[d^2:1]{h} J \xrightarrow[d^2:1]{h'} E \times E' \quad , \tag{2.2}$$

whose composition is multiplication by d. Composing the Abel-Jacobi map $C \hookrightarrow J$ with the projection to E' we get a second d-elliptic map $f' \colon C \to E'$, which we call the *complementary* d-elliptic map.

The construction that follows, which we call the Frey-Kani construction, has been described in [FK91, §1] for smooth curves, but the proof works verbatim for stable curves of compact type. Therefore one has:

Proposition 2.3 — *Let C be a stable d-elliptic curve of genus 2 of compact type, let $f \colon C \to E$ and $f' \colon C \to E'$ be complementary d-elliptic maps and let $h \colon E \times E' \to J = J(C)$ be as in (2.2). Then:*

- (i) *there exists an anti-symplectic isomorphism $\alpha \colon E[d] \to E'[d]$ such that $\ker h$ is the graph H_α of α;*
- (ii) *the principal polarization on J pulls back to $d(E \times \{0\} + \{0\} \times E')$.*

Notice that if $d = 2$, then any isomorphism α as in Proposition 2.3 is anti-symplectic. More generally, for a prime d the number of anti-symplectic isomorphisms $E[d] \to E'[d]$ is equal to $d(d^2-1)$ (cf. [FK91]).

The above proposition has a converse (see [FK91]):

Proposition 2.4 — *Let E, E' be elliptic curves and let $\alpha \colon E[d] \to E'[d]$ be an anti-symplectic isomorphism. Denote by H_α the graph of α; set $A := (E \times E')/H_\alpha$ and denote by $h \colon E \times E' \to A$ the quotient map.*

Then

- (i) *$d(E \times \{0\} + \{0\} \times E')$ descends to a principal polarization Θ on A;*
- (ii) *let C be a Theta-divisor on A; then C is a stable curve of genus 2 of compact type and the maps $f \colon C \to E$ and $f' \colon C \to E'$ induced by the natural maps $A \to E$ and $A \to E'$ are complementary d-elliptic maps.*
- (iii) *if d is odd, then there is precisely one symmetric Theta-divisor on A that is linearly equivalent to $d(E \times \{0\} + \{0\} \times E')$*

2.C. **Special geometry for small d.** It is an interesting question under what conditions the polarisation coming from the Frey-Kani construction is reducible.

We answer this question for $d = 2$.

Lemma 2.5 — *Let A be constructed as in Proposition 2.4 for $d = 2$. Then the principal polarization Θ of A is reducible if and only if there exists an isomorphism $\psi \colon E' \to E$ such that the map $E \times E' \to E \times E$ defined by $(x, y) \mapsto (x, \psi(y))$ maps H_α to the subgroup $\Delta[2] = \{(\eta, \eta) | \eta \in E[2]\}$.*

Moreover, up to isomorphism the bi-elliptic map f is given by the composition

$$C = E \times \{0\} \cup \{0\} \times E \hookrightarrow J(C) = E \times E \xrightarrow{+} E,$$

that is, it is the identity on each component of C; the complementary map f' is the identity on one component and multiplication by -1 on the other.

Proof. Denote by $\bar{E}$, $\bar{E}'$ the image of E, E' in A: we have $\Theta = (\bar{E} + \bar{E}')/2$ and $\bar{E}\bar{E}' = 4$. Assume that $\Theta = B_1 + B_2$ is reducible, with B_1, B_2 elliptic curves such that $B_1 B_2 = 1$ (recall that in this case $A \cong B_1 \times B_2$ as ppav's). Then $1 = B_1\Theta = B_1(\bar{E} + \bar{E}')/2$ implies either $B_1\bar{E} = B_1\bar{E}' = 1$ or, say, $B_1\bar{E} = 2$, $B_1\bar{E}' = 0$. The latter case cannot occur, since we would have $B_1 = \bar{E}'$, hence $2 = B_1\bar{E} = \bar{E}'\bar{E} = 4$, a contradiction.

Hence composing the map $E \to \bar{E}$ with the projections $A \to B_i$, for $i = 1, 2$ one obtains isomorphisms $\varphi_i \colon E \to B_i$. Hence we have an isomorphism $A \to E \times E$ that maps $\bar{E}$ to the diagonal in $E \times E$ and $\bar{E}'$ to the graph of an automorphism

σ of E. Since $\bar{E}\bar{E}' = 4$, σ has 4 fixed points. By the classification of the possible automorphism groups of an elliptic curve, it follows that σ is multiplication by -1 and $\bar{E}'$ is mapped to the antidiagonal. Composing $E' \to A$ with the isomorphisms $A \to E \times E$ and with the first projection $E \times E \to E$ we obtain the required isomorphism $\psi\colon E' \to E$. Indeed it is not hard to check that the following diagram commutes:

(2.6)
$$\begin{array}{ccc} E \times E' & \xrightarrow{(\mathrm{id}, \psi)} & E \times E \\ \downarrow h & & \downarrow q \\ A & \longrightarrow & E \times E \end{array},$$

where h is the quotient map and $q(x, y) = (x + y, x - y)$.

Conversely, assume that $E = E'$ and α is the identity. The map $q\colon E \times E \to E \times E$ defined by $q(x, y) = (x + y, x - y)$ has kernel $H_\alpha = \Delta[2]$, hence A is isomorphic to $E \times E$. Let $C = E \times \{0\} + \{0\} \times E$; then $q^*C = \Delta + \Delta^-$, where Δ is the diagonal and Δ^- is the antidiagonal. Since $(q^*C)^2 = 8$ by the pull-back formula and $q^*C(E \times \{0\} + \{0\} \times E) = 4$, the divisors q^*C and $2(E \times \{0\} + \{0\} \times E)$ are algebraically equivalent by the Index Theorem, hence C is the principal polarisation of Proposition 2.4. (More precisely, since q^*C and $2(E \times \{0\} + \{0\} \times E)$ restrict to the same divisor on $E \times \{0\}$ and $\{0\} \times E$ they are actually linearly equivalent). The final part of the statement follows. □

We close this section with an alternative description of bi-elliptic curves of genus 2 of compact type, which basically stems from the fact that a double cover is the quotient by an involution.

Lemma 2.7 — *Let C be a genus 2 stable curve of compact type, let $f\colon C \to E$ and $f'\colon C \to E'$ be complementary bi-elliptic maps and let σ, resp. σ', be the involution induced by f, resp. f'. Then the group $\langle \sigma, \sigma' \rangle$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\tau := \sigma\sigma'$ is the hyperelliptic involution.*

Proof. Let J be the Jacobian of C. The involution σ on J is induced by the involution $(x, y) \mapsto (x, -y)$ of $E \times E'$ (cf. (2.2)) and, similarly, σ' is induced by $(x, y) \mapsto (-x, y)$. So $\tau = \sigma\sigma'$ acts as multiplication by -1 on J and therefore, if C is smooth, is the hyperelliptic involution. If C is reducible, then τ is multiplication by -1 on both components of C.

Since τ is in the center of $\mathrm{Aut}(C)$, σ and σ' commute and $\langle \sigma, \sigma' \rangle$ has order 4. □

It is well known that bidouble covers can be reconstructed from their building data (cf. [Par91, §2]); we explain this in the case at hand.

Choose points P_1, P_2 and distinct points $Q_1, Q_2, Q_3 \in \mathbb{P}^1$ that are also distinct from P_1 and P_2 (P_1 and P_2 are allowed to coincide). Let $\pi\colon C \to \mathbb{P}^1$ be the bidouble cover branched on $D_1 = P_2$, $D_2 = P_1$ and $D_3 = Q_1 + Q_2 + Q_3$, denote by G the Galois group of π and by $\sigma \in G$ (resp. σ' and τ) the involution the fixes the preimage of D_1 (resp. D_2, D_3). Assume first that $P_1 \neq P_2$; in this case C is a smooth curve of genus 2 and for $i = 1, 2$ the quotients $E = C/\sigma$ and $E' = C/\sigma'$ are smooth curves of genus 1. The involution τ has 6 fixed points and therefore is the hyperelliptic involution.

If $P_1 = P_2$, then C has a node over $P_1 = P_2$ and the normalization is the bidouble cover of $\mathbb{P}^1$ with branch divisors $D_1 = D_2 = 0$ and $D_3 = P_1 + Q_1 + Q_2 + Q_3$. So

C is reducible and has two components, both isomorphic to the double cover of $\mathbb{P}^1$ branched on $P_1 + Q_1 + Q_2 + Q_3$.

This construction is related to the construction given in Proposition 2.4 as follows.

Let $\pi\colon C \to \mathbb{P}^1$ be as above, with C smooth, and take the preimage of P_1 as the origin $0 \in E$ and the preimage of P_2 as the origin $0' \in E'$. Denote by A_1, A_2, A_3 (resp. by B_1, B_2, B_3) the preimages of Q_1, Q_2, Q_3 in E (resp. in E'). Then the nonzero elements of $E[2]$ (resp. $E'[2]$) are $\eta_i := A_i - 0$ (resp. $\eta'_i := B_i - 0'$), $i = 1, 2, 3$); we define $\alpha\colon E[2] \to E'[2]$ as the isomorphism that maps $\eta_i \to \eta'_i$.

We claim that the bi-elliptic structure on C is obtained via the Frey-Kani construction with the above choice of α, i.e., the kernel of the pull-back map $h'^*\colon E \times E' \to J := J(C)$ is the graph H_α of α.

Indeed, since the kernel Γ of the pull-back map has order 4, it is enough to show that H_α is in contained in Γ. In addition one has $f^*A_i = f'^*B_i$ for $i = 1, 2, 3$, hence we only need to show that f^*0 and f'^*0' are linearly equivalent. The divisor f^*0 is the ramification divisor of f', hence $f^*0 \equiv K_C$; the same argument shows that $f'^*0' \equiv K_C$ and we are done.

The case C reducible is obvious.

Remark 2.8 — For tri-elliptic curves one can apply the general theory of triple covers [Mir85] to deduce the following result [FPR16, Lem. 2.8]: a stable curve C of genus 2 admits a tri-elliptic map $C \to E$ such that C embedds into the symmetric square of E as a tri-section of the Albanese map $S^2E \to E$.

Note however that a tri-elliptic map $C \to E$ cannot be a cyclic cover, since by the Hurwitz formula it would be ramified over precisely one point and this is impossible, for instance, by [Par91, Prop. 2.1]). So there is no elementary description of C just in terms of the ramification divisor.

3. (p,d)-ELLIPTIC CURVES OF GENUS 2

We consider stable curves of genus 2 admitting two distinct maps to elliptic curves.

3.A. (n,d)-elliptic curves and configurations.

Definition 3.1 — Let C be a stable curve of genus 2. An (n,d)-elliptic configuration (C, f, g) is a diagram

$$E \xleftarrow[n:1]{f} C \xrightarrow[d:1]{g} D \qquad (3.2)$$

where f is an n-elliptic map and g is a d-elliptic map such that there is no isomorphism $\psi\colon E \to D$ such that $g = \psi \circ f$. We refer to C as to an (n,d)-elliptic curve (of genus 2).

An isomorphism of (n,d)-elliptic configurations is an isomorphism of diagrams like (3.2).

Lemma 3.3 — *If C is (n,d)-elliptic stable curve of genus 2, then it is of compact type.*

Proof. As usual, we let $f\colon C \to E$ and $g\colon C \to D$ be the n-elliptic and d-elliptic maps.

The curve C cannot have rational components since it has a finite map onto an elliptic curve, hence it is enough to rule out the possibility that C is irreducible with one node. Assume by contradiction that this is the case, let $C^\nu \to C$ be the normalization map and denote by $\psi_1\colon C^\nu \to E$ (resp. $\psi_2\colon C^\nu \to D$) the map of degree n (resp. d) induced by f (resp. g). We denote by $O, P \in C^\nu$ the points of that map to the node of C and we take O as the origin; with a suitable choice of the origins in E and D we can assume that ψ_1 and ψ_2 are isogenies.

Since ψ_1 and ψ_2 factor through $C^\nu \to C$, P belongs to $\ker\psi_1 \cap \ker\psi_2$ and for $i = 1, 2$ ψ_i factors through the étale covers $C^\nu/ < P > \to E$ and $C^\nu/ < P > \to D$. It follows that f and g also factor through $C^\nu/ < P > \to E$ and $C^\nu/ < P > \to D$ hence, by the definition of d-elliptic curve, it follows that $n = d = 2$ and the two bi-elliptic structures differ by an isomorphism $E \to D$, a contradiction. □

Remark 3.4 — By the Frey-Kani construction given in §2, if C is of compact type and has a d-elliptic map $f\colon C \to E$ then, if we denote $f'\colon C \to E'$ complementary d-elliptic map, (C, f, f') is a (d, d)-elliptic configuration. We refer to this as to the *trivial (d, d)-elliptic configuration.*

3.B. **(p, d)-elliptic curves.** Assume we are given an (n, d)-elliptic configuration as in (3.2), which is non-trivial in the sense of Remark 3.4. Then both elliptic maps factor, up to isomorphism, through the Abel-Jacobi map of C and are thus uniquely determined by the subgroups $\ker f_*$ and $\ker g_*$.

To analyse such a configuration, we apply the Frey-Kani construction to the n-elliptic map given by f and then we consider $\bar F = \ker g_*$. We can extend (2.2) to a diagram

$$
\begin{array}{ccccccc}
F & \overset{h_F}{\dashrightarrow} & \bar F & & & \overset{m:1}{\searrow} & \\
{\scriptstyle (\varphi,\varphi')}\downarrow & & \downarrow & & & & \\
E \times E' & \xrightarrow{h} & J(C) & \xrightarrow{f_* \times f'_*} & E \times E' & \longrightarrow & E \\
 & & \downarrow{\scriptstyle g_*} & & & & \\
 & & D & & & &
\end{array}
\qquad (3.5)
$$

where $\bar F \Theta = d$ (Θ denoting the principal polarization) and F is the connected component of $h^{-1}F$ containing the origin. Indeed we have the following

Remark 3.6 — A genus 2 curve of compact type has an n-elliptic structure if and only if its Jacobian J contains a connected 1-dimensional subgroup $\bar E'$ such that $\bar E'\Theta = n$. Therefore an n-elliptic curve C has an (n, d)-elliptic structure if and only if J contains a second connected 1-dimensional subgroup $\bar F$ such that $\bar F\Theta = d$ and $\bar F \neq \bar E'$. So, if we exclude the trivial (d, d)-structures (cf. Remark 3.4), the Jacobian of an (n, d)-elliptic curve of genus 2 contains at least three, hence infinitely many, connected 1-dimensional subgroups. In particular the curve C has infinitely many elliptic structures, and the curves E and E' are isogenous.

For a given (n, d)-elliptic configuration (C, f, g) we denote $\bar F = \ker g_*$ and $\bar E' = \ker f_*$ and we define the *twisting number* of (C, f, g) as

$$m = m(C, f, g) := \bar F \ker f_* = \bar F \bar E' = \deg(\bar F \to E) = n^2 \frac{\deg\varphi}{\deg h_F} \qquad (3.7)$$

where φ and h_F are as in (3.5).

Remark 3.8 — One has $m > 0$, by the definition of (n,d)-elliptic curve.

Denote by $\bar{E}$ the kernel of $f'_*\colon J \to E'$, where f' is the complementary map of f. Then by the Frey-Kani construction we have $\Theta = \frac{\bar{E}+\bar{E}'}{n}$, hence $nd = n\bar{F}\Theta = m + \bar{F}\bar{E}$. It follows that $m \leq nd$, with equality holding if and only if $\bar{F}\bar{E} = 0$, namely iff we are in the trivial case $g = f'$.

We first provide three examples with $m < nd$ that fit into this general pattern and then prove that when $n = p$ is a prime these cover all possibilities for non trivial elliptic configurations.

Example 3.9 — Let n, d be integers. Let F be an elliptic curve and let $\varphi\colon F \to E$ and $\varphi'\colon F \to E'$ be isogenies such that:

- $\ker\varphi \cap \ker\varphi' = \{0\}$, hence $\varphi \times \varphi'\colon F \to E \times E'$ is injective;
- $\deg\varphi + \deg\varphi' = nd$, and n and $\deg\varphi$ are coprime.

We abuse notation and denote again by F the image of $\varphi \times \varphi'$. The subgroup $H := F[n] \subset (E \times E')[n]$ satisfies $H \cap E = H \cap E' = \{0\}$, since $EF = \deg\varphi'$ and $E'F = \deg\varphi$ are coprime to n. Hence H is the graph of an isomorphism $E[n] \to E'[n]$. The polarization $n(E \times \{0\} + \{0\} \times E')$ restricts on F to a divisor of degree $n^2 d$, which therefore is a pull back via the map $F \to F$ defined by multiplication by n. By the functorial properties of the Weil pairing (see statement (1) of [Mum74, Ch.IV, §23, p.228]) it follows that $F[n]$ is an isotropic subspace of $(E \times E')[n]$.

Let $A = (E \times E')/H$ and let Θ be the principal polarization of A (see Proposition 2.4). Denote by $\bar{F}$ be the image of F in A: then we have $n^2\bar{F}\Theta = nF(E \times \{0\} + \{0\} \times E') = n^2 d$, namely $\bar{F}\Theta = d$. By Remark 3.6 we obtain an (n,d)-elliptic configuration with twisting number $m = n^2 \frac{\deg\varphi}{\deg h_F} = \deg\varphi$ (cf. (3.7)).

From now on we will focus on the case where $n = p$ is a prime number.

Example 3.10 — Let p, d be integers and assume that p is a prime. Let F be an elliptic curve and let $\varphi\colon F \to E$ and $\varphi'\colon F \to E'$ be isogenies such that:

- $\ker\varphi \cap \ker\varphi' = \{0\}$;
- $\deg\varphi + \deg\varphi' = d$;
- $F[p] \not\subset \ker\varphi$ and $F[p] \not\subset \ker\varphi'$.

Under the above conditions, it is possible to find an antisymplectic isomorphism $\alpha\colon E[p] \to E'[p]$ such that $H_\alpha \cap F$ has order p, where H_α is the graph of α. This follows because by our assumptions there exists $0 \neq v \in F[p]$ such that $v \notin \ker\varphi \cup \ker\varphi'$. Moreover, since the Weil pairing of a product is given by the product of the Weil pairings (see statement (2) of [Mum74, Ch.IV, §23, p.228]), the annihilator W of v in $(E \times E')[p]$ does not contain $E[p] \times \{0\}$ nor $\{0\} \times E'[p]$. The linear subspace W is three dimensional, hence $\mathbb{P}(W)$ is a projective plane over $\mathbb{F}_p$. Now consider in $\mathbb{P}(W)$ the pencil $\mathcal{F}$ of lines through $[v]$: since $\mathcal{F}$ consists of $p+1$ lines, if $p > 2$ there is at least a line $l \in \mathcal{F}$ that does not intersect the lines $r := \mathbb{P}(E[p] \times \{0\})$ and $s := \mathbb{P}(\{0\} \times E'[p])$ and distinct from $t := \mathbb{P}(F[p])$. The subspace of $(E \times E')[p]$ corresponding to l is the graph of an isomorphism $\alpha\colon E[p] \to E'[p]$ and is also isotropic, hence α is antisymplectic. For $p = 2$, any isomorphism α is antisymplectic, hence it is enough to find a line in $\mathbb{P}((E \times E')[2])$ that contains $[v]$, which is distinct from t and does not intersect r and s. An elementary geometrical argument shows that there exist two lines with this property.

Therefore we can consider $A := (E \times E')/H_\alpha$ and the principal polarization Θ of A (cf. Proposition 2.4). Again we denote by $\bar{F}$ the image of F in A, obtaining

$p^2\bar F\Theta = pF(p(E\times\{0\}+\{0\}\times E')) = p^2 d$, namely $\bar F\Theta = d$, i.e. by Remark 3.6 we get a (p,d)-elliptic configuration. In this case by (3.7) we have $m = p^2\frac{\deg\varphi}{\deg h_F} = p\deg\varphi$.

Example 3.11 — Let p, d be integers such that p is a prime and d is divisible by p. Write $d = p\delta$ and let F be an elliptic curve with $\varphi\colon F\to E$ and $\varphi'\colon F\to E'$ isogenies such that:

- $\ker\varphi\cap\ker\varphi' = \{0\}$;
- $\deg\varphi + \deg\varphi' = \delta$;

We look for an antisymplectic isomorphism $\alpha\colon E[p]\to E'[p]$ such that $H_\alpha\cap F = \{0\}$, H_α being the graph of α.

To see that such α exists we argue as follows. As in Example 3.10 we identify 2-dimensional subspaces of $(E\times E')[p]$ with lines in $\mathbb{P}^3(\mathbb{F}_p) := \mathbb{P}((E\times E')[p])$. We have seen in Example 3.10 that there are $p+1$ isotropic lines through any point, hence there exist $(p+1)(p^2+1)$ isotropic lines.

The isotropic lines meeting a given line are $p(p+1)+1$ or $(p+1)^2$, according to whether the line is isotropic or not. Set $r := \mathbb{P}(E[p]\times\{0\})$, $s := \mathbb{P}(\{0\}\times E'[p])$ and $t = \mathbb{P}(F[p])$; note that r and s are not isotropic. Hence there are at most $3(p+1)^2$ isotropic lines meeting $r\cup s\cup t$. However, all the lines joining a point of r and a point of s are isotropic hence, by subtracting these lines (that we had counted twice) we get the better upper estimate $2(p+1)^2$ for the number of isotropic lines meeting $r\cup s\cup t$. For $p\geq 3$ this shows the existence of the isotropic subspace H_α that we are looking for, since $(p+1)(p^2+1) - 2(p+1)^2 = (p+1)(p^2-2p-1) > 0$.

For $p = 2$, we need only find a line that is disjoint from $r\cup t\cup s$. We observe that $\mathbb{P}^3(\mathbb{F}_2)$ contains 35 lines, that the lines intersecting a given line are 19, that the lines meeting two given skew lines are 9 and the lines meeting three mutually skew lines are 3.

If $\deg\varphi$ and $\deg\varphi'$ are odd, then the three lines r, s and t are mutually skew: then the set of lines meeting at least one of these consists of $3\cdot 19 - 3\cdot 9 + 3 = 33$ lines, hence there are 2 possibilities for H_α.

Now assume that both $\deg\varphi$ and $\deg\varphi'$ are even. In this case t meets both r and s. The number of lines intersecting $r\cup t$ is equal to $7+7-3 = 11$, since there are 7 lines in plane spanned by r and t, there are 7 lines passing through $r\cap t$, and 3 lines common to these two sets. An analogous argument shows that there are $3+3-1 = 5$ lines meeting t, r and s. So the number of lines intersecting $r\cup t\cup s$ is equal to $3\cdot 19 - 9 - 2\cdot 11 + 5 = 31$, so there are 4 possibilities for H_α.

Finally we consider the case where $\deg\varphi$ is even and $\deg\varphi'$ is odd. There are two possibilities: either $r = t$ or r and t are coplanar but distinct. In the former case, the number of lines meeting $r\cup t\cup s = r\cup s$ is equal to $2\cdot 19 - 9 = 29$, so there are 6 possibilities for H_α. In the latter case, the number of lines meeting $r\cup t\cup s$ is equal to $3\cdot 19 - 2\cdot 9 - 11 + 5 = 33$, so there are 2 possibilities for H_α.

Taking $A := (E\times E')/H_\alpha$ and Θ the principal polarization of A (cf. Proposition 2.4) and denoting by $\bar F$ the image of F in A, we get $p^2\bar F\Theta = p^2 F(p(E\times\{0\}+\{0\}\times E')) = p^2 d$, namely $\bar F\Theta = d$, i.e. by Remark 3.6 we get a (p,d)-elliptic configuration. In this case (3.7) yields $m = p^2\frac{\deg\varphi}{\deg h_F} = p^2\deg\varphi$.

Theorem 3.12 — *Let p be a prime and let d be a positive integer. Let C be a stable curve of genus 2 and let (C, f, g) be a non-trivial (cf. Remark 3.4) (p,d)-elliptic* *configuration*

$$\begin{array}{ccc} & C & \\ {}^{f}\swarrow_{p:1} & & {}_{d:1}\searrow^{g} \\ E & & D \end{array} \quad ,$$

Denote by $\bar{E}'$ (resp. $\bar{F}$) the kernel of $f_*\colon J = J(C) \to E$ (resp. $g_*\colon J \to D$) and let $m = \bar{E}'\bar{F}$ be the twisting number as in (3.7). Then

- (i) *the (p,d)-elliptic configuration arises as in Example 3.9, or 3.10, or 3.11, and thus $1 \le m \le pd-1$;*
- (ii) *the case of Example 3.11 can occur only if p divides d and p^2 divides m;*
- (iii) *the case of Example 3.9 occurs if and only if m is not divisible by p.*

Proof. By Remark 3.8 we have $0 < m \le pd$, and $m = pd$ holds only in the trivial case $g = f'$. Therefore by our assumptions it is $1 \le m \le pd-1$.

We use freely the notation of §3.B and diagram 3.5 and we denote by $\varphi\colon F \to E$ and $\varphi'\colon F \to E'$ the isogenies induced by the two projections of $E \times E'$. Note that $\ker\varphi \cap \ker\varphi' = \{0\}$ by construction. The pull-back $h^*\bar{F} \subset E \times E'$ is algebraically equivalent to νF for some integer $\nu \in \{1, p, p^2\}$ (one has $p^2 = \nu|H \cap F|$). We have $p^2 m = p^2 \bar{F}\bar{E}' = \nu F(p^2(\{0\} \times E'))$, i.e., $m = \nu F(\{0\} \times E') = \nu \deg\varphi$. In the same way, one obtains $pd - m = \nu F(E \times \{0\}) = \nu \deg\varphi'$. In particular, $\nu = 1$ if m is not divisible by p.

Consider the case $\nu = 1$, i.e, $H = F[p]$. In this case the map $E \times E' \to J(C)$ induces a degree p^2 isogeny $F \to \bar{F} \cong F$, the degree of φ is equal to m and the degree of φ' is equal to $pd - m$. Since H, being a graph, intersects $E \times \{0\}$ and $\{0\} \times E'$ only in 0, it follows that m, which is equal to the order of $(\{0\} \times E') \cap F$, is prime to p, and the same is true for $\deg\varphi' = pd - m$. So, C is constructed as in Example 3.9.

Next, assume that $\nu = p$, i.e. $H \cap F$ has order p. In this case, one has $m = p\deg\varphi$ and $\deg\varphi + \deg\varphi' = d$. Since $H \cap F$ has order p and H is a graph, it follows that $F[p] \not\subset \ker\varphi$ and $F[p] \not\subset \ker\varphi'$, hence C is constructed as in Example 3.10.

Finally consider the case $\nu = p^2$. In this case one has $m = p^2\deg\varphi$ and $d = p(\deg\varphi + \deg\varphi')$, hence C is constructed as in Example 3.11. □

3.C. **Existence of smooth (n,d)-elliptic curves.** First of all let us recall that an irreducible (n,d)-elliptic curve is smooth by Lemma 3.3.

By Lemma 2.5, for $n = 2$ a necessary condition for the irreducibility of the genus 2 curve C constructed as in Proposition 2.4 is that the curves E and E' are isomorphic, hence if E does not have complex multiplication then the constructions of Examples 3.9, 3.10 and 3.11 yield examples of smooth $(2,d)$-elliptic curves of genus 2 for every $d > 2$.

In general, it is not clear whether the constructions of Examples 3.9, 3.10 and 3.11 give rise to irreducible, hence smooth, curves. We are able to settle this point at least in a special case:

Proposition 3.13 — *Let $n \ge 2, d \ge 3$ be integers; let E be an elliptic curve without complex multiplication, $\xi \in E$ an element of order $r := dn - 1$, and $\varphi'\colon E \to E' := E/ < \xi >$ the quotient map.*

Then the (n,d)-elliptic genus 2 curve constructed as in Example 3.9 with $F = E$, $\varphi = \mathrm{Id}_E$ and φ' as above is smooth.

As an immediate consequence we obtain:

Corollary 3.14 — *For every pair of integers $n, d > 1$ there exists a smooth (n, d)-elliptic curve of genus 2 with twisting number $m = 1$.*

Proof. For $n = d = 2$ the claim follows by Lemma 2.5, for instance by using the construction of Example 3.9, and by Proposition 3.13 in the remaining cases. □

Proof of Proposition 3.13. Denote by Ξ the product polarization on $E \times E'$. Set $H := \{(\eta, \varphi'(\eta)) | \eta \in E[p]\}$ and let $h\colon E \times E' \to A := (E \times E')/H$ be the quotient map.

We argue by contradiction, so assume that the principal polarization of A induced by $n\Xi$ is reducible and denote it by $C = C_1 + C_2$. Up to a translation we may assume that the singular point of C is the origin of A. Let $\widetilde{C}_i$ be the connected component of the preimage of C_i containing the origin of $E \times E'$, $i = 1, 2$, so that h^*C_i is numerically equivalent to $\nu_i\widetilde{C}_i$ for a positive integer ν_i. One has

$$n^2 = \nu_i|H \cap \widetilde{C}_i| \quad \text{and} \quad \frac{n}{\nu_i} = \widetilde{C}_i\Xi \in \mathbb{Z}. \tag{3.15}$$

Since E does not have complex multiplication ($\operatorname{End} E = \mathbb{Z}$), the connected 1-dimensional subgroups of $E \times E$ distinct from $E \times \{0\}$ and $\{0\} \times E$ are of the form $\{(ax, bx) \mid x \in E\}$, with a, b coprime integers. This is well known, but we give a quick proof for lack of a suitable reference. Let G be such a subgroup, and denote by $\psi_i\colon G \to E$, $i = 1, 2$ the isogenies induced by the two projections. Note that $\ker\psi_1 \cap \ker\psi_2 = \{0\}$. If G is isomorphic to E, then the ψ_i are multiplication maps and G is of the form $D_{a,b}$ for some pair of coprime integers a, b. So assume that G and E are not isomorphic and consider an isogeny $\chi\colon E \to G$. Since χ is not a multiplication map, there exists an integer k and elements $u, v \in E[k]$ such that $\chi(u) = 0$ and $\chi(v) = v' \neq 0$. Now consider the maps $\mu_i := \psi_i \circ \chi\colon E \to E$, which are multiplication maps by integer t_i, $i = 1, 2$. Both t_1 and t_2 are divisible by k, since for $i = 1, 2$ we have $\mu_i(u) = 0$, hence $\psi_i(v') = \mu_i(v) = 0$ and so $v' = 0$, a contradiction.

Since $D_{a,b} = D_{-a,-b}$, we may always assume $a \geq 0$. It follows that the connected 1-dimensional subgroups of $E \times E'$ distinct from $E \times \{0\}$ and $\{0\} \times E'$ are of the form $D_{a,b} = \{(ax, b\varphi'(x)) \mid x \in E\}$. Notice that the kernel of the induced map $E \to D_{a,b}$ is the cyclic subgroup of $< \xi >$ of order $\delta := g.c.d.(a, r)$. Using this observation one computes:

$$D_{a,b}(\{0\} \times E') = \frac{a^2}{\delta}, \quad D_{a,b}(E \times \{0\}) = \frac{b^2 r}{\delta}, \quad D_{a,b}D_{1,1} = (b - a)^2\frac{r}{\delta}. \tag{3.16}$$

For $i = 1, 2$, let $a_i, b_i \in \mathbb{Z}$ be such that $\widetilde{C}_i = D_{a_i,b_i}$, with $a_i \geq 0$; set $\delta_i = g.c.d.(a_i, r)$. We will now derive a contradiction using intersection numbers.

Step 1: We have $a_i > 0$. Indeed if $a_i = 0$ we have $\widetilde{C}_i\Xi = 1$ and $|H \cap \widetilde{C}_i| = 1$, so (3.15) gives $n = \nu_i$ and $\nu_i = n^2$, against our assumptions.

Step 2: We show $(a_i, b_i) \neq (1, 1)$. Indeed, assuming $\widetilde{C}_i = D_{1,1}$ (3.15) gives

$$\frac{n}{\nu_i} = \widetilde{C}_i\Xi = D_{1,1}\Xi = 1 + r = dn,$$

which is impossible since $d > 1$. In particular, since a_i and b_i are coprime, we have $a_i \neq b_i$.

Step 3: From the above steps we derive two inequalities and a divisibility property which will lead to a contradiction.

First of all we have, for $i = 1, 2$,

$$n | \nu_i(a_i - b_i), \tag{3.17}$$

Indeed, since $D_{1,1} \cap \widetilde{C}_i$ is a subgroup containing $H \cap \widetilde{C}_i$ we have that $\widetilde{C}_i D_{1,1} = (b_i - a_i)^2 \frac{r}{\delta_i}$ is divisible by $\frac{n^2}{\nu_i}$, hence $(b_i - a_i)^2$ is divisible by $\frac{n^2}{\nu_i}$, since $\frac{r}{\delta_i}$ is an integer prime to n. So $\nu_i^2 (b_i - a_i)^2$ is divisible by n^2, and therefore $\nu_i(a_i - b_i)$ is divisible by n.

Secondly, by (3.16) we have $n = (\nu_1 \widetilde{C}_1 + \nu_2 \widetilde{C}_2)(\{0\} \times E') = \nu_1 a_1 \frac{a_1}{\delta_1} + \nu_2 a_2 \frac{a_2}{\delta_2}$ and $n = (\nu_1 \widetilde{C}_1 + \nu_2 \widetilde{C}_2)(E \times \{0\}) = \nu_1 b_1^2 \frac{r}{\delta_1} + \nu_2 b_2^2 \frac{r}{\delta_2}$. In particular, we have

$$\nu_1 a_1 + \nu_2 a_2 \leq n, \qquad d(\nu_1 b_1^2 + \nu_2 b_2^2) \leq n, \tag{3.18}$$

since $\frac{r}{\delta_i}$ is an integer and $\frac{r}{\delta_i} \geq d \frac{n}{a_i} - 1 > d - 1$.

Step 4: We cannot have $b_i > 0$. Indeed in this case, since $0 \leq \nu_i a_i, \nu_i b_i < n$ by (3.18) and n divides the difference $\nu_i a_i - \nu_i b_i$ by (3.17), then we necessarily have $a_i = b_i$ contradicting Step 2.

Step 5: We cannot have $b_i \leq 0$. Indeed the same argument as in the previous step shows that we would necessarily have $\nu_i b_i = \nu_i a_i - n$ for $i = 1, 2$. By (3.18) we may assume, say, $\nu_1 a_1 \leq \frac{n}{2}$ and thus by the above equality $\nu_1 |b_1| = -\nu_1 b_1 \geq \frac{n}{2}$. Then (3.18) gives:

$$n \geq d\nu_1 b_1^2 \geq |b_1| \frac{dn}{2},$$

a contradiction since $d > 2$.

Combining the last two steps we arrive at a contradiction and have thus proved that that the polarisation is irreducible and hence is a smooth (n, d)-elliptic curve of genus 2. □

4. BI-TRI-ELLIPTIC CURVES

For the applications to the classification of Gorenstein stable Godeaux surfaces the case of bi-tri-elliptic configurations is of particular interest. In this section we first formulate Theorem 3.12 in this case and then analyse reducible bi-tri-elliptic curves in more detail.

Indeed we have the following characterization of reducible bi-tri-elliptic configurations.

Corollary 4.1 — *Let (C, f, g) be a bi-tri-elliptic configuration on a stable curve of arithmetic genus* 2*. Then the twisting number m defined in* (3.7) *satisfies* $1 \leq m \leq 5$ *and there are the following possibilities:*

(a) m is odd and the configuration arises as in Example 3.9 with $\deg \varphi = m$*;*
(b) $m = 2\mu$ is even and the configuration arises as in Example 3.10 with $\deg \varphi = \mu$*.*

Remark 4.2 — Counting parameters we see that the space of bi-tri-elliptic configurations is one-dimensional, but we did not consider its finer structure, e.g., the number of irreducible or connected components.

Now assume that $C \cong E \cup_0 E$, where E is an elliptic curve with a degree 2 endomorphism $\psi \colon E \to E$. Then we can build a natural bi-tri-elliptic configuration

$$\begin{array}{ccccc} & & E \cup_0 E & & \\ & \swarrow^{f = \mathrm{id} \cup \mathrm{id}}_{2:1} & & \searrow^{g = \mathrm{id} \cup \psi}_{3:1} & \\ E & & & & E \end{array} . \tag{4.3}$$

We will now show that every bi-tri-elliptic configuration (C, f, g) with C reducible is of this form. Indeed, by Lemma 2.5 the bi-elliptic map f on the reducible curve C is isomorphic to the composition of horizontal arrows in the diagram

$$\begin{array}{ccccc} & & \bar{F} & & \\ & & \downarrow & & \\ C = E \times \{0\} \cup \{0\} \times E & \hookrightarrow & E \times E & \xrightarrow{+} & E \quad , \\ & \searrow^{g} & \downarrow & & \\ & & D & & \end{array}$$

and the tri-elliptic map is uniquely determined by the subgroup $\bar{F}$. Note that the covering involution of f exchanges the components of C.

We have $\bar{F}C = 3$ and without changing f we can assume that $\bar{F}(\{0\} \times E) = 1$ and $\bar{F}(E \times \{0\}) = 2$. In other words, $\bar{F}$ is the graph of a degree 2 endomorphism $\psi \colon E \to E$ and $E \times \{0\}$ is identified with the second elliptic curve D by the restriction of g. Therefore the bi-tri-elliptic configurations is as in (4.3).

An isomorphism from (C, f, g) to another bi-tri-elliptic configuration $(\widetilde{C} = \widetilde{E} \cup_0 \widetilde{E}, \widetilde{f}, \widetilde{g})$ such that $\widetilde{f}$ is the identity on each component of $\widetilde{C}$ is uniquely determined by an isomorphism $E \cong \widetilde{E}$ and thus we have proved the first part of the following

Proposition 4.4 — *The above construction induces a bijection on the set of isomorphism-classes of bi-tri-elliptic configurations (C, f, g) with C a reducible stable curve of genus 2 and the set $\{(E, \psi)\}$ of elliptic curves together with an endomorphism of degree 2.*

For every $1 \leq m \leq 5$ there are exactly two such pairs (E, ψ), which are listed in Table 1, thus in total there are 10 isomorphism classes of bi-tri-elliptic configurations with C a reducible stable curve of genus 2.

TABLE 1. Endomorphisms of degree 2 on elliptic curves

$E = \mathbb{C}/\Gamma$	$\Gamma = \mathrm{End}(E)$	ξ	m
E_1	$\mathbb{Z}\,[\mathrm{i}]$	$-1 \pm \mathrm{i}$	1
		$1 \pm \mathrm{i}$	5
E_2	$\mathbb{Z}\,[\mathrm{i}\sqrt{2}]$	$\pm\mathrm{i}\sqrt{2}$	3
E_3	$\mathbb{Z}\left[\frac{1}{2}(1 + \mathrm{i}\sqrt{7})\right]$	$-\frac{1}{2}(1 \pm \mathrm{i}\sqrt{7})$	2
		$\frac{1}{2}(1 \pm \mathrm{i}\sqrt{7})$	4

Proof. We need to recall some elementary facts about endomorphisms of elliptic curves. Details can be found for example in [Sil09, Ch.11] or [Sil94, Ch.II]. Any endomorphism ψ of an elliptic curve E is given by multiplication by a complex number ξ and this embeds $\mathrm{End}\, E \hookrightarrow \mathbb{C}$ as a maximal order in an imaginary quadratic number field $K \cong \mathrm{End}(E) \otimes \mathbb{Q}$. The degree of the endomorphism ψ coincides with the norm $N_{K/\mathbb{Q}}(\xi)$.

Thus elements inducing an endomorphism of degree 2 are characterised as those $\xi \in \mathbb{C} \setminus \mathbb{R}$ that are integral over $\mathbb{Z}$ with characteristic polynomial

$$p_\xi(t) = t^2 - \mathrm{trace}_{K/\mathbb{Q}}(\xi)t + N_{K/\mathbb{Q}}(\xi) = t^2 - 2\mathrm{Re}(\xi)t + 2 \in \mathbb{Z}[t].$$

This gives exactly the elements listed in Table 1 and each one of them is contained in a unique maximal order by [Sil09, Example 11.3.1] (see also [Sil94, Prop. 2.3.1]).

It remains to compute the invariant m, which is in our case the intersection of $\Gamma_\psi = \bar{F} \subset E \times E$ with the kernel of the addition map, that is, the anti-diagonal. Thus m equals the number of fixed points of the endomorphism $-\psi$, which by the holomorphic Lefschetz fixed-point formula [GH78, Ch. 3.4] gives

$$m = \sum_{i=0}^{2} (-1)^i \mathrm{trace}\left(-\psi_*|_{H_i(E,\mathbb{Q})}\right) = 1 - \mathrm{trace}_{K/\mathbb{Q}}(-\xi) + N_{K/\mathbb{Q}}(-\xi) = p_\xi(-1),$$

because every fixed point of ψ is simple. □

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