Origami and fractal solutions of differential systems

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Abstract

In this article we describe the fractal nature of the solutions of the Dirichlet problem associated with the definition of *origami* by an analytic point of view. In particular we introduce a new iterative algorithm to construct a solution of the differential problem when the boundary datum is not homogenous. The paper is dedicated to Michele Emmer, who few years ago invited us to give talks about this mathematical approach to origami at one of the meetings on *Mathematics and Culture* that he organized in Venezia.

1 A mathematical origami from the analytic point of view

We consider an open set $\Omega \subset \mathbb{R}^2$ which represents a *sheet of paper*, usually a rectangle in $\mathbb{R}^2$. The *origami* is a *folded paper* and lives in the three dimensional space $\mathbb{R}^3$. We identify the origami with the image of a map $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$.

A sheet of paper is rigid in tangential directions. Indeed, it cannot be stretched, compressed or sheared. If a sheet of paper is constrained on a plane, it would only be possible to achieve *rigid motions* i.e. *rotations* and *translations* of the whole sheet. Since origami is a *folded paper*, the map u cannot be everywhere smooth; it is only *piecewise smooth.* Folding creates discontinuities in the gradient. Since we do not allow to cut the sheet of paper, u is however a continuous map. The singular set $\Sigma_u \subset \Omega$, which is the set of discontinuities of the gradient Du, is called *crease pattern* in the origami context. Usually (but not necessarily in the general three dimensional case) this set is composed by straight segments. Of special interest is the case of the so called *flat origami*, which is a map u whose image is contained in a plane, and which, up to a change of coordinates, can be represented as a map $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$.

In collaboration with Bernard Dacorogna we investigated this analytic approach to origami in a series of papers [7]-[13] and [24]; we refer to these references for further details, proofs and descriptions of related aspects about origami from a mathematical point of view. In particular, the article [11] in the Notices of the American Mathematical Society contains a less technical description of our analytic approach.

We also mention some other mathematical approaches to origami, not necessarily of analytic nature: we quote the recent article by Abate [3] and, for instance, Alperin [1], Arkin-Bender-Demaine [2], Bern-Hayes [4], Haga [15], Heller [16], Huffman [17], Hull [18], Kawahata-Nishikawa [14], Kawasaki [19], Kilian *et al* [20], Lang [21],[22],[23], Robertson [25].

2 The fractal nature of the solutions of the Dirichlet problem

In the general three dimensional case, with $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ being *piecewise smooth* in Ω, the tangential rigidity can be expressed by requiring that the gradient $Du(x,y)$ of the map u is an *orthogonal* 3×2 matrix. That is, in any subdomain where u is smooth, the matrix product satisfies the condition

$$Du^t\,(x,y) \cdot Du\,(x,y) = I$$

where I is the identity matrix. In the special two dimensional case, with $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, under the notation

$$u\,(x,y) = \begin{pmatrix} u^1\,(x,y) \\ u^2\,(x,y) \end{pmatrix}$$

the same condition $Du^t\,(x) \cdot Du\,(x) = I$ equivalently gives $|Du|^2 = 2\,|\det Du| = 2$ and also $\det Du = \pm 1$. On the regions where the gradient is continuous the determinant $\det Du$ must be constant and hence has a fixed sign. If we consider a subdomain of Ω where $\det Du = 1$, the equation $|Du|^2 = 2 \det Du$ can be easily transformed into the system

$$\begin{cases} u^1_x - u^2_y = 0 \\ u^1_y + u^2_x = 0 \\ \left|Du^1\,(x,y)\right| = 1 \\ \left|Du^2\,(x,y)\right| = 1 \end{cases}$$

where the nonlinear nature of the differential equations is apparent.

The study of this kind of systems of partial differential equations is motivated by the study of elasticity and rigidity properties of materials. If we assume, as it is natural, that the elastic energy vanishes for rigid deformations, then any map with orthogonal gradient must be a minimum for the elastic energy. It is hence interesting to investigate when such maps exist.

In particular tension and compression of a material is achieved by constraining the boundary of such material in a given position. This is why we are interested in solving the Dirichlet problem:

$$\begin{cases} Du = \begin{pmatrix} u^1_x & u^1_y \\ u^2_x & u^2_y \end{pmatrix} & \text{orthogonal matrix a.e. in } \Omega, \\ u\,(x,y) = \varphi(x,y) & \text{on the boundary } \partial\Omega. \end{cases} \tag{1}$$

It is not difficult to convince oneself that if φ is a dilation problem (1) has no solution. On the other hand when φ is a strict contraction there are general abstract results [5, 6] which guarantee the existence of infinitely many solutions.

In the particular case when φ is constant we are able to find explicit solutions to this problem. From the point of view of origami we are looking for a crease pattern on a square sheet of paper (for example) such that the whole boundary of the square is sent on a single point. The set of points where the map assumes a fixed value cannot have interior, othwerise the gradient would be zero and hence not orthogonal. On the other hand in a region where the gradient is constant and orthogonal the map is locally invertible and hence there cannot be two points with the same value.

This forces the crease pattern to accumulate and become dense while approaching the boundary of the domain and explains the *fractal nature* of the solutions of our differential problems.

More precisely, by denoting by τ, ν respectively the tangent and normal unit vectors on $\partial\Omega$, up to a sign we have $\left(Du^1, \tau\right) = \left(Du^2, \nu\right)$ and $\left(Du^2, \tau\right) = \left(Du^1, \nu\right)$. Since $u^1(x, y) = u^2(x, y) = 0$ on $\partial\Omega$, we also obtain $Du^1 = Du^2 = 0$, which contradicts the fact that $\left|Du^1\right| = \left|Du^2\right| = 1$. Thus any solution to the differential problem (1), with $\varphi = 0$ is Lipschitz continuous but not of class C^1 near the boundary; therefore it assumes in a *fractal way* the homogenous boundary datum $\varphi = 0$. The map u will be explicitly defined at every $(x, y) \in \Omega$ and it will be piecewise affine, with infinitely many pieces, in accord with its fractal nature near the boundary of Ω.

3 A strategy to solve the differential problem

As usual we denote by $O(n)$ the set of $n-$dimensional orthogonal matrices; this in particular $O(2)$ is the set of 2×2 orthogonal matrices. Under this notation the Dirichlet problem (1) with $\varphi = 0$ becomes

$$\begin{cases} Du(x, y) \in O(2) & \text{a.e. } (x, y) \in \Omega \\ u(x, y) = 0 & (x, y) \in \partial\Omega \end{cases} \tag{2}$$

with Ω rectangle in $\mathbb{R}^2$. As we already pointed out in the previous section, only a fractal construction can ensure the boundary condition $u = 0$. When Ω is a rectangle we can divide it in infinitely many homothetic rectangles which are smaller and smaller while we approach to the boundary of Ω. Then it is enough to consider a *base map* u_0 defined on one of these tiles. This map will be translated rotated and rescaled to fit any other rectangles. To assure that the gluing of the rectangles gives a continuous map, we need the base map to have prescribed *recursive* boundary conditions. I.e., we require that on the right hand side of the *base rectangle* (say a square of side 1) the map is defined so that it reproduces twice the values of the left hand side, rescaled by half; i.e.

$$\begin{aligned} u_0(1, y) &= u_0(0, 2y) && \text{for } y \in [0, 1/2], \\ u_0(1, y) &= u_0(0, 2y - 1) && \text{for } y \in [1/2, 1]; \end{aligned}$$

while on the upper and lower sides we only need periodic boundary conditions $u_0(x, 0) = u_0(x, 1)$ for $x \in [0, 1]$. If the map assumes at least once the value 0 on every rectangle in the net, then by its Lipschitz continuity (every rigid map is 1-Lipschitz continuous) it can be extended to the boundary $\partial\Omega$ with the 0 value.

4 The Dirichlet problem with not homogeneous boundary condition

In this section we propose some new ideas to solve the Dirichlet problem (2) when the homogeneous boundary condition $u\,(x, y) = 0$ on $\partial\Omega$ is replaced by a not homogeneous one. From the applicative point of view the $\varphi = 0$ boundary datum is not really applicable because we are usually interested in finding solutions when a small compression is applied to the boundary of our body. The problem of finding explicit solutions becomes more difficult and for simplicity we only consider a linear datum such as

$$\varphi(x, y) = (1 - 2\lambda)(x, y) \qquad \forall (x, y) \in \partial\Omega \tag{3}$$

with $\Omega = [0, 1]^2$ and $0 < \lambda < 1$. When $\lambda = 0$ the only solution is the identity $u(x, y) = (x, y)$ while for $\lambda = 1$ the only solution is $u(x, y) = (-x, -y)$.

We build a solution to the Dirichlet problem with a recursive construction, as explained in the previous section.

In particular we start by defining the mesh of the cells as in Figure 1. Note that we approach the boundary recursively by splitting each cell into two cells of half the size.

In Figure 2 we represent the singular set of the map, i.e., the discontinuity set of the gradient. In fact our solution is Lipschitz-continuous: only the gradient can have discontinuities.

We describe the construction of the solution in each cell. We start by the diagonal cells, see Figure 3, where we emphasize the discontinuity lines of the gradient of the solution.

The main construction is described in Figure 4 where we have inserted the analytic expression of the solution in each subcell. We also inserted the cartesian equation of the discontinuity lines. Note that the solution matches continuously on each discontinuity line of the gradient. We invite the reader to check this property.

In Figure 5, we give the values of the gradient on a base cell. Note that the map has only diagonal gradient matrices. In this Figure we use precisely *six* different gradient matrices.

Finally a similar detailed analytical description is proposed in Figure 6 where we show four adjacent cells around a point in the diagonal of the square.

Up to now we have a map u whose gradient is orthogonal. To check that this map solves the boundary datum (3) we consider again Figure 4 (similarly we could consider Figure 6). In this basic cell the map assumes the same values

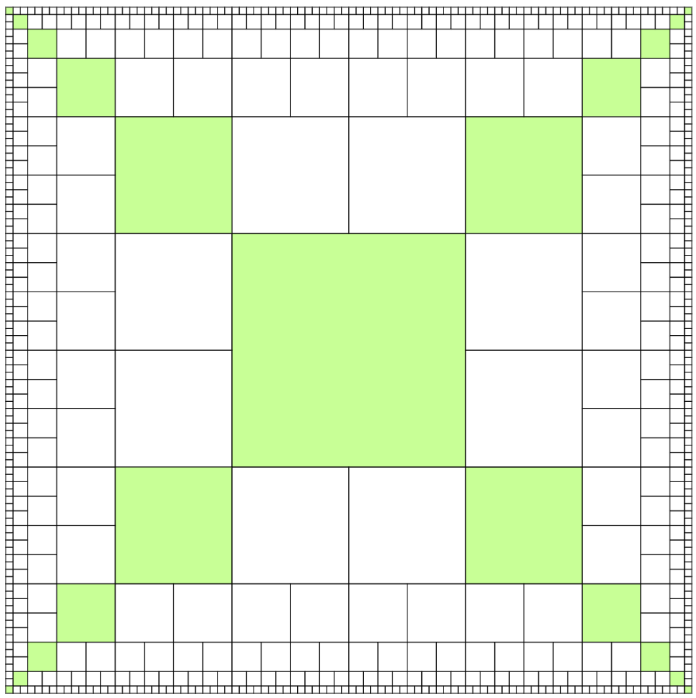

Figure 1: The scheme of the cells which approach the boundary in a fractal way. The green cells are described in Figure 3 while all other cells are obtained by a rotation of the construction with the singular set and the solution described in Figure 4 and Figure 6.

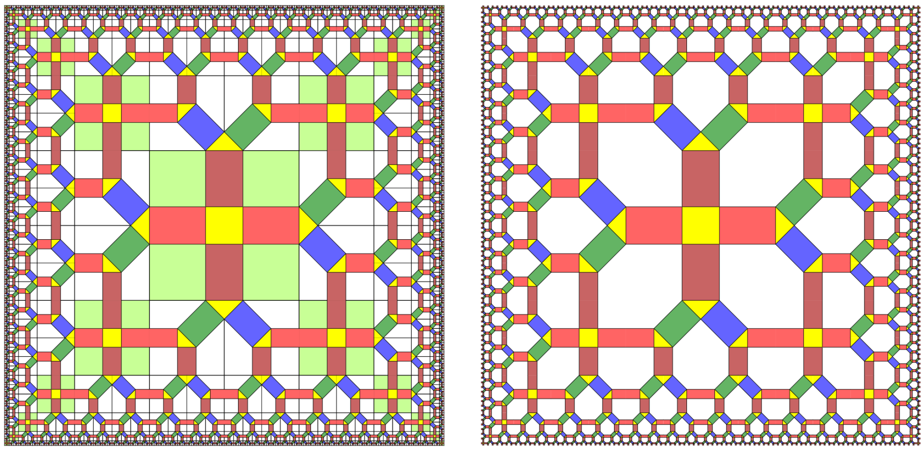

Figure 2: The singular set i.e. the discontinuity lines of the gradient. In the picture in the left-hand side the singular set is superimposed to the grid of the cells, while the picture on the right-hand side shows the singular set alone. Each different value of the gradient of the solution corresponds to each color in the picture. The identity gradient matrix $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is identified with white color, while the yellow color denotes the gradient matrix $-I$.

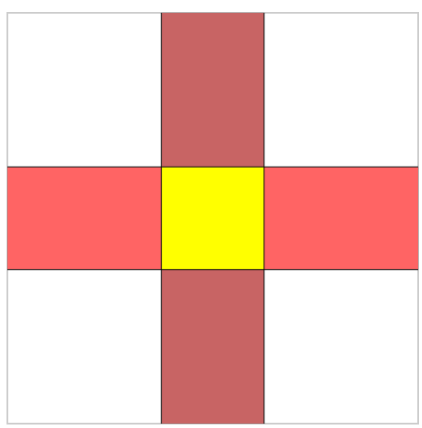

Figure 3: A detail of each green square in Figure 1. The set of discontinuity lines of the gradient of the map is respresented here. Some analytic details are shown in Figure 6.

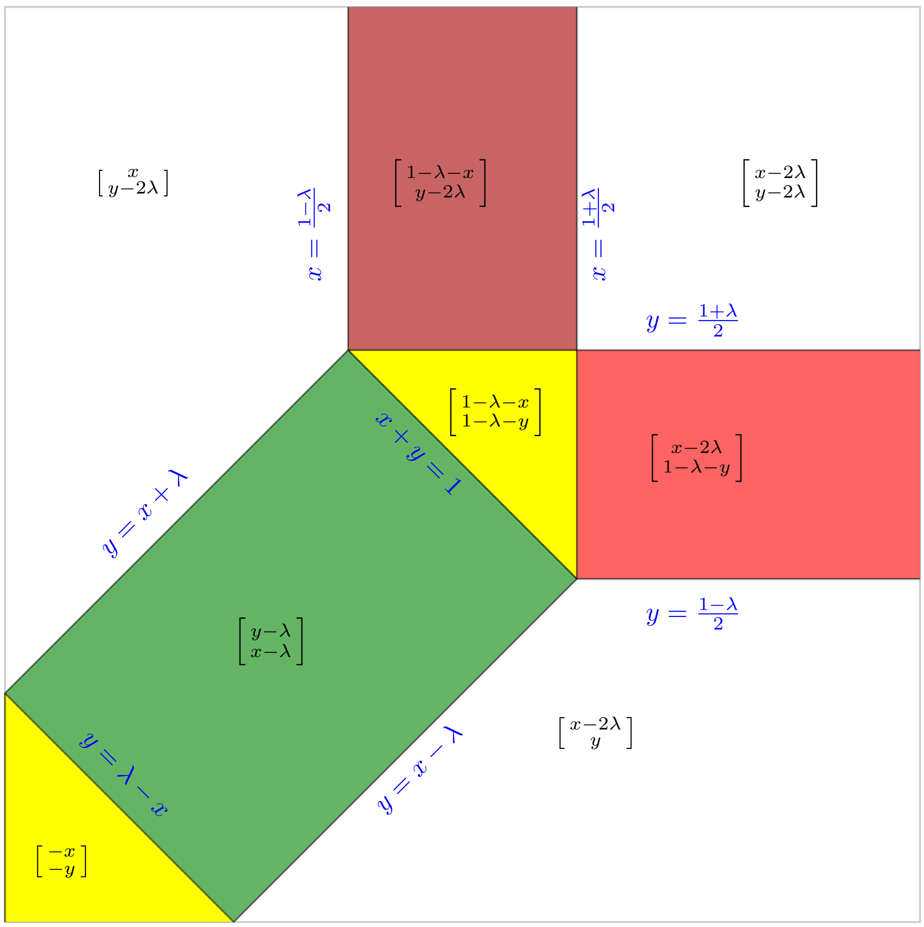

Figure 4: This is the main construction of the solution in a basic cell. Recall that our solution is a map $\mathbb{R}^2 \to \mathbb{R}^2$. We wrote the analityic expression of the two components of the solution on each subregion where the gradient is constant. The solution matches continuously on the discontinuity lines of the gradient.

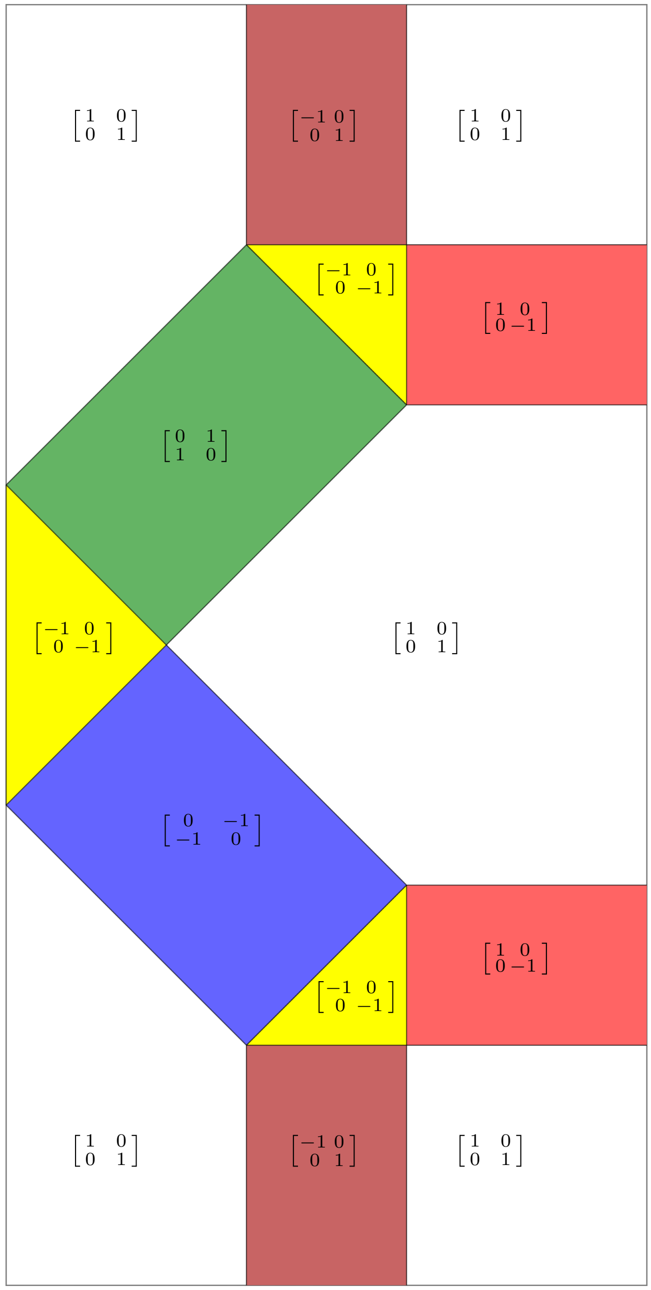

Figure 5: The gradient of the solution on two contiguous basic cells. Note that the map only uses diagonal unitary matrices. This depends on the fact that the discontinuity lines of the gradient are either parallel to the coordinate axes or are rotated by ±45 degrees. In this picture six different gradient matrices can be seen.

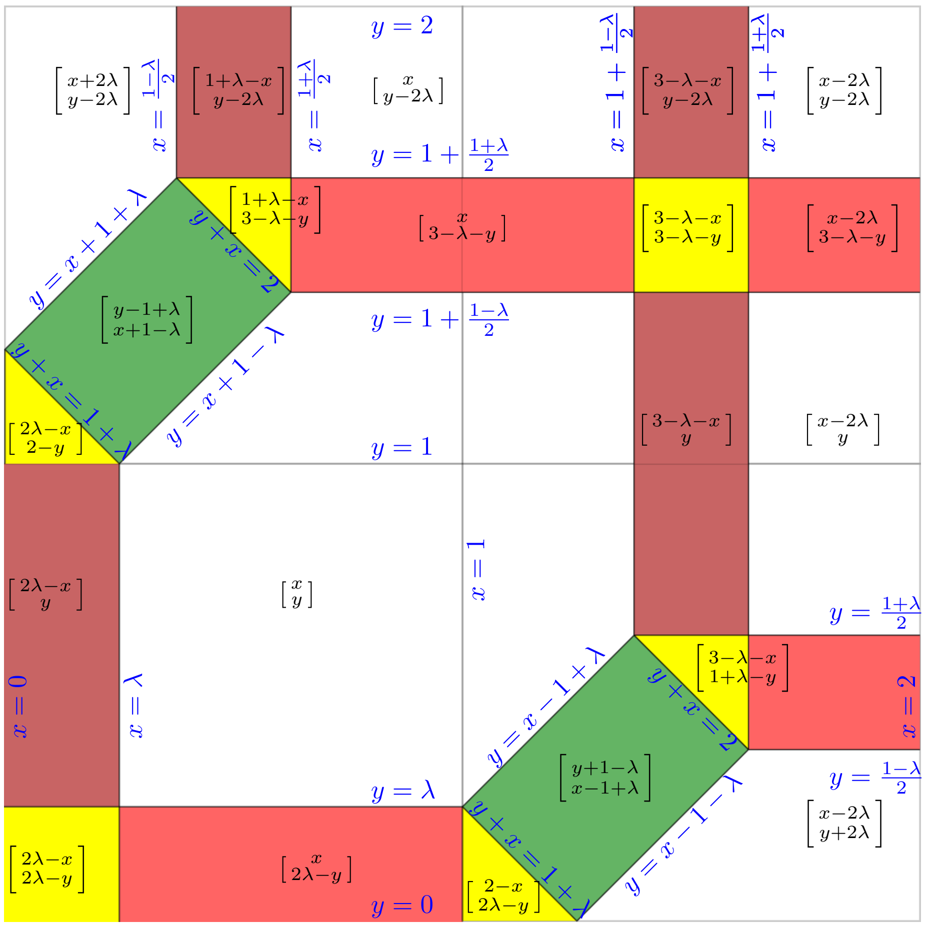

Figure 6: We represent here the analytic expression of the vector-valued solution, up to an additive constant and a rescaling, specifically around a corner cell (as in Figure 3). Of course also here the solution matches continuously on each discontinuity line of the gradient. The difference of the values of the solution at the vertex points $(2,2)$ and $(0,0)$ is equal to $2-4\lambda$; i.e. it is the same value computed similarly for the boundary value $(1-2\lambda)(x,y)$. The same is true for the other two vertices $(2,0)$ and $(0,2)$.

of the boundary datum (3) on the four vertices of the unit square. From a basic cell to another basic cell, the map is rescaled, rotated, and translated so that this property is preserved on all the vertices of the grid in Figure 1.

Since the cells have diameter which goes to zero as we approach the boundary $\partial\Omega$, and since the map is Lipschitz-continuous, we can extend it to the boundary so that the map assumes exactly the linear datum $(1-2\lambda)(x,y)$.

In conclusion our map u solves the Dirichlet problem

$$\begin{cases} Du(x,y) \in \left\{\pm\begin{bmatrix} 1 & 0 \\ 0 & \pm 1\end{bmatrix}, \pm\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\right\} \subset O(2) & \text{a.e. } (x,y) \in [0,1]^2 \\ u(x,y) = (1-2\lambda)(x,y) & \text{for all } (x,y) \in \partial[0,1]^2 \end{cases}$$

the map being orthogonal almost everywhere on $[0,1]^2$ and assuming only six gradient values.

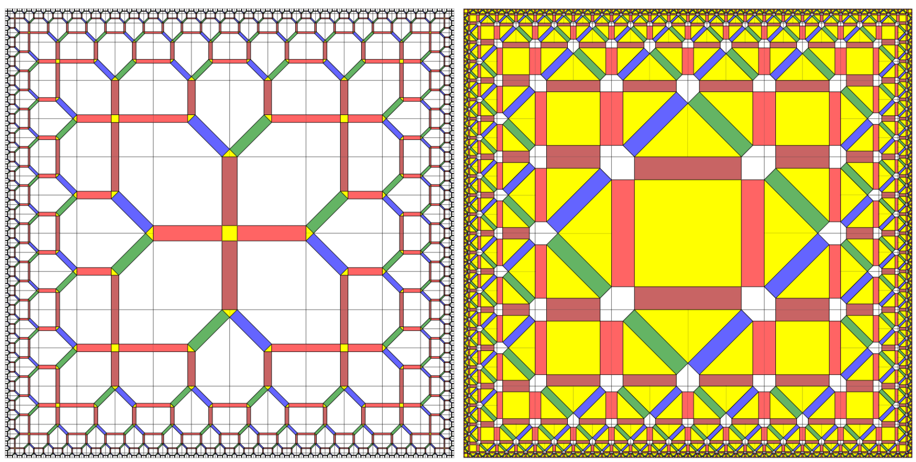

Figure 7: Two singular sets of the solution in dependence on λ: on the left-hand side $\lambda = \frac{1}{10}$ on the right-hand side $\lambda = \frac{7}{10}$. Again, the gradient matrix of the solution is equal to the identity matrix I in the white regions, while it is equal to $-I$ in the yellow regions.

When λ varies in $(0,1)$ the singular set of the solution that we have built varies in consequence. In Figure 7 we present two singular sets. The first one with a small value of λ the second one with a value of λ close to 1.

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