# GORENSTEIN STABLE SURFACES WITH $K_{X}^{2}=1$ AND $p_{g}>0$ 

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#### Abstract

In this paper we consider Gorenstein stable surfaces with $K_{X}^{2}=1$ and positive geometric genus. Extending classical results, we show that such surfaces admit a simple description as weighted complete intersection.

We exhibit a wealth of surfaces of all possible Kodaira dimensions that occur as normalisations of Gorenstein stable surfaces with $K_{X}^{2}=1$; for $p_{g}=2$ this leads to a rough stratification of the moduli space.

Explicit non-Gorenstein examples show that we need further techniques to understand all possible degenerations.


## Contents

1. Introduction 1
2. Stable surfaces and moduli spaces 3
3. Canonical ring and pluricanonical maps 4
4. Stratification of $\overline{\mathfrak{M}}_{1,3}^{(\mathrm{Gor})}$ and beyond 8
5. Bestiarium in $\overline{\mathfrak{M}}_{1,2}^{(\text {Gor })}$ and $\overline{\mathfrak{M}}_{1,2} \quad 18$

References 24

## 1. Introduction

This is the third in a series of papers studying Gorenstein stable surfaces with $K_{X}^{2}=1$. Such surfaces are parametrized by (an open part of) the moduli space of stable surfaces $\overline{\mathfrak{M}}_{K^{2}, \chi}$, a natural compactification of Gieseker's moduli space of canonical models of surfaces of general type $\mathfrak{M}_{K^{2}, \chi}$. Unlike the case of curves, the moduli space of stable surfaces is not obtained just by adding a boundary divisor but it can have extra irreducible/connected components. Also there are numerical invariants which can be realized by stable surfaces but not by minimal surfaces; most notably, $K_{X}^{2}$ may not be an integer if $X$ is only $\mathbb{Q}$-Gorenstein and the holomorphic Euler characteristic of a stable surface can be negative.

The classification of minimal surfaces with $K_{X}^{2}=1$ and positive geometric genus is a classical topic, studied for example by Enriques, Kodaira, Horikawa, Catanese, and Todorov (see [Enr49, Hor76, Cat79, Cat80, Tod80]). In this paper we consider Gorenstein stable surfaces with $K_{X}^{2}=1$ and positive geometric genus, recovering the standard embeddings in weighted projective space and the known results on pluricanonical maps on the one hand and finding a detailed description of singular ones (either normal or non-normal) on the other hand.

Our first result says that the classical descriptions extend uniformly to Gorenstein stable surfaces.

[^0]Theorem 3.3 - Let $X$ be a Gorenstein stable surface with $K_{X}^{2}=1$.
(i) If $p_{g}(X)=2$ then $X$ is canonically embedded as a hypersurface of degree 10 in the smooth locus of $\mathbb{P}(1,1,2,5)$.
(ii) If $p_{g}(X)=1$ then $X$ is canonically embedded as a complete intersection of bidegree $(6,6)$ in the smooth locus of $\mathbb{P}(1,2,2,3,3)$.

Note however that this no longer holds true if we drop the Gorenstein assumption (see Section 4.C).

As a consequence, such surfaces are smoothable, and therefore the moduli space $\overline{\mathfrak{M}}_{1,3}^{(\text {Gor })}$ of Gorenstein stable surfaces with $K^{2}=1$ and $\chi=3$ is irreducible and rational of dimension 28 , whilst the moduli space $\overline{\mathfrak{M}}_{1,2}^{(\text {Gor })}$ of Gorenstein stable surfaces with $K^{2}=1$ and $\chi=2$ is an irreducible and rational variety of dimension 18 (see Corollary 3.5).

The above explicit description entails control over the structure of pluricanonical maps, especially the bicanonical map. In case $p_{g}(X)=2$ the bicanonical map realizes $X$ as a double cover of the quadric cone in $\mathbb{P}^{3}$ branched over a quintic section. In Section 4 we make a detailed study of the possible branch divisors resulting in a (rough) stratification of the moduli space $\overline{\mathfrak{M}}_{1,3}^{(\text {Gor })}$. As a byproduct we show with explicit examples that the resolution of a Gorenstein stable surface with $K_{X}^{2}=1$ and $p_{g}=2$ can have arbitrary Kodaira dimension; this had been announced in [FPR15b]. We also give some examples of non-Gorenstein stable surfaces with $K_{X}^{2}=1$ and $p_{g}=2$ which are not canonically embedded in $\mathbb{P}(1,1,2,5)$.

It is worth remarking that these surfaces play an important role in the construction of threefolds near the Noether line, see e.g. [Che04].

The case of surfaces with $K_{X}^{2}=1$ and $p_{g}(X)=1$ was intensively studied for some time as it provided the counterexamle for the local Torelli theorem on surfaces [Kyn77] (see also [Usu00] and references therein). Since for the general such surface the bicanonical map $\varphi_{2}: X \rightarrow \mathbb{P}^{2}$ is not a Galois covering, we cannot carry out a similarly detailed analysis. In Section 5 we construct some examples, again of all possible Kodaira dimensions, which show the possible variations already in the special case where the bicanonical map is a bi-double cover.

The more challenging case of numerical Godeaux surfaces $\left(K_{X}^{2}=\chi\left(\mathcal{O}_{X}\right)=1\right)$ will be treated in a subsequent paper.

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Notations and conventions. We work exclusively with schemes of finite type over the complex numbers.

- A surface is a reduced, projective scheme of pure dimension two but not necessarily irreducible or connected.
- For a scheme $X$ which is Gorenstein in codimension 1 and $S_{2}$ we use the competing notations $m K_{X}$ and $\omega_{X}^{[m]}$ for multiples of canonical divisor, respectively reflexive powers of the canonical sheaf.
- Given a variety $Y$ and a line bundle $L \in \operatorname{Pic}(Y)$, one defines the ring of sections $R(Y, L)=\bigoplus_{m \geq 0} H^{0}(m L)$; for $L=K_{Y}$, we have the canonical ring $R\left(K_{Y}\right):=R\left(Y, K_{Y}\right)$.


## 2. Stable surfaces and moduli spaces

In this section we recall some necessary notions and establish the notation that we need throughout the text. Our main reference is [Kol13, Sect. 5.1-5.3].
2.A. Stable surfaces and log-canonical pairs. Let $X$ be a demi-normal surface, that is, $X$ satisfies $S_{2}$ and at each point of codimension one $X$ is either regular or has an ordinary double point. We denote by $\pi: \bar{X} \rightarrow X$ the normalisation of $X$. The conductor ideal $\mathscr{H} \operatorname{om}_{\mathcal{O}_{X}}\left(\pi_{*} \mathcal{O}_{\bar{X}}, \mathcal{O}_{X}\right)$ is an ideal sheaf both in $\mathcal{O}_{X}$ and $\mathcal{O}_{\bar{X}}$ and as such defines subschemes $D \subset X$ and $\bar{D} \subset \bar{X}$, both reduced and of pure codimension 1 ; we often refer to $D$ as the non-normal locus of $X$.

Definition 2.1 - The demi-normal surface $X$ is said to have semi-log-canonical (slc) singularities if it satisfies the following conditions:
(i) The canonical divisor $K_{X}$ is $\mathbb{Q}$-Cartier.
(ii) The pair ( $\bar{X}, \bar{D}$ ) has log-canonical (lc) singularities.

It is called a stable surface if in addition $K_{X}$ is ample. In that case we define the geometric genus of $X$ to be $p_{g}(X)=h^{0}\left(X, \omega_{X}\right)=h^{2}\left(X, \mathcal{O}_{X}\right)$ and the irregularity as $q(X)=h^{1}\left(X, \omega_{X}\right)=h^{1}\left(X, \mathcal{O}_{X}\right)$. A Gorenstein stable surface is a stable surface such that $K_{X}$ is a Cartier divisor.

Since a demi-normal surface $X$ has at most double points in codimension one, the map $\pi: \bar{D} \rightarrow D$ on the conductor divisors is generically a double cover and thus induces a rational involution on $\bar{D}$. Normalising the conductor loci we get an honest involution $\tau: \bar{D}^{\nu} \rightarrow \bar{D}^{\nu}$ such that $D^{\nu}=\bar{D}^{\nu} / \tau$. By [Kol13, Thm. 5.13], the triple $(\bar{X}, \bar{D}, \tau)$ determines $X$.

The log-canonical pairs ( $\bar{X}, \bar{D}$ ) that can arise normalising a Gorenstein stable surface $X$ with $K_{X}^{2}=1$ have been classified in [FPR15b, Thm. 1.1] and are the following:
(P) $\bar{X}=\mathbb{P}^{2}, \bar{D}$ is a quartic.
(dP) $\bar{X}$ is a Gorenstein Del Pezzo surface with $K_{\bar{X}}^{2}=1$ and $\bar{D} \in\left|-2 K_{\bar{X}}\right|$.
$\left(E_{-}\right) \bar{X}$ is obtained from a $\mathbb{P}^{1}$-bundle $p: Y \rightarrow E$ over an elliptic curve by contracting a section $C_{\infty}$ with $C_{\infty}^{2}=-1$ and $\bar{D}$ is the image in $\bar{X}$ of a bisection of $p$ disjoint from $C_{\infty}$.
( $E_{+}$) $\bar{X}=S^{2} E$, where $E$ is an elliptic curve and $\bar{D}$ is a trisection of the Albanese map $\bar{X} \rightarrow E$ with $p_{a}(\bar{D})=2$.
In addition, we have:
Theorem 2.2 ([FPR15a], Prop. 4.2, [FPR15b], Thm. 3.6) - Let $X$ be a Gorenstein stable surface with $K_{X}^{2}=1$ Then $0 \leq \chi(X) \leq 3$ and moreover:
(i) if $\chi(X)=0$ then $p_{g}(X)=0$ and $q(X)=1$
(ii) if $\chi(X)>0$, then $q(X)=0$ and $p_{g}(X)=\chi(X)-1$;
(iii) if $\chi(X)=3$, then $(\bar{X}, \bar{D})$ is not of type $\left(E_{+}\right)$;
(iv) if $\chi(X)=1$, then $(\bar{X}, \bar{D})$ is not of type $\left(E_{-}\right)$;

In what follows we do not use the classification of the pairs $(\bar{X}, \bar{D})$ to describe geometry of $X$ for $\chi(X)=2,3$, but we analyse instead the canonical ring and the pluricanonical maps. However, from this analysis we will be able to recover examples of all the possible types of normalisation except a surface with $p_{g}(X)=1$ and normalisation ( $E_{+}$) (see Remark 5.3), and thus keep a promise made in [FPR15b, Sect. 4].
2.B. Moduli spaces. We will discuss surfaces in the following hierarchy of open inclusions of moduli spaces of surfaces with fixed invariants $a=K_{X}^{2}$ and $b=\chi\left(\mathcal{O}_{X}\right)$ :

$$
\begin{aligned}
\int_{a, b}^{\mathfrak{M}_{a, b}} & =\text { Gieseker moduli space of surfaces of general type } \\
\overline{\mathfrak{M}}_{a, b}^{(\text {Gor })} & =\text { moduli space of Gorenstein stable surfaces } \\
\int_{\mathfrak{M}_{a, b}} & \\
& =\text { moduli space of stable surfaces }
\end{aligned}
$$

The openness of second inclusion follows from [BH93, Cor. 3.3.15]. For the time being there is no self-contained reference for the existence of the moduli space of stable surfaces with fixed invariants as a projective scheme, and we will not use this explicitly. A major obstacle in the construction is that in the definition of the moduli functor one needs additional conditions beyond flatness to guarantee that invariants are constant in a family. For Gorenstein surfaces these problems do not play a role; we refer to [Kol12] and the forthcoming book [Kol15] for details.

## 3. Canonical Ring and Pluricanonical maps

Here we compute the canonical ring of stable Gorenstein surfaces with $K^{2}=1$ and $p_{g}>0$. The upshot is that from this point of view Gorenstein stable surfaces behave exactly like smooth minimal ones. The study of minimal surfaces with $K^{2}=1$ and $p_{g}=1,2$ goes back to Enriques [Enr49, Chapter VIII] and Kodaira (compare [Hor76, (2.1)]). Later many authors studied such surfaces developing a complete picture (see [Tod80, Cat79, Cat80, Hor76]). Therefore, we give a very synthetic treatment, just stressing the points where a different argument is needed for the stable case.

Our point of view is that the canonical ring can be recovered from the restriction to a canonical curve and enables us to describe the canonical maps and to deduce some basic properties of the moduli spaces. It is thus important that in our case the canonical curves are sufficiently nice.

Lemma 3.1 ([FPR15a], Lem. 4.1) - Let $X$ be a Gorenstein stable surface such that $K_{X}^{2}=1$ and let $C \in\left|K_{X}\right|$ be a canonical curve.
Then $C$ is an integral Gorenstein curve with $p_{a}(C)=2$.
3.A. Half-canonical rings of Gorenstein curves of genus 2. Let $C$ be a reduced and irreducible Gorenstein curve of genus 2 and let $L \in \operatorname{Pic}(C)$ be a square root of $K_{C}$; in our application $C$ is a canonical curve of $X$ and $L=\left.K_{X}\right|_{C}$.

The analysis of rings of sections of line bundles on curves goes back at least to Petri, and the following result is well known on smooth curves (see for example [Rei90, Sect. 4]).
To formulate the result in a form coherent with Section 3.B we denote by $\bar{S}_{1}$ the polynomial ring $\mathbb{C}\left[x_{1}, y, z\right]$ where $x_{1}$ has degree $1, y$ has degree 2 and $z$ has degree

5 , and by $\bar{S}_{2}$ the polynomial ring $\mathbb{C}\left[y_{1}, y_{2}, z_{1}, z_{2}\right]$ where $y_{i}$ has degree 2 and $z_{i}$ has degree $3(i=1,2)$.

Proposition 3.2 - Let $C$ be an integral Gorenstein curve with $p_{a}(C)=2$ and let $L \in \operatorname{Pic}(C)$ such that $L^{\otimes 2}=\omega_{C}$.
( $i$ ) If $h^{0}(L)=1$, then $R(L) \cong \bar{S}_{1} /(f)$, where $f=z^{2}+y^{5}+x_{1}^{2} g\left(x_{1}, y\right)$ is weighted homogeneous of degree 10 .
(ii) If $h^{0}(L)=0$, then $R(L) \cong \bar{S}_{2} /\left(f_{1}, f_{2}\right)$, where $f_{1}=z_{1}^{2}+c_{1}\left(y_{1}, y_{2}\right)$ and $f_{2}=z_{2}^{2}+c_{2}\left(y_{1}, y_{2}\right)$ are weighted homogeneous of degree 6 and $c_{1}, c_{2}$ have no common factor.

Proof. The main tools to prove statements (i) and (ii) are
(a) the Riemann-Roch theorem and Serre duality, used to compute $h^{0}(m L)$, $m \geq 1$ and to determine the base points of $|m L|$;
(b) the base point free pencil trick (see [ACGH85, chap. III §3]), used to to show surjectivity of multiplication maps of the form $H^{0}(a L) \otimes H^{0}(b L) \rightarrow$ $H^{0}((a+b) L)$.
Since both (a) and (b) hold for an irreducible Gorenstein curve (see for example [Fra13]) the degree of generators and relations can be determined verbatim as in the case of a smooth curve.

For $(i)$ we can thus choose variables such that the unique relation is $z^{2}+h\left(x_{1}, y\right)$; it remains to prove that $h\left(x_{1}, y\right)$ is not divisible by $x_{1}^{2}$. So assume by contradiction that this is the case: then the point $A=(0: 1: 0)$ lies on $C$ and therefore it is singular for $C$. On the other hand, $A$ is the support of the zero locus on $C$ of the section $x_{1} \in H^{0}(L)$, but this is impossible since $L$ is a line bundle of degree 1 .

For (ii) we still need to show that $c_{1}$ and $c_{2}$ have no common factor. Assume for contradiction that both $c_{1}$ and $c_{2}$ are divisible by, say, $y_{1}$. Then the point $A=(0$ : $1: 0: 0)$ lies on the curve, is singular for $C$ and is a base point of the 1-dimensional system $|3 L|$. It follows that $A$ is a double point of $C$, the fixed part of $|3 L|$ is equal to $|2 A|$ and the moving part $|M|$ of $|3 L|$ is a linear system of dimension 1 and degree 1 , contradicting the assumption that $C$ has genus 2 .
3.B. The canonical ring. We now lift the descriptions of section rings from the previous section to Gorenstein stable surfaces, recovering in the case of a minimal surface of general type the previously known descriptions (see [BHPV04, Ch. VII, §7] and [Cat79, §1 Prop. 6]).

We denote by $S_{1}$ the polynomial ring $\mathbb{C}\left[x_{0}, x_{1}, y, z\right]$ where $x_{0}, x_{1}$ have degree $1, y$ has degree 2 and $z$ has degree 5 , and by $S_{2}$ the polynomial ring $\mathbb{C}\left[x_{0}, y_{1}, y_{2}, z_{1}, z_{2}\right]$ where $x_{0}$ has degree $1, y_{i}$ has degree 2 and $z_{i}$ has degree $3(i=1,2)$.

Theorem 3.3 - Let $X$ be a Gorenstein stable surface with $K_{X}^{2}=1$ and $p_{g}(X)>0$. Then there are the following possibilities:
(i) $q(X)=0, \chi(X)=3$ and $R\left(K_{X}\right) \cong S_{1} /(f)$, where

$$
f=z^{2}+y^{5}+g\left(x_{0}, x_{1}, y\right)
$$

is weighted homogeneous of degree 10 and $g$ does not contain $y^{5}$. Hence $X$ is canonically embedded as a hypersurface of degree 10 in (the smooth locus of) $\mathbb{P}(1,1,2,5)$.
(ii) $q(X)=0, \chi(X)=2$ and $R\left(K_{X}\right) \cong S_{2} /(f, g)$, where

$$
\begin{aligned}
& f_{1}=z_{1}^{2}+z_{2} x_{0} a_{1}\left(x_{0}, y_{1}, y_{2}\right)+b_{1}\left(x_{0}, y_{1}, y_{2}\right) \\
& f_{2}=z_{2}^{2}+z_{1} x_{0} a_{2}\left(x_{0}, y_{1}, y_{2}\right)+b_{2}\left(x_{0}, y_{1}, y_{2}\right)
\end{aligned}
$$

are weighted homogeneous of degree 6. Hence $X$ is canonically embedded as a complete intersection of bidegree $(6,6)$ in (the smooth locus of) $\mathbb{P}(1,2,2,3,3)$.

Proof. The possibilities for the invariants $p_{g}(X)$ and $q(X)$ have already been given in Theorem 2.2. Let $C \in\left|K_{X}\right|$ and set $L=\left.K_{X}\right|_{C}$, so that by adjunction we have $L^{\otimes 2}=\mathcal{O}_{C}\left(K_{C}\right)$, and let $x_{0} \in R\left(K_{X}\right)$ be a section defining $C$. By Lemma 3.1 the pair $(C, L)$ satisfies the hypothesis of Proposition 3.2.

Consider the usual restriction sequence

$$
0 \longrightarrow \mathcal{O}_{X}\left(m K_{X}-C\right) \xrightarrow{\cdot x_{0}} \mathcal{O}_{X}\left(m K_{X}\right) \longrightarrow L^{\otimes m} \longrightarrow 0
$$

Since $q(X)=0$ by Theorem 2.2 and $H^{1}\left(m K_{X}\right)=0$ for $m \geq 0$ by Kodaira vanishing [LR14, Cor. 19], we see that the map $R\left(K_{X}\right) /\left(x_{0}\right) \rightarrow R(L)$ is a surjection, hence an isomorphism. In particular, $h^{0}(L)=p_{g}(X)-1$, so that the case $p_{g}=2$ corresponds to $(i)$ of Proposition 3.2 and $p_{g}=1$ corresponds to case (ii). The claim about generators and relations is now obtained by lifting the relations of $R(L)$ to $R\left(K_{X}\right)$ and completing the squares in the lifted equations.

Assume that $p_{g}(X)=2$ : the singular points of $\mathbb{P}(1,1,2,5)$ are the points of coordinates $(0: 0: 1: 0)$ and $(0: 0: 0: 1)$. Neither of these belongs to $X$ since $f$ contains the monomials $y^{5}$ and $z^{2}$.

Assume that $p_{g}(X)=1$ : the singular points of $\mathbb{P}(1,2,2,3,3)$ are the union of the two lines $\mathbb{P}(2,2)$ and $\mathbb{P}(3,3)$, which do not meet $X$ in view of the format of the equations and of the fact that by Proposition 3.2, (ii), the polynomials $b_{1}\left(0, y_{1}, y_{2}\right)$ and $b_{2}\left(0, y_{1}, y_{2}\right)$ have no common factor.

Theorem 3.3 gives immediately:
Corollary 3.4 - Let $X$ be a stable Gorenstein surface with $K_{X}^{2}=1$ and $p_{g}(X)>0$ (equivalently, with $\chi(X)>1$, compare Theorem 2.2). Then $X$ is smoothable.

## Corollary $3.5-$

(i) The moduli space $\overline{\mathfrak{M}}_{1,3}^{(G o r)}$ of Gorenstein stable surfaces with $K^{2}=1$ and $\chi=3$ is irreducible and rational of dimension 28.
(ii) The moduli space $\overline{\mathfrak{M}}_{1,2}^{(G o r)}$ of Gorenstein stable surfaces with $K^{2}=1$ and $\chi=2$ is irreducible and rational of dimension 18.

Proof. By Corollary 3.4, the statements follow by the corresponding statements for minimal surfaces of general type (see [Hor76, §3] for $\mathfrak{M}_{1,3}$, [Cat80, Thm. 2.3] for $\mathfrak{M}_{1,2}$ ). We could not track down a reference for $\mathfrak{M}_{1,3}$ being rational, so here is a quick argument, explained to us by Christian Böhning: we fix the $y$ and $z$ coordinate so that the equation is of the form

$$
z^{2}+y^{5}+f_{4} y^{3}+f_{6} y^{2}+f_{8} y+f_{10}=0
$$

where $f_{i}$ is a polynomial in $x_{0}, x_{1}$ of degree $i$. The remaining automorphisms are given by $G=\mathrm{Gl}(2, \mathbb{C})$ acting on $x_{0}, x_{1}$.

To conclude that $\mathfrak{M}_{1,3}$ is rational it is thus sufficient to prove that the quotient of $V=H^{0}\left(\mathcal{O}_{\mathbb{P}^{1}}(4) \oplus \mathcal{O}_{\mathbb{P}^{1}}(6) \oplus \mathcal{O}_{\mathbb{P}^{1}}(8) \oplus \mathcal{O}_{\mathbb{P}^{1}}(10)\right)$ by $G$ is rational. This follows from the so-called no-name-lemma [CGR06, Lem. 4.4] by looking at the projection $V \rightarrow H^{0}\left(\mathcal{O}_{\mathbb{P}^{1}}(6)\right)$ because
(i) the quotient $H^{0}\left(\mathcal{O}_{\mathbb{P}^{1}}(6)\right) / G$ is rational by [BK85],
(ii) the action of $G$ on $H^{0}\left(\mathcal{O}_{\mathbb{P}^{1}}(6)\right)$ is generically free, i.e., the stabiliser in the generic point is trivial. [Sha94, II, §7, Ex. 1 or Thm. 7.11].
This concludes the proof.
3.C. Pluricanonical maps. Here we spell out some properties of the pluricanonical maps of Gorenstein stable surfaces with $K^{2}=1$ and $p_{g}>0$. Such properties are implicit in the description of the canonical ring given in Theorem 3.3. As in the previous section, the known results for surfaces of general type extend to our case (cf. [Tod80, Cat79, Cat80, Hor76, BHPV04]).

We denote by $\varphi_{m}$ the $m$-canonical map of the stable Gorenstein surface $X$, given by the $m$-canonical system $\left|m K_{X}\right|$. Recall that by Riemann-Roch and Serre-duality [LR13, Cor. 3.2] for $m \geq 2$ one has $h^{0}\left(m K_{X}\right)=\chi(X)+\frac{m(m-1)}{2}$.
Proposition 3.6 - Let $X$ be a Gorenstein stable surface with $K_{X}^{2}=1$ and $p_{g}(X)=$ 2. Then:
(i) $\left|K_{X}\right|$ has a simple base point $P$, which is smooth for $X$
(ii) $\varphi_{2}$ is a finite degree 2 morphism, $\varphi_{2}(X) \subset \mathbb{P}^{3}$ is the quadric cone, image of the embedding of $\mathbb{P}(1,1,2)$ defined by $|\mathcal{O}(2)|$, and the branch locus of $\varphi_{2}$ is the union of the vertex $O \in \varphi_{2}(X)$ and of a degree 5 hypersurface section of $\varphi_{2}(X)$ not containing $O$. The base point $P$ of $\left|K_{X}\right|$ is the only point mapped to $O$.
(iii) $\varphi_{3}(X)$ is the embedding of $\mathbb{F}_{2}$ as a normal ruled surface of degree 4 in $\mathbb{P}^{5}$, $\varphi_{3}$ has degree 2 and is not defined at $P$.
(iv) $\varphi_{4}$ is the composition of $\varphi_{2}$ with the degree 2 Veronese embedding of $\varphi_{2}(X)$ in $\mathbb{P}^{8}$
(v) $\varphi_{m}$ is an embedding for $m \geq 5$.

Proof. We use the notation of Theorem 3.3 for the generators of the canonical ring $R\left(K_{X}\right)$ and for the relations between them.
(i) A basis of $H^{0}\left(K_{X}\right)$ is given by $x_{0}, x_{1}$, hence the base locus of $\left|K_{X}\right|$ is the intersection of $X$ with $x_{0}=x_{1}=0$ and it consists just of the point $P=(0: 0$ : $-1: 1)$. Since $K_{X}^{2}=1$ and all the canonical curves are reduced and irreducible by Lemma 3.1, $P$ is a simple base point and it is smooth for $X$.
(ii) Let $\pi: \mathbb{P}(1,1,2) \hookrightarrow \mathbb{P}^{3}$ be the embedding given by $|\mathcal{O}(2)|$. A basis of $H^{0}\left(2 K_{X}\right)$ is given by $x_{0}^{2}, x_{0} x_{1}, x_{1}^{2}, y$, hence the bicanonical map $\varphi_{2}$ is the projection from $\mathbb{P}(1,1,2,5)$ to $\mathbb{P}(1,1,2)$ composed with $\pi$. Since the point $(0: 0: 0: 1)$ is not on $X$ by Theorem 3.3, the map $\varphi_{2}$ is a finite degree 2 morphism. The branch points of $\varphi_{2}$ different from the vertex $O=(0: 0: 1) \in \mathbb{P}(1,1,2)$ are defined by the equation $f=y^{5}+g\left(x_{0}, x_{1}, y\right)$ which corresponds to a quintic section not containing $O$ in the embedding $\pi$. The only point of $X$ that maps to $O$ is the base point $P=(0: 0:-1: 1)$ of $\left|K_{X}\right|$, hence $\varphi_{2}$ is branched also on the vertex $O$.
(iii) and (iv) can be proven in a similar way.
(v) To show that the $m$-canonical map is an embedding for $m \geq 5$ we note first that for $m \geq 5$ the system $\left|\mathcal{O}_{\mathbb{P}(1,1,2,5)}(m)\right|$ embeds the locus $\mathbb{P}(1,1,2,5) \backslash \mathbb{P}(2,5)$. The
surface $X$ intersects $\mathbb{P}(2,5)$ only at the point $P=(0: 0:-1: 1)$, which is the base point of the canonical system. It is easy to check that for $m \geq 5$ the image of $\mathbb{P}(2,5)$ via the map given by $\left|\mathcal{O}_{\mathbb{P}(1,1,2,5)}(m)\right|$ is disjoint from the image of its complement $\mathbb{P}(1,1,2,5) \backslash \mathbb{P}(2,5)$, hence $\varphi_{m}$ is injective for $m \geq 5$. In addition, for every $m \geq 5$, there exist a monomial $s$ of degree $m-1$ and a monomial $t$ of degree $m$ that do not vanish at $P$ : then the $m$-canonical map is given locally by $\left(\frac{s x_{0}}{t}, \frac{s x_{1}}{t}, \ldots\right)$ and therefore has injective differential at $P$. Indeed, the canonical curves defined by $x_{0}$ and $x_{1}$ intersect transversally at $P$, since they are distinct and irreducible (Lemma 3.1) and $K_{X}^{2}=1$.

Proposition 3.7 - Let $X$ be a Gorenstein stable surface with $K_{X}^{2}=1$ and $p_{g}(X)=$ 1. Then:
(i) $\varphi_{2}: X \rightarrow \mathbb{P}^{2}$ is a finite degree 4 morphism.
(ii) $\varphi_{3}$ and $\varphi_{4}$ are birational morphisms but do not embed.
(iii) $\varphi_{m}$ is an embedding for $m \geq 5$.

Proof. We use the notation of Theorem 3.3 for the generators of the canonical ring $R\left(K_{X}\right)$. In particular, we have an isomorphism $H^{0}\left(\mathcal{O}_{\mathbb{P}(1,2,2,3,3)}(d)\right) \cong H^{0}\left(d K_{X}\right)$ for $d \leq 5$.
(i) The bicanonical map is induced by the projection $\mathbb{P}(1,2,2,3,3) \rightarrow \mathbb{P}(1,2,2)=$ $\mathbb{P}^{2}$. Looking at the equations of $X$ given in Theorem 3.3, one sees that $\varphi_{2}$ is finite of degree four.
(ii) The base-point-freeness of $\left|d K_{X}\right|$ follows by restriction from $\left|\mathcal{O}_{\mathbb{P}(1,2,2,3,3)}(d)\right|$, for $d \geq 3$ since $X$ does not meet the lines $\mathbb{P}(2,2)$ and $\mathbb{P}(3,3)$ by Theorem 3.3. The restriction of $\left|3 K_{X}\right|$ to the canonical curve $C=\left\{x_{0}=0\right\}$ is spanned by $z_{1}$ and $z_{2}$, hence $\left.\varphi_{3}\right|_{C}$ is not birational and $\varphi_{3}$ is not an embedding.
To see that $\varphi_{3}$ is birational, consider $X_{0}:=X \cap\left\{x_{0} \neq 0\right\}$ with affine coordinates $t_{1}=\frac{y_{1}}{x_{0}^{2}}, t_{2}=\frac{y_{2}}{x_{0}^{2}}, t_{3}=\frac{z_{1}}{x_{0}^{3}}, t_{4}=\frac{z_{2}}{x_{0}^{3}}$. The functions $t_{1}, \ldots, t_{4}$ are all pull-backs from the 3-canonical image $\varphi_{3}(X)$, hence $\varphi_{3}$ is birational. One can argue exactly in the same way for $\varphi_{4}$.
(iii) Let $m \geq 5$. We have already observed in (ii) that the $m$-canonical system is base-point free. Since the linear system $\left|\mathcal{O}_{\mathbb{P}(1,2,2,3,3)}(m)\right|$ embeds the complement of $\left\{x_{0}=0\right\}$ we only have to check that every subscheme $Z$ of length two with support intersecting the canonical curve $C$ is embedded. If $Z \subset 2 C$ this follows because the restriction of the $m$-canonical system is very ample on $2 C$ by the numerical criterion [CFHR99, Thm. 1.1]. That points on $C$ can be separated from points in the complement follows as above.

## 4. Stratification of $\overline{\mathfrak{M}}_{1,3}^{(G o r)}$ and beyond

In this section we examine more closely the moduli spaces $\overline{\mathfrak{M}}_{1,3}^{(\text {Gor })} \subset \overline{\mathfrak{M}}_{1,3}$. The Gorenstein surfaces in the boundary $\overline{\mathfrak{M}}_{1,3} \backslash \mathfrak{M}_{1,3}$ are studied in detail. In addition, we show by explicit examples that a stable surface with $K_{X}^{2}=1$ and $\chi(X)=3$ need neither be Gorenstein nor be canonically embedded in $\mathbb{P}(1,1,2,5)$.
4.A. The double cover construction. In this section $X$ denotes a stable Gorenstein surface with $K_{X}^{2}=1$ and $\chi(X)=3$, or, equivalently, $p_{g}(X)=2$ and $q(X)=0$.

We denote by $\mathcal{Q} \subset \mathbb{P}^{3}$ the quadric cone, i.e., the image of the embedding of $\mathbb{P}(1,1,2)$ given by the system $\left|\mathcal{O}_{\mathbb{P}(1,1,2)}(2)\right|$, and by $O \in \mathcal{Q}$ the vertex.

By Proposition 3.6, the bicanonical map is a double cover $\varphi_{2}: X \rightarrow \mathcal{Q}$ branched on a divisor $\Delta \in\left|\mathcal{O}_{\mathcal{Q}}(5)\right|$ such that $O \notin \Delta$. The divisor $\frac{1}{2} \Delta$ is the so-called Hurwitz divisor of the cover (cf. [AP12, Def. 2.4]) and ( $\mathcal{Q}, \frac{1}{2} \Delta$ ) is a log-canonical pair by [AP12, Def. 2.5].

Here we show that it is possible to reverse this construction:
Proposition 4.1 - Let $\Delta \in\left|\mathcal{O}_{\mathcal{Q}}(5)\right|$ be such that $\left(\mathcal{Q}, \frac{1}{2} \Delta\right)$ is a log-canonical pair. Then:
(i) There exists a unique double cover $p: X \rightarrow \mathcal{Q}$ with Hurwitz divisor $\frac{1}{2} \Delta$
(ii) $X$ is a stable surface with $K_{X}^{2}=1$
(iii) the Cartier index of $X$ is equal to 1 if $O \notin \Delta$ and it is equal to 2 otherwise
(iv) $X$ is normal if and only if $\Delta$ is reduced.

Proof. (i) We write $\mathcal{Q}_{0}=\mathcal{Q} \backslash\{O\}$ and we denote by $\Delta_{0}$ the restriction of $\Delta$ to $\mathcal{Q}_{0}$. Since $\operatorname{Pic}\left(\mathcal{Q}_{0}\right)=\mathbb{Z} f$, where $f$ is the class obtained by restricting a ruling of $\mathcal{Q}$, the divisor $\Delta_{0}$ is linearly equivalent to $10 f$, hence there exists a flat double cover $X_{0} \rightarrow \mathcal{Q}_{0}$ branched on $\Delta_{0}$. The surface $X_{0}$ is demi-normal, since, by definition of $\log$ pair, all components of $\Delta$ have multiplicity $\leq 2$. By taking the $S_{2}$-closure (cf. [AP12, Lem. 1.2]) one obtains a demi-normal double cover $p: X \rightarrow \mathcal{Q}$ with Hurwitz divisor $\frac{1}{2} \Delta$.

The cover $p$ is unique, since it is determined by its restriction to $\mathcal{Q}_{0}$ and $\operatorname{Pic}\left(\mathcal{Q}_{0}\right)$ has no torsion.
(ii), (iii) One has $2 K_{X}=p^{*}\left(2 K_{\mathcal{Q}}+\Delta\right)=p^{*}\left(\mathcal{O}_{\mathcal{Q}}(1)\right)$, hence the Cartier index of $X$ is at most $2, K_{X}$ is ample and $K_{X}^{2}=1$. Notice that $K_{X_{0}}$ is Cartier by construction, hence to decide whether $K_{X}$ is Cartier it is enough to examine the point $O$. Notice also that $O$ is always in the branch locus of $p$, since a ruling of $\mathcal{Q}$ intersects $\Delta_{0}$ in 5 points and 5 is an odd number. If $O \notin \Delta$, then $X \rightarrow \mathcal{Q}$ is normal over $\Delta$, hence it is smooth there and therefore $X$ is Gorenstein. The converse follows by Proposition 3.6, (ii).
(iv) The surface $X$, being $S_{2}$ by construction, is normal if and only if it is smooth in codimension 1 , if and only if $X_{0}$ is smooth in codimension 1 , if and only if $\Delta$ is reduced.

Remark 4.2 - As it will be apparent in the sequel, the utility of Proposition 4.1 lies mainly in the fact that it reduces the analysis of the singularities of $X$ to the study of the curve $\Delta \subset \mathcal{Q}$. However the construction of $X$ as a double cover of $\mathcal{Q}$ is not easy to perform in families, since it involves taking the $S_{2}$-closure of a cover of $\mathcal{Q}_{0}$. This difficulty can be avoided by using Theorem $3.3,(i)$ : in the notation there, the divisor $\Delta \subset \mathbb{P}(1,1,2)$ is given by the term $y^{5}+g\left(x_{0}, x_{1}, y\right)$ in the equation $f$ of $X \subset \mathbb{P}(1,1,2,5)$, hence it is immediate to construct a flat family having as (repeated) fibres all the surfaces $X$ constructed as in Proposition 4.1.
4.B. Stratification of $\overline{\mathfrak{M}}_{1,3}^{(\text {Gor })} \backslash \mathfrak{M}_{1,3}$. We now describe more precisely $\overline{\mathfrak{M}}_{1,3}^{(\text {Gor })} \backslash \mathfrak{M}_{1,3}$, namely we consider ramified covers of $\mathcal{Q}$ as above where the Hurwitz divisor $\Delta$ does not contain the vertex.

If $P \in \Delta$ is an isolated singularity, then one of the following occurs:

- $P$ is a negligible singularity: $P$ is a double point or a triple point such that every point infinitely near to $P$ is at most double for the strict transform of $\Delta$. The preimage of $P$ in $X$ is a canonical singularity.
- $P$ is a quadruple point such that every point infinitely near to $P$ is at most double for the strict transform of $\Delta$. The preimage of $P$ in $X$ is an elliptic Gorenstein singularity of degree 2 .
- $P$ is a [3,3]-point, namely $P$ is a triple point with an infinitely near triple point $P_{1}$ and all the points infinitely near to $P_{1}$ are at most double. The preimage of $P$ in $X$ is an elliptic Gorenstein singularity of degree 1.
This can be checked by blowing up and analysing the cases similar to what is done in [LR12] or from the point of view of log-canonical threshold [KSC04, 6.5].

We consider the (open) strata

$$
\mathfrak{N}_{d_{1}, \ldots, d_{k}}=\left\{\begin{array}{l|l}
X \in \overline{\mathfrak{M}}_{1,3}^{(G o r)} & \begin{array}{l}
X \text { is normal and has exactly } k \text { elliptic } \\
\text { singularites of degree } d_{1} \leq \cdots \leq d_{k}
\end{array}
\end{array}\right\}
$$

TABLE 1. Irreducible strata of normal surfaces in $\overline{\mathfrak{M}}_{1,3}^{(\text {Gor })}$

| stratum | dimension | minimal resolution $\widetilde{X}$ | $\kappa(\widetilde{X})$ |
| :---: | :---: | :--- | :---: |
| $\mathfrak{N}_{\varnothing}=\mathfrak{M}_{1,3}$ | 28 | general type | 2 |
| $\mathfrak{N}_{2}$ | 20 | blow up of a K3-surface | 0 |
| $\mathfrak{N}_{1}$ | 19 | minimal elliptic surface with $\chi(\widetilde{X})=2$ | 1 |
| $\mathfrak{N}_{2,2}$ | 12 | rational surface | $-\infty$ |
| $\mathfrak{N}_{1,2}$ | 11 | rational surface | $-\infty$ |
| $\mathfrak{N}_{1,1}^{R}$ | 10 | rational surface | $-\infty$ |
| $\mathfrak{N}_{1,1}^{E}$ | 10 | blow up of an Enriques surface | 0 |
| $\mathfrak{N}_{1,1,2}$ | 2 | ruled surface with $\chi(\widetilde{X})=0$ | $-\infty$ |
| $\mathfrak{N}_{1,1,1}$ | 1 | ruled surface with $\chi(\widetilde{X})=0$ | $-\infty$ |

Proposition 4.3 - The subset of $\overline{\mathfrak{M}}_{1,3}^{(\text {Gor })}$ parametrizing normal surfaces is stratified by the irreducible and unirational strata $\mathfrak{N}_{d_{1}, \ldots, d_{k}}$ given in Table 1.
Proof. We have explained above that it is sufficient to control the singularities of the branch divisor $\Delta \subset \mathcal{Q}$. For convenience we work in $\mathbb{P}(1,1,2)$ with coordinates $x_{0}, x_{1}, y$. Recall that the branch divisor is given by a polynomial in $H^{0}\left(\mathcal{O}_{\mathbb{P}(1,1,2)}(10)\right)$, which is of dimension 36. Note that any automorphism of $\mathbb{P}(1,1,2)$ is of the form $\left(x_{0}, x_{1}, y\right) \mapsto\left(a x_{0}+b x_{1}, c x_{0}+d x_{1}, e y+q\left(x_{0}, x_{1}\right)\right)$ where $a d-b c \neq 0, e \in \mathbb{C}^{*}$ and $q$ is a quadratic polynomial; fixing the degree 2 coordinate $y$ up to a multiple corresponds in the embedding in $\mathbb{P}^{3}$ to fixing a hyperplane section not containing the vertex.

For each potential stratum we first use the automorphim group to fix the the position of non-negligible singular points as far as possible and then impose singularities at these points, that is, look at $H^{0}(\mathcal{I}(10))$ for an appropriate ideal sheaf $\mathcal{I}$. For example, for $\mathfrak{N}_{2}$ the ideal sheaf is $\mathcal{I}=\left(x_{1}, y\right)^{4}$. Then three steps are needed to conclude:
(i) Compute the dimension of the automorphism group fixing the given configuration.
(ii) Check how many conditions are imposed by the singularites, i.e., compute $h^{0}(\mathcal{I}(10))$.
(iii) Show that there is a reduced curve $\Delta$ in the linear system induced by $H^{0}(\mathcal{I}(10))$ such that the pair $\left(\mathbb{P}(1,1,2), \frac{1}{2} \Delta\right)$ is slc and has exactly the prescribed singularities.

The first is elementary. The second could be achieved either by explict computation in a computer algebra system or, in most cases, by showing $H^{1}(\mathcal{I}(10))=0$ via Kawamata-Viehweg vanishing on a blow up. The third however requires the construction of examples, which for several strata required the computation of a basis for $H^{0}(\mathcal{I}(10))$. Therefore we used the computer systematically also to determine $h^{0}(\mathcal{I}(10))$ and do not provide abstract arguments even in the cases where this is easily possible. All computations have been carried out using the computer algebra system Macaulay 2 [GS]; a file containing the commented code is included in the arxiv source code.

The list of examples is given after the proof, so in the following we only address the first two questions. In each case the structure of the minimal resolution follows from [FPR15b, Thm. 4.1] combined with [FPR15b, Lem. 4.3]) and $\chi(X)=3$, except for a surface in $\mathfrak{N}_{1,1}$, which a priori could either be rational or an Enriques surface. We will see below that both cases occur.

Note that by [FPR15b, Thm. 4.1] there are no surfaces with more that 3 elliptic singularities. Therefore we have to investigate branch curves with at most three non-negligible singular points.

First of all let us consider $\mathfrak{N}_{1}$, i.e., assume that $\Delta$ has a point $[3,3]$ at $P=(1: 0: 0)$. The ruling through $P$ cannot be tangent to $\Delta$ in $P$ because otherwise it has to be contained in $\Delta$ which is excluded by Proposition 4.1. Thus we may assume that $\Delta$ is tangent to $H=\{y=0\}$ at $P$. A point $[3,3]$ with given tangent imposes 12 independent conditions. Thus $\mathfrak{N}_{1}$ is dominated by an open subset of a 23 -dimensional linear subspace of $\mathbb{P}^{35}$ and thus it is unirational and irreducible of dimension 19, because the subgroup of automorphisms fixing a point and a tangent direction is of dimension 4.

Next let us look at $\mathfrak{N}_{2}$, i.e., assume that $\Delta$ has an ordinary quadruple point at $P$. This imposes 10 independent conditions and thus $\mathfrak{N}_{2}$ is dominated by an open subset of a $\mathbb{P}^{25}$. The subgroup of automorphisms fixing a point has dimension 5 and thus we get a unirational irreducible component of dimension 20.

For the strata with two elliptic singularities we choose coordinates such that the corresponding singularities of $\Delta$ are at $P=(1: 0: 0)$ and $Q=(0: 1: 0)$. In the presence of one point of type $[3,3]$ at $P$ we arrange in addition that $\Delta$ is tangent to $H=\{y=0\}$ at $P$. If $Q$ is of type $[3,3]$ as well then there are two cases:
(R) $\Delta$ is not tangent to $H$ at $Q$. Then we can use a further automorphism to fix the tangent to $\Delta$ at $Q$, which has to be different from $\{y=0\}$.
(E) $\Delta$ is tangent to $H$ at $Q$. Then intersecting $\Delta$ and $H$ shows that $H$ is actually contained in $\Delta$. In this case we say that the two $[3,3]$ points have a matching tangent hyperplane.
We decompose $\mathfrak{N}_{1,1}=\mathfrak{N}_{1,1}^{R} \cup \mathfrak{N}_{1,1}^{E}$ according to these two cases. It is straightforward to compute a bicanonical divisor of the minimal resolution of $X \in \mathfrak{N}_{1,1}$ and to conclude that we get as a minimal model an Enriques surface if $X$ is in $\mathfrak{N}_{1,1}^{E}$ and a rational surface if $X$ is in $\mathfrak{N}_{1,1}^{R}$.

The explicit computations show that two elliptic singularities impose independent conditions, unless we are in case $E$ where we have one condition less than expected.

The dimension of the group of automorphisms fixing $P, Q$ and a tangent direction at each point is equal to 1 , thus $\mathfrak{N}_{1,1}^{R}$ is unirational of dimension $10=35-2 \cdot 12-1$.

The dimension of the group of automorphisms fixing $P, Q$ and the hyperplane section $H$ is 2 , thus $\mathfrak{N}_{1,1}^{E}$ is unirational of dimension $10=35-2 \cdot 12+1-2$, because the two [3,3]-points with matching tangent hyperplane impose only 23 conditions.

For the study of the stratum $\mathfrak{N}_{1,2}$ we can adopt the same argument. To compute the dimension, since we do not need to fix the tangent direction at $Q$, we get $\operatorname{dim}\left(\mathfrak{N}_{1,2}\right)=35-12-10-2=11$.

For the stratum $\mathfrak{N}_{2,2}$ with two quadruple points we do not need to fix the tangent directions at all and compute $\operatorname{dim} \mathfrak{N}_{2,2}=35-2 \cdot 10-3=12$.

To treat the remaining cases with three singular points, assume that we have a surface with three elliptic singularities. We choose coordinates such that the corresponding singularities of $\Delta$ are at $P, Q$, and $R=(1: 1: 0)$. If $\Delta$ has two quadruple points at $P$ and $Q$ then an additional quadruple point or $[3,3]$ point at $R$ forces $\Delta$ to contain twice the hyperplane section $H$, because $H \cdot \Delta$ and $H \cdot(\Delta-H)$ would otherwise be too big; this is impossible.

So we may assume that $\Delta$ is reduced and has [3,3]-points at $P$ and $Q$ and a point of multiplicity at least 3 at $R$. If $H=\{y=0\}$ is tangent to $\Delta$ in $P$ then it is contained in $\Delta$, since otherwise one would have $H \cdot \Delta>10$. Computing again $H \cdot(\Delta-H)$ we conclude that $2 H \leq \Delta$, a contradiction. Thus the tangent in a [3,3]point is neither $H$ nor a ruling of the cone. Note that the group of automorphisms of $\mathbb{P}(1,1,2)$ fixing $P, Q, R$, and a tangent direction distinct from $H$ at $P$ is trivial.

For $\mathfrak{N}_{1,1,2}$ parametrize the tangent at $Q$ with a parameter $t$ and let $\mathcal{I}_{t}$ be the ideal imposing a quadruple point at $R$, a $[3,3]$-point at $P$ with a fixed tangent and a [3,3]-point at $Q$, such that the infinitely near triple point is the chosen tangent direction, and let $V_{t}$ be the linear system induced by $H^{0}(\mathcal{I}(10))$. Then for $t$ general $V_{t}$ is a pencil, which however does not contain a reduced curve. Only when the two [3, 3]-points have a matching tangent hyperplane the conditions imposed are no longer independent and we get a 2 -dimensional linear system whose general member has the correct type of singularities.

Now consider $\mathfrak{N}_{1,1,1}$. As long as no two of the [3, 3]-points have a matching tangent hyperplane the conditions imposed by the singularities are independent and thus cannot be satisfied. In fact, it turns out that the only case in which there is a reduced curve satisfying the conditions is when two pairs of points have matching tangent hyperplanes, which gives us a pencil of solutions.

We conclude the discussion of normal Gorenstein degenerations by giving an example in each stratum.

Example in $\mathfrak{N}_{2}$ : An example with exactly one elliptic singularity of degree 2 and some negligible singularities is given by $\Delta=H_{1}+H_{2}+H_{3}+H_{4}+H_{5}$ where $H_{i}$ are general hyperplane sections such that $H_{1}, \ldots, H_{4}$ have a common point of intersection.

One can see the corresponding K3 surface in the following way. Consider a plane sextic $C$ which is the sum of four general lines and a conic. Fix a general point $P$ on the conic and let $L$ be the tangent in $P$. Let $S$ be the singular K3 surface obtained as a double cover branched over $C$. The strict transform of $L$ is an elliptic curve with a node at the preimage of $P$. Blowing up the node and contracting the elliptic curve gives $X$. Alternatively, blow up twice at $P$ to separate $L$ and $C$ and then contract the strict transform of $L$, a $(-1)$-curve, to a smooth point $R$ and the exceptional $(-2)$ curve to an $A_{1}$ point. The result is the quadric cone $\mathcal{Q}$, and the strict transform of $C$ has a quadruple point at $R$.

Example in $\mathfrak{N}_{1}$ : An example with exactly one elliptic singularity of degree 1 and some negligible singularities is given by $\Delta=H_{1}+H_{2}+H_{3}+C$ where the $H_{i}$ are

Figure 1. Branch locus and surface in $\mathfrak{N}_{1,1}^{R}$

general hyperplane sections which are tangent at a point $P$ and $C$ is a general quadric section.

Indeed, the pencil of hyperplane sections that are tangent to $\Delta$ at $P$ gives rise to a pencil of elliptic curves on $\widetilde{X}$. Using the canonical bundle formula one can check that the elliptic fibration $\widetilde{X} \rightarrow \mathbb{P}^{1}$ has a unique multiple fibre of multiplicity two, which corresponds to twice the ruling through $P$ and the exceptional elliptic curve $E$ is a two-section ot the fibration.

Example in $\mathfrak{N}_{2,2}$ : An example with exactly two elliptic singularities of degree 2 and some negligible singularities is given by $\Delta=H_{1}+H_{2}+H_{3}+H_{4}+H_{5}$ where the $H_{1}, \ldots, H_{4}$ are general hyperplane sections through two points $P$ and $Q$ and $H_{5}$ is a hyperplane section not containing these points.

The pencil of hyperplane section through $P$ and $Q$ pulls back to a pencil of rational curves on $X$.

Example in $\mathfrak{N}_{1,2}$ : An example with exactly two elliptic singularities, one of degree 1 and one of degree 2 is given by $z^{2}+y^{5}+x_{1}^{4}\left(x_{0}^{6}+y^{3}\right)+2 y^{4} x_{0}^{2}=0$ in $\mathbb{P}(1,1,2,5)$. Indeed, the local equations of the branch divisor at $P$, resp. $Q$, are $x_{1}^{4}\left(1+y^{3}\right)+2 y^{4}(1+y)$, an ordinary quadruple point, resp. $x_{0}^{6}+y^{3}\left(1+y^{2}+y x_{0}^{2}\right)$, which is a point of type $[3,3]$. One can check that these are the only singular points.

Example in $\mathfrak{N}_{1,1}^{E}$ : An example with exactly two elliptic singularities of degree 1 and some negligible singularities is given by $\Delta=H_{1}+H_{2}+H_{3}+H_{4}+H_{5}$ where the $H_{i}$ are general hyperplane sections such that $H_{1}, H_{2}, H_{3}$ are tangent at a point $P$ and $H_{3}, H_{4}, H_{5}$ are tangent at a point $Q \neq P$. One can check that on the minimal resolution $\widetilde{X}$ the bicanonical divisor $2 K_{\tilde{X}}$ is linearly equivalent to the strict transform of $p^{*} H_{3}$ which is twice a $(-1)$-curve. The minimal model thus has trivial bicanonical bundle and is an Enriques surface.

Example in $\mathfrak{N}_{1,1}^{R}$ : Since the tangent directions of the two [3, 3$]$ points do not match there is a certain genericity to the branch divisor, so we only could find a fairly complicated equation for $\Delta$ :

$$
\begin{aligned}
& y^{5}+\left(5 x_{0}^{2}+2 x_{0} x_{1}\right) y^{4}+\left(19 x_{0}^{3} x_{1}+x_{0}^{2} x_{1}^{2}-x_{0} x_{1}^{3}\right) y^{3}+\left(4 x_{0}^{4} x_{1}^{2}-3 x_{0}^{2} x_{1}^{4}\right) y^{2} \\
&-3 x_{0}^{3} x_{1}^{5} y-x_{0}^{4} x_{1}^{6}=0
\end{aligned}
$$

Example in $\mathfrak{N}_{1,1,2}$ : The branch divisor contains the hyperplane section tangent to the two $[3,3]$ points $P$ and $Q$. For a particular choice of this hyperplane the equation is a product

$$
\left(x_{0} x_{1}+y\right)\left(\alpha y^{4}+\beta f+\gamma g\right)=0
$$

Figure 2. Branch locus and surface in $\mathfrak{N}_{1,1,2}$


Figure 3. Branch locus and surface in $\mathfrak{N}_{1,1,1}$

where

$$
\begin{gathered}
f=x_{0}^{3} x_{1} y^{2}-2 x_{0}^{2} x_{1}^{2} y^{2}+x_{0} x_{1}^{3} y^{2}+x_{0}^{2} y^{3}-2 x_{0} x_{1} y^{3}+x_{1}^{2} y^{3} \\
g=x_{0}^{6} x_{1}^{2}-4 x_{0}^{5} x_{1}^{3}+6 x_{0}^{4} x_{1}^{4}-4 x_{0}^{3} x_{1}^{5}+x_{0}^{2} x_{1}^{6}+2 x_{0}^{5} x_{1} y-8 x_{0}^{4} x_{1}^{2} y+12 x_{0}^{3} x_{1}^{3} y \\
-8 x_{0}^{2} x_{1}^{4} y+2 x_{0} x_{1}^{5} y+x_{0}^{4} y^{2}-2 x_{0}^{2} x_{1}^{2} y^{2}+x_{1}^{4} y^{2}+4 x_{0}^{2} y^{3}-8 x_{0} x_{1} y^{3}+4 x_{1}^{2} y^{3}
\end{gathered}
$$

For a general choice of parameters, e.g. $\alpha=\beta=\gamma=1$, the curve has exactly the required singularities.

Example in $\mathfrak{N}_{1,1,1}$ : The branch divisor contains two hyperplane sections which each pass through two of the three [3,3]-points. Fixing those two with equations $x_{0} x_{1}+y$ and $x_{0} x_{1}-x_{1}^{2}+y$, there is a pencil of cubic sections of the cone generated by

$$
f=\left(x_{0}^{2} x_{1}-x_{0} x_{1}^{2}+2 x_{0} y-x_{1} y\right)^{2}
$$

and

$$
\begin{aligned}
& g=x_{0}^{5} x_{1}+x_{0}^{4} x_{1}^{2}-5 x_{0}^{3} x_{1}^{3}+3 x_{0}^{2} x_{1}^{4}+x_{0}^{4} y+12 x_{0}^{3} x_{1} y-19 x_{0}^{2} x_{1}^{2} y+6 x_{0} x_{1}^{3} y \\
&+14 x_{0}^{2} y^{2}-13 x_{0} x_{1} y^{2}+3 x_{1}^{2} y^{2}+y^{3},
\end{aligned}
$$

whose general member has exactly the required singularities and tangencies.
Next we describe the strata containing non-normal Gorenstein surfaces. We refer to $\S 2$.A for the notation and the terminology.
Proposition 4.4 - The subset $\mathfrak{N N} \subset \overline{\mathfrak{M}}_{1,3}^{(\text {Gor })}$ of non-normal surfaces is equal to $\mathfrak{P} \sqcup \mathfrak{d} \mathfrak{P} \sqcup \mathfrak{E}$, where:
(i) $\mathfrak{P} \subset \overline{\mathfrak{M}}_{1,3}^{(\text {Gor })}$ consists of the non-normal surfaces with normalisation of type $(P)$, and it is irreducible and unirational of dimension 4. The corresponding branch locus is a double quadric section plus a hyperplane section.

TABLE 2. Irreducible strata of non-normal surfaces in $\overline{\mathfrak{M}}_{1,3}^{\text {(Gor) }}$

| stratum | dimension | minimal resolution $\widetilde{X}$ |
| :---: | :---: | :--- |
| $\mathfrak{d} \mathfrak{P}$ | 11 | del Pezzo surface of degree 1 |
| $\mathfrak{P}$ | 4 | $\mathbb{P}^{2}$ |
| $\mathfrak{E}$ | 2 | minimal ruled surface with $\chi(\widetilde{X})=0$ |

(ii) $\mathfrak{J P} \subset \overline{\mathfrak{M}}_{1,3}^{(\text {Gor })}$ consists of the non-normal surfaces with normalisation of type $(d P)$, and it is irreducible and unirational of dimension 11. The corresponding branch locus is a double hyperplane section plus a sufficiently general cubic section.
(iii) $\mathfrak{E} \subset \overline{\mathfrak{M}}_{1,3}^{\text {(Gor) }}$ consists of the non-normal surfaces with normalisation of type $\left(E_{-}\right)$, and it is irreducible and rational of dimension 2. The corresponding branch locus is as in case $(d P)$ where the cubic section acquires a $[3,3]$ point.

Proof. By Proposition 4.1 a surface $X \in \overline{\mathfrak{M}}_{1,3}^{(\text {Gor })}$ is non-normal if and only if $\Delta$ is not reduced. Since $\left(\mathcal{Q}, \frac{1}{2} \Delta\right)$ is log-canonical, every component of $\Delta$ appears with multiplicity at most 2 , hence we may write $\Delta=\Delta_{0}+2 \Delta_{1}$, with $\Delta_{1} \in\left|\mathcal{O}_{\mathcal{Q}}(k)\right|$ and $\Delta_{0} \in\left|\mathcal{O}_{\mathcal{Q}}(5-2 k)\right|$, where $k=1$ or $k=2$. The normalisation $\bar{X}$ of $X$ is also a double cover of $\mathcal{Q}$, branched over $\Delta_{0}$ and the vertex $0 \in \mathcal{Q}$.

Assume first $k=2$. Then $\Delta_{0}$ is a hyperplane section and $\bar{X}=\mathbb{P}^{2}$, namely the normalisation of $X$ is of type $(P)$.

Assume now that $k=1$. If $\Delta_{0}$ has at most negligible singularities, then $\bar{X}$ is a del Pezzo surface of degree 1 , hence $X$ is of type $(d P)$. If $\Delta_{0}$ has non-negligible singularities, then the only possibility is that $\bar{X}$ is of type ( $E_{-}$): in this case $\bar{X}$ has an elliptic singularity of degree 1 and therefore $\Delta$ has a [3, 3]-point.

For $k=2$, we may assume that the section $\Delta_{0}$ is fixed. Letting $\Delta_{1}$ vary in an appropriate open subset $U \subset\left|\mathcal{O}_{\mathcal{Q}}(2)\right|$ one obtains all the surfaces of $\mathfrak{P}$, hence $\mathfrak{P}$ is unirational and irreducible. Since the subgroup of automorphisms of $\mathcal{Q}$ that preserve $\Delta_{0}$ has dimension 4 , it follows that $\mathfrak{P}$ has dimension $8-4=4$.

In case $k=1$ we may assume that $\Delta_{1}$ is fixed. Letting $\Delta_{0}$ vary in an appropriate open subset $U \subset\left|\mathcal{O}_{\mathcal{Q}}(3)\right|$ one obtains all the surfaces of $\mathfrak{d} \mathfrak{P}$, hence $\mathfrak{d} \mathfrak{P}$ is unirational and irreducible. Since the subgroup of automorphisms of $\mathcal{Q}$ that preserve $\Delta_{1}$ has dimension 4 , it follows that $\mathfrak{d} \mathfrak{P}$ has dimension $15-4=11$.

To describe $\mathfrak{E}$, notice that in this case the $[3,3]$-point $P \in \Delta_{0}$ does not lie on $\Delta_{1}$, because $\left(\mathcal{Q}, \frac{1}{2} \Delta_{0}+\Delta_{1}\right)$ is a log-canonical pair. The subgroup of the automorphisms of $\mathcal{Q}$ that preserve a fixed plane section $\Delta_{1}$ acts transitively on $\mathcal{Q} \backslash\left(\Delta_{1} \cup\{O\}\right)$, so we may assume that the point $P$ is also fixed. In turn, the subgroup of the automorphisms of $\mathcal{Q}$ that preserve $\Delta_{1} \cup\{P\}$ fixes the infinitely near point $P_{0}$ corresponding to the ruling of $\mathcal{Q}$ containing $P$ and acts transitively on the set of points infinitely near to $P$ and distinct from $P_{0}$. So it is enough to consider the divisors $\Delta_{0} \in\left|\mathcal{O}_{\mathcal{Q}}(3)\right|$ with triple points at $P$ and at a fixed infinitely near point $P_{1}$. Arguing as in the proof of Proposition 4.3, one sees that such divisors give an open subset of a 3-dimensional linear subsystem of $\left|\mathcal{O}_{\mathcal{Q}}(3)\right|$, so $\mathfrak{E}$ is unirational and irreducible. (Actually, it is not hard to see that in this case $\Delta_{0}$ is the union of three plane sections passing through $P$ and $P_{1}$ ). The subgroup of automorphisms of $\mathcal{Q}$ that fix $\Delta_{1}, P$ and $P_{1}$ is 1-dimensional, so $\mathfrak{E}$ has dimension 2 and is rational.

Figure 4. Relation between some strata in $\overline{\mathfrak{M}}_{1,3}^{(\text {Gor })}$


In Figure 4 we give a coarse picture of the relations of the strata, where we connect two strata if the lower one is contained in the closure of the upper one. It does not give much more than the obvious relations, because our stratification is note fine enough. Indeed, a general curve singularity of type $[3,3]$ cannot degenerate to an ordinary quadruple point, for example, because its Milnor number is too large; any degeneration to a quadruple point will not be ordinary. Thus to understand the intersections of the closures of the strata given in Table 1 we would need a finer stratification, distinguishing the type of quadruple points occurring. For a description of the possible adjacencies see [Bri79].

Remark 4.5 - The most degenerate surface $X_{0}$ that we have encountered is the double cover of the quadric cone branched over $\Delta=H_{1}+2 H_{2}+2 H_{3}$ for three general hyperplane sections $H_{1}, H_{2}, H_{3}$, which is a particular surface of type $(P)$. The surface $X_{0}$ lies in the closure of $\mathfrak{E}, \mathfrak{N}_{1}, \mathfrak{N}_{2,2}, \mathfrak{P}$, and thus all strata lying above these in Figure 4. So we suspect $X_{0}$ is in the closure of every stratum that we considered. Finding a degeneration from the remaining strata with more than one elliptic singularity is however not obvious.
4.C. Some non-Gorenstein degenerations. We now discuss some increasingly general non-Gorenstein degenerations in $\overline{\mathfrak{M}}_{1,3}$. More precisely, let $\mathfrak{D C}$ be the set of surfaces arising as double covers of the quadric cone as in Proposition 4.1. This family maps onto an irreducible subset of $\overline{\mathfrak{M}}_{1,3}$, so that we have inclusions:

$$
\overline{\mathfrak{M}}_{1,3}^{(\text {Gor })} \subset \mathfrak{D C} \subset \overline{\mathfrak{M}}_{1,3}
$$

The examples given below will show that all the above inclusions are strict.
Remark 4.6 - One is tempted to consider also the subset $\mathfrak{H S}$ of slc hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$. However, it turns out that this set coincides with $\mathfrak{D C}$, that is, no hypersurface of degree 10 that passes through the $\frac{1}{5}(1,1,2)$ singularity of $\mathbb{P}(1,1,2,5)$ has slc singularities. The argument, which was explained to us by Stephen Coughlan, runs as follows:

Let $\Delta$ be a disc with parameter $t$ and let $\mathbb{P}=\mathbb{P}(1,1,2,5) \times \Delta$. A general surface passing through the $\frac{1}{5}(1,1,2)$-singularity can be viewed as the central fibre $\mathcal{X}_{0}$ of a
family of surfaces $\mathcal{X} \subset \mathbb{P}$ with equation

$$
t z^{2}+f_{5}\left(x_{0}, x_{1}, y, t\right) z+f_{10}\left(x_{0}, x_{1}, y, t\right)=0
$$

where $f_{d}$ is general of weighted degree $d$. We may assume that for $t \neq 0$ the fibres $\mathcal{X}_{t}$ are smooth. If $\mathcal{X}_{0}$ has slc singularities then by [KSB88, Thm. 5.1] the total space $\mathcal{X}$ has canonical singularities. But near $P_{0}=((0: 0: 0: 1), 0)$ the $z$-coordinate is invertible and we can isolate $t$ in the local equation. In other words, $x_{0}, x_{1}, y$ are local orbifold coordinates near $P_{0} \in \mathcal{X}$ and $\mathcal{X}$ has a singularity of type $\frac{1}{5}(1,1,2)$ at $P_{0}$. By the Tai-Reid criterion (see [Rei80] or [Kol13, Thm. 3.21]), this quotient singularity is not canonical, thus $\mathcal{X}_{0}$ cannot be slc.

Indeed, Coughlan shows that the stable limit of the above family is a double cover of $\mathcal{Q}$ with Hurwitz divisor given by $f_{5}^{2}=0$, i.e., a non-normal surface with normalisation two copies of $\mathcal{Q}$.
4.C.1. General surfaces in $\mathfrak{D C} \backslash \overline{\mathfrak{M}}_{1,3}^{(G o r)}$. As in Proposition 4.1, let $\left(\mathcal{Q}, \frac{1}{2} \Delta\right)$ be a lc pair, where $\mathcal{Q}$ is the quadric cone and $\Delta$ is a quintic section. Let $p: X \rightarrow \mathcal{Q}$ be the associated double cover, which is a stable surface. It remains to treat the non-Gorenstein case, that is, the case where $\Delta$ contains the vertex $O$ of the cone.

Let $q: \mathcal{Q}^{\prime} \cong \mathbb{F}_{2} \rightarrow \mathcal{Q}$ be the blow up of the vertex, denote by $C$ the exceptional curve and write

$$
q^{*}\left(K_{\mathcal{Q}}+\frac{1}{2} \Delta\right)=K_{\mathcal{Q}^{\prime}}+\Delta^{\prime}=K_{\mathcal{Q}^{\prime}}+\frac{1}{2}\left(q^{-1}\right)_{*} \Delta+\left(\frac{C \cdot\left(q^{-1}\right)_{*} \Delta}{4}\right) C
$$

where the coefficient of $C$ is computed by intersecting with $C$. Since $\left(\mathcal{Q}, \frac{1}{2} \Delta\right)$ is lc if and only if $\left(Q^{\prime}, \Delta^{\prime}\right)$ is lc, we see that the strict transform of $\Delta$ intersects $C$ at most with multiplicity 4 . Writing out the classes in the Néron-Severi group of $\mathcal{Q}^{\prime}$ it is easy to check that the intersection is also even, which leaves us with three cases.

We will now analyse $X$ via the commutative square

under the assumption that $\Delta$ is sufficiently general, in particular, $\left(q^{-1}\right)_{*} \Delta$ is smooth and intersects $C$ transversally.
$C .\left(q^{-1}\right)_{*} \Delta=0$ : This is the case where $\Delta$ does not contain the vertex. Note however, that since $O$ is in the branch locus of $p$, the double cover $p^{\prime}$ is branched over $\left(q^{-1}\right)_{*} \Delta+C$. The inverse image of $C$ is a $(-1)$-curve in $X^{\prime}$ that is contracted to a smooth point of $X$, hence $X$ is a smooth surface of general type.
$C .\left(q^{-1}\right)_{*} \Delta=2$ : Viewed as a degeneration of the previous case, the double cover $p^{\prime}$ should be branched over $\left(q^{-1}\right)_{*} \Delta+2 C$. However, this would not be normal. The normalisation is the double cover branched over $\left(q^{-1}\right)_{*} \Delta$ and thus the preimage of $C$ becomes a smooth rational curve of self-intersection -4 . Thus $X$ has a $\frac{1}{4}(1,1)$ singularity above the vertex of the cone and is smooth otherwise if $\Delta$ is sufficiently general.

If we let $F$ be a fibre of the ruling on $\mathcal{Q}^{\prime}$ then $F .\left(q^{-1}\right)_{*} \Delta=4$ and $2 K_{\mathcal{Q}^{\prime}}+\left(q^{-1}\right)_{*} \Delta \sim$ $2 F$. Hence $X^{\prime}$ is a properly elliptic surface of Kodaira dimension 1 with $\chi\left(X^{\prime}\right)=$ $\chi(X)=3$, because $X$ has rational singularities.
$C .\left(q^{-1}\right)_{*} \Delta=4$ : Viewed as a degeneration of the previous cases, the double cover $p^{\prime}$ should be branched over $\left(q^{-1}\right)_{*} \Delta+3 C$. Passing to the normalisation we get a double cover branched over $\left(q^{-1}\right)_{*} \Delta+C$.

Since we chose $\Delta$ generic, $\Delta+C$ has 4 ordinary nodes along $C$; let $q^{\prime \prime}: \mathcal{Q}^{\prime \prime} \rightarrow \mathcal{Q}^{\prime}$ be the blow up in these 4 points and consider the corresponding double cover. Again passing to the normalisation, we obtain a double cover whose branch divisor on $\mathcal{Q}^{\prime \prime}$ is the strict transform of $\left(q^{-1}\right)_{*} \Delta+C$. On this double cover the configuration of curves we want to contract to $X$ has dual graph

where the numbers indicate the negative self-intersection. The contraction of these curves gives a $\mathbb{Z} / 2$-quotient of an elliptic singularity on $X$ (see e.g. [Kol13, Ex. 3.28]).

Arguing as in the previous case, the minimal resolution of $X$ is a properly elliptic surface with holomorphic Euler-characteristic equal to 3.
4.C.2. Surfaces which are not canonically embedded as hypersurfaces. We now consider an example in $\overline{\mathfrak{M}}_{1,3} \backslash \mathfrak{D C}$.

Example 4.7: Consider, in the notation of Section 3.B, the ring $S_{1}[u]$ where $u$ has degree 2 .

Pick a family of surfaces depending on a parameter $t$

$$
X_{t}=\operatorname{Proj} S_{1}[u] /(f, g) \hookrightarrow \operatorname{Proj} S_{1}[u] \cong \mathbb{P}(1,1,2,2,5)
$$

where $g=x_{0} x_{1}-t u$ and $f \in S_{1}[u]$ is general of degree 10 .
Then for $t \neq 0$ the surface $X_{t}$ is a Gorenstein stable surface as in Theorem 3.3, ( $i$ ), since we can use the equation $g$ to eliminate the new variable $u$.

For $t=0$ the bicanonical map realizes $X_{0}$ as a double cover of the union of two planes $Y_{0} \subset \mathbb{P}^{3}$, branched on a curve of degree 5 and over the intersection line $r$ of the two planes. That is, $X_{0}$ consists of two singular $K 3$ surfaces with five nodes, which are double planes with branch curve a line plus a general quintic. The surfaces are glued along the strict transform of the line and every canonical curve is a nonreduced curve supported on the non-normal locus of $X_{0}$. In particular, $X_{0}$ is not canonically embedded in $\mathbb{P}(1,1,2,5)$.

The surface $X_{0}$ is not Gorenstein, and its Cartier index is equal to 2 . These surfaces give a 26 -dimensional locus inside the moduli space: the linear system of quintics in the two planes that match on the intersection line is of dimension 35 and the automorphism group of $Y_{0}$ has dimension 9 .

## 5. Bestiarium in $\overline{\mathfrak{M}}_{1,2}^{(\text {Gor })}$ and $\overline{\mathfrak{M}}_{1,2}$

In this section we consider the moduli space of Gorenstein stable surfaces with $K^{2}=1$ and $\chi=2$ (i.e. $p_{g}=1$ ). We refer to [Cat79] and [Tod80] for the analysis of the classical case.

By Proposition 3.7 in this case the bicanonical map $X$ is a degree 4 cover of the plane. The general theory of quadruple covers has been studied by Hahn-Miranda [HM99] and by Casnati-Ekedahl [CE96], but it is quite complicated and a description as detailed as the one given in section 4.B is not feasible in this case. Thus here we restrict to a much coarser analysis, just giving a cornucopia of examples, mostly in

TABLE 3. Some normal and non-normal surfaces in $\overline{\mathfrak{M}}_{1,2}^{(\text {Gor })}$

| type of cover | example | normal | minimal resolution $\widetilde{X}$ | $\kappa(\tilde{X})$ |
| :---: | :---: | :---: | :--- | :---: |
|  | $Z_{1}$ | yes | minimal elliptic surface | 1 |
|  | $Z_{1,1}^{A}$ | yes | blow up of an abelian surface | 0 |
| bi-double | $Z_{1,1}^{B}$ | yes | blow up of a bielliptic surface | 0 |
|  | $Z_{4}$ | yes | rational surface | $-\infty$ |
|  | $Z^{(d P)}$ | no | del Pezzo surface of degree 1 | $-\infty$ |
|  | $Z^{(E-)}$ | no | ruled with $\chi(\widetilde{X})=0$ | $-\infty$ |
|  | $Z^{(P)}$ | no | $\mathbb{P}^{2}$ | $-\infty$ |
| iterated double | $Z_{2}^{E}$ | yes | blow up of an Enriques surface | 0 |
|  | $Z_{2}^{R}$ | yes | rational surface | $-\infty$ |

the special case when the bicanonical map is Galois. Canonical surfaces with this property are also called Kunev surfaces after [Kyn77] and they have been studied in connection with the failure of local Torelli for surfaces. The bicanonical map of a Kunev surface factors through a double cover of a (singular) $K 3$-surface of degree two. From this point of view degenerations of such surfaces have been studied by Usui, again with applications to Torelli-type questions in mind [Usu87, Usu00].

An overview over the examples we construct can be found in Table 3.
5.A. Gorenstein bi-double covers of the plane. In most of our examples the bicanonical map is a $\mathbb{Z}_{2}^{2}$-cover ("bi-double cover"): it is not hard to show that this is the case when the terms $a_{1}$ and $a_{2}$ in the equations of Theorem 3.3, (ii) vanish (see [Cat79, §1, Prop. 10] for the smooth case).
Non-normal abelian covers are studied in [AP12]; we recall below the facts that we need in the special case of a bi-double cover of $\varphi: X \rightarrow \mathbb{P}^{2}$. If $X$ is demi-normal, then by [AP12, Cor. 1.10] $\varphi$ is uniquely determined (up to isomorphism of covers) by effective divisors $D_{i}$ of $\mathbb{P}^{2}$ of degree $d_{i}, i=0,1,2$, such that:

- $d_{i} \equiv d_{j} \bmod 2$ for every $i, j$.
- the so-called Hurwitz divisor $\Delta:=\frac{1}{2}\left(D_{0}+D_{1}+D_{2}\right)$ has no component of multiplicity $>1$.
The divisors $D_{0}, D_{1}$ and $D_{2}$ are called the branch data of $\varphi$; setting $a_{i}=\frac{d_{j}+d_{k}}{2}$, where $i, j, k$ is a permutation of $0,1,2$, one has:

$$
\varphi_{*} \mathcal{O}_{X}=\mathcal{O}_{\mathbb{P}^{2}} \oplus \mathcal{O}_{\mathbb{P}^{2}}\left(-a_{0}\right) \oplus \mathcal{O}_{\mathbb{P}^{2}}\left(-a_{1}\right) \oplus \mathcal{O}_{\mathbb{P}^{2}}\left(-a_{2}\right) .
$$

Proposition 5.1 - In the above set-up:
(i) $2 K_{X}=\varphi^{*} \mathcal{O}_{\mathbb{P}^{2}}\left(d_{0}+d_{1}+d_{2}-6\right)$
(ii) $X$ is slc if and only if $\left(\mathbb{P}^{2}, \Delta\right)$ is an lc pair
(iii) $X$ is non-normal above an irreducible curve $\Gamma$ of $\mathbb{P}^{2}$ if and only if $\Gamma$ appears in $\Delta$ with coefficient 1
(iv) $X$ is Gorenstein if and only if $D_{0} \cap D_{1} \cap D_{2}=\emptyset$.

Proof. (i) and (ii) follow by [AP12, Prop. 2.5] and (iii) follows by [AP12, Thm. 1.9, (2)].

To prove (iv) observe first of all that if a point $P \in \mathbb{P}^{2}$ lies on at most two of the $D_{i}$ then locally above $P$ the map $\varphi$ can be seen as the composition of two flat double covers, hence $X$ is Gorenstein above $P$. If $P$ lies on all the $D_{i}$ and $D_{0}+D_{1}+D_{2}$
has an ordinary triple point at $P$, then $X$ has a singular point of type $\frac{1}{4}(1,1)$ at $P$ (see[AP12, Table 1)], case 4.3). Since being Gorenstein is an open condition [BH93, Cor. 3.3.15] it follows that $X$ is not Gorenstein at $P$ for any choice of divisors $D_{i}$ through $P$.

By Proposition 5.1, to construct a stable surface with $K_{X}^{2}=1$ and $\chi(X)=2$ as a bi-double cover of $\mathbb{P}^{2}$ one has to choose as branch data a line $D_{0}$ and cubics $D_{1}$, $D_{2}$ such that $\left(\mathbb{P}^{2}, \frac{1}{2}\left(D_{0}+D_{1}+D_{2}\right)\right)$ is an lc pair: indeed, since $K^{2}=1$ we must have $d_{0}+d_{1}+d_{2}=7$ and thus all the $d_{i}$ have to be odd by the parity condition. Computing $2=\chi(X)=\chi\left(\varphi_{*} \mathcal{O}_{X}\right)=4+\frac{1}{2} \sum a_{i}\left(a_{i}-3\right)$ we see that the only possibility is $\left(d_{0}, d_{1}, d_{2}\right)=(1,3,3)$, that is, $\left(a_{0}, a_{1}, a_{2}\right)=(3,2,2)$. In particular, the projection formula implies $\left|2 K_{X}\right|=\varphi^{*}\left|\mathcal{O}_{\mathbb{P}^{2}}(1)\right|$.

Remark 5.2 - Let $\varphi: X \rightarrow \mathbb{P}^{2}$ be an slc bi-double cover with branch data $D_{i}$, $i=0,1,2$. By Proposition 5.1 the surface $X$ is not normal above an irreducible curve $\Gamma$ of $\mathbb{P}^{2}$ if and only if one of the following happens:
(a) $\Gamma$ appears with multiplicity 2 in exactly one of the $D_{i}$
(b) $\Gamma$ appears with multiplicity 1 in exactly two of the $D_{i}$.

Note that if case (b) occurs, then $X$ is not Gorenstein by Proposition 5.1, (iv). The normalisation algorithm (see [Par91, §3]) is very simple in this situation: in case (a) one subtracts $2 \Gamma$ from the only divisor containing it, while in case (b) one subtracts $\Gamma$ from the two divisors containing it and adds it to the remaining one. In both cases the divisors thus obtained are the branch data of a cover $\varphi^{\prime}: X^{\prime} \rightarrow \mathbb{P}^{2}$ such that $X^{\prime}$ is normal above $\Gamma$ and there is a birational morphism $X^{\prime} \rightarrow X$ commuting with the covering maps to $\mathbb{P}^{2}$.

More generally, if $\varphi$ is a standard bi-double cover (see [AP12, § 1] for the definition) with branch data $D_{i}$ then one first reduces the $D_{i}$ modulo 2 and then removes all the irreducible components common to all the $D_{i}$. This way one obtains a cover such that $\Delta$ has no component of multiplicity $>1$, whose normalization can be computed as in the slc case.

We now give our examples by describing the branch data $D_{i}$. We take coordinates $\left(y_{0}, y_{1}, y_{2}\right)$ in $\mathbb{P}^{2}$ so that the first branch divisor is the line $D_{0}=\left\{y_{0}=0\right\}$ and we only specify the cubics $D_{1}$ and $D_{2}$. The possible singularities of slc $(\mathbb{Z} / 2)^{r}$-covers such that the support of the Hurwitz divisor $\Delta$ has ordinary singularities have been classified in [AP12, Table 1-4]. In our restricted situation only two different normal singularities can occur, because $r=2$ and $D_{1}$ and $D_{2}$ can have at most three local branches through a point $P$ :

- $D=2 \Delta$ has an ordinary quadruple point at $P$, such that three of the local components are in the same $D_{i}$. The resulting singularity is an elliptic singularity of degree 1 (see case 4.5 in loc.cit.).
- Both $D_{1}$ and $D_{2}$ have an ordinary double point at $P$ such that $D$ has an ordinary quadruple point at $P$. The resulting singularity is an elliptic singularity of degree 4 (see case 4.6 in loc.cit.).
The low degree of the branch divisors leaves very few combinatorial possibilities. We will now describe all possible normal examples where $D$ has only ordinary singularities; the surface $Z_{d_{1}, \ldots, d_{r}}$ will have $r$ elliptic singularities with the given degrees. Often the identification of the minimal resolution $\pi: \widetilde{X} \rightarrow X$ is immediate by the restrictions found in [FPR15b, Thm. 4.1].

Example $Z_{1}$ : Let $D_{1}$ be a union of three general lines through $P \in D_{0}$ and $D_{2}$ a general cubic. Then $X$ has a unique elliptic singularity of degree 1 . Blowing up at $P$ and then changing the branch divisor to get a normal bi-double cover $\widetilde{X} \rightarrow \mathbb{P}^{2}$ as in Remark 5.2, one computes that $\left|2 K_{\tilde{X}}\right|$ is an elliptic pencil, induced by the pencil of lines passing through $P$. Thus by [FPR15b, Thm. 4.1] $\widetilde{X}$ is a minimal properly elliptic surface with $\chi(\widetilde{X})=1$.

Example $Z_{1}^{\prime}$ : If in the previous example the point $P$ is a general point on $D_{1}$ instead of on $D_{0}$, we get, by the same computation, a different family of surfaces with minimal resolution a minimal properly elliptic surface with $\chi(\widetilde{X})=1$.

Example $Z_{1,1}^{A}$ : Let $P, Q$ be two different points on the line $D_{0}$ and let $D_{1}$ be a union of three general lines through $P$ and $D_{2}$ be a union of three general lines through a $Q$. Then $X$ has two elliptic singularities of degree 1 . Blowing up at $P$ and $Q$ and then changing the branch divisor to get a normal bi-double cover as in Remark 5.2 , one computes that $2 K_{\tilde{X}}$ is linearly equivalent to the strict transform of $\varphi_{2}^{*} D_{0}$ on $\widetilde{X}$, which is twice a $(-1)$-curve. Thus $\widetilde{X}$ is a blow-up of a surface with trivial bicanonical bundle and $\chi(\widetilde{X})=0$. A formula to compute the sections of the canonical bundle of a singular bi-double cover has been given in [Cat99, Sect. 3]. Using it it is straightforward to check that $K_{\tilde{X}}$ is effective as well and hence the minimal model of $\widetilde{X}$ is an abelian surface.

Example $Z_{1,1}^{B}$ : Let $D_{1}$ be a union of three general lines through $P \in D_{0}$ and $D_{2}$ be a union of three general lines passing through a general point $Q \in D_{1}$. Then $X$ has two elliptic singularities of degree 1. Blowing up at $P$ and $Q$ and then changing the branch divisor to get a normal bi-double cover $\widetilde{X} \rightarrow \mathbb{P}^{2}$ as in Remark 5.2, one computes that $2 K_{\tilde{X}}$ is linearly equivalent to the strict transform of $\varphi_{2}^{*} D_{0}$ on $\tilde{X}$, which is twice a $(-1)$-curve. Thus $\tilde{X}$ is a blow-up of a surface with trivial bicanonical bundle and $\chi(\widetilde{X})=0$.

Using again [Cat99, Sect. 3] one checks that $K_{X}$ is not effective and hence the minimal model of $\widetilde{X}$ is a bielliptic surface.

Example $Z_{4}$ : Assume that both $D_{1}$ and $D_{2}$ have an ordinary node at a point $P$ not in $D_{0}$. Then $X$ has an elliptic singularity of degree 4 and thus by [FPR15b, Thm. 4.1] its minimal resolution is a rational surface. Indeed, the pull back of the pencil of lines through $P$ gives a free pencil of rational curves on the resolution.

We now turn to the non-normal case. By Proposition 5.1 $X$ is non-normal and Gorenstein if and only if there exists an irreducible curve $\Gamma$ that appears with multiplicity 2 in one of the $D_{i}$ and is not contained in the remaining two. In particular, $\Gamma$ must be a line. This leaves very few possibilities that we describe below. We denote by $\bar{X}$ the normalisation of $X$; the possibilities for $\bar{X}$ are listed in Section 2.A.

Example $Z^{(d P)}$ : If $D_{1}=2 L_{1}+L_{2}$ is the union of a double line and a general line and $D_{2}$ is general then $-2 K_{\bar{X}}=\varphi_{2}^{*}\left(-2 K_{\mathbb{P}^{2}}-D_{0}-L_{2}-D_{2}\right) \sim \varphi_{2}^{*} L_{2}$ is ample of square 4. Thus $\bar{X}$ is a del Pezzo surface of degree 1 and $X$ is of type $(d P)$.

Example $Z_{1}^{(d P)}=Z^{\left(E_{-}\right)}$: If $D_{1}=2 L_{1}+L_{2}$ is the union of a double line and a general line and $D_{2}$ is the union of three lines meeting $D_{0}$ at a general point, then $X$ acquires an additional elliptic singularity of degree 1 . Thus the normalisation is a singular del Pezzo of degree 1 with minimal resolution a ruled surface over an elliptic curve: $X$ is of type ( $E_{-}$).

Example $Z^{(P)}$ : If both $D_{1}$ and $D_{2}$ contain a double line then the normalisation $\bar{X}$ of $X$ is the projective plane. Indeed the induced cover $\bar{X} \rightarrow \mathbb{P}^{2}$ is given by squaring the coordinates.

Remark 5.3 - Since the normalisation $\bar{X}$ of a non-normal bi-double cover of $\mathbb{P}^{2}$ is again such a bi-double cover, the canonical divisor of $\bar{X}$ is the pullback of some $\mathcal{O}_{\mathbb{P}^{2}}(d)$ and thus either ample, anti-ample, or trivial. Thus no bi-double cover can have normalisation of type $(E+)$.

A construction of a different flavor which relies on a glueing result of Kollár has already been given in [FPR15b, Sect. 3.3]: let $E$ be an elliptic curve and $\bar{D} \subset S^{2} E$ a general 3-section of the Albanese map. Then $\bar{D}$ is a smooth curve of genus 2. To get a non-normal stable surface, we glue $\bar{D}$ to itself via the hyperelliptic involution $\tau$. The result is a surface $X$ which is non-normal along $D=\bar{D} / \tau$. At a general point of $D$ the surface $X$ has a double normal-crossing singularity while at the branch points of $\bar{D} \rightarrow D$ we find pinch points with local equation $z^{2}-y x^{2}=0$.

It would be interesting to compute the canonical ring directly from this description, thus realizing $X$ as a complete intersection in $\mathbb{P}(1,2,2,3,3)$.
5.B. Iterated double covers. We start by noting that a bi-double cover with branch data $D_{0}, D_{1}, D_{2}$ can be seen as an iterated double cover as follows. First one takes the double cover $f: Y \rightarrow \mathbb{P}^{2}$ branched on $D_{0}+D_{1}:$ if $D_{0}$ and $D_{1}$ intersect transversally, then $Y$ is a singular Del Pezzo surface of degree 2 that has ordinary double points over the three intersection points of $D_{0}$ and $D_{1}$. The cover $g: X \rightarrow Y$ is obtained by taking the double cover of $Y$ branched on the singular points and on the divisor $B:=f^{*} D_{2} \in\left|-3 K_{Y}\right|$.

More generally, the same construction can be performed taking as $B$ any element of the system $\left|-3 K_{Y}\right|$ not passing through the singular points of $Y$; in this way one obtains a Gorenstein cover $\varphi_{2}=g \circ f: X \rightarrow \mathbb{P}^{2}$ with the same numerical invariants, which in general is not Galois.

We give two variants of this construction by specifying the plane cubic $D_{1}$ and the branch divisor $B$. It is possible to impose further or different singularities either on $D_{0}+D_{1}$ or on $B$ to get other examples but we will not pursue this here.

Examples $Z_{2}^{R}$ and $Z_{2}^{E}$ : By taking $D_{1}$ general and choosing $B$ with a quadruple point a smooth point $Q$ of $Y$ such that the infinitely near points are at most double we obtain an example with an elliptic Gorenstein singularity of degree 2 . The minimal desingularization $\tilde{X}$ of $X$ has $\chi(X)=1$, hence by [FPR15b, Thm. 4.1] it is either rational or birational to an Enriques surface. By the standard formulae for double covers, the bicanonical curves of $\widetilde{X}$ correspond to the curves in $\left|-K_{Y}\right|$ with a double point at $Q$. So we have $P_{2}(\widetilde{X})>0$ if $Q$ lies on the ramification divisor of $Y \rightarrow \mathbb{P}^{2}$ and $P_{2}(\widetilde{X})=0$ otherwise. So $\widetilde{X}$ is Enriques in the former case $\left(Z_{2}^{E}\right)$ and it is rational $\left(Z_{2}^{R}\right)$ in the other one.

That such a divisor $B$ exists can be seen as follows: for $Z_{2}^{E}$ pick a general point $P$ on $D_{1}$, let $l_{0}$ be an equation of the tangent to $D_{1}$ at $P$ and let $l_{1}, l_{2}$ be the equations of general lines through $P$. Then the curve $B$ of equation $l_{0} l_{1} l_{2}=0$ has a quadruple point at the preimage $Q$ of $P$, with an infinitely near double point. Taking the double cover branched on $B$ one obtains an elliptic singularity of degree 2 , whose exceptional cycle consists of a -4-curve and a -2 -curve meeting transversally at two points.

For $Z_{2}^{R}$ one can for instance take $Y=\left\{y^{2}-x_{0}\left(x_{0}^{3}+x_{1}^{3}+x_{2}^{3}+2 x_{0} x_{2}^{2}\right)=0\right\}$ and $B \subset Y$ given by $\left\{x_{1}\left(y+x_{0}^{2}+x_{2}^{2}\right)=0\right\}$, which has the required quadruple point at $Q=(1: 0: 0:-1)$ and no other singularities.
5.C. Some non-Gorenstein examples. We conclude this section by giving two examples of non-Gorenstein Galois-covers of the plane.

The first one is a bidouble cover which occurs as a degeneration of the construction in Section 5.A but the second one is a (non-simple) cyclic cover with Galois-group $\mathbb{Z} / 4$. This cannot occur in the classical case and we do not know if it is contained in the closure of $\mathfrak{M}_{1,2}$ in $\overline{\mathfrak{M}}_{1,2}$.

Example 5.4: By Proposition 5.1, a non Gorenstein degeneration of a bi-double cover of the plane of the type analysed in Section 5.A can be obtained by letting the divisors $D_{i}$ all go through a point $P$. If the $D_{i}$ are taken to be general otherwise, then $X$ has a singularity of type $\frac{1}{4}(1,1)$ over $P$ and is smooth elsewhere. The bicanonical system of the minimal resolution $\widetilde{X}$ is a free linear pencil of elliptic curves (the strict transform of the pencil of lines through $P$ ), hence $\widetilde{X}$ is properly elliptic. More degenerate configurations can be analysed as above.

We just briefly describe here the additional possibilities for the non-normal case assuming that the components of $\Delta$ are general (we keep the previously introduced notation):

- $D_{1}$ and $D_{2}$ have a line in common. In this case the normalisation $\bar{X}$ is an Enriques surface with two $A_{1}$ points.
- $D_{1}$ and $D_{2}$ have a conic in common. In this case $\bar{X}$ is a singular del Pezzo surface of degree 1 with four $A_{1}$ points.
- $D_{0}$ is a component of $D_{1}$. In this case $\bar{X}$ is a singular $K 3$ surface with six $A_{1}$ points.
Notice that of the normalisations that we obtain in the first and the last case cannot occur in the Gorenstein case (cf. Section 2.A).

Finally we give an example such that the bicanonical map is a $\mathbb{Z}_{4}$-cover:
Example 5.5: Let $D_{1}$ and $D_{2}$ be lines and $D_{3}$ be a reduced cubic of $\mathbb{P}^{2}$. If we take $L=\mathcal{O}_{\mathbb{P}^{2}}(3)$, the equivalence relation $4 L \equiv D_{1}+2 D_{2}+3 D_{3}$ is satisfied and therefore by [Par91, Prop. 2.1] there exists a $\mathbb{Z}_{4}$-cover $\varphi: X \rightarrow \mathbb{P}^{2}$ such that:

- the preimages $R_{1}$ and $R_{3}$ of $D_{1}$ and $D_{3}$ are fixed pointwise by $\mathbb{Z}_{4}$; the group acts on the normal space to $R_{1}$ at a general point via a character $\chi$ of order 4 and on the normal space to the preimage of $R_{3}$ at a general point via the opposite character $\chi^{-1}$.
- the preimage $R_{2}$ of $D_{2}$ is fixed pointwise by the order 2 subgroup of $\mathbb{Z}_{4}$ but not by all the group.
- $\varphi_{*} \mathcal{O}_{X}=\mathcal{O}_{\mathbb{P}^{2}} \oplus \mathcal{O}_{\mathbb{P}^{2}}(-2) \oplus \mathcal{O}_{\mathbb{P}^{2}}(-2) \oplus \mathcal{O}_{\mathbb{P}^{2}}(-3)$.

In this case the Hurwitz divisor is $\Delta=\frac{3}{4}\left(D_{1}+D_{3}\right)+\frac{1}{2} D_{2}$, hence $2 K_{X}=\varphi^{*}\left(2 K_{\mathbb{P}^{2}}+\right.$ $2 \Delta)=\varphi^{*}\left(\mathcal{O}_{\mathbb{P}^{2}}(1)\right)$ and the projection formula gives that $\left|2 K_{X}\right|=\varphi^{*}\left|\mathcal{O}_{\mathbb{P}^{1}}(1)\right|$, hence $X$ is 2 -Gorenstein. For a general choice of the $D_{i}$ the singularities of $X$ are three points of type $\frac{1}{4}(1,1)$, occurring over the intersection points of $D_{1}$ and $D_{3}$, and four points of type $A_{1}$ occurring over the intersection points of $D_{2}$ with $D_{1}+D_{3}$. In particular, $X$ is not Gorenstein.

An interesting feature of this example is that by [AP12, Thm. 1.9, (2)] if $X$ is demi-normal then it is normal, namely one cannot obtain non-normal examples by degenerating this construction.

These examples give an 8-dimensional subset of $\overline{\mathfrak{M}}_{1,2}$ but we do not know whether this set lies in the closure of $\overline{\mathfrak{M}}_{1,2}^{(\text {Gor })}$.

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