Gramian-based optimal active sensing control under intermittent measurements

Olga Napolitano[1], Daniele Fontanelli[2], Lucia Pallottino[1] and Paolo Salaris[1]

Abstract— **This paper proposes an online optimal perception-aware strategy meant to maximize the information collected along the trajectory via the available measurements while simultaneously minimizing the negative effects of actuation/process noise. Indeed, in several robotic applications, the actuation/process noise is far from negligible and its negative effects are particularly relevant especially with intermittent measurements (e.g. collected by a vision system with limited Field-Of-View). New metrics are proposed as combinations of the Constructability Gramian, for measuring the amount of information collected via the available sensors, and the Reachability Gramian, for measuring the degrading effects of actuation/process noise. Control inputs that optimize those cost functions are provided. To show the effectiveness of our method, we consider one case study involving a unicycle-like vehicle, subject to Gaussian measurement noise and Gaussian or Brownian actuation noise, that estimates its state using intermittent distances from known environmental markers.**

I. INTRODUCTION

Active perception strategies [1], which consist in planning actions that collect the maximum amount of information needed for correctly accomplishing a task in spite of all uncertainty sources and variability, has been largely studied in relation to human beings. Action selection is indeed an important process highly influenced by the knowledge about the world [2]. Since the quality of this knowledge is affected by variability and noise in both muscles and sensing apparatuses, the human brain dedicates a lot of effort to reduce their negative effects, this resulting in coupled feed-forward strategies [3].

During the last 40 years a great effort has been spent by the robotic community to endow such capabilities in robots. Indeed, the problem of retrieving the best possible knowledge about the environment surrounding a robot is becoming important to increase their autonomous behavior. However, similarly to human beings, robotic systems are affected by several sources of variability and uncertainties that may negatively affect the quality of the information coming from the on-board sensors.

In [4], active perception is used to improve the accuracy of domain randomization-based pose estimation where a neural network is trained to predict the poses directly from a 2D image of the scene. In [5], a multi-robot exploration and coordination method for discovering the highest peak of an unknown environmental field has been proposed. In [6], authors present a task-oriented active sensing scheme that minimizes the uncertainty in future task-related actions instead of limiting only to the state. An algorithm that determines the yaw trajectory that jointly optimizes aggressiveness and feature co-visibility with the aim of increasing state estimation accuracy has been proposed in [7]. In [8], instead, a perception-aware model predictive control framework for quadrotors has been proposed for maximizing the visibility of a point of interest and minimizing its velocity in the image plane. In [9], authors proposed a trajectory planning method for finding online the optimal path for a differentially flat nonlinear system that maximizes the amount of information collected by the on-board sensors (and, thus, improving the accuracy and convergence speed of an observer).

Nevertheless, the above-mentioned literature results do not take explicitly into account the negative effects of actuation/process noise which is far from negligible in several robotic applications (e.g. in aerial vehicles). Some exceptions can be found in [10], [11] where a POMDP-based motion planning algorithm that minimizes the motion and the sensing uncertainties in environments with static obstacles has been proposed. In [12], an easy-to-use software toolkit, called On-line POMDP Planning Toolkit (OPPT), reduces the coding difficulty in implementing POMDP algorithms for robots, hence increasing their applicability. In [13], instead, a cost function depending also on the actuation noise has been optimized for minimizing control sensitivity. A different approach is then adopted in [14], where an online solution to the problem of minimizing the largest eigenvalue of the *a posteriori* covariance matrix, solution of the Continuous Riccati Equation (CRE), is proposed. However, this approach is only valid if an Extended Kalman Filter is used as an observer, hence limiting its application domain. In general, the above cited approaches are more devoted to an offline implementation (mainly because of the computational complexity of the solutions), while one of the objectives of this work is to provide an *online* strategy, as in [9].

An important problem when dealing with limited sensing range or Field-Of-View (FOV) is the difficulty of guaranteeing a sufficiently high number of measurements, possibly yielding to intermittent observability. To ensure that the maximum number of points of interest remains in view for the largest amount of time has been tackled in [8] and [7]. In [15], intermittent measurements availability has been cast in a minimum-time trajectory planning for a quadrotor able to reach the desired configuration with a given maximum uncertainty. Also in this case, uncertainty is measured by

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[1] Centro di Ricerca "E. Piaggio," and Dipartimento di Ingegneria dell'Informazione, Università di Pisa, Italy.

[2] Department of Industrial Engineering, Università di Trento, Italy.

the *a posteriori* covariance matrix of the CRE, limiting this methodology to Riccati observers.

In this work we propose an online optimal perception-aware strategy meant to maximize the information collected along the planned trajectory using intermittently available measurements while minimizing the negative effects of actuation/process noise. The main novelty of this paper is on the use of the Reachability Gramian as a metric for quantifying how actuation/process noise is degrading the gained state knowledge and combine it, providing effective new metrics, with the Constructability Gramian. The latter was already used in [9] where, differently from this paper, the actuation/process noise was considered negligible and measurements were always available during motion. The Reachability Gramian is usually used for quantifying and then optimize the effort required to drive a system towards a desired state [16]. In [17], the Reachability Gramian is used to find the optimal actuator location. In [18], instead, classical reachability set analysis under unknown disturbances and incomplete knowledge about the state space variables is studied in terms of a solution to the Hamilton-Jacobi-Bellman equation. Differently from the above-mentioned papers, in this paper the Reachability Gramian is used to quantify the degrading effects of actuation/process noise, also showing its relationship with the Riccati equation and hence the estimation uncertainty.

Finally, the feasibility and the induced robot behavior are discussed using a set of effective simulations on a unicycle vehicle with instances of Gaussian and Brownian actuation noises.

II. PRELIMINARIES

We here summarize the main tools and quantities, some of them already introduced in [9], adopted in this work. Let us consider a generic nonlinear system with noisy nonlinear outputs and not negligible actuation/process noise

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{f}(\boldsymbol{q}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)), \quad \boldsymbol{q}(t_0) = \boldsymbol{q_0} \tag{1}$$

$$\boldsymbol{z}(t) = \boldsymbol{h}(\boldsymbol{q}(t), \boldsymbol{\nu}(t)) \tag{2}$$

where $\boldsymbol{q}(t) \in \mathbb{R}^n$ represents the state of the system, which may include some unknown calibration/environment parameters, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the control input, $\boldsymbol{z}(t) \in \mathbb{R}^p$ represents the sensors output (i.e., the measurements available through sensors at time t), $\boldsymbol{f}(\cdot)$ and $\boldsymbol{h}(\cdot)$ are smooth C^∞ functions, and $\boldsymbol{\nu}(t) \sim \mathcal{N}(0, \boldsymbol{R}(t))$, $\boldsymbol{w}(t) \sim \mathcal{N}(0, \boldsymbol{Q}(t))$ are respectively white, normally-distributed Gaussian output and actuation (or process) noises with zero mean and covariance matrices $\boldsymbol{R}(t)$ and $\boldsymbol{Q}(t)$, respectively. The goal of this paper is to estimate the unknown state $\boldsymbol{q}(t)$ while minimizing its maximum uncertainty at a desired final time t_f given the knowledge of the sensor readings $\boldsymbol{z}(t)$ and of the applied inputs $\boldsymbol{u}(t)$ over the whole time window $[t_0, t_f]$. In order to achieve this goal, on one side it is important to maximize the amount of information collected via the available noisy sensor readings. On the other side, it is also important to minimize the negative effects of the actuation noise in degrading the above-mentioned acquired information, especially in the case where no measurements are available or they are not sufficient to completely reconstruct the whole state. We therefore need suitable metrics to capture both the information that can be retrieved from candidate trajectories $\boldsymbol{q}(t)$ over $[t_0, t_f]$ and the degrading effects of actuation noise so as to follow state space directions with higher information content.

A. Quantifying the acquired information

In [9], a suitable metric for quantifying the amount of collected information along a given trajectory has been introduced. This metric is strictly related to nonlinear observability [19] and, in particular, to the ability of estimating the *current state* $\boldsymbol{q}(t)$ from knowledge of the system outputs $\boldsymbol{z}(\bar{t})$ and inputs $\boldsymbol{u}(\bar{t})$ with $\bar{t} \in [t_0,\, t]$. This ability is represented by the so-called *Constructability Gramian* (CG) [20], defined as

$$\boldsymbol{\mathcal{G}_c}(t_0, t_f) \triangleq \int_{t_0}^{t_f} \boldsymbol{\Phi}^T(\tau, t_f)\boldsymbol{H}^T(\tau)\boldsymbol{W}_c(\tau)\boldsymbol{H}(\tau)\boldsymbol{\Phi}(\tau, t_f)\,\mathrm{d}\tau\,, \tag{3}$$

where $t_f > t_0$, $\boldsymbol{H}(\tau) = \frac{\partial \boldsymbol{h}(\boldsymbol{q}(\tau))}{\partial \boldsymbol{q}(\tau)}$, and $\boldsymbol{W}_c(\tau) \in \mathbb{R}^{p\times p}$ is a symmetric positive definite weight matrix (a design parameter), that may be used for, e.g., accounting for outputs with different units and different uncertainties. Matrix $\boldsymbol{\Phi}(t, t_f) \in \mathbb{R}^{n\times n}$, also known as *sensitivity matrix*, is defined as $\boldsymbol{\Phi}(t, t_f) = \frac{\partial \boldsymbol{q}(t)}{\partial \boldsymbol{q}_f}$ and obeys the following differential equation with *final* conditions at t_f

$$\dot{\boldsymbol{\Phi}}(t, t_f) = \frac{\partial f(\boldsymbol{q}(t), \boldsymbol{u}(t))}{\partial \boldsymbol{q}(t)}\boldsymbol{\Phi}(t, t_f)\,, \qquad \boldsymbol{\Phi}(t_f, t_f) = I. \tag{4}$$

Notice that the CG appears to be a less popular choice in the existing robotics literature on active sensing/perception than the *Observability Gramian* (OG) $\boldsymbol{\mathcal{G}}_o(t_0, t_f) \in \mathbb{R}^{n\times n}$ [21], [22].

We conclude by recalling from [9] an important link between the CG and the optimal estimation error covariance matrix $\boldsymbol{P}$, solution of the Riccati equation

$$\dot{\boldsymbol{P}}^{-1}(t) = -\boldsymbol{P}^{-1}(t)\boldsymbol{A}(t) - \boldsymbol{A}^T(t)\boldsymbol{P}^{-1}(t) + \boldsymbol{H}^T(t)\boldsymbol{R}^{-1}\boldsymbol{H}(t), \tag{5}$$

with $\boldsymbol{A}(t) = \frac{\partial \boldsymbol{f}(\boldsymbol{q},\boldsymbol{u})}{\partial \boldsymbol{q}}$. Equation (5) is linked to the linear time-varying system obtained by linearizing system (1)–(2) with $\boldsymbol{w}(t)$ approximated by its mean ($\boldsymbol{w}(t) \approx 0$) around a nominal trajectory $\boldsymbol{q}(t)$. The solution of (5) is

$$\boldsymbol{P}^{-1}(t) = \boldsymbol{\Phi}^T(t_0, t)\boldsymbol{P}_0^{-1}\boldsymbol{\Phi}(t_0, t) + \boldsymbol{\mathcal{G}_c}(t_0, t) = \boldsymbol{\mathcal{G}_c}(-\infty, t). \tag{6}$$

This expression can be interpreted as follows: the first term represents the contribution of the *a priori* information $\boldsymbol{P}^{-1}(t_0) = \boldsymbol{P}_0^{-1}$ (or the information encoded by the CG $\boldsymbol{P}_0^{-1} = \boldsymbol{\mathcal{G}_c}(-\infty, t_0)$ over the interval $[-\infty,\, t_0]$) but *shifted* at time t by the operator $\boldsymbol{\Phi}(t_0, t)$. The second term in (6) is instead the contribution of the information actually collected during the interval $[t_0,\, t]$ and encoded by the CG in (3) with $\boldsymbol{W}_c(t) = \boldsymbol{R}^{-1}(t)$ and $t_f = t$. Succinctly, (6) represents the information encoded by the CG over the whole interval $[-\infty, t]$. We can then conclude that the maximization of (some norm of) $\boldsymbol{\mathcal{G}_c}(-\infty, t)$ w.r.t. $\boldsymbol{q}(t)$ with $t \in [t_0, t_f]$ will result in a state trajectory that maximizes the performance in reconstructing the *final state* $\boldsymbol{q}_f$ from knowledge of the present and past system outputs and inputs.

B. Quantifying the degrading effects of actuation noise

In this subsection we will introduce a new metric able to measure the degrading effects of actuation (or process) noise on the acquired information. To this end, we now briefly summarize some known concepts of nonlinear reachability by means of the Reachability Gramian (RG). The ability of an input $\boldsymbol{u}(t),\ t \in [t_0,\, t_1]$ to move the state $\boldsymbol{q}(t)$ of a nonlinear system from any initial state $\boldsymbol{q}_0$ at $t_0 = 0$ (equivalently the origin) to any other final state $\boldsymbol{q}_f$ at time t_f in a finite time interval $[t_0,\, t_f]$ revolves around the notion of *Reachability* [19]. This ability is represented by the RG [20], defined as

$$\boldsymbol{\mathcal{G}_r}(t_0, t_f) \triangleq \int_{t_0}^{t_f} \boldsymbol{\Phi}(t_f,\tau)\boldsymbol{B}(\tau)\boldsymbol{W}_r(\tau)\boldsymbol{B}^T(\tau)\boldsymbol{\Phi}^T(t_f,\tau)\,\mathrm{d}\tau, \tag{7}$$

where $t_f > t_0$, $\boldsymbol{B}(\tau) = \frac{\partial \boldsymbol{f}(\boldsymbol{q}(\tau),\boldsymbol{u}(\tau))}{\partial \boldsymbol{u}(\tau)}$, and $\boldsymbol{W}_r(\tau) \in \mathbb{R}^{m\times m}$ is again a symmetric positive definite weight matrix as in (3).

There is an important link between the RG and the optimal estimation error covariance matrix $\boldsymbol{P}$, solution of the Riccati equation

$$\dot{\boldsymbol{P}}(t) = \boldsymbol{A}(t)\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{A}^T(t) + \boldsymbol{B}(t)\boldsymbol{Q}(t)\boldsymbol{B}^T(t), \tag{8}$$

which is linked to the linear time-varying system obtained by linearizing system (1), without measurement $\boldsymbol{z}(t)$, around a nominal trajectory $\boldsymbol{q}(t)$. The solution of (8), which is linear in $\boldsymbol{P}(t)$, is

$$\boldsymbol{P}(t) = \boldsymbol{\Phi}(t,t_0)\boldsymbol{P}_0\boldsymbol{\Phi}^T(t,t_0) + \boldsymbol{\mathcal{G}_r}(t_0,t) = \boldsymbol{\mathcal{G}_r}(-\infty,t). \tag{9}$$

This expression can be interpreted as follows. The first term represents the contribution of the *a priori* information $\boldsymbol{P}(t_0) = \boldsymbol{P}_0$ available at time t_0 (or the information encoded by the RG $\boldsymbol{P}_0 = \boldsymbol{\mathcal{G}_r}(-\infty,t_0)$ over the interval $[-\infty,\, t_0]$) but *shifted* at time t by the operator $\boldsymbol{\Phi}(t_0,t)$. The second term is instead the contribution of the information actually degraded during the interval $[t_0,\, t]$ by the actuation noise and encoded by the RG in (7) with $\boldsymbol{W}_r(t) = \boldsymbol{Q}(t)$ and $t_f = t$. To summarize, (9) is the information encoded by the RG over the whole interval $[-\infty,t]$ and represents the information (or equivalently the uncertainty) time evolution without measurements. We can then conclude that the maximization of (some norm of) $\boldsymbol{\mathcal{G}_r}(-\infty,t)$ w.r.t. $\boldsymbol{q}(t)$ with $t \in [t_0,t_f]$ will result in a state trajectory that minimizes the degrading effects of the actuation noise in reconstructing the *final state* $\boldsymbol{q}_f$ when no measurements are available.

III. OPTIMAL ACTIVE SENSING CONTROL WITH INTERMITTENT MEASUREMENTS

In Section II, we have shown how the CG can be used as a metric to quantify the state estimation uncertainty growth along the planned trajectory due to the output noise and when no actuation noise is considered. Moreover, we have shown that the RG is instead a suitable metric of the uncertainty growth due to the actuation in the absence of output measurements. In practical applications, measurements and actuation noises act on the system simultaneously, which calls for a combined metric on the combination of both uncertainty sources to limit the maximum estimation uncertainty. For this reason, one of the main contributions of this paper is to propose an effective combination of the two Gramians (Subsection III-A) capable of correctly handling intermittent measurements availability, e.g. limited sensing range or FOV, yielding to intermittent observability. As a result, the cost functions related to the CG and the RG can be directly used to improve state/parameters estimation for generic filters, such as Uscented Kalman Filters or Particle Filters.

To model intermittent measurement availability, as also done in [9], we will weight the measurement covariance matrix $\boldsymbol{R}^{-1}$ in the CG by a weight matrix $\boldsymbol{W}_R \in \mathbb{R}^{p\times p}$ so that $\boldsymbol{R}_W^{-1} = \boldsymbol{W}_R^T\boldsymbol{R}^{-1}\boldsymbol{W}_R$ where $\boldsymbol{W}_R = \mathrm{diag}(w_1, w_2, \dots, w_k, \dots)$. A possible shape for weight w_k in case of a range sensor can be found in [9] (cf. Figure 9 therein). This is equivalent to have a measurement noise that increases near the sensor limits, becoming infinite once violated.

A. Information-aware cost functions

Based on the analysis in Section II-A, the smallest eigenvalue of CG is associated to the state space direction with the highest uncertainty. Therefore, to actually limit the estimation uncertainty, the goal is to synthesize a trajectory that maximizes the sensor information of the lowest eigenvalue, i.e. we design an optimization problem that maximizes an infinity norm on the eigenvalues space. Similarly, as reported in Section II-B, the smallest eigenvalue of the RG is related to the direction in the state space with minimum uncertainty. Since the effect of the actuation noise is to increase the uncertainty and it is unavoidable, the basic idea is to control this uncertainty growth along the direction of the state space that has minimum uncertainty. This way, this detrimental effect is minimized in the direction with maximum uncertainty (i.e. the largest eigenvalue direction), thus imposing again an infinity norm problem in the eigenvalues space.

While the CG computed on the future trajectory is aware about the number of available measurements by weighting the covariance matrix $\boldsymbol{R}$ with $\boldsymbol{W}_R$, the RG is not aware about the degree of Constructability. For this reason and to consider the future contribution of the actuation/process noise only when complete observability (equivalently Constructability) is not guaranteed, in the following we will also weight the actuation covariance matrix $\boldsymbol{Q}$ in the RG by a weight matrix $\boldsymbol{W}_Q \in \mathbb{R}^{m\times m}$ so that $\boldsymbol{Q}_W = \boldsymbol{W}_Q\boldsymbol{Q}\boldsymbol{W}_Q^T$ when $\boldsymbol{W}_Q = diag(1-w_p, 1-w_p, \dots, 1-w_p, \dots)$ with $w_p = 0$ if $\sum_i w_i = 0$ (i.e. no measurements available), $w_p = 1$ if $\sum_i w_i \geq \bar{p}-1$ with $\bar{p}$ the minimum number of measurement that guarantee complete Constructability and finally[1], $w_p = A(1+\sin(\alpha\sum_i w_i + \beta))$ if $0 < \sum_i w_i < \bar{p}-1$

[1] Based on the definition of weight w_k, $\sum_i w_i \geq \bar{p}-1$ implies to already have the minimum number of measurements for complete observability

with $A = 0.5$, $\alpha = \frac{\pi}{\bar{p}-1}$ and $\beta = -\frac{\pi}{2}$ (i.e. partial observability). The choice of parameters allows to smoothly increase w_p from 0 to 1 as the number of available measurements increase from 0 to $\bar{p}$. Based on the above discussion, CG and RG can be combined in a weighted (with weight $\boldsymbol{W}_R$ and $\boldsymbol{W}_Q$) sum of their smallest eigenvalues (evaluated at t_f)

$$J_0(\boldsymbol{u}, \boldsymbol{q}, t, t_f) = \lambda_{min}(\boldsymbol{\mathcal{G}_c}(-\infty, t, t_f)) + \lambda_{min}(\boldsymbol{\mathcal{G}_r}(-\infty, t, t_f)) \tag{10}$$

with $\boldsymbol{\mathcal{G}_c}(-\infty, t, t_f)$=$\boldsymbol{\Phi}^T(t, t_f)\boldsymbol{\mathcal{G}_c}(-\infty, t)\boldsymbol{\Phi}(t, t_f)$+$\boldsymbol{\mathcal{G}_c}(t, t_f)$ and $\boldsymbol{\mathcal{G}_r}(-\infty, t, t_f)$=$\boldsymbol{\Phi}(t_f, t)\boldsymbol{\mathcal{G}_r}(-\infty, t)\boldsymbol{\Phi}^T(t_f, t)$+$\boldsymbol{\mathcal{G}_r}(t, t_f)$. Matrices $\boldsymbol{W}_R$ and $\boldsymbol{W}_Q$ appear in $\boldsymbol{\mathcal{G}_c}(t, t_f)$ and $\boldsymbol{\mathcal{G}_r}(t, t_f)$, respectively. $\boldsymbol{\mathcal{G}_c}(-\infty, t)$ and $\boldsymbol{\mathcal{G}_r}(-\infty, t)$ can be computed online or can be freely obtained from an EKF, if this were the adopted observer. By maximizing J_0 *online* at any time t, trajectories that simultaneously increase the sensor information and reduces the negative actuation effects along the highest uncertainty direction at the final time t_f, depending on the number of available measurements and hence the degree of Constructability, are obtained.

For better cope with partial observability, a better choice for the contribution of RG is to control the actuation uncertainty toward the direction with minimum uncertainty into the unobservable space, i.e. where, however, the sensor information can not be beneficial. As a consequence, the information content about the observable space can freely increase towards its maximum value. The modified cost function

$$J_1(\boldsymbol{u}, \boldsymbol{q}, t, t_f) = \\ = \lambda_{min}(\boldsymbol{\mathcal{G}_c}(-\infty, t, t_f)) + \lambda_{min}(\mathcal{P}_{N_{\mathcal{G}_c}}(\boldsymbol{\mathcal{G}_r}(-\infty, t, t_f))), \tag{11}$$

considers the smallest eigenvalue of the sub-matrix of the RG (still weighted by $\boldsymbol{W}_Q$) projected along the null space of the CG by means of the projector $\mathcal{P}_{N_{\mathcal{G}_c}}(\cdot)$.

A complementary choice w.r.t. J_1 is to control the actuation uncertainty toward the direction with minimum uncertainty but still observable, i.e. where the sensor information can still be beneficial. As a consequence, the information content into the unobservable space remains as much as possible unchanged until the next phase when complete observability is re-established. The modified cost function

$$J_2(\boldsymbol{u}, \boldsymbol{q}, t, t_f) = \\ = \lambda_{min}(\boldsymbol{\mathcal{G}_c}(-\infty, t, t_f)) + \lambda_{min}(\mathcal{P}_{obs(\mathcal{G}_c)}(\boldsymbol{\mathcal{G}_r}(-\infty, t, t_f)), \tag{12}$$

considers the smallest eigenvalue of the sub-matrix of the RG (still weighted by $\boldsymbol{W}_Q$) projected along the observable space of the CG by means of the projector $\mathcal{P}_{obs(\mathcal{G}_c)}(\cdot)$.

Instrumental to the definition of the optimal control problem, that is to maximize the above-defined cost functions, we conclude this subsection by recalling that the smallest eigenvalue is not differentiable in case of repeated eigenvalue, i.e. if the smallest eigenvalue has algebraic multiplicity larger than one. For this reason also in this paper we will use the Schatten Norm (cf. [9] for its definition).

B. Online Optimal Active Sensing Control with Intermittent Measurements

Let us use the subscripts 0 and f for variables evaluated at the initial and final time t_0 and t_f, respectively. We are now able to state the online (because of the state updating process) optimal control problem.

Problem 1 *For all $t \in [t_0, t_f]$, find the optimal control*

$$\boldsymbol{u}^*(t) = \arg\max_{\boldsymbol{u}} J_i(\boldsymbol{u}(t), \hat{\boldsymbol{q}}(t), t, t_f), \; i = 0, 1, 2$$

over the interval $[t, t_f], s.t.$

1) $\dot{\hat{\boldsymbol{q}}}(t) = f_{filter}(\hat{\boldsymbol{q}}(t), \boldsymbol{u}(t), \boldsymbol{z}(t)), \; \hat{\boldsymbol{q}}(t_0) = \hat{q}_0$

2) $\underline{\boldsymbol{u}} \leq \boldsymbol{u}(t) \leq \bar{\boldsymbol{u}}$;

3) $\hat{\boldsymbol{q}}_f = \bar{\boldsymbol{q}}_f$;

4) $c(\boldsymbol{u}(t), \hat{\boldsymbol{q}}(t)) \leq 0$;

where 1) represents the dynamic of the employed observer , $\underline{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}$ are the minimum and maximum admissible value for the control inputs, $\bar{\boldsymbol{q}}_f$ is the final desired configuration while $c(\boldsymbol{u}, \hat{\boldsymbol{q}}(t)) \leq 0$ are possible linear/nonlinear equality/inequality constraints (e.g. for obstacle avoidance).

Problem 1 will be solved in CasADi tool [23] with direct single shooting method by adopting the ma57 ipopt solver. We have hence rewritten Problem 1 as a NonLinear Program (NLP) problem assuming control inputs as piecewise constant functions over the whole interval $[t_0, t_f]$. The length of each piece is $\frac{t_f - t_0}{N}$, where N is the total number of control values to be optimized online, as discussed in the next section. Due to the local nature of our method, the optimization could converge to a local maxima depending on the initial guess which consists in a random control sequence. Nonetheless, they will certainly improve the estimation performance w.r.t. the initial guess. A globally optimal solution can be obtained using a multi-Search or a Multi-start methods.

IV. SIMULATION RESULTS

We show here the results obtained by solving Problem 1 for the particular case of a unicycle vehicle, which is endowed with an EKF for state estimation and has to reach a desired goal while intermittently (because of the on-board limited range sensor) measuring distances from markers. While for modeling measurements noise we will use the classical Gaussian hypothesis, the actuation uncertainty also assumes a more degrading Brownian distribution.

Let us assume the unicycle moves on a plane where a frame $\mathcal{W}$ is defined with the origin $\mathcal{O}_{\mathcal{W}}$ and axes $X_{\mathcal{W}}$, $Y_{\mathcal{W}}$. The configuration of the vehicle is described by $q(t) = (x(t), y(t), \theta(t))$ where $(x(t), y(t))$ is the position on the plane and $\theta(t)$ is the heading with respect to the $X_{\mathcal{W}}$ axes. Let $\boldsymbol{\xi}(t) = [\rho(t), \psi(t), \beta(t)] = [\sqrt{x(t)^2 + y(t)^2 +}, \arctan(y(t)/x(t)), \psi - \theta(t) + \pi]$ the polar coordinates. The kinematic in Cartesian coordinate is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}, \; \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} r/2 & r/2 \\ r/b & -r/b \end{bmatrix} (\boldsymbol{u} + \boldsymbol{w}) \tag{13}$$

where $\boldsymbol{u} = [\omega_R, \quad \omega_L]$ are the angular velocities of the right and left wheels, respectively, while r and b are the wheels radius and the axle length, respectively. The robot is equipped with a range sensor measuring the distance from 8 fixed markers located at known positions (x_{Mi}, y_{Mi}), $i = 1, \ldots, 8$, i.e. $\rho_i = \sqrt{(x - x_{Mi})^2 + (y - y_{Mi})^2}$. As customary for this class of sensors, we assume as sensory data $z_i = \rho_i$ if $d \leq \rho_i \leq D$, unavailable otherwise, where D and d model the limited sensing range. The outputs can then be expressed as $\boldsymbol{z} = [z_1, z_2, \ldots, z_8]^T + \boldsymbol{\nu}$.

The state of a unicycle measuring distances from markers is completely observable (apart from "bad input") if the number of markers within the sensor range is equal or greater than 2. If only one marker is visible, there is an unobservable space related to ψ. Cost function J_1 and J_2 requires the computation of a basis of the unobservable space of CG, if any, and, from it the projectors $\mathcal{P}_{N_{\mathcal{G}_c}}$ and $\mathcal{P}_{obs(\mathcal{G}_c)}$. Since the unobservable space is known and state-independent if the unicycle kinematics are expressed in polar coordinate, in order to correctly project RG in the unobservable space, it is convenient to rewrite RG in these coordinates as ${}^{\xi}\boldsymbol{\mathcal{G}}_r = \boldsymbol{J}_q {}^{q}\boldsymbol{\mathcal{G}}_r \boldsymbol{J}_q^{-1}$, and $J_q = \frac{\partial \mathcal{S}(q)}{\partial q}$ is the Jacobian of the change of coordinates. The projection of RG can be done as: $\boldsymbol{V}^T {}^{\xi}\boldsymbol{\mathcal{G}}_r \boldsymbol{V}$ where the projector matrix $\boldsymbol{V} = [\boldsymbol{0}_{3\times 2},\ \text{Null}(\mathcal{G}_c)]$ for J_1 and $\boldsymbol{V} = [\text{Obs}(\mathcal{G}_c),\ \boldsymbol{0}_{3\times 1}]$ for J_2 with $\text{Null}(\mathcal{G}_c) = [0\ 0\ 1]^T$ and $\text{Obs}(\mathcal{G}_c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$.
For all simulations, $\boldsymbol{q}_0 = [-17 \text{ m}, 0 \text{ m}, 0 \text{ rad}]^T$ while $\hat{\boldsymbol{q}}_0 = [-16.5 \text{ m}, -0.3 \text{ m}, 0.03 \text{ rad}]^T$ with covariance matrix (used to initialize the EKF) $\boldsymbol{P}_0 = 0.4\boldsymbol{I}$. Finally, $0 \leq v \leq 3$ m/s and $-3 \leq \omega \leq 3$ rad/s.

A. Methodology validation with Gaussian actuation noise

We generated 100 random paths that were optimized using our proposed cost functions with $D = 5$ m e $d = 1.5$ m. The final configuration $\boldsymbol{q}_f$ is not fixed. Moreover, we assumed normally distributed Gaussian output and actuation/process noises with zero mean and covariance matrices $\boldsymbol{R} = \boldsymbol{Q} = 0.3\boldsymbol{I}$. For each optimized path, the smallest eigenvalue of P_f^{-1} ($\lambda_{min}(P_f^{-1})$) and the Root Mean Square of the estimation error $RMS(e_f)$ are considered. Then, their average values and standard deviations are computed and a statistical analysis is provided in order to show the effectiveness of our proposed cost functions J_i, $i = 0, 1, 2$ w.r.t. the one proposed in [9] by using a Wilcoxon rank sum test. A significance level of 5% was assumed and p-values less than 10^{-4} were considered equal to zero. This test has shown that, only the J_1 p-values are zero, meaning that in terms of uncertainty and estimation error, J_1 is able to find more informative paths. However, this is obtained with a longest computation time. Indeed, the average computation times for offline optimization, i.e. the time needed for finding the optimal path starting from the initial guess, and their standard deviations are also evaluated, obtaining 17 ± 0.002 s for CG, 29 ± 0.002 s for J_0, 104 ± 0.8 s for J_1 and 36 ± 0.003 s for J_2. However, the average times (with their standard deviations) needed for updating online the optimal path as soon as a new estimate is available, are smaller and equal to 0.2 ± 0.2 s for CG, 0.3 ± 0.2 s for J_0, 0.7 ± 1.2 s for J_1 and 0.5 ± 0.4 s for J_2. Future works will be dedicated to further reduce these times, e.g. by providing the analytical Gradient/Hessian to the solver or by parametrizing inputs as B-Splines.

	CG		J_0		J_1		J_2	
D	RMS	λ_{min}	RMS	λ_{min}	RMS	λ_{min}	RMS	λ_{min}
30	0.018	39.83	0.024	39.85	**0.016**	**40.39**	0.026	40.21
7	0.019	17.2	0.020	17.082	**0.013**	**17.25**	0.020	17
6	0.036	1.32	0.133	1.82	**0.033**	**2.43**	0.017	2.44
5	0.080	0.72	0.033	0.75	**0.021**	**1.53**	0.036	0.84
3	3.43	0.039	3.202	0.033	**0.086**	**0.91**	0.17	0.25

TABLE I

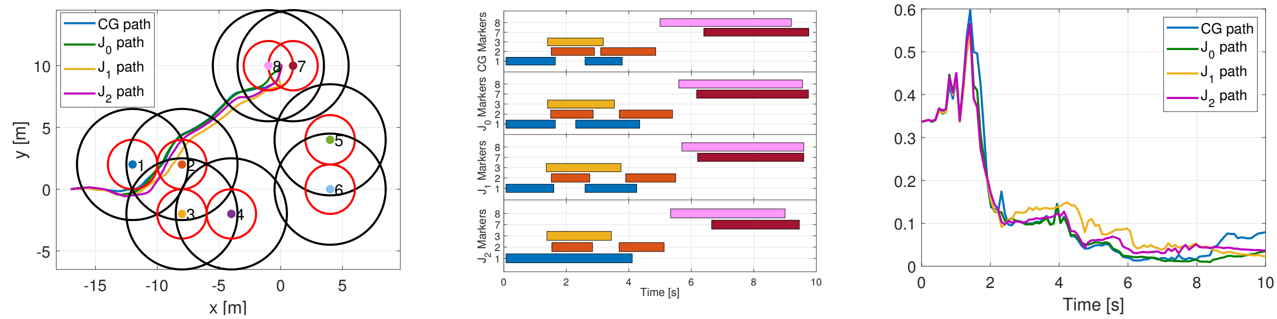
$\lambda_{min}(P_f^{-1})$ AND $RMS(e_f)$ WITH DIFFERENT VALUES OF D.

B. Active sensing control with Gaussian actuation noise

In this section, we compare the performance obtained with our proposed cost functions as the range D varies from $D = 30$ m (all markers in view) to $D = 3$ m (no more then 2 marker in view). The vehicle needs to reach a final state $\boldsymbol{q}_f = [0 \text{ m}, 10 \text{ m}, \pi/2 \text{ rad}]^T$ in $T = 10$ s. Moreover, d, R and Q are as in previous section. TABLE I summarizes results in terms of $RMS(e_f)$ and $\lambda_{min}(P_f^{-1})$. As soon as sensor range D reduces, estimation performance along the CG-based optimal path degrades in terms of both maximum estimation uncertainty and estimation error (especially for $D = 3$). The similarity in performance between CG-based and J_0-based optimal path can be due to two possible reasons: 1) the smallest eigenvalue of RG is quite smaller w.r.t. the one of CG; 2) for the simple case of unicycle it is difficult to increase one eigenvalue and at the same time decrease the other. J_2 indeed outperforms J_0 even if, also for J_2, both eigenvalues belong to the same sub-space and hence may still be highly constrained. However, reason 1) is no longer valid for it. Finally, J_1 outperforms all the other cost function as both reasons 1) and 2) are no longer valid for it. Fig. 1 shows the real trajectories, as well as the marker visibility and the RMS of the estimation errors along the path for the particular case of $D = 5$ m. Trajectories look quite similar. Main differences can be seen at the end. However, these small differences are enough to increase performances of the EFK along the J_i-based, $i = 0, 1, 2$ optimal path w.r.t. the CG one (see Fig. 1(c)). Fig. 1(b) shows the marker visibility along the optimal path. The percentage of total time during which the system is completely observable is 43.3% for CG, 52.8% for J_0, 48.3% for J_1, 45.1% for J_2, partially observable is 50.7% for CG, 41.7% for J_0, 45.9% for J_1, 46.4% for J_2 and unobservable is 6% for CG, 5.5% for J_0, 5.8% for J_1, 8.5% for J_2. In spite of similar percentage of total time where only 1 marker or no marker in view (this may be due to the limited total time T and the disposition of markers), J_1-based optimal path seems to better cope with the actuation noise by better adjusting (with the aim of limiting the actuation noise effects) the unavoidable intermittence of measurements during the path.

C. Active sensing control with Brownian noise

In this section, we consider a Brownian unicycle kinematics as described in [24]. The Brownian noise is a random Gaussian variable whose variance increases linearly with the

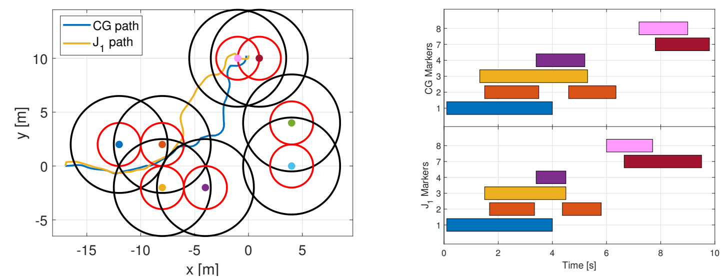


(a) Real trajectories for J_i, $i = 0, 1, 2$ and CG. (b) Marker visibility along the path. (c) RMS of the estimation error along the paths.

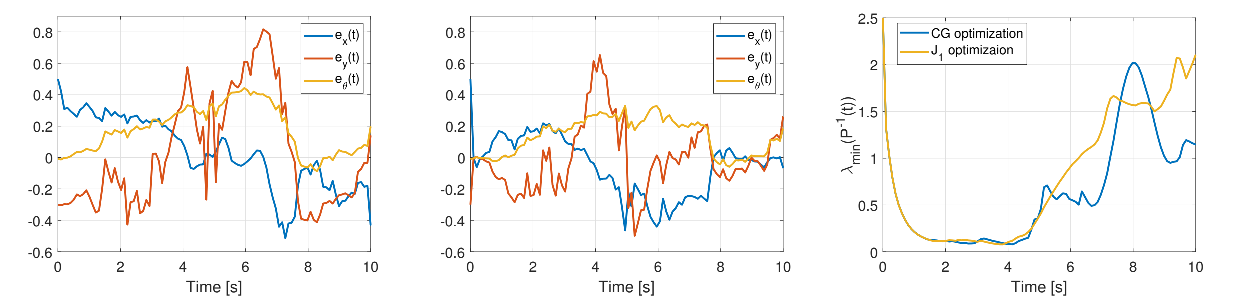
Fig. 1. Comparison in terms of estimation performance for the particular case of $D = 5$ m reported in TABLE I. J_1-based optimal path outperforms all the other cost functions as $\lambda_{min}(P_f)$ reaches a value that is almost two times the one reached at the end of the CG-based optimal path. This is beneficial also in terms of estimation error.

	CG		J_1	
D	RMS	λ_{min}	RMS	λ_{min}
30	0.063	6.005	**0.053**	**6.581**
7	0.037	1.933	**0.036**	**2.106**
5	0.196	1.148	**0.188**	**2.103**
4	0.593	0.518	**0.366**	**0.873**

markers	CG	J_1
0	11.1 %	8.1 %
1	30.7 %	56.1 %
≥ 2	58.2 %	35.8 %



(a) Performance estimation comparison and markers in view. (b) Real trajectories for J_1 and CG. (c) Marker visibility along the path.



(d) Estimation error along CG-based optimal path. $e_f = (-0.428, 0.139, 0.195)^T$ (e) Estimation error along J_1-based optimal path. $e_f = (-0.065, 0.260, 0.186)^T$ (f) $\lambda_{min}(P_f^{-1})$ is 2.1 for J_1 and 1.15 for CG.

Fig. 2. Comparison in terms of estimation performance in case of Brownian unicycle dynamic with $D = 5$ m Notice how J_1 outperforms CG in terms of both estimation error and maximum estimation uncertainty.

traveled distance. For the sake of space, only CG-based optimal path and J_1-based one (the one that previously provided the best performance) are compared. Moreover, q_f is as in previous section. In Fig. 2(a) several numerical results for different values of range D are reported. For all values of range, the J_1-based optimal paths allow to reach the final configuration with less error and minimum estimation uncertainty. Fig. 2(b)-(f) shows the results for $D = 5$ m. Notice how the trajectory of the CG-based optimal path and the J_1-based one are now completely different. The CG-based optimal path crosses a wider region where no marker is visible. In term of estimation error, the J_1-based optimal path slightly outperforms the CG-based one even if the vehicle spends much more time in partial observability (see Fig. 2(a)). The same holds for $\lambda_{min}(P_f^{-1})$ (see Fig. 2(f)) which is two time greater at the end of J_1-based optimal path.

V. CONCLUSIONS

An active sensing control scheme able to simultaneously cope with the negative effects of actuation noise (often neglected in the literature) as well as with intermittent measurements has been proposed. We have shown the effectiveness of our method comparing state estimation performance of an EKF for a unicycle-like vehicle subject to Gaussian or Brownian actuation noise. Future works will aim at finding an effective combination of the different cost indices proposed based on the current robot configuration and implementing it on an actual robotic system in real time. We are also planning to apply our methodology to more complex robots (e.g. a quadrotor).

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