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Research Article

Mohamed Abdelwahed, Luigi C. Berselli*, and Nejmeddine Chorfi

On the uniqueness for weak solutions of steady double-phase fluids

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Abstract: We consider a double-phase non-Newtonian fluid, described by a stress tensor which is the sum of a p-Stokes and a q-Stokes stress tensor, with $1 < p < 2 < q < \infty$. For a wide range of parameters (p, q), we prove the uniqueness of small solutions. We use the $p < 2$ features to obtain quadratic-type estimates for the stress-tensor, while we use the improved regularity coming from the term with $q > 2$ to justify calculations for weak solutions. Results are obtained through a careful use of the symmetries of the convective term and are also valid for rather general (even anisotropic) stress-tensors.

Keywords: Uniqueness, double-phase, steady motion, non-Newtonian fluid

MSC: 76A05, 35J62, 35Q30, 35J25, 35J55

1 Introduction

In this paper we study the uniqueness of "small" (in an appropriate sense) solutions to a family of double-phase steady problems, arising in the analysis of non-Newtonian fluids. The interest in double-phase problems started with the celebrated result by Zhikov [29] concerning the Lavrentiev phenomenon. The problem was set in the framework of functionals with (p, q)-growth, for which we refer also to the pioneering paper by Marcellini [22]; recent results can be found in the works of Esposito, Leonetti, and Mingione [15], Baroni, Colombo, and Mingione [3], and Colombo and Mingione [9], especially in the context of regularity of minimizers. For results regarding the applications to spectral analysis and multiplicity of solutions, see Chorfi and Radulescu [8], Baraket, Chebbi, Chorfi, and Radulescu [2], and the review in Radulescu [25]. In the context of fluid mechanics, problems with more than one phase arise especially in the case of fluids with complex rheologies, as introduced by Růžička [23], where the modeling leads to a problem with variable exponent. See also the review in Rădulescu and Repovš [26] for general problems involving partial differential equations with variable exponents.

Here, instead that regularity or multiplicity of solutions, we focus on some specific problems concerning uniqueness in the class of small weak solutions.

Mohamed Abdelwahed, Department of Mathematics, College of Sciences, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia, E-mail:mabdelwahed@ksu.edu.sa
***Corresponding Author: Luigi C. Berselli,** Dipartimento di Matematica Università di Pisa Via F. Buonarroti 1/c, Pisa I-56127, Italy, E-mail:luigi.carlo.berselli@unipi.it
Nejmeddine Chorfi, Department of Mathematics, College of Sciences, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia, E-mail:nchorfi@ksu.edu.sa

In order to introduce the problem, we recall that a basic result for the steady Navier-Stokes equations in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{aligned} -\nu_0 \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla\pi = \mathbf{f} \qquad & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \qquad & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

is that of uniqueness of weak solutions, under the assumption of smallness of the $L^3(\Omega)$-norm of $\mathbf{u}$. (In this case the possible non-uniqueness is neither a special feature of the problem in $\mathbb{R}^3$, nor coming from the fact that it is a nonlinear system with the divergence-free constraint. Possible multiplicity of solutions is a common result even for semi-linear elliptic scalar equations.) More precisely, for the steady Navier-Stokes equations (1.1), it holds that if $\mathbf{u}_1, \mathbf{u}_2 \in W_0^{1,2}(\Omega)$ are weak solutions corresponding to the same external force $\mathbf{f} \in W^{-1,2}(\Omega)$, then there exists $\epsilon_0 > 0$ such that if $\|\mathbf{u}_1\|_3 \le \epsilon_0$, then $\mathbf{u}_1 = \mathbf{u}_2$; see the review in Galdi [17, Ch. IX.2]. The proof is based on writing the system satisfied by the difference $\mathbf{U} = \mathbf{u}_1 - \mathbf{u}_2$ and testing by $\mathbf{U}$ itself (a procedure which is legitimate in the steady case also for weak solutions). Next, one uses the inequality

$$\left| \int_\Omega (\mathbf{U}\cdot\nabla)\,\mathbf{U}\cdot\mathbf{u}_1\,\mathrm{d}\mathbf{x} \right| \le \|\mathbf{U}\|_6 \|\nabla\mathbf{U}\|_2 \|\mathbf{u}_1\|_3 \le C\|\nabla\mathbf{U}\|_2^2 \|\mathbf{u}_1\|_3, \tag{1.2}$$

with a constant C depending only on Ω, obtained by application of the Hölder inequality and of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$.

In this way from (1.1) one easily gets, after integration by parts, that

$$\nu_0 \|\nabla\mathbf{U}\|_2^2 \le C\epsilon_0 \|\nabla\mathbf{U}\|_2^2,$$

which implies $\mathbf{U} = \mathbf{0}$, provided that $\epsilon_0 < \nu_0/C$. Observe that from the energy estimate valid for weak solutions one knows a priori just that

$$\|\nabla\mathbf{u}_1\|_2 \le \frac{1}{\nu_0}\|\mathbf{f}\|_{W^{-1,2}},$$

hence, uniqueness for small forces/large viscosities follows by using a Sobolev embedding to ensure smallness in $L^3(\Omega)$, at least when Ω is bounded.

We stress the critical role played by the exponent “two” of $\|\nabla\mathbf{U}\|_2$ in the energy estimate (1.2). It appears at the same time as exponent in: i) the lower bound for the dissipative term; and ii) the upper bound for the quantity obtained by estimating of the convective term. If there is a mismatch in the powers, then this easy but powerful argument may fail.

The same argument can be also applied if $\|\nabla\mathbf{u}_1\|_{3/2} \le \epsilon_1$ (for a possibly different small constant $\epsilon_1 > 0$), by using a similar estimate for the convective term and observing that $W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega)$. Note that the two alternative assumptions have the same scaling.

In the case of *steady* non-Newtonian fluids, the situation becomes more complex. If one considers for $1 < p < \infty$ the following system, describing a family of shear dependent fluids,

$$\begin{aligned} -\nu_0\Delta\mathbf{u} - \operatorname{div}(\delta + |\mathbf{D}\mathbf{u}|)^{p-2}\mathbf{D}\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla\pi = \mathbf{f} \qquad & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \qquad & \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

then, for any $\delta \ge 0$ and for all $\nu_0 > 0$, the same argument valid for the Navier-Stokes equations can be applied. In fact, the additional stress tensor is monotone and the following inequality holds:

$$\int_\Omega \Big((\delta + |\mathbf{D}\mathbf{u}_1|)^{p-2}\mathbf{D}\mathbf{u}_1 - (\delta + |\mathbf{D}\mathbf{u}_2|)^{p-2}\mathbf{D}\mathbf{u}_2 \Big) : \mathbf{D}(\mathbf{u}_1 - \mathbf{u}_2)\,\mathrm{d}\mathbf{x} \ge 0.$$

The above inequality and the fact that weak solutions still belong to $W_0^{1,2}(\Omega)$ will be enough to disregard the effect of the additional non-linear stress tensor. It is possible to treat problem (1.3) in the same way as for the unperturbed Navier-Stokes equations, at least for what concerns uniqueness. The argument applies also to fluids such that the stress tensor has, in addition to the linear part, a nonlinear stress tensor $\mathbf{S}(\mathbf{Du}) = \mathbf{S}_{p,\delta}(\mathbf{Du})$ with a (p, δ)-structure (see Section 2). We recall that the stress tensor

$$\mathcal{S}_{p,\delta}(\mathbf{Du}) := (\delta + |\mathbf{Du}|)^{p-2}\mathbf{Du} \qquad \delta \geq 0,\ 1 < p < \infty, \tag{1.4}$$

is just the prototypical example of a stress tensor with (p, δ)-structure. These results are reviewed for instance in [16, Sec. 2.2.1(c)].

A similar approach has been recently used by Gasiński and Winkert [18] to treat the following anisotropic double-phase scalar problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(\mathbf{x})|\nabla u|^{q-2}\nabla u\right) &= F(\mathbf{x}, u, \nabla u) \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

in the case $p = 2 < q < 3$, with $\mu \geq 0$ Lipschitz continuous, under appropriate growth conditions on f. Our results improve also those in [18], since we consider a vector valued problem, with the constraint of divergence-free, and with term $\mathbf{F}(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u}) = \mathbf{f}(\mathbf{x}) - (\mathbf{u}\cdot\nabla)\,\mathbf{u}$.

In addition to the results above, we observe that the problem of uniqueness for non-Newtonian fluids becomes more complex when the linear part is missing (that is system (1.3) with $\nu_0 = 0$ and, in the case $p > 2$, also $\delta = 0$), since one cannot take advantage of the classical estimates. In fact, the convective term is still quadratic as in (1.1); this has to be balanced in some way by a non-quadratic term. Moreover, in the context also of electro-rheological fluids uniqueness for small (and smooth) solutions is proved in Crispo and Grisanti [10, 11], at least in the non-degenerate case.

Note that for a stress tensor with (p, δ)-structure (see precise definition in the Assumption 2.1) as in the example (1.4), it is well-known that the following point-wise estimate holds:

$$\big(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})\big) : (\mathbf{A} - \mathbf{B}) \geq c_1(\delta + |\mathbf{A}| + |\mathbf{A} - \mathbf{B}|)^{p-2}|\mathbf{A} - \mathbf{B}|^2,$$

for all $\mathbf{A}, \mathbf{B}$ symmetric matrices. For a review see for instance Růžička [24]. Hence, at least if $\delta > 0$ and $p > 2$, it follows that

$$\big(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})\big) : (\mathbf{A} - \mathbf{B}) \geq c_1\,\delta^{p-2}|\mathbf{A} - \mathbf{B}|^2,$$

which allows us to employ the same machinery.

On the other hand, the case $1 < p < 2$ presents some peculiarities as exploited in Blavier and Mikelić [6], which permits to prove uniqueness for small solutions if $\frac{9}{5} \leq p < 2$. (Note that $\frac{9}{5}$ is the critical value to apply the classical monotonicity argument and to use the solution itself as a test function in the weak formulation.) We will review these results and provide more details in Section 2.1, explaining how we improve them.

Here, we will then consider as basic example the following boundary value problem for a double-phase non-Newtonian fluid

$$\begin{aligned} -\operatorname{div}(|\mathbf{Du}|^{p-2}\mathbf{Du}) - \operatorname{div}(|\mathbf{Du}|^{q-2}\mathbf{Du}) + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla\pi &= \mathbf{f} \qquad && \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

with $1 < p < 2 < q$, proving uniqueness of small solutions (recall again that if at least one among p or q is equal to 2 the result is trivial). Our main aim is to consider the degenerate case and to exploit the interplay between the p-growth and the q-growth in double-phase problems, as a way to enforce uniqueness, which cannot be proved in the single-phase cases. In addition, a non-standard estimation of the convective term using its symmetries is also employed.

The main result we prove, see Theorem 3.1, is the uniqueness of small solutions, for $\frac{6}{5} < p < 2$ and $q = q(p)$ large enough. Then, some extensions to an anisotropic problem are proved in Theorem 4.2.

2 Notation and preliminary results

In the sequel, $\Omega \subset \mathbb{R}^3$ will be a smooth and bounded open set. As usual, we write $\mathbf{x} = (x_1, x_2, x_3) = (x', x_3)$, for all $\mathbf{x} \in \mathbb{R}^3$. If the boundary $\partial\Omega$ is at least of class $C^{0,1}$, then the normal unit vector $\mathbf{n}$ at the boundary is well defined. We recall that a domain is of class $C^{k,1}$, if for each point $P \in \partial\Omega$, there are local coordinates such that in these coordinates we have $P = \mathbf{0}$ and $\partial\Omega$ is locally described by a $C^{k,1}$-function, i.e., there exist $R_P,\ R'_P \in (0,\infty)$, $r_P \in (0,1)$ and a $C^{k,1}$-function $a_P : B^2_{R_P}(0) \to B^1_{R'_P}(0)$ such that

i) $\mathbf{x} \in \partial\Omega \cap (B^2_{R_P}(0) \times B^1_{R'_P}(0)) \iff x_3 = a_P(x')$,
ii) $\Omega_P := \{x \in \mathbb{R}^3 :\ x' \in B^2_{R_P}(0),\ a_P(x') < x_3 < a_P(x') + R'_P\} \subset \Omega$,
iii) $\nabla a_P(0) = \mathbf{0}$, and $\forall\, x' \in B^2_{R_P}(0) \quad |\nabla a_P(x')| < r_P$,

where $B^k_r(0)$ denotes the k-dimensional open ball with center 0 and radius $r > 0$.

For our analysis, we will use the customary Lebesgue $(L^p(\Omega), \|\,.\,\|_p)$ and Sobolev spaces $(W^{k,p}(\Omega), \|\,.\,\|_{k,p})$ of integer index $k \in \mathbb{N}$, with $1 \le p \le \infty$. As usual, $p' = \frac{p}{p-1}$ denotes the conjugate exponent. We do not distinguish between scalar and vector valued function spaces, we just use boldface for vectors and tensors. We recall that $L^p_0(\Omega)$ denotes the subspace with zero mean value, while $W^{1,p}_0(\Omega)$ is the closure of smooth and compactly supported functions with respect to the $\|\,.\,\|_{1,p}$ norm. We denote by $W^{-1,p'}(\Omega) := (W^{1,p}_0(\Omega))^\star$ its dual space, with norm $\|\,.\,\|_{-1,p'}$.

If Ω is bounded and if $1 < p < \infty$, the following two relevant inequalities hold:
1) the Poincaré inequality

$$\exists\, C_P(p,\Omega) > 0 : \qquad \|\mathbf{u}\|_p \le C_P \|\nabla \mathbf{u}\|_p \qquad \forall\, \mathbf{u} \in W^{1,p}_0(\Omega); \tag{2.1}$$

2) the Korn inequality

$$\exists\, C_K(p,\Omega) > 0 : \qquad \|\nabla \mathbf{u}\|_p \le C_K \|\mathbf{D}\mathbf{u}\|_p \qquad \forall\, \mathbf{u} \in W^{1,p}_0(\Omega), \tag{2.2}$$

where $\mathbf{D}\mathbf{u}$ denotes the symmetric part of the matrix of derivatives $\nabla\mathbf{u}$.

For $1 \le p < 3$ we have, as a combination of (2.1)-(2.2), also the Sobolev-type inequality

$$\exists\, C_S > 0 : \qquad \|\mathbf{u}\|_{p^\star} \le C_S \|\mathbf{D}\mathbf{u}\|_p \qquad \forall\, \mathbf{u} \in W^{1,p}_0(\Omega),$$

where $p^\star := \frac{3p}{3-p}$.

When working with incompressible fluids it is natural to incorporate the divergence-free constraint directly in the definition of the function spaces. These spaces are built upon completing the space of solenoidal smooth functions with compact support (here denoted by $C^\infty_{0,\sigma}(\Omega)$) in an appropriate topology. For $1 < p < \infty$, we define

$$L^p_\sigma(\Omega) := \overline{\{\boldsymbol{\phi} \in C^\infty_{0,\sigma}(\Omega)\}}^{\|\boldsymbol{\phi}\|_p},$$
$$W^{1,p}_{0,\sigma}(\Omega) := \overline{\{\boldsymbol{\phi} \in C^\infty_{0,\sigma}(\Omega)\}}^{\|\nabla\boldsymbol{\phi}\|_p}.$$

For the nonlinear stress tensors, we make the following assumption of being with (p,δ)-structure, which is a generalization of the example from (1.4).

Assumption 2.1. *We assume that* $\mathbf{S}\colon \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ *belongs to* $C^0(\mathbb{R}^{3\times3}, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \cap C^1(\mathbb{R}^{3\times3} \setminus \{\mathbf{0}\}, \mathbb{R}^{3\times3}_{\mathrm{sym}})$*, satisfies* $\mathbf{S}(\mathbf{P}) = \mathbf{S}(\mathbf{P}^{\mathrm{sym}})$*, and* $\mathbf{S}(\mathbf{0}) = \mathbf{0}$*. Moreover, we assume that* $\mathbf{S}$ *has* (p,δ)-structure, *i.e., there exist* $p \in (1,\infty)$, $\delta \in [0,\infty)$*, and constants* $C_0, C_1 > 0$ *such that*

$$\sum\nolimits_{i,j,k,l=1}^{3} \partial_{kl} S_{ij}(\mathbf{P}) Q_{ij} Q_{kl} \ge C_0 \big(\delta + |\mathbf{P}^{\mathrm{sym}}|\big)^{p-2} |\mathbf{Q}^{\mathrm{sym}}|^2, \tag{2.3a}$$

$$\big|\partial_{kl} S_{ij}(\mathbf{P})\big| \le C_1 \big(\delta + |\mathbf{P}^{\mathrm{sym}}|\big)^{p-2}, \tag{2.3b}$$

are satisfied for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times3}$ *with* $\mathbf{P}^{\mathrm{sym}} \neq \mathbf{0}$ *and all* $i,j,k,l = 1,\ldots,3$. *The constants* C_0, C_1, *and* p *are called the* characteristics *of* $\mathbf{S}$.

Remark 2.1. We would like to emphasize that, if not otherwise stated, the constants depend only on the characteristics of $\mathbf{S}$, but are independent of $\delta \geq 0$.

Defining for $t \geq 0$ a special N-function φ by

$$\varphi(t) := \int_0^t \varphi'(s)\,ds \qquad \text{with} \quad \varphi'(t) := (\delta + t)^{p-2}t\,, \tag{2.4}$$

we can replace $C_i\left(\delta + |\mathbf{P}^{\mathrm{sym}}|\right)^{p-2}$ on the right-hand side of (2.3a) and (2.3b) by $\widetilde{C}_i\,\varphi''\left(|\mathbf{P}^{\mathrm{sym}}|\right)$, for $i = 0, 1$. Next, the shifted functions are defined for $t \geq 0$ by

$$\varphi_a(t) := \int_0^t \varphi_a'(s)\,ds \qquad \text{with} \quad \varphi_a'(t) := \varphi'(a+t)\frac{t}{a+t}.$$

In the following proposition, we recall several useful results, which will be frequently used in the paper. The proofs of these results and more details can be found in [4, 13, 14, 27]. Many inequalities can be written in a compact form by means of the following tensor valued function

$$\mathbf{F}(\mathbf{A}) := \left(\delta + |\mathbf{A}^{\mathrm{sym}}|\right)^{\frac{p-2}{2}}\mathbf{A}^{\mathrm{sym}}\,, \tag{2.5}$$

and we write $f \sim g$ if there exist constants $c_1, c_2 > 0$ such that $c_1 g \leq f \leq c_2 f$.

Proposition 2.1. *Let* $\mathbf{S}$ *satisfy Assumption 2.1, let* φ *be defined in* (2.4), *and let* $\mathbf{F}$ *be defined in* (2.5).

(i) *For all* $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}$

$$\begin{aligned}
\big(\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})\big) : \big(\mathbf{P} - \mathbf{Q}\big) &\sim \left|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})\right|^2,\\
&\sim \varphi_{|\mathbf{P}^{\mathrm{sym}}|}(|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|),\\
&\sim \varphi''\left(|\mathbf{P}^{\mathrm{sym}}| + |\mathbf{Q}^{\mathrm{sym}}|\right)|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|^2,\\
\mathbf{S}(\mathbf{Q}) : \mathbf{Q} &\sim |\mathbf{F}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}^{\mathrm{sym}}|),\\
|\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})| &\sim \varphi'_{|\mathbf{P}^{\mathrm{sym}}|}\left(|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|\right),
\end{aligned}$$

and the constants depend only on the characteristics of $\mathbf{S}$.

(ii) *For all* $\epsilon > 0$, *there exists a constant* $c_\epsilon > 0$ *(depending only on* $\epsilon > 0$ *and on the characteristics of* $\mathbf{S}$*) such that for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W^{1,p}(\Omega)$

$$\begin{aligned}
\big(\mathbf{S}(\mathbf{Du}) - \mathbf{S}(\mathbf{Du}), \mathbf{Dw} - \mathbf{Du}\big) \leq \epsilon\,\|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Du})\|_2^2&\\
+ c_\epsilon\,\|\mathbf{F}(\mathbf{Dw}) - \mathbf{F}(\mathbf{Du})\|_2^2\,.&
\end{aligned}$$

Since the range of the allowed $p \in (1, 2)$ will play a relevant role, we first recall that the restriction $p > 6/5$ is quite natural for the problem, at least for what concerns existence of weak solutions. In the case $\nu_0 = 0$, the weak formulation of (1.3) is in fact: find $\mathbf{u} \in W^{1,p}_{0,\sigma}(\Omega)$ such that

$$\int_\Omega \mathbf{S}_{p,\delta}(\mathbf{Du}) : \mathbf{D}\boldsymbol{\phi} - \mathbf{u} \otimes \mathbf{u} : \nabla\boldsymbol{\phi}\,\mathrm{d}\mathbf{x} = \langle \mathbf{f}, \boldsymbol{\phi}\rangle \qquad \forall\,\boldsymbol{\phi} \in C^\infty_{0,\sigma}(\Omega).$$

To properly define the quadratic term, one needs (at least) that $\mathbf{u} \in L^2_{loc}(\Omega)$, which follows, for instance, if $\mathbf{u} \in W^{1,p}(\Omega)$ for $p \geq \frac{6}{5}$. The basic a priori estimate obtained testing with $\mathbf{u}$ itself shows that, if $\mathbf{f} \in W^{-1,p'}(\Omega)$, then $\|\mathbf{Du}\|_p \leq C$. Next, Korn inequality and the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^2(\Omega)$ (which holds for $p \geq 6/5$ in three space-dimensions) give that $\mathbf{u} \otimes \mathbf{u} \in L^1(\Omega)$. This restriction on p is intrinsic to the problem, due to the growth of the convective term. The limiting case $p = 6/5$ is excluded from the existence theorem due to some technical compactness arguments which are used in the proof.

The following existence result holds true.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be smooth and bounded, let $\nu_0 = 0$, and let the stress tensor $\mathbf{S}_{p,\delta}$ satisfy Assumption 2.1 for some $\delta \geq 0$ and for some $p > 6/5$. Then, for any given $\mathbf{f} \in W^{-1,p'}(\Omega)$, there exists at least a weak solution $\mathbf{v} \in W^{1,p}_{0,\sigma}(\Omega)$ of problem* (1.3), *which satisfies the estimate*

$$\|\mathbf{Du}\|_p^p \leq C\|\mathbf{f}\|_{-1,p'}^{p'}, \tag{2.6}$$

for some constant C depending on the domain and on the characteristics of the stress tensor.

The proof of existence of weak solutions requires a precise use of the monotonicity of the operator, but for $p < \frac{9}{5}$ the usual Browder-Minty approach (as employed by Lions [21] and Ladyžhenskaya [20] also for values of $p > 3$) is not directly applicable and one has to resort to more technical arguments with bounded or even Lipschitz truncation, plus an appropriate divergence correction, see the reviews in Breit [7] and Růžička [24].

Here, we do not discuss the existence of weak solutions, which easily follows also for the problem (1.6), by employing similar arguments without any relevant change. Hence, we can state the following theorem.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^3$ be smooth and bounded, let the stress tensors $\mathbf{S}_{p,\delta_p}(\mathbf{Du})$, $\mathbf{S}_{q,\delta_q}(\mathbf{Du})$ satisfy Assumption 2.1 for some $\delta_p \geq 0$, $p > 6/5$ and for some $\delta_q \geq 0$, $q > p$. Then, for all $\mathbf{f} \in W^{-1,q'}(\Omega)$, the problem*

$$\begin{aligned} -\operatorname{div} \mathbf{S}_{p,\delta_p}(\mathbf{Du}) - \operatorname{div} \mathbf{S}_{q,\delta_q}(\mathbf{Du}) + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla\pi &= \mathbf{f} && \text{in } \Omega,\\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega,\\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned} \tag{2.7}$$

has at least a weak solution $\mathbf{u} \in W^{1,q}_{0,\sigma}(\Omega)$, which satisfies

$$\int_\Omega \mathbf{S}_{p,\delta_p}(\mathbf{Du}) : \mathbf{D}\boldsymbol{\phi} + \mathbf{S}_{q,\delta_q}(\mathbf{Du}) : \mathbf{D}\boldsymbol{\phi} - \mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\phi}\,\mathrm{d}\mathbf{x} = \langle \mathbf{f}, \boldsymbol{\phi}\rangle \qquad \forall\,\boldsymbol{\phi} \in C^\infty_{0,\sigma}(\Omega),$$

and which satisfies the following inequality:

$$\|\mathbf{Du}\|_p^p + \|\mathbf{Du}\|_q^q \leq C\|\mathbf{f}\|_{-1,q'}^{q'}, \tag{2.8}$$

for some constant C depending only on the characteristics of $\mathbf{S}_q$ and Ω.

2.1 On problems without linear part: some known uniqueness results

When considering the problem (1.3) in the case $\nu_0 = \delta = 0$ and $p > 2$, the problem of uniqueness (even of small solutions) is completely open, since the (sharp) estimates from Proposition 2.1 seem to be not suitable to “absorb” the convective term. This is determined by a “gap” between the powers in the lower bound for the stress tensors and those in the upper bound for the convective term.

On the other hand, in the case $\nu_0 = 0$ and for some $p < 2$, the following result of uniqueness for weak solutions is proved in Blavier and Mikelić [6], under appropriate smallness of both weak solutions.

Theorem 2.3. *Let $\mathbf{u}_i \in W^{1,p}_{0,\sigma}(\Omega)$ be weak solutions of* (1.3) *in the case $\nu_0 = 0$ and let $p \in [9/5, 2[$. Then, there exists a constant $\epsilon_0 > 0$ (which depends on p and on the data of the problem) such that, if $\|\nabla\mathbf{u}_i\|_p \leq \epsilon_0$, then $\mathbf{u}_1 = \mathbf{u}_2$.*

The available proofs of uniqueness go through considering the equation satisfied by the difference and testing with the difference of the solutions, $\mathbf{U} := \mathbf{u}_1 - \mathbf{u}_2$. To this end, one needs to be able to rigorously write the integrals

$$\int_\Omega \big((\mathbf{u}_1\cdot\nabla)\,\mathbf{u}_1 - (\mathbf{u}_2\cdot\nabla)\,\mathbf{u}_2\big)\cdot(\mathbf{u}_1 - \mathbf{u}_2)\,\mathrm{d}\mathbf{x} = \int_\Omega \big((\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla\big)\,\mathbf{u}_1\cdot(\mathbf{u}_1 - \mathbf{u}_2)\,\mathrm{d}\mathbf{x},$$

(where the equality derives from integration by parts and from the divergence-free constraint) and to prove proper upper bounds (similar to (1.2), already used in the Newtonian case). By using the information $\mathbf{u}_i \in W^{1,p}_{0,\sigma}(\Omega)$, and by the Hölder inequality, we get that if

$$\frac{2}{p^\star} + \frac{1}{p} = \frac{2}{p} - \frac{2}{3} + \frac{1}{p} \leq 1 \quad \Longleftrightarrow \quad p \geq \frac{9}{5},$$

then the following estimate holds:

$$\left| \int_\Omega (\mathbf{U} \cdot \nabla)\, \mathbf{u}_1 \cdot \mathbf{U}\, \mathrm{d}\mathbf{x} \right| \leq \|\mathbf{U}\|^2_{p^\star} \|\nabla \mathbf{u}_1\|_p. \tag{2.9}$$

Hence, by using Sobolev and Korn inequalities, we get

$$\left| \int_\Omega (\mathbf{U} \cdot \nabla)\, \mathbf{u}_1 \cdot \mathbf{U}\, \mathrm{d}\mathbf{x} \right| \leq C(p, \Omega) \|\mathbf{D}\mathbf{U}\|^2_p \|\nabla \mathbf{u}_1\|_p.$$

The uniqueness result from [6] exploits this estimation, together with the lower bound

$$\int_\Omega \big(\mathbf{S}_{p,0}(\mathbf{D}\mathbf{u}_1) - \mathbf{S}_{p,0}(\mathbf{D}\mathbf{u}_2)\big) : \mathbf{D}(\mathbf{u}_1 - \mathbf{u}_2)\, \mathrm{d}\mathbf{x} \geq c(p) \frac{\|\mathbf{D}\mathbf{u}_1 - \mathbf{D}\mathbf{u}_2\|^2_p}{\|\mathbf{D}\mathbf{u}_1\|^{2-p}_p + \|\mathbf{D}\mathbf{u}_2\|^{2-p}_p}, \tag{2.10}$$

valid for all $1 < p < 2$, for some $c(p) > 0$, when $\mathbf{u}_i \in W^{1,p}_0(\Omega)$.

If the smallness is assumed on one solution, namely, if

$$\|\nabla \mathbf{u}_1\|_p \leq \epsilon_0,$$

then the estimates for the convective term and the stress tensors imply that (for any $\frac{9}{5} \leq p < 2$),

$$c(p) \frac{\|\mathbf{D}\mathbf{U}\|^2_p}{2 \max_{i=1,2} \|\mathbf{D}\mathbf{u}_i\|^{2-p}_p} \leq C\epsilon_0 \|\mathbf{D}\mathbf{U}\|^2_p.$$

From this inequality, by using (2.6), one gets

$$c(p, \|\mathbf{f}\|_{-1,p'}) \|\mathbf{D}\mathbf{U}\|^2_p \leq C\epsilon_0 \|\mathbf{D}\mathbf{U}\|^2_p, \tag{2.11}$$

and uniqueness will follow if $\epsilon_0 > 0$ is small enough, that is, if $\epsilon_0 < \frac{c(p, \|\mathbf{f}\|_{-1,p'})}{C}$.

3 On the double-phase problem

In this section, we consider the problem (2.7) with $\delta_p = \delta_q = 0$ and we observe that –on one hand– the "good" estimates (lower bound) for the stress tensor of the difference are valid if $1 < p < 2$, hence one would like to have one phase with this properties; On the other hand, to handle the convective term larger (than 2) values of the exponent q will provide the estimates needed to rigorously define the integrals involving the convective term.

First, we have the existence Theorem 2.2 for weak solutions. Next, we observe that the same argument as in [6] can be directly adapted to the problem (2.7) (of which system (1.6) represents a particular case), to produce the following elementary –but original– uniqueness result.

Proposition 3.1. *Let the same assumption as in Theorem 2.2 be satisfied. Let be given* $\frac{6}{5} < p < \frac{9}{5}$ *and let* $\mathbf{f} \in W^{-1,q'}(\Omega)$, *for some* $q > 2$ *such that*

$$q \geq \frac{3p}{5p - 6}. \tag{3.1}$$

Let $\mathbf{u}_1, \mathbf{u}_2 \in W^{1,q}_{0,\sigma}(\Omega)$ *be weak solutions to* (1.6) *corresponding to the same* $\mathbf{f}$. *Then, there exists* $\epsilon_0 = \epsilon_0(q, \Omega, \|\mathbf{f}\|_{-1,q'})$ *such that if*

$$\|\nabla \mathbf{u}_1\|_q \leq \epsilon_0, \tag{3.2}$$

then $\mathbf{u}_1 = \mathbf{u}_2$.

Remark 3.1. *In the range* $\frac{6}{5} < p < \frac{9}{5}$ *the inequality* $\frac{3p}{5p-6} > \frac{9}{5}$ *holds and, to have results which are not included in Theorem 2.3, we need to require* $q > 2$; *next, when* $p > \frac{6}{5}$, *the following inequality holds:*

$$\frac{3p}{5p-6} \leq 2 \quad \text{for} \quad p \geq \frac{12}{7} \sim 1.714\ldots$$

In the light of the above observations, Proposition 3.1 can be restated as follows: The uniqueness of weak solutions can be proven under condition (3.2) *for some* q *such that*

$$\begin{aligned} q &\geq \frac{3p}{5p-6} && \text{in the case} \quad \frac{6}{5} < p < \frac{12}{7}, \\ q &> 2 && \text{in the case} \quad \frac{12}{7} \leq p < \frac{9}{5}. \end{aligned}$$

Proof of Proposition 3.1. The proof is a simple adaption of results in [6]. Taking the difference of two solutions $\mathbf{u}_i \in W^{1,q}_{0,\sigma}(\Omega)$ corresponding to the same force $\mathbf{f} \in W^{-1,q'}(\Omega)$, one gets for the difference $\mathbf{U}$ the following estimates:

$$\begin{aligned} c(p)\frac{\|\mathbf{DU}\|_p^2}{\|\mathbf{Du}_1\|_p^{2-p} + \|\mathbf{Du}_2\|_p^{2-p}} &\leq \left| \int_\Omega (\mathbf{U}\cdot\nabla)\,\mathbf{u}_1 \cdot \mathbf{U}\,d\mathbf{x} \right| \\ &\leq \|\mathbf{U}\|_{p^\star}^2 \|\nabla \mathbf{u}_1\|_q \\ &\leq C\|\mathbf{DU}\|_p^2 \|\nabla \mathbf{u}_1\|_q, \end{aligned}$$

obtained by using Hölder, Korn, and Sobolev inequalities, and also (2.10). The calculations are justified if

$$\frac{2}{p^\star} + \frac{1}{q} \leq 1,$$

which follows if q is as in (3.1). Then, if one assumes the smallness of the L^q-norm of the gradient or $\mathbf{u}_1$, the following estimate holds:

$$c(p)\frac{\|\mathbf{DU}\|_p^2}{\|\mathbf{Du}_1\|_p^{2-p} + \|\mathbf{Du}_2\|_p^{2-p}} \leq C\epsilon_0 \|\mathbf{DU}\|_p^2,$$

which allows again to conclude uniqueness (exactly as in the previous case) by the same argument developed in [6]. □

In order to improve this result, here we follow a slightly different approach which is based on more precise point-wise estimates for the stress tensor corresponding to the “p-phase.” This will allow us to require less restrictive conditions on q. The main result we prove is the following one:

Theorem 3.1. *Let* $\delta_p \geq 0$ *and* $\delta_q = 0$. *Let* $p \in]6/5, 12/7[$ *and* $q > 2$ *such that*

$$\begin{aligned} &(i) && q > \frac{3p(2-p)}{5p-6} && \text{for} \quad \frac{6}{5} < p \leq \frac{\sqrt{33}-3}{2}, \\ &(ii) && q > \frac{3p(4-p)}{7p-6} && \text{for} \quad \frac{\sqrt{33}-3}{2} < p \leq \frac{\sqrt{37}-1}{3}, \\ &(iii) && q > 2 && \text{for} \quad \frac{\sqrt{37}-1}{3} < p < \frac{12}{7}. \end{aligned} \tag{3.3}$$

Let $\mathbf{u}_1, \mathbf{u}_2$ *be weak solutions of* (2.7) *corresponding to the same* $\mathbf{f} \in W^{-1,q'}(\Omega)$. *Then, there exists a constant* $\epsilon_0 = \epsilon_0(p, q, \Omega, \|\mathbf{f}\|_{-1,q'}) > 0$ *such that if at least one solution satisfies*

$$\|\nabla \mathbf{u}_1\|_q \leq \epsilon_0,$$

then $\mathbf{u}_1 = \mathbf{u}_2$.

Before proving the theorem, we observe that the significant case is when the "q-phase" is such that $\delta_q = 0$. Also the possible degeneracy $\delta_p = 0$ of the p-phase improves previously known results. Moreover, we think that an interesting further development would be the study of the uniqueness for a single-phase fluid by using the results of "higher regularity" from Crispo and Maremonti [12]. This will hold provided that: a) the results from [12] can be adapted to the non-modified p-Stokes system; and b) an explicit estimation of some constants related with singular integrals is available.

We make some remarks to explain the improvements in the various ranges, with respect to the results in Proposition 3.1. A first significant difference comes into play if q is smaller or larger than the space dimension, since this will imply estimates involving the Sobolev exponent $q^\star = \frac{3q}{3-q}$ or, alternatively, estimates in $L^\infty(\Omega)$.

Remark 3.2. *Concerning the case* (i) *in* (3.3)

$$\frac{6}{5} < p \leq \frac{\sqrt{33}-3}{2} \sim 1.2 < p \leq 1.37228\dots,$$

we can see that the restriction on the range of p is made to enforce $q > 3$, since the calculations leading to this estimate are valid for $q > 3$ (cf. the proof of Lemma 3.2).

Moreover, we also have that

$$\frac{3p(2-p)}{5p-6} < \frac{3p}{5p-6} \qquad \textit{within the range} \quad \frac{6}{5} < p \leq \frac{\sqrt{33}-3}{2},$$

hence the condition (i) *is less restrictive than* (3.1), *coming from Proposition 3.1 (in the same range of p).*

Remark 3.3. *Concerning the case* (ii) *in* (3.3)

$$\frac{\sqrt{33}-3}{2} < p \leq \frac{\sqrt{37}-1}{3} \sim 1.37228\cdots < p \leq 1.69425\dots,$$

the restrictions are made to impose that $2 \leq \frac{3p(4-p)}{7p-6} < 3$, hence to make possible the choice of some $2 < q < 3$. The lower bound on q is requested to have a non-trivial result, and the upper bound is requested to use Lemma 3.2.

Moreover, we have again

$$\frac{3p(4-p)}{7p-6} < \frac{3p}{5p-6}, \qquad \textit{within the range} \quad \frac{\sqrt{33}-3}{2} < p \leq \frac{\sqrt{37}-1}{3},$$

and this condition is less restrictive than that from Proposition 3.1 (in the same range of p).

Remark 3.4. *The restriction $p < 12/7$ is already present in Proposition 3.1 since beyond that value the result is included in Proposition 3.1. Again, the condition* (iii) *is less restrictive than the one previously obtained, since the following inequality holds:*

$$\frac{3p}{5p-6} > 2, \qquad \textit{within the range} \quad \frac{\sqrt{37}-1}{3} < p < \frac{12}{7}.$$

The proof of Theorem 3.1 is heavily based on the following inequality coming from an application of Proposition 2.1:

$$\int_\Omega \big(\mathbf{S}_{p,\delta_p}(\mathbf{Du}_1) - \mathbf{S}_{p,\delta_p}(\mathbf{Du}_2)\big) : \mathbf{DU}\,\mathrm{d}\mathbf{x} \geq c \int_\Omega \big(\delta_p + |\mathbf{Du}_1| + |\mathbf{DU}|\big)^{p-2} |\mathbf{DU}|^2\,\mathrm{d}\mathbf{x},$$

and then using the following two lemmas about the convective term.

The estimation of the tri-linear term we use is not the direct one as in (2.9). We bound the convective term exploiting some of its symmetries. This fact is relevant since in the stress tensors present in the equations (1.6) the velocity enters only through the symmetric gradient. We first have a result which follows by integrating by parts:

Lemma 3.1. *The following equalities hold:*

$$\int_\Omega (\mathbf{u}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{u}\,d\mathbf{x} = \int_\Omega \mathbf{u}\cdot\mathbf{D}\mathbf{u}_1\cdot\mathbf{u}\,d\mathbf{x} = -2\int_\Omega \mathbf{u}\cdot\mathbf{D}\mathbf{u}\cdot\mathbf{u}_1\,d\mathbf{x} \qquad \forall\,\mathbf{u},\mathbf{u}_1\in C^\infty_{0,\sigma}(\Omega),$$

where

$$\mathbf{v}\cdot\mathbf{A}\cdot\mathbf{w} := \sum_{ij=1}^{3} v^i A^{ij} w^j \qquad \forall\,\mathbf{v},\mathbf{w}\in\mathbb{R}^3,\ \mathbf{A}\in\mathbb{R}^{3\times3}.$$

Proof. The proof of the first equality is simply obtained by interchanging the dummy variable in the double summation

$$(\mathbf{u}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{u} = \sum_{i,j=1}^{3} u^i\partial_i u_1^j u^j,$$

and using that $\mathbf{D}\mathbf{u}_1$ is, by definition, a symmetric tensor.

The second equality follows by integrating by parts: due to the vanishing of the boundary trace of $\mathbf{u}$ we obtain

$$\int_\Omega (\mathbf{u}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{u}\,d\mathbf{x} = -\int_\Omega (\mathbf{u}\cdot\nabla)\,\mathbf{u}\cdot\mathbf{u}_1\,d\mathbf{x}.$$

Then adding the null term

$$-\int_\Omega \mathbf{u}\cdot[\nabla\,\mathbf{u}]^T\cdot\mathbf{u}_1\,d\mathbf{x} = -\int_\Omega u^j\partial_i u^j u_1^i\,d\mathbf{x} = -\frac{1}{2}\int_\Omega \partial_i|\mathbf{u}|^2 u_1^i\,d\mathbf{x} = 0,$$

the second identity follows.

The same result clearly holds also for $\mathbf{u},\mathbf{u}_1$ in spaces in which $C^\infty_{0,\sigma}(\Omega)$ is dense, provided that the integrals are well-defined (as an example Lemma 3.1 is valid for $\mathbf{u},\mathbf{u}_1\in W^{1,q}_{0,\sigma}(\Omega)$, with $q\ge 9/5$). □

The nonlinear term is now estimated with the following inequalities:

Lemma 3.2. *Let $\mathbf{U},\mathbf{u}_1\in W^{1,q}_{0,\sigma}(\Omega)$ for some $q\ge\frac{9}{5}$ and let $\delta>0$ be any positive number. If $q<3$, (hence if $W^{1,q}(\Omega)\subset L^{q^*}(\Omega)$ for $q^\star=\frac{3q}{3-q}<\infty$), then the following inequality holds:*

$$\Big|\int_\Omega (\mathbf{U}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{U}\,d\mathbf{x}\Big| \le 2\|\mathbf{U}\|_{p^\star}\big\|(\delta+|\mathbf{D}\mathbf{u}_1|+|\mathbf{D}\mathbf{U}|)^{\frac{p-2}{2}}\mathbf{D}\mathbf{U}\big\|_2\|\mathbf{u}_1\|_{q^\star}\big\|(\delta+|\mathbf{D}\mathbf{u}_1|+|\mathbf{D}\mathbf{U}|)^{\frac{2-p}{2}}\big\|_{\frac{2q}{2-p}},$$

with

$$q\ge\frac{3p(4-p)}{7p-6};$$

Observe that $\frac{3p(4-p)}{7p-6}<3$, for $\frac{\sqrt{33}-3}{2}\le p<2$.

If $q>3$, then $W^{1,q}(\Omega)\subset L^\infty(\Omega)$, and the following estimate holds:

$$\Big|\int_\Omega (\mathbf{U}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{U}\,d\mathbf{x}\Big| \le 2\|\mathbf{U}\|_{p^\star}\big\|(\delta+|\mathbf{D}\mathbf{u}_1|+|\mathbf{D}\mathbf{U}|)^{\frac{p-2}{2}}\mathbf{D}\mathbf{U}\big\|_2\|\mathbf{u}_1\|_\infty\big\|(\delta+|\mathbf{D}\mathbf{u}_1|+|\mathbf{D}\mathbf{U}|)^{\frac{2-p}{2}}\big\|_{\frac{2q}{2-p}},$$

with

$$q\ge\frac{3p(2-p)}{5p-6};$$

observe that $\frac{3p(2-p)}{5p-6}>3$, for $\frac{6}{5}<p<\frac{\sqrt{33}-3}{2}$.

Finally, if $q=3$, then $W^{1,q}(\Omega)\subset L^s(\Omega)$ for all $s<\infty$, and the following estimate holds:

$$\Big|\int_\Omega (\mathbf{U}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{U}\,d\mathbf{x}\Big| \le 2\|\mathbf{U}\|_{p^\star}\big\|(\delta+|\mathbf{D}\mathbf{u}_1|+|\mathbf{D}\mathbf{U}|)^{\frac{p-2}{2}}\mathbf{D}\mathbf{U}\big\|_2\|\mathbf{u}_1\|_s\big\|(\delta+|\mathbf{D}\mathbf{u}_1|+|\mathbf{D}\mathbf{U}|)^{\frac{2-p}{2}}\big\|_{\frac{2q}{2-p}},$$

with

$$q>\frac{3p(2-p)}{5p-6}.$$

Proof. We observe that the integral $\int_\Omega (\mathbf{U}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{U}\,\mathrm{d}\mathbf{x}$ is finite by (2.9) since all functions belong to $W^{1,q}_{0,\sigma}(\Omega)$, hence all calculations we perform are completely justified.

We start from the case $q<3$ and we observe that the estimate involves a "natural quantity" related to a stress tensor with (p,δ)-structure. We use directly Lemma 3.1, and since $\delta>0$, we can freely multiply and divide by $\big(\delta+|\mathbf{Du}_1|+|\mathbf{DU}|\big)^{\frac{2-p}{2}}\neq 0$. We use the Hölder inequality which is valid if

$$\frac{1}{p^\star}+\frac{1}{2}+\frac{1}{q^\star}+\frac{2-p}{2q}\le 1,$$

and Sobolev inequalities to get

$$\begin{aligned}-\int_\Omega(\mathbf{U}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{U}\,\mathrm{d}\mathbf{x}&=2\int_\Omega \mathbf{U}\cdot\mathbf{DU}\cdot\mathbf{u}_1\,\mathrm{d}\mathbf{x}\\&=2\int_\Omega \mathbf{U}\cdot\big(\delta+|\mathbf{Du}_1|+|\mathbf{DU}|\big)^{\frac{p-2}{2}}\mathbf{DU}\cdot\mathbf{u}_1\big(\delta+|\mathbf{Du}_1|+|\mathbf{DU}|\big)^{\frac{2-p}{2}}\,\mathrm{d}\mathbf{x}\\&\le 2\|\mathbf{U}\|_{p^\star}\big\|\big(\delta+|\mathbf{Du}_1|+|\mathbf{DU}|\big)^{\frac{p-2}{2}}\mathbf{DU}\big\|_2\|\mathbf{u}_1\|_{q^\star}\big\|\big(\delta+|\mathbf{Du}_1|+|\mathbf{DU}|\big)^{\frac{2-p}{2}}\big\|_{\frac{2q}{2-p}}.\end{aligned}$$

The proof in the case $q>3$ follows more or less the same lines and if the following inequality is satisfied

$$\frac{1}{p^\star}+\frac{1}{2}+\frac{2-p}{2q}\le 1,$$

we can write

$$\begin{aligned}-\int_\Omega(\mathbf{U}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{U}\,\mathrm{d}\mathbf{x}&=2\int_\Omega \mathbf{U}\cdot\mathbf{DU}\cdot\mathbf{u}_1\,\mathrm{d}\mathbf{x}\\&=2\int_\Omega \mathbf{U}\cdot\big(\delta+|\mathbf{Du}_1|+|\mathbf{DU}|\big)^{\frac{p-2}{2}}\mathbf{DU}\cdot\mathbf{u}_1\big(\delta+|\mathbf{Du}_1|+|\mathbf{DU}|\big)^{\frac{2-p}{2}}\,\mathrm{d}\mathbf{x}\\&\le 2\|\mathbf{U}\|_{p^\star}\big\|\big(\delta+|\mathbf{Du}_1|+|\mathbf{DU}|\big)^{\frac{p-2}{2}}\mathbf{DU}\big\|_2\|\mathbf{u}_1\|_{\infty}\big\|\big(\delta+|\mathbf{Du}_1|+|\mathbf{DU}|\big)^{\frac{2-p}{2}}\big\|_{\frac{2q}{2-p}}.\end{aligned}$$

□

We can now give the proof of the main result of this paper.

Proof of Theorem 3.1. We write the difference between the two weak solutions of (2.7) and we use $\mathbf{U}=\mathbf{u}_1-\mathbf{u}_2\in W^{1,q}_{0,\sigma}(\Omega)$ as test function to get

$$\begin{aligned}&\int_\Omega\big(\mathbf{S}_{p,\delta_p}(\mathbf{Du}_1)-\mathbf{S}_{p,\delta_p}(\mathbf{Du}_2)\big):\mathbf{D}(\mathbf{u}_1-\mathbf{u}_2)\,\mathrm{d}\mathbf{x}\\&\qquad+\int_\Omega\big(\mathbf{S}_{q,0}(\mathbf{Du}_1)-\mathbf{S}_{q,0}(\mathbf{Du}_2)\big):\mathbf{D}(\mathbf{u}_1-\mathbf{u}_2)\,\mathrm{d}\mathbf{x}=\int_\Omega(\mathbf{U}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{U}\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Since the operator $\mathbf{S}_{q,0}(\cdot)$ is monotone, we get

$$\int_\Omega\big(\mathbf{S}_{p,\delta_p}(\mathbf{Du}_1)-\mathbf{S}_{p,\delta_p}(\mathbf{Du}_2)\big):\mathbf{DU}\,\mathrm{d}\mathbf{x}\le\Big|\int_\Omega(\mathbf{U}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{U}\,\mathrm{d}\mathbf{x}\Big|,\tag{3.4}$$

and the right-hand side is finite due to the fact that $q>2$.

We now estimate the left-hand side by Proposition 2.1 and, for $\mathbf{u}_i$ at least in $W^{1,p}_{0,\sigma}(\Omega)$, the following inequality holds:

$$\int_\Omega\big(\delta_p+|\mathbf{Du}_1|+|\mathbf{DU}|\big)^{p-2}|\mathbf{DU}|^2\,\mathrm{d}\mathbf{x}\le C\int_\Omega\big(\mathbf{S}_{p,\delta_p}(\mathbf{Du}_1)-\mathbf{S}_{p,\delta_p}(\mathbf{Du}_2)\big):\mathbf{DU}\,\mathrm{d}\mathbf{x}.$$

Next, the last term from the right-hand side of the inequalities proved in Lemma 3.2 can be estimated for $\delta = \delta_p$ by observing that since

$$\text{for } \alpha \in (0,1) \qquad (x+y)^\alpha \leq x^\alpha + y^\alpha \qquad \forall\, x, y \geq 0,$$

by the Minkowski inequality the following holds:

$$\begin{aligned}\big\|\big(\delta_p + |\mathbf{Du}_1| + |\mathbf{DU}|\big)^{\frac{2-p}{2}}\big\|_{\frac{2q}{2-p}} &\leq \|\delta_p^{\frac{2-p}{2}}\|_{\frac{2q}{2-p}} + \||\mathbf{Du}_1|^{\frac{2-p}{2}}\|_{\frac{2q}{2-p}} + \||\mathbf{DU}|^{\frac{2-p}{2}}\|_{\frac{2q}{2-p}} \\ &\leq \delta_p^{\frac{2-p}{2}}|\Omega|^{\frac{2-p}{2q}} + \|\mathbf{Du}_1\|_q^{\frac{2-p}{2}} + \|\mathbf{DU}\|_q^{\frac{2-p}{2}} \\ &\leq \delta_p^{\frac{2-p}{2}}|\Omega|^{\frac{2-p}{2q}} + 2\|\mathbf{Du}_1\|_q^{\frac{2-p}{2}} + \|\mathbf{Du}_2\|_q^{\frac{2-p}{2}},\end{aligned}$$

and, if $\delta_p \in [0, \delta_0]$ for some $\delta_0 > 0$, then

$$\begin{aligned}\big\|\big(\delta_p + |\mathbf{Du}_1| + |\mathbf{DU}|\big)^{\frac{2-p}{2}}\big\|_{\frac{2q}{2-p}} &\leq C(\delta_0, p, q, |\Omega|)(1 + \max_{i=1,2}\|\mathbf{Du}_i\|_q^{\frac{2-p}{2}}) \\ &\leq C(\delta_0, p, q, |\Omega|)\Big[1 + \|\mathbf{f}\|_{-1,q'}^{\frac{2-p}{2(q-1)}}\Big].\end{aligned}$$

Hence, in the case of the problem with a non-degenerate stress tensor $\mathbf{S}_{p,\delta_p}$, that is if $\delta_p > 0$, by collecting the lower bound for the left-hand side of (3.4) and the upper bound for the right-hand side (using Lemma 3.2 with $\delta = \delta_p$) and simplifying similar terms, we get

$$\left[\int_\Omega \big(\delta_p + |\mathbf{Du}_1| + |\mathbf{DU}|\big)^{p-2}|\mathbf{DU}|^2\,d\mathbf{x}\right]^{1/2} \leq 2C\|\mathbf{Du}\|_p\|\nabla\mathbf{u}_1\|_q\|\mathbf{f}\|_{-1,q'}^{\frac{2-p}{2(q-1)}}.$$

We use now the hypothesis $\|\nabla\mathbf{u}_1\|_q \leq \epsilon_0$ and, after squaring both sides,w we get

$$\int_\Omega \big(\delta_p + |\mathbf{Du}_1| + |\mathbf{DU}|\big)^{p-2}|\mathbf{DU}|^2\,d\mathbf{x} \leq C\epsilon_0^2\|\mathbf{Du}\|_p^2,$$

where C depends on the data of the problem, since we used the a priori estimate in $W_0^{1,q}(\Omega)$ for both solutions. The proof follows now as in the previous case. In fact, by using (2.10) and (2.11), we get

$$c(p, \|\mathbf{f}\|_{-1,q'})\|\mathbf{DU}\|_p^2 \leq c(p)\frac{\|\mathbf{DU}\|_p^2}{\|\mathbf{DU}\|_p^{2-p} + \|\mathbf{Du}_2\|_p^{2-p}} \leq C\epsilon_0^2\|\mathbf{DU}\|_p^2, \tag{3.5}$$

implying uniqueness, provided that ϵ_0 is small enough.

In the degenerate case $\delta_p = 0$, we observe that from the $(p, 0)$-structure we get

$$\int_\Omega \big(|\mathbf{Du}_1| + |\mathbf{DU}|\big)^{p-2}|\mathbf{DU}|^2\,d\mathbf{x} \leq C\Big|\int_\Omega (\mathbf{U}\cdot\nabla)\,\mathbf{u}_1\cdot\mathbf{U}\,d\mathbf{x}\Big| \leq C_1,$$

hence the integral from the left-hand side is well defined. Then, we show that the inequalities from Lemma 3.2 are valid also with $\delta = 0$. This can be seen by writing the estimates in Lemma 3.2 for some $0 < \delta \leq 1$ and taking the inferior limit of the right-hand side as $\delta \to 0$. In fact, since $p < 2$ we have the following monotonic increasing convergence

$$\big(\delta + |\mathbf{Du}_1| + |\mathbf{DU}|\big)^{p-2}|\mathbf{DU}|^2 \underset{\delta\to 0}{\uparrow} \big(|\mathbf{Du}_1| + |\mathbf{DU}|\big)^{p-2}|\mathbf{DU}|^2 \qquad \text{a.e in } \Omega,$$

which implies convergence of the corresponding integrals. This shows convergence of the second term from the right-hand side of estimates from Lemma 3.2.

Moreover, since $p < 2$, we have the uniform bound

$$\big(\delta + |\mathbf{Du}_1| + |\mathbf{DU}|\big)^{\frac{2-p}{2}} \leq \big(1 + |\mathbf{Du}_1| + |\mathbf{DU}|\big)^{\frac{2-p}{2}} \in L^{\frac{2q}{2-p}}(\Omega),$$

which allows us to use the Lebesgue dominated convergence, to handle the last term in the right-hand side of estimates from Lemma 3.2. After having justified the limiting step $\delta \to 0$, one obtains again (3.5) and the proof proceeds as in the non-degenerate case. □

4 Some remarks for anisotropic (weighted) double-phase problems

In the analysis of double-phase problems, it is also interesting to consider, as done in [9, 18, 25], families of anisotropic double-phase problems. We also adapt our uniqueness results to the following problem

$$\begin{aligned} -\operatorname{div}\left(|\mathbf{Du}|^{p-2}\mathbf{Du}\right) - \operatorname{div}\left(\mu(\mathbf{x})|\mathbf{Du}|^{r-2}\mathbf{Du}\right) + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla\pi = \mathbf{f} \qquad & \text{in } \Omega,\\ \operatorname{div}\mathbf{u} = 0 \qquad & \text{in } \Omega,\\ \mathbf{u} = \mathbf{0} \qquad & \text{on } \partial\Omega, \end{aligned}$$

where μ is a non-negative regular function and $r > 2$.

Stress-tensors of this type occur in fluid mechanics in certain turbulence models introduced by Baldwin and Lomax [1]. The mathematical analysis in terms of existence of weak solutions has been recently developed by one of the authors and D. Breit in [5]. In this case, it makes sense (from the modeling point of view based on Prandtl mixing length), to use as function μ the distance of $\mathbf{x}$ from the boundary of Ω

$$\mu(\mathbf{x}) = d(\mathbf{x}) := d(\mathbf{x}, \partial\Omega),$$

or one of its powers. The original model of Baldwin and Lomax concerns the tensor $\mathcal{BL}(\mathbf{x}, \boldsymbol{\omega}) = d^2(\mathbf{x})|\boldsymbol{\omega}|\boldsymbol{\omega}$ and it enters in the momentum equations through its curl

$$\operatorname{curl}\mathcal{BL}(\mathbf{x}, \boldsymbol{\omega}) = \operatorname{curl}\left(d^2(\mathbf{x})|\boldsymbol{\omega}|\boldsymbol{\omega}\right) \qquad \text{with } \boldsymbol{\omega} = \operatorname{curl}\mathbf{v}, \tag{4.1}$$

being written in rotational form. The system resembles the p-curl system studied in certain problems in electromagnetism, especially in mathematical model for superconductors, see Yin [28]. The value 2 for the power of the distance function is critical in terms of being able to recover estimates on the full gradient from those valid for the weighted curl, by means of the theory of weighted Sobolev spaces and Muckenhoupt weights. The expression (4.1) is the rotational form of the stress tensor

$$-\operatorname{div}(d^2\left(\mathbf{x})|\mathbf{Du}(\mathbf{x})|\mathbf{Du}(\mathbf{x})\right),$$

which is a *weighted version* of a stress tensor with "(3, 0)-structure," within the notation of Assumption 2.1.

In this section, we will see how to generalize the uniqueness results when $\mathbf{S}_{q,\delta}(\mathbf{Du})$, the stress tensor in (1.6), is replaced by

$$-\operatorname{div}\left(d^\alpha(\mathbf{x})|\mathbf{Du}(\mathbf{x})|^{r-2}\mathbf{Du}(\mathbf{x})\right) \qquad \text{for some } r > 2, \;\; \text{with } 0 < \alpha < r-1.$$

To handle this term we first recall a well-known lemma about the distance function $d(\mathbf{x})$, see for instance Kufner [19].

Lemma 4.1. *Let Ω be a domain of class $C^{0,1}$. There exist constants $0 < c_0, c_1 \in \mathbb{R}$ such that*

$$c_0\, d(\mathbf{x}) \le |a(x') - x_3| \le c_1\, d(\mathbf{x}) \qquad \forall\, \mathbf{x} = (x', x_3) \in \Omega_P.$$

From the above result we get the following weighted estimate

Lemma 4.2. *Let f be measurable such that $\int_\Omega d^\alpha |f|^r \,\mathrm{d}\mathbf{x} < \infty$. Then, for $0 < \alpha < r-1$ it follows that $f \in L^{\frac{r}{1+\alpha}}(\Omega)$.*

Proof. The proof follows by Hölder inequality and Lemma 4.1. In fact we can write

$$\begin{aligned} \int_\Omega |f|^s \,\mathrm{d}\mathbf{x} &= \int_\Omega d^{-\alpha s/r}\, d^{\alpha s/r} |f|^s \,\mathrm{d}\mathbf{x} \\ &\le \left(\int_\Omega d^{-\frac{\alpha s}{r-s}} \,\mathrm{d}\mathbf{x}\right)^{(r-s)/r} \left(\int_\Omega d^\alpha\, |f|^r \,\mathrm{d}\mathbf{x}\right)^{s/r} \\ &\le c\left(\int_\Omega d^\alpha\, |f|^r \,\mathrm{d}\mathbf{x}\right)^{s/r}, \end{aligned}$$

where the last estimate on the integral of the distance function follows immediately from Lemma 4.1, if $\frac{\alpha s}{r-s} < 1$. □

We can now state the following existence theorem:

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be smooth and bounded and let $\mathbf{S}_{p,\delta_p}$ satisfy assumption 2.1 for some $\delta_p \geq 0$ and for some $p > 6/5$. Let $r > 2$ be given and $\alpha \geq 0$. Then, for all $\mathbf{f} \in W^{-1,p'}(\Omega)$, the problem*

$$\begin{aligned} -\operatorname{div}\mathbf{S}_{p,\delta_p}(\mathbf{Du}) - \operatorname{div}\left(d^{\alpha}(\mathbf{x})|\mathbf{Du}|^{r-2}\mathbf{Du}\right) + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla\pi &= \mathbf{f} \qquad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \qquad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \qquad \text{on } \partial\Omega, \end{aligned} \tag{4.2}$$

has at least a weak solution $\mathbf{u} \in W^{1,p}_{0,\sigma}(\Omega)$, such that

$$\|\mathbf{Du}\|_p^p + \int_\Omega d^{\alpha}(\mathbf{x})|\mathbf{Du}(\mathbf{x})|^r \,\mathrm{d}\mathbf{x} \leq C\|\mathbf{f}\|_{-1,p'}^{p'},$$

for some constant depending only on p and Ω, but not on the solution. Moreover, if $0 < \alpha < r-1$, then

$$\|\mathbf{Du}\|_q^r \leq C\|\mathbf{f}\|_{-1,p'}^{p'} \qquad \forall\, q < \frac{r}{1+\alpha}.$$

Proof. The proof of existence can be obtained by a standard perturbation argument, as the one employed in [5]. The L^q-estimate for the deformation tensor $\mathbf{Du}$ follows from Lemma 4.2. □

We can now state the uniqueness result for the anisotropic double-phase problem.

Theorem 4.2. *Let be given $p \in]6/5, 12/7[$, $r > 2$, and $0 < \alpha < r-1$ such that*

$$\begin{aligned} &(i) \quad \frac{r}{1+\alpha} > \frac{3p(2-p)}{5p-6} && \text{for } \frac{6}{5} < p \leq \frac{\sqrt{33}-3}{2}, \\ &(ii) \quad \frac{r}{1+\alpha} > \frac{3p(4-p)}{7p-6} && \text{for } \frac{\sqrt{33}-3}{2} < p \leq \frac{\sqrt{37}-1}{3}, \\ &(iii) \quad \frac{r}{1+\alpha} > 2 && \text{for } \frac{\sqrt{37}-1}{3} < p < \frac{12}{7}. \end{aligned}$$

Let $\mathbf{u}_1, \mathbf{u}_2$ be a weak solutions of (4.2) with $\mathbf{f} \in W^{-1,p'}(\Omega)$. Then, there exists a constant $\epsilon_0 = \epsilon_0(p, r, \alpha, \Omega, \|\mathbf{f}\|_{-1,p'}) > 0$ such that if at least one solution has a small enough weighted gradient, that is, if

$$\left[\int_\Omega d^{\alpha}(\mathbf{x})|\mathbf{Du}_1(\mathbf{x})|^r \,\mathrm{d}\mathbf{x}\right]^{1/r} \leq \epsilon_0,$$

then $\mathbf{u}_1 = \mathbf{u}_2$.

Proof. The proof follows the same lines of that presented in Theorem 3.1, where $\frac{r}{1+\alpha}$ plays the same role as q (with the caveat that equality is not valid in the estimates, due to the strict inequality in Lemma 4.2). □

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References

[1] B. Baldwin and H. Lomax, Thin-layer approximation and algebraic model for separated turbulent flows, *16th Aerospace Sciences Meeting* (Huntsville, AL), U.S.A, Aerospace Sciences Meetings, (1978).

[2] S. Baraket, S. Chebbi, N. Chorfi, and V. D. Rădulescu, Non-autonomous eigenvalue problems with variable (p_1, p_2)-growth, *Adv. Nonlinear Stud.* **17** (2017), no. 4, 781–792.

[3] P. Baroni, M. Colombo, and G. Mingione, Regularity for general functionals with double phase, *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, Art. 62, 48.

[4] L. Belenki, L. C. Berselli, L. Diening, and M. Růžička, On the finite element approximation of p-Stokes systems, *SIAM J. Numer Anal.* **50** (2012), no. 2, 373–397.

[5] L. C. Berselli and D. Breit, On the existence of weak solutions for the steady baldwin-lomax model and generalizations, *J. Math. Anal. Appl.* **501** (2021), 124643.

[6] E. Blavier and A. Mikelić, On the stationary quasi-Newtonian flow obeying a power-law, *Math. Methods Appl. Sci.* **18** (1995), no. 12, 927–948.

[7] D. Breit, *Existence theory for generalized Newtonian fluids*, Mathematics in Science and Engineering, Elsevier/Academic Press, London, 2017.

[8] N. Chorfi and V. D. Rădulescu, Standing wave solutions of a quasilinear degenerate Schrödinger equation with unbounded potential, *Electron. J. Qual. Theory Differ. Equ.* (2016), Paper No. 37, 12.

[9] M. Colombo and G. Mingione, Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.* **215** (2015), no. 2, 443–496.

[10] F. Crispo and C. R. Grisanti, On the existence, uniqueness and $C^{1,\gamma}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ regularity for a class of shear-thinning fluids, *J. Math. Fluid Mech.* **10** (2008), no. 4, 455–487.

[11] ———, On the $C^{1,\gamma}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ regularity for a class of electro-rheological fluids, *J. Math. Anal. Appl.* **356** (2009), no. 1, 119–132.

[12] F. Crispo and P. Maremonti, A high regularity result of solutions to modified p-Stokes equations, *Nonlinear Anal.* **118** (2015), 97–129.

[13] L. Diening and F. Ettwein, Fractional estimates for non-differentiable elliptic systems with general growth, *Forum Math.* **20** (2008), no. 3, 523–556.

[14] L. Diening and Ch. Kreuzer, Linear convergence of an adaptive finite element method for the p–Laplacian equation, *SIAM J. Numer. Anal.* **46** (2008), 614–638.

[15] L. Esposito, F. Leonetti, and G. Mingione, Sharp regularity for functionals with (p, q) growth, *J. Differential Equations* **204** (2004), no. 1, 5–55.

[16] G. P. Galdi, *Mathematical problems in classical and non-Newtonian fluid mechanics*, Hemodynamical flows, Oberwolfach Semin., vol. 37, Birkhäuser, Basel, 2008, pp. 121–273.

[17] ———, *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2011.

[18] L. Gasiński and P. Winkert, Existence and uniqueness results for double phase problems with convection term, *J. Differential Equations* **268** (2020), 4183–4193.

[19] A. Kufner, *Weighted Sobolev spaces*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1985.

[20] O. A. Ladyžhenskaya, *The mathematical theory of viscous incompressible flow*, Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2, Gordon and Breach Science Publishers, New York, 1969.

[21] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[22] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Differential Equations* **90** (1991), no. 1, 1–30.

[23] M. Rųžička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, vol. 1748, Springer-Verlag, Berlin, 2000.

[24] ———, Analysis of generalized Newtonian fluids, *Topics in mathematical fluid mechanics*, Lecture Notes in Math., vol. 2073, Springer, Heidelberg, (2013), pp. 199–238.

[25] V. D. Rădulescu, Isotropic and anistropic double-phase problems: old and new, *Opuscula Math.* **39** (2019), no. 2, 259–279.

[26] V. D. Rădulescu and D. D. Repovš, *Partial differential equations with variable exponents*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015, Variational methods and qualitative analysis.

[27] M. Růžička and L. Diening, Non–Newtonian fluids and function spaces, *Proceedings of NAFSA* (Prague 2006), vol. 8 (2007), 95–144.

[28] H.-M. Yin, Regularity of weak solution to a p-curl-system, *Differential Integral Equations* **19** (2006), no. 4, 361–368.

[29] V. V. Zhikov, On Lavrentiev's phenomenon, *Russian J. Math. Phys.* **3** (1995), no. 2, 249–269.