Almost everywhere uniqueness of blow-up limits for the lower dimensional obstacle problem

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We answer a question left open in [4] and [3], by proving that the blow-up of minimizers u of the lower dimensional obstacle problem is unique at generic point of the free boundary.

1. Introduction

Let $s \in (0, 1)$, let B_1 be the unit ball in $\mathbb{R}^d$, where $d \geqslant 2$, and let $B_1' := B_1 \cap \{x_d = 0\}$. For any point $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we denote by x' the vector of the first $(d-1)$ coordinates, $x' = (x_1, \ldots, x_{d-1})$. We consider the class of admissible functions

$$\mathcal{A} := \big\{ u \in H^1\big(B_1, |x_d|^{1-2s} \mathcal{L}^d\big) \, : \, u \geqslant 0 \text{ on } B_1' \,,$$
$$u(x', x_d) = u(x', -x_d) \text{ for every } (x', x_d) \in B_1 \big\}.$$

We say that $u \in \mathcal{A}$ is a *solution of the lower dimensional obstacle problem* if

$$\int_{B_1} |x_d|^{1-2s} |\nabla u|^2 \, dx \leqslant \int_{B_1} |x_d|^{1-2s} |\nabla v|^2 \, dx \quad \text{for every } v \in \mathcal{A} \text{ such that } u - v \in H_0^1(B_1).$$
(1)

For a solution $u \in \mathcal{A}$ of the lower dimensional obstacle problem, we define the *coincidence set* as

$$\Delta(u) := \big\{ (x', 0) \in B_1' \, : \, u(x', 0) = 0 \big\},$$

and the *free boundary* Γ_u *of* u as the topological boundary of $\Delta(u)$ in B_1'.

We say that u has a *unique blow-up limit at x_0*, if the family of functions

$$u_{x_0,r} : B_r \to \mathbb{R}, \quad u_{x_0,r}(x) = \|u(r \cdot + x_0)\|^{-1}_{L^2(\partial B_1)} u(rx + x_0),$$

converges weakly in $H^1(B_1, |x_d|^{1-2s} \mathcal{L}^d)$ to an admissible function u_{x_0}.

Here, building on the rectifiability of the free boundary, recently proved by Focardi and Spadaro (see Theorem 7 below), and the classification of the two-dimensional homogeneous solutions, we prove that, at almost-every point of the free boundary, the blow-up is unique and corresponds to certain two-dimensional profiles with homogeneities $2m, 2m - 1 + s$, or $2m + 2s$. In particular, we answer a question left open in a recent paper of Focardi and Spadaro (see [3, 4]).

Our main result is the following.

Theorem 1 *Let u be a solution of the lower dimensional obstacle problem* (1). *Then, for $\mathcal{H}^{d-2}$-almost every $x_0 \in \Gamma(u)$, the following does hold:*

(i) *u has a unique blow-up limit u_{x_0} at x_0;*
(ii) *such blow up is either $2m, 2m - 1 + s$, or $2m + 2s$ homogeneous, for some $m \in \mathbb{N}$;*
(iii) *the blow-up limit $u_{x_0} : \mathbb{R}^d \to \mathbb{R}$ is of the form*

$$u_{x_0}(x', x_d) = \bar{u}(x' \cdot e, x_d) \text{ for some vector } e \in \mathbb{S}^{d-2} \subset \mathbb{R}^{d-1},$$

and $\bar{u} : \mathbb{R}^2 \to \mathbb{R}$ is a homogeneous solution of the lower dimensional obstacle problem (1) *in dimension two.*

REMARK 2 (Lower dimensional obstacle problem VS minimal surfaces/harmonic maps) Our proof of Theorem 1 is based on a very general dimension-reduction lemma (Lemma 5), which allows to reduce the question of the uniqueness of the blow-up limit to the analysis of the blow-up limits with a maximal number of symmetries. In fact, our argument is very general and can be applied in different contexts, for example, to the singular sets of minimal surfaces and harmonic maps. On the other hand, we notice that, in the case of the lower-dimensional (thin) obstacle problem, the blow-up limits with a maximal number of symmetries are completely described (for instance, in the case of the thin-obstacle problem, the homogeneous two-dimensional solutions are explicit), while for minimal surfaces and harmonic maps the singular blow-ups of minimal dimension (that is, with maximal number of symmetries) are not classified. However, combining the analogous version of Lemma 5 for minimal surfaces and harmonic maps with the work of L. Simon [10], it is still possible to deduce uniqueness of the blow-up at almost every point of the singular set from its rectifiability (that is from Naber–Valtorta's result [8]). This is precisely the content of [10, Remark 1.14] and we will briefly explain it in Appendix A.

REMARK 3 The same technique can be applied to the case when the Dirichlet integral in (1) is replaced by the area term $\int_{B_1} \sqrt{1 + |\nabla u|^2} \, dx$ (we refer to [5] for further discussions on this problem) and also to the case when the obstacle is nonzero (see [7] and [6]).

REMARK 4 We notice that recently in [2] it was shown that the blow-up limit at every point of frequency $2m + 1 (= 2m + 2s$ for $s = {}^1/2)$ is unique.

2. Main lemma and proof of Theorem 1

For every point $x_0 \in B_1$, we define the Almgren's frequency function

$$N(u, x_0, r) := \frac{r \int_{B_r(x_0)} |x_d|^{1-2s} |\nabla u|^2 \, dx}{\int_{\partial B_r(x_0)} |x_d|^{1-2s} u^2 \, d\mathcal{H}^{d-1}}.$$

The function $r \mapsto N(u, x_0, r)$ is monotone non-decreasing in r (see [1]), so that it is well defined the limit

$$N(u, x_0, 0) := \lim_{r \to 0} N(u, x_0, r). \tag{2}$$

In particular, the free boundary can be decomposed according to the value of the frequency function at $r = 0$. We denote the set of points of frequency $\lambda \in \mathbb{R}$ by

$$S_\lambda(u) := \{x_0 \in \Gamma(u) : N(u, x_0, 0) = \lambda\}.$$

Our main lemma is the following.

Lemma 5 (Splitting lemma) *Let u be a solution of the lower dimensional obstacle problem. Let $\lambda \in \mathbb{R}$ and $x_0 \in S_\lambda(u)$ be a point of frequency λ for which there exists a linear subspace T_{x_0} of $\mathbb{R}^d$ satisfying the following property:*

(SP) *For every $y_0 \in T_{x_0}$ and sequence of radii r_n converging to 0, there is a sequence of points y_n converging to y_0 such that $y_n \in S_\lambda(u_{x_0, r_n})$, for every n.*

Then, any blow-up limit b of u at x_0 is invariant in the direction of T_{x_0}, that is,

$$b(x + y_0) = b(x) \quad \text{for every } x \in \mathbb{R}^d \text{ and every } y_0 \in T_{x_0}. \tag{3}$$

REMARK 6 We notice that in the proof of Lemma 5, we use only the following properties of the frequency function N:

- *Monotonicity.* For every $x_0 \in B_1$, the function $r \mapsto N(u, x_0, r)$ is non-decreasing.
- *Scaling.* For $y_0 \in B_1$, $s > 0$ and $r > 0$, such that $u_{x_0, r}$ is defined on the ball $B_s(y_0)$, we have

$$N(u_{x_0, r}, y_0, s) = N(u, x_0 + ry_0, sr). \tag{4}$$

- *Continuity.* For every fixed $r > 0$, the function $(u, x) \mapsto N(u, x, r)$, defined on $H^1(B_1) \times \mathbb{R}^d$ is continuous in the strong $H^1(B_1) \times \mathbb{R}^d$ topology.
- *Characterization of the homogeneous functions.* Suppose that the point $x_0 \in \mathbb{R}^d$ and the function $u : \mathbb{R}^d \to \mathbb{R}$ are such that

$$N(u, x_0, r) = \lambda \quad \text{for every } r > 0.$$

Then u is λ-homogeneous with respect to x_0, that is,

$$u(x_0 + rx) = r^\lambda u(x_0 + x) \quad \text{for every } x \in \mathbb{R}^d \text{ and } r > 0.$$

We also notice that the *monotonicity* property gives the existence of $N(u, x_0, 0)$ (see (2)). Moreover, the *continuity* property implies the following:

- *Upper semicontinuity.* Suppose that $u_n : B_1 \to \mathbb{R}$ is a sequence of functions converging strongly in $H^1(B_1)$ to a function $u_\infty \in H^1(B_1)$. Suppose that $x_n \in B_1$ be a sequence converging to some $x_\infty \in B_1$. Then we have that

$$N(u_\infty, x_\infty, 0) \geqslant \limsup_{n \to \infty} N(u_n, x_n, 0). \tag{5}$$

Indeed, using the monotonicity of the function $r \mapsto N(u, x, r)$, we have

$$N(u_\infty, x_\infty, r) = \lim_{n \to \infty} N(u_n, x_n, r) \geqslant \limsup_{n \to \infty} N(u_n, x_n, 0).$$

Taking, the limit as $r \to 0$, we get (5).

Proof of Lemma 5. Let b be any blow-up limit of u at x_0. Then, there is a sequence $r_n \to 0$ such that u_{r_n, x_0} converges to b both strongly in H^1_{loc} and in $C^1_{\mathrm{loc}}(\{x_d \geqslant 0\})$.

We first claim that

$$N(b, y_0, 0) = \lambda \quad \text{for every } y_0 \in T_{x_0}. \tag{6}$$

Indeed let $y_0 \in T_{x_0}$ be fixed and let $S_\lambda(u_{x_0, r_n}) \ni y_n \to y_0$ be the sequence of points whose existence is guaranteed by (SP). In particular, since $u_{x_0, r_n}(y_n) = 0$ and u_{x_0, r_n} converges uniformly to b, we have that $b(y_0) = 0$. By the upper semi-continuity of N we have that $N(b, y_0, 0) \geqslant \lambda$. Indeed, since $y_n \in S_\lambda(u_{x_n, r_n})$ and u_{x_0, r_n} converges to b strongly in $H^1(B_1)$, we have

$$N(b, y_0, 0) \geqslant \limsup_{n \to \infty} N(u_{x_0, r_n}, y_n, 0) = \lambda.$$

On the other hand, $N(b, y_0, 0) \leqslant \lambda$. Indeed, by (4) and the fact that b is homogeneous, we have that

$$N(b, y_0, 0) = \lim_{s \to 0} N(b, y_0, s) = \lim_{s \to 0} N(b_{0,r}, y_0, s) = \lim_{s \to 0} N(b, ry_0, rs) = N(b, ry_0, 0),$$

for every $r > 0$. In particular, this means that

$$N(b, y_0, 0) = \lim_{r \to 0} N(b, ry_0, 0) \leqslant N(b, 0, 0) = \lambda,$$

where the inequality follows by the upper semi-continuity of the frequency function. This concludes the proof of (6).

We next prove that the function b is invariant in any direction $y \in T_{x_0}$, that is

$$b(x + y) = b(x) \quad \text{for every } x \in \mathbb{R}^d. \tag{7}$$

Using the homogeneity of b and (4), for every $r > 0$ we have that

$$N(b, y, R) = N\left(b_{0,R}, \frac{y}{R}, 1\right) = N\left(b, \frac{y}{R}, 1\right).$$

Taking the limit as $R \to \infty$, we get that

$$\lim_{R \to \infty} N(b, y, R) = \lim_{R \to \infty} N\left(b, \frac{y}{R}, 1\right) = N(b, 0, 1) = \lambda.$$

In particular, together with (6), this implies that

$$N(b, y, r) = \lambda \quad \text{for every } r > 0,$$

and so, b is homogeneous with respect to y:

$$b(y + rx) = r^\lambda b(y + x) \text{ for every } r > 0.$$

Hence, for every $x \in \mathbb{R}^d$ we can use the homogeneity with respect to 0 and y to obtain

$$b(x + y) = 2^\lambda b\left(\frac{x + y}{2}\right) = 2^\lambda b\left(y + \frac{x - y}{2}\right) = b(x).$$

This concludes the proof of (7). □

In the proof of Theorem 1 we will use Lemma 5 and the following recent result by Focardi and Spadaro, which we report here for the reader's convenience.

Theorem 7 (Focardi–Spadaro; see Theorem 1.2 and Theorem 1.3 of [4]) *Let u be a solution of the lower dimensional obstacle problem* (1) *in B_1. Then $\Lambda(u)$ is a set of finite perimeter and there exists $\Sigma(u) \subseteq \Gamma(u)$ with Hausdorff dimension at most $n - 2$ such that*

$$N(u, x_0, 0) \in \{2m, 2m - 1 + s, 2m + 2s\}_{m \in \mathbb{N} \setminus \{0\}} \quad \text{for every } x_0 \in \Gamma(u) \setminus \Sigma(u).$$

Proof of Theorem 1. Let

$$\Sigma(u) := \Gamma(u) \setminus \left(\left(\bigcup_{m=1}^\infty \mathcal{S}_{2m}\right) \cup \left(\bigcup_{m=1}^\infty \mathcal{S}_{2m-1+s}\right) \cup \left(\bigcup_{m=1}^\infty \mathcal{S}_{2m+2s}\right)\right).$$

By [4, Theorem 1.3], we have that $\mathcal{H}^{d-2}(\Sigma(u)) = 0$. Thus, it is sufficient to prove the claim for almost-every $x_0 \in \mathcal{S}_\lambda$, where $\lambda = 2m, 2m - 1 + s$ or $2m + 2s$. Moreover, by [4, Theorem 1.2], we have that the free boundary $\Gamma(u)$ is C^1-rectifiable and so is each of the sets $\mathcal{S}_{2m-1+s}$, $\mathcal{S}_{2m}$ and $\mathcal{S}_{2m+2s}$ (for every $m \in \mathbb{N}$). In particular, this means that for almost every point x_0 of these sets, there exists a unique $(d - 2)$-dimensional approximate tangent plane $T_{x_0} \subseteq \mathbb{R}^{d-1} \times \{0\}$, namely

$$\mathcal{H}^{d-1}\left|(\mathcal{S}_\lambda(u_r) \cap B_1) \rightharpoonup \mathcal{H}^{d-1}\right|(T_{x_0} \cap B_1) \tag{8}$$

as locally finite measures. Hence, the splitting property hypothesis (SP) of Lemma 5 is satisfied. Then Lemma 5 implies that every blow-up limit b of u at x_0 is invariant with respect to a $(d - 2)$-dimensional plane T_{x_0}. This means, that b depends only on two variables: $x \cdot e$ and the last coordinate x_d, e being (one of) the normal vector to T_{x_0} in the hyperplane $\mathbb{R}^{d-1}$. Precisely, b is of the form

$$b(x) = \bar{b}(x \cdot e, x_d), \tag{9}$$

where $\bar{b}$ is a homogeneous solution of the lower dimensional obstacle problem in dimension two.

We now consider the three cases $\lambda = 2m$, $\lambda = 2m - 1 + s$ and $\lambda = 2m + 2s$ separately. Indeed, we first notice that there is only one (up to a multiplicative constant) two-dimensional solution of the lower-dimensional obstacle problem of homogeneity $2m$. In particular, if $\lambda = 2m$, then the blow-up is unique and two-dimensional.

Let now $\lambda = 2m - 1 + s$. In this case there are two two-dimensional homogeneous solutions (see for instance [7]) and so, two possible blow up limits of u at x_0. We call them b_1 and b_2. In order to prove the uniqueness of the blow-up as in statement (i) we have to exclude that, for two

different sequences $r_j \to 0$ and $t_j \to 0$, the blow-up is b_1 and b_2, respectively. Indeed, taking the scalar product of $u_{x_0,r}$ with b_1 we see that

$$\lim_{j \to \infty} \int_{\partial B_1} u_{x_0,r_j} b_1 = 1 \quad \text{and} \quad \lim_{j \to \infty} \int_{\partial B_1} u_{x_0,t_j} b_1 = \int_{\partial B_1} b_1 b_2 < 1;$$

hence, for every j, there exists $q_j \in (r_j, t_j)$ such that

$$\lim_{j \to \infty} \int_{\partial B_1} u_{x_0,q_j} b_1 = \frac{1}{2} \Big(1 + \int_{\partial B_1} b_1 b_2 \Big).$$

This gives a contradiction. Indeed, up to a subsequence, u_{x_0,q_j} converges to a blow-up limit, which by Lemma 5 should be b_1 or b_2.

It now remains the case $\lambda = 2m + 2s$. Fix $x_0 \in \mathcal{S}_{2m+2s}(u)$ that admits a $(d-2)$-dimensional approximate tangent plane $T_{x_0} \subset \mathbb{R}^{d-1} \times \{0\}$ and such that, by (9), every blow-up limit b is of the form $b(x) = \bar{b}(x \cdot e, x_d)$, where $\bar{b}$ is a $(2m + 2s)$-homogeneous solution in dimension two. It is sufficient to prove that e is a normal vector to T_{x_0}. Let

$$H_b := \{ x' \in \mathbb{R}^{d-1} : x' \cdot e = 0 \},$$

and suppose that there is a point $y_0 \in T_{x_0} \setminus H_b \subset \mathbb{R}^{d-1}$. Without loss of generality, we may assume that $|y_0| = 1/2$. Let u_{x_0,r_n} be a blow-up sequence converging to b. By definition of the tangent plane, there is a sequence of points $y_n \in B_1 \cap \mathcal{S}_{2m+2s}(u_{r_n,x_0})$ such that $y_n \to y_0$. Since, y_n are on the free boundary, there is a sequence of points z_n in the non-contact set of u_{x_0,r_n} (that is, $u_{r_n,x_0}(z_n, 0) > 0$ and, as a consequence, $\frac{\partial u_{r_n,x_0}}{\partial x_d}(z_n, 0) = 0$) such that $z_n \to y_0$.

When $s = 1/2$, we use the classification of the solutions in dimension two (see [7]), which implies that $\bar{b} : \mathbb{R}^2 \to \mathbb{R}$ can be written (up to a positive multiplicative constant) in polar coordinates as

$$\bar{b}(r, \theta) = r^{2m+1} \sin\big(-(2m+1)\theta \big) \quad \text{in } \{x_2 \geq 0\}, \tag{10}$$

and it is reflected in an even way in the half-plane $\{x_2 < 0\}$. In particular, $\frac{\partial b}{\partial x_d}(y_0, 0) < 0$. On the other hand, the blow-up sequence u_{x_0,r_n} converges in C^1 to the blow-up b (see [1]). Thus,

$$\frac{\partial b}{\partial x_d}(y_0, 0) = \lim_{n \to \infty} \frac{\partial u_{x_0,r_n}}{\partial x_d}(z_n, 0) = 0,$$

which is a contradiction. In conclusion, $T_{x_0} = H_b$, so the vector e and the blow-up b are uniquely determined by the tangent plane T_{x_0}. This concludes the analysis of $\mathcal{S}_{2m+2s}$ in the case $s = 1/2$.

For general s, a nice formula as (10) is not available but the two-dimensional solutions are described in detail in [4, Appendix A.1] and the uniqueness of the blow-up follows by a similar argument. Indeed, by [4, Equation (A.4)], up to a multiplicative constant, we have that

$$\bar{b}(x_1, x_2) = |x_2|^{2s} \big(-1 + O(|x - y_0|^2) \big).$$

This means that, at the point y_0, we have

$$|x_d|^{1-2s} \frac{\partial b}{\partial x_d}(y_0, 0) := \lim_{x_d \to 0} |x_d|^{1-2s} \frac{\partial b(y_0, x_d)}{\partial x_d} < 0.$$

On the other hand, by [4, Theorem 2.1] we have that

$$|x_d|^{1-2s} \frac{\partial b}{\partial x_d}(y_0, 0) = \lim_{n \to \infty} |x_d|^{1-2s} \frac{\partial u_{x_0, r_n}}{\partial x_d}(z_n, 0) = 0,$$

where the last inequality is due to the fact that z_n is not on the contact set (see for instance [4, Corollary 2.4]). This is a contradiction. Thus, also in the case $s \neq 1/2$ and $\lambda = 2m + 2s$, the blow-up is unique (as it is uniquely determined by T_{x_0}). This concludes the proof. □

Appendix A. About Remark 2

In this section we elaborate a bit more on Remark 2 in the particular case of minimal surfaces (although the same holds for harmonic maps). Following the notations of [10], we denote with $\mathfrak{M}$ a multiplicity one class of n-dimensional minimal surfaces and we denote with Sing M, the singular set of $M \in \mathfrak{M}$. Moreover, we let

$$m := \max\{\dim \operatorname{Sing} M \; : \; M \in \mathfrak{M}\}.$$

Thanks to a result of Naber–Valtorta [8], we know that Sing M has finite $\mathcal{H}^m$-volume and it is locally $\mathcal{H}^m$-rectifiable. Next, let us denote with $\Theta_M(x)$ the density of $M \in \mathfrak{M}$ at a point x, and recall that a consequence of Łojasiewicz inequality for minimal surfaces is that the set of admissible densities is discrete, that is,

$$\{\Theta_{\mathbf{C}}(0) \; : \; \mathbf{C} \text{ stationary cone with } \dim(\operatorname{Sing} \mathbf{C}) = m\} = \{\alpha_1, \ldots, \alpha_N\},$$

with $\alpha_1 < \cdots < \alpha_N$ (see [10, 4.3 Lemma]). Consider the sets

$$\mathcal{S}_j := \{x \in \operatorname{Sing} M \; : \; \Theta_M(x) = \alpha_j\} \quad j = 1, \ldots, N,$$

and notice that, by standard stratification arguments,

$$\mathcal{H}^m\left(\operatorname{Sing} M \setminus \left(\bigcup_{j=1}^{N} \mathcal{S}_j\right)\right) = 0. \tag{A1}$$

As a consequence (of the analogous) of Lemma 5, applied to this case, we know that

(MS) for every point $x \in \mathcal{S}_j$ for which the approximate tangent space T_x to $\mathcal{S}_j$ at x exists, all the tangent cones $\mathbf{C}$ to M at x are such that $\dim(\operatorname{Sing} \mathbf{C}) = m$ and moreover $T_x \subset \mathbf{C}$.

Thanks to the Naber–Valtorta rectifiability result, this is the case for $\mathcal{H}^m$-a.e. point of $\mathcal{S}_j$, that is

(MS′) for $\mathcal{H}^m$-a.e. $x \in \mathcal{S}_j$, all the tangent cones $\mathbf{C}$ to M at x are such that

$$\dim(\operatorname{Sing} \mathbf{C}) = m \quad \text{and} \quad T_x \subset \operatorname{Sing} \mathbf{C}.$$

It follows from (MS′) and (A1), combined with standard arguments that, for $\mathcal{H}^m$-a.e. $x \in \operatorname{Sing} M$, there is an m-dimensional subspace L_x such that, for every $\varepsilon > 0$,

$$B_1(0) \cap \eta_{x,\sigma}(\operatorname{Sing} M) \subset \text{the } \varepsilon\text{-neighborhood of } L_x, \tag{A2}$$
$$B_1(0) \cap L_x \subset \text{the } \varepsilon\text{-neighborhood of } \eta_{x,\sigma}(\operatorname{Sing} M), \tag{A3}$$

where $\eta_{x,\sigma}(y) := \sigma^{-1}(y - x)$. Indeed, if $L_x = T_x$ is as in (MS′), then (A3) follows immediately by the definition of approximate tangent, while (A2) follows from (MS'), the upper semicontinuity of the density and a simple blow-up argument.

Now, the main content of [10] is precisely to show that at $\mathcal{H}^m$-a.e. $x \in \operatorname{Sing} M$, for which (A2) and (A3) do hold, the blow-up is unique (see the second part of [10, Proof of Remark 1.14]). Indeed, these are the points where no δ-gap nor δ-tilt happens.

Finally, we notice that, for the thin obstacle problem and the minimal surfaces, the set of points at which the blow-up limit is unique is characterized differently. In the case if the lower-dimensional (thin) obstacle problem, the blow-up is unique at every point at which the free boundary admits an approximate tangent plane. On the other hand, for minimal surfaces, the blow-up is unique at almost-every point satisfying the conditions (A2) and (A3) (this is due to the fact that the uniqueness is achieved by an averaging process), which (as we noticed above) turn out to be fulfilled whenever the singular set admits an approximate tangent plane. In particular, for minimal surfaces we cannot characterize the points with unique blow-up as the ones at which the approximate tangent plane to $\mathcal{S}_j$ exists. However, this would be the case if we knew a priori that the $(n-m)$-dimensional minimal cones are integrable. Precisely, if the $(n - m)$-dimensional minimal cones were integrable, then the blow-up would be unique at every point satisfying (A2) and (A3) (we refer to [9] for more details on the integrability and its relation with the uniqueness of the blow-up limits).

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