BLOWING-UP SOLUTIONS FOR SUPERCRITICAL YAMABE PROBLEMS ON MANIFOLDS WITH UMBILIC BOUNDARY

MARCO G. GHIMENTI AND ANNA MARIA MICHELETTI

Abstract. We build blowing-up solutions for a supercritical perturbation of the Yamabe problem on manifolds with umbilic boundary provided the dimension of the manifold is $n \geq 8$ and that the Weyl tensor W_g is not vanishing on ∂M.

1. Introduction

Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with a smooth boundary ∂M. A well known problem in differential geometry is whether (M,g) can be conformally deformed to a constant scalar manifold with boundary of constant mean curvature. When the boundary is empty this is called the Yamabe problem (see [5, 34]) which has been completely solved by Aubin [5], Schoen [31], and Trundinger [33]. In 1992, Escobar [16] studied the problem in the context of manifolds with boundary, when the background metric is scalar flat, and gave an affirmative solution to the question in some cases. The remaining cases were studied by Marques [27], Almaraz [1], Brendle and Chen [7], Mayer and Ndiaye [29]. Another branch of literature studies when both the scalar and the mean curvature of the target metric are constant and non zero. An exhaustive list of works in the field is beyond the scope of this introduction: we limit ourselves to cite [23, 4, 9] and the references therein.

The context of scalar flat metrics is particularly interesting since it leads to the study of a linear equation in the interior with a critical nonlinear Neumann-type boundary condition

$$\begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} R_g u = 0 & \text{on } M, \\ \frac{\partial}{\partial \nu} u + \frac{n-2}{2} h_g u = (n-2) u^{\frac{n}{n-2}} & \text{on } \partial M, \end{cases} \tag{1}$$

where R_g is the scalar curvature of M, h_g is the mean curvature on ∂M, and ν is the outward normal to the boundary. The geometric meaning of (1) is that if u is a positive solution of (1), the scalar curvature of the conformal metric $\tilde{g} = u^{\frac{4}{n-2}} g$ is zero and the mean curvature of $\tilde{g}$ on the boundary of M is $n-2$. The boundary Yamabe problem in the case of scalar flat metrics can be also seen as the multidimensional version of the Riemann Mapping Theorem.

Once it is known that the Yamabe problem admits a solution, a natural question arises about the compactness of the full set of solutions. Concerning manifolds without boundary, a necessary condition is that the manifold is not conformally equivalent to the standard sphere $\mathbb{S}^n$, since the set of conformal transformation of the round sphere is not compact. The problem of compactness has been studied by Schoen in 1988 [32], then by Li and Zhu [24], Druet [10], Li and Zhang [25, 26],

2000 *Mathematics Subject Classification.* 35J65, 53C21.

Key words and phrases. Umbilic boundary, Yamabe problem, Compactness, Stability.

The first authors was supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of Istituto Nazionale di Alta Matematica (INdAM) and by project PRA from University of Pisa.

Marques [28], Brendle [6], Brendle and Marques [8], Khuri Marques and Schoen [15] in the last years.

When the boundary of the manifold is not empty, a necessary condition is that M is not conformally equivalent to the standard ball $\mathbb{B}^n$. Compactness for the boundary Yamabe problem has been studied firstly by Felli and Ould Ahmedou [12], Han and Li [22], and Almaraz [3]. These papers produce a large offspring of works concerning the compactness of solutions for boundary Yamabe problems.

Felli and Ould Ahmedou, in the cited paper [12], have proved compactness for solutions of (1), when M is locally conformally flat and the boundary is umbilic, and Almaraz in [3] has proved compactness when $n \geq 7$ and the trace free second fundamental form is non zero everywhere on ∂M, that is any point of the boundary is non umbilic. In [14], Kim Musso and Wei showed that compactness continues to hold when $n = 4$ and when $n = 6, 7$ and the trace free second fundamental form is non zero everywhere on the boundary.

Very recently, compactness has been proved by the authors in [17] for manifolds with umbilic boundary when $n > 8$ and the Weyl tensor of M is everywhere non zero on the boundary ∂M. In [19], the authors extend the compactness result to manifolds of dimension $n = 6, 7, 8$, when the boundary is umbilic and the Weyl tensor of M is everywhere non zero on ∂M.

An example of non compactness is given for $n \geq 25$ and manifold with umbilic boundary in [2]. We recall that the boundary of M is called umbilic if the trace free second fundamental form of ∂M is zero everywhere on ∂M. The problem of compactness of solutions for manifolds with umbilic boundary when $n < 25$ and the Weyl tensor vanishes on the boundary is still open, both for the scalar flat and scalar constant manifolds (see [22])

One can ask, also, whether or not the compactness property is preserved under perturbation of the equation. This is called in literature *stability* of the problem (see [10]). The problem is equivalent to having or not uniform a priori estimates for solutions of the perturbed problem.

In the following, we consider the problem

$$\begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} R_g u = 0 & \text{on } M, \\ \frac{\partial}{\partial \nu} u + \frac{n-2}{2} h_g u = (n-2) u^{\frac{n}{n-2}+\varepsilon} & \text{on } \partial M. \end{cases}$$

This equation can be recast, using the conformal Laplacian $L_g := \Delta_g - \frac{n-2}{4(n-1)} R_g$ and the conformal boundary operator $B_g := -\frac{\partial}{\partial \nu} - \frac{n-2}{2} h_g$, in the compact form

$$\begin{cases} L_g v = 0 & \text{on } M, \\ B_g v + (n-2) v^{\frac{n}{n-2}+\varepsilon} = 0 & \text{on } \partial M. \end{cases} \tag{2}$$

In the following we often use $a(x) := \frac{n-2}{4(n-1)} R_g$ to simplify the notation.

We study the question of stability of the problem (1), for perturbation from above of the exponent. It is clear that the problem is not stable with respect to supercritical perturbation of the nonlinearity if we are able to build solutions v_ε of the perturbed problem (2) which blow up at one point of the manifold as the parameter ε goes to zero.

Our main result is the following

Theorem 1. *Let M be a n-dimensional manifold of positive type with umbilic boundary ∂M. Suppose $n \geq 8$ and that the Weyl tensor W_g is not vanishing on ∂M.*

Then there exists a family of solutions v_ε of (2) such that v_ε blows up at a point on ∂M when $\varepsilon \to 0^+$.

Here M of positive type means that there exists $C>0$ such that

$$Q(u):=\frac{\int\limits_M\left(|\nabla u|^2+\frac{n-2}{4(n-1)}R_g u^2\right)dv_g+\int\limits_{\partial M}\frac{n-2}{2}h_g u^2 d\sigma_g}{\left(\int\limits_{\partial M}|u|^{\frac{2(n-1)}{n-2}}d\sigma_g\right)^{\frac{n-2}{n-1}}}\geq C \text{ for any } u\in H^1(M)\smallsetminus\{0\}\,.$$

We remark that this assumption on the positivity of Q is natural when we address to compactness questions in Yamabe problems since if $-\infty<\inf\limits_{u\in H^1(M)\smallsetminus\{0\}}Q(u)\leq 0$, then the solution of the Yamabe problem is unique.

The stability of problem (1) with respect to the principal quantity of the boundary term has been studied in a series of papers by the authors and by Pistoia, both in the case of non umbilic boundary and in the case of umbilic boundary with Weyl tensor non vanishing on the boundary. Firstly, they studied what happens linearly perturbing the mean curvature term. This problem present a strong analogy to the classical Yamabe problem when perturbing the scalar curvature term (see, on this topic, [10, 11] and the references therein). In fact, we have that the set of solutions is compact -and hence (1) is stable- perturbing the mean curvature from below while we construct a blowing up sequence when the perturbation is everywhere positive on the boundary, and for a class of perturbation which are positive in at least one point on the boundary. The result of compactness is dealt with in [18] both in umbilic and non umbilic cases, while for the construction of blowing up sequences for umbilic boundary manifold we refer to [21].

Concerning the exponent of the nonlinearity, all the compactness results hold for $p\leq\frac{n}{n-2}$, so the boundary Yamabe problem is stable for perturbations from below of the critical exponent. In the present paper we have that small perturbations above the critical exponent imply blowing up solutions when $n\geq 8$, the boundary ∂M is umbilic and the Weyl tensor is non vanishing on ∂M, so problem (1) is not stable for perturbation from above of the critical exponent.

As a final remark, we notice that in [21] we assumed that the manifold is umbilic, the Weyl tensor is never vanishing and that $n\geq 11$, in order to construct a blowing up sequence. The assumption on the dimension in that paper is technical, since we required some integrability condition when performing the Ljapounov Schmidt procedure. Indeed, in the present paper we perform more effective computations in Lemma 6: this method could be applied verbatim in the cited paper, so it is possible to reformulate Theorem 1 of [21] with the assumption $n\geq 8$, instead of $n\geq 11$.

Theorem 1 is obtained by the widely used Ljapounov Schmidt reduction. Firstly, we reduce the problem to a finite dimensional one, then we solve the finite dimensional problem evaluating a functional which depends on the point on the boundary and on a rescaling parameter. The finite dimensional reduction is usually based on the approximation of a rescaled solution around an isolated simple blowup point by the standard bubble. In the context of umbilic boundary manifold, this approximation is not sufficient, and a crucial step is the introduction of a sharp correction term to the usual approximation. This term is a solution of a suitable linear differential equation (17), whose right hand side is determined by the particular form of the expansion of the metric when the boundary is umbilic. To calculate the expansion with regards to ε of the reduced functional, we need the fine estimate obtained in Proposition 7 via fixed point theorem. This estimate is a consequence of the evaluation given in Lemma 6, and this computation strongly depends on the definition of the mentioned correction term (see (31)). Notice that (31) contains the second order expansion of the metric tensor: this determines the right hand side terms of (17).

Once performed the finite dimensional reduction, we look for a critical point of the finite dimensional functional $I_\varepsilon(\lambda, q)$ defined in (40). This functional has as a leading term the function $I(\lambda, q) = \lambda^4\varphi(q) + C\log\lambda$ with $(\lambda, q) \in (0, +\infty) \times \partial M$ (see Proposition 9). Theorem 1 is proved by finding a maximum point (λ_0, q_0) of $I(\lambda, q)$. Also, in the theorem we prove that the solutions v_ε that we have found blow up at the point $q_0 \in \partial M$, which is a critical point of φ. This function depends on the aforementioned correction. Since we do not have an explicit expression of the correction term, also we lack an explicit expression for the critical point q_0.

2. Preliminaries

We recall here a series of preliminary result which are useful for our result.

Since the manifold is of positive type, then

$$\langle\langle u, v\rangle\rangle_g = \int_M (\nabla_g u \nabla_g v + auv) d\mu_g + \frac{n-2}{2}\int_{\partial M} h_g uv d\sigma_g$$

is an equivalent scalar product in H^1_g, which induces to the equivalent norm $\|\cdot\|_g$.

We define the exponent

$$s_\varepsilon = \frac{2(n-1)}{n-2} + n\varepsilon$$

and the Banach space $\mathcal{H}_g := H^1(M) \cap L^{s_\varepsilon}(\partial M)$ endowed with norm $\|u\|_{\mathcal{H}_g} = \|u\|_g + |u|_{L^{s_\varepsilon}_g(\partial M)}$.By trace theorems, we have the following inclusion $W^{1,\tau}(M) \subset L^t(\partial M)$ for $t \leq \tau\frac{n-1}{n-\tau}$.

We recall the following result, by Nittka [30, Th. 3.14]

Remark 2. Let $\frac{2n}{n+2} \leq q < \frac{n}{2}$, $r > 0$. Then there exists a constant c such that if $f_0 \in L^{q+r}(\Omega)$, β bounded and measurable and $f \in L^{\frac{(n-1)q}{n-q}+r}(\partial\Omega)$ and $u \in H^1(\Omega)$ is the unique weak solution of

$$\begin{cases} Lu = f_0 & \text{on } \Omega \\ \frac{\partial}{\partial\nu}u + \beta u = f & \text{on } \partial\Omega \end{cases}$$

where L is an uniformly strict elliptic second order operator, then

$$u \in L^{\frac{nq}{n-2q}}(\Omega),\ \ u|_{\partial\Omega} L^{\frac{(n-1)q}{n-2q}}(\partial\Omega) \text{ and}$$

$$|u|_{L^{\frac{nq}{n-2q}}(\Omega)} + |u|_{L^{\frac{(n-1)q}{n-2q}}(\partial\Omega)} \leq c\left(|f_0|_{L^{q+r}(\Omega)} + |f|_{L^{\frac{(n-1)q}{n-q}+r}(\partial\Omega)}\right).$$

With this remark, chosen suitably r and q, we have the necessary estimates to work in the functional space $\mathcal{H}_g$. We consider $i : H^1(M) \to L^{\frac{2(n-1)}{n-2}}(\partial M)$ and its adjoint with respect to $\langle\langle\cdot,\cdot\rangle\rangle_g$

$$i^*_g : L^{\frac{2(n-1)}{n}}(\partial M) \to H^1(M)$$

defined by

$$\left\langle\left\langle \varphi, i^*_g(f)\right\rangle\right\rangle_g = \int_{\partial M} \varphi f d\sigma_g \text{ for all } \varphi \in H^1$$

so that $v = i^*_g(f)$ is the weak solution of the problem

$$\begin{cases} L_g v = 0 & \text{on } M \\ B_g v + f = 0 & \text{on } \partial M \end{cases}. \tag{3}$$

By [30, Th. 3.14] (see Remark 2), we can recast Problem (2) using i^*_g. In fact, if $v \in H^1$ is a solution of (3), then for $\frac{2n}{n+2} \leq q < \frac{n}{2}$ and $r > 0$ it holds

$$|v|_{L^{\frac{(n-1)q}{n-2q}}(\partial M)} = |i^*_g(f)|_{L^{\frac{(n-1)q}{n-2q}}(\partial M)} \leq c|f|_{L^{\frac{(n-1)q}{n-q}+r}(\partial M)}. \tag{4}$$

By this result, we can choose q, r such that

$$\frac{(n-1)q}{n-2q} = \frac{2(n-1)}{n-2} + n\varepsilon \text{ and } \frac{(n-1)q}{n-q} + r = \frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\varepsilon} \tag{5}$$

that is

$$q = \frac{2n + n^2\left(\frac{n-2}{n-1}\right)\varepsilon}{n+2+2n\left(\frac{n-2}{n-1}\right)\varepsilon} \text{ and } r = \frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\varepsilon} - \frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\left(\frac{n}{n-1}\right)\varepsilon}.$$

Set $f_\varepsilon(v) = (n-2)\left(v^+\right)^{\frac{n}{n-2}+\varepsilon}$, we have that, if $v \in L_g^{\frac{2(n-1)}{n-2}+n\varepsilon}(\partial M)$, then $f_\varepsilon(v) \in L_g^{\frac{2(n-1)+n(n-2)\varepsilon}{n+\varepsilon(n-2)}}(\partial M)$ and, in light of (4), also $i_g^*\left(f_\varepsilon(v)\right) \in L_g^{\frac{2(n-1)}{n-2}+n\varepsilon}(\partial M)$.

Thus we can recast then Problem (2) as

$$v = i_g^*\left(f_\varepsilon(v)\right),\ v \in \mathcal{H}_g. \tag{6}$$

The problem has also a variational structure: we can associate to Problem (2) the following functional, which is well defined on $\mathcal{H}_g$.

$$\begin{aligned} J_{\varepsilon,g}(v) :=& \frac{1}{2}\int_M |\nabla_g v|^2 + av^2 d\mu_g + \frac{n-2}{4}\int_{\partial M} h_g v^2 d\sigma_g \\ &- \frac{(n-2)^2}{2(n-1)+\varepsilon(n-2)}\int_{\partial M}\left(v^+\right)^{\frac{2(n-1)}{n-2}+\varepsilon} d\sigma_g. \end{aligned} \tag{7}$$

Notation. We collect here our main notation. We will use the indices $1 \leq i,j,k,l,s \leq n-1$ and $1 \leq a,b,c,d \leq n$. Moreover we use the Einstein convention on repeated indices. We denote by g the Riemannian metric, by R_{abcd} the full Riemannian curvature tensor, by R_{ab} the Ricci tensor and by R_g the scalar curvature of (M,g); moreover the Weyl tensor of (M,g) will be denoted by W_g. The bar over an object (e.g. $\bar{W}_g$) will means the restriction to this object to the metric of ∂M. Finally, on the half space $\mathbb{R}^n_+ = \{y = (y_1,\ldots,y_{n-1},y_n) \in \mathbb{R}^n,\ y_n \geq 0\}$ we set $B_r(y_0) = \{y \in \mathbb{R}^n,\ |y-y_0| \leq r\}$ and $B_r^+(y_0) = B_r(y_0) \cap \{y_n > 0\}$. When $y_0 = 0$ we will use simply $B_r = B_r(y_0)$ and $B_r^+ = B_r^+(y_0)$. On the half ball B_r^+ we set $\partial' B_r^+ = B_r^+ \cap \partial\mathbb{R}^n_+ = B_r^+ \cap \{y_n = 0\}$ and $\partial^+ B_r^+ = \partial B_r^+ \cap \{y_n > 0\}$. On $\mathbb{R}^n_+$ we will use the following decomposition of coordinates: $(y_1,\ldots,y_{n-1},y_n) = (\bar{y},y_n) = (z,t)$ where $\bar{y}, z \in \mathbb{R}^{n-1}$ and $y_n, t \geq 0$.

Finally, fixed a point $q \in \partial M$, we denote by $\psi_q : B_r^+ \to M$ the Fermi coordinates centered at q. We denote by $B_g^+(q,r)$ the image of $\psi_q(B_r^+)$. When no ambiguity is possible, we will denote $B_g^+(q,r)$ simply by B_r^+, omitting the chart ψ_q.

Remark 3. Since ∂M is umbilic for any $q \in \partial M$, there exists a metric $\tilde{g}_q = \tilde{g}$, conformal to g, $\tilde{g}_q = \Lambda_q^{\frac{4}{n-2}} g_q$ such that

$$|\det\tilde{g}_q(y)| = 1 + O(|y|^N), \tag{8}$$

$$|\tilde{h}_{ij}(y)| = o(|y^3|), \tag{9}$$

$$
\begin{aligned}
\tilde g^{ij}(y) =& \delta_{ij} + \frac{1}{3}\bar R_{ikjl}y_k y_l + R_{ninj}y_n^2 \\
&+ \frac{1}{6}\bar R_{ikjl,m}y_k y_l y_m + R_{ninj,k}y_n^2 y_k + \frac{1}{3}R_{ninj,n}y_n^3 \\
&+ \left(\frac{1}{20}\bar R_{ikjl,mp} + \frac{1}{15}\bar R_{iksl}\bar R_{jmsp}\right) y_k y_l y_m y_p \\
&+ \left(\frac{1}{2}R_{ninj,kl} + \frac{1}{3}\mathrm{Sym}_{ij}(\bar R_{iksl}R_{nsnj})\right) y_n^2 y_k y_l \\
&+ \frac{1}{3}R_{ninj,nk}y_n^3 y_k + \frac{1}{12}\left(R_{ninj,nn} + 8R_{nins}R_{nsnj}\right) y_n^4 + O(|y|^5),
\end{aligned} \tag{10}
$$

$$
\bar R_{\tilde g_q}(y) = O(|y|^2) \text{ and } \partial^2_{ii}\bar R_{\tilde g_q}(q) = -\frac{1}{6}|\bar W(q)|^2, \tag{11}
$$

$$
\bar R_{kl}(q) = R_{nn}(q) = R_{nk}(q) = 0, \tag{12}
$$

uniformly with respect to $q \in \partial M$ and $y \in T_q(M)$. Also, we have $\Lambda_q(q) = 1$ and $\nabla\Lambda_q(q) = 0$. These results are contained in [27, 13]. Here $\tilde h_{ij}$ is the tensor of the second fundamental form referred to the metric $\tilde g$.

The conformal Laplacian and the conformal boundary operator transform as follows under the change of metric $\tilde g_q = \Lambda_q^{\frac{4}{n-2}} g_q$:

$$
\begin{aligned}
L_{\tilde g_q}\varphi &= \Lambda_q^{-\frac{n+2}{n-2}} L_g(\Lambda_q\varphi) \\
B_{\tilde g_q}\varphi &= \Lambda_q^{-\frac{n}{n-2}} B_g(\Lambda_q\varphi).
\end{aligned}
$$

By these transformations we have that $v := \Lambda_q u$ is a positive solution of (2), if and only if u is a positive solution of

$$
\begin{cases}
L_{\tilde g_q} u = 0 & \text{in } M \\
B_{\tilde g_q} u + (n-2)\Lambda_q^{\varepsilon} u^{\frac{n}{n-2}+\varepsilon} = 0 & \text{on } \partial M
\end{cases} \tag{13}
$$

From now on we set $\tilde f_\varepsilon(u) = (n-2)\Lambda_q^{\varepsilon}\,(u^+)^{\frac{n}{n-2}+\varepsilon}$.

Furthermore we have

$$
\langle\langle \Lambda_q u, \Lambda_q v\rangle\rangle_g = \langle\langle u, v\rangle\rangle_{\tilde g}
$$

and, consequently,

$$
\|\Lambda_q u\|_g = \|u\|_{\tilde g}.
$$

In addition, we have that $\Lambda_q u \in L_g^{s_\varepsilon}$ if and only if $u \in L_{\tilde g}^{s_\varepsilon}$, so $\Lambda_q u \in \mathcal H_g$ if and only if $u \in \mathcal H_{\tilde g}$. Finally, we can define the functional $J_{\varepsilon,\tilde g}$ associated to (13), as

$$
\begin{aligned}
J_{\varepsilon,\tilde g}(u) :=& \frac{1}{2}\int_M |\nabla_{\tilde g} u|^2 + \tilde a u^2 d\mu_{\tilde g} + \frac{n-2}{4}\int_{\partial M} h_{\tilde g} v^2 d\sigma_{\tilde g} \\
&- \frac{(n-2)^2}{2(n-1)+\varepsilon(n-2)}\Lambda_q \int_{\partial M} \left(u^+\right)^{\frac{2(n-1)}{n-2}+\varepsilon} d\sigma_{\tilde g},
\end{aligned}
$$

where $\tilde a = \frac{n-2}{4(n-1)} R_{\tilde g}$, and we get

$$
J_{\varepsilon,g}(\Lambda_q u) = J_{\varepsilon,\tilde g}(u).
$$

so we can always switch between metrics g and $\tilde g$, and this will be useful in the next. In the Section 3 we emphasize other equivalences of the same kind. As a last remark, we notice also that a solution of (13) can be expressed by means of $i^*_{\tilde g}$, in fact u solves (13) if and only if

$$
u = i^*_{\tilde g}(\tilde f_\varepsilon(u)).
$$

3. The finite dimensional reduction.

Given $q\in\partial M$ and $\psi_q^\partial:\mathbb{R}^n_+\to M$ the Fermi coordinates in a neighborhood of q; we define

$$W_{\delta,q}(\xi)=U_\delta\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right)\chi\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right)=$$
$$=\frac{1}{\delta^{\frac{n-2}{2}}}U\left(\frac{y}{\delta}\right)\chi(y)=\frac{1}{\delta^{\frac{n-2}{2}}}U(x)\chi(\delta x)$$

where $y=(z,t)$, with $z\in\mathbb{R}^{n-1}$ and $t\geq 0$, $\delta x=y=\left(\psi_q^\partial\right)^{-1}(\xi)$ and χ is a radial cut off function, with support in ball of radius R.

Here $U_\delta(y)=\frac{1}{\delta^{\frac{n-2}{2}}}U\left(\frac{y}{\delta}\right)$ is the one parameter family of solution of the problem

$$\begin{cases}-\Delta U_\delta=0 & \text{on } \mathbb{R}^n_+;\\ \frac{\partial U_\delta}{\partial t}=-(n-2)U_\delta^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+.\end{cases}\tag{14}$$

and $U(z,t):=\dfrac{1}{\left[(1+t)^2+|z|^2\right]^{\frac{n-2}{2}}}$ is the standard bubble in $\mathbb{R}^n_+$.

Now, let us consider the linearized problem

$$\begin{cases}-\Delta\phi=0 & \text{on } \mathbb{R}^n_+,\\ \frac{\partial\phi}{\partial t}+nU^{\frac{2}{n-2}}\phi=0 & \text{on } \partial\mathbb{R}^n_+,\\ \phi\in H^1(\mathbb{R}^n_+).\end{cases}\tag{15}$$

and it is well know that every solution of (15) is a linear combination of the functions $j_1,\dots,j_n$ defined by .

$$j_i=\frac{\partial U}{\partial x_i},\ i=1,\dots n-1 \qquad j_n=\frac{n-2}{2}U+\sum_{i=1}^n y_i\frac{\partial U}{\partial y_i}.\tag{16}$$

Given $q\in\partial M$ we define, for $b=1,\dots,n$

$$Z^b_{\delta,q}(\xi)=\frac{1}{\delta^{\frac{n-2}{2}}}j_b\left(\frac{1}{\delta}\left(\psi_q^\partial\right)^{-1}(\xi)\right)\chi\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right),$$

we decompose $H^1(M)$ in the direct sum of the following two subspaces

$$\tilde K_{\delta,q}=\operatorname{Span}\left\langle\Lambda_q Z^1_{\delta,q},\dots,\Lambda_q Z^n_{\delta,q}\right\rangle$$
$$\tilde K^\perp_{\delta,q}=\left\{\varphi\in H^1(M)\ :\ \left\langle\left\langle\varphi,\Lambda_q Z^b_{\delta,q}\right\rangle\right\rangle_g=0,\ b=1,\dots,n\right\}$$

and we define the projections

$$\tilde\Pi=H^1(M)\to\tilde K_{\delta,q}\text{ and }\tilde\Pi^\perp=H^1(M)\to\tilde K^\perp_{\delta,q}.$$

In order to give a good ansatz on the shape of the solution we need to introduce the function $v_q:\mathbb{R}^n_+\to\mathbb{R}$ which is a solution of the linear problem

$$\begin{cases}-\Delta v_q=\left[\frac{1}{3}\bar R_{ijkl}(q)y_ky_l+R_{ninj}(q)y_n^2\right]\partial^2_{ij}U & \text{on } \mathbb{R}^n_+\\ \frac{\partial v_q}{\partial y_n}=-nU^{\frac{2}{n-2}}v_q & \text{on } \partial\mathbb{R}^n_+\end{cases}\tag{17}$$

This function is a key tool for several estimates in what follows. In fact, a good choice of v_q will allow us to get the correct size of the remainder term in the finite dimensional reduction (Lemma 6).

Remark 4. There exists a unique $v_q:\mathbb{R}^n_+\to\mathbb{R}$ solution of the problem (17) $L^2(\mathbb{R}^n_+)$-orthogonal to j_b for all $b=1,\dots,n$. Moreover it holds

$$|\nabla^\tau v_q(y)|\leq C(1+|y|)^{4-\tau-n}\text{ for }\tau=0,1,2,\tag{18}$$

$$\int_{\partial\mathbb{R}^n_+}U^{\frac{n}{n-2}}(t,z)v_q(t,z)dz=0\tag{19}$$

and

$$\int_{\partial\mathbb{R}^n_+} v_q(t,z)\Delta v_q(t,z)dz \le 0, \tag{20}$$

where $y \in \mathbb{R}^n_+$, $y=(t,z)$ with $t \ge 0$ and $z \in \mathbb{R}^{n-1}$. In addition, the map $q \mapsto v_q$ is in $C^2(\partial M)$. The proof of this remark can be found in [21, Lemma 3] and will be omitted.

At this point, given $q \in \partial M$ we define, similarly to $W_{\delta,q}$, the function

$$V_{\delta,q}(\xi) = \frac{1}{\delta^{\frac{n-2}{2}}} v_q\left(\frac{1}{\delta}\left(\psi_q^\partial\right)^{-1}(\xi)\right)\chi\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right)$$

and

$$\left(v_q\right)_\delta(y) = \frac{1}{\delta^{\frac{n-2}{2}}} v_q\left(\frac{y}{\delta}\right),$$

where v_q is chosen as in Remark 4.

We look for solution of (6) having the form

$$v = \Lambda_q u = \tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q} + \tilde{\phi} \text{ with } \tilde{\phi} \in \tilde{K}^\perp_{\delta,q} \cap \mathcal{H}.$$

where we used the intuitive notation

$$\tilde{W}_{\delta,q} = \Lambda_q W_{\delta,q} \quad \tilde{V}_{\delta,q} = \Lambda_q V_{\delta,q} \text{ and } \tilde{\phi} = \Lambda_q \phi$$

We can rewrite, in light of the previous orthogonal decomposition, Problem (6) (and so Problem (2)) as

$$\tilde{\Pi}\left\{\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi} - i_g^*\left[f_\varepsilon(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi})\right]\right\} = 0 \tag{21}$$

$$\tilde{\Pi}^\perp\left\{\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi} - i_g^*\left[f_\varepsilon(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi})\right]\right\} = 0. \tag{22}$$

We stress than we can proceed in analogous way in the manifold $(M, \tilde{g})$. In this case we should define

$$K_{\delta,q} = \text{Span}\left\langle Z^1_{\delta,q}, \dots, Z^n_{\delta,q}\right\rangle$$
$$K^\perp_{\delta,q} = \left\{\varphi \in H^1(M) \ : \ \left\langle\left\langle \varphi, Z^b_{\delta,q}\right\rangle\right\rangle_{\tilde{g}} = 0,\ b = 1,\dots,n\right\},$$

and we should ask that $u = W_{\delta,q} + \delta^2 V_{\delta,q} + \phi$, recasting (13) as the couple of equations

$$\Pi\left\{W_{\delta,q} + \delta^2 V_{\delta,q} + \phi - i_{\tilde{g}}^*\left[\tilde{f}_\varepsilon(W_{\delta,q} + \delta^2 V_{\delta,q} + \phi)\right]\right\} = 0 \tag{23}$$

$$\Pi^\perp\left\{W_{\delta,q} + \delta^2 V_{\delta,q} + \phi - i_{\tilde{g}}^*\left[\tilde{f}_\varepsilon(W_{\delta,q} + \delta^2 V_{\delta,q} + \phi)\right]\right\} = 0. \tag{24}$$

Roughly speaking, we are allowed to move the tilde symbol from solutions to problems and vice versa, so in any moment we can choose in which metric and with which functional it is more convenient to work.

Coming back to problem (22), we define the linear operator $L : \tilde{K}^\perp_{\delta,q} \cap \mathcal{H}_g \to \tilde{K}^\perp_{\delta,q} \cap \mathcal{H}_g$ as

$$L(\tilde{\phi}) = \tilde{\Pi}^\perp\left\{\tilde{\phi} - i_g^*\left(f'_\varepsilon(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q})[\tilde{\phi}]\right)\right\}, \tag{25}$$

a nonlinear term $N(\tilde{\Phi})$ and a remainder term R as

$$N(\tilde{\phi}) = \tilde{\Pi}^\perp\left\{i_g^*\left(f_\varepsilon(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi}) - f_\varepsilon(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q}) - f'_\varepsilon(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q})[\tilde{\phi}]\right)\right\} \tag{26}$$

$$R = \tilde{\Pi}^\perp\left\{i_g^*\left(f_\varepsilon(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q})\right) - \tilde{W}_{\delta,q} - \delta^2\tilde{V}_{\delta,q}\right\}, \tag{27}$$

so equation (22) becomes

$$L(\tilde{\phi}) = N(\tilde{\phi}) + R.$$

The rest of the section is devoted to show that, for any choice of δ, q, a solution $\tilde{\phi}$ of (22) exists.

Lemma 5. *Let $\delta = \varepsilon^{\frac{1}{4}}\lambda$. For $a, b \in \mathbb{R}$, $0 < a < b$ there exists a positive constant $C_0 = C_0(a,b)$ such that, for ε small, for any $q \in \partial M$, for any $\lambda \in [a,b]$ and for any $\phi \in K^{\perp}_{\delta,q} \cap \mathcal{H}$ there holds*

$$\|L_{\delta,q}(\phi)\|_{\mathcal{H}} \geq C_0 \|\phi\|_{\mathcal{H}}.$$

Proof. The proof of this Lemma is very similar to the proof of [20, Lemma 2] and will be omitted. $\square$

Lemma 6. *It holds*

$$\|R\|_{\mathcal{H}_g} = \begin{cases} \delta^{-O^+(\varepsilon)} \left\{ O\left(\delta^3 \log \delta\right) + O(\varepsilon \log \delta) + O(\varepsilon) \right\} & \text{if } n = 8 \\ O\left(\delta^3\right) + \delta^{-O^+(\varepsilon)} \left\{ O(\varepsilon \log \delta) + O(\varepsilon) \right\} & \text{if } n > 8 \end{cases}$$

where $0 < O^+(\varepsilon) < C\varepsilon$ for some positive constant C. In addition, with the choice $\delta = \varepsilon^{\frac{1}{4}}\lambda$ we have that

$$\|R\|_{\mathcal{H}_g} = \begin{cases} O\left(\varepsilon^{\frac{3}{4}} \log \varepsilon\right) & \text{if } n = 8 \\ O\left(\varepsilon^{\frac{3}{4}}\right) & \text{if } n > 8 \end{cases}.$$

Proof. **Step 1.** It holds

$$\|R\|_g = \begin{cases} O\left(\delta^3 \log \delta\right) + O(\varepsilon \log \delta) + O(\varepsilon) & \text{if } n = 8 \\ O\left(\delta^3\right) + O(\varepsilon \log \delta) + O(\varepsilon) & \text{if } n > 8 \end{cases} \tag{28}$$

We have

$$\begin{aligned} \|R\|_g \leq & \left\| i_g^* \left(f_\varepsilon(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) \right) - i_g^* \left(f_0(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) \right) \right\|_g \\ & + \left\| i_g^* \left(f_0(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) \right) - \tilde{W}_{\delta,q} - \delta^2 \tilde{V}_{\delta,q} \right\|_g, \end{aligned}$$

and we start by estimating the second term. By definition of i_g^* there exists $\Gamma = i_g^* \left(f_0(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) \right)$, that is a function Γ solving

$$\begin{cases} -\Delta_g \Gamma + a(x)\Gamma = 0 & \text{on } M \\ \frac{\partial}{\partial \nu}\Gamma + \frac{n-2}{2} h_g(x)\Gamma = f_0(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) & \text{on } \partial M \end{cases}. \tag{29}$$

So we have

$$
\begin{aligned}
&\left\| i_g^*\left(f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q}\right)-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q}\right\|_g^2=\|\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q}\|_g^2\\
&=\int_M\left[-\Delta_g(\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})+a(\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})\right](\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})d\mu_g\\
&\qquad+\int_{\partial M}h_g(\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})^2d\sigma_g\\
&\qquad+\int_{\partial M}\left[\frac{\partial}{\partial\nu}(\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})\right](\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})d\sigma_g\\
&=\int_M\left[\Delta_g(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})-a(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right](\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})d\mu_g\\
&\qquad-\int_{\partial M}h_g(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})(\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})d\sigma_g\\
&\quad+\int_{\partial M}\left[f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})-\frac{\partial}{\partial\nu}(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right](\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})d\sigma_g\\
&\qquad\qquad=:I_1+I_2+I_3.
\end{aligned}
$$

We have

$$
\begin{aligned}
I_2&=\int_{\partial M}h_{\tilde g}(W_{\delta,q}+\delta^2V_{\delta,q})(\Lambda_q^{-1}R)d\sigma_{\tilde g}\\
&\le C|h_{\tilde g}(W_{\delta,q}+\delta^2V_{\delta,q})|_{L_{\tilde g}^{\frac{2(n-1)}{n}}(\partial M)}\|\Lambda_q^{-1}R\|_{\tilde g}
\end{aligned}
$$

Set $B_{1/\delta}^{n-1}=\left\{z\in\mathbb{R}^{n-1},\ |z|\le 1/\delta\right\}$, we have

$$
|h_{\tilde g}(W_{\delta,q}+\delta^2V_{\delta,q})|_{L_{\tilde g}^{\frac{2(n-1)}{n}}(\partial M)}=O(\delta)\left|h_{\tilde g}(\delta z)(U(z)-\delta^2v_q(z))\right|_{L^{\frac{2(n-1)}{n}}(B_{1/\delta}^{n-1})}.
$$

Since $z\le 1/\delta$ we have that $\delta(1+|z|)=O(1)$. We recall that $|\nabla^\tau v_q(y)|\le C(1+|y|)^{4-\tau-n}$ by (18) and that $|\nabla^\tau U(y)|\le C(1+|y|)^{2-\tau-n}$ for $\tau=0,1,2$. By 9 we have also that $h_{\tilde g_q}(q)=h_{\tilde g_q,i}(q)=h_{\tilde g_q,ik}(q)=0$, so,

$$
\begin{aligned}
|h_{\tilde g}(W_{\delta,q}+\delta^2V_{\delta,q})|_{L_{\tilde g}^{\frac{2(n-1)}{n}}(\partial M)}&=O(\delta)\left|h_{\tilde g}(\delta z)(1+|z|)^{2-n}\right|_{L^{\frac{2(n-1)}{n}}(B_{1/\delta}^{n-1})}\\
&=O(\delta^3)\left||z|^2(1+|z|)^{2-n}\right|_{L^{\frac{2(n-1)}{n}}(B_{1/\delta}^{n-1})}\\
&=\begin{cases}O(\delta^3\log\delta) & \text{if } n=8\\ O(\delta^3) & \text{if } n>8\end{cases},
\end{aligned}\tag{30}
$$

since $|z|^2(1+|z|)^{2-n}\le(1+|z|)^{4-n}$ and $\left|(1+|z|)^{4-n}\right|_{L^{\frac{2(n-1)}{n}}(B_{1/\delta}^{n-1})}$ is bounded when $n>8$ or $\left|(1+|z|)^{4-n}\right|_{L^{\frac{2(n-1)}{n}}(B_{1/\delta}^{n-1})}=O(\log\delta)$ when $n=8$. Thus

$$
I_2=O(\delta^3)\|\Lambda_q^{-1}R\|_{\tilde g}=\begin{cases}O(\delta^3\log\delta)\|R\|_g & \text{if } n=8\\ O(\delta^3)\|R\|_g & \text{if } n>8\end{cases}.
$$

For I_1 we proceed in a similar way, having

$$
\begin{aligned}
I_1&=\int_M\left[\Delta_{\tilde g}(W_{\delta,q}+\delta^2V_{\delta,q})-\tilde a(W_{\delta,q}+\delta^2V_{\delta,q})\right](\Lambda_q^{-1}R)d\mu_{\tilde g}\\
&\le\left|\Delta_{\tilde g}(W_{\delta,q}+\delta^2V_{\delta,q})-\tilde a(W_{\delta,q}+\delta^2V_{\delta,q})\right|_{L_{\tilde g}^{\frac{2n}{n+2}}(M)}\|\Lambda_q^{-1}R\|_{\tilde g}.
\end{aligned}
$$

Set $B^n_{1/\delta}=\{z\in\mathbb{R}^n,\ |z|\le 1/\delta\}$. By [27, page 1609], we have $R_{\tilde g}(0)=0$, so we get

$$\left|\tilde a(W_{\delta,q}+\delta^2V_{\delta,q})\right|_{L^{\frac{2n}{n+2}}_{\tilde g}(M)}=O(\delta^2)\left|R_{\tilde g}(\delta x)(U(x)+\delta^2 v_q(x))\right|_{L^{\frac{2n}{n+2}}_{\tilde g}(B^n_{1/\delta})}$$
$$O(\delta^3)\left||x|(1+|x|)^{2-n}\right|_{L^{\frac{2n}{n+2}}_{\tilde g}(B^n_{1/\delta})}=\begin{cases} O(\delta^3\log\delta) & \text{if } n=8\\ O(\delta^3) & \text{if } n>8\end{cases}$$

since $\left|(1+|x|)^{3-n}\right|_{L^{\frac{2n}{n+2}}_{\tilde g}(B^n_{1/\delta})}$ is bounded when $n>8$ and $\left|(1+|x|)^{3-n}\right|_{L^{\frac{2n}{n+2}}_{\tilde g}(B^n_{1/\delta})}=O(\log\delta)$ when $n=8$.

For the Laplacian term, in local charts we have

$$\Delta_{\tilde g_q}=\Delta_{\mathrm{euc}}+[\tilde g_q^{ij}(y)-\delta_{ij}]\partial^2_{ij}+\left[\partial_i\tilde g_q^{ij}(y)+\frac{\tilde g_q^{ij}(y)\partial_i|\tilde g_q|^{\frac12}(y)}{|\tilde g_q|^{\frac12}(y)}\right]\partial_j+\frac{\partial_n|\tilde g_q|^{\frac12}(y)}{|\tilde g_q|^{\frac12}(y)}\partial_n.$$

Thus by the expansion of the metric given in (8), (10) and since v_q solves (17) and $\Delta_{\mathrm{euc}}U=0$, we have that

$$\left|\Delta_{\tilde g}(W_{\delta,q}+\delta^2V_{\delta,q})\right|_{L^{\frac{2n}{n+2}}_{\tilde g}(M)}=O(1)\left|\Delta_{\tilde g}(U+\delta^2v_q)\right|_{L^{\frac{2n}{n+2}}(B^n_{1/\delta})} \tag{31}$$
$$=O(1)\Bigg|\Delta U+[\tilde g_q^{ij}(\delta x)-\delta_{ij}]\partial^2_{ij}U+\delta^2\Delta_{\mathrm{euc}}v_q+\delta^2[\tilde g_q^{ij}(\delta x)-\delta_{ij}]\partial^2_{ij}v_q$$
$$+\left[\partial_i\tilde g_q^{ij}(\delta x)+\frac{\tilde g_q^{ij}(\delta x)\partial_i|\tilde g_q|^{\frac12}(\delta x)}{|\tilde g_q|^{\frac12}(\delta x)}\right]\partial_j(U+\delta^2v_q)+\frac{\partial_n|\tilde g_q|^{\frac12}(\delta x)}{|\tilde g_q|^{\frac12}(\delta x)}\partial_n(U+\delta^2v_q)\Bigg|_{L^{\frac{2n}{n+2}}(B^n_{1/\delta})}$$
$$=O(1)\left|\delta^3|x|^3\partial^2_{ij}U+\delta^4|x|^2\partial^2_{ij}v_q+\delta^3|x|^2\partial_j(U+\delta^2v_q)+\delta^3|x|^2\partial_n(U+\delta^2v_q)\right|_{L^{\frac{2n}{n+2}}(B^n_{1/\delta})}$$
$$=O(1)\left|\delta^3(1+|x|)^{3-n}+\delta^4(1+|x|)^{4-n}\right|_{L^{\frac{2n}{n+2}}(B^n_{1/\delta})}$$
$$=O(\delta^3)\left|(1+|x|)^{3-n}\right|_{L^{\frac{2n}{n+2}}(B^n_{1/\delta})}=\begin{cases} O(\delta^3\log\delta) & \text{if } n=8\\ O(\delta^3) & \text{if } n>8\end{cases},$$

and we conclude that

$$I_1=\begin{cases} O(\delta^3\log\delta)\|R\|_g & \text{if } n=8\\ O(\delta^3)\|R\|_g & \text{if } n>8\end{cases}.$$

For the last integral I_3 we have

$$I_3\le C\left|(n-2)\left((W_{\delta,q}+\delta^2V_{\delta,q})^+\right)^{\frac{n}{n-2}}-\frac{\partial}{\partial\nu}(W_{\delta,q}+\delta^2V_{\delta,q})\right|_{L^{\frac{2(n-1)}{n}}_{\tilde g}(\partial M)}\|R\|_g$$
$$\le C(n-2)\left|\left((W_{\delta,q}+\delta^2V_{\delta,q})^+\right)^{\frac{n}{n-2}}-(W_{\delta,q})^{\frac{n}{n-2}}-\delta^2\frac{\partial}{\partial\nu}V_{\delta,q}\right|_{L^{\frac{2(n-1)}{n}}_{\tilde g}(\partial M)}\|R\|_g$$
$$+C\left|(n-2)\,(W_{\delta,q})^{\frac{n}{n-2}}-\frac{\partial}{\partial\nu}W_{\delta,q}\right|_{L^{\frac{2(n-1)}{n}}_{\tilde g}(\partial M)}\|R\|_g. \tag{32}$$

Since U is a solution of (14) one can easily obtain

$$\left|(n-2)\,(W_{\delta,q})^{\frac{n}{n-2}}-\frac{\partial}{\partial\nu}W_{\delta,q}\right|_{L^{\frac{2(n-1)}{n}}_{\tilde g}(\partial M)}=O(\delta^3). \tag{33}$$

Finally we have, using (17), and expanding $\left((U+\delta^2 v_{\delta,q})^+\right)^{\frac{n}{n-2}}$ near U,

$$\left|\left((W_{\delta,q}+\delta^2 V_{\delta,q})^+\right)^{\frac{n}{n-2}}-(W_{\delta,q})^{\frac{n}{n-2}}-\delta^2\frac{\partial}{\partial\nu}V_{\delta,q}\right|_{L_{\tilde g}^{\frac{2(n-1)}{n}}(\partial M)} \tag{34}$$
$$=O(1)\left|\left((U+\delta^2 v_{\delta,q})^+\right)^{\frac{n}{n-2}}-U^{\frac{n}{n-2}}-\delta^2\frac{\partial}{\partial\nu}v_q\right|_{L^{\frac{2(n-1)}{n}}(B^{n-1}_{1/\delta})}$$
$$=O(\delta^2)\left|\left((U+\theta\delta^2 v_{\delta,q})^+\right)^{\frac{2}{n-2}}v_q-U^{\frac{n}{n-2}}v_q\right|_{L^{\frac{2(n-1)}{n}}(B^{n-1}_{1/\delta})}.$$

By the decay estimates (18) we have that $U+\theta\delta v_q>0$ in $B^{n-1}_{1/\delta}$ provided δ small enough. So, expanding again we have

$$O(\delta^2)\left|\left((U+\theta\delta^2 v_{\delta,q})^+\right)^{\frac{2}{n-2}}v_q-U^{\frac{n}{n-2}}v_q\right|_{L^{\frac{2(n-1)}{n}}(B^{n-1}_{1/\delta})} \tag{35}$$
$$=O(\delta^3)\left|\delta\left((U+\theta_1\delta^2 v_{\delta,q})^+\right)^{\frac{4-n}{n-2}}v_q^2\right|_{L^{\frac{2(n-1)}{n}}(B^{n-1}_{1/\delta})}$$
$$=O(\delta^3)\left|\delta\,(1+|y|)^{4-n}\right|_{L^{\frac{2(n-1)}{n}}(B^{n-1}_{1/\delta})}$$
$$=O(\delta^3)\left|(1+|y|)^{3-n}\right|_{L^{\frac{2(n-1)}{n}}(B^{n-1}_{1/\delta})}=O(\delta^3)$$

since $n\geq 8$ and we get $I_3=O(\delta^3)\left\|R\right\|_g$ and, consequently,

$$\left\|i_g^*\left(f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q}\right)-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q}\right\|_g^2=O(\delta^3)\|R\|_g.$$

To conclude the first part of the proof we estimate the term

$$\left\|i_g^*\left(f_\varepsilon(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right)-i_g^*\left(f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right)\right\|_g$$

It is useful to recall the following Taylor expansions with respect to ε

$$U^\varepsilon=1+\varepsilon\ln U+\frac{1}{2}\varepsilon^2\ln^2 U+o(\varepsilon^2) \tag{36}$$

$$\delta^{-\varepsilon\frac{n-2}{2}}=1-\varepsilon\frac{n-2}{2}\ln\delta+\varepsilon^2\frac{(n-2)^2}{8}\ln^2\delta+o(\varepsilon^2\ln^2\delta) \tag{37}$$

We have that, recalling that $\Lambda_q(0)=0$,

$$\left\|i_g^*\left(f_\varepsilon(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q}\right)-i_g^*\left(f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q}\right)\right\|_g \tag{38}$$
$$=\left\|i_{\tilde g}^*\left(\tilde f_\varepsilon(W_{\delta,q}+\delta^2 V_{\delta,q}\right)-i_{\tilde g}^*\left(f_0(W_{\delta,q}+\delta^2 V_{\delta,q}\right)\right\|_{\tilde g}$$
$$\leq C\left|\Lambda_q^\varepsilon\left(W_{\delta,q}+\delta^2 V_{\delta,q}\right)^{\frac{n}{n-2}+\varepsilon}-\left(W_{\delta,q}+\delta^2 V_{\delta,q}\right)^{\frac{n}{n-2}}\right|_{L_{\tilde g}^{\frac{2(n-1)}{n}}(\partial M)}$$
$$=O(1)\left|\Lambda_q^\varepsilon(\delta y)\left(U+\delta^2 v_q\right)^{\frac{n}{n-2}+\varepsilon}-\left(U+\delta^2 v_q\right)^{\frac{n}{n-2}}\right|_{L^{\frac{2(n-1)}{n}}(B^{n-1}_{1/\delta})}$$
$$=O(1)\left|\left(\frac{\Lambda_q^\varepsilon(\delta y)}{\delta^{\varepsilon\frac{n-2}{2}}}(U+\delta^2 v_q)^\varepsilon-1\right)\left(U+\delta^2 v_q\right)^{\frac{n}{n-2}}\right|_{L^{\frac{2(n-1)}{n}}(B^{n-1}_{1/\delta})}$$
$$\leq O(1)\left|\left(-\frac{n-2}{2}\varepsilon\ln\delta+\varepsilon\ln(U+\delta^2 v_q)+O(\varepsilon^2\ln\delta)\right)U^{\frac{n}{n-2}}\right|_{L^{\frac{2(n-1)}{n}}(B^{n-1}_{1/\delta})}$$
$$=O(\varepsilon\log\delta)+O(\varepsilon),$$

and we have proved (28).

Step 2. It holds

$$|R_{\varepsilon,\delta,q}|_{L^{s_\varepsilon}(\partial M)}=\left\{\begin{array}{ll}\delta^{-O^+(\varepsilon)}\left\{O\left(\delta^3\log\delta\right)+O(\varepsilon\log\delta)+O(\varepsilon)\right\} & \text{if } n=8\\ O\left(\delta^3\right)+\delta^{-O^+(\varepsilon)}\left\{O(\varepsilon\log\delta)+O(\varepsilon)\right\} & \text{if } n>8\end{array}\right. . \tag{39}$$

As in the previous case we consider

$$\begin{aligned}|R|_{L_g^{s_\varepsilon}(\partial M)}\leq&\left|i_g^*\left(f_\varepsilon(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right)-i_g^*\left(f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right)\right|_{L_g^{s_\varepsilon}(\partial M)}\\&+\left|i_g^*\left(f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right)-\tilde W_{\delta,q}(x)-\delta^2\tilde V_{\delta,q}\right|_{L_g^{s_\varepsilon}(\partial M)}\end{aligned}$$

and we start estimating the second term. Taking again $\Gamma=i_g^*\left(f_0(W_{\delta,q}+\delta V_{\delta,q}\right)$ the solution of (29), the function $\Gamma-W_{\delta,q}-\delta V_{\delta,q}$ solves the problem

$$\left\{\begin{array}{ll}-\Delta_g(\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})+a(x)(\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q}) & \\ =-\Delta_g(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})+a(x)(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q}) & \text{on } M\\ & \\ \frac{\partial}{\partial\nu}(\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q})=f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})-\frac{\partial}{\partial\nu}(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q}) & \text{on } \partial M\end{array}\right. .$$

We choose $q=\frac{2n+n^2\left(\frac{n-2}{n-1}\right)\varepsilon}{n+2+2n\left(\frac{n-2}{n-1}\right)\varepsilon}$ and $r=\varepsilon$, so, by Remark 2, we get

$$\begin{aligned}|\Gamma-\tilde W_{\delta,q}-\delta^2\tilde V_{\delta,q}|_{L_g^{s_\varepsilon}(\partial M)}\leq&|-\Delta_g(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})+a(x)(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})|_{L_g^{q+\varepsilon}(M)}\\&+\left|f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})-\frac{\partial}{\partial\nu}(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right|_{L_g^{\frac{(n-1)q}{n-q}+\varepsilon}(\partial M)}.\end{aligned}$$

We remark that with our choice we can write $q=\frac{2n}{n+2}+O^+(\varepsilon)$, $\frac{1}{q+\varepsilon}=\frac{n+2}{2n}-O^+(\varepsilon)$ and $\frac{(n-1)q}{n-q}+\varepsilon=\frac{2(n-1)}{n}+O^+(\varepsilon)$ where $0<O^+(\varepsilon)<C\varepsilon$ for some positive constant C.

We proceed as in the first part, obtaining

$$|a(x)(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})|_{L_g^{q+\varepsilon}(M)}=O\left(\delta^{3-O^+(\varepsilon)}\right)\left|(1+|x|)^{3-n}\right|_{L_{\tilde g}^{\frac{2n}{n+2}+O^+(\varepsilon)}(B^n_{1/\delta})}.$$

At this point we have that, if $n>8$, $\left|(1+|x|)^{3-n}\right|_{L_{\tilde g}^{\frac{2n}{n+2}+O^+(\varepsilon)}(B^n_{1/\delta})}=O(1)$, while, if $n=8$, $\left|(1+|x|)^{3-n}\right|_{L_{\tilde g}^{\frac{2n}{n+2}+O^+(\varepsilon)}(B^n_{1/\delta})}=O\left(\delta^{-O^+(\varepsilon)}\log\delta\right)$. So we get

$$|a(x)(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})|_{L_g^{q+\varepsilon}(M)}=\left\{\begin{array}{ll}O\left(\delta^{3-O^+(\varepsilon)}\log\delta\right) & \text{if } n=8\\ O\left(\delta^3\right) & \text{if } n>8\end{array}\right. .$$

In the same spirit one can check that

$$|-\Delta_g(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})|_{L_g^{q+\varepsilon}(M)}=\left\{\begin{array}{ll}O\left(\delta^{3-O^+(\varepsilon)}\log\delta\right) & \text{if } n=8\\ O\left(\delta^3\right) & \text{if } n>8\end{array}\right. ;$$

$$\left|f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})-\frac{\partial}{\partial\nu}(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right|_{L_g^{\frac{(n-1)q}{n-q}+\varepsilon}(\partial M)}=\left\{\begin{array}{ll}O\left(\delta^{3-O^+(\varepsilon)}\log\delta\right) & \text{if } n=8\\ O\left(\delta^3\right) & \text{if } n>8\end{array}\right. .$$

To finish the proof we estimate

$$\left|i_g^*\left(f_\varepsilon(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right)-i_g^*\left(f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right)\right|_{L_g^{s_\varepsilon}(\partial M)}$$

Again, by Remark 2, we have

$$\left|i_g^*\left(f_\varepsilon(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right)-i_g^*\left(f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right)\right|_{L_g^{s_\varepsilon}(\partial M)}$$
$$\leq\left|f_\varepsilon(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})-f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right|_{L_g^{\frac{2(n-1)}{n}+O^+(\varepsilon)}(\partial M)}$$

and, proceeding as in (38) we obtain

$$\left|f_\varepsilon(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})-f_0(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\right|_{L_g^{\frac{2(n-1)}{n}+O^+(\varepsilon)}(\partial M)}$$
$$=\delta^{-O^+(\varepsilon)}\left\{O(\varepsilon\,|\ln\delta|)+O(\varepsilon)\right\},$$

and we have proved (39).

The last claim follows by direct computation. □

Proposition 7. *Let $\delta=\lambda\varepsilon^{\frac14}$ For $a,b\in\mathbb{R}$, $0<a<b$ there exists a positive constant $C=C(a,b)$ such that, for ε small, for any $q\in\partial M$ and for any $\lambda\in[a,b]$ there exists a unique $\phi_{\delta,q}$ which solves (21) with*

$$\|\phi_{\delta,q}\|_{\mathcal{H}_g}=\begin{cases}O\left(\varepsilon^{\frac34}\log\varepsilon\right) & \text{if } n=8\\ O\left(\varepsilon^{\frac34}\right) & \text{if } n>8\end{cases}.$$

Moreover the map $q\mapsto\phi_{\delta,q}$ is a $C^1(\partial M,\mathcal{H}_g)$ map.

Proof. First we prove that the nonlinear operator N defined (26) is a contraction on a suitable ball of $\mathcal{H}$. Recalling that

$$\|N(\phi_1)-N(\phi_2)\|_{\mathcal{H}}=\|N(\phi_1)-N(\phi_2)\|_H+|N(\phi_1)-N(\phi_2)|_{L^{s_\varepsilon}(\partial M)}$$

we estimate the two right hand side terms separately.

By the continuity of $i^*:L^{\frac{2(n-1)}{n}}(\partial M)\to H$, and by Lagrange theorem we have

$$\|N(\phi_1)-N(\phi_2)\|_H$$
$$\leq\|\left(f'_\varepsilon\left(W_{\delta,q}+\theta\phi_1+(1-\theta)\phi_2+\delta V_{\delta,q}\right)-f'_\varepsilon(W_{\delta,q}+\delta V_{\delta,q})\right)[\phi_1-\phi_2]\|_{L^{\frac{2(n-1)}{n}}(\partial M)}$$

and, since $|\phi_1-\phi_2|^{\frac{2(n-1)}{n}}\in L^{\frac{n}{n-2}}(\partial M)$ and $|f'_\varepsilon(\cdot)|^{\frac{2(n-1)}{n}}\in L^{\frac n2}(\partial M)$, we have

$$\|N(\phi_1)-N(\phi_2)\|_H$$
$$\leq\|\left(f'_\varepsilon\left(W_{\delta,q}+\theta\phi_1+(1-\theta)\phi_2+\delta V_{\delta,q}\right)-f'_\varepsilon(W_{\delta,q})+\delta V_{\delta,q}\right)\|_{L^{\frac{2(n-1)}{2}}(\partial M)}\|\phi_1-\phi_2\|_H$$
$$=\gamma\|\phi_1-\phi_2\|_H$$

where we can choose

$$\gamma:=\|\left(f'_\varepsilon\left(W_{\delta,q}+\theta\phi_1+(1-\theta)\phi_2+\delta V_{\delta,q}\right)-f'_\varepsilon(W_{\delta,q}+\delta V_{\delta,q})\right)\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)}<1,$$

provided $\|\phi_1\|_H$ and $\|\phi_2\|_H$ sufficiently small.

For the second term we argue in a similar way and, recalling that, by (4), $|i^*(g)|_{L^{s_\varepsilon}(\partial M)}\leq|g|_{L^{\frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\varepsilon}}(\partial M)}$, we have

$$|N(\phi_1)-N(\phi_2)|_{L^{s_\varepsilon}(\partial M)}$$
$$\leq|\left(f'_\varepsilon\left(W_{\delta,q}+\theta\phi_1+(1-\theta)\phi_2+\delta V_{\delta,q}\right)-f'_\varepsilon(W_{\delta,q}+\delta V_{\delta,q})\right)[\phi_1-\phi_2]|_{L^{\frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\varepsilon}}(\partial M)}$$

Since $\phi_1, \phi_2, W_{\delta,q}V_{\delta,q} \in L^{s_\varepsilon}$ we have that $|\phi_1 - \phi_2|^{\frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\varepsilon}} \in L^{\frac{n+(n-2)\varepsilon}{n-2}}(\partial M)$ and $|f'(\cdot)|^{\frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\varepsilon}} \in L^{\frac{n+(n-2)\varepsilon}{2+(n-2)\varepsilon}}(\partial M)$. So, as above, we can choose $|\phi_1|_{L^{s_\varepsilon}(\partial M)}$, $|\phi_2|_{L^{s_\varepsilon}(\partial M)}$ sufficiently small in order to get

$$|N(\phi_1) - N(\phi_2)|_{L^{s_\varepsilon}(\partial M)} \leq \gamma\, |\phi_1 - \phi_2|_{L^{s_\varepsilon}(\partial M)} .$$

So

$$\|N(\phi_1) - N(\phi_2)\|_{\mathcal{H}} \leq \gamma \|\phi_1 - \phi_2\|_{\mathcal{H}}$$

with $\gamma < 1$, provided $\|\phi_1\|_{\mathcal{H}}$, $\|\phi_2\|_{\mathcal{H}}$ small enough.

With the same strategy it is possible to prove that if $\|\phi\|_{\mathcal{H}}$ is sufficiently small there exists $\bar{\gamma} < 1$ such that $\|N(\phi)\|_{\mathcal{H}} \leq \bar{\gamma}\|\phi\|_{\mathcal{H}}$.

At this point, recalling Lemma 5 and Lemma 6, it is not difficult to prove that there exists a constant $C > 0$ such that, if $n > 8$ and $\|\phi\|_{\mathcal{H}} \leq C\varepsilon^{\frac{3}{4}}$ then the map

$$T(\phi) := L^{-1}(N(\phi) + R)$$

is a contraction from the ball $\|\phi\|_{\mathcal{H}} \leq C\varepsilon^{\frac{3}{4}}$ in itself. We proceed analogously for $n = 8$ and we get the first claim by the Contraction Mapping Theorem. The regularity claim can be proven via the Implicit Function Theorem. □

4. The reduced problem

For any choice of (δ, q), Proposition 7 states the we can solve the infinite dimensional problem (22). Now, set $\delta = \lambda\varepsilon^{\frac{1}{4}}$, we look for a critical point for the functional $J_{\varepsilon,g}$ having the form $\tilde{W}_{\lambda\varepsilon^{\frac{1}{4}},q} + \lambda^2\varepsilon^{\frac{1}{2}}\tilde{V}_{\lambda\varepsilon^{\frac{1}{4}},q} + \tilde{\phi}_{\lambda\varepsilon^{\frac{1}{4}},q}$.We define the function

$$I_\varepsilon(\lambda, q) := J_{\varepsilon,g}\left(\tilde{W}_{\lambda\varepsilon^{\frac{1}{4}},q} + \lambda^2\varepsilon^{\frac{1}{2}}\tilde{V}_{\lambda\varepsilon^{\frac{1}{4}},q} + \tilde{\phi}_{\lambda\varepsilon^{\frac{1}{4}},q}\right) \tag{40}$$

$$I_\varepsilon : [a, b] \times \partial M \to \mathbb{R} \tag{41}$$

Which will be useful in the next

Lemma 8. *Assume $n \geq 8$ and $\delta = \lambda\varepsilon^{\frac{1}{4}}$. It holds*

$$\left|I_\varepsilon(\lambda, q) - J_{\varepsilon,g}\left(\tilde{W}_{\lambda\varepsilon^{\frac{1}{4}},q} + \lambda^2\varepsilon^{\frac{1}{2}}\tilde{V}_{\lambda\varepsilon^{\frac{1}{4}},q}\right)\right| \\ \leq \left\|\tilde{\phi}_{\lambda\varepsilon^{\frac{1}{4}},q}\right\|^2_{\mathcal{H}_g} + C\left(\varepsilon\,|\log\varepsilon| + \varepsilon^{\frac{1}{2}}\right)\left\|\tilde{\phi}_{\lambda\varepsilon^{\frac{1}{4}},q}\right\|_{\mathcal{H}_g} = o(\varepsilon)$$

C^0-uniformly for $q \in \partial M$ and λ in a compact set of $(0, +\infty)$.

The proof of this result is similar to prove of [21, Lemma 6] and will be postponed in the Appendix.

At this point we can prove the main result of this section.

Proposition 9. *Assume $n \geq 8$ and $\delta = \lambda\varepsilon^{\frac{1}{4}}$. It holds*

$$J_{\varepsilon,g}(\tilde{W}_{\lambda\varepsilon^{\frac{1}{4}},q} + \lambda^2\varepsilon^{\frac{1}{2}}\tilde{V}_{\lambda\varepsilon^{\frac{1}{4}},q}) = A + B(\varepsilon) + \varepsilon\lambda^4\varphi(q) + C\varepsilon\ln\lambda + o(\varepsilon),$$

C^0-uniformly for $q \in \partial M$ and λ in a compact set of $(0, +\infty)$, where

$$\begin{aligned}
A =& \frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla U(t,z)|^2 dtdz - \frac{(n-2)^2}{2(n-1)}\int_{\mathbb{R}^{n-1}} U(0,z)^{\frac{2(n-1)}{(n-2)}}dz\\
B(\varepsilon) =& \varepsilon\left[\frac{(n-2)^3}{2(n-1)}\int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}}(z,0)dz - \frac{(n-2)^2}{2(n-1)}\int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}}(z,0)\ln U(z,0)dz\right]\\
& - \varepsilon|\ln\varepsilon|\frac{(n-2)^3}{16(n-1)}\int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}}(z,0)dz\\
\varphi(q) =& \frac{1}{2}\int_{\mathbb{R}^n_+} v_q\Delta v_q dtdz - \frac{n-2}{96(n-1)}|\bar W(q)|^2\int_{\mathbb{R}^n_+}|z|^2U^2(t,z)dtdz\\
& - \frac{(n-2)(n-8)}{2(n^2-1)}R^2_{ninj}(q)\int_{\mathbb{R}^n_+}\frac{t^2|z|^4}{((1+t)^2+|z|^2)^n}dtdz.\\
C =& \frac{(n-2)^3}{4(n-1)}\int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}}dz > 0.\\
R^2_{ninj}(q) =& R_{ninj}(q)R_{ninj}(q) = \sum_{i,j=1}^{n-1} R^2_{ninj}(q)
\end{aligned}$$

Proof. We write

$$\begin{aligned}
J_{\varepsilon,g}(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q}) = & J_{0,g}(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\\
& - \frac{(n-2)^2}{2(n-1)+\varepsilon(n-2)}\int_{\partial M}\left((\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})^+\right)^{\frac{2(n-1)}{n-2}+\varepsilon}d\sigma_g\\
& + \frac{(n-2)^2}{2(n-1)}\int_{\partial M}\left((\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})^+\right)^{\frac{2(n-1)}{n-2}}d\sigma_g,
\end{aligned}$$

where

$$\begin{aligned}
J_{0,g}(v) :=& \frac{1}{2}\int_M |\nabla_g v|^2 + av^2 d\mu_g + \frac{n-2}{4}\int_{\partial M} h_g v^2 d\sigma_g\\
& - \frac{(n-2)^2}{2(n-1)}\int_{\partial M}\left(v^+\right)^{\frac{2(n-1)}{n-2}}d\sigma_g.
\end{aligned}$$

Since $\frac{(n-2)^2}{2(n-1)+\varepsilon(n-2)} = \frac{(n-2)^2}{2(n-1)} - \varepsilon\frac{(n-2)^3}{2(n-1)} + o(\varepsilon)$ we have

$$\begin{aligned}
J_{\varepsilon,g}(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q}) = & J_{0,g}(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})\\
- \frac{(n-2)^2}{2(n-1)}\int_{\partial M} & \left((\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})^+\right)^{\frac{2(n-1)}{n-2}+\varepsilon} - \left((\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})^+\right)^{\frac{2(n-1)}{n-2}}d\sigma_g\\
& + \left[\varepsilon\frac{(n-2)^3}{2(n-1)}+o(\varepsilon)\right]\int_{\partial M}\left((\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})^+\right)^{\frac{2(n-1)}{n-2}+\varepsilon}d\sigma_g.
\end{aligned}$$

For $J_{0,g}$ we proceed as in [21, Lemma 8] (see also [17]) obtaining that

$$J_{0,g}(\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q}) = A + \delta^4\varphi(q) + o(\delta^4)$$

where

$$\begin{aligned}
A =& \frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla U(t,z)|^2 dtdz - \frac{(n-2)^2}{2(n-1)}\int_{\mathbb{R}^{n-1}} U(0,z)^{\frac{2(n-1)}{(n-2)}}dz\\
\varphi(q) =& \frac{1}{2}\int_{\mathbb{R}^n_+} v_q\Delta v_q dtdz - \frac{n-2}{96(n-1)}|\bar W(q)|^2\int_{\mathbb{R}^n_+}|z|^2U^2(t,z)dtdz\\
& - \frac{(n-2)(n-8)}{2(n^2-1)}R^2_{ninj}(q)\int_{\mathbb{R}^n_+}\frac{t^2|z|^4}{((1+t)^2+|z|^2)^n}dtdz.
\end{aligned}$$

Using again (36) and (37), proceeding similarly to (38), and recalling that $\delta = \lambda\varepsilon^{\frac{1}{4}}$ we have

$$
\begin{aligned}
&\int_{\partial M} \left((\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})^+\right)^{\frac{2(n-1)}{n-2}+\varepsilon} - \left((\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})^+\right)^{\frac{2(n-1)}{n-2}} d\sigma_g\\
&\quad= \int_{\partial M} \Lambda_q^\varepsilon \left((W_{\delta,q}+\delta^2 V_{\delta,q})^+\right)^{\frac{2(n-1)}{n-2}+\varepsilon} - \left((W_{\delta,q}+\delta^2 V_{\delta,q})^+\right)^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde g}\\
&\quad= (1+o(1))\int_{|z|<\frac{1}{\delta}} \frac{\Lambda_q^\varepsilon(\delta y)}{\delta^{\varepsilon\frac{n-2}{2}}}\left((U+\delta^2 v_q)^\varepsilon-1\right)(U+\delta^2 v_q)^{\frac{2(n-1)}{n-2}}dz\\
&\quad= \int_{\mathbb{R}^{n-1}}\left(-\frac{n-2}{8}\varepsilon\ln\varepsilon - \frac{n-2}{2}\varepsilon\ln\lambda+\varepsilon\ln(U)+o(\varepsilon)\right)U^{\frac{2(n-1)}{n-2}}dz\\
&\quad= \frac{n-2}{6}\varepsilon|\ln\varepsilon|\int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}}dz+\varepsilon\int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}}\ln(U)dz\\
&\qquad\qquad - \frac{n-2}{2}\varepsilon\ln\lambda\int_{\mathbb{R}^{n-1}} U^{\frac{2(n-1)}{n-2}}dz+o(\varepsilon).
\end{aligned}
$$

Finally, with the same technique,

$$
\begin{aligned}
&\left[\varepsilon\frac{(n-2)^3}{2(n-1)}+o(\varepsilon)\right]\int_{\partial M}\left((\tilde W_{\delta,q}+\delta^2\tilde V_{\delta,q})^+\right)^{\frac{2(n-1)}{n-2}+\varepsilon}d\sigma\\
&\quad=\left[\varepsilon\frac{(n-2)^3}{2(n-1)}+o(\varepsilon)\right]\int_{|z|<\frac{1}{\delta}}\frac{\Lambda_q(\delta y)}{\delta^{\varepsilon\frac{n-2}{2}}}(U+\delta v_q)^{\frac{2(n-1)}{n-2}}(U+\delta v_q)^\varepsilon dz+o(\delta^3)\\
&\qquad\qquad=\varepsilon\frac{(n-2)^3}{2(n-1)}\int_{\mathbb{R}^{n-1}}U^{\frac{2(n-1)}{n-2}}+o(\varepsilon).
\end{aligned}
$$

□

5. Proof of Theorem 1.

By the following result we prove that once we have a critical point of the reduced functional $I_\varepsilon(\lambda,q)$, we solve Problem (2). The proof of this result is very similar to the proof of [20, Claim (i) of Prop. 5], and we will omit it for the sake of brevity.

Lemma 10. *If $(\bar\lambda,\bar q)\in(0,+\infty)\times\partial M$ is a critical point for the reduced functional $I_\varepsilon(\lambda,q)$, then the function $\tilde W_{\bar\lambda\varepsilon^{\frac14},\bar q}+\bar\lambda^2\varepsilon^{\frac12}\tilde V_{\lambda\varepsilon^{\frac14},\bar q}+\tilde\phi_{\lambda\varepsilon^{\frac14},q}$ is a solution of (2). Here $\tilde\phi_{\lambda\varepsilon^{\frac14},q}$ is defined in Proposition 7.*

Lemma 11. *Assume $n\geq 8$ and that the Weyl tensor W_g is not vanishing on ∂M. Then the function $\varphi(q)$ defined in Proposition 9 is strictly negative on ∂M.*

Proof. By Proposition 9 we have that

$$\varphi(q)=\frac12\int_{\mathbb{R}^n_+} v_q\Delta v_q dtdz - C_1|\bar W(q)|^2-(n-8)C_2R^2_{ninj}(q),$$

where C_1, C_2 are positive constants. We recall (see [27, page 1618]) that when the boundary is umbilic $W(q)=0$ if and only if $\bar W(q)$ and $R_{ninj}(q)$ are both zero, so, by our assumption, we have that at least one among $|\bar W(q)|$ and $R^2_{nlnj}(q)$ which is strictly positive. This, combined with (20), implies that, $\varphi(q)$ is strictly negative for $n>8$.

When $n=8$ the same strategy leads only to the weak inequality $\varphi(q)\leq 0$. To overcome this difficulty, a delicate analysis of the term $\int_{\mathbb{R}^n_+} v_q\Delta v_q dtdz$ is needed. An improvement of the estimate (20) is performed in [19], where the authors give a more precise description of the function v_q as a sum of an harmonic function with explicit rational functions. This description, for $n=8$, leads to the inequality, proved in [19, Lemma 19],

$$\int_{\mathbb{R}^8_+} v_q \Delta v_q dy \leq -C_3 R^2_{8i8j}(q),$$

where $C_3>0$. Thus for $n=8$ we have

$$\varphi(q) \leq -C_1 |\bar{W}(q)|^2 - C_3 R^2_{8i8j}(q) < 0,$$

which completes the proof. □

Proof of Theorem 1. By Lemma 11 we have that the function $\varphi(q)$ defined in Proposition 9 is strictly negative on ∂M. We recall as well, that the number C defined in the same proposition is positive. Defined

$$I : [a,b] \times \partial M \to \mathbb{R}$$
$$I(\lambda, q) = \lambda^4 \varphi(q) + C \log \lambda$$

we notice that $\max_{\partial M} I(\lambda, \cdot)$ tends to $-\infty$ as $\lambda \to 0$ or $\lambda \to +\infty$. It follows that there exist $0<a<b<+\infty$ such that the function I admits a absolute maximum on $(a,b)\times \partial M$. This maximum is also C^0-stable. in other words, if (λ_0, q_0) is the maximum point for I, for any function $f \in C^1([a,b]\times\partial M)$ with $\|f\|_{C^0}$ sufficiently small, then the function $I+f$ on $[a,b]\times \partial M$ admits a maximum point $(\bar\lambda, \bar q)$ close to (λ_0, q_0).

Then, taken an ε sufficiently small, in light of Lemma 8 and Proposition 9, there exists a pair $(\lambda_\varepsilon, q_\varepsilon)$ maximum point for $I_\varepsilon(\lambda, q)$. Thus, by Lemma 10, $v_\varepsilon := \tilde{W}_{\bar\lambda\varepsilon^{\frac14}, \bar q} + \bar\lambda^2 \varepsilon^{\frac12} \tilde{V}_{\lambda\varepsilon^{\frac14}, \bar q} + \tilde\phi_{\lambda\varepsilon^{\frac14}, q} \in \mathcal{H}_g$ is a solution of (2). By construction v_ε blows up at $q_\varepsilon \to q_0$ when $\varepsilon \to 0$. □

6. Appendix

Proof of Lemma 8. We have, for some $\theta \in (0,1)$

$$\begin{aligned}
\tilde J_{\varepsilon,\tilde g_q}(W_{\delta,q} + \delta^2 V_{\delta,q} + \phi_{\delta,q}) - \tilde J_{\varepsilon,\tilde g_q}(W_{\delta,q} + \delta^2 V_{\delta,q}) &= \tilde J'_{\varepsilon,\tilde g_q}(W_{\delta,q} + \delta^2 V_{\delta,q})[\phi_{\delta,q}] \\
&\quad + \frac12 \tilde J''_{\varepsilon,\tilde g_q}(W_{\delta,q} + \delta^2 V_{\delta,q} + \theta\phi_{\delta,q})[\phi_{\delta,q}, \phi_{\delta,q}] \\
= \int_M \left(\nabla_{\tilde g_q} W_{\delta,q} + \delta^2 \nabla_{\tilde g_q} V_{\delta,q}\right) \nabla_{\tilde g_q}\phi_{\delta,q} &+ \tilde a(x)\left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)\phi_{\delta,q} d\mu_{\tilde g_q} \\
- (n-2)\int_{\partial M} \Lambda_q^\varepsilon \left(\left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}+\varepsilon} \phi_{\delta,q} d\sigma_{\tilde g_q} \\
+ \frac{n-2}{2}\int_{\partial M} h_{\tilde g_q}\left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)\phi_{\delta,q} d\sigma_{\tilde g_q} \\
+ \frac12 \int_M |\nabla_{\tilde g_q}\phi_{\delta,q}|^2 + \tilde a(x)\phi^2_{\delta,q} d\mu_{\tilde g_q} + \frac{n-2}{4}\int_{\partial M} h_{\tilde g_q}\phi^2_{\delta,q} d\sigma_{\tilde g_q} \\
- \frac{n+\varepsilon(n-2)}{2}\int_{\partial M} \Lambda_q^\varepsilon \left(\left(W_{\delta,q} + \delta^2 V_{\delta,q} + \theta\phi_{\delta,q}\right)^+\right)^{\frac{2}{n-2}+\varepsilon} \phi^2_{\delta,q} d\sigma_{\tilde g_q}.
\end{aligned}$$

Immediately we have, by definition of $\|\cdot\|_{\tilde g}$,

$$\int_M |\nabla_{\tilde g_q}\phi_{\delta,q}|^2 + \tilde a \phi^2_{\delta,q} d\mu_{\tilde g_q} + \int_{\partial M} \frac{n-2}{4} h_{\tilde g_q}\phi^2_{\delta,q} d\sigma = C\|\phi_{\delta,q}\|^2_{\tilde g_q} = o(\varepsilon).$$

By Holder inequality one can easily obtain

$$\int_M \tilde a W_{\delta,q}\phi_{\delta,q} d\mu_{\tilde g_q} \leq C|W_{\delta,q}|_{L_{\tilde g}^{\frac{2n}{n+2}}} |\phi_{\delta,q}|_{L_{\tilde g}^{\frac{2n}{n-2}}} \leq C\delta^2 \|\phi_{\delta,q}\|_{\tilde g} = o(\varepsilon)$$

and

$$\delta^2\int_M \tilde a V_{\delta,q}\phi_{\delta,q}d\mu_{\tilde g_q}\le C\delta^2|V_{\delta,q}|_{L^2_{\tilde g}}|\phi_{\delta,q}|_{L^2_{\tilde g}}\le C\delta^2\|\phi_{\delta,q}\|_{\tilde g}=o(\varepsilon).$$

In addition, notice that $\left(\left(W_{\delta,q}+\delta^2V_{\delta,q}+\theta\phi_{\delta,q}\right)^+\right)^{\frac{2}{n-2}+\varepsilon}$ belongs to $L_{\tilde g}^{\frac{2(n-1)+n(n-2)\varepsilon}{2+(n-2)\varepsilon}}$ and that $2\left(\frac{2(n-1)+n(n-2)\varepsilon}{2+(n-2)\varepsilon}\right)'=\frac{4(n-1)+2n(n-2)\varepsilon}{2(n-2)+(n-1)(n-2)\varepsilon}<s_\varepsilon$, so, again by Holder inequality we have

$$\begin{aligned}\int_{\partial M}\Lambda_q^\varepsilon\left(\left(W_{\delta,q}+\delta^2V_{\delta,q}+\theta\phi_{\delta,q}\right)^+\right)^{\frac{2}{n-2}+\varepsilon}\phi^2_{\delta,q}d\sigma_{\tilde g_q}&\\ \le C\left(\left|W_{\delta,q}+\delta^2V_{\delta,q}+\theta\phi_{\delta,q}\right|^{\frac{2}{n-2}}_{L^{s_\varepsilon}_{\tilde g}}\right)\|\phi_{\delta,q}\|^2_{\tilde g}=o(\varepsilon).\end{aligned}$$

and by(30)

$$\begin{aligned}&\int_{\partial M}h_{\tilde g_q}(W_{\delta,q}+\delta^2V_{\delta,q})\phi_{\delta,q}d\sigma_{\tilde g_q}\\ &\qquad\le|h_{\tilde g}(W_{\delta,q}+\delta^2V_{\delta,q})|_{L_{\tilde g}^{\frac{2(n-1)}{n}}(\partial M)}\|\phi_{\delta,q}\|_{\tilde g}=O(\delta^3)\|\phi_{\delta,q}\|_{\tilde g}=o(\varepsilon).\end{aligned}$$

By integration by parts we have

(42)
$$\begin{aligned}\int_M\left(\nabla_{\tilde g_q}W_{\delta,q}+\delta^2\nabla_{\tilde g_q}V_{\delta,q}\right)\nabla_{\tilde g_q}\phi_{\delta,q}d\mu_{\tilde g_q}&=-\int_M\Delta_{\tilde g_q}\left(W_{\delta,q}+\delta^2V_{\delta,q}\right)\phi_{\delta,q}d\mu_{\tilde g_q}\\&+\int_{\partial M}\left(\frac{\partial}{\partial\nu}W_{\delta,q}+\delta^2\frac{\partial}{\partial\nu}V_{\delta,q}\right)\phi_{\delta,q}d\sigma_{\tilde g_q}.\end{aligned}$$

and, as in (31), we get

$$\begin{aligned}&\int_M\Delta_{\tilde g_q}\left(W_{\delta,q}+\delta^2V_{\delta,q}\right)\phi_{\delta,q}d\mu_{\tilde g_q}\\&\qquad\le|\Delta_{\tilde g_q}(W_{\delta,q}+\delta^2V_{\delta,q})|_{L_{\tilde g}^{\frac{2n}{n+2}}}\|\phi_{\delta,q}\|_{\tilde g}=O(\delta^2)\|\phi_{\delta,q}\|_{\tilde g}=o(\varepsilon),\end{aligned}$$

and for the boundary term in (42), we have, in light of (32), (33), (34) and (35), that

$$\begin{aligned}&\int_{\partial M}\left[\left(\frac{\partial}{\partial\nu}W_{\delta,q}+\delta^2\frac{\partial}{\partial\nu}V_{\delta,q}\right)-(n-2)\left(\left(W_{\delta,q}+\delta^2V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}}\right]\phi_{\delta,q}d\sigma_{\tilde g_q}\\&\quad=\left|(n-2)\left(\left(W_{\delta,q}+\delta^2V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}}-\frac{\partial}{\partial\nu}W_{\delta,q}\right|_{L_{\tilde g}^{\frac{2(n-1)}{n}}}|\phi_{\delta,q}|_{L_{\tilde g}^{\frac{2(n-1)}{n-2}}}\\&\quad=O(\delta^3)\|\phi_{\delta,q}\|_{\tilde g}=o(\varepsilon).\end{aligned}$$

At this point it remains to estimate

$$\int_{\partial M}\left[\Lambda_q^\varepsilon\left(\left(W_{\delta,q}+\delta^2V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}+\varepsilon}-\left(\left(W_{\delta,q}+\delta^2V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}}\right]\phi_{\delta,q}d\sigma_{\tilde g_q}$$

and we proceed as in (38) to get

$$\begin{aligned}&\int_{\partial M}\left[\Lambda_q^\varepsilon\left(\left(W_{\delta,q}+\delta^2V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}+\varepsilon}-\left(\left(W_{\delta,q}+\delta^2V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}}\right]\phi_{\delta,q}d\sigma_{\tilde g_q}\\&\le\left|\Lambda_q^\varepsilon\left(\left(W_{\delta,q}+\delta^2V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}+\varepsilon}-\left(\left(W_{\delta,q}+\delta^2V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}}\right|_{L_{\tilde g}^{\frac{2(n-1)}{n}}}\|\phi_{\delta,q}\|_{\tilde g}=o(\varepsilon),\end{aligned}$$

ending the proof. $\square$

References

[1] S. Almaraz, *An existence theorem of conformal scalar-flat metrics on manifolds with boundary.* Pacific J. Math. **248** (2010), 1–22.

[2] S. Almaraz, *Blow-up phenomena for scalar-flat metrics on manifolds with boundary*, J. Differential Equations **251** (2011), 1813–1840.

[3] S. Almaraz, *A compactness theorem for scalar-flat metrics on manifolds with boundary*, Calc. Var. Partial Differential Equations **41** (2011), 341–386.

[4] A. Ambrosetti, Y.Y. Li, A. Malchiodi, *On the Yamabe problem and the scalar curvature problems under boundary conditions*, Math. Ann. **322**, (2002) 667–699

[5] T. Aubin, *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976), 269–296.

[6] S. Brendle, *Blow-up phenomena for the Yamabe equation*, J. Amer. Math. Soc. **21** (2008), 951–979.

[7] S. Brendle, S. S. Chen, *An existence theorem for the Yamabe problem on manifolds with boundary*, J. Eur. Math. Soc. **16** (2014), 991–1016.

[8] S. Brendle, F. Marques, *Blow-up phenomena for the Yamabe equation II*, J. Differential Geom. **81** (2009), 225–250.

[9] X. Chen, Y. Ruan, L. Sun, *The Han-Li conjecture in constant scalar curvature and constant boundary mean curvature problem on compact manifolds.* Adv. Math. **358** (2019), 56 pp.

[10] O. Druet, *Compactness for Yamabe metrics in low dimensions*, Int. Math. Res. Not. 23 (2004), 1143–1191.

[11] O. Druet and E. Hebey, *Blow-up examples for second order elliptic PDEs of critical Sobolev growth*, Trans. Amer. Math. Soc. 357 (2005), no. 5, 1915–1929 (electronic).

[12] V. Felli, M. Ould Ahmedou, *Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries*, Math. Z. **244** (2003), 175–210.

[13] S. Kim, M. Musso, J. Wei, Juncheng *Existence theorems of the fractional Yamabe problem.* Anal. PDE **11** (2018), 75-113.

[14] S. Kim, M. Musso, J. Wei, *Compactness of scalar-flat conformal metrics on low-dimensional manifolds with constant mean curvature on boundary*, in press on Ann. Institut H. Poincaré C, An. non lin.

[15] M. Khuri, F. Marques, R. Schoen, *A compactness theorem for the Yamabe problem*, J. Differential Geom. **81** (2009), 143–196.

[16] J. F. Escobar, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary.* Ann. of Math. **136** (1992), no. 1, 1–50.

[17] M. Ghimenti, A.M. Micheletti, *Compactness for conformal scalar-flat metrics on umbilic boundary manifolds*, Nonlinear Analysis, **200** (2020).

[18] M. Ghimenti, A. M. Micheletti, *Compactness results for linearly perturbed Yamabe problem on manifolds with boundary*, Discr. Cont. Dyn. Syst. Series S,**14** (2021) 1757–1778

[19] M. Ghimenti, A.M. Micheletti, A compactness result for scalar-flat metrics on low dimensional manifolds with umbilic boundary, in press on Calc Var PDE

[20] M. Ghimenti, A.M. Micheletti, A. Pistoia, *On Yamabe type problems on Riemannian manifolds with boundary*, Pacific Journal of Math. **284** (2016), 79–102.

[21] M. Ghimenti, A.M. Micheletti, A. Pistoia, *Blow-up phenomena for linearly perturbed Yamabe problem on manifolds with umbilic boundary* , J. Diff. Eq. **267** (2019) 587–618.

[22] Z.C. Han, Y. Li, *The Yamabe problem on manifolds with boundary: existence and compactness results.* Duke Math, J. **99** (1999), 489–542.

[23] Z.C. Han, Y. Li, *The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature.* Comm. Anal. Geom. **8** (2000), 809–869.

[24] Y.Y. Li, L. Zhu, *Yamabe type equations on three-dimensional Riemannian manifolds*, Commun. Contemp. Math. **1** (1999) 1–50

[25] Y.Y. Li, L. Zhang, *Compactness of solutions to the Yamabe problem II*, Calc. Var. PDE **24** (2005) 185–237.

[26] Y.Y. Li. L. Zhang, *Compactness of solutions to the Yamabe problem III*, J. Funct. Anal. **245** (2007) 438–474,

[27] F.C. Marques, *Existence results for the Yamabe problem on manifolds with boundary*, Indiana Univ. Math. J. **54** (2005) 1599–1620.

[28] F.C. Marques, *A-priori estimates for the Yamabe problem in the non-locally conformally flat case*, J. Differential Geom. **71** (2005) 315–346.

[29] M. Mayer, C.B. Ndiaye, *Barycenter technique and the Riemann mapping problem of Cherrier-Escobar.* J. Differential Geom. **107** (2017), 519–560.

[30] R. Nittka, *Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains.* J. Diff. Eq. **251** (2011), 860–880.

[31] R.M. Schoen,: *Conformal deformation of a Riemannian metric to constant scalar curvature* J. Differ. Geom. **20** (1984), 479–495.
[32] R.M. Schoen, Notes from graduates lecture in Stanford University (1988). http://www.math.washington.edu/pollack/research/Schoen-1988-notes.html.
[33] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Annali Scuola Norm. Sup. Pisa **22** (1968), 265–274.
[34] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21–37.

M. Ghimenti,
Dipartimento di Matematica Università di Pisa Largo B. Pontecorvo 5, 56126 Pisa, Italy
E-mail address: marco.ghimenti@unipi.it

A. M. Micheletti,
Dipartimento di Matematica Università di Pisa Largo B. Pontecorvo 5, 56126 Pisa, Italy
E-mail address: a.micheletti@dma.unipi.it.