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Reducing the computational effort of MPC with closed-loop optimal sequences of affine laws

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Abstract: We consider the classical infinite-horizon constrained linear-quadratic regulator (CLQR) problem and its receding-horizon variant used in model predictive control (MPC). If the terminal constraints are inactive for the current initial condition, the optimal input signal sequence that results for the open-loop CLQR problem is equal to the closed-loop optimal sequence that results for MPC. Consequently, the closed-loop optimal solution is available from solving only one CLQR problem instead of the usual infinite number of CLQR problems solved on the receding horizon. In the presence of disturbances or because of plant-model mismatch, the system will eventually leave the predicted optimal trajectory. Consequently, the solution of the single open-loop CLQR problem is no longer optimal, and the receding horizon problem must resume. We show, however, that the open-loop solution is also robust. Robustness essentially is given, because the solution of the CLQR problem not only provides the sequence of nominally optimal input signals, but a sequence of optimal affine laws along with their polytopes of validity. We analyze the degree of robustness by computational experiments. The results indicate the degree of robustness is practically relevant.

1. INTRODUCTION

Linear-quadratic MPC is known to result in a piecewise-affine optimal feedback law. Because this law quickly grows complex as the problem dimensions or the horizon is increased, it is often preferable to solve the underlying optimal control problem online, thus evaluating the optimal feedback law point by point without ever calculating it explicitly.

We propose to use the piecewise-affine structure (and another property, called persistency, explained below) without calculating the full explicit solution. Essentially, we show how to infer a sequence of optimal control laws for time points $k+1, k+2, \ldots$ from the pointwise solution at time point k. We stress we do *not* use the *open-loop optimal input sequence* that results at time point k, but our approach results in the same *closed-loop* optimal feedback as standard receding horizon MPC. Because we use optimal laws instead of optimal points, the approach is robust with respect to disturbances to a certain degree. It is the purpose of the paper to present the idea and to analyze its robustness with computational experiments.

Other ideas have been explored on how to use the structure of the solution without calculating it explicitly. By storing regions of activity of the constraints instead of the affine pieces, the storage requirements can be reduced to grow only linearly in the number of constraints (Jost and Mönnigmann, 2013; Jost and Mönnigmann, 2013). Methods that determine and reuse the polytope for time step k have been proposed before; larger regions on which optimality is lost but stability is guaranteed can also be constructed (Jost et al., 2015). Another class of methods accelerates the online computations by storing the most frequent active sets (Pannocchia et al., 2007, 2011). Reachability analysis has been used to remove irrelevant regions (Kvasnica et al., 2019). Since neighboring polytopes sometimes have the same feedback law in common, the number of regions relevant for MPC can be reduced (Kvasnica and Fikar, 2012).

Problem statement

We consider the problem of solving, on a receding horizon of length N, $V_N^\star(x(0)) :=$

$$\min_{\substack{u(k),\, k=0,\ldots,N-1\\ x(k),\, k=1,\ldots,N}} \tfrac{1}{2}\|x(N)\|_P^2 + \tfrac{1}{2}\sum_{k=0}^{N-1}\left(\|x(k)\|_Q^2 + \|u(k)\|_R^2\right) \tag{1a}$$

subject to

$$\begin{aligned} x(k+1) = Ax(k) + Bu(k),\ k = 0,\ldots,N-1\\ u(k) \in \mathcal{U},\ k = 0,\ldots,N-1\\ x(k) \in \mathcal{X},\ k = 0,\ldots,N-1\\ x(N) \in \mathcal{T}, \end{aligned} \tag{1b}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are the state and input variables at time point k, respectively, and $x(0)$ is a known initial condition. We assume all matrices to have appropriate dimensions, (A, B) to be stabilizable, and $Q \succeq 0$, $R \succ 0$, and $\mathcal{X}$, $\mathcal{U}$ to be compact polytopes that contain the origin in their interiors. Furthermore, P is the

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solution to the discrete time algebraic Riccati equation and $\mathcal{T}$ is chosen to be the largest invariant set such that the solution to (1) and to its unconstrained counterpart are equal whenever both exist.

Let $\mathcal{F}_N$ be the set of initial states $x(0)$ for which (1) has a solution and note $\mathcal{F}_N \neq \emptyset$ since $\mathcal{F}_N \supseteq \cdots \supseteq \mathcal{F}_1 \supseteq \mathcal{T} \neq \emptyset$.

2. KNOWN RESULTS

2.1 Piecewise-affine optimal solution

The CLQR problem (1) is a quadratic program with a strictly convex cost function. There exist $H \in \mathbb{R}^{mN\times mN}$, $H \succ 0$, $F \in \mathbb{R}^{n\times mN}$, $Y \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{q\times mN}$, $E \in \mathbb{R}^{q\times n}$, $w \in \mathbb{R}^q$ such that (1) is equivalent to

$$\min_{\underline{u}} \frac{1}{2}\underline{u}' H \underline{u} + x' F \underline{u} + \frac{1}{2} x' Y x \tag{2a}$$

subject to

$$G\underline{u} - Ex - w \leq 0 \tag{2b}$$

where q is the number of inequality constraints in (1) and (2), $\underline{u} = (u(0)^\top, \ldots, u(N-1)^\top)^\top$ and x refers to the initial condition from hereon by a slight (but common) abuse of notation. For any x such that a solution to (1), or equivalently to (2), exists, let

$$\underline{u}^\star = (u^{\star\top}(0), u^{\star\top}(1), \ldots, u^{\star\top}(N-1))^\top \tag{3}$$

refer to the optimal input sequence and let

$$\mathcal{A} = \{i | G_i \underline{u}^\star - E_i x - w_i = 0\} \tag{4a}$$
$$\mathcal{I} = \{i | G_i \underline{u}^\star - E_i x - w_i < 0\} \tag{4b}$$

be the active and inactive set, respectively, where G_i (E_i, w_i, etc.) refers to the ith line of G (E_i, w_i, etc.). $G_\mathcal{A}$ ($E_\mathcal{A}$, $w_\mathcal{A}$ etc.) refers to the submatrix of G (E, w etc.) with the rows indicated by $\mathcal{A}$. We say $\mathcal{A}$ *is an active set of* (2) (or equivalently of (1)) if there exists an $x \in \mathcal{F}_N$ such that the constraints $i \in \mathcal{A}$ are active at its optimal solution and all other constraints are inactive. Then the active and inactive set immediately result from (4).

The following lemma is an immediate consequence of the statements in (Bemporad et al., 2002).

Lemma 1. Let $x \in \mathcal{F}_N$ be arbitrary and assume the quadratic program (2) has been solved. Let $\mathcal{A}$ and $\mathcal{I}$ refer to the active and inactive set at the optimal solution. Assume the matrix $G_\mathcal{A}$ to have full rank. Let

$$\begin{aligned} K &= H^{-1}(G_\mathcal{A})'\Gamma S_\mathcal{A} - H^{-1}F', \\ b &= H^{-1}(G_\mathcal{A})'\Gamma w_\mathcal{A}, \\ T &= \begin{pmatrix} G_\mathcal{I} H^{-1}(G_\mathcal{A})'\Gamma S_\mathcal{A} - S_\mathcal{I} \\ \Gamma S_\mathcal{A} \end{pmatrix}, \\ d &= -\begin{pmatrix} G_\mathcal{I} H^{-1}(G_\mathcal{A})'\Gamma w_\mathcal{A} - w_\mathcal{I} \\ \Gamma w_\mathcal{A} \end{pmatrix}, \end{aligned} \tag{5}$$

where $\Gamma = (G_\mathcal{A} H^{-1}(G_\mathcal{A})')^{-1}$ and $S = E + GH^{-1}F'$, $S \in \mathbb{R}^{q\times n}$. Let

$$\mathcal{P}^\star = \{x \in \mathbb{R}^n \,|\, T^\star x \leq d^\star\}. \tag{6}$$

Then, $\underline{u}^\star(\cdot) : \mathcal{P}^\star \to \mathbb{R}^{Nm}$ defined by

$$x \to K^\star x + b^\star \tag{7}$$

is the optimal input sequence introduced in (3) for all $x \in \mathcal{P}^\star$ and the first m components of $\underline{u}^\star(\cdot)$ define the optimal feedback law $u^\star(\cdot) : \mathcal{P}^\star \to \mathbb{R}^m$.

Lemma 1 essentially provides an optimal feedback *law* from *a single optimal* point. More specifically, assume (2) has been solved for an arbitrary but fixed point $x \in \mathcal{F}_N$ and let $\underline{u}^\star \in \mathbb{R}^{mN}$ refer to the resulting point in $\mathbb{R}^{mN}$. Then the active and inactive set immediately result from evaluating (4) and the optimal feedback law $\underline{u}^\star(\cdot)$ and its polytope of validity $\mathcal{P}^\star$ can be determined with Lemma 1 without solving an optimization or otherwise costly problem. The computational effort for calculating the matrices in (1) is known to be smaller than for solving a QP (2) (see Berner and Mönnigmann (2019) for details).

We stress again we never determine *all* polytopes and affine laws that constitute the parametric solution to (2), but Lemma 1 will be applied to very few polytopes only.

3. DETERMINING SEQUENCES OF CLOSED-LOOP OPTIMAL FEEDBACK LAWS

It is important to sort the constraints in a stagewise order for the results to follow. We assume the following order in (1) and (2) without restriction

$$\begin{array}{lll} u(0) \in \mathcal{U}, & x(0) \in \mathcal{X} & \text{(stage 0)} \\ u(1) \in \mathcal{U}, & x(1) \in \mathcal{X} & \text{(stage 1)} \\ \vdots & & \\ u(N-1) \in \mathcal{U}, & x(N-1) \in \mathcal{X} & \text{(stage } N-1) \\ & x(N) \in \mathcal{T} & \text{(stage } N) \end{array} \tag{8}$$

There exist $q_\mathcal{U} + q_\mathcal{X}$ constraints in stages $0, \ldots, N-1$ and $q_\mathcal{T}$ constraints in stage N, where $q_\mathcal{U}$, $q_\mathcal{X}$ and $q_\mathcal{T}$ are the number of halfspaces that define $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{T}$, respectively.

It furthermore proves to be convenient to state active sets as sequences of bits. More precisely, for any $\mathcal{A} \subset \{1, \ldots, q\}$, let the sequence α of q bits α_k be defined by

$$\alpha_i = \begin{cases} 1 & \text{if } i \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

We use $\mathcal{A}$ and α interchangeably (e.g, $G_\alpha = G_\mathcal{A}$ if α is as in (9) for $\mathcal{A}$). By $\alpha^1\alpha^2$ we denote the concatenation of the two bit sequences α^1 and α^2. We split the bit sequences α into stages for ease of interpretation. For example, the active set $\mathcal{A} = \{2, 15\}$ corresponds to

$$\alpha = \underbrace{010000}_{\substack{q_\mathcal{U}+q_\mathcal{X} \\ (k=0)}} \cdot \underbrace{000000}_{\substack{q_\mathcal{U}+q_\mathcal{X} \\ (k=1)}} \cdot \underbrace{0010}_{\substack{q_\mathcal{T} \\ (k=2)}},$$

where k enumerates stages, appears in the example with $N = 2$, $q_\mathcal{U} = 2$, $q_\mathcal{X} = 4$ and $q_\mathcal{T} = 4$ treated below. More generally, α^k in $\alpha = \alpha^0\,\alpha^1 \cdots \alpha^{N-1}\,\alpha^N$ denotes the subsequence of bits that belongs to stage k.

An optimal feedback law and its polytope that result for horizon N are in general not an optimal law and polytope for horizon $N+1$ (see, e.g., Fig. 2 in (Muñoz de la Peña et al., 2004) or Fig. 1 in (Mönnigmann, 2019)). There exist feedback laws and polytopes, however, that remain the same when the horizon is increased. We say an optimal control law and its polytope are *persistent from horizon N on* if they exist for all $N + l$, $l \geq 0$. In fact, it is easy to find persistent polytopes by analyzing active sets assuming the stated constraint order and the α-notation, as stated by the next result.

Lemma 2. Let α be an active set for (1) with horizon N. If the terminal constraints are inactive, i.e., there exists an $\tilde{\alpha}$ such that

$$\alpha = \tilde{\alpha}\underbrace{0\cdots 0}_{q_{\mathcal{T}}} \tag{10}$$

then the affine law and polytope that α defines according to Lemma 1 is persistent from N on. Moreover, the optimal feedback law and polytope that result with Lemma 1 for α and horizon N are equal to the optimal feedback law and polytope that result with Lemma 1 for

$$\tilde{\alpha}\ \underbrace{0\cdots 0}_{l\cdot(q_{\mathcal{U}}+q_{\mathcal{X}})}\ \underbrace{0\cdots 0}_{q_{\mathcal{T}}} \tag{11}$$

and horizon $N+l$ for any $l \geq 0$.

The proof of Lemma 2 can be found in (Mönnigmann, 2019, Lemma 3 and Proposition 4). Lemma 2 is quite technical but based on a simple idea: Assume the CLQR problem (1) resulted in an optimal sequence $x^\star(1), \ldots, x^\star(N)$ such that $x^\star(N)$ is in the interior of the terminal set $\mathcal{T}$ (equivalent to the $q_{\mathcal{T}}$ zeroes in (10) and (11)). We can extend the optimal sequence by an arbitrary number of additional inactive stages (those with the $l\cdot(q_{\mathcal{U}}+q_{\mathcal{X}})$ zeros in (11)). These additional inactive stages correspond to steps taken with the optimal solution to the unconstrained LQR controller for which the terminal set is control invariant (Chmielewski and Manousiouthakis, 1996; Scokaert and Rawlings, 1998). This is illustrated with Figure 1, which shows the solution to the CLQR problem 1 for Example 1 stated in Section 4. The label '0' marks the initial condition ($x(0) = (-0.33, 0.45)^\top$). The points marked '0', '1' and '2' result for the CLQR problem with $N = 2$. The points marked '0' to '8' result for $N = 8$, where '0', '1', and '2' coincide for both horizons. The initial condition results in

$$\underbrace{000001}_{q_{\mathcal{X}}+q_{\mathcal{U}}=6}.000001.\underbrace{000000000000}_{q_{\mathcal{T}}=12} \tag{12}$$

for $N = 2$ and

$$000001.000001.\underbrace{000000.\cdots.000000}_{6\text{ stages}}.000000000000 \tag{13}$$

for $N = 8$, where colors are used to highlight the stages both active sets have in common. In accordance with Lemma 1, $l = 6$ stages with inactive constraints are inserted in (13). Since $q_{\mathcal{X}} = 4$ and $q_{\mathcal{U}} = 2$, each stage corresponds to 6 constraints. The terminal set $\mathcal{T}$ is defined by $q_{\mathcal{T}} = 12$ constraints in the example, which correspond to the 12 trailing zeroes in (12) and (13). The additional 6 inactive stages in (13) correspond to the application of the optimal feedback of the unconstrained LQR problem and result in points '2' to '7' in Figure 1. The number of additional stages, $l = 6$, is chosen arbitrarily in the example. The results for $l = 1, \ldots, 5$, which correspond to (13) with the respective number of inserted inactive stages, is evident from Figure 1 and $l > 6$ results in additional inactive stages, since the terminal set $\mathcal{T}$ is invariant under the unconstrained optimal feedback.

Based on Lemma 2, we can state the result required for inferring infinite sequences of optimal MPC feedback laws from the solution of just one optimal control problem.

Proposition 3. Consider (1) for a fixed but arbitrary horizon N. Let $x(0) \in \mathcal{F}_N$ be arbitrary and let $x^{+,\mathrm{MPC}}$ refer

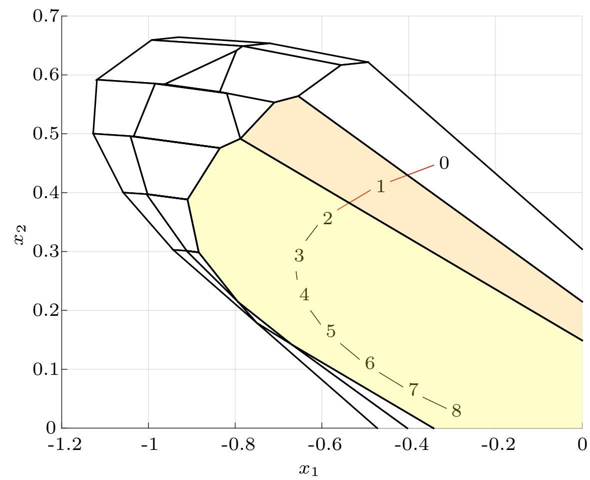

Fig. 1. Illustration of Lemma 2. Lines are added as a guide to the eye only. Polytopes belong to the explicit solution for $N = 2$ are added for illustration only. The explicit solution is point-symmetric with respect to the origin; the figure shows only the upper left part of the solution.

to the successor state to $x(0)$ that results under MPC assuming no plant-model mismatch.

If

$$\alpha = \alpha^{(0)}\,\alpha^{(1)}\,\cdots\,\alpha^{(N-1)}\,\alpha^{(N)} \tag{14}$$

with $\alpha^N = 0\cdots 0$ is the optimal active set for x, then

$$\alpha^+ = \alpha^{(1)}\,\cdots\,\alpha^{(N-1)}\,\underbrace{0\cdots 0}_{q_{\mathcal{X}}+q_{\mathcal{U}}}\,\alpha^{(N)} \tag{15}$$

is the optimal active set for $x^{+,\mathrm{MPC}}$.

Proof. Note that α^+ in (15) results from (14) in two steps, (i) dropping the first stage $\alpha^{(0)}$, and (ii) inserting a penultimate zero stage. The proof proceeds in two corresponding steps. Part (i) shows the intermediate active set that results after (i) but before (ii)

$$\alpha^{(1)}\cdots\alpha^{(N)}\,\alpha^{(N)}. \tag{16}$$

defines the solution to the CLQR problem (1) with reduced horizon $N-1$ and initial condition $x^{+,\mathrm{MPC}}$. Part (ii) then shows the intermediate active set (16) can be extended to α^+ from (15) to yield the desired result.

Part (i): The claim actually holds by the principle of optimality. We show this in more detail for clarity. Let

$$\begin{aligned} &u^\star(0), u^\star(1), \ldots,\ u^\star(N-1) \\ &x(0), x^\star(1), \ldots, x^\star(N-1), x^\star(N) \end{aligned} \tag{17}$$

refer to the optimal input sequence and state sequence that result for the initial condition $x(0)$ from solving the CLQR for horizon N and note $x^{+,\mathrm{MPC}} = x^\star(1)$. Now consider the CLQR for the reduced horizon $N-1$ and the initial condition $x^\star(1)$ and decorate all quantities related to this new problem with a tilde (e.g., $\tilde{x}(0) = x^\star(1)$). We need to show that the optimal input and state sequences

$$\begin{aligned} &\tilde{u}^\star(0), \tilde{u}^\star(1), \ldots, \tilde{u}^\star(N-2) \\ &\tilde{x}(0), \tilde{x}^\star(1), \ldots, x^\star(N-2), x^\star(N-1) \end{aligned} \tag{18}$$

for the new problem are equal to (17) with the first entries removed, i.e.,

$$\begin{aligned} &u^\star(1), \ldots, u^\star(N-1) \\ &x^\star(1), \ldots, x^\star(N-1), x^\star(N), \end{aligned} \tag{19}$$

which can be seen by contradiction: Assume (18) is optimal and not equal to (19), which implies (18) results in a lower cost function value than (19) in the CLQR problem with horizon $N-1$. Since $u^\star(0)$ from (17) steers the system from $x(0)$ to $\tilde{x}(0) = x^\star(1)$, the sequences

$$\begin{aligned} &u^\star(0), \tilde{u}^\star(0), \ldots, \tilde{u}^\star(N-2) \\ &x(0), \tilde{x}(0), \tilde{x}^\star(1), \ldots, \tilde{x}^\star(N-1), \end{aligned} \tag{20}$$

respect the constraints of the CLQR with horizon N and results in a lower cost function value than (17), since the first term is equal and the remaining term smaller for (20) than (17) by the assumption. This is a contradiction to the optimality of (17) and proves (18) and (19) are equal, or equivalently, (19) are the optimal sequences that result for $\tilde{x}(0) = x^\star(1) = x^{+,\mathrm{MPC}}$.

Now recall the active set can, for a given initial condition and optimal sequence of input signals, be determined by substituting the initial condition and optimal input sequence into the constraints (cf. (4a)). By assumption, the optimal input sequence from (17) results in $\alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(N-1)}$ from (16). Because of the constraint order (8), the optimal input sequence without the first term from (19) therefore results in the same active set without the first term $\alpha^{(2)}\cdots\alpha^{(N-1)}$, which proves part (i).

Part (ii): Since $\alpha^{(1)}\cdots\alpha^{(N-1)}\,\alpha^{(N)}$ is the active set for $x^{+,\mathrm{MPC}}$ taken as an initial condition for horizon $N-1$, α^+ defined in (15) is an active set for $x^{+,\mathrm{MPC}}$ taken as an initial condition for horizon N according to Lemma 2. Note that we need $\alpha^{(N)} = 0\cdots 0$ in order for Lemma 2 to apply, which holds by assumption. □

We summarize three important implications of Proposition 3 in the following remark for ease of reference.

Remark 4. (i) The active set α^+ from Proposition 3 defines the optimal feedback law for $x^{+,\mathrm{MPC}}$ and therefore no optimal control problem needs to be solved for $x^{+,\mathrm{MPC}}$.

(ii) Proposition 3 can obviously be applied repeatedly until the system has entered the terminal set $\mathcal{T}$. More specifically, just as the active set that defines the optimal feedback for $x(0)$

$$\alpha^{(0)}\alpha^{(1)}\alpha^{(2)}\alpha^{(3)}\cdots\alpha^{(N-1)}\alpha^{(N)}$$

yields the active set

$$\alpha^{(1)}\alpha^{(2)}\alpha^{(3)}\cdots\alpha^{(N-1)}\underbrace{0\cdots0}_{q_\mathcal{X}+q_\mathcal{U}}\alpha^{(N)} \tag{21a}$$

for $x^{+,\mathrm{MPC}}$, repeated application of Proposition 3 yields the active sets

$$\alpha^{(2)}\alpha^{(3)}\cdots\alpha^{(N-1)}\ \underbrace{0\cdots0}_{2\cdot(q_\mathcal{X}+q_\mathcal{U})}\ \alpha^{(N)} \tag{21b}$$

$$\alpha^{(3)}\cdots\alpha^{(N-1)}\ \underbrace{0\cdots0}_{3\cdot(q_\mathcal{X}+q_\mathcal{U})}\ \alpha^{(N)} \tag{21c}$$

$$\vdots$$

$$\underbrace{0\cdots0}_{N\cdot(q_\mathcal{X}+q_\mathcal{U})}\ \alpha^{(N)} \tag{21d}$$

that define the optimal feedback for the subsequent successor states that result under MPC feedback until the empty active set that defines the terminal set $\mathcal{T}$ and the optimal feedback law of the unconstrained LQR appear.

(iii) Proposition 3 applies to the *nominal* successor states $x^{+,\mathrm{MPC}}$, $x^{++,\mathrm{MPC}}, \ldots$ and therefore is in general not useful in any practical appliation. However, since α^+ and its successor define polytopes $\mathcal{P}^+$, $\mathcal{P}^{++}$ (with $x^{+,\mathrm{MPC}} \in \mathcal{P}^+$, $x^{++,\mathrm{MPC}} \in \mathcal{P}^{++}, \ldots$) and the optimal control law on these polytopes, no optimal control problem needs to be solved for any actual state in these successor polytopes.

We assess the robustness explained in Remark 4 (iii) in Section 4.

Proposition 3 and the sequence (21) of active sets explained in Remark 4(ii) can be illustrated with Figure 1. We already stated the active set (12)

$$\underbrace{000001}_{\text{dropped}}.000001.000000000000,$$

repeated here for convenience, that result for the initial condition marked '0' in Figure 1. Applying Proposition 3 yields, without solving an optimization problem, the active set

$$000001.\underbrace{000000}_{\text{new}}.000000000000,$$

for the closed-loop successor state marked '1', where colors merely point out the relations between the stages. According to Lemma 1, this active set defines the polytope highlighted in light red in Figure 1 and the optimal control law that holds on this polytope (not shown). Note that Proposition 3 not only yields the optimal feedback signal for the successor state, but a control law that is optimal on full-dimensional polytope; this is the robustness claimed in Remark 4(iii). Repeated application of Proposition 3 yields the active set

$$000000.\underbrace{000000}_{\text{new}}.000000000000,$$

for the next and all subsequent closed-loop successor states marked '2', '3', ... This active set defines the terminal set $\mathcal{T}$ highlighted in light yellow in Figure 1 and the optimal control law that holds on $\mathcal{T}$ (not shown) according to Lemma 1.

Finally, we stress that the assumption $\alpha^{(N)} = 0\cdots0$, i.e., inactive terminal constraints, is necessary in Proposition 3. In other words, the active set for the closed-loop successor state *can in general not* be determined with (15) if the terminal constraints are active in (14). This is illustrated in Figure 2. The initial condition marked '0' in this figure results in the active set

$$000001.000000.001000000000$$

with an active terminal constraint. Consequently, Lemma 2 does not apply and we cannot obtain the solution for a longer horizon by adding inactive stages, but the solution for $N = 3$ (black lines in the figure) departs from that for $N = 2$ (red lines in the figure). As a result, part (ii) of the proof of Proposition 3, which uses Lemma 2, does not apply. This shows that Proposition 3 does in general not apply if its condition $\alpha^{(N)} = 0\cdots0$ is not fulfilled. .

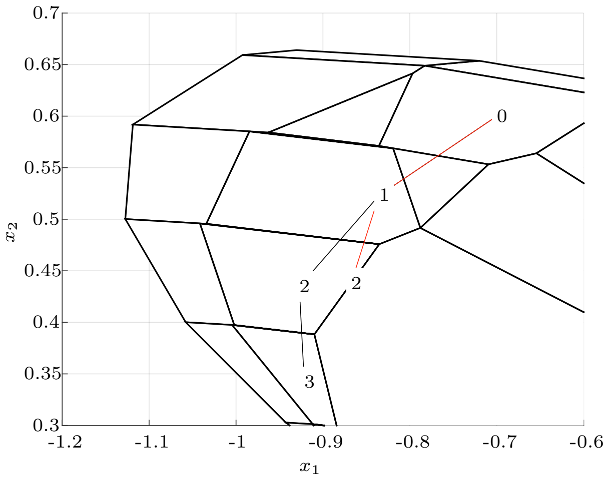

Fig. 2. The requirement $\alpha^{(N)} = 0 \cdots 0$, i.e., the inactivity of all terminal constraints, is necessary in Proposition 3.

4. EXAMPLES

We illustrate the proposed approach with two examples that differ with respect to their sizes.

Example 1. The first example results from discretizing the state-space representation of $G(s) = \frac{1}{s^2+s+2}$ with a sampling time of 0.1 seconds. This results in

$$A = \begin{pmatrix} 0.89550 & -0.18969 \\ 0.09485 & 0.99034 \end{pmatrix}, \; B = \begin{pmatrix} 9.4846 \cdot 10^{-2} \\ 4.8294 \cdot 10^{-3} \end{pmatrix}.$$

The example is similar to the one used by Seron et al. (2003), but we here impose state and input constraints

$$x(k) \in \mathcal{X} = \{x \in \mathbb{R}^2; -2 \leq x_i \leq 2, i = 1, 2\},$$
$$u(k) \in \mathcal{U} = \{u \in \mathbb{R}; -1 \leq u_1 \leq 1\}$$

and choose $Q = \text{diag}(0, 1)$, $R = 0.01$ and $N = 30$. The number of constraints amounts to $q = N(q_{\mathcal{X}} + q_{\mathcal{U}}) + q_{\mathcal{T}} = 192$, where $q_{\mathcal{X}} = 4$, $q_{\mathcal{U}} = 2$ and $q_{\mathcal{T}} = 12$. The terminal cost matrix P and constraint set $\mathcal{T}$ are as introduced in Sect. 1 and not reported here in the sake of space.

Example 2. The second example results from discretizing the minimal state-space representation of

$$G(s) = \begin{pmatrix} \dfrac{-5s+1}{36s^2+6s+1} & \dfrac{0.5}{8s+1} & 0 \\ 0 & \dfrac{0.1(-10s+1)}{s(8s+1)} & \dfrac{-0.1}{(64s^2+6s+1)s} \\ \dfrac{-2s+1}{12s^2+3s+1} & 0 & \dfrac{2(-5s+1)}{16s^2+2s+1} \end{pmatrix}$$

with a sampling time of 1 second. A state-space model with $n = 10$ states and $m = 3$ inputs results after removing the uncontrollable states. We enforce the state and input constraints

$$x(k) \in \mathcal{X} = \{x \in \mathbb{R}^{10}; -10 \leq x_i \leq 10, i = 1, \ldots, 10\},$$
$$u(k) \in \mathcal{U} = \{u \in \mathbb{R}^3; -10 \leq u_i \leq 10, i = 1, 2, 3\}$$

and we choose Q to be the identity matrix, $R = \text{diag}(0.25, 0.25, 0.25)$ and $N = 30$. This results in $q = N(q_{\mathcal{X}} + q_{\mathcal{U}}) + q_{\mathcal{T}} = 996$ constraints, where $q_{\mathcal{X}} = 20$, $q_{\mathcal{U}} = 6$ and $q_{\mathcal{T}} = 216$. The terminal cost matrix P and constraint set $\mathcal{T}$ are as introduced in Sect. 1.

The optimal inputs are applied in closed loop to the additively disturbed system

$$x(k+1) = Ax(k) + Bu(k) + d(k) \tag{22}$$

for both Example 1 and 2, where the components of $d(k) \in \mathbb{R}^n$ are drawn from independent zero-mean normal distributions. We vary the standard deviations, which are specified in Tables. 1 and 2, over a large range to investigate the robustness of the proposed approach.

The procedure is summarized in Algorithm 1, which essentially checks if the actual successor state of the disturbed system is in the successor polytope of the nominal system. If this is the case, no OCP needs to be solved, but the optimal feedback law on the successor polytope can be applied. Whenever an OCP is solved, the algorithm must check whether the resulting active set has inactive terminal constraints, because this is required for Proposition 3 to apply. If this condition is not fulfilled, an OCP must be solved in the subsequent step.

1 Initialization: set x to initial condition;
2 Solve QP (2) for x and horizon N; set $u \leftarrow u^\star(0)$;
3 Apply u;
4 Measure or estimate x^+;
5 **if** *terminal constraints are active* **then**
6   set $x \leftarrow x^+$ and goto 2
7 **else**
8   Determine $\mathcal{P}^+$;
9   **if** $x^+ \in \mathcal{P}^+$ **then**
10   determine K^+, b^+;
11   set $u \leftarrow K^+x^+ + b^+$ and goto 3;
12   **else**
13   set $x \leftarrow x^+$ and goto 2.
14   **end**
15 **end**

Algorithm 1: Summary of the proposed procedure

We apply the proposed approach to 10^4 standard MPC runs with random initial conditions that are uniformly distributed in $\mathcal{X}$ of the respective example. We carry out 30 MPC steps for each random initial condition. The number of steps is arbitrary and merely appears to be reasonable for the horizons $N = 30$.

Tables 1 and 2 list the fraction of input signals that can be generated without solving an OCP, because the disturbed successor state is located in the successor polytope of the nominal system. An OCP must be solved in the proposed approach if the the successor state of the additively disturbed system does not lie in the predicted polytope of the nominal system.

A considerable number of OCPs can be avoided in both examples, even if the standard deviations of the disturbances are fairly large (see column 1 in Tables 1 and 2). This number increases as the standard deviation is decreased in both examples as expected. It is also evident from the examples that the proposed method does not per se work better for small systems.

5. OUTLOOK

Because the proposed method anticipates optimal control laws that are valid on full-dimensional polytopes, it is

Table 1. Results for Example 1. The column 'relative std' states the standard deviation (std) of the normal distributions $d(k)$ in (22). Values in this column are in percent of the width of the intervals $-2 \leq x_i \leq 2$ that define $\mathcal{X}$ for Example 1. The columns 'fraction reused' and 'fraction OCP' are the fraction of the optimal inputs are calculated without solving an OCP and with solving an OCP, respectively.

relative std	absolute std	fraction reused	fraction OCP
0.1%	0.004	85.1%	14.9%
0.3%	0.012	74.7%	25.3%
1%	0.04	60.7%	39.3%
3%	0.12	33.3%	66.7%
10%	0.4	12.4%	87.6%

Table 2. Results for Example 2. Relative standard deviations are in percent of the width of the intervals $-5 \leq x_i \leq 5$ that define $\mathcal{X}$ for Example 2. All other columns are as in Table 1.

relative std	absolute std	fraction reused	fraction OCP
0.1%	0.01	99.9%	0.1%
0.3%	0.03	99.8%	0.2%
1%	0.1	99.7%	0.3%
3%	0.3	99.5%	0.5%
10%	1	98.8%	1.2%

inherently robust to some extent. Future work has to address the development of criteria that permit anticipating the extent of this robustness for disturbed systems like (22), for other forms of disturbances and for plant-model-mismatch.

Neighboring polytopes often are defined by active sets that differ only with respect to one active constraint (Ahmadi-Moshkenani et al., 2018). It is an obvious idea to extend the proposed method to not only using the anticipated optimal successor polytope, but also its neighbors. Note this will increase the robustness further even if not all but only some of the neighboring polytopes can be identified. Since storing the active sets requires much less memory than storing the geometric explicit solution, this extension is likely to be feasible for large systems.

Finally, there exist opportunities to reduce the computational effort of finding the optimal successor polytope and affine law compared to the effort that results with Proposition 3. While Proposition 3 expresses all successor active sets as active sets for the horizon N of the original problem, there always exists a shorter horizon (due to persistency (Mönnigmann, 2019)). This implies the number operations required to determine the polytope and law from the active set with Lemma 1 can be reduced, because the lemma can be applied for a shorter horizon.

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