Population dynamics and economic development

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Abstract

This research develops a continuous-time optimal growth model that accounts for population dynamics resembling the historical pattern of the demographic transition. The Ramsey model then becomes able to generate multiple determinate or indeterminate stationary equilibria and explain the process of the transition from a state with high fertility and low income per capita to a state with low fertility and high income per capita. The article also investigates the emergence of damped or persistent cyclical dynamics.

Keywords   Economic development; Population dynamics; Indeterminacy

JEL Classification   C61; C62; J1, J22; O41

1   Introduction

For several centuries - until about 1750 AD - the similar trend in (high) birth and death rates determined a relatively stable or moderately growing population worldwide (Fogel, 2004; Lorentzen et al., 2008; Galor, 2011; Livi-Bacci, 2017).[1] Following the abandonment of nomadic

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[1]Of course, changes in death (or mortality) rates modify the individual incentives to give birth to children. In a society where mortality rates are high, the marginal benefit of having an additional child (in a context where the number of children is still high) is larger than in a society where mortality rates are low. This is because a high number of children is indeed required to guarantee that some of them will survive at the onset of childhood and let themselves be in the condition of having their children.

life in favour of permanent settlements, the invention of the wheel, the development of agricultural tools (e.g., the hoe, the plough) as well as new techniques and technologies for farming land and raising animals, allowing for an improvement in both the quantity and quality of foods, there was an initial increase in birth rates (with death rates continuing to fluctuate around high levels) that eventually produced a first - though moderate - acceleration in population growth in Europe. Then, before the beginning of the first Industrial Revolution people in Europe experienced a remarkable increase in life expectancy due to another (substantial) improvement in food supply and, most importantly, an improvement in sanitation that contributed to reducing diseases (the second agricultural revolution). However, the lengthening in the lifetime was counterbalanced by the high birth rates, essentially because of the strengthened socio-cultural traditions of people. The second phase of the Demographic Transition (i.e., the transition from high birth and death rates to low birth and death rates if a country develops) begun some years after the Industrial Revolution (see Galor, 2011, and references therein). At that time, a downward trend in birth rates (initially caused by the rising costs of upbringing large families) and a general change in the society as a whole (migration from the countryside to cities) followed the drop-in mortality rates. The total population started increasing dramatically and there was an acceleration in income per person. This contributed to promoting investments in education, thus favouring the switch in children's demand from "quantity" to "quality". The subsequent increases in human capital, female labour participation and wages, the process of urbanization, the development of contraceptive methods, the scientific research, the improvement in private and public health services, the development of welfare states (child care policies, public pensions, diseases control and so on), the reduction in the value of children's work and perhaps other reasons not mentioned here, contributed to explain the observed drop in birth and death rates that started settling down around low levels, as well as a tremendous increase in income per person following the process of industrialization.[2] This story is well documented and summarised in Livi-Bacci (2017) and it is an essential part of the development economics on the side of demo-economic outcomes (Fogel, 2004; Galor, 2011). To sum up, along the process of economic growth and development of nations, especially in the Western world, fertility first increased and then decreased with income and the behaviour of total population over time has the well-known S-shaped course depicted by the demographic transition pattern.[3] This is the stylised fact considered in this article, whose purpose is to build on a continuous-time optimal growth (Ramsey) model in which individuals choose the consumption path in a context where the population (fertility) evolves according to the demographic transition rules. The model then considers the number of children as an exogenous variable following a sort of biological

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[2]See the recent works of Bhattacharya and Chakraborty (2017) and Prettner and Strulik (2017) for an interesting analysis on contraception and the fertility transition in industrialised countries.

[3]An exception is represented by Sub-Saharan Africa (SSA), where the coexistence of several infectious diseases, especially HIV/AIDS, contributed to halt the trend in mortality decline in the 1980s and reverse the long-term positive trend in life expectancy (see Gori et al., 2020, and references therein).

behaviour determined by natural constraints and social and economic environment but not by individual choices.

We pinpoint that in the development economics literature the terms exogenous population and endogenous population are often used with different meanings. To avoid confusion and clarify the relevant definitions, we recall that in several contributions (since the pioneering works by Ramsey, 1928, Solow, 1956, and Diamond, 1965) demographic dynamics implicitly follow some given biological rules (e.g., sexual drive leading to reproduction), *but they depend neither on agents' choices nor on economic variables*. In this case, demographic dynamics are exogenous. However, Solow (1956), in the final part of his work, speculates on the dependence of the fertility rate on some economic variables determined in the model (e.g., aggregate income) concluding that multiple equilibria and then multiple development regimes can be explained in his framework. Even though the demographic variables are not determined by individual choices based on economic constraints and incentives, often this kind of models is defined as "models with endogenous population" (e.g., Fanti and Manfredi, 2003, and references cited therein). We refer to this literature as models with "semi endogenous population".

The development economic literature following the pioneering contributions of Leibenstein (1957) and Becker (1960), that have originated the so-called New Home Economics, considers the number of children to be chosen by rational individuals (possibly with a different degree of selfishness) by comparing (marginal) benefits and costs of having an additional child. In this case, the literature refers to endogenous population variables. This literature has seen a tremendous improvement growing rapidly until the works belonging to the Unified Growth Theory (Galor and Weil, 2000; Blackburn and Cipriani, 2002; Kalemli-Ozcan, 2002; Tabata, 2003; Bloom et al., 2003, 2009; Yakita, 2010; Lagerlöf, 2006; Galor, 2011; Gori et al., 2020). This theory aims at explaining the reasons why some countries follow development trajectories, where capital accumulation is high and fertility and death rates are low and others remain instead entrapped in stagnation or poverty following under-development paths, where capital accumulation is low, and fertility and death rates are high. The most common conclusion of this kind of models is that history matters. Therefore, what eventually determines the reasons why some countries have moved to a lower level of economic activity - leading also to an impoverishment of sociocultural conditions - is the initial state of an economy. The reasons to follow development or underdevelopment trajectories based on path dependence (history) do not allow explaining why some countries with similar initial values of a state variable (e.g., the capital stock) have experienced very different scenarios. From a theoretical point of view, the overcoming of this lacuna was achieved by the formulation of models of economic development characterised by indeterminacy, i.e., the possibility of observing different trajectories driven by the expectations of economic agents.[4] Amongst the contributions with endogenous population for which indeterminacy becomes a theoretical reason to explain the development of some

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[4]See the seminal works of Krugman (1991) and Matsuyama (1991) for a debate about history versus expectations.

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countries and the lack of development of others (coordination failures) are the works of Palivos (1995) and Gori and Sodini (2021) that specifically analyse growth models with endogenous fertility, and the work of Gori and Sodini (2020) that analyses a growth model with endogenous longevity and health expenditures.

However, it remains difficult to understand whether demographics affects economics or vice versa when inquiring into the causes of poverty or prosperity of nations. Nonetheless, population dynamics has certainly relevant consequences on the economic activity and then encompassing a theoretical view in which fertility is exogenous or endogenous may depend on the subject dealt with by scholars (Azariadis and Drazen, 1990; Galor and Weil, 1996, 2000). The main aim of this work is to account for a population growth function that resembles the demographic transition pattern in an optimal growth framework, i.e., total population (for simplicity represented only by the fertility rate) first increases and then decreases with income. With this simple assumption, the one-sector optimal growth model à la Ramsey becomes able to characterise different paths of economic development and indeterminacy matters. Specifically, though the present work strictly belongs to the literature encompassing models with *semi endogenous* population variables it also accounts for some characteristics allowing us to obtain a wide range of dynamic results typical of the literature with *endogenous* population variables, including local and global indeterminacy.

Surprisingly, even at the time of writing, there is are few articles belonging to the economic growth literature dealing explicitly with population dynamics in continuous-time models (e.g., Accinelli et al., 2007; Canton and Meijdam, 1997; Brida and Accinelli, 2007; Marsiglio and La Torre, 2012).[5] None of the aforementioned works, however, has analysed the influence of the pattern of the demographic transition for economic development in the basic Ramsey-type growth set-up. The present article points out that this pattern of demo-economic outcomes dramatically matters for explaining the reasons why some countries accumulate a large amount of capital and enter an impressive development trajectory and others instead accumulate a small amount of capital entering a phase of moderate growth or poor development.

The rest of the article is organised as follows. Sections 2 outlines the model. Section 3 studies the optimality conditions and the dynamics of the continuous-time system. Section 4 concentrates on an economy where consumers have preferences captured by a Constant Intertemporal Elasticity of Substitution (CIES) utility function and firms produce with a Cobb-Douglas (CD) technology. Section 5 concludes the article.

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[5]The article by Accinelli et al. (2007) is the most closely related to ours in terms of modelling, perspectives and aims. Specifically, the authors' will is to consider a Ramsey-like model (augmented with a non-linear process in the dynamics of capital accumulation) to study the different stages of the historical pattern of the economic transition (abstaining, however, from including the stages of the demographic transition) to describe essentially the economic transition of Western countries. Unlike them, the present work concentrates on the analysis of the demo-economic transition by considering a non-linear dynamic evolution of the population growth rate.

2 The model

The model we are herewith considering describes an economy comprised of a continuum of infinite-lived identical agents whose population size is $L(t)$. There is no unemployment so that population entirely belongs to the labour force.[6] At every time $t \in [0, \infty)$, the representative agent produces output $Y(t)$ by using the following technology satisfying the properties of constant returns to scale, positive and diminishing returns to labour and capital and Inada conditions:

$$Y(t) = F(K(t), L(t)), \tag{1}$$

where $K(t)$ is the stock of physical capital, $L(t)$ is the labour input employed in production and function F satisfies the usual neoclassical properties.

About the law governing population dynamics, two alternatives are generally considered in the literature (Canton and Meijdam, 1997; Guerrini, 2006; Brida and Accinelli, 2007; Bucci and Guerrini, 2009; Marsiglio and La Torre, 2012): constant population growth and logistic population growth. In the former case, the instantaneous change in the labour force ($\dot{L}(t)$) is proportional to the existing labour force,

$$\dot{L}(t) = gL(t), \tag{2}$$

where $g \geq 0$ is the constant growth rate of population (fertility or birth rate) and $gL(t)$ is the number of births per unit of time.[7] By assuming an initial value $L(0) > 0$, the expression in (2) can be solved to give the exponential function $L(t) = L(0)e^{gt}$. This equation says that the population steadily increases with no upper bound at rate g. However, this kind of behaviour is at odds with the observed pattern of the demographic transition, so that it cannot be satisfactorily included in the analysis (see Roser et al., 2013, for details about the time evolution of the population growth rate and its main characteristics).[8] In the latter case (logistic law), the differential equation governing population growth has the form introduced first by the Belgian mathematician Verhulst in the 19th century. Given two positive parameters, a and b, this law can be written as follows:

$$\dot{L}(t) = L(t)[a - bL(t)]. \tag{3}$$

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[6]In addition, we assume a fixed relationship between labor supply and population. This means that we consider no changes in labour-force participation or working hours and effort. Then, we use the terms population, labour force or labour supply interchangeably.

[7]Without loss of generality, we abstract from including mortality rates into the analysis. Therefore, the growth rate g represents the birth rate net of mortality rates, which however is not explicitly included into the analysis. If one decided to distinguish between birth rates and mortality rates, the expression in (2) would change becoming $\dot{L}(t) = (\widetilde{g} - \pi)L(t)$, where $\widetilde{g} \geq 0$ is the gross birth rate, $\pi > 0$ is the mortality rate and the term $\widetilde{g} - \pi$ is the net growth rate of population.

[8]"A common question we're asked is: is the global population growing exponentially? The answer is no [...] since the 1960s the growth rate has been falling. This means the world population is not growing exponentially – for decades now, growth has been more similar to a linear trend." (Roser et al., 2013: https://ourworldindata.org/world-population-growth).

The linear term in (3) acts as a positive device allowing to capture the trend in total population resembling the second and third stages of the demographic transition. The quadratic term, instead, introduces negative feedback capturing the effects of the quantity-quality trade-off and resembling the subsequent trends of the (moderately growing or declining) total population of the demographic transition pattern. Given a positive initial value $L(0)$, by integrating the expression in (3) by parts, verifying $a - bL(0) > 0$ and defining $M := L(0)/[a - bL(0)]$, one gets the solution $L(t) = aM/(bM + e^{-at})$. Then, the total population asymptotically takes the upper bound a/b as $t \to +\infty$. This threshold represents a sort of "carrying capacity" of the system due, for instance, to the overcrowding of a given territory. The logistic law has then the merit of being much more prone to describe the actual behaviour of the total population compared to the constant population growth function in (2). However, it represents only a part of the story as the demographic transition is a long-term phenomenon relating the trend in population variables with the historical pattern of GDP per capita. Then, to build on a demographically founded formulation for the dynamics of $L(t)$ one should make the rate of change in population (labour force) dependent also on income per person, allowing for a non-monotonic behaviour resembling the Malthusian epoch (positive relationship between fertility and income) and the modern regime of growth (negative relationship between fertility and income).[9]

The idea that the growth rate of the labour force g should not be constant but rather a function depending on current income (endogenous process) already existed in the seminal work of Solow (1956). This idea is formalised in the following way:

$$\dot{L}(t)/L(t) = g(y(t)), \tag{4}$$

where $y(t) = f(k(t))$ is the neoclassical intensive-form production function that comes from (1), with $f(k(t))$ satisfying the conditions $\lim_{k \to 0^+} f'(k(t)) = +\infty$ and $\lim_{k \to +\infty} f'(k(t)) = 0$, $k(t) := K(t)/L(t)$ is the capital-labour ratio and $y(t) := Y(t)/L(t)$ is income per person. The function $f(k(t))$ satisfies the usual Inada conditions, that is $\lim_{k \to 0^+} f'(k(t)) = +\infty$ and $\lim_{k \to +\infty} f'(k(t)) = 0$. Solow gave some qualitative insights about the macroeconomic effects of a general non-monotonic function g that can potentially generate multiple long-term stationary states. Along the line of this intuition, several decades later Fanti and Manfredi (2003) analysed a Solowian growth model with an exogenous saving rate and considered a Cobb-Douglas version for the intensive-form production function f, i.e., $f(k(t)) = k(t)^{\alpha}$ ($0 < \alpha < 1$), and a linear and increasing (with respect to income) version of function g, i.e., $g(f(k(t))) = \beta k(t)^{\alpha}$, where $\beta > 0$ is a constant parameter.[10] From these assumptions, they found that $\dot{L}(t)/L(t) = \beta k(t)^{\alpha}$. However, this specification for the growth rate of population (used essentially for reasons of

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[9]See Galor and Weil (2000).

[10]Their main aim was to include a demographic delay into the analysis, thus letting the rate of change in the supply of labour depend on past (rather than current) values of fertility and wage. This is because they wanted to model out demographic changes in a more accurate way than in the standard formulation of Solow (1956).

analytical tractability) is a bit unsatisfying as it does not allow having a complete description of the pattern of the demographic transition by capturing only a part of it.

To overcome the lacunae of each of the previous versions for the law of motion of population and the criticism of a macroeconomic framework with an exogenous saving rate (Solow, 1956), in this work we assume that the fertility rate is not a choice variable for the rational agent, being instead dictated by natural constraints and social and economic environment. This is done by considering a model with consumers that optimise their utility coming from the stream of inter-temporal consumption over an infinite time horizon (Ramsey, 1928) by considering a population dynamic equation in which the evolution in the labour force is related to the shape of both the demographic variables and the economic activity. This equation is given by the following generic dynamic expression:

$$\dot{L}(t) = L(t)n(f(k(t)), L(t)), \tag{5}$$

where $n(f(k), L)$ is a C^2 function on the positive orthant with $\partial n(f(k), L)/\partial L < 0$ (the growth rate is decreasing with respect to the population size due to a natural saturation effect: the larger the population size, the lower its growth rate; this is in line with the data on world population growth, see Roser et al., 2013) and $\lim_{L \to +\infty} n(f(k), L) = -\infty \ \forall k > 0$, $\lim_{f(k) \to +\infty} n(f(k), L) = \overline{n} < +\infty \ \forall L > 0$. The latter two conditions reveal that the growth rate of the population is negative when its size (L) is large (due, for instance, to scarcity of resources) and cannot exceed a given threshold though it is positively affected by the accumulation of capital.

The expression in (5) implies that the growth rate of the population at any time depends on the size of the population and the intensive form production function (i.e., income per capita). The choice of employing a version for the dynamics of a population that includes the value of the per capita stock of capital instead of its level is due to analytical convenience. However, it is in line with the choice made by Solow (1956) as well as the historical pattern of the demographic transition contrasting the evolution of total population to the historical pattern of per capita income.

The instantaneous utility function at time t of the representative agent is

$$U(C(t)/L(t)), \tag{6}$$

where $C(t)$ is the consumption bundle, $U'(\cdot) > 0$ and $U''(\cdot) < 0$ and U satisfies the Inada conditions. Lifetime welfare is determined by the infinite discounted flow of the instantaneous utility times the population size weighted by the degree of inter-temporal altruism towards future generations, i.e., $L(t)^\varepsilon$, where $\varepsilon \in [0, 1]$. This flow is formalised by considering the following integral:

$$\int_0^{+\infty} U(C(t)/L(t))L(t)^\varepsilon e^{-\rho t} dt, \tag{7}$$

where $\rho > 0$ is the inter-temporal discount rate. The case $\varepsilon = 0$ (resp. $\varepsilon = 1$) boils down to average (resp. total) utilitarianism, i.e., lifetime welfare accords with the Millian (resp. Benthamite) criterion. Intermediate values of ε represent impure altruism with different degrees. We will see that the degree of inter-temporal altruism will not affect stability outcomes and the possibility of having determinate or indeterminate stationary state equilibria.

By assuming that capital depreciates at rate $\delta > 0$, the dynamics of $K(t)$ is given by the following differential equation

$$\dot{K}(t) = Y(t) - \delta K(t) - C(t), \tag{8}$$

where $\dot{K}(t)$ is the time derivative of the state variable $K(t)$ and production $Y(t)$ is determined by the neoclassical technology in (1).

The representative agent faces the following inter-temporal maximisation problem:

$$\max_{C(t)} \int_0^\infty U(C(t)/L(t))L(t)^\varepsilon e^{-\rho t}dt, \tag{9}$$

subject to

$$
\begin{aligned}
\dot{K}(t) &= F(K(t), L(t)) - \delta K(t) - C(t), \\
\dot{L}(t) &= L(t)n(f(k(t)), L(t)), \\
L(0) &= L_0 > 0, K(0) = K_0 > 0.
\end{aligned} \tag{10}
$$

It is important to recall that in what follows we will not characterise the dynamics of the optimal choices of a social planner that considers the dynamics of both state variables, $K(t)$ and $L(t)$. Instead, we study the inter-temporal dynamics defined by the choices of the representative agent about his consumption path, which considers the dynamics of $L(t)$ as given. In other words, the dynamics of $L(t)$ are not the result of optimising behaviour, being dictated by biological and macroeconomic rules. At the same time, however, the dynamics of $L(t)$ affect the consumption pattern over time and then they may have relevant consequences for the outcomes of the model.

The constrained maximisation problem expressed in (9) and (10) can be reformulated in terms of per capita variables in the following way:

$$\max_{c(t)} \int_0^\infty U(c(t))L(t)^\varepsilon e^{-\rho t}dt, \tag{11}$$

subject to

$$
\begin{aligned}
\dot{k}(t) &= f(k(t)) - c(t) - \left[\delta + \dot{L}(t)/L(t)\right]k(t), \\
\dot{L}(t) &= L(t)n(f(k(t)), L(t)), \\
L(0) &= L_0 > 0, K(0) = K_0 > 0,
\end{aligned} \tag{12}
$$

where $c(t) := C(t)/L(t)$. This reformulation allows us to clarify that the dynamics of the stock of capital per person, $k(t)$, depends on the growth rate of population, which is in turn a generic function of both the size of population, $L(t)$, and income per person, $f(k(t))$.

3   Optimality conditions and dynamics

The model is solved by introducing the current-value Hamiltonian function (we omit the time index for convenience henceforth):

$$H = U(c)L^{\varepsilon} + \lambda[f(k) - c - (\delta + n(f(k), L))k], \tag{13}$$

from which we get the necessary and sufficient conditions for optimisation[11]

$$
\begin{aligned}
H_c' &= 0 \iff U'(c)L^{\varepsilon} - \lambda = 0, \\
\dot{k} &= f(k) - \delta k - c - n(f(k), L)k, \\
\dot{\lambda} &= \lambda\left(\rho - f'(k) + \delta + n_f'(f(k), L)f'(k)k + n(f(k), L)\right),
\end{aligned} \tag{14}
$$

and the transversality condition

$$\lim_{t \to +\infty} k\lambda e^{-\rho t} = 0. \tag{15}$$

The Hamiltonian function in (13) includes one and only one multiplier (λ). This is because the dynamics of the population is not affected by the choices of the representative agent. In other words, each agent cannot modify the growth rate of L, which is instead determined by aggregate (macro)economic and demographic conditions. In addition, the agent does not consider how his own choices about consumption smoothing affect the evolution of the population in this context. By eliminating the multiplier, we get the following dynamic system[12]:

$$
\begin{cases}
\dot{k} &= f(k) - \delta k - c - n(f(k), L)k \\
\dot{c} &= \sigma c\left[(\varepsilon - 1)n(f(k), L) - \rho - \delta + f'(k) - n_f'(f(k), L)f'(k)k\right] \\
\dot{L} &= Ln(f(k), L)
\end{cases}, \tag{16}
$$

where $\sigma := -U'(c)/[U''(c)c]$ represents the reciprocal of the elasticity of marginal utility or, alternatively, the elasticity of inter-temporal substitution.

A stationary state equilibrium of (16) (k^*, c^*, L^*) is a solution of the following system:

$$
\begin{cases}
f(k) = \delta k + c \\
\rho + \delta = f'(k) - n_f'(f(k), L)f'(k)k \\
0 = n(f(k), L)
\end{cases}. \tag{17}
$$

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[11] Given a generic function $X = X(M, P)$, we use X_M' and $X_{M,P}''$ to denote $\partial X/\partial M$ and $\partial^2 X/\partial M \partial P$, respectively.

[12] By the regularity hypotheses on the functions to get involved in the model, it follows the existence and uniqueness of the solution of the dynamical system (16), given a positive initial condition (k_0, c_0, L_0).

As $\partial n(f(k), L)/\partial L < 0$, from the third equation in (17) we obtain L as a function of k, that is

$$L = T(f(k)). \tag{18}$$

Substitution of (18) in the second equation of (17) for L allows us to get the stationary state equilibrium value of k as a solution of the following:

$$\rho + \delta = R(k) := f'(k) - n'_f(f(k), T(f(k)))f'(k)k. \tag{19}$$

Once this value has been determined, the stationary state values of c and L are obtained from the first and third equations of the system in (17), respectively. To study the existence and number of solutions of (19) we now introduce the following assumption.

Assumption 1 *We assume that*

$$\lim_{k \to 0^+} \frac{n'_f(f(k), L)f'(k)k}{n(f(k), L)}$$

exists and is finite $\forall L > 0$.

Assumption 1 requires that the elasticity of the growth rate of the population n with respect to the stock of capital k takes a finite value as k approaches 0 from the right. In economic terms, this means that the increase in the growth rate of population is not infinite as $k \to 0^+$. In this regard, the following lemma holds:

Lemma 2 *Let R be the function defined in (19). We have that $\lim_{k \to 0^+} R(k) = +\infty$; $\lim_{k \to +\infty} R(k) = +\infty$ if $\lim_{k \to +\infty} n'_f(f(k), T(f(k))) < 0$; $\lim_{k \to +\infty} R(k) = -\infty$ if $\lim_{k \to +\infty} n'_f(f(k), T(f(k))) > 0$.*

Under Assumption 1, the behaviour of R as $k \to 0^+$ is independent of the specifications on utility, production and growth rate of population. Differently, the behaviour of R as $k \to +\infty$ depends on the functional forms of f and n (but not on the functional form of U). We can now state a result on the existence and multiplicity of the stationary state equilibria of the model. Generally, by neglecting the cases in which the stationary points coincide, we have the following proposition.

Proposition 3 *Let Assumption 1 hold. If $\lim_{k \to +\infty} n'_f(f(k), T(f(k))) > 0$, the number of stationary state equilibria is odd. If $\lim_{k \to +\infty} n'_f(f(k), T(f(k))) < 0$, then the number of stationary state equilibria is even (and can be zero).*

Proof. As already shown in the text, the existence and number of stationary state equilibria are determined by the study of the number of intersection points of the graph of function R with the horizontal line whose ordinate is $\rho + \delta$. As $\lim_{k \to 0^+} R(k) = +\infty$, if $\lim_{k \to +\infty} n'_f(f(k), T(f(k))) >$

0 then at least one intersection point exists and in general there is an odd number. If $\lim\limits_{k \to +\infty} n'_f(f(k), T(f(k))) < 0$ then an intersection point may not exist, but if it does, an even number of intersection points will exist. ■

By considering the expressions in (17), the Jacobian matrix evaluated at a generic stationary state equilibrium (c^*, k^*, L^*) reads as:

$$J^* := J(k^*, c^*, L^*) = \begin{bmatrix} \rho & -1 & J^*_{13} \\ J^*_{21} & 0 & J^*_{23} \\ J^*_{31} & 0 & J^*_{33} \end{bmatrix}, \tag{20}$$

where

$$J^*_{13} := -n'_L(f(k^*), L^*)k^*, \tag{21}$$

$$J^*_{21} := \sigma c^* (((\varepsilon - 2)f'(k^*) - k^*f''(k^*))n'_f(f(k^*), L^*) - n''_{f,f}(f(k^*), L^*)\left[f'(k^*)\right]^2 k^* + f''(k^*)), \tag{22}$$

$$J^*_{23} := \sigma c^* (\varepsilon n'_L(f(k^*), L^*) - n''_{f,L}(f(k^*), L^*)f'(k^*)k^* - n_L(f(k^*), L^*)), \tag{23}$$

$$J^*_{3,1} := n'_f(f(k^*), L^*)f'(k^*)L^*, \tag{24}$$

$$J^*_{3,3} := n'_L(f(k^*), L^*)L^*. \tag{25}$$

The dynamic properties of the model around (c^*, k^*, L^*) (local dynamics) are characterised by the eigenvalues of (20), that is the roots of the characteristic polynomial

$$P(z) = z^3 - Tr(J^*)z^2 + w(J^*)z - \det(J^*), \tag{26}$$

where

$$Tr(J^*) = \rho + J^*_{33}, \tag{27}$$

$$\det(J^*) = \begin{vmatrix} J^*_{21} & J^*_{23} \\ J^*_{31} & J^*_{33} \end{vmatrix} = L^* \left(J^*_{21} n'_L(f(k^*), L^*) - J^*_{23} n'_f(f(k^*), L^*)f'(k^*) \right), \tag{28}$$

are the trace and determinant of (20), whereas

$$w(J^*) = \begin{vmatrix} \rho & -1 \\ J^*_{21} & 0 \end{vmatrix} + \begin{vmatrix} 0 & J^*_{23} \\ 0 & J^*_{33} \end{vmatrix} + \begin{vmatrix} \rho & J^*_{13} \\ J^*_{31} & J^*_{33} \end{vmatrix} = J^*_{21} + J^*_{33} \left[\rho + n'_f(f(k^*), L^*)f'(k^*)k^* \right]. \tag{29}$$

We recall that in our model we have two state variables and one control variable. Then, a stationary equilibrium is determinate if there exists a unique two-dimensional manifold on which the dynamics converge to the steady-state. This occurrence is guaranteed when the Jacobian matrix (evaluated at the stationary state) exhibits one positive eigenvalue and two negative (or negative real part) eigenvalues. Differently, a stationary state equilibrium is locally indeterminate if, for initial values of k and L close to the stationary state values of the same variables, there exists a continuum of initial conditions of the control variable such that the dynamics of the system converge to the stationary state equilibrium. This result occurs when

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the Jacobian matrix (at the stationary state) has three negative real eigenvalues or one negative real eigenvalue and a pair of conjugate complex eigenvalues with a negative real part. A stationary state equilibrium is completely unstable if no trajectory starting from initial conditions different from the stationary state equilibrium converges towards the equilibrium. This result holds if the Jacobian matrix evaluated at the stationary state has three positive (or positive real part) eigenvalues. The stationary state equilibrium is saddlepoint unstable if the stable manifold is one-dimensional. This configuration is classified as unstable because, given the initial values of the two state variables, a level of the control variable allowing the generated trajectory to converge towards the equilibrium does not generally exist. This result holds if the Jacobian matrix evaluated at the stationary state has one negative real eigenvalue and two positive (real or real part) eigenvalues.

Regarding the classifications introduced above the following lemmas hold (see Wirl, 1997):

Lemma 4 *If* $\det(J^*) > 0$ *and* $w(J^*) < 0$ *then the stationary state equilibrium is saddlepoint stable.*

Lemma 5 *If* $\det(J^*) < 0$ *then the stationary state equilibrium is indeterminate or saddlepoint unstable.*

From Lemma 5 the following remark holds.

Remark 6 *A necessary condition for indeterminacy of the stationary state equilibrium is* $\det(J^*) < 0$.

As we will show in the next sections, Lemma 4 and Lemma 5 will play a critical role in defining the dynamic properties of the model under appropriate specifications of the functions f, n and U. In addition, they allow us to state the following proposition.

Proposition 7 *The stationary state equilibria with an odd index are saddlepoint stable or repulsive, whereas the stationary state equilibria with an even index are indeterminate or saddlepoint unstable.*

Proof. We show that the study of the sign of $\det(J^*)$ can be reduced to the study of the derivative of R as the graph of R intersects the horizontal line $\rho + \delta$, that is at the stationary state values of k (k^*). By applying the Implicit Function Theorem, from the equilibrium condition (18) we get $T'(f(k)) = -\frac{n'_f(f(k),L)}{n'_L(f(k),L)}$. By direct calculations, it follows that

$$\left.\frac{dR(k)}{dk}\right|_{k=k^*} = R'_k(k^*) = \frac{1}{n'_L(f(k^*), L^*)}M,$$

and

$$\det(J^*) = \sigma c^* L^* M,$$

where

$$M \quad : \quad = n'_f(f(k^*), L^*)(f'(k^*))^2 n''_{f,L}(f(k^*), L^*)k^* + \tag{30}$$
$$+n'_L(f(k^*), L^*)[(f'(k^*))^2 - n''_f(f'(k^*))^2 k^* - n'_f(f(k^*), L^*)(f'(k^*) + f''(k^*)k^*)].$$

Therefore, as $n'_L(f(k), L) < 0$ we have that

$$sgn(\det(J^*)) = -sgn(R'_k(k^*)).$$

Consequently, from Proposition 3 every stationary state equilibrium with odd index, showing $R'_k(k^*) < 0$, is generically either saddlepoint stable or repulsive. Instead, every stationary state equilibrium with an even index, showing $R'_k(k^*) > 0$, is generically either indeterminate or saddlepoint unstable. ∎

4 The CIES-CD economy

In the general context studied so far, it is difficult to interpret the conditions on the functions to get stable outcomes and give simple and clear economic intuitions. This is even more important in the light of the results stated in Lemma 5 and Remark 6. In addition, though it is not difficult to give a simple and clear interpretation to the conditions that guarantee the existence of a stationary state equilibrium, it is not possible to rule out the case of an indeterminate equilibrium. Another relevant feature of the model is the possibility to observe cycles in both the short term (transition) and long term (steady-state). In this regard, we can go one step further and study how the population growth rate reacts to changes in both k and L as well as how these changes interact with the dynamics of capital accumulation and the consumption path. To show these results more clearly, we consider specific functional forms regarding preferences, technology and population dynamics. We follow the common practice (Spataro and Fanti, 2011; Fanti and Gori, 2013; Gori et al., 2019) and use the Constant Inter-temporal Elasticity of Substitution (CIES) formulation for the instantaneous utility function capturing the preferences of the representative agent and the standard Cobb-Douglas (CD) technology on the production side of the economy. Therefore, the instantaneous utility function takes the form:

$$U(c) = \frac{c^{1-1/\sigma} - 1}{1 - 1/\sigma}, \tag{31}$$

where $\sigma > 0$ ($\sigma \neq 1$) is the constant elasticity of inter-temporal substitution. When $\sigma = 1$, the CIES utility function in (31) boils down to the standard log-utility $U(c) = \ln(c)$. The higher is σ, the larger the decline in the marginal utility of consumption following an increase in the consumption bundle c. Then, an increase in the elasticity of substitution has the consequence of letting consumption smoothing more desirable. On the side of production, we use the standard CD technology, so that:

$$y = f(k) = k^\alpha, \tag{32}$$

where $0 < \alpha < 1$ is the usual output elasticity of capital, whose value is around 1/3 (Krueger 1999; Gollin, 2002, Jones, 2004).

In the following three subsections we will consider three different specifications of the population growth rate n as a function of population size, L, and income per capita, $f(k)$ to generalise as much as possible the main findings of the paper. Considering the negative relationship between population growth rate and population size still holds, we will start by assuming the existence of a monotonic relationship between n and $f(k)$ (Subsection 4.1). In this case, the model accounts only for the positive effect of economic growth on the standard of living of the overall population. This allows us describing only the initial phase of the demographic transition (i.e., the Malthusian phase), occurring before the quantity/quality switch. The analysis will proceed further (Subsection 4.2) considering an inverted U-shaped relationship between n and $f(k)$ to allow for the phases of demographic transition after the Malthusian one to be included in the analysis. In these phases, the average birth rate tends to become negatively related to income per capita due to the occurrence of the quantity/quality switch, according to which individuals prefer to give birth to a lower number of more educated children as their income rises to depart from the subsistence level. In these two cases, we assume – for reasons of analytical tractability – that the growth rate of the population is (i) an additively separable function of its arguments, $f(k)$ and L, and (ii) a linear function of L. Then, we consider the following formulation for the growth rate of the population:

$$\dot{L}/L = a - bL + G\left(f(k)\right),\tag{33}$$

where the first term $a - bL$ describes the evolution of the population growth rate (in line with Brida and Accinelli, 2007) neglecting however the role of the economic transition within the demographic transition. Based on this term, the population will approach a/b in the long term. The second term G of the expression in (33) captures the relationship between the growth rate of population and income per capita. It will be specified as a positive monotonic function of $f(k)$ in Subsection 4.1 and an inverted U-shaped function of $f(k)$ in Subsection 4.2. The expression in (33) combines two different approaches: the logistic population growth function and the idea of Solow of letting population dynamics be endogenous to the system, i.e., dependent on a variable explained in the model.

Unlike the previous assumptions, Subsection 4.3 relaxes the hypothesis of separability between $f(k)$ and L, and considers the population growth rate is a non-linear function of $f(k)$ and L. This will be useful to show how the inter-relationship between these two variables can indeed generate cycles both in the evolution of population and the evolution of capital and income.

4.1 The Malthusian case (saddle-path stability)

Let us now specify $G(f(k))$ as a monotonic increasing function of $f(k)$. Specifically, we assume that

$$G(f(k)) = \beta k^\alpha, \quad \beta > 0. \tag{34}$$

In this case, both the (unique) equilibrium and the dynamics of the model can be easily characterised, and the following proposition holds.

Proposition 8 *The system admits a unique stationary state equilibrium, which is saddle point stable.*

Proof. By recalling Proposition 3, the number of stationary state equilibria are determined by the intersections between the graph of function R, defined in (19), and the horizontal line $\rho + \delta$. As $\lim_{k \to 0^+} R(k) = +\infty$ from Lemma 2 and

$$\frac{dR(k)}{dk} = -2\alpha k^\alpha \left[\left(\alpha - \frac{1}{2} \right) (\alpha\beta k^\alpha - 1) \right] < 0,$$

the graph of R intersects once the horizontal line $\rho + \delta$ and the stationary state equilibrium is then unique. In terms of stability, $\frac{dR(k)}{dk} < 0$ implies that $\det(J^*) > 0$. In addition, by writing the Jacobian matrix J^* in terms of k^* (using the conditions in 17) we have that

$$J_{21}^* = -\frac{1}{\sigma} \left(\alpha c^* \left[\beta(1 + \alpha - \varepsilon)(k^*)^{\alpha-1} + (1 - \alpha)(k^*)^{\alpha-2} \right] \right) < 0,$$

where $c^* = (k^*)^\alpha - \delta k^*$, and

$$J_{33}^* = -(a + \beta(k^*)^\alpha) < 0.$$

Consequently, $w(J^*) < 0$. Then, the result follows by Lemma 4. ∎

Figure 1 shows a portion of the stable manifold, obtained through a shooting algorithm, on which the trajectories converging to the unique stationary state equilibrium of the system (P^*) develop. From an economic point of view, this characterisation allows us to show the existence of a positive relationship between population and income, as in the Malthusian phase of the demographic transition (Fanti and Manfredi, 2003), which, however, gives only a partial description of the overall trend of economic and demographic variables. To overcome this gap, in the next section we will introduce a more general formulation of $G(f(k))$ for a thorough characterisation of the demographic transition allowing for a more accurate description of the various phases of the overall process.
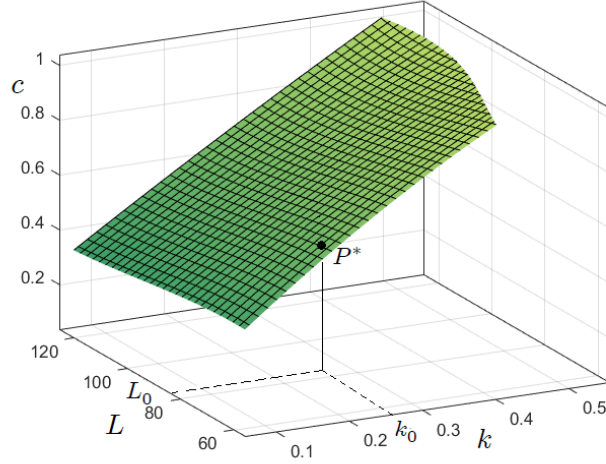
Figure 1. A portion of the stable manifold on which the trajectories converging to the steady state P^* occur. Parameter set: $\alpha = 0.18$, $\beta = 0.84$, $\delta = 0.13$, $\varepsilon = 0.4$, $\rho = 0.3$, $\sigma = 1.25$, $a = 40.5$, $b = 0.5$. Stationary state equilibrium: $P^* := (k^*, c^*, L^*) = (0.257, 0.749, 82.315)$.

4.2 The non-monotonic case (multiple equilibria)

This section assumes the existence of a non-monotonic relationship (inverted U-shaped) between the population growth rate and income per capita. Function G now takes the form:

$$G\left(f(k)\right) = \beta k^\alpha (x - k^\alpha), \quad \beta > 0, \quad x > 0, \tag{35}$$

Eq. (35) reveals that G is a function that modifies the natural long-term level of population a/b, as dictated by the logistic law. In addition, the constant x determines the threshold $(x/2)^{1/\alpha}$ below (resp. above) which there exists a positive (resp. negative) relationship between the growth rate of population and the accumulation of capital. The thresholds $(x/2)^{1/\alpha}$ and a/b contribute defining the carrying capacity of the system. In the literature, there are several points of view around the concept of carrying capacity. It seems more appropriate to consider it not as an exogenously given threshold but rather a variable determined by endogenous variables (see Seidl and Tisdell, 1999). The rationale for the use of the formulation in (35) stems from the fact that the demo-economic transition was characterised by a first phase experiencing dramatic growth of both population and income per capita due essentially to technological changes (for example, the use of new tools in agriculture) and improved hygiene practices (allowing to reduce mortality substantially), the Malthusian epoch, followed by a phase of stagnation that eventually ended up in the Modern Growth Regime, in which technological progress allowed to greatly increase income and the quantity/quality switch contributed to reduce the demand for the number of children and increase the demand for their education, eventually pushing down birth and death rates and up the rate of income growth. The dynamic equation for the

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evolution population that we propose here takes account of this fact. The expression in (33), where $G(f(k))$ is specified as in (35), accords with the historical pattern of the demographic transition allowing for a non-monotonic behaviour resembling the Malthusian epoch (positive relationship between total population and income, and high birth rate with death rates falling rapidly) and the Modern Growth Regime (negative relationship between total population and income, and low birth and death rates). Indeed, it may also be in line with the theoretical speculation about *stage five* of the demographic transition implying birth rate declining and low death rates (the birth rate declines to below-replacement levels becoming lower than the death rate), causing negative growth rates in the overall population that starts then declining. This could potentially be due to population ageing, the widespread use of contraception, the high costs of raising children in several cities of the Western world and the phenomenon of postponing the entry into the labour market because of education activities, implying delaying giving birth to the first child (Lorentzen et al., 2008; Bhattacharya and Chakraborty, 2017; Prettner and Strulik, 2017). However, the *stage five* trend is currently merely speculative and greatly depends on population dynamics in Sub-Saharan Africa. Speculations in this direction may show a different *stage five* implying increasing birth rates coexisting with low death rates eventually producing a further sharp increase in total population worldwide. However, this view could support the idea that fertility rates should rise also at very high levels of economic development, which is at odds with the main view of the demographic literature (e.g., Gaddy, 2021) unless considering the catastrophic effects of infectious diseases, with specific regard to the fertility reversal effect of HIV/AIDS in SSA (Kalemli-Ozcan and Turan, 2011; Kalemli-Ozcan, 2012; Gori et al., 2020).

By using the versions of utility function, production function and population dynamics as expressed in (31), (32) and (33) – where $G(f(k))$ is specified in (35) –, respectively, we obtain the following dynamic system:

$$
\begin{cases}
\dot{k} &= k^\alpha - \delta k - c - k\left[a - bL + \beta k^\alpha(x - k^\alpha)\right] \\
\dot{c} &= \sigma c \left\{ (\varepsilon - 1)\left[a - bL + \beta k^\alpha(x - k^\alpha)\right] - \rho - \delta + \alpha k^{\alpha - 1} - \left[\alpha\beta k^\alpha(x - k^\alpha) - \alpha\beta k^{2\alpha}\right] \right\} \\
\dot{L} &= L\left[a - bL + \beta k^\alpha(x - k^\alpha)\right]
\end{cases}.
$$
(36)

Given the expression in (35), it is not possible to state in explicit form the non-trivial stationary state equilibria of the system, i.e., those with positive values of L and c.[13] However, we can characterise the number of stationary state equilibria by concentrating on the second equation in (36) having previously used the third one, i.e., $a - bL + \beta k^\alpha(x - k^\alpha) = 0$. This is because once one has obtained the stationary state values of k, the corresponding stationary state values of L and c are respectively given by $L^* = [a + \beta(k^*)^\alpha(x - (k^*)^\alpha)]/b$ and $c^* = (k^*)^\alpha - \delta k^*$. Then, the values of k^* are solutions of the following equation:

$$
-\rho - \delta + k^{\alpha - 1}\alpha - \left[\beta k^\alpha\alpha(x - k^\alpha) - \alpha\beta k^{2\alpha}\right] = 0.
$$
(37)

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[13] The case $c^* = 0$ is ruled out by the transversality condition.

It is important to note that the classical result of uniqueness of the stationary state equilibrium of the Ramsey model does not hold when one augments the setting by including a demographically founded formulation for the dynamics of population, i.e. when $\dot{L}/L$ also depends on income per capita in a way compatible with the trend of the demographic transition. In fact, for every $\beta > 0$, the system produces zero or two equilibria. This result changes the perspective we had until now of the Ramsey set up and it is independent of the degree of altruism of individuals. This is because this modified version can explain the coexistence of countries with high accumulation of capital and countries with a low accumulation of capital. These findings are summarised in the following proposition.

Proposition 9 *In this modified version of the Ramsey model, there exist two thresholds $\underline{\beta}$ and $\overline{\beta}$, with $\underline{\beta} < \overline{\beta}$, such that for every $0 < \beta < \underline{\beta}$ there exist two stationary state equilibria $P_1^* := (k_1^*, c_1^*, L_1^*)$ and $P_2^* := (k_2^*, c_2^*, L_2^*)$, for every $\underline{\beta} < \beta < \overline{\beta}$ there exist no equilibria and for every $\beta > \overline{\beta}$ there exist two stationary state equilibria P_1^* and P_2^*. We denote with P_1^* the stationary point with the lowest values of the stock of capital.*

Proof. The proof follows from the characterisation of the shape of the graph of the function defined by the equation:

$$h(k) = 2\alpha\beta k^{2\alpha+1} - \alpha\beta x k^{\alpha+1} + \alpha k^{\alpha} - \delta k - \rho k. \tag{38}$$

■

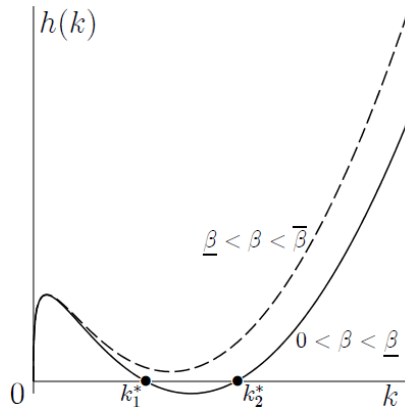Results of Proposition 9 are qualitatively illustrated in Figure 2.

Figure 2. Existence of multiple equilibria depending on β.

Given the result on the number of stationary state equilibria, it is important to study their stability properties, that is the possibility for the economic system to approach or to move away from them. The following proposition aims at clarifying this problem.

Proposition 10 *The stationary state equilibrium P_1^* can be saddlepoint stable or unstable. The stationary state equilibrium P_2^* can be saddlepoint unstable or locally indeterminate.*

Proof. These results directly follow by the general statement in Proposition 7. ∎

Proposition 10 shows that the stationary state equilibria P_1^* and P_2^* stated in Proposition 9 can indeed be reached by the economic system. This opens the route for the Ramsey model to explain the existence of different development scenarios with different trajectories of capital accumulation and population. To illustrate these results, we consider the following parameter set: $\alpha = 0.18$, $\beta = 0.84$, $\delta = 0.13$, $\varepsilon = 0.4$, $\rho = 0.3$, $\sigma = 1.25$, $a = 40.5$, $b = 0.5$ and $x = 0.7$. There exist two stationary state equilibria, P_1^* (saddlepoint stable) and P_2^* (locally indeterminate). Therefore, the non-linear relationship between population growth rate and capital accumulation as that given in the expression in (35) allows for the existence of trajectories converging towards two distinct long-term scenarios characterised by different development paths: one characterised by high population size, low capital accumulation and consumption per capita (P_1^*) – explaining a development trap – and the other characterised by lower population size, and larger values of capital accumulation and consumption per capita (P_2^*) – explaining a development trajectory. This can just happen also to two economies starting from the same initial conditions of the state variables (population and capital). Observing different long-term outcomes starting from identical economic conditions is referred to as global indeterminacy (Figure 3). Ending up in one or another scenario is a matter of individual consumption decisions. This makes expectations (namely, the coordination of individuals) important for obtaining a long-term outcome, which is therefore not history-dependent. Nonetheless, the convergence towards P_1^* seems quite unlikely as there exists one and only one value of the initial value of c (c_0^1) driving the economy to P_1^*. We note, however, that there is an infinite number of c_0 allowing for an infinite number of trajectories leading to P_2^*, each one giving rise to (and thus characterised by) a different development path (to this purpose Figure 3 depicts five initial values of c_0 each with its trajectory). The existence of different transition outcomes leading to the same long-term stationary state equilibrium is referred to as local indeterminacy.

Finally, we pinpoint that the expression of the growth rate of population $n(f(k), L)$ adopted in this section allows us observing non-permanent cycles of the trajectories convergent towards the long-term development scenario P_2^* (Figure 4).
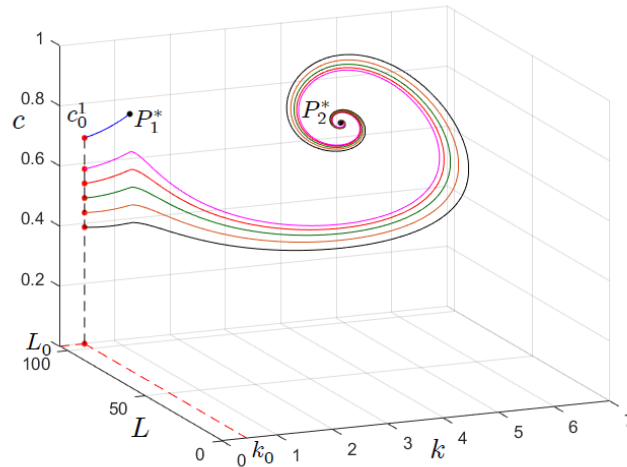
Figure 3. Global indeterminacy. Starting from the same initial conditions $k_0 = 0.455$ and $L_0 = 104.843$, an infinity of initial choices on c_0 lead to the indeterminate stationary state equilibrium P_2^*, while there exists a unique choice ($c_0^1 = 0.833$) leading to the saddle P_1^*. Parameter set: $\alpha = 0.18$, $\beta = 0.84$, $\delta = 0.13$, $\varepsilon = 0.4$, $\rho = 0.3$, $\sigma = 1.25$, $a = 40.5$, $b = 0.5$ and $x = 0.7$. Stationary state equilibria: $P_1^* := (k_1^*, c_1^*, L_1^*) = (0.594, 0.833, 80.677)$ and $P_2^* := (k_2^*, c_2^*, L_2^*) = (4.372, 2.925, 79.676)$.
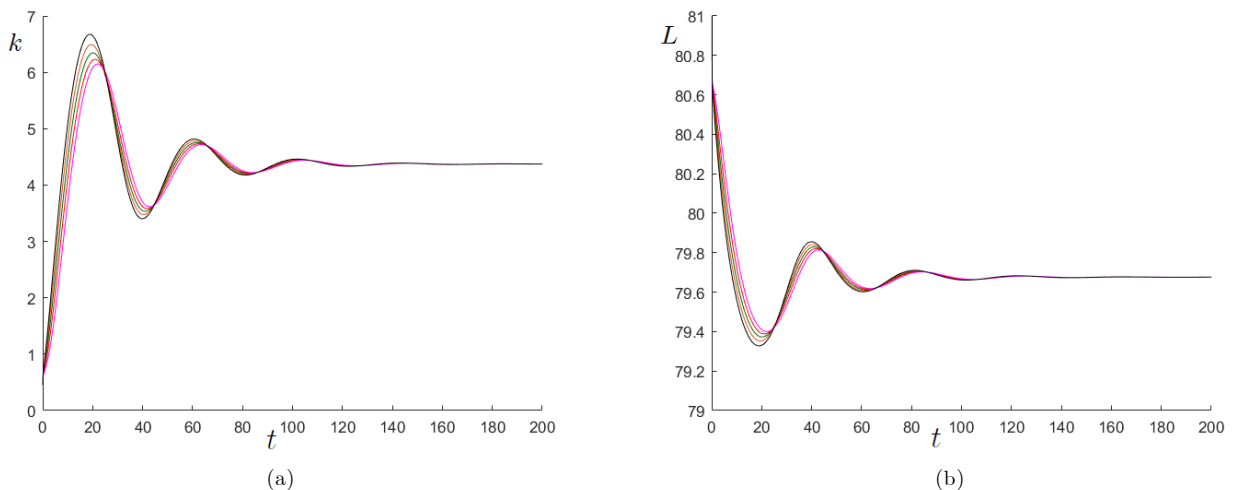
Figure 4. Local indeterminacy scenario in terms of capital (Panel A) and population size (Panel B).

4.3 The non-linear case (permanent cycles)

The previous sections considered specific functional forms for the population growth rate n to capture additive separability with respect to its arguments, that is $f(k)$ and L. This section

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relaxes this assumption and introduces a non-linear specification of n with respect to $f(k)$ and L by assuming the following formulation:

$$n(f(k), L) = \gamma \arctan\left[a - bL + \beta k^\alpha (x - k^\alpha)\right]. \tag{39}$$

From a qualitative point of view, the expression in (39) describes the same reactions of the population growth rate as those described in Subsection 4.2 when $f(k)$ and L change. This means that, given the levels of $f(k)$ and L, the sign of $\dot{L}/L$ coincides with the sign of $\dot{L}/L$ as specified in the previous section (indeed, the arctan function is an increasing monotonic transformation of the argument). The nonlinearity in (39) allows us to greatly enrich the dynamics of the system. In this regard, in addition to the phenomena of local and global indeterminacy, the new system can generate oscillatory dynamics in the two state variables k and L and the control variable c. Starting from a situation analogous to the one described in the previous section with one locally indeterminate stationary state equilibrium and the other determined stationary state equilibrium, an increase in γ let the indeterminate equilibrium undergo a supercritical Hopf bifurcation. Increasing γ further gives rise to an attracting limit cycle, as shown in Figure 5.
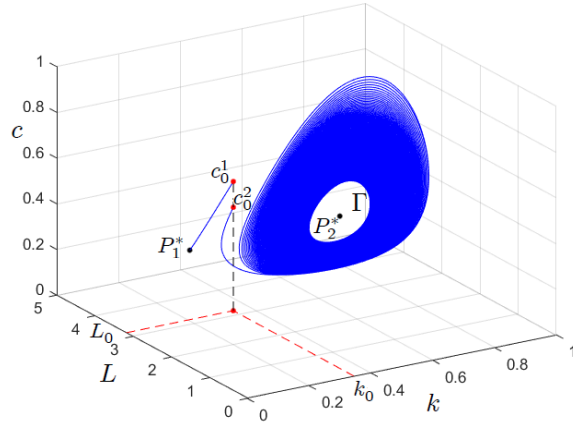

Figure 5. Three-dimensional phase portrait in the space (k, c, L) showing: (i) the trajectory converging to P_1^* starting from the initial condition $(k_0, c_0^1, L_0) = (0.346, 0.565, 3.178)$, and (ii) a trajectory converging to a limit cycle Γ around P_2^* starting from the initial condition $(k_0, c_0^2, L_0) = (0.346, 0.452, 3.178)$. Economic and demographic variables permanently fluctuate around P_2^*. Parameter set: $\alpha = 0.54$, $\beta = 1$, $\gamma = 1.5672$, $\delta = 0.7$, $\varepsilon = 0.4$, $\rho = 0.3$, $\sigma = 7.04$, $a = 0.37$, $b = 0.2$ and $x = 1.1$. Stationary state equilibria: $P_1^* := (k_1^*, c_1^*, L_1^*) = (0.221, 0.288, 3.305)$ and $P_2^* := (k_2^*, c_2^*, L_2^*) = (0.673, 0.336, 3.03)$.

5   Conclusions

"The concept of development is by no means unproblematic. The different problems underlying the concept have become clearer over the years on the basis of conceptual discussions as well as from insights emerging from empirical work" (Sen, 1988, p. 23). The Ramsey growth model is a classical set-up to study problems of economic growth. In its basic formulation, it does not allow, however, getting insights about economic development, as there exists a unique saddle path the economy may follow to approach the stationary state equilibrium, that is countries starting with a low stock of capital grow faster than countries starting with a high stock of capital but eventually they all will end up in the same stationary state. This is the result formerly based on the work of Ramsey (1928), who considered a (benevolent) central planner aiming at maximising the consumers' stream of consumption bundle over an infinite time horizon. Only some decades later, was the model extended by Cass (1965) and Koopmans (1965) to a decentralised competitive market context, showing that the market solution is Pareto optimal. The concept of economic growth is substantially different from the concept of economic development. The former is essentially a phenomenon of market productivity related to the change in GDP per person over time. The latter has a deeper meaning and relates economic, institutional, demographic and environmental issues amongst them. To explain economic development, therefore, it is necessary to include at least demographic variables in an economic model, which should also be able to produce a multiplicity of long-term stationary state equilibria. In this way, a model may be prone to explain a long-term (or also very long-term, Spolaore and Wacziarg, 2013) phenomenon such as the demographic transition by which the change in income per capita is accompanied by a dramatic change in birth and death rates, according to the well-known non-monotonic historical pattern.

Amongst the several extensions of the Ramsey model studied by scholars over time, only a few of them has dealt with demographic changes (Canton and Meijdam, 1997; Brida and Accinelli, 2007; Marsiglio and La Torre, 2012). The basic Ramsey model, whose formulation was also popularised by Barro and Sala-i-Martin (2003) in their economic growth textbook (using essentially the Benthamite criterion of total utilitarianism), does generally account for the non-realistic assumption of exponential growth by which population (alternatively, the labour force in a full-employment context) grows with no upper bounds at a constant rate. Unfortunately, this is at odds with the empirical evidence. A first solution was given by including the logistic population growth function, which is a law that can more accurately describe the historical (non-monotonic) behaviour of birth and death rates. However, also this solution is not completely satisfying as the demo-economic transition is not (entirely) characterised by the trend in population variables, but it does also incorporate the trend in income, allowing to range from the Malthusian epoch (positive relationship between total population and income) to the modern regime of growth (negative relationship between total population and income) and beyond.

To overcome this (surprising) lacuna, this work has included a demographically founded formulation for the time-varying population process within the standard Ramsey set-up. As the demographic transition is a long-term macro phenomenon, the fertility rate has been assumed as an exogenous variable (i.e., a variable not dictated by economic constraints and incentives). This apparent flaw is in line with the growth literature with exogenous fertility and aims at emphasising that reproduction is a biological phenomenon not necessarily driven by the behaviour of rational agents. However, the whole demographic transition pattern should necessarily be endogenous to the model as it depends - amongst other things - on the economic activity. Then, the growth rate of the population comprises two wings: the current level of the population (that evolves according to the logistic law), and a non-monotonic dependency on income per person (following the well-known idea of Solow, 1956). Then, the Ramsey model becomes able to produce multiple (determinate or indeterminate) stationary states and explain the reasons why some countries develop whereas others remain entrapped in poverty. From a theoretical perspective, the results of this work complement those of the Unified Growth Theory.

Conflict of Interest   The authors declare that they have no conflict of interest.

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