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Regularity of the free boundary for the two-phase Bernoulli problem

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Abstract We prove a regularity theorem for the free boundary of minimizers of the two-phase Bernoulli problem, completing the analysis started by Alt, Caffarelli and Friedman in the 80s. As a consequence, we also show regularity of minimizers of the multiphase spectral optimization problem for the principal eigenvalue of the Dirichlet Laplacian.

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1 Introduction

We consider the two-phase functional J_{tp} defined, for every open set $D \subset \mathbb{R}^d$ and every function $u : D \to \mathbb{R}$, as

$$J_{tp}(u, D) := \int_D |\nabla u|^2\,dx + \lambda_+^2|\Omega_u^+ \cap D| + \lambda_-^2|\Omega_u^- \cap D|, \tag{TP}$$

where the constants $\lambda_+ > 0$ and $\lambda_- > 0$ are given and fixed, and the two phases

$$\Omega_u^+ = \{u > 0\} \qquad \text{and} \qquad \Omega_u^- = \{u < 0\}$$

are the positivity sets of the functions $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$.

We say that a function $u : D \to \mathbb{R}$ is a *local minimizer of* J_{tp} *in* D if

$$J_{tp}\big(u, \Omega\big) \leq J_{tp}\big(v, \Omega\big)$$

for all open sets Ω and functions $v : D \to \mathbb{R}$ such that $\overline{\Omega} \subset D$ and $v = u$ on $D \setminus \Omega$.

In this paper we aim to study the regularity of the free boundary $\partial\Omega_u^+ \cup \partial\Omega_u^- \cap D$ for local minimizers of J_{tp} in D. Our main result is a full description of $\partial\Omega_u^+$ and $\partial\Omega_u^-$ around two-phase points:

$$\Gamma_{tp} := \partial\Omega_u^+ \cap \partial\Omega_u^- \cap D.$$

More precisely, we prove that, in a neighborhood of a two-phase point, the sets Ω_u^+ and Ω_u^- are $C^{1,\eta}$-regular domains touching along the closed set Γ_{tp}.

Theorem 1.1 (Regularity around two-phase points). *Let* $u : D \to \mathbb{R}$ *be a local minimizer of* J_{tp} *in the open set* $D \subset \mathbb{R}^d$*. Then, for every two-phase point* $x_0 \in \Gamma_{tp} \cap D$*, there exists a radius* $r_0 > 0$ *(depending on* x_0*) such that* $\partial\Omega_u^\pm \cap B_{r_0}(x_0)$ *are* $C^{1,\eta}$ *graphs for some* $\eta > 0$.

Combining Theorem 1.1 with the known regularity theory for *one-phase problem*, one obtains the following result, which provides a full description of the free boundary of local minimizers of J_{tp}.

Corollary 1.2 (Regularity of the free boundary). *Let* $u : D \to \mathbb{R}$ *be a local minimizer of* J_{tp} *in the open set* $D \subset \mathbb{R}^d$*. Then, each of the sets* $\partial\Omega_u^+ \cap D$ *and* $\partial\Omega_u^- \cap D$ *can be decomposed as a disjoint union of a regular and a (possibly empty) singular part*

$$\partial\Omega_u^\pm \cap D = \mathrm{Reg}(\partial\Omega_u^\pm) \cup \mathrm{Sing}(\partial\Omega_u^\pm),$$

with the following properties:

(i) *The regular part* $\mathrm{Reg}(\partial\Omega_u^{\pm})$ *is a relatively open subset of* $\partial\Omega_u^{\pm} \cap D$ *and is locally the graph of a* $C^{1,\eta}$*-regular function, for some* $\eta > 0$. *Moreover, the two-phase free boundary is regular, that is,*

$$\Gamma_{tp} \cap D \subset \mathrm{Reg}(\partial\Omega_u^{\pm}).$$

(ii) *The singular set* $\mathrm{Sing}(\partial\Omega_u^{\pm})$ *is a closed subset of* $\partial\Omega_u^{\pm} \cap D$ *of Hausdorff dimension at most* $d-5$. *Precisely, there is a critical dimension*[1] $d^* \in [5, 7]$ *such that*
  - *if* $d < d^*$, *then* $\mathrm{Sing}(\partial\Omega_u^{\pm}) = \emptyset$*;*
  - *if* $d = d^*$, *then* $\mathrm{Sing}(\partial\Omega_u^{\pm})$ *is locally finite in* D*;*
  - *if* $d > d^*$, *then* $\mathrm{Sing}(\partial\Omega_u^{\pm})$ *is a closed* $(d-d^*)$*-rectifiable subset of* $\partial\Omega_u^{\pm} \cap D$ *with locally finite* $\mathcal{H}^{d-d^*}$ *measure.*

As a second corollary of our analysis, by applying the same type of arguments as in [43] we obtain a complete regularity results for the following shape optimization problem, studied in [6,8,45], where the optimal sets have the same qualitative behavior as the sets Ω_u^+ and Ω_u^- in Corollary 1.2, contrary to the classical optimal partition problem studied in [13,14,19–21] (which corresponds to the case of zero weights $m_i = 0$, for every i).

Corollary 1.3 (Regularity for a multiphase shape optimization problem). *Let* D *be a* $C^{1,\gamma}$*-regular bounded open domain in* $\mathbb{R}^d$*, for some* $\gamma > 0$ *and* $d \geq 2$. *Let* $n \geq 2$ *and* $m_i > 0$, $i = 1, \dots, n$ *be given. Let* $(\Omega_1, \dots, \Omega_n)$ *be a solution of the following optimization problem:*

$$\min\Big\{\sum_{i=1}^{n}\big(\lambda_1(\Omega_i) + m_i|\Omega_i|\big)\ :\ \Omega_i \subset D \ \text{open};\ \Omega_i \cap \Omega_j = \emptyset \ \text{for}\ \ i \neq j\Big\}. \tag{SOP}$$

where $\lambda_1(\Omega_i)$ *is the first eigenvalue for the Dirichlet Laplacian in* Ω_i.

Then, the free boundary $\partial\Omega_i$ *of each of the sets* Ω_i, $i = 1, \dots, n$, *can be decomposed as the disjoint union of a regular part* $\mathrm{Reg}(\partial\Omega_i)$ *and a (possibly empty) singular part* $\mathrm{Sing}(\partial\Omega_i)$*, where:*

(i) *The regular part* $\mathrm{Reg}(\partial\Omega_i)$ *is a relatively open subset of* $\partial\Omega_i$ *and is locally the graph of a* $C^{1,\eta}$*-regular function, for some* $\eta > 0$. *Moreover, both the*

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[1] The critical dimension d^* is the first dimension, for which there exists a one-homogeneous non-negative local minimizer of the one-phase functional with a singular free boundary. Currently, it is only known that $5 \leq d^* \leq 7$, [12,30,34].

contact set with the boundary of the box and the two-phase free boundaries are regular, that is,

$$\partial\Omega_i \cap \partial D \subset \mathrm{Reg}(\partial\Omega_i) \quad \textit{and} \quad \partial\Omega_i \cap \partial\Omega_j \subset \mathrm{Reg}(\partial\Omega_i) \quad \textit{for every} \quad j \in \{1, \dots, n\} \setminus \{i\}.$$

(ii) *The singular set* $\mathrm{Sing}(\partial\Omega_i)$ *is a closed subset of* $\partial\Omega_i$ *of Hausdorff dimension at most* $d-5$. *Precisely,*
- *if* $d < d^*$, *then* $\mathrm{Sing}(\partial\Omega_i) = \emptyset$,
- *if* $d = d^*$, *then* $\mathrm{Sing}(\partial\Omega_i)$ *is locally finite in* D,
- *if* $d > d^*$, *then* $\mathrm{Sing}(\partial\Omega_i)$ *is a closed* $(d-d^*)$*-rectifiable subset of* $\partial\Omega_i$ *with locally finite* $\mathcal{H}^{d-d^*}$ *measure,*

where $d^* \in \{5, 6, 7\}$ *is the critical dimension from Corollary* 1.2.

1.1 Regularity of local minimizers of the Bernoulli functional

The study of the regularity of minimizers of J_{tp} started in the seminal paper of Alt and Caffarelli [1], which was dedicated to the one-phase case, in which u is non-negative. In this case, it is sufficient to work with the one-phase functional

$$J_{op}(u, D) := \int_D |\nabla u|^2\,dx + \lambda_+^2 |\Omega_u^+ \cap D|, \tag{OP}$$

as the negative phase Ω_u^- is empty. In [1] it was proved that for a local minimizer u of J_{op}, the free boundary $\partial\Omega_u^+ \cap D$ decomposes into a $C^{1,\eta}$-regular set $\mathrm{Reg}(\partial\Omega_u^+)$ and a closed singular set $\mathrm{Sing}(\partial\Omega_u^+)$ of zero $\mathcal{H}^{d-1}$-Hausdorff measure. A precise estimate on the Hausdorff dimension of $\mathrm{Sing}(\partial\Omega_u^+)$ was then given by Weiss [46] as a consequence of his monotonicity formula and its rectifiability was established by Edelen and Engelstein [31]. In fact, the results in Corollary 1.2 are an immediate consequence of Theorem 1.1 and the known regularity for the one-phase parts

$$\Gamma_{op}^+ := \left(\partial\Omega_u^+ \setminus \partial\Omega_u^-\right) \cap D \qquad \text{and} \qquad \Gamma_{op}^- := \left(\partial\Omega_u^- \setminus \partial\Omega_u^+\right) \cap D.$$

Indeed:

- the regularity of $\mathrm{Reg}(\partial\Omega_u^\pm)$ (Corollary 1.2 (i)) follows by Theorem 1.1 and [1, Theorem 8.1];
- the estimates on the dimension of the singular set $\mathrm{Sing}(\partial\Omega_u^\pm)$ (Corollary 1.2 (ii)) are again a consequence of Theorem 1.1 (which shows that singularities can appear only on the one-phase parts of the free boundary) and the results in [31,46].

The regularity of local minimizers with two-phases (that is, local minimizers of J_{tp} which change sign) was first addressed by Alt, Caffarelli and Freedman [2], where the authors consider free boundary functionals that weight also the zero level set of u:

$$\begin{aligned} J_{acf}(u, D) := \int_D |\nabla u|^2\,dx + \lambda_+^2\big|\Omega_u^+ \cap D\big| \\ + \lambda_-^2\big|\Omega_u^- \cap D\big| + \lambda_0^2\big|\{u = 0\} \cap D\big|, \end{aligned} \tag{ACF}$$

where $\lambda_+ \geq \lambda_0 \geq 0$ and $\lambda_- \geq \lambda_0 \geq 0$. When $D \subset \mathbb{R}^2$ is a planar domain, and under the additional assumptions

$$\lambda_+ \neq \lambda_- \qquad \text{and} \qquad \lambda_0 = \lambda_+ \ \text{or}\ \lambda_-,$$

they showed that the free boundaries $\partial\Omega_u^+ \cap D$ and $\partial\Omega_u^- \cap D$ are C^1-regular curves. The key observation here is that the additional assumption

$$\lambda_0 = \lambda_+ \ \text{ or }\ \lambda_0 = \lambda_-\,, \tag{1.1}$$

forces the level set $\{u = 0\}$ to have zero Lebesgue measure. Thus, the two boundaries $\partial\Omega_u^+ \cap D$ and $\partial\Omega_u^- \cap D$ coincide and the solution u satisfies the transmission condition

$$|\nabla u^+|^2 - |\nabla u^-|^2 = \lambda_+^2 - \lambda_-^2 \qquad \text{on} \qquad \partial\Omega_u^+ = \partial\Omega_u^-. \tag{1.2}$$

The free boundary regularity for local minimizers of J_{acf} in the case (1.1) is already known in any dimension. Indeed, the regularity of the free boundary $\partial\Omega_u^+ = \partial\Omega_u^-$, for functions which are harmonic (or solve an elliptic PDE) in $\Omega_u^+ \cup \Omega_u^-$ and satisfy the transmission condition (1.2), is today well-understood, after the seminal work of Caffarelli [9–11] (see also the book [15]) and the more recent results of De Silva, Ferrari and Salsa [27–29], which are based on the techniques introduced by De Silva [26] and which are central also in the present paper.

On the other hand, in the general case,

$$\lambda_+ > \lambda_0 \ \text{ and }\ \lambda_- > \lambda_0\,, \tag{1.3}$$

one can easily construct solutions of (ACF) for which the zero set $\{u = 0\}$ has positive measure, preventing the application of the existing results and techniques about two-phase free boundary problems, as for instance [9–11,27–29], which rely on the transmission condition (1.2).

To the best of our knowledge, the only known regularity result for minimizers of (ACF) under the condition (1.3) is due to the second and third authors in

[42], where it is proved that, in dimension $d = 2$, the free boundaries $\partial\Omega_u^+$ and $\partial\Omega_u^-$ are $C^{1,\eta}$ regular. The proof relies on a novel epiperimetric type inequality which applies only in dimension two and it was recently extended (still in dimension two) to almost-minimizers by the same two authors and Trey [43].

In this paper, we complete the analysis started by Alt, Caffarelli and Freedman [2], by proving a regularity result for the free boundaries of local minimizers of (ACF), in the general case (1.3) and in any dimension $d \geq 2$. Indeed, Theorem 1.1 and Corollary 1.2 apply directly to (ACF) as the local minimizers of (ACF), corresponding to the parameters λ_0, λ_+ and λ_-, are local minimizers of (TP) with parameters

$$\lambda'_+ = \sqrt{\lambda_+^2 - \lambda_0^2} \quad \text{and} \quad \lambda'_- = \sqrt{\lambda_-^2 - \lambda_0^2}\,.$$

1.2 One-phase, two-phase and branching points on the free boundary

Let $u : B_1 \to \mathbb{R}$ be a (local) minimizer of J_{tp} in B_1 and let, as above, $\Omega_u^\pm = \{\pm u > 0\}$. Notice that, the zero level set $\{u = 0\}$ might have positive Lebesgue measure in B_1 and also non-empty interior, contrary to what happens with the minimizers of (ACF) with $\lambda_+ = \lambda_0$. This introduces a new element in the analysis of the free boundary, which can now switch from one-phase to two-phase at the so-called branching points, at which the zero level set looks like a cusp. Precisely, this means that the free boundary $\partial\Omega_u^+ \cap B_1$ (the same holds for the negative phase $\partial\Omega_u^- \cap B_1$) can be decomposed into:

- a set of one-phase points $\Gamma_{op}^+ := \partial\Omega_u^+ \setminus \partial\Omega_u^- \cap B_1$, and
- a set of two-phase points $\Gamma_{tp} := \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$.

By definition the set of one-phase points Γ_{op}^+ is relatively open in $\partial\Omega_u^+$. Precisely, if $x_0 \in \Gamma_{op}^+$, then there is a ball $B_r(x_0)$ which does not contain points from the negative phase, $B_r(x_0) \cap \Omega_u^- = \emptyset$. Thus, u is a minimizer of the one-phase functional J_{op} in $B_r(x_0)$ and the regularity of $\partial\Omega_u^+ \cap B_r(x_0)$ follows from the results in [1,46].

For what concerns the two-phase points, we can further divide them into *interior* and *branching* points:

- we say that x_0 is an *interior* two-phase point, $x_0 \in \Gamma_{tp}^{\text{int}}$, if $x_0 \in \Gamma_{tp}$ and

$$\big|B_r(x_0) \cap \{u = 0\}\big| = 0 \quad \text{for some} \quad r > 0\,;$$

- conversely, we say that x_0 is a *branching* point, $x_0 \in \Gamma_{tp}^{\text{br}}$, if $x_0 \in \Gamma_{tp}$ and

$$\big|B_r(x_0) \cap \{u = 0\}\big| > 0 \quad \text{for every} \quad r > 0\,.$$

By definition, $\Gamma_{tp}^{\mathrm{int}}$ is an open subset of $\partial\Omega_u^+\cap B_1$. In particular, u is a minimizer of the Alt–Caffarelli–Friedman functional (ACF) with $\lambda_+=\lambda_0$ in a small ball $B_r(x_0)$ and the regularity of $\Gamma_{tp}^{\mathrm{int}}$ is a consequence of the results in [2,9–11,15,27–29].

In order to complete the study of the regularity of the free boundaries one has then to focus on the branching points. Note that by the previous discussion $|\nabla u^+|$ is a Hölder continuous function on $\Gamma_{op}^+\cup\Gamma_{tp}^{\mathrm{int}}$. By relying on the results of [26], to prove Theorem 1.1 one has to show that $|\nabla u^+|:\partial\Omega_u^+\cap B_1\to\mathbb{R}$ is Hölder continuous across the branching points

$$\Gamma_{tp}^{\mathrm{br}}=\left(\partial\Omega_u^+\cap B_1\right)\setminus\left(\Gamma_{op}^+\cup\Gamma_{tp}^{\mathrm{int}}\right).$$

By following [42,43] this will be consequence of

uniform "flatness" decay at the two-phase points $x_0\in\Gamma_{tp}$,

which is the main result of our paper.

1.3 Flatness decay at the two-phase points

By the Weiss' monotonicity formula (see [46]), at every two-phase point $x_0\in\Gamma_{tp}$, the limits of blow-up sequences

$$u_{x_0,r_k}(y)=\frac{u(x_0+r_k y)}{r_k}$$

are *two-plane solutions* of the form

$$H_{\alpha,\boldsymbol{e}}(x)=\alpha(x\cdot\boldsymbol{e})^+-\beta(x\cdot\boldsymbol{e})^-,\ \text{where}\quad \boldsymbol{e}\in\mathbb{S}^{d-1},\quad \alpha^2-\beta^2=\lambda_+^2-\lambda_-^2,\quad\text{and}\quad\alpha\geq\lambda_+,\quad\beta\geq\lambda_-. \tag{TPS}$$

However, *a priori* the limiting profile might depend on the chosen sequence. As it is usual in this type of problems, uniqueness of the blow-up profile (and thus regularity of u) is a consequence of a uniform flatness (or excess) decay.

Given u, its flatness in $B_r(x_0)$ with respect to $H=H_{\alpha,\boldsymbol{e}}$ is defined as

$$\mathrm{flat}_{B_r(x_0)}(u,H)=\frac{1}{r}\|u-H\|_{L^\infty(B_r(x_0))}.$$

In particular, we can assume that the flatness becomes small at a uniform scale in a neighborhood of any $x_0\in\Gamma_{tp}$. Precisely, for every $\varepsilon>0$ and $x_0\in\Gamma_{tp}$, there is $r > 0$ and a neighborhood $\mathcal{U}$ of x_0, such that

$$\operatorname{flat}_{B_r(y_0)}(u, H) \le \varepsilon \quad \text{for every} \quad y_0 \in \mathcal{U} \cap \Gamma_{tp}.$$

Our aim is to prove that there is a universal threshold $\varepsilon > 0$ such that

$$\operatorname{flat}_{B_r(x_0)}(u, H) \le \varepsilon \quad \text{for some two-plane solution} \quad H = H_{\alpha,\boldsymbol{e}},$$

then it improves in the ball $B_{r/2}(x_0)$, which means that there exists another two-plane solution $\widetilde{H} = H_{\tilde{\alpha},\tilde{\boldsymbol{e}}}$ such that

$$\operatorname{flat}_{B_{r/2}(x_0)}(u, \widetilde{H}) \le 2^{-\gamma} \operatorname{flat}_{B_r(x_0)}(u, H), \tag{1.4}$$

for some small, but universal, $\gamma > 0$.

In order to prove (1.4), we argue by contradiction. That is, there is a sequence of minimizers u_k and a sequence of two-plane solutions H_k, such that

$$\varepsilon_k := \|u_k - H_k\|_{L^\infty(B_1)} \to 0 \qquad \text{but} \qquad \inf_{\widetilde{H}} \|u_k - \widetilde{H}\|_{L^\infty(B_{1/2})} \ge 2^{-\gamma}\varepsilon_k\,,$$

where the infimum is taken over all $\widetilde{H}$ of the form (TPS).

Now, the two key points of the argument are to show that the sequence

$$v_k := \frac{u_k - H_k}{\varepsilon_k}$$

is (pre-)compact in a suitable topology and that any limit point v_∞ is a solution of a suitable "linearized" problem (that turns out to be a non-linear one); then the regularity theory for the limiting problem allows to obtain the desired contradiction.

While the linearized problem can be guessed by formal computations, in order to transfer back regularity estimates for v_∞ to v_k, it is crucial to establish the uniform convergence of v_k (which is a priori only bounded in L^∞) to v_∞. In our case, as well as for other variational problems, this is not just a technical issue but is where the proof of ε-regularity-type theorems may actually fail, even if the "formal" linearized problem enjoys all the desired estimates. Instances of this phenomena are well known in literature. For example, while an ε-regularity theorem for minimizers of quasi-convex functions holds true (as proved by Evans [32]), a similar result is false for critical points, as shown by Muller and Sverak [36], eventhough the two problems share the same formal linearization. A similar issue happens for harmonic maps: an ε-regularity theorem is true for minimizers but it is false for critical points as it was proved by Rivière [37]; also in this case the formal linearized problem is the same namely the Laplace equation. In general, in many variational problems which exhibit singular behavior (as in our situation), the linearization is well-understood, but the compactness is still out of reach.

Let us now briefly analyze these two main steps of the proof.
The "linearized" problem. The nature of the limiting problem depends on the type of free boundary point one is considering. At branching points (the ones that we are most interested in), v_∞ turns out be the the solution of a *two-membrane problem*, (3.10). At interior two-phase points Γ_{tp}^{int}, we instead recover a transmission problem as in [27].

Note that in the first case, the "linearized" problem is actually non-linear. Similar phenomena have been already observed in a number of related situation: in this same context, a derivation of the limiting problem was done in [4], while for Bernoulli type problems a similar fact appears in studying regularity close to the boundary of the container, [18]. See also [33,41] for similar issues in studying the singular set of obstacle type problems. Heuristically linearizing to an "obstacle" type problem is due to the fact that there is a natural "ordering" between the negative and the positive phases of any possible competitor. Note instead if one linearizes the plain one phase problem, the natural linearized problem is the Neumann one, this was observed in [3] (in the parabolic case) and fully exploited in [26], see also [16,17] where other non-local type problems appear as linearization.
Compactness of the linearizing sequence v_k. We follow the approach introduced by De Silva [26], which is based on a partial Harnack type inequality, introduced in different context by Savin [39,40]. This is a weaker form of the flatness decay estimate (1.4) that does not take into account the scaling of the functional (which means that it cannot be used to obtain the regularity of the free boundary in a direct way). The rough idea is that if $\|u - H\|_{L^\infty(B_r(x_0))}$ falls below a certain (universal) threshold, then u is closer to H in the ball $B_{r/2}(x_0)$, precisely:

$$\|u - H\|_{L^\infty(B_{r/2}(x_0))} \leq 2^{-\delta}\|u - H\|_{L^\infty(B_r(x_0))}, \tag{1.5}$$

for some $\delta > 0$. This estimate implies the compactness of the sequence v_k by a classical (Ascoli-Arzelà type) argument.

For local minimizers of the one-phase functional (OP) or the two-phase functional (ACF) with coefficients satisfying the condition (1.1), the functions H can be chosen in the respective class of blow-up limits. In fact, for the one-phase problem, it is sufficient to take H to be the (possibly translated and rotated) one-homogeneous global one-phase solution $H(x) = \lambda_+ x_d^+$ (as in [26]); for the two-phase problem in the case (1.1), it is sufficient to take H in the class of two-plane solutions (TPS), precisely as in [27]. However, in our case, it turns out that the class of two-plane solutions is not large enough. The reason is that there exist solutions which are arbitrarily close to a two-plane

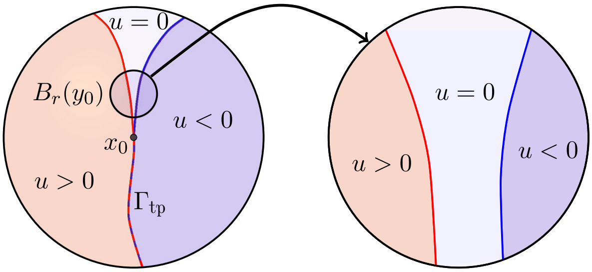

Fig. 1 In a neighborhood of a branching point x_0 the solution u might get closer to the solution of the form(1.6)

solution of the form $H_{\lambda_+,\boldsymbol{e}_d}$ but which are not a smooth perturbation of it. For instance the function,

$$H(x) = \lambda_+(x_d + \varepsilon_1)^+ - \lambda_-(x_d - \varepsilon_2)^-, \tag{1.6}$$

is $\max\{\varepsilon_1, \varepsilon_2\}$-close to the two-plane solution $H_{\lambda_+,\boldsymbol{e}_d}$, but (1.5) fails for it.

This is not just a technical difficulty. In fact, in order to get the compactness of the linearizing sequence, the partial improvement of flatness (1.5) is not needed just at one two-phase point x_0, but in all the points in a neighborhood of x_0. Now, since at a branching point, the behavior of the free boundary switches from two-phase (which roughly speaking corresponds to the case when the two free boundaries $\partial\Omega_u^+$ and $\partial\Omega_u^-$ coincide) to one-phase (in which the two free boundaries $\partial\Omega_u^+$ and $\partial\Omega_u^-$ are close to each other but separate, as on Fig. 1 below), the class of reference functions H has to contain both the two-plane solutions (TPS) and the solutions of the form (1.6).

Structure of the paper. This paper is organized as follows: in Sect. 2 we recall some basic properties of minimizers and we fix the notation; in Sect. 3 we establish the excess decay lemma; in Sect. 4 we prove our main results; in "Appendices A and B" we collect the proofs of some technical facts.

At the final stage of the preparation of this work, the authors have been informed by personal communication that two other groups are working on similar problems, namely in [4] the authors aim to establish a result analogous to the ours via variational techniques, while in [24] the goal is to prove the same result for almost minimizers in the spirit of [22,23,25]. At our knowledge, at this date, none of these works has appeared in a journal or on arxiv.

2 Basic properties of minimizers

In this section we recall (mostly without proof) some basic properties of local minimizers of J_{tp}. In particular, in Sect. 2.1 we recall Lipschitz-regularity and non-degeneracy property of u; Sect. 2.2 is dedicated to the study of blow-up limits of u at two-phase points and in Sect. 2.3 we show that u satisfies an optimality condition in viscosity sense.

2.1 Regularity of minimizers

Let u be a local minimizer of J_{tp}. Then, it is well-known that u is locally Lipschitz continuous and non-degenerate.

Throughout this paper, we will assume that the weights in (TP) are ordered as follows:

$$\lambda_+ \geq \lambda_- > 0. \tag{2.1}$$

Notice that this is not restrictive as one can always replace u by $-u$ in J_{tp}.

Proposition 2.1 (Lipschitz regularity and non-degeneracy of local minimizers). *Let $D \subset \mathbb{R}^d$ be an open set, $\lambda_+ \geq \lambda_- > 0$, and u be a local minimizer of J_{tp}. Then the following properties hold:*

(i) Lipschitz continuity. $u \in C^{0,1}_{\mathrm{loc}}(D)$.
(ii) Non-degeneracy. *There is constant $\alpha = \alpha(d, \lambda_\pm) > 0$ such that*

$$⨍_{\partial B_r(x_0)} u^\pm \geq \alpha\, r \quad \textit{for every } x_0 \in \overline{\Omega_u^\pm} \cap D \textit{ and every } 0 < r < \operatorname{dist}(x_0, \partial D).$$

Proof The second claim was first proved in [2, Theorem 3.1] and depends only on the fact that each of the two phases Ω_u^+ and Ω_u^- is optimal with respect to one-sided inwards perturbations (see for instance [8] and [44, Section 4]). The Lipschitz continuity of u is more involved and requires the use of the Alt–Caffarelli–Friedman monotonicity formula and the non-degeneracy of u^+ and u^-. It was first proved in [2, Theorem 5.3], see also the recent paper [23] for quasi-minimizers. □

2.2 Blow-up sequences and blow-up limits

Let u be a local minimizer of J_{tp} in the open set $D \subset \mathbb{R}^d$. For every $x_0 \in \partial\Omega_u \cap D$ and every $0 < r < \operatorname{dist}(x_0, \partial D)$, we consider the function

$$u_{x_0,r}(x) := \frac{u(x_0 + rx)}{r},$$

which is well-defined for $|x| < \frac{1}{r}\operatorname{dist}(x_0, \partial D)$ and vanishes at the origin. Given a sequence $r_k > 0$ such that $r_k \to 0$, we say that the sequence of functions u_{x_0,r_k} is a *blow-up sequence*. Note that, for every $R > 0$, and $k \gg 1$, the functions u_{x_0,r_k} are defined on the ball B_R, vanish at zero and are uniformly Lipschitz in B_R. Hence, there is a Lipschitz continuous function $v : \mathbb{R}^d \to \mathbb{R}$ and a (non-relabeled) subsequence of u_{x_0,r_k} such that u_{x_0,r_k} converges to v uniformly on every ball $B_R \subset \mathbb{R}^d$. We say that v is a blow-up limit of u at x_0. Notice that v might depend not only on x_0 and u but also on the (sub-)sequence r_k. We will denote by $\mathcal{BU}(x_0)$ the collection of all possible blow-up limits of u at x_0.

The following lemma classifies all the possible elements of $\mathcal{BU}(x_0)$ when $x_0 \in \Gamma_{tp}$.The result is well-known and we only sketch the proof for the sake of completeness.

Lemma 2.2 (Classification of the blow-up limits). *Let u be a local minimizer of J_{tp} in the open set $D \subset \mathbb{R}^d$, and let v be a blow-up limit of u at the two-phase point $x_0 \in \Gamma_{tp}$. Then, v is of the form*

$$v(x) = H_{\alpha,\boldsymbol{e}}(x) = \alpha(x \cdot \boldsymbol{e})^+ - \beta(x \cdot \boldsymbol{e})^-,$$

where $\boldsymbol{e} \in \mathbb{S}^{d-1}$, and α, β are such that

$$\alpha^2 - \beta^2 = \lambda_+^2 - \lambda_-^2 \quad \text{and} \quad \alpha \geq \lambda_+, \quad \beta \geq \lambda_-.$$

Proof Let v be a blow-up limit of u at x_0 and let u_{x_0,r_k} be a blow-up sequence converging to v (locally uniformly in $\mathbb{R}^d$). First, notice that the non-degeneracy of u, Lemma 2.1 (ii), implies that v is non trivial and changes sign: $v^+ \not\equiv 0$ and $v^- \not\equiv 0$. Moreover, since every u_{x_0,r_k} is a local minimizer of J_{tp} (it is standard to infer that v is also a local minimizer of J_{tp} in $\mathbb{R}^d$ (see for instance [44, Section 6]). Thus, v is harmonic on Ω_v^+ and Ω_v^-. On the other hand, by the Weiss monotonicity formula, [46], v is one-homogeneous, in polar coordinates:

$$v(\rho, \theta) = \rho V(\theta).$$

In particular V is an eigenfunction of the spherical Laplacian $\Delta_{\mathbb{S}}$ on the spherical sets $\Omega_v^\pm \cap \mathbb{S}^{d-1}$:

$$-\Delta_{\mathbb{S}} V^\pm = (d-1)V^\pm \quad \text{in} \quad \Omega_v^\pm \cap \mathbb{S}^{d-1}. \tag{2.2}$$

We now choose $c > 0$ such that

$$\int_{\mathbb{S}^{d-1}} (V^+ - cV^-)\, d\mathcal{H}^{d-1} = 0.$$

Using (2.2) and integrating by parts, we get that

$$\int_{\mathbb{S}^{d-1}} |\nabla_\theta (V^+ - cV^-)|^2 \, d\mathcal{H}^{d-1} = (d-1) \int_{\partial B_1} |V^+ - cV^-|^2 \, d\mathcal{H}^{d-1}.$$

This means that $V^+ - cV^-$ is an eigenfunction of the spherical Laplacian on $\mathbb{S}^{d-1}$, corresponding to the eigenvalue $(d-1)$. Since the $(d-1)$-eigenspace contains only linear functions one easily deduce that v is of the form (TPS).

Conditions (2.2) can be obtained by a smooth variation of the free boundary $\{v = 0\}$. Indeed, if considering competitors of the form $v_t(x) = v(x + t\xi(x))$ for smooth compactly vector fields ξ, and taking the derivative of $J_{tp}(v_t, B_1)$ at $t \to 0$, we get that

$$\int_{\{v=0\}\cap B_1} (\boldsymbol{e} \cdot \xi)\Big(|\nabla v^+|^2 - |\nabla v^-|^2 - \big(\lambda_+^2 - \lambda_-^2\big)\Big) \, d\mathcal{H}^{d-1} = 0,$$

which by the arbitrariness of ξ is precisely the first part of (2.2). The second part of (2.2) is analogous and follows by considering competitors of the form $v_t(x) = v^+(x) - v^-(x + t\xi(x))$ for vector fields with $\xi \cdot \boldsymbol{e} \leq 0$ so that it moves negative phase only inwards, that is, $\{v_t < 0\} \subset \{v < 0\}$. Taking the derivative of the energy at $t > 0$, we get

$$\int_{\{v=0\}\cap B_1} (\xi \cdot \boldsymbol{e})\big(|\nabla v^-|^2 - \lambda_-^2\big) \, d\mathcal{H}^{d-1} \leq 0,$$

which gives $\beta \geq \lambda_-$. The estimate on α is analogous. □

The following consequence of Lemma 2.2 says that the "flatness" can be chosen uniformly small in a neighborhood of a two-phase point.

Corollary 2.3 *Let u be a local minimizer of J_{tp} in the open set $D \subset \mathbb{R}^d$, and let x_0 be a two-phase point $x_0 \in \Gamma_{tp}$. Then, for every $\varepsilon > 0$ there are $r > 0$ and $\rho > 0$, and a function $H_{\alpha,\boldsymbol{e}}$ of the form* (TPS) *such that*

$$\|u_{y_0,r} - H_{\alpha,\boldsymbol{e}}\|_{L^\infty(B_1)} \leq \varepsilon \quad \textit{for every} \quad y_0 \in B_\rho(x_0).$$

Proof By Lemma 2.2, there exists $r > 0$ and a function H of the form (TpS) such that $\|u_{x_0,r} - H\|_{L^\infty(B_1)} \leq \varepsilon/2$. On the other hand, by the Lipschitz continuity of u

$$\|u_{x_0,r} - u_{y_0,r}\|_{L^\infty(B_1)} \leq \frac{L}{r}|x_0 - y_0|.$$

Choosing ρ small enough (such that $\frac{L\rho}{r} \leq \varepsilon/2$), we get the claim. □

2.3 Optimality conditions at the free boundary

Let $u : D \to \mathbb{R}$ be a local minimizer of J_{tp}. In this section, we will show that u satisfies the following optimality conditions at two-phase free boundary points:

$$|\nabla u^+|^2 - |\nabla u^-|^2 = \lambda_+^2 - \lambda_-^2 \qquad \text{and} \qquad |\nabla u^{\pm}| \geq \lambda_{\pm} \qquad \text{on} \qquad \Gamma_{tp}. \tag{2.3}$$

We notice that if u was differentiable at $x_0 \in \Gamma_{tp}$, that is,

$$\begin{cases} u^+(x) = (x - x_0) \cdot \nabla u^+(x_0) + o(|x - x_0|) & \text{for every} \quad x \in \Omega_u^+, \\ u^-(x) = (x - x_0) \cdot \nabla u^-(x_0) + o(|x - x_0|) & \text{for every} \quad x \in \Omega_u^-, \end{cases} \tag{2.4}$$

then (2.3) would be an immediate consequence of Lemma 2.2. Of course, differentiability of u^+ and u^- (and the uniqueness of the blow-up limits)[2] is not a priori known, so we will use the optimality condition in some weak (viscosity) sense, based on comparison with (more regular) test functions.

Definition 2.4 Let D be an open set.

(i) We say that a function $Q : D \to \mathbb{R}$ *touches* a function $w : D \to \mathbb{R}$ from below (resp. from above) at a point $x_0 \in D$ if $Q(x_0) = w(x_0)$ and

$$Q(x) - w(x) \leq 0 \quad \big(\text{resp. } Q(x) - w(x) \geq 0\big)$$

for every x in a neighborhood of x_0. We will say that Q touches w strictly from below (resp. above), if the above inequalities are strict for $x \neq x_0$.

(ii) A function Q is an admissible *comparison function* in D if
  (a) $Q \in C^1\big(\overline{\{Q > 0\}} \cap D\big) \cap C^1\big(\overline{\{Q < 0\}} \cap D\big)$;
  (b) $Q \in C^2\big(\{Q > 0\} \cap D\big) \cap C^2\big(\{Q < 0\} \cap D\big)$;
  (c) $\partial\{Q > 0\}$ and $\partial\{Q < 0\}$ are smooth manifolds in D.

The optimality conditions on u are given in the next lemma. Before we give the precise statement, we recall that $\partial\Omega_u \cap D = \Gamma_{op}^+ \cup \Gamma_{op}^- \cup \Gamma_{tp}$, where

$$\Gamma_{op}^+ := \partial\Omega_u^+ \setminus \partial\Omega_u^- \cap D, \quad \Gamma_{op}^- := \partial\Omega_u^- \setminus \partial\Omega_u^+ \cap D$$
$$\text{and} \quad \Gamma_{tp} := \partial\Omega_u^- \cap \partial\Omega_u^+ \cap D.$$

---

[2] It is immediate to check that, if the blow-up is unique at $x_0 \in \Gamma_{tp}$, that is,

$$\lim_{r \to 0} \|u_{x_0,r} - H\|_{L^\infty(B_1)} = 0 \qquad \text{for some } H \text{ as in (TPS)},$$

then $\alpha = |\nabla u^+|(x_0)$, $\beta = |\nabla u^-|(x_0)$ and (2.4) does hold.

Lemma 2.5 (The local minimizers are viscosity solutions). *Let u be a local minimizer of J_{tp} in the open set $D \subset \mathbb{R}^d$. Then, u in harmonic in $\Omega_u^+ \cup \Omega_u^-$ and satisfies the following optimality conditions on the free boundary $\partial\Omega_u \cap D$.*

(A) *Suppose that Q is a comparison function that touches u from below at x_0.*
 (A.1) *If $x_0 \in \Gamma_{op}^+$, then $|\nabla Q^+(x_0)| \le \lambda_+$;*
 (A.2) *if $x_0 \in \Gamma_{op}^-$, then $Q^+ \equiv 0$ in a neighborhood of x_0 and $|\nabla Q^-(x_0)| \ge \lambda_-$;*
 (A.3) *if $x_0 \in \Gamma_{tp}$, then $|\nabla Q^-(x_0)| \ge \lambda_-$ and*

$$|\nabla Q^+(x_0)|^2 - |\nabla Q^-(x_0)|^2 \le \lambda_+^2 - \lambda_-^2.$$

(B) *Suppose that Q is a comparison function that touches u from above at x_0.*
 (B.1) *If $x_0 \in \Gamma_{op}^+$, then $Q^- \equiv 0$ in a neighborhood of x_0 and $|\nabla Q^+(x_0)| \ge \lambda_+$;*
 (B.2) *if $x_0 \in \Gamma_{op}^-$, then $|\nabla Q^-(x_0)| \le \lambda_-$;*
 (B.3) *if $x_0 \in \Gamma_{tp}$, then $|\nabla Q^+(x_0)| \ge \lambda_+$ and*

$$|\nabla Q^+(x_0)|^2 - |\nabla Q^-(x_0)|^2 \ge \lambda_+^2 - \lambda_-^2.$$

Proof If x_0 is a one-phase point, then the gradient bounds in (A.1), (A.2), (B.1) and (B.2) follow by [44, Proposition 7.1], the claims $Q^+ \equiv 0$ in (A.2) and $Q^- \equiv 0$ in (B.2) being trivially true. Suppose now that $x_0 \in \Gamma_{tp}$ and that Q touches u from below at x_0. Let u_{x_0,r_k} and Q_{x_0,r_k} be blow-up sequences of u and Q at x_0. Then, up to extracting a subsequence, we can assume that u_{x_0,r_k} converges uniformly to a blow-up limit $H_u \in \mathcal{BU}(x_0)$ of the form

$$H_u(x) = \alpha(x \cdot \boldsymbol{e})_+ - \beta(x \cdot \boldsymbol{e})_-.$$

On the other hand, since Q^+ and Q^- are differentiable at x_0 (respectively in $\overline{\Omega}_Q^+$ and $\overline{\Omega}_Q^-$), we get that Q_{x_0,r_k} converges to the function

$$H_Q(x) = |\nabla Q^+(x_0)|(x \cdot \boldsymbol{e}')_+ - |\nabla Q^-(x_0)|(x \cdot \boldsymbol{e}')_-,$$

where $\boldsymbol{e}' = |\nabla Q^+(x_0)|^{-1}\nabla Q^+(x_0) = -|\nabla Q^-(x_0)|^{-1}\nabla Q^-(x_0)$. Now since, H_Q touches H_u from below (and since $\alpha \neq 0$ and $\beta \neq 0$), we have that $\boldsymbol{e}' = \boldsymbol{e}$,

$$\begin{gathered} |\nabla Q^+(x_0)|^2 - |\nabla Q^-(x_0)|^2 \le \alpha^2 - \beta^2 \\ \text{and} \qquad |\nabla Q^+(x_0)| \le \alpha, \quad |\nabla Q^-(x_0)| \ge \beta. \end{gathered}$$

Combined with (2.2), this gives (A.3). The proof of (B.3) is analogous. □

In particular, if $u : D \to \mathbb{R}$ is a continuous function such that the claims (A) and (B) hold for every comparison function Q, then we say that u satisfies the following overdetermined condition on the free boundary in viscosity sense:

$$\begin{cases} |\nabla u^+|^2 - |\nabla u^-|^2 = \lambda_+^2 - \lambda_-^2, \ |\nabla u^+| \ge \lambda_+ \text{ and } |\nabla u^-| \ge \lambda_- & \text{on} \quad \Omega_u^+ \cap \Omega_u^- \cap D\,; \\ |\nabla u^+| = \lambda_+ & \text{on} \quad D \cap \Omega_u^+ \setminus \Omega_u^-\,; \\ |\nabla u^-| = \lambda_- & \text{on} \quad D \cap \Omega_u^- \setminus \Omega_u^+\,. \end{cases} \tag{2.5}$$

Thus, Lemma 2.5 can be restated as follows: *If u is a local minimizer of J_{tp} in D, then it satisfies* (2.5) *in viscosity sense.*

We conclude this section by recording the following straightforward consequence of definition of viscosity solution, where we consider what happens when a function is touching only one of the two phases (note that in the second item we are restricting the touching points only to the one-phase free boundaries).

Lemma 2.6 *Let $u : D \to \mathbb{R}$ be a continuous function which satisfies* (2.5).

(i) *Assume that Q is a comparison function touching u^+ from above at the point $x_0 \in \partial\Omega_u^+$ (resp. $-u^-$ from below at $x_0 \in \partial\Omega_u^-$), then*

$$|\nabla Q^+|(x_0) \ge \lambda_+ \qquad \Big(\textit{resp.} \ \ |\nabla Q^-|(x_0) \ge \lambda_-\Big).$$

(ii) *Assume that Q is a comparison function touching u^+ from below at the point $x_0 \in \Gamma_{\mathrm{OP}}^+$ (resp. $-u^-$ from above at $x_0 \in \Gamma_{\mathrm{OP}}^-$), then*

$$|\nabla Q^+|(x_0) \le \lambda_+ \qquad \Big(\textit{resp.} \ \ |\nabla Q^-|(x_0) \le \lambda_-\Big).$$

Proof The claim (i) simply follows by, for instance, noticing that the assumption implies that $Q \ge u^+ \ge 0$ so that Q touching u from above and thus one can apply $B.1$ and the first part of $B.3$ in the definition of viscosity solution and that a symmetric argument holds for u^-.

Concerning claim (ii), we note that since $x_0 \in \Gamma_{\mathrm{OP}}^+$, $u \ge 0$ in a neighborhood of x_0. In particular, the function Q^+ is touching u from below at x_0 and thus the conclusion follows by (B.2) in the definition of viscosity solution. □

3 Flatness decay

In this section we prove that, at two-phase points, the flatness decays from one scale to the next. Our main result is the following theorem, which applies to any viscosity solution of the two-phase problem.

Theorem 3.1 (Flatness decay for viscosity solutions). *For every* $L \geq \lambda_+ \geq \lambda_- > 0$ *and* $\gamma \in (0, 1/2)$, *there exist* $\varepsilon_0 > 0$, $C > 0$ *and* $\rho \in (0, 1/4)$ *such that the following holds. Suppose that the function* $u : B_1 \to \mathbb{R}$ *satisfies:*

(a) *u is L-Lipschitz continuous;*
(b) *zero is on the two-phase free boundary,* $0 \in \Gamma_{tp} = \partial\Omega_u^+ \cap \partial\Omega_u^-$;
(c) *u is harmonic in* $\Omega_u^+ \cup \Omega_u^-$;
(d) *u satisfies the optimality condition* (2.5) *in viscosity sense;*
(e) *u is* ε_0*-flat in* B_1, *that is,*

$$\|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)} \leq \varepsilon_0 \qquad \text{for some} \qquad L \geq \alpha \geq \lambda_+ . \tag{3.1}$$

Then, there are $\boldsymbol{e} \in \mathbb{S}^{d-1}$ *and* $\tilde{\alpha} \geq \lambda_+$ *such that*

$$|\boldsymbol{e} - \boldsymbol{e}_d| + |\tilde{\alpha} - \alpha| \leq C\, \|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)}, \tag{3.2}$$

and

$$\|u_\rho - H_{\tilde{\alpha},\boldsymbol{e}}\|_{L^\infty(B_1)} \leq \rho^\gamma\, \|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)}. \tag{3.3}$$

Proof of Theorem 3.1 follows easily combining the two upcoming lemmas. In the first one we deal with the situation where the two-plane solution is, roughly, H_{λ_+}. Note that this is the situation where one might expect the presence of branching points and it is indeed in this setting that we will obtain the two membrane problem as "linearization". In the second lemma, we deal with the case when the closest half-plane solution has a gradient much larger than λ_+. We will later show that in this case the origin is an interior two-phase point.

Lemma 3.2 (Improvement of flatness: branching points). *For every* $L \geq \lambda_+ \geq \lambda_- > 0$, $\gamma \in (0, 1/2)$, *and* $M > 0$, *there exist* $\varepsilon_1 = \varepsilon_1(\gamma, d, L, M)$, $C_1 = C_1(\gamma, d, L, M)$ *and* $\rho = \rho(\gamma, d, L, M)$ *such that the following holds. For every function* $u : B_1 \to \mathbb{R}$ *satisfying (a)–(d) of Theorem* 3.1 *and such that*

$$\|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)} \leq \varepsilon_1 , \quad \text{with} \quad 0 \leq \alpha - \lambda_+ \leq M\|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)},$$

there exist $\boldsymbol{e} \in \mathbb{S}^{d-1}$ *and* $\tilde{\alpha} \geq \lambda_+$, *for which* (3.2) *and* (3.3) *hold.*

Lemma 3.3 (Improvement of flatness: non-branching points). *For every* $L \geq \lambda_+ \geq \lambda_- > 0$ *and* $\gamma \in (0, 1)$, *there exist* $\varepsilon_2 = \varepsilon_2(\gamma, d, L)$, $\overline{M} = \overline{M}(\gamma, d, L)$ *and* $\rho = \rho(\gamma, d, L)$ $C_2 = C_2(\gamma, d, L)$ *such that the following holds. For every function* $u : B_1 \to \mathbb{R}$ *satisfying (a)–(d) of Theorem* 3.1 *and such that*

$$\|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)} \leq \varepsilon_2 , \quad \text{with} \quad \alpha - \lambda_+ \geq \overline{M}\|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)},$$

there exist $\boldsymbol{e} \in \mathbb{S}^{d-1}$ *and* $\tilde{\alpha} \geq \lambda_+$, *for which* (3.2) *and* (3.3) *hold.*

Let us first show that Theorem 3.1 follows from Lemmas 3.2 and 3.3.

Proof of Theorem 3.1 Fix $\gamma \in (0, 1/2)$ and notice that $\alpha < 2L$, where L is the Lipschitz constant of u. Next choose $M = 2\overline{M}$ in Lemma 3.2, where $\overline{M}$ is as in Lemma 3.3. Let $\varepsilon_0 = \min\left\{\varepsilon_2(2\overline{M}), \varepsilon_1/2\right\}$. Then, we can apply either Lemmas 3.2 or 3.3. □

In order to prove Lemmas 3.2 and 3.3, we will argue by contradiction. Hence in the following we consider a sequence u_k of minimizers such that

$$\varepsilon_k := \|u_k - H_{\alpha_k, \boldsymbol{e}_d}\|_{L^\infty(B_1)} \to 0 \qquad \text{and} \qquad \lambda_+ \leq \alpha_k \leq L, \tag{3.4}$$

where

$$\|\nabla u_k\|_{L^\infty(B_1)} \leq L \qquad \text{for every} \qquad k \geq 1.$$

We also set

$$\ell := \lambda_+^2 \lim_{k\to\infty} \frac{\alpha_k^2 - \lambda_+^2}{2\alpha_k^2 \varepsilon_k} = \lambda_-^2 \lim_{k\to\infty} \frac{\beta_k^2 - \lambda_-^2}{2\beta_k^2 \varepsilon_k} \tag{3.5}$$

which we can assume to exists up to extracting a subsequence. It might be useful to keep in mind that $\ell = \infty$ will correspond to Lemma 3.3 while $0 \leq \ell \leq M < \infty$ to Lemma 3.2.

In order to prove Lemmas 3.3 and 3.2, we will first show that the sequence

$$v_k(x) = \begin{cases} v_{+,k}(x) := \dfrac{u_k(x) - \alpha_k x_d^+}{\alpha_k \varepsilon_k} & x \in \Omega_{u_k}^+ \cap B_1 \\[2ex] v_{-,k}(x) := \dfrac{u_k(x) + \beta_k x_d^-}{\beta_k \varepsilon_k} & x \in \Omega_{u_k}^- \cap B_1 \end{cases} \tag{3.6}$$

is compact in some suitable sense; we give the precise statement in Corollary 3.4 below and we postpone the proof to Sect. 3.1. We then establish in Lemma 3.5 the limiting problem solved by its limit v. Note that this problem depends on the value of ℓ which is distinguishing whether we are or not at branching points.

Finally, in Sect. 3.3 we show how to deduce Lemmas 3.3 and 3.2 from Corollary 3.4 and Lemma 3.5. In the rest of the paper we will use the notation

$$B_r^\pm := B_r \cap \{x_d^\pm > 0\} \qquad \text{for every } r > 0\,.$$

Lemma 3.4 (Compactness of the linearizing sequence v_k). *Let* u_k *be a sequence of functions satisfying (a), (b), (c) and (d) of Theorem* 3.1 *uniformly* *in k and let ε_k and α_k be as in* (3.4) *and let v_k be defined by* (3.6)*. Then there are Hölder continuous functions*

$$v_+ : \overline{B^+_{1/2}} \to \mathbb{R} \quad \text{and} \quad v_- : \overline{B^-_{1/2}} \to \mathbb{R},$$

with

$$v_+ \le v_- \text{ on } B_{1/2} \cap \{x_d = 0\}, \qquad v_+(0) = v_-(0) = 0,$$

and such that the sequences of closed graphs

$$\Gamma^\pm_k := \Big\{(x, v_{\pm,k}(x)) \ : \ x \in \overline{\Omega^\pm_{u_k} \cap B_{1/2}}\Big\},$$

converge, up to a (non-relabeled) subsequence, in the Hausdorff distance to the closed graphs

$$\Gamma_\pm = \Big\{(x, v_\pm(x)) \ : \ x \in \overline{B^\pm_{1/2}}\Big\}.$$

In particular, the following claims hold:

(i) *For every $\delta > 0$, $v_{\pm,k}$ converges uniformly to $v\pm$ on $B_{1/2} \cap \{\pm x_d > \delta\}$.*

(ii) *For every sequence $x_k \in \overline{\Omega^\pm_{u_k}} \cap B_1$ converging to $x \in \overline{B^\pm_{1/2}}$, we have*

$$v_\pm(x) = \lim_{k\to\infty} v_{\pm,k}(x_k).$$

(iii) *For every $x \in \{x_d = 0\} \cap B_{1/2}$, we have*

$$v_\pm(x) = \mp \lim_{k\to\infty} \frac{x_k \cdot \boldsymbol{e}_d}{\alpha_k \varepsilon_k} \quad \text{for any sequence} \quad \partial\Omega^\pm_{u_k} \ni x_k \to x.$$

In particular, $\{x_d = 0\} \cap \overline{B}_{1/2}$ decomposes into a open jump set

$$\mathcal{J} = \{v_+ < v_-\} \cap \{x_d = 0\} \cap \overline{B}_{1/2},$$

and its complementary contact set

$$\mathcal{C} = \{v_+ = v_-\} \cap \{x_d = 0\} \cap \overline{B}_{1/2}.$$

Furthermore, if $x \in \mathcal{J}$, then

$$\liminf_{k\to\infty} \operatorname{dist}\big(x, \partial\Omega^+_{u_k} \cap \partial\Omega^-_{u_k}\big) > 0. \tag{3.7}$$

In particular for all $x \in \mathcal{J}$, there exists two sequences $x_k^{\pm} \in \Gamma_{k,\mathrm{OP}}^{\pm}$ such that $x_k^{\pm} \to x$.

In the next lemma we determine the limiting problem solved by the function v defined as

$$v(x) = \begin{cases} v_+(x) & \text{for} \quad x \in B_{1/2}^+, \\ v_-(x) & \text{for} \quad x \in B_{1/2}^-, \end{cases} \tag{3.8}$$

where v_+ and v_- are as in Corollary 3.4.

Lemma 3.5 (The "linearized" problem). *Let u_k, ε_k and α_k be as in (3.4), v_k be defined by (3.6) and ℓ as in (3.5). Let also $v_\pm$ be as in Corollary 3.4:*
$\ell = \infty$: *Then $\mathcal{J} = \emptyset$ and $v_\pm$ are viscosity solutions of the transmission problem:*

$$\begin{cases} \Delta v_\pm = 0 & \text{in } B_{1/2}^{\pm} \\ \alpha_\infty^2 \partial_d v_+ = \beta_\infty^2 \partial_d v_- & \text{on } B_{1/2}^{\pm} \cap \{x_d = 0\} \end{cases} \tag{3.9}$$

where $\alpha_\infty = \lim_k \alpha_k$ and $\beta_\infty = \lim_k \beta_k$, which we can assume to exist up to extracting a further subsequence.
$0 \le \ell < \infty$: *Then v is a viscosity solution of the two membrane problem:*

$$\begin{cases} \Delta v_\pm = 0 & \text{in } B_{1/2}^{\pm}, \\ \lambda_\pm^2 \partial_d v_\pm + \ell \ge 0 & \text{in } B_{1/2} \cap \{x_d = 0\}, \\ \lambda_\pm^2 \partial_d v_\pm + \ell = 0 & \text{in } \mathcal{J}, \\ \lambda_+^2 \partial_d v_+ = \lambda_-^2 \partial_d v_- & \text{in } \mathcal{C}, \\ v_+ \le v_- & \text{in } B_{1/2} \cap \{x_d = 0\}. \end{cases} . \tag{3.10}$$

Remark 3.6 Here by *viscosity solution of* (3.9) *and* (3.10) we mean a function v as in (3.8) such that $v_\pm$ are continuous in $\overline{B_{1/2}^{\pm}}$, $\Delta v_\pm = 0$ in $B_{1/2}^{\pm}$ and such that the following holds:

- If we are in case (3.9), let $p, q \in \mathbb{R}$ and let $\tilde{P}$ be a smooth function such that $\partial_d \tilde{P} = 0$. Suppose that $\tilde{P}$ is subharmonic (superharmonic) and that the function

  $$P := px_d^+ - qx_d^- + \tilde{P}$$

  touches v strictly from below (above) at $x_0 \in B_{1/2} \cap \{x_d = 0\}$, then

  $$\alpha_\infty^2 p \le \beta_\infty^2 q \qquad \left(\alpha_\infty^2 p \ge \beta_\infty^2 q\right).$$

- If we are in case (3.10) then

(1) if $P_\pm$ is a smooth superharmonic function in $B^\pm_{1/2}$ touching $v_\pm$ strictly from above at $x_0 \in B_{1/2} \cap \{x_d = 0\}$, then $\lambda^2_\pm \partial_d P_\pm \geq 0$;
(2) if $P_\pm$ is a smooth subharmonic function in $B^\pm_{1/2}$ touching $v_\pm$ strictly from below at $x_0 \in \mathcal{J}$, then $\lambda^2_\pm \partial_d P_\pm \leq 0$;
(3) if $p, q \in \mathbb{R}$ and $\tilde{P}$ is a smooth subharmonic (superharmonic) function such that $\partial_d \tilde{P} = 0$ and such that the function

$$P := px_d^+ - qx_d^- + \tilde{P}$$

touches v strictly from below (above) at $x_0 \in B_{1/2} \cap \{x_d = 0\}$, then

$$\lambda^2_+ p \leq \lambda^2_- q \qquad \left(\lambda^2_+ p \geq \lambda^2_- q\right).$$

3.1 Compactness of the linearizing sequence: Proof of Corollary 3.4

The key point in establishing a suitable compactness for v_k is a "partial Harnack" inequality, in the spirit of [26,27]. As explained in the introduction, in dealing with branching points one needs to work separately on the positive and negative part. An additional difficulties arise also at pure two-phase points since we want also to deal with the case $\lambda_- = \lambda_+$. Let us briefly explain the ideas of the proof.

If u is close in B_1 to a global solution of the form $H_{\alpha,\boldsymbol{e}_d}$ with $\alpha > \lambda_+$, then we expect that in a small neighborhood B_ρ of the origin the level set $\{u = 0\}$ has zero Lebesgue measure and that all the free boundary points in B_ρ are "interior" two-phase points (indeed, at the end, this will be a consequence of the C^1 regularity of u and of the free boundary). In this case one expects to be able to do the same argument as in [27]. This is true except for the following caveat, if one wants to deal with the case $\lambda_- = \lambda_+$ then the sliding arguments used in [26,27] (see also [9,10]) does not yield the desired contradiction since the positive term might actually be zero. For this reason one has first to "increase" the slope of the trapping solution, so that the sliding argument would give the desired contradiction. Namely if u is trapped between two translation of a two-plane solution:

$$H_{\alpha,\boldsymbol{e}_d}(x + b) \leq u \leq H_{\alpha,\boldsymbol{e}_d}(x + a)$$

in say B_1 and at the point $P = (0, \dots, 0, 1/2)$ u is closer to $H_{\alpha,\boldsymbol{e}_d}(\cdot + a)$ then to $H_{\alpha,\boldsymbol{e}_d}(\cdot + b)$, we can increase in a quantitative way the slope of the positive part of the lower two-plane solution in half ball, i.e.

$$u \geq \alpha'(x + b)^+ - \beta(x + b)^+, \qquad \alpha' > \alpha,$$

see Lemma 3.7. The sliding argument of [26,27] then allows to translate this to a a (quantitative) increase of b, yielding the partial decay of flatness of the free boundary. This is the situation studied in Lemma 3.9.

If instead u is close to $H_{\lambda_+,\mathbf{e}_d}$ then the free boundary can behave in several different ways. Indeed, in this case the origin can be either an interior two-phase point, a branching two-phase point but it might also happen that

$$u(x) \approx \lambda_+(x_d + \varepsilon_1)_+ - \lambda_-(x_d - \varepsilon_2)_- \quad \text{with} \quad 0 < \varepsilon_1, \varepsilon_2 \ll 1.$$

Since as explained in the introduction we have to deal with all the of the above situations we have to prove a decay in this situation is to improve separately the positive and the negative parts of u. More precisely if in B_1

$$\begin{aligned} \lambda_+\big(x_d + b_+\big)^+ \le u^+(x) \le \lambda_+\big(x_d + a_+\big)^+, \\ -\lambda_-\big(x_d + b_-\big)^- \le -u^-(x) \le -\lambda_-\big(x_d + a_-\big)^-, \end{aligned}$$

for suitable $a_\pm$, $b_\pm$, one wants to find new constants $\bar{a}_\pm$, $\bar{b}_\pm \in$ with

$$(\bar{b}_- - \bar{a}_-) < (b_- - a_-), \qquad (\bar{b}_+ - \bar{a}_+) < (b_+ - a_+),$$

and for which, in half the ball,

$$\begin{aligned} \lambda_+\big(x_d + \bar{b}_+\big)^+ \le u^+(x) \le \lambda_+\big(x_d + \bar{a}_+\big)^+, \\ -\lambda_-\big(x_d + \bar{b}_-\big)^- \le -u^-(x) \le -\lambda_-\big(x_d + \bar{a}_-\big)^-. \end{aligned}$$

Here one has to distinguishes the case in which, say, the lower function

$$\lambda_+\big(x_d + \bar{b}_+\big)^+ - \lambda_-\big(x_d + \bar{b}_-\big)^-$$

looks like a two plane solution, i.e $b_+ - b_- \ll 1$, or not and to perform different comparisons according to the situation. This dealt in Lemma 3.8.

We start with the following simple lemma which allows to "increase" the slope of the comparison functions.

Lemma 3.7 *There is a dimensional constants $\tau = \tau(d) > 0$ such that the following hold. Assume that $v : B_1 \to \mathbb{R}$ is a continuous function with $\Delta v = 0$ on $\{v > 0\}$ and such that*

$$\lambda\big(x_d + b\big)^+ \le v \le \lambda\big(x_d + a\big)^+,$$

for some $a, b \in \big(-{}^1\!/_{100}, {}^1\!/_{100}\big)$. Let $P = (0, \dots, 0, {}^1\!/_2)$, then for all $\varepsilon \in (0, \frac{1}{2})$

$$v(P) \le \lambda(1-\varepsilon)\Big(\frac{1}{2} + a\Big)^+ \quad \Longrightarrow \quad v \le \lambda(1 - \tau\varepsilon)\big(x_d + a\big)^+ \quad \text{in} \quad B_{1/4}(0),$$

and

$$v(P) \geq \lambda(1+\varepsilon)\Big(\frac{1}{2}+b\Big)^+ \quad \Longrightarrow \quad v \geq \lambda(1+\tau\varepsilon)\big(x_d+b\big)^+ \quad \textit{in} \quad B_{1/4}(0).$$

Proof We prove only the first implication, since the second one can be obtained by the same arguments. First, we notice that, since $b \leq 1/100$, both v and $\lambda(x_d+a)_+$ are positive and harmonic in $B_{1/4}(P)$. Thus,

$$\lambda(x_d+a)^+ - v \geq 0 \quad \text{in} \quad B_{1/4}(P)$$

and

$$\lambda\Big(\frac{1}{2}+a\Big)^+ - v(P) \geq \lambda\varepsilon\Big(\frac{1}{2}+a\Big)^+ \geq \frac{49}{100}\lambda\varepsilon.$$

Hence, by Harnack inequality and the bound $|a| \leq 1/100$ there are dimensional constants $\bar{c}$ and c such that

$$v(x) \leq \lambda\big(x_d+a\big)^+ - \lambda\bar{c}\varepsilon \leq \lambda(1-c\varepsilon)\big(x_d+a\big)^+ \qquad \text{for all } x \in B_{1/8}(P).$$

We now let w be the solution of the following problem:

$$\begin{cases} \Delta w = 0 & \text{in } B_1(0) \setminus B_{1/8}(P) \cap \{x_d > -a\}, \\ w = 0 & \text{on } B_1 \cap \{x_d = -a\}, \\ w = \lambda\big(x_d+a\big)^+ & \text{on } \partial B_1(0) \cap \{x_d > -a\}, \\ w = \lambda(1-c\varepsilon)\big(x_d+a\big)^+ & \text{on } \partial B_{1/8}(P) \cap \{x_d > -a\}. \end{cases}$$

By the Hopf Boundary Lemma,

$$w(x) \leq (1-\tau\varepsilon)(x_d+a)^+ \quad \text{for every } x \text{ in} \quad B_{1/4} \cap \{x_d > -a\},$$

for a suitable constant $\tau = \tau(d)$. Since, by the comparison principle, $u \leq w$, this concludes the proof. □

We next prove the two partial Harnack inequalities.

The proof is based on comparison with suitable test functions. In order to build these "barriers", we will often use the following function φ. Let $Q = (0,\ldots,0,1/5)$ and we let $\varphi : B_1 \to R$ be defined by:

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in B_{1/100}(Q), \\ \kappa_d\Big(|x-Q|^{-d} - \big(3/4\big)^{-d}\Big) & \text{if } x \in B_{3/4}(Q) \setminus \overline{B}_{1/100}(Q), \\ 0 & \text{otherwise,} \end{cases} \tag{3.11}$$

where the dimensional constant κ_d is chosen in such a way that φ is continuous.

It is immediate to check that φ has the following properties:

(φ.1) $0 \leq \varphi \leq 1$ in $\mathbb{R}^d$, and $\varphi = 0$ on ∂B_1;
(φ.2) $\Delta\varphi \geq c_d > 0$ in $\{\varphi > 0\} \setminus \overline{B}_{1/100}(Q)$;
(φ.3) $\partial_d\varphi > 0$ in $\{\varphi > 0\} \cap \{|x_d| \leq 1/100\}$;
(φ.4) $\varphi \geq c_d > 0$ in $B_{1/6}$.

where c_d is a dimensional constant. We distinguish two cases.

Lemma 3.8 (Partial Boundary Harnack I). *Given $\lambda_+ \geq \lambda_- > 0$ there exist constants $\bar{\varepsilon} = \bar{\varepsilon}(d, \lambda_\pm) > 0$ and $\bar{c} = \bar{c}(d, \lambda_\pm) \in (0, 1)$ such that, for every function $u : B_4 \to \mathbb{R}$ satisfying (a), (c) and (d) in Theorem 3.1, the following property holds true.*

Let $a_\pm, b_\pm \in \big(-1/100, 1/100\big)$ be such that

$$b_- \leq a_-, \qquad b_+ \leq a_+, \qquad b_- \leq b_+, \qquad a_- \leq a_+,$$

and

$$(a_- - b_-) + (a_+ - b_+) \leq \bar{\varepsilon}.$$

Assume that for $x \in B_4$:

$$\lambda_+\big(x_d + b_+\big)^+ \leq u^+(x) \leq \lambda_+\big(x_d + a_+\big)^+$$

and

$$-\lambda_-\big(x_d + b_-\big)^- \leq -u^-(x) \leq -\lambda_-\big(x_d + a_-\big)^-.$$

Then, one can find new constants $\bar{a}_\pm, \bar{b}_\pm \in \big(-1/100, 1/100\big)$, with

$$\bar{b}_- \leq \bar{a}_-, \qquad \bar{b}_+ \leq \bar{a}_+, \qquad \bar{b}_- \leq \bar{b}_+, \qquad \bar{a}_- \leq \bar{a}_+,$$

and

$$\bar{a}_- - \bar{b}_- \leq \bar{c}(a_- - b_-) \qquad \bar{a}_+ - \bar{b}_+ \leq \bar{c}(a_+ - b_+)$$

such that for $x \in B_{1/6}$:

$$\lambda_+\big(x_d + \bar{b}_+\big)^+ \leq u^+(x) \leq \lambda_+\big(x_d + \bar{a}_+\big)^+$$

and

$$-\lambda_-\big(x_d + \bar{b}_-\big)^- \leq -u^-(x) \leq -\lambda_-\big(x_d + \bar{a}_-\big)^-.$$

Proof Let us show how to improve the positive part. More precisely we show how given a_+, a_-, b_+, b_- as in the statement we can find $\bar{a}_+$ and $\bar{b}_+$. The proof for $\bar{b}_-$ and $\bar{a}_-$ works in the same way and is left to the reader.

We set

$$P = (0, \dots, 0, 2)$$

and we distinguish two cases:

- *Case 1. Improvement from above.* Assume that, at the point P, u^+ is closer to $\lambda_+(2+b_+)^+$ than to the upper barrier $\lambda_+(2+a_+)^+$. Precisely that

$$u^+(P) \le \lambda_+(2+a_+)^+ - \frac{\lambda_+(a_+ - b_+)}{2}.$$

In this case, we will show that u is below $\lambda_+(x+\bar{a}_+)^+$ in a smaller ball centered at the origin for $\bar{a}_+$ strictly smaller than a_+.

We start by setting

$$\varepsilon := a_+ - b_+ \le \bar{\varepsilon}.$$

Then

$$u^+(P) \le \lambda_+(2+a_+)^+ - \frac{\lambda\varepsilon}{2} \le \lambda_+(1-c\varepsilon)(2+a_+)^+$$

for a suitable (universal) constant c. We can thus apply (the scaled version of) Lemma 3.7 to u^+, to infer the existence of a dimensional constant τ such that

$$u^+ \le \lambda_+(1-\tau\varepsilon)\big(x_d + a_+\big)^+ \quad \text{in} \quad B_1. \tag{3.12}$$

For φ as in (3.11) and $t \in [0,1]$ we set

$$f_t = \lambda_+(1-\tau\varepsilon/2)\big(x_d + a_+ - tc\varepsilon\varphi\big)^+,$$

where $c = c(d)$ is a small constant chosen such that for all $x \in B_{1/100}(Q)$ and $t \in [0,1)$,

$$\begin{aligned} u(x) &\le \lambda_+(1-\tau\varepsilon)\big(x_d + a_+\big)^+ \\ &\le \lambda_+(1-\tau\varepsilon/2)\big(x_d + a_+ - c\varepsilon\big)^+ < f_t(x), \end{aligned} \tag{3.13}$$

where we used that $(x_d + a_+)$ is within two universal constants for $x \in B_{1/100}(Q)$.

We now let $\bar{t} \in (0, 1]$ the largest t such that $f_t \geq u$ in B_1 and we claim that $\bar{t} = 1$. Indeed assume that $\bar{t} < 1$, then there exists $\bar{x} \in B_1$ such that

$$u(x) - f_{\bar{t}}(x) \leq u(\bar{x}) - f_{\bar{t}}(\bar{x}) = 0 \quad \text{for all} \quad x \in B_1. \tag{3.14}$$

Note that by (3.13), $\bar{x} \notin B_{1/100}(Q)$, while, by ($\varphi$.1) and (3.12), $\bar{x} \in \{\varphi > 0\}$. Moreover $\bar{x} \in \{f_{\bar{t}} = 0\}$. In fact, if this was not the case, then, by (φ.2), $\Delta f_{\bar{t}}(\bar{x}) < 0$ and $\Delta u(\bar{x}) = 0$, a contradiction with (3.14). Assume now $\bar{x} \in \{f_{\bar{t}} = 0\}$, since u is a viscosity solution we get that, by (φ.3),

$$\lambda_+^2 \leq |\nabla f_{\bar{t}}(\bar{x})|^2 = \lambda_+^2(1 - \tau\varepsilon/2)^2 - 2c\varepsilon\bar{t}\lambda_+\partial_d\varphi(\bar{x}) + O(\varepsilon^2) < \lambda_+^2$$

provided $\varepsilon \leq \bar{\varepsilon}(d, \lambda_+) \ll 1$ (note that necessarily $u(\bar{x}) = 0$ which gives that $\bar{x} \in \{|x_d| \leq 1/100\}$). This contradiction implies that $\bar{t} = 1$. Hence, by (φ.4), we get for all $x \in B_{1/6}$.

$$u(x) \leq \lambda_+(1 - \tau\varepsilon/2)\big(x_d + a_+ - c\varepsilon\varphi\big)^+ \leq \lambda_+\big(x_d + a_+ - \bar{c}\varepsilon\big)^+$$

for a suitable dimensional constant $\bar{c}$. Setting

$$\bar{a}_+ = a_+ - \bar{c}\varepsilon, \qquad \bar{b}_+ = b_+$$

and recalling that $\varepsilon = (b_+ - a_+)$ allows to conclude the proof in this case.

- *Case 2. Improvement from below.* We now assume that, at the point P, u^+ is closer to $\lambda_+(2 + a_+)^+$ than to $\lambda_+(2 + b_+)^+$. Hence, we have

$$u^+(P) \geq \lambda_+(2 + b_+)^+ + \frac{\lambda_+(a_+ - b_+)}{2}$$

and we set again

$$\varepsilon := a_+ - b_+ \leq \bar{\varepsilon}.$$

Arguing as in Case 1, by Lemma 3.7, there exists a dimensional constant τ such that

$$u^+ \geq \lambda_+(1 + \tau\varepsilon)\big(x_d + b_+\big)^+ \quad \text{in} \quad B_1. \tag{3.15}$$

We need now to distinguish two further sub-cases:

- *Case 2.1:* Suppose that

$$0 \leq b_+ - b_- \leq \eta\varepsilon$$

where $\eta \ll \tau$ is a small universal constant which we will choose at the end of the proof. In this case, for $x \in B_1$,

$$\begin{aligned} u &\geq \lambda_+(1+\tau\varepsilon)\big(x_d+b_+\big)^+ - \lambda_-\big(x_d+b_-\big)^- \\ &\geq \lambda_+(1+\tau\varepsilon)\big(x_d+b_+\big)^+ - \lambda_-(1-c_1\eta\varepsilon)\big(x_d+b_+\big)^- \end{aligned} \tag{3.16}$$

for a suitable universal constant c_1. We now take φ as in (3.11) and we set, for $t \in [0, 1]$,

$$f_t(x) = \lambda_+(1+\tau\varepsilon/2)\big(x_d+b_++c_2t\varphi\big)^+ - \lambda_-(1-c_1\eta\varepsilon)\big(x_d+b_++c_2t\varphi\big)^-$$

for a suitably small universal constant $0 < c_2 \ll \tau$, chosen so that for all $x \in B_{1/100}(Q)$:

$$\big(1+\tau\varepsilon\big)\big(x_d+b_+\big)^+ \geq \big(1+\tau\varepsilon/2\big)\big(x_d+b_++c_2\varepsilon\big)^+.$$

This together with (3.15) implies that

$$\begin{aligned} u(x) \geq \lambda_+(1+\tau\varepsilon)\big(x_d+b_+\big)^+ &\geq \lambda_+(1+\tau\varepsilon/2)\big(x_d+b_++c_2\big)^+ \\ &\geq f_1(x) \geq f_t(x) \end{aligned} \tag{3.17}$$

for all $x \in B_{1/100}(Q)$, $t \in [0, 1]$. Furthermore $u \geq f_0$ in B_1 thanks to (3.16).

As in Case 1 we let $\bar{t}$ the biggest t such that $f_t \leq u$ in B_1 and $\bar{x}$ the first contact point, so that

$$u(x) - f_{\bar{t}}(x) \geq u(\bar{x}) - f_{\bar{t}}(\bar{x}) = 0 \qquad \text{for all } x \in B_1.$$

Since $\Delta f_{\bar{t}} > 0$ on $\{f_t \neq 0\} \cap B_{1/100}(Q)$, as in Case 1, $\bar{x}$ is a free boundary point. Moreover, since $f_{\bar{t}}$ changes sign in a neighborhood of $\bar{x}$:

$$\begin{aligned} &\text{either} \quad \bar{x} \in \Gamma_{\mathrm{OP}}^+ = \partial\Omega_u^+ \setminus \partial\Omega_u^-, \\ &\text{or} \quad \bar{x} \in \Gamma_{\mathrm{TP}} = \partial\Omega_u^+ \cap \partial\Omega_u^-. \end{aligned}$$

In the first case, by definition of viscosity solution and (φ.3),

$$\lambda_+^2 \geq |\nabla f_{\bar{t}}^+(\bar{x})|^2 = \lambda_+^2(1+\tau\varepsilon/2)^2 + 2c\varepsilon\bar{t}\lambda_+\partial_d\varphi(\bar{x}) + O(\varepsilon^2) > \lambda_+^2,$$

a contradiction for $\varepsilon \ll 1$. In the second case we have a contradiction as well, provided $\eta \ll \tau$, since (recall also that $\lambda_+ \geq \lambda-$, (2.1)):

$$\begin{aligned} \lambda_+^2 - \lambda_-^2 &\geq |\nabla f_{\bar{t}}^+|^2 - |\nabla f_{\bar{t}}^-|^2 \\ &= \lambda_+^2(1+\tau\varepsilon/2)^2 - \lambda_-^2(1-c_1\eta\varepsilon)^2 \end{aligned}$$

$$
\begin{aligned}
&+ 2c_2\varepsilon\bar{t}(\lambda_+ - \lambda_-)\partial_d\varphi(\bar{x}) + O(\varepsilon^2)\\
&> \lambda_+^2 - \lambda_-^2
\end{aligned}
$$

provided $\eta = \eta(d) \ll \tau$ and $\varepsilon \ll 1$ (only depending on d and λ_+). Hence, $\bar{t} = 1$, $u \geq f_1$ which implies the desired conclusion by setting

$$\bar{a}_+ = a_+, \qquad \bar{b}_+ = b_+ + \bar{c}_2\varepsilon$$

and by recalling that $\varepsilon = (a_+ - b_+)$.

- *Case 2.2:* Assume instead that

$$b_+ - b_- \geq \eta\varepsilon,$$

where $\eta = \eta(d)$ has been chosen according to Case 2.1. In this case we consider the family of functions

$$f_t(x) = \lambda_+(1 + \tau\varepsilon/2)\big(x_d + b_+ + \eta t\varphi\big)^+ - \lambda_-\big(x_d + b_-\big)^-.$$

Being $\varphi \leq 1$, this is well defined since $b_+ \geq b_- + \eta$. Moreover $u \geq f_0$ and, thanks, to (3.15) and by possibly choosing η smaller depending only on the dimension,

$$u(x) \geq f_1(x) \geq f_t(x) \qquad \text{for all } x \in B_{1/100}(Q),\, t \in [0, 1].$$

We consider again the first touching time $\bar{t}$ and the first touching point $\bar{x}$. Note that this can not happen where $u \neq 0$. Moreover, by the very definition of $f_{\bar{t}}$, $\bar{x} \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$. However, again by arguing as in Case 2.1, this is in contradiction with u being a viscosity solution. We now conclude as in the previous cases.

Since either the assumption of Case 1 or the one of Case 2 is always satisfied, this concludes the proof. □

The next lemma deals with the case in which the origin is not a branching point.

Lemma 3.9 (Partial Boundary Harnack II). *Given $L \geq \lambda_+ \geq \lambda_- > 0$ there exist constants $\bar{\varepsilon} = \bar{\varepsilon}(d, \lambda_\pm, L) > 0$, $M = M(d, \lambda_\pm, L)$ and $c = c(d, \lambda_\pm, L) \in (0, 1)$ such that for every function $u : B_4 \to \mathbb{R}$ satisfying (a), (c) and (d) in Theorem 3.1 the following property holds true. If there are constants $a, b \in \left(-1/100, 1/100\right)$ with*

$$0 \leq a - b \leq \bar{\varepsilon}$$

such that for $x \in B_4$

$$H_{\alpha,\boldsymbol{e}_d}(x + b\boldsymbol{e}_d) \le u(x) \le H_{\alpha,\boldsymbol{e}_d}(x + a\boldsymbol{e}_d)$$

and

$$\lambda_+ + M\varepsilon \le \alpha \le 2L,$$

then there are constants $\bar{a}, \bar{b} \in \left(-{}^{1}/_{100}, {}^{1}/_{100}\right)$ *with*

$$0 \le \bar{b} - \bar{a} \le c(b - a)$$

such that for $x \in B_{1/6}$

$$H_{\alpha,\boldsymbol{e}_d}(x + \bar{b}\boldsymbol{e}_d) \le u(x) \le H_{\alpha,\boldsymbol{e}_d}(x + \bar{a}\boldsymbol{e}_d).$$

Proof We consider the point $P = (0, \dots, 0, 2)$ and we distinguish the two cases (note that one of the two is always satisfied):

$$\text{either } H_{\alpha,\boldsymbol{e}_d}\Big(P + b\boldsymbol{e}_d\Big) + \frac{\alpha(a-b)}{2} \le u(P),$$
$$\text{or } H_{\alpha,\boldsymbol{e}_d}\Big(P + a\boldsymbol{e}_d\Big) - \frac{\alpha(a-b)}{2} \ge u(P).$$

Since the argument in both cases is completely symmetric we only consider the second one. If we set

$$\varepsilon = (a - b),$$

by Lemma 3.7 and by arguing as in Lemma 3.8 we deduce the existence of a dimensional constant τ such that

$$u \le \alpha(1 - \tau\varepsilon)\big(x_d + a\big)^+ - \beta\big(x_d + a\big)^-$$

in B_1. We let φ as in (3.11) and we set

$$f_t(x) = \alpha(1 - \tau\varepsilon/2)\big(x_d + a - ct\varphi\big)^+ - \beta\big(x_d + a - ct\varphi\big)^-$$

where c is a dimensional constant chosen such that

$$u(x) \le f_1(x) \le f_t(x) \qquad \text{for all } x \in B_{1/100}(Q), t \in [0, 1].$$

where, again, $Q = (0, \dots, 0, {}^{1}/_{5})$. As in Lemma 3.8 we let $\bar{t}$ and $\bar{x}$ be the first contact time and the first contact point and we aim to show that $\bar{t} = 1$. For, we note that, by the same arguments as in Lemma 3.8, necessarily $\bar{x} \in \{u = 0\}$. We claim that

$$\bar{x} \in \Gamma_{\mathrm{TP}} = \partial\Omega_u^+ \cap \partial\Omega_u^-.$$

Indeed otherwise $\bar{x} \in \partial\Omega_u^- \setminus \partial\Omega_u^+$, the case $\bar{x} \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$ being impossible since $f_{\bar{t}}$ is negative in a neighborhood of $\bar{x}$. By definition of viscosity solution this would imply

$$\lambda_-^2 \geq |\nabla f_{\bar{t}}^-(\bar{x})|^2 = \beta^2 - O(\varepsilon) \geq \lambda_-^2 + 2M\lambda_+\varepsilon - O(\varepsilon), \tag{3.18}$$

where the implicit constants in $O(\varepsilon)$ depends on $\lambda_\pm$, L and d and we exploited that, since $\alpha \geq \lambda_+ + M\varepsilon$,

$$\beta^2 = \alpha^2 - \lambda_+^2 + \lambda_-^2 \geq \lambda_-^2 + 2M\lambda_+\varepsilon.$$

Inequality (3.18) is impossible if M is chosen sufficiently large. Hence $\bar{x} \in \Omega_u^- \cap \partial\Omega_u^+$. This however implies:

$$\begin{aligned}
\lambda_+^2 - \lambda_-^2 &\leq |\nabla f_{\bar{t}}^+(\bar{x})|^2 - |\nabla f_{\bar{t}}^-(\bar{x})|^2 \\
&= \alpha^2(1 - \tau\varepsilon/2)^2 - \beta^2 - 2c\bar{t}\varepsilon(\alpha - \beta)\partial_d\varphi(\bar{x}) + O(\varepsilon^2) \\
&\leq \lambda_+^2 - \lambda_-^2 - \alpha^2\tau\varepsilon + O(\varepsilon^2),
\end{aligned}$$

where we have used (φ.3), the equality

$$\lambda_+^2 - \lambda_-^2 = \alpha^2 - \beta^2$$

and that since $\lambda_+ \geq \lambda_-$, $\alpha \geq \beta$. This is a contradiction provided $\bar{\varepsilon}$ is chosen small enough. Hence $\bar{t} = 1$ and , as in Lemma 3.8, this concludes the proof. □

With Lemmas 3.9 and 3.8 at hand we can use the same arguments as in [26,27] to prove Corollary 3.4.

Proof of Corollary 3.4 We distinguish two cases:
$0 \leq \ell < +\infty$: By triangular inequality we have

$$\|u_k - H_{\lambda_+,\boldsymbol{e_d}}\|_{L^\infty(B_1)} \leq (2\ell + 1)\varepsilon_k$$

for k sufficiently large. In particular we can repeatedly apply Lemma 3.8 as in [26], see also [44, Lemma 7.14 and Lemma 7.15] for a detailed proof, to deduce that if we define the sequence $(w_k)_k$ by

$$w_k(x) = \begin{cases} w_{+,k}(x) := \dfrac{u_k(x) - \lambda_+ x_d^+}{\alpha_k \varepsilon_k} & x \in \Omega_{u_k}^+ \cap B_1, \\ w_{-,k}(x) := \dfrac{u_k(x) + \lambda_- x_d^-}{\beta_k \varepsilon_k} & x \in \Omega_{u_k}^- \cap B_1, \end{cases}$$

then the sets

$$\tilde{\Gamma}_k^{\pm} := \left\{(x, w_{\pm,k}(x)) \ : \ x \in \overline{\Omega_{u_k}^{\pm} \cap B_{1/2}}\right\}$$

converge, up to a not relabeled subsequence, in the Hausdorff distance to the closed graphs

$$\tilde{\Gamma}_{\pm} = \left\{(x, w_{\pm}(x)) \ : \ x \in \overline{B_{1/2}^{\pm}}\right\},$$

where $w \in C^{0,\alpha}$ for a suitable α. Since

$$h_k(x) := \frac{H_{\alpha_k,\boldsymbol{e_d}} - H_{\lambda^+,\boldsymbol{e_d}}}{\varepsilon_k} \to \begin{cases} \lambda_+^{-1} \ell x_d & \text{if } x_d > 0, \\ \lambda_-^{-1} \ell x_d & \text{if } x_d < 0, \end{cases}$$

the original sequence v_k satisfies that their graphs,

$$\tilde{\Gamma}_{\pm} = \left\{(x, v_{\pm}(x)) \ : \ x \in \overline{B_{1/2}^{\pm}}\right\},$$

converges to the graph of a limiting function v as we wanted, this in particular proves (i), (ii) and (iii).

Since $0 \in \partial\Omega_{u_k}^+ \cap \partial\Omega_{u_k}^-$ then 0 is in the domain of $v_{\pm,k}$ and

$$v_{\pm,k}(0) = 0,$$

which implies that $v_{\pm}(0) = 0$. To show that $v_+(x) \leq v_-(x)$ for $x = (x', 0) \in \{x_d = 0\} \cap B_{1/2}$ we simply exploit (iii) at the points $x_k^{\pm} = (x', t_k^{\pm})$ where

$$t_k^+ = \sup\left\{t : (x', t) \in \partial\Omega_{u_k}^+\right\} \qquad \text{and} \qquad t_k^- = \inf\left\{t : (x', t) \in \partial\Omega_{u_k}^-\right\}$$

and by noticing that $-t_k^+ \leq -t_k^-$. Finally to show the last claim it is enough to note that if $x_k \in \partial\Omega_{u_k}^+ \cap \partial\Omega_{u_k}^-$ is converging to x then $v_{+,k}(x_k) = v_{-,k}(x_k)$ and thus $v_+(x) = v_-(x)$, yielding $x \in \mathcal{C}$.

$\boldsymbol{\ell = \infty}$**:** In this case the conclusion follows exactly as in [27] by using Lemma 3.9 and noticing that its assumptions are satisfied since $\ell = \infty$. □

3.2 The linearized problem: Proof of Lemma 3.5

The following technical lemma is instrumental to the proof of Lemma 3.5. We defer its proof to "Appendix A" below.

Lemma 3.10 *Let u_k, ε_k and α_k be as in the statement of Corollary 3.4, v_k be defined by* (3.6) *and $v_\pm$ be as in Corollary* 3.4. *Then:*

(1) *Let P_+ a strictly subharmonic (superharmonic) function on $B^+_{1/2}$ touching v_+ strictly from below (above) at a point $x_0 \in \{x_d = 0\} \cap B_{1/2}$. Then, there exists a sequence of points $\partial\Omega^+_{u_k} \ni x_k \to x_0$ and a sequence of comparison functions Q_k such that Q_k touches from below (above) u_k^+ at x_k, and such that*

$$\nabla Q_k^+(x_k) = \alpha_k \boldsymbol{e}_d + \alpha_k\varepsilon_k \nabla P_+(x_0) + o(\varepsilon_k). \tag{3.19}$$

(2) *Let P_- be a strictly subharmonic (superharmonic) function on $B^-_{1/2}$ and touching v_- strictly from below (above) at a point $x_0 \in \{x_d = 0\} \cap B_{1/2}$. Then, there exists a sequence of points $\partial\Omega^-_{u_k} \ni x_k \to x_0$ and a sequence of comparison functions Q_k such that Q_k touches from below (above) $-u_k^-$ at x_k, and such that*

$$\nabla Q_k^-(x_k) = -\beta_k \boldsymbol{e}_d + \beta_k\varepsilon_k \nabla P_-(x_0) + o(\varepsilon_k). \tag{3.20}$$

(3) *Let $p, q \in \mathbb{R}$ and $\tilde P$ be a function on $B_{1/2}$ such that $\partial_d \tilde P = 0$. Suppose that $\tilde P$ is subharmonic (superharmonic) and that the function*

$$P := px_d^+ - qx_d^- + \tilde P$$

touches v strictly from below (above) at a point $x_0 \in \mathcal{C}$. Then, there exists a sequence of points $x_k \to x_0$ and a sequence of comparison functions Q_k such that Q_k touches from below (above) the function u_k at $x_k \in \partial\Omega_{u_k}$, and such that

$$\begin{aligned}\nabla Q_k^+(x_k) &= \alpha_k \boldsymbol{e}_d + \alpha_k\varepsilon_k p + o(\varepsilon_k)\\ \nabla Q_k^-(x_k) &= -\beta_k \boldsymbol{e}_d + \beta_k\varepsilon_k q + o(\varepsilon_k).\end{aligned} \tag{3.21}$$

In particular, if $p > 0$ and Q_k touches u_k from below then $x_k \notin \partial\Omega^-_{u_k} \setminus \partial\Omega^+_{u_k}$, while if $q < 0$ and Q_k touches u_k from above then $x_k \notin \partial\Omega^+_{u_k} \setminus \partial\Omega^-_{u_k}$.

Proof of Lemma 3.5 We note that $v_k^\pm$ converge uniformly to $v_\pm$ on every compact subset of $\{\pm x_d > 0\} \cap B_{1/2}$. Since these functions are harmonic there,

by elliptic estimates the convergence is smooth and in particular $v_\pm$ are harmonic on the (open) half balls $B^\pm_{1/2}$. Hence we only have to check the boundary conditions on $\{x_d = 0\}$. We distinguish two cases.
$\boldsymbol{\ell = \infty}$. In this case we first want to show that $\mathcal{J} = \emptyset$. Assume not, since the set $\{v_- > v_+\}$ is open in $\{x_d = 0\}$, it contains a $(d-1)$-dimensional ball

$$B'_\varepsilon(y') := B_\varepsilon((y', 0)) \cap \{x_d = 0\} \subset \mathcal{J}\,.$$

Next let P be the polynomial

$$P(x) = A\big((d - 1/2)x_d^2 - |x' - y'|^2\big) - Bx_d\,, \quad \text{where} \quad x = (x', x_d)\,,$$

for some constants A, B. We first choose $A \gg 1$ large enough so that

$$P < v^+ \qquad \text{on} \quad \{|x' - y'| = \varepsilon\} \cap \{x_d = 0\}$$

and then we choose $B \gg A$ so that

$$P < v^+ \quad \text{on} \quad B_\varepsilon((y', 0)).$$

Now we can translate P first down and then up to find that there exists C such that $P + C$ is touching v^+ from below at a point $x_0 \in B_\varepsilon((y', 0)) \cap \{x_d \geq 0\}$. Since $\Delta P > 0$, the touching point can not be in the interior of the (half) ball and thus $x_0 \in B'_\varepsilon(y') \subset \mathcal{J}$.

By using Lemma 3.10, there exists a sequence of points $\partial\Omega^+_{u_k} \ni x_k \to x_0$ and of functions Q_k touching u_k^+ from below at x_k and such that

$$\nabla Q_k^+(x_k) = \alpha_k \boldsymbol{e}_d + \alpha_k \varepsilon_k \nabla P(x_0) + o(\varepsilon_k).$$

Since $x_0 \in \mathcal{J}$, by (3.7) in Lemma 3.10, $x_k \in \partial\Omega^+_{u_k} \setminus \partial\Omega^-_{u_k}$. Hence, by (ii) in Lemma 2.6

$$\lambda_+^2 \geq |\nabla Q_k^+(x_k)|^2 \geq \alpha_k^2 + 2\alpha_k^2 \varepsilon_k \partial_d P(x_0) + o(\varepsilon_k)$$

Hence, recalling the definition of ℓ,

$$-B = \partial_d P(x_0) \leq \frac{\lambda_+^2 - \alpha_k^2}{2\alpha_k^2 \varepsilon_k} + o(1) \to -\infty.$$

This contradiction proves that $\mathcal{J} = \emptyset$.

We next prove the transmission condition in (3.9). Let us show that

$$\alpha_\infty^2 \partial_d v_+ - \beta_\infty^2 \partial_d v_- \leq 0,$$

the opposite inequality can then be proved by the very same argument. Suppose that there exist p and q with $\alpha_\infty^2 p > \beta_\infty^2 q$ and a strictly sub-harmonic function $\tilde{P}$ with $\partial_d \tilde{P} = 0$ such that

$$P = px_d^+ - qx_d^- + \tilde{P}$$

touches v strictly from below at a point $x_0 \in \{x_d = 0\} \cap B_{1/2}$ (note that the last set coincide with $\mathcal{C}$ by the previous step). By Lemma 3.10 there exists a sequence of points $\partial\Omega_{u_k} \ni x_k \to x_0$ and a sequence of comparison functions Q_k touching u_k from below at x_k and satisfying (3.21). In particular $x_k \notin \partial\Omega_{u_k}^- \setminus \partial\Omega_{u_k}^+$. We claim that $x_k \in \partial\Omega_{u_k}^+ \cap \partial\Omega_{u_k}^-$. Indeed, if this was not the case, then by (A.1) in Lemma 2.5,

$$\lambda_+^2 \geq |\nabla Q_k^+(x_k)|^2.$$

Arguing as above, this contradicts $\ell = +\infty$. Hence, by Lemma 2.5 (A.3)

$$\begin{aligned}\lambda_+^2 - \lambda_-^2 &\geq |\nabla Q_k^+(x_k)|^2 - |\nabla Q_k^-(x_k)|^2 \\ &= \alpha_k^2 - \beta_k^2 + 2\varepsilon_k(\alpha_k^2 p - \beta_k^2 q) + o(\varepsilon_k) \\ &= \lambda_+^2 - \lambda_-^2 + 2\varepsilon_k(\alpha_k^2 p - \beta_k^2 q) + o(\varepsilon_k).\end{aligned}$$

Dividing by ε_k and letting $k \to \infty$, we obtain the desired contradiction.

$\mathbf{0 \leq \ell < \infty}$. We start by showing that $\lambda_\pm^2 \partial_d v_\pm \geq -\ell$ on $B_{1/2} \cap \{x_d = 0\}$. We focus on v_- since the argument is symmetric. Let us assume that there exists $q \in \mathbb{R}$ with $\lambda_-^2 q < -\ell$ and a strictly subharmonic function $\tilde{P}$ with $\partial_d \tilde{P} = 0$ such that function

$$P = qx_d + \tilde{P}$$

touches v_- strictly from below at a point $x_0 \in \{x_d = 0\} \cap B_{1/2}$. Let now x_k and Q_k be as in Lemma 3.10 (2). By the optimality conditions

$$\lambda_-^2 \leq |\nabla Q_k^-(x_k)|^2 = \beta_k^2 + 2\varepsilon_k \beta_k^2 q + o(\varepsilon_k).$$

Since $\ell < \infty$, we have $\beta_k = \lambda_- + O(\varepsilon_k)$ and so the above inequality leads to

$$-\frac{\ell}{\lambda_-^2} = \lim_{k\to\infty} \frac{\lambda_-^2 - \beta_k^2}{2\varepsilon_k \beta_k^2} \leq q < -\frac{\ell}{\lambda_-^2},$$

which is a contradiction.

We now show that $\lambda_\pm^2 \partial_d v_\pm = -\ell$ on $\mathcal{J}$ and again we focus on v_-. By the previous step it is enough to show that if there exists a strictly superharmonic polynomial $\tilde{P}$ with $\partial_d \tilde{P} = 0$ such

$$P = qx_d + \tilde{P}$$

touches v_- strictly from above at a point $x_0 \in \mathcal{J}$, then $\lambda_-^2 q \leq -\ell$. Again, by Lemma 3.10, we find points $x_k \to x_0$ and functions Q_k satisfying (3.20) and touching $-u_k^-$ from below at x_k. Since $x_0 \in \mathcal{J}$, by (3.7) in Corollary 3.4, $x_k \in \partial\Omega_{u_k}^- \setminus \partial\Omega_{u_k}^+$. Hence, by Lemma 2.5,

$$\lambda_-^2 \geq |\nabla Q_k^-(x_k)|^2 = \beta_k^2 + 2\beta_k^2 \varepsilon_k q + o(\varepsilon_k),$$

which by arguing as above implies that $\lambda_-^2 q \leq -\ell$.

It then remain to show the transmission condition in (3.10) at points in $\mathcal{C}$. Again by symmetry of the arguments we will only show that

$$\lambda_+^2 \partial_d v_+ - \lambda_-^2 \partial_d v_- \leq 0 \qquad \text{on } \mathcal{C}.$$

Let us hence assume that there exist p and q with $\lambda_+^2 p > \lambda_-^2 q$ and a strictly subharmonic polynomial $\tilde{P}$ with $\partial_d \tilde{P} = 0$ such that

$$P = px_d^+ - qx_d^- + \tilde{P}$$

touches v^+ and v^- strictly from below at $x_0 \in \mathcal{C}$. By Lemma 3.10, we find points $x_k \to x_0$ and functions Q_k satisfying (3.21). In particular $x_k \notin \partial\Omega_{u_k}^- \setminus \partial\Omega_{u_k}^+$. By the previous step we know that $\lambda_-^2 q \geq -\ell$ and thus $\lambda_+^2 p > -\ell$, since we are assuming $\lambda_+^2 p+ > \lambda_-^2 q \geq 0$. We now distinguish two cases:

(1) x_k are one-phase points, namely $x_k \in \partial\Omega_{u_k}^+ \setminus \partial\Omega_{u_k}^-$. In this case

$$\lambda_+^2 \geq |\nabla Q_k^+(x_k)|^2 = \alpha_k^2 + 2\alpha_k^2 \varepsilon_k p + o(\varepsilon_k),$$

which implies that

$$\lambda_+^2 p + \ell = \lambda_+^2 \lim_{k\to\infty} \left(p + \frac{\alpha_k^2 - \lambda_+^2}{2\alpha_k^2 \varepsilon_k} \right) \leq 0$$

in contradiction with $\lambda_+^2 p > -\ell$.

(2) x_k are two-phase points, namely $x_k \in \partial\Omega_{u_k}^+ \cap \partial\Omega_{u_k}^-$. Arguing as in Case 1, we have that, by Lemma 2.5,

$$\begin{aligned}\lambda_+^2 - \lambda_-^2 &\geq |\nabla Q_k^+(x_k)|^2 - |\nabla Q_k^-(x_k)|^2 \\ &= \alpha_k^2 - \beta_k^2 + 2\varepsilon_k(\alpha_k^2 p - \beta_k^2 q) + o(\varepsilon_k)\end{aligned}$$

$$= \lambda_+^2 - \lambda_-^2 + 2\varepsilon_k(\lambda_+^2 p - \lambda_-^2 q) + o(\varepsilon_k),$$

which gives a contradiction with $\lambda_+^2 p > \lambda_-^2 q$ as $\varepsilon_k \to 0$.

□

3.3 Proof of Lemmas 3.3 and 3.2

We recall the following regularity results for the limiting problems.

Lemma 3.11 (Regularity for the transmission problem). *There exists a universal constant* $C = C(\alpha_\infty, \beta_\infty, d) > 0$ *such that if* $v \in C^0(B_{1/2})$ *is a viscosity solution of* (3.9) *with* $\|v\|_{L^\infty(B_{1/2})} \leq 1$ *then there exists* $\boldsymbol{v} \in \mathbb{R}^{d-1}$, $p, q \in \mathbb{R}$ *with* $\alpha_\infty^2\, p = \beta_\infty^2\, q$ *such that*

$$\sup_{x \in B_r} \frac{\left|v(x) - v(0) - (\boldsymbol{v} \cdot x' + p\, x_d^+ - q\, x_d^-)\right|}{r^2} \leq C. \tag{3.22}$$

The proof of this fact can be found in [27, Theorem 3.2]. A similar result holds for the linearized problem (3.10).

Lemma 3.12 (Regularity for the two-membrane problem). *There exists a universal constant* $C = C(\lambda_\pm, d) > 0$ *such that if* $v \in C^0(B_{1/2})$ *is a viscosity solution of* (3.10) *with* $\|v\|_{L^\infty(B_{1/2})} \leq 1$ *then there exists* $\boldsymbol{v} \in \mathbb{R}^{d-1}$, $p, q \in \mathbb{R}$ *satisfying* $\lambda_+^2\, p = \lambda_-^2\, q \geq -\ell$ *such that*

$$\sup_{x \in B_r} \frac{\left|v(x) - v(0) - (\boldsymbol{v} \cdot x' +\, p\, x_d^+ - q\, x_d^-)\right|}{r^{3/2}} \leq C(1 + \ell). \tag{3.23}$$

The proof of the above lemma reduces easily to the one of the thin obstacle problem, since we were not able to find the statement of this fact in the literature, we sketch its proof in "Appendix B".

It is by now well known that the regularity theory fo the limiting problems and a classical compactness argument prove Lemmas 3.3 and 3.2. We sketch their arguments here:

Proof of Lemma 3.2 We argue by contradiction and we assume that for fixed $\gamma \in (0, 1/2)$ and M we can find a sequences of functions u_k and numbers α_k such that

$$\varepsilon_k = \|u_k - H_{\alpha_k, \boldsymbol{e}_d}\|_{L^\infty(B_1)} \to 0 \quad \text{and} \quad 0 \leq \alpha_k - \lambda_+ \leq M\varepsilon_k,$$

but for which (3.2) and (3.3) for any choice of ρ and C. Note that by the second assumption above

$$\ell < \frac{M}{\lambda_+}.$$

We let $(v_k)_k$ be the sequence of functions defined in (3.6) and we assume that they converge to a function v as in Corollary 3.4, note that $\|v\|_{L^\infty(B_{1/2})} \leq 1$. By Lemma 3.5, v solves (3.10) and thus by Lemma 3.12 there exists $\boldsymbol{v} \in \mathbb{R}^{d-1}$, $p, q \in \mathbb{R}$ satisfying $\lambda_+^2\, p = \lambda_-^2\, q \geq -\ell$ such that for all $r \in (0, 1/4)$

$$\sup_{x \in B_\rho} \frac{\left|v(x) - v(0) - (\boldsymbol{v} \cdot x' + \, p\, x_d^+ - q\, x_d^-)\right|}{r^\gamma} \leq r^{3/2-\gamma} C(1 + M). \quad (3.24)$$

Hence we can fix $\rho = \rho(\lambda_\pm, \gamma, M)$ such that

$$\sup_{x \in B_\rho} \left|v(x) - v(0) - (\boldsymbol{v} \cdot x' + \, p\, x_d^+ - q\, x_d^-)\right| \leq \frac{\rho^\gamma}{2}. \quad (3.25)$$

We now set

$$\tilde{\alpha}_k := \alpha_k(1 + \varepsilon_k p) + \delta_k \varepsilon_k \qquad \text{and} \qquad \boldsymbol{e}_k := \frac{\boldsymbol{e}_d + \varepsilon_k \boldsymbol{v}}{\sqrt{1 + \varepsilon_k^2\, |\boldsymbol{v}|^2}},$$

where $\delta_k \to 0$ is chosen so that $\tilde{\alpha}_k \geq \alpha_k$, note that the existence of such a sequence is due to the condition $\lambda_+^2 p \geq -\ell$ since

$$\alpha_k(1 + \varepsilon_k p) = \Big(\lambda_+ + \frac{\ell}{\lambda_+}\varepsilon_k + o(\varepsilon_k)\Big)(1 + \varepsilon_k p) \geq \lambda_+ + o(\varepsilon_k).$$

We let $H_k := H_{\tilde{\alpha}_k, \boldsymbol{e}_k}$ and we note that

$$|\alpha_k - \alpha| + |\boldsymbol{e}_k - \boldsymbol{e}_d| \leq C\, \varepsilon_k$$

for a universal constant $C > 0$, hence the proof will be concluded if we can show that

$$\sup_{B_\rho} |u_k(x) - H_k(x)| \leq \rho^\gamma \varepsilon_k,$$

where ρ is defined so that (3.25) holds. This however easily follows from the convergence of v_k to v in the sense of Corollary 3.4 since the sequence of functions defined by

$$\begin{cases} \frac{H_k(x)-H_{\alpha_k,\boldsymbol{e}_d}}{\alpha_k\varepsilon_k} & x_d > 0 \\ \frac{H_k(x)-H_{\alpha_k,\boldsymbol{e}_d}}{\beta_k\varepsilon_k} & x_d < 0 \end{cases}$$

converges (again in the sense of Corollary 3.4) to the function

$$\boldsymbol{v}\cdot x' + px_d^+ - qx_d^-. \qquad \square$$

Proof of Lemma 3.3 Arguing by contradiction one assume for fixed $\gamma \in (0,1)$ the existence of a sequence of of functions u_k and numbers $\alpha_k, M_k \to \infty$ such that

$$\varepsilon_k = \|u_k - H_{\alpha_k,\boldsymbol{e}_d}\|_{L^\infty(B_1)} \to 0 \quad \text{and} \quad \frac{\alpha_k - \lambda_+}{\varepsilon_k} \geq M_k \to \infty,$$

but for which (3.2) and (3.3) for any choice of ρ and C. This implies that $\ell = \infty$ and that the limiting functions v obtained in Corollary 3.4 are solutions of (3.9). One then concludes the proof as above by using (3.11). $\square$

4 Proof of the main results

4.1 Proof of Theorem 1.1 and Corollary 1.2

The final step to obtain the desired regularity result is to show that $|\nabla u^\pm|$ are C^η for a suitable $\eta > 0$ up to the boundary. This indeed implies that $u^\pm$ are solutions of the classical one-phase free boundary problem in its viscosity formulation and the regularity will follows form [26]. The argument is similar to the one in [42], therefore we only sketch the main steps and refer the reader to that paper for more details.

Lemma 4.1 *Suppose that u is a local minimizer of J_{TP} in D. Then at every point of Γ_{tp} there is a unique blow-up, that is,*

$$\mathcal{BU}(x_0) = \{H_{\alpha(x_0),\boldsymbol{e}(x_0)}\}.$$

Moreover there exists $\eta > 0$ such that for every open set $D' \Subset D$ there is a constant $C(D', \lambda_\pm, d) > 0$ such that, for every $x_0, y_0 \in \Gamma_{tp} \cap D'$, we have

$$|\alpha(x_0) - \alpha(y_0)| \leq C|x_0 - y_0|^\eta \quad \text{and} \quad |\boldsymbol{e}(x_0) - \boldsymbol{e}(y_0)| \leq C_0|x_0 - y_0|^\eta, \tag{4.1}$$

where $H_{\boldsymbol{e}(x_0),\alpha(x_0)}$ and $H_{\boldsymbol{e}(x_0),\alpha(x_0)}$ are the blow-ups at x_0 and y_0 respectively. In particular, $\Gamma_{tp} \cap D'$ is locally a closed subset of the graph of a $C^{1,\eta}$ function.

Proof We first notice that by Corollary 2.3 and the definition of $\mathcal{BU}(x_0)$, given $\varepsilon_0 > 0$ as in Theorem 3.1 we can find $r_0 > 0$ and ρ_0 such that (3.1) is satisfied by u_{y_0,r_0} for some $H_{\alpha,\boldsymbol{e}} \in \mathcal{BU}(x_0)$ and for all $y_0 \in B_{\rho_0}(x_0)$.

We can thus repeatedly apply Theorem 3.1 together with standard arguments to infer that for all $y_0 \in B_{\rho_0}(x_0)$ there exists a unique $H_{\boldsymbol{e}(y_0),\alpha(y_0)}$ such that

$$\|u_{r,x_0} - H_{\boldsymbol{e}(y_0),\alpha(y_0)}\|_{L^\infty(B_r(y_0))} \le C_0 r^\gamma \tag{4.2}$$

where $\gamma \in (0, 1/2)$. A covering argument implies the validity of the above estimate for all $x_0 \in \Gamma_{tp} \cap D'$. Next, for $x_0, y_0 \in \Gamma_{tp} \cap D'$ set $r := |x_0 - y_0|^{1-\eta}$ and $\eta := \gamma/(1+\gamma)$, and recall that u is L-Lipschitz (with constant depending on D') to get

$$\begin{aligned}
&\|H_{\boldsymbol{e}(x_0),\alpha(x_0)} - H_{\boldsymbol{e}(y_0),\alpha(y_0)}\|_{L^\infty(B_1)} \\
&\quad \le \|u_{r,x_0} - H_{\boldsymbol{e}(x_0),\alpha(x_0)}\|_{L^\infty(B_1)} \\
&\qquad + \|u_{r,x_0} - u_{r,y_0}\|_{L^\infty(B_1)} + \|u_{r,y_0} - H_{\boldsymbol{e}(y_0),\alpha(y_0)}\|_{L^\infty(B_1)} \\
&\quad \le \left(C_0 r^\gamma + \frac{L}{r}|x_0 - y_0| + C_0 r^\gamma \right) = (L + 2C_0)\ |x_0 - y_0|^\eta \ .
\end{aligned}$$

The conclusion now follows easily from this inequality. □

Lemma 4.2 *Under the same assumptions of Lemma 4.1, there are $C^{0,\eta}$ continuous functions $\alpha\colon \partial\Omega_u^+ \to \mathbb{R}$, $\beta\colon \partial\Omega_u^- \to \mathbb{R}$ such that $\alpha \ge \lambda_+$, $\beta \ge \lambda-$, and $u^\pm$ are viscosity solutions of the one-phase problem*

$$\Delta u^+ = 0 \quad \text{in} \quad \Omega_u^+ , \qquad |\nabla u^+| = \alpha \quad \text{on} \quad \partial\Omega_u^+$$

and

$$\Delta u^- = 0 \quad \text{in} \quad \Omega_u^- , \qquad |\nabla u^-| = \beta \quad \text{on} \quad \partial\Omega_u^- .$$

Proof We will sketch the argument for u^+, u^- being the same. Clearly $\Delta u^+ = 0$ in Ω_u^+. By (4.2) we have that, if $x_0 \in \Gamma_{tp} \cap D'$, then

$$\left|u^+(x) - \alpha(x_0)(x - x_0) \cdot \boldsymbol{e}(x_0)\right| \le C_0 |x - x_0|^{1+\gamma} \tag{4.3}$$

for every $x \in B_{r_0}(x_0) \cap \Omega_u^+$, where r_0 and C_0 depends only on D'. In particular, u^+ is differentiable on Ω_u^+ up to x_0 and $|\nabla u^+(x_0)| = \alpha(x_0)$. On the other hand if $x_0 \in \Gamma_{\text{OP}}^+ := \Omega_u^+ \setminus \partial\Omega_u^-$, then $|\nabla u^+(x_0)| = \lambda_+$ is constant, in the viscosity sense.

To conclude we only need to prove that $\alpha \in C^{0,\eta}(\partial\Omega_+)$. Since α is η Hölder continuous on Γ_{tp} by Lemma 4.1 and constant on Γ_{OP}^+, we just need to show that if $x_0 \in \Gamma_{tp}$ is such that there is a sequence $x_k \in \Gamma_{\text{OP}}^+$ converging to x_0, then $\alpha(x_0) = \lambda_+$. To this end, let $y_k \in \Gamma_{tp}$ be such that

$$\mathrm{dist}(x_k, \Gamma_{tp}) = |x_k - y_k| \,.$$

Let us set

$$r_k = |x_k - y_k| \qquad \text{and} \qquad u_k(x) = \frac{1}{r_k} u^+(x_k + r_k x),$$

and note that u_k is a viscosity solution of the free boundary problem

$$\Delta u_k = 0 \quad \text{in} \quad \Omega^+_{u_k} \cap B_1 \,, \qquad |\nabla u_k| = \lambda_+ \quad \text{on} \quad \partial\{u_k > 0\} \cap B_1 \,.$$

Since u_k are uniformly Lipschitz they converge to a function u_∞ which is also a viscosity solution of the same problem, [26]. On the other hand, by (4.3), we have that

$$u_\infty(x) = \alpha(x_0)(x \cdot \boldsymbol{e}(x_0))^+,$$

which gives that $\alpha(x_0) = \lambda_+$. □

Proof of Theorem 1.1 Let $x_0 \in \Gamma_{tp} = \partial\Omega^+_u \cap \partial\Omega^-_u$ and let $\bar\varepsilon$ be the constant in [26, Theorem 1.1]. Thanks to the classification of blow-ups at points of Γ_{tp}, we can choose $r_0 > 0$, depending on x_0, such that

$$\|u_{x_0,r_0} - H_{\alpha,\boldsymbol{e}}\|_{L^\infty(B_1)} < \bar\varepsilon$$

so that thanks to Lemma 4.2, we can apply [26, Theorem 1.1] to conclude that locally at $x_0 \in \Gamma_{tp}$ the free boundaries $\partial\Omega^\pm_u$ are $C^{1,\eta}$ graphs. By the arbitrariness of x_0 this concludes the proof. □

Proof of Corollary 1.2 The proof of the corollary is straightforward. Indeed by Theorem 1.1 there exits an open neighborhood W of the two-phase free boundary Γ_{TP} such that $\partial\Omega^\pm_u \cap W \subset \mathrm{Reg}(\partial\Omega^\pm_u)$. Outside W, $u^\pm$ are (local) minimizers of the one-phase problem and thus the desired decomposition and the stated properties follows by the results in [1,31,46]. □

4.2 Proof of Theorem 1.3

In this section we prove the regularity of the solutions to the shape optimization problem (SOP). The proof is a consequence of Theorem 1.1 and the analysis in [43]. Indeed, the existence of an optimal (open) partition $(\Omega_1, \ldots, \Omega_n)$ was proved in [8] and (in dimension two) in [6]. Moreover, in [8,45], it has been shown that each of the eigenfunctions u_i on Ω_i is Lipschitz continuous as a function defined on $\mathbb{R}^d$ (extended as zero outside Ω_i). Furthermore, there are no triple points inside the box D and no two-phase points on the boundary ∂D, that is,

- $\partial\Omega_i \cap \partial\Omega_j \cap \partial\Omega_k = \emptyset$ for every set $\{i, j, k\} \subset \{1, \dots, n\}$ of different coefficients;
- $\partial\Omega_i \cap \partial\Omega_j \cap \partial D = \emptyset$ for every $i \neq j \in \{1, \dots, n\}$.

The regularity of $\partial\Omega_i$ can then be obtained as follows.

- By [43, Lemma 7.3], the function $u = u_i - u_j$ is a almost of (OP) with $\lambda_+^2 = m_i$ and $\lambda_-^2 = m_j$, in the sense that
$$J_{\mathrm{TP}}(u, B_r) \le J_{\mathrm{TP}}(v, B_r) + Cr^{d+2} \qquad \text{for all } v = u \text{ on } \partial B_r\,,$$
provided r is sufficiently small.
- By the classification of the blow up limits in [43, Proposition 4.3] and the arguments in Sect. 2.3, u is a viscosity solution of
$$\begin{cases} \Delta u = -\lambda_1(\Omega_i)u_i + \lambda_1(\Omega_j)u_j & \text{on } \{u \neq 0\}, \\ |\nabla u^+|^2 - |\nabla u^-|^2 = m_i - m_j,\ |\nabla u^+| \ge \sqrt{m_i} \ \text{ and } \ |\nabla u^-| \ge \sqrt{m_j} & \text{on } \partial\Omega_u^+ \cap \partial\Omega_u^-\,; \\ |\nabla u^+| = \sqrt{m_i} & \text{on } \partial\Omega_u^+ \setminus \partial\Omega_u^-\,; \\ |\nabla u^-| = \sqrt{m_j} & \text{on } \partial\Omega_u^- \setminus \partial\Omega_u^+\,. \end{cases}$$
- C^∞ regularity of the one-phase part $\partial\Omega_i \setminus \big(\partial D \cup \big(\bigcup_{i\neq j} \partial\Omega_j\big)\big)$ follows by techniques in [1], see [7];
- $C^{1,\eta}$-regularity of $\partial\Omega_i$ in a neighborhood of $\partial\Omega_i \cap \partial D$ was proved in [38]; the main argument boils down to the regularity result from [18];
- $C^{1,\eta}$-regularity of $\partial\Omega_i$ in a neighborhood of $\partial\Omega_i \cap \partial\Omega_j$ follows by using the same arguments[3] in the proof of Theorem 1.1, using Theorem 4.3 in place of Theorem 3.1.

Theorem 4.3 *Let $0 \le \lambda_+ \le \lambda_- \le L$, $f \in C^0(B_1)$ and let $u : B_1 \to \mathbb{R}$ be a L-Lipschitz viscosity solution of*
$$\begin{cases} \Delta u = f & \text{on } \{u \neq 0\} \\ |\nabla u^+|^2 - |\nabla u^-|^2 = \lambda_+^2 - \lambda_-^2,\ |\nabla u^+| \ge \lambda_+ \ \text{ and } \ |\nabla u^-| \ge \lambda_- & \text{on } \partial\Omega_u^+ \cap \partial\Omega_u^-\,; \\ |\nabla u^+| = \lambda_+ & \text{on } \partial\Omega_u^+ \setminus \partial\Omega_u^-\,; \\ |\nabla u^-| = \lambda_- & \text{on } \partial\Omega_u^- \setminus \partial\Omega_u^+\,. \end{cases}$$
Then for every $\gamma \in (0, 1/2)$, there exist $\varepsilon_0 > 0$, $C > 0$ and $\rho \in (0, 1/4)$ depending only on $\lambda_\pm$, L and γ such that if
$$\|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)} \le \varepsilon_0 \qquad \textit{for some} \qquad L \ge \alpha \ge \lambda_+\,.$$

---

[3] Note that $\Delta u_r(x) = r\Delta u(rx)$. Hence, since Δu is uniformly bounded in L^∞, $\|\Delta u_r\|_{L^\infty} = O(r)$ and thus this does not interfere with the iteration argument.

then, there are $\boldsymbol{e} \in \mathbb{S}^{d-1}$ *and* $\tilde{\alpha} \geq \lambda_+$ *such that*

$$|\boldsymbol{e} - \boldsymbol{e}_d| + |\tilde{\alpha} - \alpha| \leq C\big(\|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\big) \tag{4.4}$$

and

$$\|u_\rho - H_{\tilde{\alpha},\boldsymbol{e}}\|_{L^\infty(B_1)} \leq \rho^\gamma \, \|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)} + C\|f\|_{L^\infty(B_1)}$$

Proof Note that (4.3) is satisfied with $\tilde{\alpha} = \alpha$ and $\boldsymbol{e} = \boldsymbol{e}_d$, $\rho = 1/4$ and $C = C(\varepsilon)$ if

$$\|f\|_{L^\infty(B_1)} \geq \varepsilon \|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)}.$$

Hence it is enough to show that there exists ε_0 universal such that the conclusion of the theorem holds provided

$$\|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)} \leq \varepsilon_0 \qquad \text{for some} \qquad L \geq \alpha \geq \lambda_+$$

and

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon_0 \|u - H_{\alpha,\boldsymbol{e}_d}\|_{L^\infty(B_1)}. \tag{4.5}$$

We can then argue by contradiction as in the proof of Theorem 3.1 by noticing that, thanks to (4.5) the contradicting sequence satisfies

$$\Delta u_k = o(\varepsilon_k).$$

This allows to almost verbatim repeat the proofs in Sect. 3, see for instance [26,27]. □

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Appendix A. Proof of Lemma 3.10

Here we prove Lemma 3.10. The idea to construct the comparison functions is to perform (the inverse of) the changed of variable used in [35] (and attributed to Friedrichs) which maps, for smooth solutions, the free boundary problem, to a (non linear) problem on a fixed domain, see [35, Section 3].

Proof of Lemma 3.10 We divide the proof into several steps:

- *Step 1*: Given $\alpha > 0$ function $P \in C^1(\overline{B^+_{1/2}}) \cap C^2(\overline{B^+_{1/2}})$ there exist $\bar{\varepsilon} \ll 1$, depending only on the C^1 norm of P such that for all $\varepsilon \leq \bar{\varepsilon}$ there exists a function $Q \in C^1(\overline{\{Q > 0\}}) \cap C^2(\{Q > 0\})$ such that

$$Q_\varepsilon(y', y_d - \varepsilon\alpha P(y', y_d))) = \alpha y_d \qquad \text{for all } y = (y', y_d) \in \overline{\{Q > 0\}} \tag{A.1}$$

To this end we define the following map $\boldsymbol{T}_\varepsilon : \overline{B^+_{1/2}} \to \mathbb{R}^d$:

$$\boldsymbol{T}_\varepsilon(x', x_d) = (x', x_d - \varepsilon\alpha P(x', x_d)) \qquad x = (x', x_d) \in B^+_{1/2}.$$

Note that if $\varepsilon \ll \|P\|^{-1}_{C^1}$, $\boldsymbol{T}_{\boldsymbol{\varepsilon}}$ induces a bijection between the sets $B^+_{1/2}$ and $U_\varepsilon := \boldsymbol{T}_\varepsilon(B^+_{1/2}) \subset B_1$. We let $\boldsymbol{Q}_\varepsilon$ be its inverse and we define Q_ε as its d-th component times α, namely

$$Q_\varepsilon := \alpha(\boldsymbol{Q}_\varepsilon \cdot \boldsymbol{e}_d) : U \to (0, 1/2),$$

and we extend it to zero on $B_{1/2} \setminus \overline{\{Q > 0\}}$. It is now immediate to verify that (A.1) is satisfied. Furthermore, with the notation $y_\varepsilon = \boldsymbol{T}\varepsilon(x)$,

$$\nabla Q_\varepsilon(y_\varepsilon) = \alpha \boldsymbol{e_d} + \alpha\varepsilon\nabla P(x) + O(\varepsilon^2) \tag{A.2}$$

and

$$\Delta Q(y_\varepsilon) = \alpha^2\varepsilon\Delta P(x) + O(\varepsilon^2). \tag{A.3}$$

- *Step 2*: Let us now prove item (i) of the statement, item (ii) can be obtained by a symmetric argument. Let α_k, ε_k a be as in the statement. Let us assume that P_+ is a strictly subharmonic function touching v_+ strictly form below at x_0. By assumption, for all $\delta \ll 1$ the function $v_+ - P_- + \delta$ has a strictly positive minimum at x_0 as $\delta \to 0$. Let Q^δ_k be the functions constructed in Step 1 with $\varepsilon = \varepsilon_k$, $\alpha = \alpha_k$ and $P = P_- - \delta$. Let us define

$$P^\delta_k(x) = \frac{Q^\delta_k - \alpha_k x^+_d}{\alpha_k \varepsilon_k}$$

and

$$\tilde{\Gamma}_k = \left\{(x, P_k^{\delta}(x)) \qquad x \in \overline{\{Q_k^{\delta} > 0\} \cap B_{1/2}}\right\}.$$

One easily checks that they converge in the Hausdorff distance to

$$\tilde{\Gamma} = \left\{(x, P_+(x) - \delta), \quad x \in \overline{B_{1/2}^+}\right\}.$$

By using that the graphs Γ_k^+ defined in Corollary 3.4 converges in the Hausdorff distance to

$$\tilde{\Gamma} = \left\{(x, v_+(x)), \quad x \in \overline{B_{1/2}^+}\right\}.$$

We claim that

$$\{Q_k^{\delta} > 0\} \cap B_{1/2} \Subset \{u_k > 0\} \cap B_{1/2}.$$

Indeed otherwise one would find a sequence of points x_k such that $Q_k^{\delta}(x_k) > 0$ and $u^+(x_k) = 0$ which implies that

$$P_k^{\delta}(x_k) \geq v_{+,k}(x_k),$$

where $v_{+,k}$ is define in Corollary 3.4. Assuming that $x_k \to \bar{x}$ we get $P^+(\bar{x}) - \delta \geq v_+(\bar{x})$ in contradiction with $P - \delta < v_+$.

In particular there exists $\sigma = O(\delta)$ such that $Q_k^{\delta}(\cdot - \sigma \boldsymbol{e}_d)$ touches u^+ from below at some point x_k^{δ}. Note also that, arguing as above) $x_k^{\delta} \to x_0$ as k goes to infinity. By (A.3) and the strict subharmonicity of P one has that

$$\Delta Q_k^{\delta} > 0 \qquad \text{on } Q_k^{\delta} > 0.$$

Hence the touching point lies on the free boundary $\partial \Omega_{u_k}^+$. Furthermore by (A.2)

$$\begin{aligned}\nabla Q_{\varepsilon}^{\delta}(x_k^{\delta}) &= \alpha \boldsymbol{e_d} + \alpha \varepsilon \nabla P_+(\boldsymbol{Q}_{\varepsilon_k}(x_{\delta}^k)) + O(\varepsilon^2)\\ &= \alpha \boldsymbol{e_d} + \alpha \varepsilon \nabla P_+(x_0) + \varepsilon_k O(|x_{\delta}^k - x_0|) + O(\varepsilon^2).\end{aligned}$$

Choosing a sequence $\delta_k \to 0$ we obtain the desired conclusion.

• *Step 3*: We now prove item (iii). The proof goes exactly as above, more precisely we let P be as in the statement and we define $P^{\pm}$ as P restricted to $B_{1/2}^{\pm}$. We let also $\boldsymbol{T}^{\pm} : B_{1/2}^{\pm}$ be the corresponding transformations as in Step 1(with $\boldsymbol{T}^-$ defined in the obvious way on $B_{1/2}^-$). The key point is to note that

$$\boldsymbol{T}^+(B_{1/2}^+) \cap \boldsymbol{T}^-(B_{1/2}^-) = \emptyset.$$

Hence, with obvious notation, the function[4]

$$Q = Q^+ + Q^-$$

is a well defined comparison function. Arguing as in Step 2 gives the desired sequence. □

Appendix B. Proof of Lemma 3.12

Give a solution v we define w

$$w_\pm(x) = v_\pm(x) - \frac{\ell}{\lambda_\pm^2} x_d, \qquad x \in B^\pm_{1/2}.$$

It is straightforward to check it is a viscosity solution of

$$\begin{cases} \Delta w_\pm = 0 & \text{in } B^\pm_{1/2}, \\ \partial_d w_\pm \geq 0 & \text{in } B_{1/2} \cap \{x_d = 0\}, \\ \partial_d w_\pm = 0 & \text{in } \mathcal{J}, \\ \lambda_+^2 \partial_d w_+ = \lambda_-^2 \partial_d w_- & \text{in } \mathcal{C}, \\ w_+ \leq w_- & \text{in } B_{1/2} \cap \{x_d = 0\}. \end{cases}$$

Furthermore one can easily check that

$$w_\pm(x', x_d) = \frac{1}{\lambda_\pm^2} w_N(x', \mp x_d) - w_S(x', \mp x_d)\,,$$

where w_N solves the Neumann problem

$$\begin{cases} \Delta w_N = 0 & \text{on } B^-_{1/2}, \\ \partial_d w_N = 0 & \text{on } B^-_{1/2} \cap \{x_d = 0\}, \end{cases}$$

and w_S is a solution of the thin obstacle problem

$$\begin{cases} \Delta w_S = 0 & \text{on } B_{1/2^-}, \\ w_S \geq 0 & \text{on } B^-_{1/2} \cap \{x_d = 0\}, \\ \partial_d w_S \geq 0 & \text{on } B^-_{1/2} \cap \{x_d = 0\}, \\ w_S\, \partial_d w_S = 0 & \text{on } B^-_{1/2} \cap \{x_d = 0\}. \end{cases}$$

---

[4] Note that if Q^- is the d-th component of the inverse of $\boldsymbol{T}^-$ then it is *negative*!

Clearly $w_N \in C^\infty(\overline{B^+_{1/4}})$ with

$$\|w_N\|_{C^k(B_{1/4})} \le C_k \|w_N\|_{L^\infty(B_{1/2})}.$$

On the other hand, by [5, Corollary pg. 58], $w_S \in C^{1,1/2}(\overline{B^+_{1/4}})$ with

$$\|w_S\|_{C^{1,1/2}(B_{1/4})} \le C \|w_S\|_{L^\infty(B_{1/2})}.$$

From the last two estimates and the definition of w it is easy to deduce the conclusion of the Lemma. □

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