The Gallager Bound in Fiber Optical MIMO

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***Abstract*—The Multiple Input Multiple Output (MIMO) technique in fiber optical networks is a promising technology for up scaling networks' capabilities. Therefore, effective bounds on the error probability of finite length codewords are increasingly important. In this paper, we use random matrix techniques to obtain an analytic result for the Gallager bound error exponent for the fiber optical MIMO channel in the limit that the size of the codeword increases to infinity at a fixed ratio with the transmitter array dimensions. We assume zero backscattering inside the fiber which makes the transmission coefficients between the modes, elements of a unitary matrix. Moreover, the channel can be modelled a a random Haar unitary matrix between N transmitting and K receiving modes respectively, due to the scattering between modes.**

***Index Terms*—Error bound, Gallager, Large Deviation, MIMO, Fiber Optical Communication**

I. Introduction

It is almost a common practice nowadays accessing the internet from our handheld devices. In fact, more people are constantly connected than ever before. Moreover, new services such as video on demand or online gaming, and the upcoming technologies of internet of things (IoT) [1] and the wireless 5G [2], are adding a significant burden to the throughput demand that the telecommunication networks struggle to meet. Thus, a capacity crunch on the telecommunication networks, by the year 2020 is eminent [3].

In order to avoid this event the backbone telecommunication networks must increase their capacity capabilities fast and cost efficiently. A candidate technology for this upgrade is the multiple–input–multiple–output (MIMO) [4] which is already a reality in the wireless domain and recently it was proven as a liable option for fiber optical communications [5], [6]. For the evaluation and the quantification of the performance of such fiber optical systems, the error probability can be proposed as a metric.

Gallager [7], proposed a simple bound to the probability of error as a function of rate and codeword length T. More recently, Gallager's idea, which addressed only single links, was expanded to include also MIMO systems [8]–[10]. Therefore, it is logical, since MIMO fiber optical systems are becoming a reality, to investigate their performance using the well–known and well–trusted Gallager bound. The main drawback in the above methods, is that the expression of the Gallager bound is cumbersome to be analyzed. Therefore it makes sense to take the asymptotic limit for large length T of codewords and numbers N and K of transmitting and receiving modes respectively.

In this paper we evaluate the error probability exponent of the Gallager bound for large T and large numbers N and K and fixed ratios, using random matrix theory. That way, the calculation becomes simpler but the outcome is still valid: we invoke the large deviations theory to examine the tails of the Gallager bound which correspond to regions with low outage probability which is the region where the fiber optical networks operate. Our approach was first introduced in the context of random matrix theory by Dyson [11] and recently in [12] and [13] in the context of information theory and communications.

A. Outline

In the next section we formulate the problem and show that the appropriate channel matrix is a random Haar unitary matrix, while in Section III we present the main results. In Section IV we discuss our findings and in Section V we conclude.

B. Notations

We use upper case letters in bold font to denote matrices, e.g., $\mathbf{X}$, with entries given by X_{ab} and lower case letter in bold to denote vectors. The superscript † denotes the hermitian transpose operation and $\mathbf{I}_N$ represents the N-dimensional identity matrix.

II. Problem Formulation

A. System Model

In this paper we consider a single-segment N_{tot}-channel lossless optical fiber system, with $N \leq N_{tot}$ transmitting channels excited and $K \leq N_{tot}$ receiving channels coherently excited in the input (left) and output (right) side of the fiber. The propagation through the fiber may be analyzed through its $2N_{tot} \times 2N_{tot}$ scattering matrix given by [6], [14]

$$\mathbf{S} = \begin{bmatrix} \mathbf{r}_t & \mathbf{t} \\ \mathbf{t}^T & \mathbf{r}_r \end{bmatrix} \tag{1}$$

Due to time-reversal symmetry, $\mathbf{S}$ is a complex unitary *symmetric* matrix (with $\mathbf{S} = \mathbf{S}^T$) [15]. The $N_{tot} \times N_{tot}$ blocks on the diagonal $\mathbf{r}_t$, $\mathbf{r}_r$ represent the reflection matrices of the left and right ingoing channels respectively. The unitary aspects of the transmission channel have been introduced by [14], in a somewhat ad-hoc fashion, the so called Jacobi MIMO channel. There, the channel corresponding matrix is rectangular submatrix from a Haar distributed random matrix from $U(N_{tot})$.

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In our case we are interested only in transmission from the left to right. We also assume no backscattering, i.e. $\mathbf{r}_t = \mathbf{r}_r = 0$ and strong scattering between right-moving channels. Therefore we may model $\mathbf{t}$ as a complex Haar-distributed matrix with $\mathbf{t}^\dagger\mathbf{t} = \mathbf{t}\mathbf{t}^\dagger = \mathbf{I}_{N_{tot}}$. Particularly, we are interested in a segment of this matrix corresponding to the N columns and the K rows, which are coupled to the transceiver. We denote this $K \times N$ matrix $\mathbf{U}$ and without loss of generality we assume it is the upper left corner of $\mathbf{t}$. The behaviour in the large deviation regime of such a truncated Haar matrix has been studied in [16]. It should be noted here that the remaining $N_{tot} - \max(K, N)$ "untapped" channels in $\mathbf{t}$ can be used to model the loss in the fiber propagation [17], since in the limit of large $N_{tot} \gg N, K$ the channel converges to a Gaussian distributed channel where the signal loss is significant.

So, the corresponding MIMO channel for this system reads

$$\mathbf{y} = \mathbf{U}\mathbf{x} + \mathbf{z} \tag{2}$$

with coherent detection and channel state information only at the receiver [18], [19]. $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are the $N \times 1$ input, the $K \times 1$ output signal vectors and the $K \times 1$ unit variance noise vector, respectively, all assumed for simplicity to be complex Gaussian. We also assume no differential delays between channels, which effectively leads to frequency flat fading [6]. We also assume no mode-dependent loss. As a result, the mutual information can be expressed as

$$C = \log\det\left(\mathbf{I}_K + \rho\, \mathbf{U}\mathbf{U}^\dagger\right), \tag{3}$$

where ρ is the SNR. The total transmission rate is $R_{erg} = N r_{erg}$ where r_{erg} is the ergodic rate per transmitter. The value of the mutual information per transmitter $C(\rho, \mathbf{U})/N$ converges weakly to a deterministic value in the large N limit: the ergodic average of the mutual information [20]. In addition it is important to note that the empirical eigenvalue density of $\mathbf{U}\mathbf{U}^\dagger$ converges weakly, almost surely so, to the well-known Marčenko-Pastur distribution [21]

$$p_0(x) = \begin{cases} \frac{\sqrt{(b_0-x)(x-a_0)}}{2\pi x}, & \text{for } x \in [a_0, b_0] \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

where $a_0, b_0 = (\sqrt{\beta} \pm 1)^2$ are the endpoints of its support.

B. Gallager Exponent

In the infinite codeword limit, the effect of the channel fading is captured through the optimal outage error probability [22], which in the large N, K limit has been analyzed in [12]. On the other hand, for finite codewords, one can use the Gallager bound: For *Maximum Likelihood* (ML) decoding for a discrete memoryless, fixed channel without feedback the error probability $\mathbb{P}(\mathcal{E})$, is bounded by

$$\begin{aligned} \mathbb{P}(\mathcal{E}|\mathbf{U}) &\le e^{-N^2 E(R|\mathbf{U})}, \\ E(R|\mathbf{U}) &= \frac{1}{N}\max_{\kappa\in[0,1]}\left\{E_0[\kappa\,|\mathbf{U}] - \kappa\, r\right\} \end{aligned} \tag{5}$$

where $E_0(\kappa\,|\mathbf{U})$ is Gallager's error exponent defined as

$$\begin{aligned} &E_0(\kappa\,|\mathbf{U}) = \\ &\log\int d\mathbf{Y}\left[\int d\mathbf{X}\ \mu(\mathbf{X})\left[\mu(\mathbf{Y}|\mathbf{X},\mathbf{U})\right]^{\frac{1}{1+\kappa}}\right]^{1+\kappa} \\ &= \log\det\left(1 + \frac{\rho}{1+\kappa}\mathbf{U}\mathbf{U}^\dagger\right), \end{aligned} \tag{6}$$

where the last line follows [23], for independent Gaussian input. So, plugging (6) in (5) we have

$$\begin{aligned} &E(R|\mathbf{U}) = \\ &\frac{1}{N}\max_{\kappa\in[0,1]}\left\{\alpha\,\kappa\left[\log\det\left(1 + \frac{\rho}{1+\kappa}\mathbf{U}\mathbf{U}^\dagger\right) - Nr\right]\right\}, \end{aligned} \tag{7}$$

where $\alpha = \frac{T}{N}$, r is the rate. $\mathbb{P}(\mathcal{E})$ is error rate when we decode the message, therefore

$$\begin{aligned} \mathbb{P}(\mathcal{E}) = \mathbb{E}_{\mathbf{U}}\left[\mathbb{P}\left(\mathcal{E}|\mathbf{U}\right)\right] &\le \mathbb{E}_{\mathbf{U}}\left(e^{-N^2 E(r|\mathbf{U})}\right) \\ &\equiv e^{-N^2 E_N(r)}. \end{aligned} \tag{8}$$

In this paper, we calculate the closed form expression of the error exponent $E_N(r)$ as $N, K, T \to \infty$ while $\beta = \frac{K}{N} > 1$ is kept constant. We further define $N_0 = N_{tot} - N - K$ and $n_0 = \frac{N_0}{N}$ the losses inside the fiber. In the case of $N_0 < 0$, [14] showed that we may recover the form of β, N_0 and n_0 by substituting $N \to N_{tot} - K$, $K \to N_{tot} - N$ and $N_{tot} \to -N_{tot}$. Then, the mutual information becomes also $C \to C + n_0 \log(1+\rho)$.

The joint distribution of eigenvalues of $\mathbf{U}\mathbf{U}^\dagger$ is

$$\begin{aligned} P_{\boldsymbol{\lambda}}(\lambda_1 \ldots \lambda_N) = \frac{1}{\mathcal{Z}_N}\prod_{N<i<j\le K}|\lambda_i - \lambda_j|^2 \times \\ \times \prod_i \lambda_i^{K-N}(1-\lambda_i)^{N_0}, \end{aligned} \tag{9}$$

where $\mathcal{Z}_N$ is the normalizing constant. We can assume that when N_{tot} is large, then the eigenvalues will coalesce to a smooth density $p(x)$, which will be such that the energy $\lim_{N\to\infty} E_N(r) = E(r)$ will be minimum and (9) corresponds to the most probable eigenvalue distribution. In this limit, we can write the minimum energy of the eigenvalues as

$$E_0(r) = \left\{\sup_c \inf_p \mathcal{L}_0[c, p]\right\} \tag{10}$$

$$\begin{aligned} \mathcal{L}_0 = \max_{\kappa\in[0,1]}\Bigg\{ &- n_0\int p(x)\log(1-x)dx \\ &-(\beta-1)\int p(x)\log(x)dx \\ &-\int\int p(x)p(y)\log|x-y|dydx\Bigg\} \\ &-c\left(\int p(x)dx - 1\right), \end{aligned} \tag{11}$$

where we have added a Lagrange multiplier c to ensure that $p(x)$ is properly normalized, while implicitly assuming that $p(x)$ is continuous in $x \in (0,1)$. In order to incorporate the constraint on the rate r, we introduce another Lagrange multiplier so that

$$E(r) = \sup_c \inf_p \mathcal{L}_1[c,\kappa,p] \quad (12)$$

$$\mathcal{L}_1[p,c,\kappa] = \mathcal{L}_0[p,c] + \alpha\,\kappa \left(\int \log\left(1+\frac{\rho x}{1+\kappa}\right) p(x)dx - r\right). \quad (13)$$

It is implied in (12) that first we maximize with respect to κ and then search for the infimum in p. But this is not an easy task because the maximization will depend on the specific distribution of $p(x)$. Hence, since it can be shown that $\mathcal{L}_1$ is continuous and convex, it is very useful to point out the Minimax theorem ([24]-Theorem 2) which allows in our case, to exchange places between the $\max - \inf$. Therefore, we can re–write

$$E(r) = \max_{\kappa\in[0,1]} \left\{\sup_c \inf_p \mathcal{L}_1[c,p]\right\} \quad (14)$$

and continue with our calculations. It is easy to show that the minimum of the above function is unique since we can reach $E_0(r)$ from $E(r)$ by maximizing over $\mathcal{L}_1$, while keeping $\kappa = 0$. The minimization of $\mathcal{L}_1$ with respect to p is done by taking the functional derivative and setting it to zero. Because of space constraints, the calculations will be omitted. It can be shown also that the minimum of $\mathcal{L}_1$ is unique [12]. So, taking the functional derivative with respect to $p(x)$ and setting to zero and differentiating with respect to x, gives us the following integral equation

$$2PV\int \frac{p(x)}{x-y}dx = \frac{n_o}{1-x} - \frac{\beta-1}{x} + \frac{\alpha\,\kappa\,\rho}{1+\kappa+\rho x} \quad (15)$$

which gives

$$\begin{aligned} p(x) =& \frac{1}{2\pi\sqrt{(x-a)(x-b)}} \\ &\times\left[\frac{n_0\sqrt{(1-a)(1-b)}}{1-x} - (\beta-1)\frac{\sqrt{ab}}{x}\right. \\ &\left. + \frac{\alpha\,\kappa\,\sqrt{(z+a)(z+b)}}{x+z} + C'\right] \end{aligned} \quad (16)$$

We search for solution among continuous, non–negative, normalized functions over $x \in (0,\infty)$. Continuity at $x=b$ results to the constraint that the expression in the square bracket vanishes at $x=b$:

$$\begin{aligned} p(x) =& \frac{\sqrt{b-x}}{2\pi\sqrt{x-a}}\left[n_0\sqrt{\frac{1-a}{1-b}}\right. \\ &\left. - \frac{(\beta-1)\sqrt{a}}{x\sqrt{b}} + \frac{\alpha\,\kappa\,\sqrt{z+a}}{(x+z)\sqrt{z+b}}\right]. \end{aligned} \quad (17)$$

The a is obtained through the continuity condition $p(a) = 0$:

$$\frac{n_0}{\sqrt{(1-a)(1-b)}} = \frac{\beta-1}{\sqrt{ab}} - \frac{\alpha\,\kappa}{\sqrt{(z+a)(z+b)}}, \quad (18)$$

and back to $p(x)$ we have

$$p(x) = \frac{(x-a)(b-x)}{2\pi(1-x)}\left[\frac{\beta-1}{x\sqrt{ab}} - \frac{\alpha\,\kappa}{\sqrt{(z+a)(z+b)}}\frac{1+z}{x+z}\right]. \quad (19)$$

The b will be evaluated through the normalization condition of $p(x)$,

$$\begin{aligned} &\frac{\beta-1}{2\sqrt{ab}}\left(1-\sqrt{ab}-\sqrt{(1-a)(1-b)}\right) \\ &-\frac{\alpha\,\kappa}{2\sqrt{(z+a)(z+b)}}\left[z+1-\sqrt{(1-a)(1-b)}\right. \\ &\left.-\sqrt{(z+a)(z+b)}\right] = 1. \end{aligned} \quad (20)$$

The value of κ will be determined by the saddle point equation,

$$r = \int_a^b p(x)\left[\log\left(1+\frac{x}{z}\right) - \frac{\kappa}{1+\kappa}\frac{x}{x+z}\right]dx. \quad (21)$$

Therefore, we integrate (21) to obtain

$$\begin{aligned} r =& \log\frac{\Delta\rho}{1+\kappa} - \frac{\Delta\alpha\,\kappa}{2\sqrt{(z+a)(z+b)}}G\left(\frac{z+a}{\Delta},\frac{z+b}{\Delta}\right) \\ &+\frac{\Delta}{2}\left(1+\frac{\alpha\,\kappa}{\sqrt{(z+a)(z+b)}}\right)G\left(\frac{z+a}{\Delta},\frac{a}{\Delta}\right) \\ &+\frac{\kappa(\beta-1)}{(1+\kappa)(z+1)\sqrt{ab}}\times \\ &\times\left(1-\sqrt{(1-a)(1-b)}-\sqrt{(z+a)(z+b)}\right) \\ &-\frac{\alpha\,\kappa^2(z+1)}{(1+\kappa)\sqrt{(z+a)(z+b)}}\left[\vphantom{\frac{1}{1}}\right. \\ &\times\frac{z+1}{2z^2+1}\left[\frac{b+a}{2}+2z-\sqrt{\frac{z+a}{z+b}}\left(\frac{b-a}{2}+2z+b\right)\right. \\ &\left.+\frac{\Delta a}{2\sqrt{(z+a)(z+b)}}\right] \\ &+\frac{1}{2z^2+1}\left[z\sqrt{(z+a)(z+b)}-z^2-\frac{\Delta}{2}(a(a+b)-3a+1)\right. \\ &\left.\left.-(1-a)^2-(1-a+a\Delta)\sqrt{(1-a)(1-b)}\right]\right], \end{aligned} \quad (22)$$

where $\Delta = b-a$, $z = \frac{1+\kappa}{\rho}$ and

$$\begin{aligned} G(x,y) =& \frac{1}{\pi}\int_0^1 \sqrt{t(1-t)}\frac{\log(t+x)}{t+y}\,dt \\ =& -2\sqrt{y(1+y)}\log\left[\frac{\sqrt{x(1+y)}+\sqrt{y(1+x)}}{\sqrt{1+y}+\sqrt{y}}\right] \\ &+(1+2y)\log\left[\frac{\sqrt{1+x}+\sqrt{x}}{2}\right] - \frac{1}{2}\left(\sqrt{1+x}-\sqrt{x}\right)^2. \end{aligned} \quad (23)$$

III. Results

Finally, we integrate over $p(x)$ for $\beta > 1$ and $n_0 > 0$:

$$\begin{aligned} E(r) =& -\kappa\,\alpha r \\ &+\frac{1}{2}\kappa\,\alpha\log\left(1+\frac{b}{z(1+\kappa)}\right) \\ &-\frac{1}{2}(\beta+1+n_0)\log(b-a) - \frac{\beta-1}{2}\log(b) \\ &-\frac{n_0}{2}\log(1-b) - \frac{n_0^2(b-a)}{4\sqrt{(1-a)(1-b}} \end{aligned}$$

$$\times\left[G\left(\frac{1-b}{b-a},\frac{1-b}{b-a}\right)-G\left(\frac{1-b}{b-a},\frac{z+b}{b-a}\right)\right]$$
$$+\frac{n_0(\beta-1)(b-a)}{4\sqrt{ab}}$$
$$\times\left[G\left(\frac{1-b}{b-a},-\frac{b}{b-a}\right)-G\left(\frac{1-b}{b-a},-\frac{z+b}{b-a}\right)\right]$$
$$+\frac{n_0(\beta-1)(b-a)}{4\sqrt{(1-a)(1-b)}}$$
$$\times\left[G\left(\frac{a}{b-a},-\frac{1-a}{b-a}\right)-G\left(\frac{a}{b-a},\frac{z+a}{b-a}\right)\right]$$
$$-\frac{(\beta-1)^2(b-a)}{4\sqrt{ab}}$$
$$\times\left[G\left(\frac{a}{b-a},\frac{a}{b-a}\right)-G\left(\frac{a}{b-a},\frac{z+a}{b-a}\right)\right]$$
$$-\frac{n_0(b-a)}{2\sqrt{(1-a)(1-b)}}$$
$$\times\left[G\left(0,\frac{1-b}{b-a}\right)-G\left(0,-\frac{z+b}{b-a}\right)\right]$$
$$+\frac{(\beta-1)(b-a)}{2\sqrt{ab}}$$
$$\times\left[G\left(0,-\frac{b}{b-a}\right)-G\left(0,-\frac{z+b}{b-a}\right)\right]$$
$$+\frac{\kappa\,\alpha}{2}\log\left(\frac{b-a}{z(1+\kappa)}\right)+\frac{n_0(b-a)}{2\sqrt{(1-a)(1-b)}}$$
$$\times\left[G\left(\frac{z+a}{b-a},\frac{z+a}{b-a}\right)-G\left(\frac{z+a}{b-a},-\frac{1-a}{b-a}\right)\right]$$
$$+\frac{(\beta-1)(b-a)}{2\sqrt{ab}}$$
$$\times\left[G\left(\frac{z+a}{b-a},\frac{a}{b-a}\right)-G\left(\frac{z+a}{b-a},\frac{z+a}{b-a}\right)\right]$$
$$-\frac{1}{2}\Big[(\beta+n_0+1)^2\log(\beta+n_0+1)$$
$$-(\beta+n_0)^2\log(\beta+n_0)-\beta^2\log(\beta)$$
$$+(\beta-1)^2\log(\beta-1)-(n_0+1)^2\log(n_0+1)$$
$$+n_0^2\log(n_0)\Big]. \tag{24}$$

Special Cases

We distinguish two different cases: while it is $\beta=\frac{K}{N}>1$ the losses inside the fiber can be either $n_0>0$ or $n_0=0$.

Therefore, for $n_0=0$ the density of eigenvalues becomes

$$p(x)=\frac{1}{2\pi\sqrt{(x-a)(x-b)}}\times\left[-\frac{(\beta-1)\sqrt{ab}}{x}+\frac{\alpha\,\kappa\,\sqrt{(z+a)(z+b)}}{x+z}+C\right]. \tag{25}$$

A. $0\le\kappa\le1$ and $\beta>1\to\alpha>0$.

From the constraint $p(a)=0$ we have

$$p(x)=\frac{\sqrt{x-a}}{2\pi\sqrt{b-x}}\left[\frac{\beta-1}{x}\sqrt{\frac{b}{a}}-\frac{\alpha\,\kappa\,\sqrt{z+b}}{\sqrt{z+a}}\frac{1}{x+z}\right]. \tag{26}$$

There are two options $b=1$ and $b<1$.

1) $b=1$: In the first case of $b=1$, which is stable close to $\kappa=0$, we have the additional constraint that the first term cannot appear since it will lead to non-integrable singularity. Hence,

$$p_1(x)=\frac{\sqrt{x-a}}{2\pi\sqrt{1-x}}\left(\frac{\beta-1}{x\sqrt{a}}-\frac{\alpha\,\kappa\,\sqrt{z+1}}{(z+x)\sqrt{z+a}}\right), \tag{27}$$

and the normalization conditions gives

$$\beta+1-\kappa\,\alpha=\frac{\beta-1}{\sqrt{a}}-\kappa\,\alpha\frac{\sqrt{z+1}}{z+a}. \tag{28}$$

To obtain κ we need to solve the fixed point equation for $r(\kappa)$:

$$r_1(\kappa)=\int_a^1 p_1(x)\log\left(1+\frac{x}{z}\right)dx-\frac{\kappa}{1+\kappa}\int_a^1 p_1(x)\frac{x}{x+z}dx. \tag{29}$$

Sparing with the calculations, we have

$$r_1=\log\left(\frac{1-a}{z}\right)-\frac{\alpha\,\kappa(1-a)}{2\sqrt{(z+1)(z+a)}}$$
$$\times\left[I_3\left(\frac{z+a}{1-a}\right)+G\left(\frac{z+a}{1-a},\frac{z+a}{1-a}\right)\right]$$
$$+\frac{(\beta-1)(1-a)}{2\sqrt{a}}\left[I_3\left(\frac{z+a}{1-a}\right)+G\left(\frac{z+a}{1-a},\frac{a}{1-a}\right)\right]$$
$$-\frac{\kappa}{1+\kappa}\Bigg[1-\frac{z(1-a)}{4(1+z)^2}\left(\frac{(\beta-1)z}{\sqrt{a}}+\beta+1-\alpha\,\kappa\right)$$
$$-\frac{(\beta-1)}{4\sqrt{a}}(1-\sqrt{a})^2$$
$$+\frac{(\sqrt{z+1}-\sqrt{z+a})^2}{4(z+1)^2}$$
$$\times\left(\frac{(\beta-1)(z+a)}{\sqrt{a}}+\frac{\alpha\,\kappa\,\sqrt{z+1}}{\sqrt{z+a}}+\frac{\alpha\,\kappa\,z+1}{z+a}\right)\Bigg], \tag{30}$$

where

$$I_3(x)=-G(x,-1). \tag{31}$$

From here, following the method we saw in previous section, the calculation of E is straight forward:

$$E(r_1)=\frac{\kappa\,\alpha}{2}\log\left(1+\frac{1}{z(1+\kappa)}\right)-\kappa\,\alpha r_1$$
$$-\frac{\beta-1}{2}\log(a)-\frac{\beta+1}{2}\log(1-a)$$
$$-\frac{(\beta-1)^2(1-a)}{4\sqrt{a}}$$
$$\times\left[G\left(\frac{a}{1-a},\frac{a}{1-a}\right)-G\left(\frac{a}{1-a},-1\right)\right]$$
$$+\frac{\alpha\,\kappa(\beta-1)(1-a)}{4\sqrt{(z(\kappa+1)+1)(z(\kappa+1)+a)}}$$
$$\times\left[G\left(\frac{a}{1-a},\frac{z(\kappa+1)+a}{1-a}\right)-G\left(\frac{a}{1-a},-1\right)\right]$$
$$-\frac{(\beta-1)(1-a)}{2\sqrt{a}}\left[G\left(0,\frac{a}{1-a}\right)-G\left(0,-1\right)\right]$$
$$+\frac{\kappa\,\alpha(1-a)}{2\sqrt{(z(\kappa+1)+1)(z(\kappa+1)+a)}}$$

$$\times\left[G\left(0,\frac{z(\kappa+1)+a}{1-a}\right)-G(0,-1)\right]$$
$$+\frac{\kappa\,\alpha}{2}\log\left(\frac{1-a}{z(\kappa+1)}\right)$$
$$-\frac{\kappa\,\alpha(1-a)}{2\sqrt{(z(\kappa+1)+1)(z(\kappa+1)+a)}}$$
$$\times\left[G\left(\frac{z(\kappa+1)+a}{1-a},\frac{z(\kappa+1)+a}{1-a}\right)\right.$$
$$\left.-G\left(\frac{z(\kappa+1)+a}{1-a},-1\right)\right]$$
$$+\frac{(\beta-1)(1-a)}{\sqrt{a}}$$
$$\times\left[G\left(\frac{z(\kappa+1)+a}{1-a},\frac{a}{1-a}\right)\right.$$
$$\left.-G\left(\frac{z(\kappa+1)+a}{1-a},-1\right)\right]$$
$$-\frac{1}{2}\left[(\beta+1)^2\log(\beta+1)\right.$$
$$\left.-2\beta^2\log(\beta)+(\beta-1)^2\log(\beta-1)\right]. \tag{32}$$

The above expressions are valid for when $p_1(x)>0$ for all $a<x<1$. When $p_1(x)<0$, it breaks down and the upper limit becomes $b<1$. The stability condition for that, is

$$\frac{\beta-1}{\sqrt{a}}-\frac{\alpha\,\kappa}{\sqrt{(z_k+1)(z_k+a)}}, \tag{33}$$

where $z_k=z(\kappa+1)$. As α grows, this will not be valid at $\kappa=1$ and hence, there will be a breakdown. The critical value of $\kappa=\kappa_c$ is the solution of

$$\kappa_c\,\alpha=\beta+1+\frac{\beta-1}{\sqrt{a_c}}z_k, \tag{34}$$

where $a_c=\left(\frac{(\beta-1)z_k}{\kappa\,\alpha-\beta-1}\right)^2$. Because $a<1$ we also have

$$\kappa>\frac{(\beta+1)(z+1)}{\alpha-z(\beta+1)}, \tag{35}$$

and we finally reach

$$\kappa_c=\frac{\alpha(z+1)(\beta+1)-z(\beta^2+1)-z^2(\beta+1)^2}{(\alpha-z(\beta+1)^2+4\beta z}$$
$$+\frac{2\sqrt{\beta z}\sqrt{z(\alpha+\beta)(\alpha+1)+\alpha(\alpha+1+\beta)}}{(\alpha-z(\beta+1)^2+4\beta z}. \tag{36}$$

So, we calculate the $E(r)$ for the two different sub–cases

For $\kappa\le\kappa_c<1$:

$$E_{\kappa\le\kappa_c}(r_1)=\frac{\kappa\,\alpha}{2}\log\left(1+\frac{1}{z(1+\kappa)}\right)-\kappa\,\alpha r_1$$
$$-\frac{\beta-1}{2}\log(a)-\frac{\beta+1}{2}\log(1-a)$$
$$-\frac{(\beta-1)^2(1-a)}{4\sqrt{a}}$$
$$\times\left[G\left(\frac{a}{1-a},\frac{a}{1-a}\right)-G\left(\frac{a}{1-a},-1\right)\right]$$
$$+\frac{\alpha\,\kappa(\beta-1)(1-a)}{4\sqrt{(z(\kappa+1)+1)(z(\kappa+1)+a)}}$$
$$\times\left[G\left(\frac{a}{1-a},\frac{z(\kappa+1)+a}{1-a}\right)-G\left(\frac{a}{1-a},-1\right)\right]$$
$$-\frac{(\beta-1)(1-a)}{2\sqrt{a}}\left[G\left(0,\frac{a}{1-a}\right)-G(0,-1)\right]$$
$$+\frac{\kappa\,\alpha(1-a)}{2\sqrt{(z(\kappa+1)+1)(z(\kappa+1)+a)}}$$
$$\times\left[G\left(0,\frac{z(\kappa+1)+a}{1-a}\right)-G(0,-1)\right]$$
$$+\frac{\kappa\,\alpha}{2}\log\left(\frac{1-a}{z(\kappa+1)}\right)$$
$$-\frac{\kappa\,\alpha(1-a)}{2\sqrt{(z(\kappa+1)+1)(z(\kappa+1)+a)}}$$
$$\times\left[G\left(\frac{z(\kappa+1)+a}{1-a},\frac{z(\kappa+1)+a}{1-a}\right)\right.$$
$$\left.-G\left(\frac{z(\kappa+1)+a}{1-a},-1\right)\right]$$
$$+\frac{(\beta-1)(1-a)}{\sqrt{a}}\left[G\left(\frac{z(\kappa+1)+a}{1-a},\frac{a}{1-a}\right)\right.$$
$$\left.-G\left(\frac{z(\kappa+1)+a}{1-a},-1\right)\right]$$
$$-\frac{1}{2}\left[(\beta+1)^2\log(\beta+1)\right.$$
$$\left.-2\beta^2\log(\beta)+(\beta-1)^2\log(\beta-1)\right]. \tag{37}$$

For $\kappa\ge\kappa_c<1$:

$$E_{\kappa\ge\kappa_c}(r_1)=\frac{\kappa\,\alpha}{2}\log\left(1+\frac{a'}{b'}\right)-\frac{(\beta-1)}{2}\log(a')$$
$$-\frac{\beta+1}{2}\log(b'-a')-\frac{(\beta-1)^2(b'-a')}{4}$$
$$\times\left[G\left(\frac{a'}{b'-a'},\frac{a'}{b'-a'}\right)\right.$$
$$\left.-G\left(\frac{a'}{b'-a'},\frac{z(1+\kappa)+a'}{b'-a'}\right)\right]$$
$$-\frac{(\beta-1)(b'-a')}{\sqrt{a'b'}}$$
$$\times\left[G\left(0,\frac{a'}{b'-a'}\right)-G\left(0,\frac{z(1+\kappa)+a'}{b'-a'}\right)\right]$$
$$+\frac{\kappa\,\alpha}{2}\left[\log\left(\frac{b'-a'}{z(1+k)}\right)\right.$$
$$+\frac{(\beta-1)(b'-a')}{2}\log(a'b')$$
$$\times\left[G\left(\frac{z(1+\kappa)+a'}{b'-a'},\frac{a'}{b'-a'}\right)\right.$$
$$\left.\left.-G\left(\frac{z(1+\kappa)+a'}{b'-a'},\frac{z(1+\kappa)+a'}{b'-a'}\right)\right]\right]$$
$$-\kappa\,\alpha r_1-\frac{1}{2}\left[(\beta+1)^2\log(\beta+1)\right.$$
$$\left.-2\beta^2\log(\beta)+(\beta-1)^2\log(\beta-1)\right]. \tag{38}$$

where

$$a' = z(\kappa+1)\left(\frac{\sqrt{\beta(\kappa\,\alpha-\beta)}-\sqrt{\kappa\,\alpha-1}}{\kappa\,\alpha-1-\beta}\right)^2 \tag{39}$$

and

$$b' = z(\kappa+1)\left(\frac{\sqrt{\beta(\kappa\,\alpha-\beta)}+\sqrt{\kappa\,\alpha-1}}{\kappa\,\alpha-1-\beta}\right)^2 \tag{40}$$

B. $\kappa > 1$

Finally, we analyse the case where $\kappa > \kappa_c$, so that the support of $p(x)$ does not extend to 1:

$$p_2(x) = \frac{\sqrt{(x-a)(x-b)}}{2\pi}\frac{(\beta-1)z}{x\sqrt{ab}(x+z)}, \tag{41}$$

with the constraints

$$\frac{\beta-1}{\sqrt{ab}} = \frac{\kappa\,\alpha}{\sqrt{(z+a)(z+b)}}, \tag{42}$$

and

$$\beta+1-\kappa\,\alpha = \frac{\beta-1}{\sqrt{ab}} - \frac{\kappa\,\alpha(z+1)}{\sqrt{(z+a)(z+b)}}. \tag{43}$$

So, we have

$$a = z\left(\frac{\sqrt{\beta\,\kappa\,\alpha-\beta^2}-\sqrt{\kappa\,\alpha-1})^2}{\kappa\,\alpha-\beta-1}\right), \tag{44}$$

and

$$b = z\left(\frac{\sqrt{\beta\,\kappa\,\alpha-\beta^2}+\sqrt{\kappa\,\alpha-1})^2}{\kappa\,\alpha-\beta-1}\right). \tag{45}$$

and finally,

$$\begin{aligned} r_2 =& \log\left(\frac{\Delta}{z}\right) + \frac{(\beta-1)\Delta}{2\sqrt{ab}} \\ &\times\left[G\left(\frac{z+a}{\Delta},\frac{a}{\Delta}\right) - G\left(\frac{z+a}{\Delta},\frac{z+a}{\Delta}\right)\right] \\ &-\frac{\kappa}{1+\kappa}\frac{(\beta-1)z}{\sqrt{ab}}\frac{(\sqrt{z+b}-\sqrt{z+a})^2}{4\sqrt{(z+a)(z+b)}}. \end{aligned} \tag{46}$$

The evaluation of $E(r_2)$ is then straight forward:

$$\begin{aligned} E(r_2) =& \frac{\kappa\,\alpha}{2}\log\left(1+\frac{a}{z(\kappa+1)}\right) \\ &-\frac{\beta-1}{2}\log(a) - \frac{\beta+1}{2}\log(1-a) \\ &-\frac{(\beta-1)^2(1-a)}{4\sqrt{a}}\left[G\left(\frac{a}{1-a},\frac{a}{1-a}\right)\right. \\ &\left.-G\left(\frac{a}{1-a},-1\right)\right] \\ &+\frac{\kappa\,\alpha(\beta-1)(1-a)}{4\sqrt{(z(\kappa+1)+1)(z(\kappa+1)+a)}} \\ &\times\left[G\left(\frac{a}{1-a},\frac{z(1+\kappa)+a}{1-a}\right) - G\left(\frac{a}{1-a},-1\right)\right] \\ &-\frac{(\beta-1)(1-a)}{\sqrt{a}}\left[G\left(0,\frac{a}{1-a}\right) - G\left(0,-1\right)\right] \\ &+\frac{\kappa\,\alpha(1-a)}{\sqrt{z(\kappa+1)+a}}\left[G\left(0,\frac{z(1+\kappa)+a}{1-a}\right) - G\left(0,-1\right)\right] \\ &+\frac{\kappa\,\alpha}{2}\left[\log\left(\frac{1-a}{z(1+\kappa)}\right)\right. \\ &-\frac{\kappa\,\alpha(1-a)}{2\sqrt{(z(\kappa+1)+1)(z(\kappa+1)+a)}} \\ &\times\left[G\left(\frac{z(\kappa+1)+a}{1-a},\frac{z(1+\kappa)+a}{1-a}\right)\right. \\ &\left.-G\left(\frac{z(\kappa+1)+a}{1-a},-1\right)\right] \\ &+\frac{(\beta-1)(1-a)}{\sqrt{a}}\left[G\left(\frac{z(1+\kappa)+a}{1-a},\frac{a}{1-a}\right)\right. \\ &\left.\left.-G\left(\frac{z(1+\kappa)+a}{1-a},-1\right)\right]\right] \\ &-\kappa\,\alpha r_2 - \frac{1}{2}\left[(\beta+1)^2\log(\beta+1)\right. \\ &\left.-2\beta^2\log(\beta) + (\beta-1)^2\log(\beta-1)\right]. \end{aligned} \tag{47}$$

C. $\kappa \approx 0$

Another issue is the behaviour close to $\kappa = 0$. We already have seen that $\frac{dE(r)}{dr} = \kappa(r)$. At the ergodic point it is $\kappa(r_{erg}) = 0$. Therefore, assuming that the first derivative exists, we can write

$$\kappa(r) = (r-r_{erg})\frac{d\,\kappa(r)}{dr}\bigg|_{r=r_{erg}} + o((r-r_{erg})), \tag{48}$$

and we define

$$\left(\frac{d\,\kappa}{dr}\right)^{-1}\bigg|_{\kappa=0} = v_{erg}\alpha. \tag{49}$$

Thus, by differentiating $r(\kappa)$ and expressing their values and the values of their derivatives at $\kappa = 0$ and $r = r_{erg}$, we have

$$E(r) = \frac{(r-r_{erg})^2}{2v_{erg}} + o\left((r-r_{erg})^2\right) \tag{50}$$

where

$$v_{erg} = v_{opt} + \frac{\delta v}{\alpha}, \tag{51}$$

where

$$v_{opt} = -\log\left[\frac{\left(\sqrt{1+z}+\sqrt{z+a}\right)^2}{4\sqrt{(z+1)(z+a)}}\right], \tag{52}$$

$$\delta v = -(\beta+1)\left(1-\sqrt{\frac{z+a}{z+1}}\right). \tag{53}$$

Strictly speaking, the Gaussian approximation is valid for values of r closer to r_{erg}, specifically for $r - r_{erg} = \mathcal{O}(\frac{1}{N})$, there we can neglect in (50) terms of order higher than 2.

IV. Analysis

We evaluated the exponent of the Gallager bound for $\beta > 1$ and the corresponding sub–cases. Values away from the optimal curve, means worst behavior. We can see that as we increase α we come closer to the optimal limit [13] which corresponds to infinite codelength ($\alpha \to \infty$). In Fig. 1 we can see the behaviour of the error exponent while there is some loss inside the fiber ($n_0 \neq 0$) and in Fig. 2 we can see the corresponding behaviour for zero loss ($n_0 = 0$). In most of the curves we identify a phase transition which is indicated with small, black circles. Since, $\kappa \in [0, 1]$, there are two major regions in the analysis of the error exponent $0 \leq \kappa \leq 1$ and $\kappa > 1$. For these two regions, we have respectively $r_{ph} < r \leq r_{erg}$ and $r < r_{ph}$, where r_{erg} is the ergodic r and $r_{ph} > r_{erg}$.

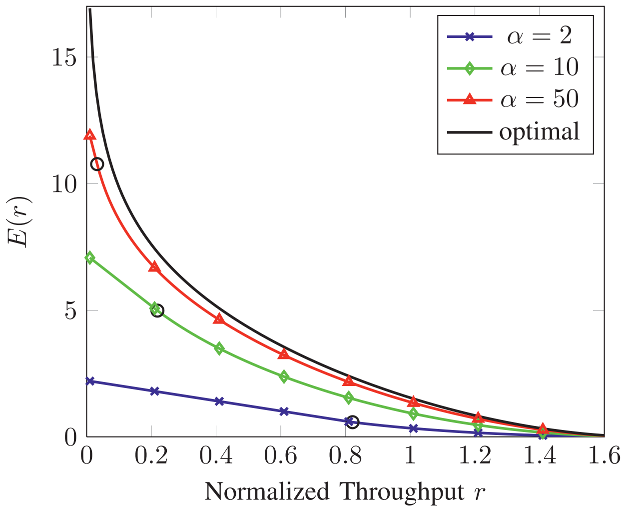

Fig. 1: The error exponent for the Gallager bound at the limit of $N \to \infty$ and for some loss inside the fiber, $n_0 \neq 0$. As we increase in α, we get closer to the optimum value; $\beta = 3$, $\rho = 10, n_0 = 2$. The small circles indicate the points of phase transition.

V. Conclusion

In this paper we used the large deviation method to calculate an analytic expression of the Gallager bound for low error rates in the fiber optical MIMO channel. Although, this method is very accurate in the limit of big $\alpha = \frac{T}{N}$, where T the codeword length, N the number of transmitting modes excited, it is also valid for smaller values of α. Moreover, we compare our method with the optimal curve which corresponds to $\alpha \to \infty$. Finally, as a future work the behavior of various error bounds like the Massey bound [25] can be investigated and compared with each other.

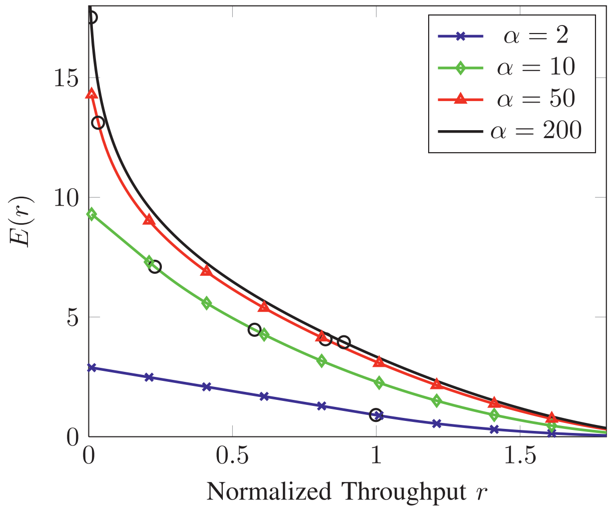

Fig. 2: The error exponent for the Gallager bound at the limit of $N \to \infty$ and for zero loss inside the fiber, $n_0 = 0$. As we increase in α, we get closer to the optimum value; $\beta = 3$, $\rho = 10, n_0 = 0$. The small circles indicate the points of phase transition.

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