Lectures on Differential Topology

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Preface

Over the years, I have taught several courses on differential topology in the master's degree program in mathematics at the University of Pisa. The class was usually attended by students who had accomplished (or were accomplishing) a first three years degree in mathematics, together with a few peer physicists and a few beginner Ph.D. students. Considering the initial knowledge of these students, time after time, a collection of different topics, in different combinations, as well as a certain way to present them, has emerged. This textbook summarizes such teaching experiences, therefore it presents itself more as "lecture notes" than as a complete and systematic treatise. Sometimes, in a class, a "short cut" to an interesting application is chosen over broader generality. Similarly, in this text we will focus, for example, on compact manifolds (especially when we consider the sources of smooth maps), allowing simplifications in dealing, for instance, with function spaces or with certain "globalization procedure" of maps. There are already a lot of interesting facts concerning compact manifolds, so we will do it without remorse.

There are several classical well-known references (such as [**M1**], [**GP**], [**H**], [**M2**], [**M3**], [**Mu**], . . .) which I used in preparing the courses and which have strongly influenced these pages. So, why another textbook on differential topology? An important motivation came to me from the students, looking at their notes and from their remark that "they had not been able to find some of the topics addressed in the course anywhere". It would be very hard to claim any 'originality' in dealing with such a classical matter. However, that remark, at least in reference to textbooks addressed mainly to undergraduate readers, has some truth to it. Let's make an example. A theme of this text (similarly, for example, to [**H**]) is the synergy between *bordism* and *transversality*. One of the limits imposed by the students' presumed initial knowledge, as mentioned before, is that we can't assume any familiarity with algebraic topology or homological algebra (besides, perhaps, the very basic facts about homotopy groups); on the other hand, it is very useful and meaningful to dispose of a (co)-homology theory suited to support several differential topology constructions. We will show that (oriented or non-oriented) bordism provides instances of so-called (covariant) "generalized" homology theories for arbitrary pairs (X, A) of topological spaces, constructed via geometric means. Then, by specializing X to be a smooth compact manifold, and after a re-idexing of the bordism modules by the *codimension* (so that they are now called cobordism modules), transversality allows to incorporate the bordism modules into a *contravariant cobordism functor* with the category of *graded rings* as the target; the product on cobordism modules is also defined by direct geometric means. This multiplicative structure is a substantial enhancement and it will lead to several important and often very classical applications. For example, it is the natural context for unavoidable topics such as the degree theory or the Poincaré-Hopf index theorem. The verification that several constructions are well-defined is eventually reduced to the fact that the cobordism product is well-defined. Moreover, when possible, the "invariance up to bordism" is emphasized rather than the "invariance up to homotopy", compared to most of the established references. Not assuming any familiarity with algebraic topology, this presentation could also be useful as an intuitive, geometrically based introduction to some topics of that discipline. Overall, this book is a collection of themes, in some cases advanced and of historical importance and whose choice was certainly due in part to personal preferences, with the common characteristic that they can be treated with "bare hands", meaning by combining specific differential-topological cut-and-paste procedures and applications of transversality, mainly through the cobordism multiplicative structure. The trait of geometric construction sets the "tone" of this textbook, intended to be accessible and useful to motivated master undergraduate students and Ph.D. students, but also to a more expert reader to recognize very basic reasons for some facts already known as the result of more advanced theories or technologies.

Dedico tutto questo ai miei nipoti Pietro(lino), Martin(in)a e Andrea bibo-chicco

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Introduction

These lecture notes were conceived with a typical class of rather good and motivated students in mind, who have accomplished (or are accomplishing) a first three years degree in mathematics and whose mathematical background is likely limited. For example, besides very basic facts about homotopy groups, no familiarity with algebraic topology or homological algebra is assumed. Concerning general topology, some knowledge is assumed about compactness in Hausdorff second-countable topological spaces, but not about para-compactness.

In some sense, the most natural way to read this text is from the beginning to the end. Nonetheless, different reading paths and various combinations of subjects are also possible and meaningful. These pages have originated from teaching experiences. Whereas not a single course has covered the whole content of the book, parts of each chapter have been treated during some of the classes.

The text (as much as the lectures it derives from) intends to give accurate definitions, statements, and descriptions of the main constructions; it also aims to develop an articulated and coherent exposition. On the other hand, proofs are intentionally not uniformly detailed (and sometimes even omitted). The text is addressed to an actively involved and motivated reader. The active participation of the reader is often required to complete some arguments or to check some statements, especially in the last two chapters. For this reason, we considered a list of exercises at the end of each chapter unnecessary.

The use of figures is limited; pictures containing substantial, not only allusive information have been introduced. Drawing pictures following geometric reasoning is often useful, but this is left to the reader's initiative.

The bibliography is far from being exhaustive; besides a few classical references which have certainly influenced these pages (such as [**M1**], [**GP**], [**H**], [**M2**], [**M3**], [**Mu**]), we just list the texts which have been cited.

The language of categories is moderately used, in the same way, for example, it is used in textbooks of algebraic topology like [**Hatch**], [**Mu2**]. The few necessary notions are collected in an appendix. Differential topology concerns the category of *smooth manifolds* and *smooth maps*; this includes the study of smooth manifolds considered up to *diffeomorphism* that is the equivalence in that category. A first necessary task is to define these objects and morphisms. We do it from scratch in Chapters 1, 2 and 4 by progressively extending the category, from the category of open sets in Euclidean spaces and smooth maps, through the category of *embedded* smooth manifolds in some Euclidean space and ending with the category of "abstract" smooth manifolds defined by the abstraction of some properties of embedded ones. Along with these generalizations, the notions of *sub-manifold*, manifold with *boundary*, and (orientable) *oriented* manifold with *oriented boundary* are developed. Basic notions such as immersion, submersion, embedding, smooth homotopy, isotopy or diffeotopy between smooth maps are also introduced.

In most applications, we will focus on *compact* manifolds, especially when we consider the sources of smooth maps. We will not present the most general version of many results; there are already a lot of interesting facts concerning compact manifolds and the assumption of compactness simplifies many arguments in dealing, for example, with function space topology or with cut-and-paste constructions where one can use only *finite* partitions of unity, avoiding any reference to para-compactness.

Moreover, we will show that every compact manifold is diffeomorphic to an embedded one. Then several important facts, such as a tubular neighbourhood theory, can be developed by exploiting the embedding in some Euclidean space, but they eventually hold for arbitrary compact manifolds.

Let us describe the content of each chapter.

In Chapter 1, we assume the knowledge of basic differential calculus in several variables and we collect some facts concerning smooth maps between open sets of Euclidean spaces. Some of these facts (such as the *inverse map theorem* and its geometric applications to the local normal forms of immersions and submersions) should be familiar to the reader. Other facts are presumably less familiar, such as Morse's lemma, the linearization of diffeomorphisms of $\mathbb{R}^n$ up to isotopy, bump functions, and the smooth homogeneity of connected open sets (which later extends to arbitrary connected manifolds). Two characteristic features of differential topology already emerge. On one hand, there is a sort of "local rigidity": up to a local change of smooth coordinates, linear algebra provides the actual local models in many 'generic and stable' smooth situations. But on the other hand, smooth maps are very "flexible", the existence of bump functions being a typical example of it. This will be the key for globalization procedures and cut-and-paste constructions. Flexibility is a quality expected from a topological theory, but this is moderated by that sort of local rigidity which allows having a good geometric control; this moderate flexibility eliminates too "wild" phenomena that occur in general topology, even dealing with merely topological manifolds, or allows simple proofs of facts (such as the invariance of dimension up to diffeomorphism) whose topological counterparts hold as well but are more demanding. Moreover, the homogeneity property, in particular, indicates that the true questions in differential topology concern the *global* structure of manifolds.

In Chapter 2, we extend the notions of smooth maps and diffeomorphisms to *arbitrary* topological subspaces of some Euclidean spaces; then an embedded smooth manifold M of dimension m is defined as a topological subspace of some $\mathbb{R}^n$ which is locally diffeomorphic to open subsets of $\mathbb{R}^m$. Although not so demanding, this extension leads to many embedded manifolds beyond the open sets, including very familiar objects like the graphs of smooth maps between open sets, which ultimately are the local models for any embedded smooth manifold.

Chapter 3 is dedicated to a detailed presentation of two distinguished families of manifolds, that is *Stiefel and Grassmann manifolds*, including projective spaces. Stiefel manifolds are naturally embedded; we provide embedded models also for the Grassmann manifolds. Besides the fact that they are nontrivial examples of (embedded) smooth manifolds, they shall be crucial in the study of vector (and frame) bundles on arbitrary manifolds. This chapter is essentially self-contained; it can be read independently and at a later stage when it is necessary.

Every embedded smooth manifold is naturally endowed with a maximal *atlas* of smooth *charts* (with corresponding smooth *local coordinates*) and smooth maps between embedded manifolds have natural *representations in local coordinates*. These notions are the key to the final abstraction made in Chapter 4.

After having stressed in Chapter 1 the functorial nature of the elementary *chain rule*, following the progressive generalizations of the concept of manifold, we build the fundamental covariant *tangent functor* which associates each manifold with its *tangent bundle* and each smooth map with its *tangent map*. This incorporates the notion of a tangent vector space at each point of a smooth manifold, of which we provide different interpretations. The tangent functor is an important source of invariants of smooth manifolds. For embedded manifolds, tangent bundles and maps are constructed as a direct generalization of the elementary case of open sets in Euclidean spaces. For abstract manifolds, tangent bundles and maps must be somehow "invented", with the constraint that they must be compatible with what is already done in the embedded category. This is probably the most demanding extension, passing from the embedded to the abstract category. Eventually, this leads us, in Section 4.4 of Chapter 4, to the general notion of *principal bundle* with a given *structural group* G and *associated fibre bundles*, governed by a suitably defined G-valued *cocycle*, and we elaborate on different notions of fibred bundle equivalence. The principal *frame bundle* of a smooth manifold with the associated *tensor bundles* (including the tangent bundle) shall be a fundamental example (see Section 4.5).

Our typical student is probably already aware of the topology of the uniform convergence on compact sets of continuous maps between open sets of Euclidean spaces. This directly extends to $\mathcal{C}^r$ maps, $r \geq 0$, in terms of the uniform convergence on compact sets of the maps and their partial derivatives up to the order r. These topologies restrict to the set of smooth maps, for which we can also consider the union topology over $r \in \mathbb{N}$. Using the representation in local coordinates, the definition of these function spaces extends to smooth maps $f : M \to N$ between smooth manifolds, giving us the spaces $\mathcal{E}^r(M, N)$ endowed with the so-called $\mathcal{C}^r$ *weak topology* and $\mathcal{E}(M, N)$ endowed with the union topology. The adjective "weak" alludes to further function space topologies, the so-called *strong topologies*. These coincide with the weak ones if the source manifold is compact; otherwise, they are much finer and aimed to have a control 'at infinity'. We will not deal with the strong topology because in the relevant applications considered in this text, the source manifold M will be *compact*. For example, in Section 4.11.2 we show that if M is compact, $f : M \to N$ is an embedding if and only if it is an injective immersion and that immersions, submersions and embeddings respectively form (possibly empty) *open sets* in $\mathcal{E}(M, N)$.

At the end of Chapter 4, we show that every abstract compact smooth manifold can be embedded in some $\mathbb{R}^n$. Then, considered up to diffeomorphism, it is not restrictive to assume that compact manifolds are embedded. As we are mainly concerned with compact manifolds, the abstraction of Chapter 4 might sound a bit superfluous. However, we will point out natural constructions to build new abstract compact manifolds, starting from given ones, even embedded. It would be artificial to force these constructions in the embedded setting. It is more convenient to use the embedding result *a posteriori*, to exploit the facts that we will establish for compact embedded manifolds.

In Chapter 5, we introduce the *pull-back* construction on fibred bundles, then we apply it to the so-called *tautological (vector or frame) bundles* over the Grassmann manifolds. This construction can be compared to a powerful machine that produces vector bundles (and the associated frame bundles) over smooth manifolds and naturally incorporates the tangent bundles of embedded manifolds and their tensorial relatives. We show that, up to equivalence, every vector bundle over a compact manifold arises in this way. After having constructed, via a suitable limit procedure, the *infinite Grassmannian* $\mathfrak{G}_{\infty,k}$ of k planes in $\mathbb{R}^\infty$, with its limit tautological bundles, an important result of the chapter is the classification of these vector bundles over a *compact* manifold M, partitioned by the rank k, up to 'strict' equivalence. The classifying space is $[M, \mathfrak{G}_{\infty,k}]$: the set of *homotopy classes* of smooth maps from M to $\mathfrak{G}_{\infty,k}$. A typical way to get algebraic topological invariants is to construct functors from some sub-category of topological spaces to some category of algebraic structures (groups, rings, vector spaces, etc). At the end of Chapter 5, we present a nontrivial implementation of this idea based on this family of vector bundles. By augmenting the strict equivalence to a suitable *stable equivalence*, we realize that the quotient set $\mathbf{K}_0(M)$ of the whole collection of vector bundles (all ranks confused) carries a natural *ring structure*; combined with the pull-back construction, this eventually builds a *contravariant functor* from the (sub) category of compact manifolds to the category of Abelian rings which satisfies the *homotopy invariance property.*

In Chapter 6, we focus on embedded *compact* manifolds; that is, following the above considerations, on compact manifolds exploiting the existence of an embedding in some Euclidean space. We develop a theory of *tubular neighbourhoods* of sub-manifolds and of *collars* for the boundary of a manifold with boundary. We present some applications of this technology. For simplicity, let us consider here boundaryless manifolds. If M and N are both compact, then we prove that smooth maps are *dense* in $\mathcal{C}^r(M,N)$ for every $r \geq 0$. Primary topological invariants, as the fundamental group or higher homotopy groups, are defined in general in terms of homotopy classes of *continuous* maps defined on spheres. As an application of the density theorem, we see that they can be equivalently defined in terms of smooth homotopy between smooth maps $f : S^n \to N$. We use this fact to classify vector bundles on spheres. Another important application, for every $r \geq 1$, is the approximation of every compact $\mathcal{C}^r$-manifold $M \subset \mathbb{R}^h$ by smooth embedded manifolds and the existence and uniqueness up to diffeomorphism of a smooth structure on each such a $\mathcal{C}^r$-manifold. We state the *Sard-Brown* theorem, which is the base of *transversality* that shall be more systematically developed in Chapter 8; here, we anticipate some manifestation. By using the restriction to $M \subset \mathbb{R}^h$ of 'generic' linear projections of $\mathbb{R}^h$ to lines, we show that *Morse functions* form an *open and dense* subset of $\mathcal{E}(M,\mathbb{R})$. We also study some instances of generic linear projections to hyperplanes and, eventually, prove *'easy' Whitney's immersion/embedding theorem*: every m-dimensional compact smooth manifold M can be immersed in $\mathbb{R}^{2m}$ and embedded in $\mathbb{R}^{2m+1}$. In the last section of the chapter, we discuss a huge refinement of the approximation theorem by smooth manifolds. Exploiting the fact that Grassmann manifolds are not only embedded smooth manifolds but actually *regular real algebraic sets*, and that the tautological bundles are also real algebraic, we outline Nash's celebrated result that every embedded smooth manifold $M \subset \mathbb{R}^h$ can be approximated by a regular sheet of a real algebraic set of $\mathbb{R}^h$ (shortly by a *Nash manifold*) and that every compact embedded smooth manifold admits a Nash manifold structure, unique up to Nash diffeomorphism. We also discuss a version of the Sard-Brown Theorem in the category of Nash manifolds. In the general setting, the result is expressed in measure-theoretic terms, while in the Nash case it is purely a geometric statement, as well as its proof.

Chapters 1 to 6, with the exceptions of the end of Chapter 5 about the rings $\mathbf{K}_0(*)$ and the digression on Nash's manifolds, form the strict foundation part of this text. The following chapters articulate a more advanced discourse.

In Chapter 7, we collect several constructions that produce new compact manifolds by modifying given ones. At first, we prove the so-called *Thom's lemma* about the extension of any isotopy defined on a compact source manifold to an ambient diffeotopy; this is the main tool to prove that such constructions are well defined up to diffeomorphism. Among cut-and-paste procedures, we recall *gluing along diffeomorphic boundary components*, *connected sum* with a discussion about the related notion of *twisted spheres*, and *attaching a p-handle* (i.e. a standard handle $D^p \times D^{m-p}$ of index p to an m-manifold M via an embedding in ∂M of the *attaching tube* $S^{p-1} \times D^{m-p}$). In many cases, the immediate result is a *smooth manifold with corners*. Corners also arise by taking the product of two manifolds with nonempty boundary. We discuss a standard procedure of *smoothing the corners* that produces ordinary smooth manifolds well defined up to diffeomorphism. We also discuss the *strong Whitney immersion/embedding theorem* of any m-dimensional compact manifold M in $\mathbb{R}^{2m-1}$ and in $\mathbb{R}^{2m}$, respectively. The main difference compared with the 'easy' Whitney theorems is that the strong ones are not entirely based on 'generic position arguments' (i.e. transversality). The strong embedding is achieved by performing a robust alteration of a 'generic' immersions in $\mathbb{R}^{2m}$, the strong immersion by modifying certain 'generic' maps of M in $\mathbb{R}^{2m-1}$. The proof of the strong embedding theorem introduces the so-called *Whitney's trick* to eliminate pairs of self-intersection points in the image of a generic immersion in $\mathbb{R}^{2m}$; this 'trick' will be considered again in Chapter 18 and in Chapter 20. By elaborating on the strong immersion theorem, we present *Rohlin's embedding theorem in* $\mathbb{R}^{2m-1}$ *up to surgery*. This shows that for every orientable manifold M as above, there is M' such that the disjoint union $M \amalg M'$ is the boundary of a compact orientable $(m+1)$-manifold W, and M' can be *embedded* in $\mathbb{R}^{2m-1}$. In the last section of the chapter, we describe the modification obtained by *blowing up a manifold M along a smooth centre $X \subset M$*; this replaces X with its *projectivized normal bundle* in M.

In Chapter 8, we develop the *transversality* concept in a more systematic way. As usual, the source manifold M is compact, possibly with a nonempty boundary, and for simplicity we assume here that the target manifold N is also compact and boundaryless; Z is a boundaryless compact submanifold of N. There are two kinds of *basic transversality theorems*. The first kind concerns certain geometric tameness under the transversality hypothesis: if $f : M \to N$ is transverse to Z, then $(Y, \partial Y) := (f^{-1}(Z), (\partial f)^{-1}(Z))$ is a 'proper sub-manifold' of $(M, \partial M)$ of the same *codimension* of Z in N. There is also a specialization within the category of oriented manifolds. The second kind of basic transversality theorem states that transverse maps are generic and stable; that is, they form an *open and dense* set in $\mathcal{E}(M, N)$. There is also a relative version, concerning maps which coincide on ∂M, provided that this restriction is already transverse to Z by itself. The bridge between the two kinds of theorems is represented by the so-called *parametric transversality*, whose proof is substantially based on the Sard-Brown theorem. These basic transversality theorems suffice for most applications later in the text. However, transversality (i.e. 'general position' reasoning) is a profound, potent and pervasive paradigm beyond such basic results. Without any pretension of completeness, in the second part of the chapter we collect a few examples of further applications (including the notion of 'generic immersion', already employed while discussing Whitney's strong embedding theorem).

In Chapter 9, we formalize the notion of a *smooth triad* (M, V_0, V_1), where M is a compact smooth m-manifold and V_0 and V_1 are unions of connected components of ∂M in such a way that the boundary is the disjoint union $\partial M = V_0 \amalg V_1$; M might be boundaryless, so that the triad $(M, \emptyset, \emptyset)$ is allowed. In some way, a triad realizes a "transition" from V_0 to V_1. We define generic Morse functions $f : M \to [0,1]$ on a triad, meaning: $f^{-1}(j) = V_j$, $j = 0, 1$, f has only non-degenerate critical points placed outside a neighbourhood of ∂M, and they have distinct critical values. Density and stability of these functions are assured by the results of Chapter 8. An important achievement of Chapter 9 is that every Morse function carries a *handle decomposition of the triad*; that is, a way to reconstruct the triad (up to diffeomorphism) from a collar of V_0 in M, by attaching successively a handle of index p for every non-degenerate critical point of index p of f. Associated with every decomposition of a triad (M, V_0, V_1), there is a *dual decomposition* of the triad (M, V_1, V_0) where every p-handle is converted into an $(m-p)$-handle and these are attached backward starting from a collar of V_1 in M. If the initial decomposition is carried by a Morse function f, then the dual decomposition is carried by the function $1 - f$. In a sense, Morse functions are used as a tool to prove the *existence* of handle decompositions. Then, handle decompositions are used as they are and eventually modified, not addressing the issue of whether the new decompositions are carried by a Morse function. We point out two *basic moves* which modify a given decomposition without changing the triad (up to diffeomorphism): the so-called *sliding handles* (which is nothing other than the possibility of modifying any attaching map up to isotopy, already treated in Chapter 7) and the *elimination/insertion of pairs of complementary handles*. We show some elementary specialization ('reordering') or simplification ('elimination of 0- and m-handles') of handle decompositions obtained by using the basic moves. As a simple but important application, we get the classification up to diffeomorphism of compact 1-dimensional manifolds, confirming the intuition: a connected compact 1-manifold is diffeomorphic either to S^1 or to the 1-disk $[-1, 1]$.

In Chapter 10, we develop *bordism*. There is an unoriented version and an oriented one. Two (unoriented) compact boundaryless m-manifold M_0 and M_1 are bordant manifolds if $M_0 \amalg M_1$ is the boundary of a compact $(m+1)$-manifold W. In the oriented case, the manifolds M_0, M_1 and W are oriented and $M_0 \amalg -M_1$ is the oriented boundary of W. Case by case, the quotient set of the relation generated by 'being bordant' and (oriented) diffeomorphisms is denoted by Ω_m in the oriented case and is a $\mathbb{Z}$-module, while it is denoted by η_m in the nonoriented case and is a $\mathbb{Z}/2\mathbb{Z}$-vector space.

The operation is induced by the *disjoint union*. If X is any topological space, a continuous map $f : M \to X$ is called a *singular* smooth m-manifold in X and we can extend the definition of bordism to such singular manifolds and, consequently, define the modules $\Omega_m(X)$ or $\eta_m(X)$, sometimes denoted by $\mathcal{B}_m(X;R)$, $R = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$. When X consists of a single point, we recover the earlier modules because the maps f are immaterial in this case. We can also define relative versions $\mathcal{B}_m(X, A; R)$ for topological pairs (X, A) (X being, as usual, identified with $(X, \emptyset)$). We prove that in this way we define a *covariant functor* from the category of topological pairs to the category of R-modules, which turns out to be a *generalized homology theory*. This means that all *Eilenberg-Steenrod axioms* are satisfied with the possible exception of 'dimension'; its failure depends on the nontriviality of $\mathcal{B}_m(X;R)$, $m \geq 1$, when X consists of a single point. This issue will be considered along the rest of the text. We discuss some relationships between bordism and homotopy group functors.

In Chapter 11, we specialize bordism assuming that X is a compact boundaryless smooth manifold. Alike the homotopy groups, thanks to the approximation theorems of Chapter 6, it is not restrictive to deal only with smooth maps $f : M \to X$. The bordism modules $\mathcal{B}_m(X;\mathbb{Z}/2\mathbb{Z})$ are indexed over $\mathbb{Z}$, by postulating that they are the trivial module 0 if $m < 0$. We formally re-index them by the *codimension*, by setting $\mathcal{B}^r(X;\mathbb{Z}/2\mathbb{Z}) = \mathcal{B}_m(X;\mathbb{Z}/2\mathbb{Z})$, $r = \dim X - m$, so that they are trivial if $r > \dim X$ and are now called *cobordism modules*. The key point is that by combining a slight extension of the basic transversality theorems of Chapter 8 with variations on the *pull-back construction*, we incorporate $X \Rightarrow \oplus_r \mathcal{B}^r(X;\mathbb{Z}/2\mathbb{Z})$ into a *contravariant functor* from the (sub)category of compact boundaryless smooth manifolds to the category of *graded rings*; this means that $\oplus_r \mathcal{B}^r(X;\mathbb{Z}/2\mathbb{Z})$ is endowed with a multiplicative structure which distributes itself in a family of $\mathbb{Z}/2\mathbb{Z}$-bilinear maps

$$\sqcup : \mathcal{B}^r(X;\mathbb{Z}/2\mathbb{Z}) \times \mathcal{B}^s(X;\mathbb{Z}/2\mathbb{Z}) \to \mathcal{B}^{r+s}(X;\mathbb{Z}/2\mathbb{Z})$$

defined geometrically via transversality and implementation of the pull-back construction. If X is oriented we can perform all the construction within the oriented category, using the $\mathbb{Z}$-modules $\mathcal{B}^r(X;\mathbb{Z})$. If $\alpha = [M_1]$ and $\beta = [M_2]$ are represented by sub-manifolds of X, then $\alpha \sqcup \beta$ is represented by any transverse intersection $M'_1 \cap M'_2$ where M'_j is a suitable small perturbation of M_j, $j = 1, 2$. Over both $R = \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}$, the product satisfies the relation

$$\alpha \sqcup \beta = (-1)^{rs} \beta \sqcup \alpha$$

which can be checked in a geometric way. If X consists of a single point, then the product reduces to $[M] \sqcup [N] = [M \times N]$. If $r + s = \dim X = n$ and X is connected (possibly oriented), then $\mathcal{B}^n(X;R) = R$ and the product $\sqcup$ induces a linear map $\phi^r : \mathcal{B}^r(X;R) \to \mathrm{Hom}(\mathcal{B}_r(X;R), R)$; in many situations it is convenient to consider the quotient module

$$\mathcal{H}^r(X;\mathbb{R}) := \mathcal{B}^r(X;R)/\ker(\phi^r)$$

with the induced linear injection

$$\hat{\phi}^r : \mathcal{H}^r(X;R) \to \mathrm{Hom}(\mathcal{H}_r(X;R),R) \ .$$

In particular, if X is oriented, then $\mathcal{H}^r(X;\mathbb{Z})$ is torsion free. If X is connected (possibly oriented) and $\dim X = 2m$, then we have the *intersection form*

$$\sqcup : \mathcal{H}^m(X;R) \times \mathcal{H}^m(X;R) \to R$$

which is symmetric if either $R = \mathbb{Z}/2\mathbb{Z}$ or $R = \mathbb{Z}$ and m is even; it is antisymmetric otherwise. Sometimes it is expressed as

$$\bullet : \mathcal{H}_m(X;R) \times \mathcal{H}_m(X;R) \to R \ .$$

The cobordism multiplicative structure is a substantial enhancement of the theory. In Chapter 12, we collect a few classical applications: the *fundamental class* $[X] \in \mathcal{H}^0(X;R)$ when X is connected and possibly oriented; *Brouwer's fixed point theorem* for continuous maps $f : D^n \to D^n$, $n \geq 1$; a *separation theorem* for hypersurfaces in S^n, $n > 1$; *intersection and linking numbers*; the *R-degree* of continuous maps $f : M \to N$ between (possibly oriented) compact, connected, boundaryless smooth manifolds of the same dimension; a proof of the *fundamental theorem of algebra*; *Borsuk-Ulam theorem*. We also define the *Euler class* $\omega(\xi) \in \mathcal{B}^k(X;R)$ of a rank-k vector bundle ξ over X (everything possibly suitably oriented), defined by the transverse self-intersection of the zero section of ξ in its total space. A non-vanishing Euler class is a primary obstruction to the existence of a nowhere vanishing section of ξ.

In Chapter 13, we focus on line (i.e. rank-1) bundles on X, on oriented rank-2 vector bundles (provided that also X is oriented), and on their Euler classes in $\mathcal{B}^1(X;\mathbb{Z}/2\mathbb{Z})$, $\mathcal{B}^1(X;\mathbb{Z})$ or $\mathcal{B}^2(X;\mathbb{Z})$. A key point here is that $\mathbf{P}^\infty(\mathbb{R})$ is a $\mathbf{K}(1,\mathbb{Z}/2\mathbb{Z})$ space, S^1 is a $\mathbf{K}(1,\mathbb{Z})$ space, and $\mathbf{P}^\infty(\mathbb{C})$ is a $\mathbf{K}(2,\mathbb{Z})$ space. This eventually gives precise information, case by case, about $\mathcal{H}^1(X;R)$ and $\mathcal{H}^2(X;\mathbb{Z})$. For example, every class in $\mathcal{H}^1(X;R)$ is the Euler class of a line bundle over X (oriented if $R = \mathbb{Z}$). It can be represented by an embedded hypersurface S of X (oriented if $R = \mathbb{Z}$); moreover, $[S_0] = [S_1]$ (in the appropriate bordism module) is equivalent to the fact that the associated bundles are strictly equivalent, and is also equivalent to the fact that a bordism between S_0 and S_1 is realized through a triad (W, S_0, S_1) properly embedded in $X \times [0,1]$ (all manifolds being oriented if $R = \mathbb{Z}$). Similarly for $\mathcal{B}^2(X;\mathbb{Z})$.

In Chapter 14, we focus on the Euler class in $\mathcal{B}^m(M;\mathbb{Z}) = \mathbb{Z}$ of the tangent bundle of a compact oriented connected boundaryless smooth m-manifold M. This integer is denoted by $\chi(M)$ and called the *Euler-Poincaré characteristic* of M. Essentially by definition, it can be computed using any section of $T(M)$ transverse to the zero section, that is, using any tangent vector field on M with only non-degenerate zeros. This can be extended to any tangent vector field on M with only isolated (not necessarily non-degenerate) zeros. This is the content of the *Index Theorem*; the key point is the reformulation of the sign of a non-degenerate zero in terms of the $\mathbb{Z}$-degree of a suitably map $f : S^{m-1} \to S^{m-1}$, defined locally at the zero using the vector field; this reformulation by the degree makes sense also for any isolated zero and well defines its *index*. Then $\chi(M)$ is eventually equal to the sum of such indices. Invariance of the degree up to bordism plays a crucial role in this achievement. The Euler-Poicaré characteristic is multiplicative with regards to the product of compact boundaryless manifolds. The value of $\chi(X)$ does not depend on the choice of the orientation of X; eventually $\chi(M) := \frac{1}{2}\chi(\tilde{M})$ is well defined also if M is not orientable, $\tilde{M} \to M$ being the orientation 2-to-1 covering map. We extend the index formula to define the *relative characteristic* $\chi(M, V_0)$ of a triad (M, V_0, V_1) by using suitable tangent vector fields on M, transverse to the boundary and with only isolated zeros. The characteristic has certain homotopy invariance properties so that, for example, if B is the total space of a disk bundle over a boundaryless M, then $\chi(M) = \chi(B, \emptyset, \partial B)$. In the special case when M is embedded in $\mathbb{R}^h$ and B is a tubular neighbourhood of M in $\mathbb{R}^h$, this leads to the classical fact that $\chi(M)$ coincides with the degree of the *Gauss map* $\partial B \to S^{h-1}$. The extended characteristic also has remarkable additive properties concerning the composition of triads. Moreover, $\chi(M, V_0)$ can be computed using any gradient vector field of any Morse function $f : M \to [0, 1]$ on the triad. By combining these facts, we obtain, for example, that if M is boundaryless and *odd*-dimensional, then $\chi(M) = 0$ (use both f and $1 - f$ to compute $\chi(M)$ in two ways); if V is even-dimensional and it is the boundary of some M, then $\chi(V) \equiv 0 \mod (2)$. It follows that for every even m, η_m is nontrivial because $\chi(\mathbf{P}^m(\mathbb{R})) = 1$. At the end of the chapter, we shortly discuss other ways (combinatorial or algebraic/topological) to recover the Euler-Poincaré characteristic.

In Chapter 15, we apply several tools developed in the previous Chapters to classify compact surfaces (i.e. compact smooth 2-manifolds) up to diffeomorphism and also to determine both bordism moduli η_2 and Ω_2. If M is a connected boundaryless compact surface, we show that $\eta_1(M)$ is a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space and that the symmetric intersection form $\bullet : \eta_1(M) \times \eta_1(M) \to \mathbb{Z}/2\mathbb{Z}$ is non-degenerate. We focus on its isometry class as the main invariant up to diffeomorphism. After having established the abstract algebraic classification, up to isometry, of non-degenerate symmetric bilinear forms on finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-spaces, we show, step by step, that there is a perfect 2D topological counterpart. Finally, every isometry class can be realized as the intersection form of some surface M, and two surfaces are diffeomorphic if and only if they have isometric intersection forms. In particular, from that isometry class, we can derive whether M is orientable or not and the value of $\chi(M)$. If M is orientable, then it is the connected sum of S^2 with g copies of $S^1 \times S^1$, where $\chi(M) = 2 - 2g$; if M is not orientable, then M is a connected sum of copies of $\mathbf{P}^2(\mathbb{R})$ whose number is determined by $\chi(M)$. As for the bordism, $\Omega_2 = 0$, while $\eta_2 = \mathbb{Z}/2\mathbb{Z}$ generated by $[\mathbf{P}^2(\mathbb{R})]$. We also discuss some aspect of the *stable equivalence* generated by diffeomorphisms and the elementary stabilization that consists of performing the connected sum with $\mathbf{P}^2(\mathbb{R})$; in particular, we refer to the relationship with the so-called *Nash's rational model question* in dimension 2. The theme of the *quadratic enhancements* of the intersection form associated with the immersion of a surface in a higher-dimensional manifold will emerge later in the text. At the end of Chapter 15, we develop the abstract theory of these quadratic enhancements, including the introduction of the *Arf* and the *Arf-Brown* invariants.

The Euler-Poincaré characteristic mod(2) is a first example of characteristic number for the nonoriented bordism modules η_m; that is, for every $m \geq 0$, it defines a homomorphism $\chi_{(2)} : \eta_m \to \mathbb{Z}/2\mathbb{Z}$ which is surjective for each even m. Pontryagin remarked that a huge family of characteristic numbers can be produced using the cohomology ring with $\mathbb{Z}/2\mathbb{Z}$-coefficients of the infinite Grassmann manifolds $\mathfrak{G}_{\infty,\bullet}$ and the classifying map $M \to \mathfrak{G}_{\infty,m+1}$ of the stable tangent bundle $T(M) \oplus \epsilon^1$ of each compact boundaryless m-manifold M. These are called *SW-characteristic numbers* as they are incorporated in the theory of (cohomological) Stiefel-Whitney *characteristic classes*. We do not dispose of cohomology, but in Chapter 16, it is easy to reformulate the definition using the cobordism rings, which we have defined from scratch, instead of the cohomology rings. We call (stable) *η-characteristic numbers* the ones obtained in this way. In [**T**], Thom determined the ring $\eta^\bullet$, using the Pontryagin-Thom construction (that we treat in Chapter 17) and combining geometric tools and homotopy theory. A byproduct of Thom's work is the *completeness* of the SW- characteristic numbers: *$\beta \in \eta_m$ is equal to zero if and only if every SW- characteristic number vanishes on β.* Later, the authors obtained in [**BH**] a nice geometric proof of this remarkable result, ultimately based on transversality and simple cohomological computations. In Chapter 16, we show that this proof can be entirely performed using the cobordism rings and eventually get the completeness of the η-characteristic numbers. At the end of the chapter, we extend η to Ω-characteristic numbers and we briefly discuss why they are not sufficient to completely detect the oriented boundaries. Cohomology cannot be avoided to deal with the oriented bordism. However, any characteristic number for $\Omega_\bullet$, whatever it is defined, should vanish on $[M]$ if the m-manifold M is *parallelizable*. At least in this special case we prove that $[M] = 0 \in \Omega_m$ for any choice of the orientation of M, using similar geometric tools.

Chapter 17 is dedicated to the *Pontryagin-Thom* construction. The original Pontryagin construction was invented to rephrase the homotopy groups of spheres $\pi_{n+k}(S^n)$, $k \geq 0$, $n > 1$, in terms of a certain more geometric (therefore presumably more accessible at that time, around 1938) codimension n embedded oriented bordism theory with target S^{n+k}. This so-called *framed bordism* makes sense for arbitrary compact target M of dimension $n+k$ and recovers $[M, S^n]$. *Viceversa*, Thom's extension of Pontryagin construction was mainly intended as a way to rephrase the cobordism rings

$\eta^\bullet$ or $\Omega^\bullet$ in terms of the homotopy groups (more accessible at that time, around 1954, after the impressive progress in homotopy theory since Serre's Thesis [**Se**]) of the so-called Thom's spaces. Concerning the determination of $\pi_{n+k}(S^n)$, Pontryagin succeeded for $k \leq 2$; in Chapter 17, we outline these results. For $k = 0$, the $\mathbb{Z}$-degree establishes an isomorphism between $\pi_n(S^n)$ and $\mathbb{Z}$. As a corollary, we show that a compact connected boundaryless manifold M is *combable* (i.e. it admits a nowhere vanishing tangent vector field) if and only if $\chi(M) = 0$; in particular, every odd-dimensional M is combable. Difficulty increases by k. For $k = 2$, a key ingredient is the Arf invariant of the quadratic enhancement of the intersection form of every framed orientable surface in S^6. The hardest application of this geometric way is for $k = 3$ and is due to Rohlin. We limit to state the result. This is of major importance for its consequences in the theory of 4-manifolds and will be considered again in Chapter 20.

In differential topology, there is a precise distinction between 'high' (6 or greater) dimensions and 'low' (4 or fewer) dimensions, 5 being on the border. The main reason is that for $d \geq 6$, Smale's (simply connected) *h-cobordism theorem* holds; moreover there is a "stable proof" that works uniformly for all high dimensions. This is an important application of handle decomposition theory. This proof does not apply to lower dimensions. In some cases the theorem fails, and in some cases it is still an open question. In Chapter 18, we briefly discuss this issue. We do not prove the whole stable h-cobordism theorem; rather we focus on an important step (the cancellation of *algebraically* complementary handles) where the high dimension assumption is crucial. This is related to the possibility of applying Whitney's trick (firstly introduced for the strong embedding theorem) to eliminate pairs of intersection points of opposite sign between transverse sub-manifolds of complementary dimension of a simply connected manifold of dimension greater or equal to 5.

For 'very low' dimensions $0 \leq d \leq 2$, we have achieved a complete classification of compact manifolds up to diffeomorphism. This is essentially 'hopeless' for $d > 2$, even for $d = 3, 4$. In Chapters 19 and 20, we address some aspects of these low-dimensional theories. We stress that, in both cases, we do not touch the mainstream themes of the last decades (the *geometrization conjecture* - now a theorem - of 3-manifolds or the use of powerful *gauge theories* applied to the study of 4-manifolds). We limit to develop a few primary differential topological results, combining several tools established in the previous chapters.

In Chapter 19, we present a few elementary and self-contained proofs that compact orientable boundaryless 3-manifolds are parallelizable and we study combing and framing. An important amount of the chapter is devoted to several proofs of "$\Omega_3 = 0$" and of the equivalent *Lickorish-Wallace theorem* about 3-manifolds considered up to 'longitudinal' Dehn surgery equivalence. Each proof will enlighten different facets of the subject. The last two sections of the chapter are more advanced. We determine the bordism semigroup (which turns out to be a group) of immersions of surfaces in a given compact connected boundaryless 3-manifold M. If M is orientable, a key ingredient will be the Arf-Brown invariant of the quadratic enhancement of the intersection form associated with every immersion of a surface in M (endowed with an auxiliary framing). We also classify compact boundaryless 3-manifolds up to certain equivalence relations generated by diffeomorphisms and blow-up-down along smooth centers (a notion introduced in Chapter 7). The subtler so-called 'tear' equivalence, in the nonorientable case, also involves instances of quadratic enhancement of the intersection form of characteristic surfaces (i.e. representing the Euler class of the determinant bundle of the ambient 3-manifold). We also discuss an application to a solution of the so-called *Nash's rational model question* in dimension 3.

In Chapter 20, in analogy to the case of surfaces, we focus on the isometry class of the intersection form $\sqcup_M : \mathcal{H}^2(M;\mathbb{Z}) \times \mathcal{H}^2(M;\mathbb{Z}) \to \mathbb{Z}$ as the main invariant of every compact connected oriented boundaryless 4-manifold M. It is a symmetric unimodular $\mathbb{Z}$-bilinear form on the finite rank free $\mathbb{Z}$-module $\mathcal{H}^2(M;\mathbb{Z})$. We prove Rohlin's theorem that the signature σ of the intersection form determines an isomorphism $\sigma : \Omega_4 \to \mathbb{Z}$, so that Ω_4 is generated by $[\mathbf{P}^2(\mathbb{C})]$. We follow his original geometric proof. Trying to pursue the analogy with surfaces, we address the abstract arithmetic classification of such symmetric unimodular forms. An important difference is that it is complete only in the *indefinite* case. We try to develop, as much as possible, a parallel 4D counterpart, at least in the indefinite case, by restricting, in fact, to *simply connected* 4-manifolds. We establish a classification up to *odd stabilization*, the elementary ones being the connected sum with $\pm\mathbf{P}^2(\mathbb{C})$. We outline a more subtle classification up to *even stabilization* (i.e. up to connected sum with $S^2 \times S^2$). Arithmetic tells us that there are *characteristic elements* $\beta \in \mathcal{H}^2(M;\mathbb{Z})$ such that for every $\alpha \in \mathcal{H}^2(M;\mathbb{Z})$, $\alpha \sqcup \alpha = \beta \sqcup \alpha$ mod(2), and that $\sigma = \beta \sqcup \beta$ mod(8). Every β can be represented by an oriented surface F embedded in M, called a *characteristic surface*. We prove the congruence

$$\sigma - \beta \sqcup \beta = 8\alpha(F) \text{ mod}(16) ,$$

where $\alpha(F) \in \mathbb{Z}/2\mathbb{Z}$ is the Arf invariant of a quadratic enhancement of the intersection form of F which represents an obstruction to surgery F within M to an embedded 2-sphere. If the intersection form is even, we can take $F = \emptyset$, so that we recover the original celebrated Rohlin congruence $\sigma = 0$ mod(16) (originally obtained as a corollary of the fact that $\pi_{n+3}(S^n) = \mathbb{Z}/24\mathbb{Z}$ for n big enough). This implies, in particular, that there are unimodular symmetric forms which cannot be realized as the intersection form of any simply connected 4-manifold. We propose an elementary proof of the congruence due to [**Mat**] and based on the classification up to odd stabilization. Following [**A3**] and [**Roh**], we illustrate an application of these four-dimensional congruences mod(16) to Hilbert's 16th problem.

We end the chapter with an informative and discursive section about more recent achievements in the realm of 4-manifolds.

In conclusion, this book is a collection of themes, in some cases advanced and of historical importance, with the common characteristic that they can be treated with “bare hands”, meaning by combining specific differential topological cut-and- paste procedures and applications of transversality, mainly through the cobordism multiplicative structure. Of course, the choice of the topics was due in part to personal preferences. The book is intended to be accessible and useful to motivated master undergraduate students, and to Ph.D. students, but also to a more expert reader to recognize very basic reasons for some facts already known as the result of more advanced theories or technologies.

CHAPTER 1

The smooth category of open subsets of Euclidean spaces

In this book, we will deal with manifolds. Roughly, a manifold is a topological space locally modeled on a Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$. Let's recall a few facts about these local models. Since many of them should be familiar to the reader, we will at times omit the proofs or just sketch them out.

1.1. Basic structures on $\mathbb{R}^n$

Every space $\mathbb{R}^n$, $n \in \mathbb{N}$, is endowed with a variety of structures that will be employed case by case .

$\mathbb{R}^n$ is the *vector space* of column vectors (with n rows). We stipulate that if $v \in \mathbb{R}^n$ occurs as a vector in any linear algebra formula, then it is considered as a column.

The space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$ coincides with the space of $m \times n$ matrices, $M(m, n, \mathbb{R})$, so that for every $v \in \mathbb{R}^n$, $v \to Lv$ via the usual "rows by column" product. Using the lexicographic order on the entries of any matrix $L = (l_{i,j})_{i=1,\dots,m;j=1,\dots,n}$, we also fix the identification of $M(m, n, \mathbb{R})$ with $\mathbb{R}^{mn}$.

As every vector space, $\mathbb{R}^n$ has a canonical structure of *affine space* over itself determined by the map that associates to every pair of *points* $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ the *vector* $\overrightarrow{xy} := y - x$. Every element of $\mathbb{R}^n$ can be considered either as a *point* x of the space or as a *vector* v. For every point x and every vector v, there is a unique point $y := x + v$ such that $v = \overrightarrow{xy}$; for every point x, the map $v \to x + v$ is a bijection. Consider $T(\mathbb{R}^n) := \mathbb{R}^n \times \mathbb{R}^n$ where the first (second) factor consists of points (vectors), and the natural projection onto the space factor $\pi_n : T(\mathbb{R}^n) \to \mathbb{R}^n$, $\pi_n(x, v) = x$. For every point x, the fibre $T_x\mathbb{R}^n = \pi^{-1}(x)$ is a copy of the vector space $\mathbb{R}^n$, called the *tangent vector space* to the affine space at the point x. We call $(T(\mathbb{R}^n), \pi_n)$ the *affine tangent vector bundle* of the affine space $\mathbb{R}^n$. The space of points $\mathbb{R}^n$ can be included in $T(\mathbb{R}^n)$ via the *zero section* $s_0(x) = (x, 0)$, so that for every point x, $\pi \circ s_0(x) = x$. For every *affine map* $f : \mathbb{R}^n \to \mathbb{R}^m$, there is a unique linear map $L_f : \mathbb{R}^n \to \mathbb{R}^m$, called the *linear part* of the affine map, such that for every point x and every vector v, $f(x + v) = f(x) + L_f v$. The map f is an affine isomorphism if and only if L_f is a linear isomorphism (in such a case, $n = m$). In other words, the *tangent map* of f,

$$Tf := (f, L_f) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^m,\ (x, v) \to (f(x), L_f v)\ ,$$

verifies $\pi_m \circ Tf = f \circ \pi_n$, so that for every point x, the tangent space $T_x\mathbb{R}^n$ goes to the tangent space $T_{f(x)}\mathbb{R}^m$ via the linear map L_f.

$\mathbb{R}^n$ is a *complete metric space* endowed with the *Euclidean distance* $d = d_n$ defined by

$$d(x, y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}\ .$$

The *standard positive definite scalar product* $(*, *) = (*, *)_n$ on $\mathbb{R}^n$ is defined by

$$(x, y) := \sum_{j=1}^{n} x_j y_j = x^t I y$$

with the associated *norm* $||x|| = \sqrt{(x, x)}$. We note that

$$d^2(x, y) = (x - y, x - y)$$

and that the familiar formula

$$(x, y) = ||x|| \cdot ||y|| \cos\theta$$

allows us to recover the measure of the angle formed by the ordered and oriented lines spanned by two non-zero vectors x, y; in particular, they are orthogonal iff $(x, y) = 0$. We see that many basic objects of elementary geometry can be expressed analytically using the standard scalar product.

$\mathbb{R}^n$ is a *topological space* endowed with the topology $\tau = \tau_n$ induced by the distance d_n. As for any metrizable topological space, a subset U of $\mathbb{R}^n$ is *open* (i.e. it belongs to τ) if and only if for every $x \in U$, there is $r > 0$ such that the *"open" n-ball of centre x and radius r*

$$B^n(x, r) := \{y \in \mathbb{R}^n;\ d(x, y) < r\}$$

is contained in U. We will denote by

$$D^n = \overline{B}^n(0, 1)$$

the *closed unitary n-ball* also called the unitary *n-disk*, and by

$$S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n;\ d(0, x) = ||x|| = 1\}$$

the *unitary sphere*. It is not hard to verify that "open" balls are indeed open sets and that the open balls with the centre in $\mathbb{Q}^n \subset \mathbb{R}^n$ and rational radius form a *countable basis of open sets of* τ (every open set is a union of such balls). Any other scalar product

$$(x, y)_A := x^t A y$$

defined by a positive definite symmetric matrix $A = A^t$ determines (by the same formulas as above) a norm $||.||_A$, a distance d_A and an associated topology τ_A. All these distances are *topologically equivalent*; that is, every $\tau_A = \tau$.

This can be proved using the version of the elementary *spectral theorem* stating that there exists a basis of $\mathbb{R}^n$ which is simultaneously orthonormal for $(*,*)$ and orthogonal for $(*,*)_A$. Another topologically equivalent distance on $\mathbb{R}^n$ is defined by $\delta(x,y) := \max\{|x_j - y_j|;\ j = 1,\dots,n\}$.

Accordingly to general topological definitions, for every $X \subset \mathbb{R}^n$, $\tau \cap X = \{U \cap X;\ U \in \tau\}$ is the topology on X that makes it a *topological subspace* of $(\mathbb{R}^n, \tau)$; given subspaces $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, a map $f : X \to Y$ is *continuous* if for every open set $U \subset Y$, the *inverse image* $f^{-1}(U) = \{x \in X;\ f(x) \in U\}$ is an open set of X. A continuous map $f : X \to Y$ is a *homeomorphism* if it is bijective and also the *inverse map* $f^{-1} : Y \to X$ is continuous.

Every subspace $X \subset \mathbb{R}^n$ is metrizable (hence, in particular, *Hausdorff*) by the restriction to X of the distance d (or of any distance topologically equivalent to d); the restriction of any (countable) basis of open sets of τ is a (countable) basis of $\tau \cap X$.

As for every Hausdorff space with a countable basis, a subspace X of $\mathbb{R}^n$ is *compact* (i.e. every open covering of X admits a *finite* sub-covering) if and only if it is *sequentially compact* (i.e. every sequence a_n of points of X admits a sub-sequence a_{j_n} converging to some point x of X). A subspace is compact if and only if it is *closed* (i.e. the complementary is open) and *bounded* (i.e. it is contained in some ball $B^n(0,r)$). $\mathbb{R}^n$ is *locally compact* (for every $x \in \mathbb{R}^n$ the family of *closed balls* $\overline{B}^n(x,r) = \{y \in \mathbb{R}^n;\ d(x,y) \leq r\}$, when $r > 0$ varies, is a basis of compact neighbourhoods of x). The same holds for every subspace X which is a closed subset of $\mathbb{R}^n$.

We recall that a nonempty open subset $U \subset \mathbb{R}^n$ is *connected* if U is the only open-and-closed nonempty subset of U; U is *path-connected* if for every two points x_0, x_1 of U, there is a continuous path $\alpha : [0,1] \to U$ such that $\alpha(0) = x_0$, $\alpha(1) = x_1$. We have:

PROPOSITION 1.1. *A nonempty open set U of $\mathbb{R}^n$ is connected if and only if it is path-connected.*

Proof : The "if" implication holds in general for arbitrary topological spaces and it is due to the basic fact that intervals in the real line are connected; for "only if", note that "being connected by a continuous path" defines an equivalence relation on U. The equivalence classes are called the *path-connected components* of U. As every open ball $B^n(x,r) \subset U$ is contained in the path-connected component of U which contains $x \in U$, then every path-connected component of U is open, hence there is only one if U is connected.

■

1.2. Differential calculus

Another fundamental structure carried by the spaces $\mathbb{R}^n$ is the *differential calculus*. Let $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ be open sets. A map

$$f = (f_1, \dots, f_m) : U \to W$$

is said to be $\mathcal{C}^0$ if it is continuous. The map is *differentiable* at $x \in U$ if there is a (necessarily unique) linear map $d_x f \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ that "well" approximates $g(h) = f(x+h) - f(x)$ in a neighbourhood of $h = 0$. Precisely, for every $\epsilon > 0$, there is $\delta > 0$ such that for every h such that $||h|| < \delta$, $x + h \in U$ and

$$||g(h) - d_x f(h)|| \leq \epsilon ||h|| \ .$$

The linear map $d_x f$ is called the *differential of* f *at* x. The map f is (globally) *differentiable* if it is differentiable at every point $x \in U$. In such a case it is defined the *differential map*

$$df : U \to M(m, n, \mathbb{R}), \ df(x) := d_x f \ .$$

We say that f is $\mathcal{C}^1$ if it is differentiable and df is continuous (being $M(m, n, \mathbb{R})$, that is $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, identified with $\mathbb{R}^{mn}$ as we said above). Every $\mathcal{C}^1$ map is $\mathcal{C}^0$. By induction, for every $r \geq 1$, we say that f is $\mathcal{C}^r$ if df is $\mathcal{C}^{r-1}$. In practice, f is $\mathcal{C}^r$, $r \geq 1$, if and only if it is $\mathcal{C}^0$ and for every multi-index $J = j_1 \dots j_n$ of *order* $|J| := j_1 + \dots + j_m \leq r$, for every $i = 1, \dots, m$, the *partial derivative function*

$$\frac{\partial^J f_i}{\partial^{j_1} x_1 \dots \partial^{j_n} x_n} : U \to \mathbb{R}$$

is defined and continuous. For every $x \in U$, the partial derivatives of the first order can be organized in an $m \times n$ matrix so that

$$d_x f := \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{i=1,\dots,m; \ j=1,\dots,n} \in M(m, n, \mathbb{R}) \ .$$

A map f is $\mathcal{C}^\infty$ or, equivalently, *smooth* if it is $\mathcal{C}^r$ for every $r \geq 0$. If f is smooth, then df is also smooth. So we can define inductively for every $r \geq 1$, $d^r f = d(d^{r-1} f)$.

If f is (at least) $\mathcal{C}^1$, we have the following *uniform* version of the above property that defines the differentials $d_x f$: for every $x \in U$ there exists a neighbourhood W of x in U (we can take as W a compact closed ball $\bar{B}^n(x, \rho) \subset U$), such that for every $\epsilon > 0$, there is $\delta > 0$ such that for every $y \in W$ and for every h, $||h|| < \delta$, we have $y + h \in U$ and

$$||f(y+h) - f(y) - d_y f(h))|| \leq \epsilon \delta \ ;$$

in other words

$$\lim_{h \to 0} \frac{g(y, h) - d_y f(h)}{||h||} = 0$$

uniformly on W.

From now on we will deal mainly with smooth maps.

Taylor polynomials. A *homogeneous polynomial map of degree* $k \geq 1$, $\mathfrak{p} : \mathbb{R}^n \to \mathbb{R}^m$, is by definition of the form $\mathfrak{p}(x) = \phi(x, \dots, x)$, where $\phi : (\mathbb{R}^n)^k \to \mathbb{R}^m$ is a (necessarily unique) *symmetric* k-linear map (ϕ is called the "polarization" of $\mathfrak{p}$). It follows that the set $\mathcal{P}_k(n, m)$ of these homogeneous polynomial maps has the natural structure of a finite-dimensional real vector space. A *polynomial map of degree d*, $p : \mathbb{R}^n \to \mathbb{R}^m$, is of the form

$$p = p_0 + p_1 + \cdots + p_d$$

where $p_0 \in \mathbb{R}^m$, p_j is a homogeneous polynomial of degree j for $j \geq 1$, and p_d is not zero.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Then for every $k \geq 1$ there is a smooth map

$$T_k(f) : U \to \mathcal{P}_k(n,m)$$

such that for every $k \geq 1$, for every $x \in U$, there is a neighbourhood W of x in U such that for every $\epsilon > 0$, there is $\delta > 0$ such for every $y \in W$ and every h, $||h|| < \delta$, we have $y + h \in U$ and

$$||f(y+h) - (f(y) + T_1(f)(y)(h) + \cdots + T_k(f)(y)(h))|| \leq \epsilon ||h||^k \ .$$

The maps $T_k(f)$ are uniquely determined by these conditions. Clearly

$$T_1(f)(x) = d_x f \ .$$

More generally, every $\mathfrak{p} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_m) \in \mathcal{P}_k(n,m)$ is of the form

$$\mathfrak{p}_i(h) = \sum_{|J|=k} a_i^J h_1^{j_1} \cdots h_n^{j_n} \ ,$$

where the coefficients $a_i^J \in \mathbb{R}$. Then we can verify that $T_k(f)(x)$ is uniquely determined by the formulas

$$a_i^J = \frac{1}{k!} \frac{\partial^J f_i}{\partial^{j_1} x_1 \ldots \partial^{j_n} x_n}(x) \ .$$

In other words, $T_k(f)(x)$ is determined in terms of $\frac{1}{k!} d_x^k(f)$. $T_k(f)(x)$ is the *homogeneous order-k Taylor polynomial of f at x*. Setting $f(x) = T_0(f)(x)$, the polynomial map (of the variable h)

$$\mathcal{T}_k(f)(x) := T_0(f)(x) + T_1(f)(x) + \cdots + T_k(f)(x)$$

is called the *Taylor polynomial* of f at x of degree $\leq k$.

1.3. An elementary division theorem

By definition, a *convex* subset C of $\mathbb{R}^n$ has the property that for every $x_0, x_1 \in C$, the (parametric) segment $\gamma : [0,1] \to \mathbb{R}^n$, $\gamma(t) = (1-t)x_0 + tx_1$ is entirely contained in C.

THEOREM 1.2. (Elementary division theorem) *Let $f = (f_1, \ldots, f_m) \in \mathcal{C}^\infty(U, \mathbb{R}^m)$ where $U \subset \mathbb{R}^n$ is a convex open subset. Assume that $0 \in U$ and $f(0) = 0$. Then there are smooth maps $g_j = (g_{j1}, \ldots, g_{jm}) : U \to \mathbb{R}^m$, $j = 1, \ldots, n$, such that for every $x \in U$, $f(x) = \sum_j x_j g_j(x)$ and (necessarily) $g_{ji}(0) = \frac{\partial f_i}{\partial x_j}(0)$.*

Proof : It is a basic property of elementary integration that for every smooth function $h : U \to \mathbb{R}$, the function $\tilde{h} : U \to \mathbb{R}$ defined by $\tilde{h}(x) = \int_0^1 h(tx)dt$ is smooth. By the fundamental theorem of elementary integration for continuous functions, we have that

$$f(x) = \int_0^1 \frac{df(tx)}{dt}dt = (\int_0^1 \frac{df_1(tx)}{dt}dt, \dots, \int_0^1 \frac{df_m(tx)}{dt}dt) \ .$$

By the chain rule, for every $i = 1, \dots, m$,

$$\int_0^1 \frac{df_i(tx)}{dt}dt = \int_0^1 (\sum_j x_j \frac{\partial f_i}{\partial x_j}(tx))dt = \sum_j x_j \int_0^1 \frac{\partial f_i}{\partial x_j}(tx))dt \ .$$

We achieve the proof by setting

$$g_{ji}(x) := \int_0^1 \frac{\partial f_i}{\partial x_j}(tx))dt$$

■

The above proof still holds if we assume only that U is *starred* with centre at 0.

REMARK 1.3. In the setting of the division theorem, if $n = m = 1$, we have that $f(x) = xg(x)$; that is, the coordinate function x divides f. Assume now that $m = 1$, f is defined on an open set of the form $U = A \times (-1, 1) \subset \mathbb{R}^{n-1} \times \mathbb{R}$, and that $\{f = 0\} = U \cap \{x_n = 0\}$. Then by applying fibre by fibre the same construction of the above proof, we get that $f(x) = x_n g(x)$. Moreover, if f is a submersion, then $g(x)$ is nowhere vanishing.

We will see several applications of the division theorem.

1.4. Bump functions and partitions of unity

Consider the function $\alpha : \mathbb{R} \to \mathbb{R}$ defined by $\alpha(x) = 0$ if $x \leq 0$, $\alpha(x) = e^{-\frac{1}{x}}$ if $x > 0$. It is easy to verify that α is smooth and that, for every $k \geq 1$, $\frac{d^k f}{dx^k}(0) = 0$. We say that α is *flat* at 0, although α is not locally constant at 0. This is an important feature of the "flexibility" of smooth functions that makes them suited for topological applications. For example, analytic functions are much more rigid: an analytic function on $\mathbb{R}$ which is flat at some points is constant.

Let us fix two real numbers $0 < a < b$. Define $\beta = \beta_{a,b} : \mathbb{R} \to \mathbb{R}$,

$$\beta(x) = \alpha(x - a)\alpha(b - x) \ .$$

The function β is smooth, $\beta(x) = 0$ on $\{x \leq a\} \cup \{x \geq b\}$, and it is strictly positive on $\{a < x < b\}$ with a unique maximum; β is flat at a and b.

Define $\gamma = \gamma_{a,b} : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(x) = \frac{\int_{|x|}^b \beta(t)dt}{\int_a^b \beta(t)dt} \ .$$

Then γ is smooth, $\gamma(x) = 1$ if $|x| \leq a$, $\gamma(x) = 0$ if $|x| \geq b$, $0 \leq \gamma(x) \leq 1$ and it is monotone on each connected interval of $\{a < |x| < b\}$; γ is flat at $\pm a$ and $\pm b$. For every $n \geq 1$, we can define $\gamma_n : \mathbb{R}^n \to \mathbb{R}$, $\gamma_n = \gamma_{n,a,b}(x) = \gamma_{a,b}(||x||)$. We will omit the index n whenever the dimension is clear by the context. Such a function $\gamma_{a,b} : \mathbb{R}^n \to \mathbb{R}$ is called a *bump function* on $\mathbb{R}^n$ with centre 0 and rays a, b. If $\tau_p(x) = x - p$, then

$$\gamma_{p,a,b} = \gamma_{a,b} \circ \tau_p$$

is a bump function with centre p; the centre will also be omitted when it is clear from the context.

Recall that the *support* of a function is the closure of the set where it is not zero. Hence $\overline{B}^n(p, b)$ is the support of $\gamma_{p,a,b}$.

We also introduce bump functions "at infinity" as follows. Let $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ as the hyperplane with equation $x_{n+1} = 0$. Denote by

$$\pi^+ : S^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$$

$(e_{n+1} = (0, \ldots, 0, 1))$ the *stereographic projection* defined geometrically by

$$\pi^+(x) = r(x, e_{n+1}) \cap \mathbb{R}^n$$

where $r(x, e_{n+1})$ is the straight line passing through the two points. Similarly, define the projection $\pi^- : S^n \setminus \{-e_{n+1}\} \to \mathbb{R}^n$. We easily verify by direct computation that

$$\rho := \pi^- \circ (\pi^+)^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$$

is a diffeomorphism. A *bump function at infinity* is, by definition, of the form $\gamma_\infty(x) = \gamma \circ \rho(x)$ if $x \in \mathbb{R}^n \setminus \{0\}$, $\gamma_\infty(0) = 0$. Clearly, it is smooth.

We now extend the definition to bump functions at an arbitrary compact $K \subset \mathbb{R}^n$, as follows. Let $K \subset \mathbb{R}^n$ be a compact set and U an open neighbourhood of K. Then we can find $W_0 := U_{\infty,a_\infty} := \mathbb{R}^n \setminus \overline{B}^n(0, a_\infty)$, some $W_j := B^n(p_j, b_j)$, $j = 1, \ldots, k$, and some $0 < a_j < b_j$, $a_\infty < b_\infty$ such that:

(1) $\overline{W}_0 \cap K = \emptyset$;
(2) The open balls $U_j := B^n(p_j, a_j)$ together with $U_0 := U_{\infty,b_\infty}$ make a finite open covering $\mathcal{U}$ of $\mathbb{R}^n$;
(3) The union of the above open balls that intersect K is an open neighbourhood $U' \subset U$ of K.

Denote by γ_0 the bump function at infinity with support equal to $\overline{W}_0$ and constantly equal to 1 on U_0; by γ_j the bump function at p_j with rays a_j, b_j. For every $j = 0, \ldots, k$, define the smooth function

$$\lambda_j := \frac{\gamma_j}{\sum_j \gamma_j} \ .$$

By the properties of the covering $\mathcal{U}$ and of the bump functions, the denominator is strictly positive everywhere. Clearly, for every $x \in \mathbb{R}^n$,

$$\sum_j \lambda_j(x) = 1 \ .$$

Such a family of functions $\{\lambda_j\}$ is called a *partition of unity subordinate to the (finite) covering* $\mathcal{U}$. We define "local" constant functions $c_j : W_j \to \mathbb{R}$, such that $c_j = 1$ if $U_j \cap K$ is nonempty and $c_j = 0$ otherwise. Finally set

$$\gamma_K = \sum_j \lambda_j c_j \ .$$

By construction γ_K is smooth, it is constantly equal to 1 on U' and has compact support contained in U. Any γ_K constructed in this way is called a *bump function at* K.

Bump functions are an important device. A basic use is the following: let U be an open neighbourhood of a compact set K as above and $f : U \to \mathbb{R}$ be a smooth function *locally* defined at K. Sometimes, it is useful to find a *globally* defined smooth function $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ with compact support and which locally agrees with f at K; that is, there is an open neighbourhood $K \subset U' \subset U$ such that $f(x) = \hat{f}(x)$ for $x \in U'$. Take any bump function $\gamma = \gamma_K$ at K constructed as above; then $\hat{f}$ defined by $\hat{f}(x) = \gamma(x)f(x)$ if $x \in U$, $\hat{f}(x) = 0$ if $x \in \mathbb{R}^n \setminus U$, does the job.

1.5. The smooth category of open sets in Euclidean spaces

Let $f : U \to W$, $g : U' \to W'$ be smooth maps between open subsets of some (possibly variable) Euclidean spaces. The composition $g \circ f$ is defined when $W \subset U'$. The fundamental well known *chain rule* for the composition of differentiable maps states that for every $x \in U$, $y = f(x)$, $g \circ f$ is differentiable at x and

$$d_x(g \circ f) = d_y g \circ d_x f \ .$$

It follows immediately that if f and g are smooth, then $g \circ f$ is also smooth. We can consider the category whose *objects* are the open subsets of Euclidean spaces and for every pair (U, W) of objects, the *"arrows"* (that is the *morphisms*) are the smooth maps $\mathcal{C}^\infty(U, W)$.

For every object $U \subset \mathbb{R}^n$, the unit map 1_U is the *identity*

$$\mathrm{id}_U : U \to U, \ \mathrm{id}_U(x) = x$$

which is obviously smooth. For every $x \in U$,

$$d_x \mathrm{id}_U = \mathrm{id}_{\mathbb{R}^n} = I_n \in \mathrm{End}(\mathbb{R}^n) = M(n, \mathbb{R}) \ .$$

If $U' \subset U$, then the inclusion $i : U' \to U$ is smooth and for every $f \in \mathcal{C}^\infty(U, W)$, the *restriction* $f|_{U'} = f \circ i$ is smooth.

The *equivalences* in this category are the *diffeomorphisms*. Let $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be open sets. Then $f \in C^\infty(U, W)$ is a diffeomorphism if it is a homeomorphism and also the inverse map $f^{-1} : W \to U$ is smooth. In such a case, by applying again the chain rule, we have that for every $x \in U$, $y = f(x)$, $d_y f^{-1} \circ d_x f = I_n$, $d_x f \circ d_y f^{-1} = I_m$. Then, by elementary linear algebra, both inequalities $n \leq m$ and $m \leq n$ hold, so that $m = n$; finally, $d_x f \in \mathrm{GL}(n, \mathbb{R})$ is invertible and

$$d_y f^{-1} = (d_x f)^{-1} \ .$$

In this way, we have proved the *invariance of the dimension up to diffeomorphism*, reducing it to the basic invariance of dimension up to linear isomorphism.

Here is another consequence of these considerations based on the chain rule:

If a smooth homeomorphism $f : U \to W$ has differentiable inverse map f^{-1} then it is a diffeomorphism (i.e. f^{-1} is smooth indeed).

1.6. The chain rule and the tangent functor

The chain rule can be rephrased in the language of *functors* between categories. One way is to consider the category of *pointed* open subsets of some Euclidean spaces and pointed smooth maps. Then, by setting

$$(U, x),\ U \subset \mathbb{R}^n \Longrightarrow \mathbb{R}^n$$

$$f \in \mathcal{C}^\infty((U, x), (W, y)),\ U \subset \mathbb{R}^n,\ W \subset \mathbb{R}^m \Longrightarrow d_x f \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

we define a *covariant* functor from the smooth pointed category to the category of finite-dimensional real vector spaces and linear maps.

Avoiding to deal with the pointed category, another way is by defining the so-called *tangent functor* which is a covariant functor from our favourite category to itself. Set

$$U \subset \mathbb{R}^n \Longrightarrow T(U) := U \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$$

$$f \in \mathcal{C}^\infty(U, W) \Longrightarrow Tf \in \mathcal{C}^\infty(T(U), T(W)),\ Tf(x, v) := (f(x), d_x f(v)) \ .$$

The chain rule can be rewritten as

$$T(g \circ f) = Tg \circ Tf$$

$$T\mathrm{id}_U = \mathrm{id}_{T(U)} \ .$$

If $f \in \mathcal{C}^\infty(U, W)$ is a diffeomorphism, then Tf is also a diffeomorphism (of a special kind indeed, see below). The map Tf is called the *tangent map* of f.

To explain the meaning of this construction, we use some notions recalled above concerning every $\mathbb{R}^n$ considered as an affine space over itself. U is an open set of the affine space $\mathbb{R}^n$. Hence, at each point $x \in U$, we have the tangent vector space $T_x U := T_x \mathbb{R}^n = \mathbb{R}^n$ determined by the affine structure. Precisely, we can consider $T(U) := U \times \mathbb{R}^n$ as an open set of $T(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$ and the restriction

$$\pi_U : U \times \mathbb{R}^n \to U,\ \pi_U(x, v) = x$$

of the projection of $\mathbb{R}^n \times \mathbb{R}^n$ onto the space factor. For every point $x \in U$, the fibre $T_x U := \pi_U^{-1}(x)$ is a copy of the vector space $\mathbb{R}^n$, called the *tangent space* to U at the point x. Every $v \in T_x U$ is a tangent vector at x. $(T(U), \pi_U)$ is called the *tangent vector bundle of* U . If $f : U \to \mathbb{R}^m$ is the restriction of an affine map $f : \mathbb{R}^n \to \mathbb{R}^m$, then it is smooth and its differential map is the *constant map* which associates to every point $x \in U$, $d_x f = L_f$, the linear part of f. Then the tangent map of an arbitrary smooth map defined so far generalizes the tangent map of an affine map already considered. For every smooth map $f : U \to W$, $\pi_W \circ Tf = f \circ \pi_U$, that is Tf sends each fibre $T_x U$ to the fibre $T_{f(x)} W$ by means of the linear map $d_x f$ which varies smoothly when x varies in U. In particular, if f is a diffeomorphism, then Tf is a diffeomorphism of a special type: it preserves the fibres of the tangent bundles, the family $\{d_x f\}_{x \in U}$ being a smooth 'field' of linear isomorphisms. We can say that the vector space structure of each tangent space $T_x U$, arising *a priori* from the affine space structure of $\mathbb{R}^n$, is preserved by the tangent map of any smooth map and eventually acquires substance within the smooth category.

1.6.1. Differential interpretations of the tangent spaces. The tangent functor defined so far is consistent and suitable for all future applications and generalizations. However, the smooth meaning of the tangent spaces could sound to someone too indirect. We are going to present a few direct differential interpretations of them. Nevertheless, the reader will eventually realize that everything is based on the functorial nature of the chain rule which we have already incorporated in the definition of the tangent functor.

As usual, let U be an open set of $\mathbb{R}^n$, $p \in U$. There are at least two natural ways of thinking about the tangent vectors at p. The first one is probably the most familiar to the reader and identifies the tangent space to U at p as the space of punctual velocities of the parametric smooth curves in U passing through the point p. Precisely, let $\gamma : \mathbb{R} \to U$ be a smooth parametric curve in U such that $\gamma(0) = p$. The differential $d_0\gamma \in M(n, 1, \mathbb{R}) = \mathbb{R}^n$ and this last space coincides with $T_p U$ according with the previous definition; $d_0\gamma$ is also called the *velocity* of γ at $p = \gamma(0)$. Let us consider on the set of all smooth curves as above, the equivalence relation such that $\gamma \sim \tau$ if and only if $d_0\gamma = d_0\tau$. The quotient set is denoted by $\mathcal{V}_p U$ and we have defined a map $\mathfrak{v}_p : \mathcal{V}_p U \to T_p U$. Now we list some properties of this map. The details of the justifications are left to the reader as an exercise.

(1) The equivalence class of a curve γ only depends on its *local* behaviour at 0. Moreover, we could extend the definition of $\mathcal{V}_p U$ by using curves defined on any open interval $(-\epsilon, \epsilon)$. However, this is not a true generalization; by using the bump functions and the partitions of unity, we readily see that there is a diffeomorphism $\phi : \mathbb{R} \to (-\epsilon, \epsilon)$ such that $\phi(t) = t$ for every $t \in (-\epsilon/2, \epsilon/2)$, hence every equivalence class admits a representative defined on the whole real line.

(2) If $d_0\gamma = v$, then γ is equivalent to a locally affine curve $\rho : \mathbb{R} \to U$ such that $\rho(t) = p + tv$ for every t belonging to a small interval $(-\delta, \delta) \subset \mathbb{R}$. Two such locally affine curves are equivalent if and only if they locally

coincide at 0. For every $v \in T_pU$, there is a locally affine curve ρ such that $d_0\rho = v$. It follows that the map $\mathfrak{v}_p : \mathcal{V}_pU \to T_pU$ is bijective.

(3) The set $\mathcal{V}_pU$ has the natural structure of a real vector space defined as follows. If $U = \mathbb{R}^n$, for every curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ passing through p, set $\gamma_0(t) = \gamma(t) - p$, $d_0\gamma_0 = d_0\gamma$; for every couple of curves and every scalar λ, set $\gamma \tilde{+} \tau := p + (\gamma_0 + \tau_0)$, $\lambda \cdot \gamma = p + \lambda\gamma_0$. By elementary properties of the derivatives, the operations pass to the quotient set $\mathcal{V}_p\mathbb{R}^n$ giving it the required linear structure. In general, by composing with the inclusion of U in $\mathbb{R}^n$, it is no longer true that $\gamma \tilde{+} \tau$ and $\lambda \cdot \gamma$ are curves in U. However, using the partitions of unity as in (1), this holds for suitable representatives of the equivalence classes and the quotient set inherits a vector space structure as well. Once $\mathcal{V}_pU$ is endowed with this intrinsic linear structure, the map $\mathfrak{v}_p : \mathcal{V}_pU \to T_pU$ becomes a linear isomorphism.

(4) If $f : U \to W$ is a smooth map such that $f(p) = q$, then we well define a map $Vf_p : \mathcal{V}_pU \to \mathcal{V}_qW$ by associating to the class of each smooth curve $\gamma : \mathbb{R} \to U$ passing through the point p, the class of the composite curve $f \circ \gamma : \mathbb{R} \to W$, passing through q. It is a consequence of the chain rule that the map Vf_p is linear and that $\mathfrak{v}_q \circ Vf_p = d_pf \circ \mathfrak{v}_p$.

The second way of thinking about the tangent space to U at p is by identifying each tangent vector v as the operator of derivative in the direction of v. Consider the set of smooth functions $f : U' \to \mathbb{R}$ defined on some open neighbourhood U' of p in U. On this set, put the equivalence relation such that $(U_1, f_1) \sim (U_2, f_2)$ if and only if there is (U_3, f_3) such that $U_3 \subset U_1 \cap U_2$ and for every $y \in U_3$, $f_3(y) = f_1(y) = f_2(y)$. Denote by $\mathcal{E}_p$ the quotient set. Note that U is immaterial for this purely local definition, as we would get the same $\mathcal{E}_p$ by taking, for instance, the whole of $\mathbb{R}^n$ instead of U. On the other hand, using bump functions and partitions of unity as above, we see that every equivalence class has representatives defined on the whole open set U. Denote by $[f] = [f]_p$ an equivalence class. It is called *the germ of f at p.* The usual sum and product defined on every $\mathcal{C}^\infty(U', \mathbb{R})$ induce well-defined sum and product on $\mathcal{E}_p$ which make it a *commutative ring* as well as a real *vector space* with compatible operations. The translation $x \to x - p$ determines a canonical isomorphism between $\mathcal{E}_p$ and $\mathcal{E}_0$, so the considerations for $\mathcal{E}_0$ can be straightforwardly transported to $\mathcal{E}_p$ by this translation. Note also that $T_p = T_pU$ and $\mathcal{V}_p = \mathcal{V}_pU$ do not really depend on the choice of the open set containing the point p. Let $v = (v_1, \dots, v_n)^t \in T_0 = \mathbb{R}^n$. By the usual *derivative at* 0 *in the direction* v, we define the function

$$\delta_v : \mathcal{E}_0 \to \mathbb{R},\ \delta_v([f]) = \sum_j \frac{\partial f}{\partial x_j}(0)v_j \ .$$

We easily verify that δ_v is well defined (it does not depend on the choice of the representative f), it is $\mathbb{R}$-linear, and satisfies the *Leibniz identity*:

$$\delta_v([f][g]) = f(0)\delta_v([g]) + g(0)\delta_v([f]) \ .$$

If we prefer the interpretation of T_0 in terms of $\mathcal{V}_0$, the map δ_v can be obtained equivalently as follows. Let $\gamma : \mathbb{R} \to U$ be a curve passing through $p = 0$ such that $d_0\gamma = v$. Let $f : U \to \mathbb{R}$ be a representative of $[f]$ defined on the whole open set U. Then $\delta_v([f]) = d_0(f \circ \gamma)$; this is again a consequence of the chain rule.

Let us call a *derivation on* $\mathcal{E}_0$ any map $\delta : \mathcal{E}_0 \to \mathbb{R}$ that satisfies the same properties as above. Set $\mathrm{Der}(\mathcal{E}_0)$ the set of these derivations. It has the natural structure of a real vector space, so that the map

$$L : T_0 \to \mathrm{Der}(\mathcal{E}_0),\ L(v) = \delta_v$$

is $\mathbb{R}$-linear. Let us prove that L *is a linear isomorphism*. For every derivation δ, we will find a unique $v \in T_0$ such that $\delta = \delta_v$. It follows immediately from the derivation properties that, for every constant germ $[f]$ (i.e. with a constant representative), $\delta([f]) = 0$. For every $[f]$, we can take a representative f defined on a small open ball $B^n(0, \epsilon)$ (which is convex). By the division theorem, for every x in such a ball,

$$f(x) - f(0) = \sum_j g_j(x)x_j$$

for some smooth functions g_j. Then, by using again the derivation properties, we have

$$\delta([f]) = \sum_j \frac{\partial f}{\partial x_j}(0)\delta([x_j]) \ .$$

We conclude by taking $v = (\delta([x_1]), \dots, \delta([x_n]))^t$.

If $f : U \to W$ is a smooth map such that $f(p) = q$, then we well define a map $\mathfrak{d}f_p : \mathrm{Der}(\mathcal{E}_p) \to \mathrm{Der}(\mathcal{E}_q)$ by associating to every derivation δ on $\mathcal{E}_p$ the derivation $\mathfrak{d}f_p(\delta)$ on $\mathcal{E}_q$ such that $\mathfrak{d}f_p(\delta)([g]) = \delta([g \circ f])$. If $\delta([h]) = d_0(h \circ \gamma)$ for some curve γ passing through p, then $\mathfrak{d}f_p(\delta)([g]) = d_0(g \circ f \circ \gamma)$; again this is a consequence of the chain rule .

1.7. Tangent vector fields, Riemannian metrics, gradient fields

A *tangent vector field* on U (sometimes, we will omit "tangent") is a smooth map of the form

$$X : U \to T(U),\ X(x) = (x, v_X(x))$$

so that $\pi_U \circ X = \mathrm{id}_U$. Such a map is also called a (smooth) *section* of the tangent bundle. Hence X selects a family (a "field") of vectors $\{v_X(x) \in T_xU\}_{x \in U}$ which vary smoothly with the point $x \in U$. One could be tempted to confuse X with the smooth map $v_X : U \to \mathbb{R}^n$; however, they are objects of a different nature: if $\phi : U \to W$ is a diffeomorphim, as a map $v = v_X$ is transported on W by the composition $v \circ \phi^{-1}$, while the vector field X is transported to ϕ_*X on W by the composition $T\phi \circ X$; that is, for every $y = \phi(x) \in W$

$$\phi_*X(y) = (y, d_x\phi(v_X(x))) \ .$$

Denote by $\Gamma(T(U))$ the set of vector fields on U. For every $X, Y \in \Gamma(T(U))$, every $f \in \mathcal{C}^\infty(U, \mathbb{R})$, and every $x \in U$, set

$$X + Y(x) = (x, v_X(x) + v_Y(x)),\ fX(x) = (x, f(x)v_X(x)) \ .$$

This defines on $\Gamma(T(U))$ the structure of a *module* over the commutative ring $\mathcal{C}^\infty(U, \mathbb{R})$ which induces (by restriction to the constant functions) the structure of an $\mathbb{R}$-vector space. Let us denote by $\mathfrak{e}_i(x) = (x, e_i)$, $i = 1, \dots, n$, the *constant vector field* on U such that $e_i = (0, 0, \dots, 1, \dots, 0)^t$ is the ith-vector of the canonical basis of $\mathbb{R}^n$. Sometimes, $\mathfrak{e}_i$ is also denoted by $\frac{\partial}{\partial x_i}$. For every $X \in \Gamma(T(U))$,

$$X = \sum_i v_{X,i} \mathfrak{e}_i \ ;$$

that is, the fields $\mathfrak{e}_i$ form a basis of such a module.

The above discussion about tangent vector spaces and derivations can be *globalized* by replacing T_p with the set $\Gamma(T(U))$ and $\mathcal{E}_p$ with the commutative ring $\mathcal{C}^\infty(U, \mathbb{R})$ with the induced compatible structure of an $\mathbb{R}$-vector space. For every vector field $X \in \Gamma(T(U))$, define

$$\delta_X : \mathcal{C}^\infty(U, \mathbb{R}) \to \mathcal{C}^\infty(U, \mathbb{R}),\ \delta_X(f)(x) = \delta_{X(x)}([f]_x) \ .$$

The map δ_X is linear and satisfies the Leibniz rule

$$\delta_X(fg) = f\delta_X(g) + \delta_X(f)g$$

hence, by definition, it is a *derivation* on $\mathcal{C}^\infty(U, \mathbb{R})$. Finally the map

$$L : \Gamma(T(U)) \to \mathrm{Der}(\mathcal{C}^\infty(U, \mathbb{R})),\ L(X) = \delta_X$$

establishes an isomorphism of $\mathcal{C}^\infty$-modules. Note that if $\delta, \delta' \in \mathrm{Der}(\mathcal{E}_0)$ (resp. $\in \mathrm{Der}(\mathcal{C}^\infty(U, \mathbb{R}))$), then $\delta\delta'$ is not in general a derivation, while $\delta\delta' - \delta'\delta$ is a derivation. In particular, for every couple $X, Y \in \Gamma(T(U))$, there is a unique vector field $[X, Y]$ such that

$$L([X, Y]) = L(X)L(Y) - L(Y)L(X) \ .$$

A Riemannian metric on $U \subset \mathbb{R}^n$ is a smooth map

$$g : U \to M(n, \mathbb{R})$$

such that for every $x \in U$, the matrix $g(x)$ is symmetric and positive definite. Then $\{g(x)\}_{x \in U}$ is a smooth *field of positive definite scalar products* $(*, *)_{g(x)}$ defined on each tangent space T_xU. Denoted by $S(n, \mathbb{R})$, the space of symmetric $n \times n$ matrices, it can be identified with $\mathbb{R}^{\frac{n(n+1)}{2}}$. By setting

$$U \to U \times S(n, \mathbb{R}),\ x \to (x, g(x)) \ ,$$

the Riemannian metric can be interpreted as a section of the "product bundle" $U \times S(n, \mathbb{R}) \to U$.

If g is a Riemannian metric on U and $X, Y \in \Gamma(T(U))$, then

$$x \to (v_X(x), v_Y(x))_{g(x)}$$

defines a smooth function on U.

If g_0 and g_1 are Riemannian metrics on U, then $g_t = (1-t)g_0 + tg_1$, $t \in [0,1]$, is a path of Riemannian metrics.

An *isometry* $\phi : (U,g) \to (W,h)$ $(g,h$ being Riemannian metrics) is by definition a diffeomorphism such that for every $x \in U$, every $v, w \in T_xU$,

$$(v,w)_{g(x)} = (d_x\phi(v), d_x\phi(w))_{h(\phi(x))} \ .$$

Given (U,g) and a diffeomorphism $\phi : U \to W$, this transports g to the Riemannian metric $\phi_* g$ on W such that ϕ is *tautologically* an isometry. If $y \in W$, set $P(y) = d_y\phi^{-1}$, then

$$\phi_* g : W \to M(n,\mathbb{R}), \ y \to P(y)^t g(\phi^{-1}(y))P(y) \ .$$

If $f \in \mathcal{C}^\infty(U,\mathbb{R})$, its differential function $df : U \to M(1,n)$ can be considered as a smooth *field of linear functionals* $\{d_xf : T_xU \to \mathbb{R}\}_{x\in U}$ with d_xf belonging to the dual space T^*_xU; in other words, it is identified with the section $x \to (x, d_xf)$ of the *cotangent bundle*

$$\pi^* : T^*(U) = U \times M(1,n) \to U \ .$$

Every section $\Omega(x) = (x, \omega(x))$ is called a *differential form* on U. For every form Ω and every vector field X on U,

$$x \to \omega(x)(v(x))$$

defines a smooth function on U. If $\phi : U \to W$ is a diffeomorphism, Ω a differential form on U, then ϕ transports Ω to the form $\phi_*\Omega$ on W such that for every $y \in W$, every $w \in T_yW$, then

$$\phi_*\Omega(y) = (y, \omega(d_y\phi^{-1}(w))) \ .$$

Denote by $\mathfrak{e}^j$, $j = 1, \dots, n$, the field constantly equal to the functional e^j such that

$$(e^i(e_j))_{i,j} = I_n \in M(n,\mathbb{R}) \ .$$

Then every $\Omega \in \Gamma(T^*(U))$ is a unique linear combination

$$\Omega = \sum_j a_j \mathfrak{e}^j \ ,$$

the a_j being smooth functions on U. Sometimes, we write ∂x_j instead of $\mathfrak{e}^j$.

If g is a Riemannian metric on U, then by setting for every $v, w \in T_xU$,

$$\psi_v(w) := g(x)(v,w) \in \mathbb{R}$$

we define a smooth *field of linear isomorphisms* $\Psi_g := \{\Psi_{g,x} : T_x(U) \to T^*_x(U)\}_{x\in U}$. This transforms vector fields into differential forms. For every $f \in \mathcal{C}^\infty(U,\mathbb{R})$, let $\nabla_g f$ be the unique vector field on U such that $\Psi_g(\nabla_g f) = df$, so that for every $x \in U$, $v \in T_x(U)$, then

$$d_xf(v) = (\nabla_g(x), v)_{g(x)} \ .$$

The field $\nabla_g f$ is called the *gradient of f with respect to the metric g* . Clearly, for every $x \in U$, $d_xf(\nabla_g(x)) = (\nabla_g(x), \nabla_g(x))_{g(x)} \geq 0$ and is strictly positive if and only if $d_xf \neq 0$.

Every U admits Riemannian metrics; for example, any constant one $g_A(x) = A$, where A is a symmetric positive definite matrix. In particular, $g_0 := g_{I_n}$ is called the *standard Riemannian metric*. For every smooth function f on U,

$$\nabla_{g_0} f(x) = d_x f^t \ .$$

The partitions of unity provide a very flexible way to construct Riemannian metrics on $\mathbb{R}^n$, hence on every $U \subset \mathbb{R}^n$. Here we adopt the notations of the end of Section 1.4. Fix on every U_j an arbitrary Riemannian metric g_j (for instance a constant one varying with j). Then

$$g = \sum_j \lambda_j g_j$$

is a well defined Riemannian metric on the whole of $\mathbb{R}^n$.

1.8. Inverse function theorem and applications

Let $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be a linear map of *maximal rank* r. There are a few possibilities and using elementary linear algebra, for every case there is a *normal form* up to pre or post composition with linear isomorphisms.

- If $r = n = m$, then $L \in GL(n, \mathbb{R})$ is invertible and the normal form is I_n obtained as
$$I_n = L \circ L^{-1} = L^{-1} \circ L \ .$$
- If $n < m$, then the rank r is equal to n and L is *injective*. Let us fix a direct sum decomposition
$$\mathbb{R}^m = L(\mathbb{R}^n) \oplus V$$
and a basis $\mathcal{B} = \mathcal{B}' \oplus \mathcal{B}$" of $\mathbb{R}^m$ adapted to the decomposition. This determines a linear isomorphism $\phi_{\mathcal{B}} : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ such that for every $x = (x_1, \dots, x_n)^t \in \mathbb{R}^n$, we have
$$\phi_{\mathcal{B}} \circ L(x) = (x_1, \dots, x_n, 0, \dots, 0)^t$$
that is the standard inclusion $j = j_{n,m} : \mathbb{R}^n \to \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$. This is the normal form in this case.
- If $n > m$, then the rank r is equal to m and L is *surjective*. Fix a direct sum decomposition
$$\mathbb{R}^n = V \oplus \ker(L)$$
and an adapted basis $\mathcal{B} = \mathcal{B}' \oplus \mathcal{B}$" of $\mathbb{R}^n$. This determines a linear isomorphism (in fact the inverse of the above defined $\phi_{\mathcal{B}}$) $\psi_{\mathcal{B}} : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ such that for every $x = (x_1, \dots, x_n)^t \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, we have
$$L \circ \psi_{\mathcal{B}}(x) = (x_1, \dots, x_m)$$
that is the natural projection $\pi_{n,m} : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$. This is the normal form in this case.

Let us consider now $f \in \mathcal{C}^\infty(U,W)$, $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$, and $p \in U$. Assume that $d_p f$ has maximal rank r. The following fundamental theorems state that locally in a neighbourhood of p in U, the map f takes the same normal form of the linear map $d_p f$, up to pre or post composition with smooth diffeomorphisms. As a first step, let us remark that the *punctual* hypothesis has in fact a *local* valence. By a well known criterion, $d_p f$ has maximal rank r if and only if there is an $r \times r$ submatrix $A(p)$ of $d_p f$ such that $\det A(p) \neq 0$. By taking the same submatrix $A(x)$ of $d_x f$ for every $x \in U$, we define the smooth function

$$\det A : U \to \mathbb{R},\ x \to \det A(x)\ .$$

Then, there exists an open neighbourhood $U' \subset U$ of p in U, such that for every $x \in U'$, $d_x f$ has maximal rank r.

A map $f \in \mathcal{C}^\infty(U,W)$ such that for every $x \in U$, $d_x f$ is injective is called an *immersion* . If $d_x f$ is surjective for every $x \in U$, then f is called a *submersion* . If $n = m$ the two notions coincide. We can state now the theorem mentioned in the title.

THEOREM 1.4. (Inverse function theorem) *Let $f \in \mathcal{C}^\infty(U,W)$, $U,W \subset \mathbb{R}^n$, such that for every $p \in U$, the differential $d_p f$ is invertible. Then f is a* local diffeomorphism*, that is for every $p \in U$ there is an open neighbourhood U' of p in U such that $W' = f(U')$ is an open subset of W and the restriction $f|_{U'} \in \mathcal{C}^\infty(U',W')$ is a diffeomorphism.*

COROLLARY 1.5. (Immersion local form) *Let $f \in \mathcal{C}^\infty(U,W)$, $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$, $n < m$, be an immersion. Then for every $p \in U$ there is*

- *An open neighbourhood U' of 0 in $\mathbb{R}^n$;*
- *An open neighbourhood W' of $q = f(p)$ in W;*
- *An open neighbourhood W" of 0 in $\mathbb{R}^m$ and a diffeomorphism*

$$\phi : (W', q) \to (W", 0)$$

such that for every $x \in U'$, $x + p \in U$, $f(x+p) \in W'$ and

$$\phi \circ f(x+p) = j_{n,m}(x)\ .$$

COROLLARY 1.6. (Submersion local form) *Let $f \in \mathcal{C}^\infty(U,W)$, $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$, $n > m$, be a submersion. Then for every $p \in U$ there is*

- *An open neighbourhood U' of p in U;*
- *An open neighbourhood U" of 0 in $\mathbb{R}^n$ and a diffeomorphism*

$$\psi : (U", 0) \to (U', p)$$

such that $f(U') \subset W$ and

$$f \circ \psi(x) - f(p) = \pi_{n,m}(x)\ .$$

Proof of the corollaries. In both cases, it is not restrictive to assume that $p = 0$ and $f(0) = 0$. We will use the notations introduced at the beginning of the section, by replacing L with $d_0 f$.

(Immersions) Given a direct sum decomposition of $\mathbb{R}^m = d_0 f(\mathbb{R}^n) \oplus V$, with adapted basis $\mathcal{B} = \mathcal{B}' \oplus \mathcal{B}$" and associated linear isomorphism

$$\psi_{\mathcal{B}} : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^m \ ,$$

consider the smooth map

$$g : U \times \mathbb{R}^{m-n} \to \mathbb{R}^m,\ g(x,h) = f(x) + \psi_{\mathcal{B}}(0,h) \ .$$

It is easy to verify that $d_{(0,0)}g$ is invertible and we can apply the inverse function theorem to g on a neighbourhood $U' \times A$ of $(0,0)$. By construction, for every $x \in U'$, $f(x) = g \circ j_{n,m}(x)$, so that $g^{-1} \circ f(x) = j_{n,m}(x)$.

(Submersions) Given a direct sum decomposition $\mathbb{R}^n = V \oplus \ker(d_0 f)$ with adapted basis $\mathcal{B} = \mathcal{B}' \oplus \mathcal{B}$" and associated linear isomorphism

$$\phi_{\mathcal{B}} : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^{n-m} \ ,$$

set $p : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ the natural projection. Define

$$g : U \to \mathbb{R}^m \times \mathbb{R}^{n-m},\ g(x) = (f(x), p(\phi_{\mathcal{B}}(x))) \ .$$

The differential $d_0 g$ is invertible, so we can apply the inverse function theorem to g on a neighbourhood U' of 0. By construction, for every $x \in U'$, $f(x) = \pi_{n,m} \circ g(x)$, and we conclude similarly to the case of immersions.

■

Corollaries 1.5 and 1.6 are examples of the following general *constant rank theorem*. The proof, based again on the Inverse Function Theorem, is left to the reader as an exercise.

Theorem 1.7. (Constant rank theorem) *Let $f : U \to W$ be a smooth map, $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ be open sets. Assume that $d_x f$ is of constant rank $k \leq \min\{n,m\}$. Then for every $p \in U$, $q = f(p)$ up to pre- and post-composition with local diffeomorphisms $\psi : U' \to U$, $\psi(0) = p$, $\phi : W' \to W$, $\phi(0) = p$ we have that*

$$\rho := \phi^{-1} \circ f \circ \psi : U' \to W',\ \rho(u_1, \dots, u_n) = (u_1, \dots, u_k) \ .$$

■

Strictly related to the local submersion theorem, there is another corollary which is known as the *implicit function theorem*. A consequence of the proof of Corollary 1.6 is that there is a diffeomorphism $\rho : A \times B \to U'$, where $A \times B \subset \mathbb{R}^{n-m} \times \mathbb{R}^m$ is an open neighbourhood of $(x_0, y_0) = (0,0)$ and U' is an open neighbourhood of 0 in $U \subset \mathbb{R}^n$, such that restriction of $g = f \circ \rho$ to $A \times B$ satisfies:

(1) $g(x_0, y_0) = 0$;
(2) The restriction $\tilde{g}$ of g to $\mathbb{R}^m = \{x_0\} \times \mathbb{R}^m$ has invertible differential $d_{y_0}\tilde{g}$ at y_0

We take such a situation as the hypotheses of the implicit function theorem.

COROLLARY 1.8. (Implicit function theorem) *Let $A \times B \subset \mathbb{R}^k \times \mathbb{R}^m$ be an open set. Let $g : A \times B \to \mathbb{R}^m$ be a smooth map and $(x_0, y_0) \in A \times B$ such that $g(x_0, y_0) = 0$. Let $\tilde{g}$ be the restriction of g to $\mathbb{R}^m = \{x_0\} \times \mathbb{R}^m$. Assume that $d_{y_0}\tilde{g}$ is invertible. Then there exists an open neighbourhood $A' \times B'$ of (x_0, y_0) in $A \times B$, and a smooth map $h : A' \to B'$ such that*

$$\mathrm{Graph}(h) = f^{-1}(0) \cap A' \times B' \ .$$

It follows that $f(x, h(x)) = 0$ for every $x \in A'$ and h is said to be (locally) *implicitly defined by the equation* $f(x, y) = 0$.

Sketch of proof. Consider the smooth map

$$G : A \times B \to \mathbb{R}^k \times \mathbb{R}^m, \ G(x, y) = (x, g(x, y)) \ .$$

It is immediate that $d_{(x_0,y_0)}G$ is invertible, so we can apply the inverse function theorem to G in a neighbourhood $A_1 \times B'$ of (x_0, y_0), and the inverse map is necessarily of the form

$$G^{-1}(x, y) = (x, l(x, y))$$

for a suitable smooth map

$$l : G(A_1 \times B') \to B' \ .$$

Take

$$A' = \{x \in A; \ (x, 0) \in G(A_1 \times B')\}$$

and define $h(x) = l(x, 0)$. The reader can verify as an exercise that $A' \subset A_1$ and this eventually achieves the proof.

■

A proof of the inverse function theorem should be known to the reader. A current proof is based on *Banach's principle* for contractions on complete metric spaces. This is suited for generalizations to infinite-dimensional Banach spaces. We will just sketch a proof in our finite-dimensional situation, based on elementary properties of continuous functions on compact spaces.

Sketch of a proof of the inverse function theorem. We can assume for simplicity, and it is not restrictive, that $p = 0$ and $f(p) = 0$. Possibly by composing f with $(d_0 f)^{-1}$, we can also assume that $d_0 f = I_n$.

The proof is achieved by following the next sequence of claims.

Claim 1. There is a sufficiently small closed ball $\overline{B} = \overline{B}^n(0, \epsilon) \subset U$ such that

(1) For every $x \in \overline{B}$, $d_x f$ is invertible;
(2) For every $x \in \overline{B}$, $x \neq 0$, then $f(x) \neq 0$;
(3) For every $x, z \in \overline{B}$, $2||f(x) - f(z)|| \geq ||x - z||$.

Assuming these facts, by the continuity of the function and the compactness of $\partial\overline{B}$, there is $\delta > 0$ such that for every $x \in \partial\overline{B}$, $||f(x)|| \geq \delta$. Consider the open ball $B' = B^n(0, \delta/2)$.

Claim 2. Set $A = B \cap f^{-1}(B')$. Then the restriction $\phi : A \to B'$ of f to the open set A is bijective.

Claim 3. ϕ is a homeomorphism.

Claim 4. ϕ is a diffeomorphism.

Proof of Claim 1. The first point is evident. Assuming that the second point fails, there would be a sequence x_n in U, converging to 0, such that $f(x_n) = 0$ for every n. Hence $||\frac{f(x_n)-x_n}{x_n}|| = 1$ against the fact that $d_0 f = I_n$. As for the third point, consider the function $g(x) = f(x) - x$, so that

$$||x - z|| - ||f(x) - f(z)|| \leq ||g(x) - g(z)|| \ .$$

As

$$\frac{\partial g_i}{\partial x_j}(x) = \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(0)$$

we can take ϵ to make $|\frac{\partial g_i}{\partial x_j}(x)| < \frac{1}{2n^2}$ uniformly on $\overline{B}$. Then the conclusive inequality

$$||g(x) - g(z)|| \leq \frac{1}{2}||x - z||$$

is obtained by applying several times the Main Value Theorem for functions of one variable.

Proof of Claim 2. It is enough to prove that for every $y \in B'$ there is a unique $x \in U$ such that $f(x) = y$. The smooth function $h(x) = ||y - f(x)||^2$ has a minimum point p on the compact set $\overline{B}$ and by construction p belongs necessarily to the open ball B. A simple computation shows that $d_p f(y - f(p)) = 0$, hence $y - f(p) = 0$ because $d_p f$ is invertible. As for the uniqueness, this follows by the inequality $||p_1 - p_2|| \leq 2||f(p_1) - f(p_2)||$, so that $p_1 = p_2$ if $f(p_1) = f(p_2) = y$.

Proof of Claim 3. The same inequality implies that $||\phi^{-1}(y_1) - \phi^{-1}(y_2)|| \leq 2||y_1 - y_2||$ and the continuity of ϕ^{-1} follows.

Proof of Claim 4. As we know, it is enough to show that ϕ^{-1} is differentiable. By using directly the definition of the differential, you can prove that $d_y\phi^{-1} = (d_x\phi)^{-1}$, where $y = \phi(x)$.

■

1.9. Topologies on spaces of smooth maps

Let $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ be open sets. For every $k \geq 0$, we define a topology δ_k on $\mathcal{C}^k(U, W)$; we will denote by $\mathcal{E}^k(U, W)$ the set $\mathcal{C}^\infty(U, W) \subset \mathcal{C}^k(U, W)$ endowed with the subspace topology. We determine δ_k by giving, for every $f = (f_1, \dots, f_m) \in \mathcal{C}^k(U, W)$, a basis of open neighbourhoods $\mathcal{U}_k(f, K, \epsilon)$ where the varying arguments are a compact subset $K \subset U$ and a real $\epsilon > 0$. Then, by definition, $g \in \mathcal{C}^k(U,W)$ belongs to $\mathcal{U}_k(f,K,\epsilon)$ if and only if

(1) For every $x \in K$, $||g(x) - f(x)|| < \epsilon$;
(2) For every multi-index J such that $|J| = r \leq k$, for every $i = 1,\dots,m$, for every $x \in K$, we have

$$||\frac{\partial^J (g_i - f_i)}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}(x)|| < \epsilon \ .$$

We omit the verification that these define bases of neighbourhoods of a topology.

We denote by $\mathcal{E}(U,W)$ the set $\mathcal{C}^\infty(U,W)$ endowed with the *union topology* $\delta = \cup_k \delta_k$.

All these are called *weak topologies* . There are other so-called *strong* topologies, σ_k, on the same sets of maps. By considering, for example, $\mathcal{E}(\mathbb{R}^n,\mathbb{R})$, we can control the difference of two functions, up to an arbitrarily prescribed order, on an arbitrarily given compact set K, but we have not any control "at infinity". The strong topologies σ_k, which contain δ_k being *heavily finer*, allow instead such control at infinity. On another hand, the weak topologies δ_k have nice properties; for example, we can prove that they are metrizable, hence every f has a countable basis of open neighbourhoods. On the contrary, this is not the case for the strong topologies; for example, if we have a converging sequence $g_n \to f$ in $\mathcal{C}^k(\mathbb{R}^n,\mathbb{R})$ for the strong topology, then there exists a compact set K in $\mathbb{R}^n$ such that g_n definitely equals f on the complement of K. *We do not define the strong topologies.* To our aims, the control at compact sets will suffice.

Let us recall (a particular case of) the classical *Stone-Weierstrass theorem* (see [**Stone**]).

Theorem 1.9. *For every $f \in \mathcal{C}^k(U,\mathbb{R}^m)$, for every $k \geq 0$, for every neighbourhood $\mathcal{U} = \mathcal{U}_k(f,K,\epsilon)$, there exists a polynomial map $p : \mathbb{R}^n \to \mathbb{R}^m$ such that the restriction of p to U belongs to $\mathcal{U}$. In other words, the polynomial maps are* dense *in $\mathcal{C}^k(U,\mathbb{R}^m)$ for every $k \geq 0$ and in $\mathcal{E}(U,W)$.*

1.10. Stability of submersions and immersions at a compact set

Let $f \in \mathcal{C}^k(U,W)$ be as above, $k \geq 1$, $K \subset U$ a compact set. We say that f is *a submersion (resp. an immersion) at* K if, for every $x \in K$, $d_x f$ is surjective (resp. injective). This is equivalent to the fact that there exists an open neighbourhood $U' \subset U$ of K such that the restriction of f to U' is a submersion (immersion). Here are the stability results.

Proposition 1.10. *If f is either (1) a submersion, (2) an immersion, or (3) an injective immersion at K, then there is a neighbourhood $\mathcal{U} = \mathcal{U}_1(f,K,\epsilon)$ such that every $g \in \mathcal{U}$ shares the same properties of f, respectively.*

Proof : If $n \geq m$ (resp. $n < m$), then every $m \times n$ matrix A has $\binom{n}{m}$ $m \times m$ (resp. $\binom{m}{n}$) submatrices A_j; in any case define

$$\delta(A) = \sum_j (\det A_j)^2 \ .$$

In both cases (1) and (2) the hypothesis is equivalent to $d(x) := \delta(d_x f) > 0$ for every $x \in K$. As d is continuous and K is compact, then

$$\sup_{x \in K}\{d(x)\} = \max_{x \in K}\{d(x)\} = d_0 > 0 \ .$$

It is clear that if g is close enough to f at K in $\mathcal{C}^1(U, W)$, then $\delta(d_x g) > 0$ for every $x \in K$. As for (3), assume that the thesis fails. Then there exist a sequence $g_n \in \mathcal{C}^1(U, W)$ and sequences of points x_n, y_n in the compact set K such that:

(1) Every g_n is an immersion at K (by (2) as above);
(2) $g_n \to f$ and $dg_n \to df$ uniformly on K;
(3) $x_n \to x$, $y_n \to y$ in K, $x_n \neq y_n$ and $g_n(x_n) = g_n(y_n)$ for every n;
(4) $v_n := \frac{x_n - y_n}{||x_n - y_n||} \to v \in S^{n-1}$ (by the compactness of the unitary sphere S^{n-1}).

Then $g_n(x_n) \to f(x)$, $g_n(y_n) \to f(y)$, hence $x = y$ because f is injective. Hence

$$||g_n(x_n) - g_n(y_n) - d_{y_n} g_n(x_n - y_n))||/||x_n - y_n|| \to 0$$

uniformly, so that

$$||d_{y_n} g_n(v_n)|| \to ||d_x f(v)|| = 0 \ .$$

This is absurd because $d_x f$ is injective.

■

1.11. Morse lemma

Let $f \in \mathcal{C}^\infty(U, \mathbb{R})$, U an open set of $\mathbb{R}^n$. A point $p \in U$ is *regular* for f if $d_p f \neq 0$ (that is f is a submersion near p); we say that p is *critical* (or also *singular*) otherwise. We are interested in the local behaviour of f at p (actually to the germ $[f]_p$). Up to pre or post composition with a translation, we can normalize the situation so that $p = 0$ and $f(0) = 0$. Moreover, we can assume that $U = B^n(0, \epsilon_0)$ for some $\epsilon_0 > 0$ and, case by case, we can restrict f to any $U_\epsilon = B^n(0, \epsilon)$, $0 < \epsilon \leq \epsilon_0$. For every ϵ, the commutative ring $\mathcal{C}^\infty(U_\epsilon, \mathbb{R})$ has a canonical ideal

$$\mathfrak{m}_\epsilon = \{g \in \mathcal{C}^\infty(U_\epsilon, \mathbb{R});\ g(0) = 0\}$$

so that we are assuming that $f \in \mathfrak{m}_\epsilon$. It is an immediate corollary of the division theorem that $\mathfrak{m}_\epsilon$ is generated by the coordinate functions x_j, $j = 1, \ldots, n$. Hence we have that for $x \in U$,

$$f(x) = \sum_j g_j(x) x_j,\ d_0 f = (g_1(0), \ldots, g_n(0)) \ .$$

If 0 is a regular point for f, the particular case of Theorem 1.6 can be rephrased by saying that, up to pre-composition with a local diffeomorphism, f locally coincides with $d_0 f$ that is its first Taylor polynomial $T_1(f)(0)$.

If 0 is critical, then all the smooth functions g_j vanish at 0, and we can apply again to each of them the division theorem. Eventually, we get that on U

$$f(x) = \sum_{|J|=2} g_J(x)x^J,\ x^J := x_1^{j_1} \dots x_n^{j_n} \ ;$$

that is, it has the form of a homogeneous polynomial of degree 2 whose coefficients are smooth functions. Moreover,

$$T_2(f)(0) = \sum_{|J|=2} g_J(0)x_1^{j_1} \dots x_n^{j_n} \ .$$

We can express $T_2(f)(0)$ in the form

$$T_2(f)(0) = \frac{1}{2}x^t H_0(f)x := Q_0(f)(x)$$

where $H_0(f)$ is the *symmetric* (by Schwartz Lemma) *Hessian matrix* of f at 0

$$H_0(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)_{i,j=1,\dots,n}$$

while $Q_0(f)$ is the associated *quadratic form.* We can organize the above functions $g_{i,j}$ to rewrite f on U as

$$f(x) = x^t G(x)x$$

where

$$G : U \to M(n, \mathbb{R})$$

is a smooth map such that $G(x) = G(x)^t$ is symmetric for every $x \in U$, and $G(0) = T_2(f)(0)$.

We say that the critical point $x = 0$ is *non-degenerate* if

$$\det H_0(f) \neq 0 \ .$$

We have the following characterization of non-degenerate critical points. For every U_ϵ, denote by $J(f,\epsilon)$ the *Jacobian ideal* of $\mathcal{C}^\infty(U_\epsilon, \mathbb{R})$ generated by the partial derivative functions $\frac{\partial f}{\partial x_j}$; that is, the ideal of the $\mathcal{C}^\infty(U_\epsilon, \mathbb{R})$-linear combinations $\sum_j h_j \frac{\partial f}{\partial x_j}$, $h_j \in \mathcal{C}^\infty(U_\epsilon, \mathbb{R})$. If 0 is a critical point, then $J(f,\epsilon) \subset \mathfrak{m}_\epsilon$.

LEMMA 1.11. *The point* 0 *is a non-degenerate critical point of* $f \in \mathcal{C}^\infty(U, \mathbb{R})$, $f(0) = 0$, *if and only if there exists* $0 < \epsilon \leq \epsilon_0$ *such that* $J(f,\epsilon) = \mathfrak{m}_\epsilon$.

Proof : It is enough to prove the inclusion "$\supseteq$", then it is enough to show that the generating coordinate functions x_j belong to $J(f,\epsilon)$ for some ϵ. As 0 is non-degenerate, the smooth map

$$x \to (\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x))$$

has invertible differential at 0, thus we can apply the inverse map theorem locally on a neighbourhood U_ϵ of 0, so that there are smooth functions F_j such that for every $j = 1, \dots, n$, $F_j(0) = 0$ and

$$x_j = F_j(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) \ .$$

Again by the division theorem we finally get

$$x_j = \sum_i G_{j,i}(x)\frac{\partial f}{\partial x_i}(x)$$

and the lemma is proved.

■

Assume that 0 is a non-degenerate critical point for f. We are going to prove that, up to pre-composition with local diffeomorphisms at 0, f locally coincides with $T_2(f)(0)$. More precisely, the Hessian matrix $H_0(f)$ has a certain *index of negativity* $0 \leq \lambda \leq n$ (i.e. the maximal dimension of the linear subspaces of $\mathbb{R}^n$ on which the restriction of the quadratic form $Q_0(f)$ is negative). By definition, λ *is the index of the non-degenerate critical point* 0. This notion is stable under local diffeomorphism.

LEMMA 1.12. *If* 0 *is a non-degenerate critical point of index* λ *of* $f \in \mathcal{C}^\infty(U_\epsilon, \mathbb{R})$, $f(0) = 0$, *and* $\phi : W \to U_\epsilon$ *is a diffeomorphism,* $\phi(0) = 0$, *then* 0 *is a non-degenerate critical point of* $f' := f \circ \phi$ *of index* λ.

Proof : By direct computation, using the chain rule and the fact that $d_0 f = 0$, we have

$$H_0(f') = d_0\phi^t H_0(f) d_0\phi$$

hence the symmetric matrices $H_0(f')$ and $H_0(f)$ are congruent, they are both non-singular, and have the same signature.

■

Let 0 be a non-degenerate critical point of f of index λ. Up to composition with a linear isomorphism $x = Pu$, we have that

$$Q_0(f)(Pu) = -(\sum_{j=1}^{\lambda} u_j^2) + (\sum_{j=\lambda+1}^{n} u_j^2) = u^t I_{n,\lambda} u$$

where $I_{n,\lambda}$ is the suitable diagonal matrix with ± 1 entries.

THEOREM 1.13. (Morse Lemma) *Let* 0 *be a non-degenerate critical point of index* $0 \leq \lambda \leq n$ *of* $f \in \mathcal{C}^\infty(U, \mathbb{R})$, $f(0) = 0$. *Then there is a local* *diffeomorphism* $x = \phi(u)$, $0 = \phi(0)$, *such that* $\psi := \phi^{-1}$ *is defined on some* U_ϵ *and*

$$f(\phi(u)) = u^t I_{n,\lambda} u \ .$$

Proof : It is not restrictive to assume that $H_0(f) = I_{n,\lambda}$. As above, let us take on U an expression of f of the form

$$f(x) = x^t G(x) x \ .$$

If $\epsilon > 0$ is small enough, on U_ϵ every symmetric matrix $G(x)$ has negativity index λ. By applying to the canonical basis of $\mathbb{R}^n$ the usual algorithm producing a normalized orthogonal basis for the scalar product $(*,*)_{G(x)}$, we get a smooth map

$$P : U_\epsilon \to \mathrm{GL}(n, \mathbb{R})$$

such that:

(1) $P(0) = I_n$;
(2) For every $x \in U_\epsilon$, the linear isomorphism $x = P(x)u$ is such that

$$P(x)^t G(x) P(x) = I_{n,\lambda} \ .$$

Consider the smooth map

$$\psi : U_\epsilon \to \mathbb{R}^n, \ \psi(x) = P(x)^{-1} x \ .$$

The map ψ has invertible differential at 0, so by the inverse map theorem, possibly by shrinking ϵ, $u = \psi(x)$ is a diffeomorphism to its open image and finally

$$f(x) = x^t G(x) x = u^t I_{n,\lambda} u$$

as desired.

■

Let us state, without proof, an interesting generalization of Morse's Lemma. With the usual notation, for every $k \geq 1$, define $\mathfrak{m}_\epsilon^k$ as the ideal of $\mathcal{C}^\infty(U_\epsilon, \mathbb{R})$ generated by the monomials $x^J = x_1^{j_1} \dots x_n^{j_n}$, J being an arbitrary multi-index with $|J| = k$. Clearly $\mathfrak{m}_\epsilon = \mathfrak{m}_\epsilon^1 \subset \mathfrak{m}_\epsilon^2 \subset \dots$.

PROPOSITION 1.14. *Let* $f \in \mathcal{C}^\infty(U, \mathbb{R})$, $f(0) = 0$, *be such that* 0 *is a critical point, and there is* $k \geq 1$ *such that* $\mathfrak{m}_\epsilon^k \subset J(f, \epsilon)$. *Then up to pre-composition with a local diffeomorphism at* 0, f *locally coincides with the Taylor polynomial* $T_k(f)(0)$.

1.12. Homotopy, isotopy, diffeotopy

Here we fix a few notions and terminology that shall be widely employed and developed. U and V are open sets in Euclidean spaces. A map

$$F : U \times [0,1] \to V$$

is *smooth* if it is the restriction of a smooth map defined on the open set $U \times J$, J being an open interval and $[0,1] \subset J$. For every $t \in [0,1]$, set f_t the restriction of F to $U \times \{t\}$. Then F is called a (smooth) homotopy between

f_0 and f_1. It can be considered as a continuous path in $\mathcal{E}(U,V)$ joining f_0 and f_1.

We say that $f : U \to V$ is an *embedding* if f is an injective immersion and is a homeomorphism between U and its image $f(U)$. If f_t is an embedding for every $t \in [0,1]$, then F is called an *isotopy* between f_0 and f_1.

If $U = V$ and f_t is a diffeomorphism for every $t \in [0,1]$, then F is called a *diffeotopy* . In this case F can be reconsidered as follows: consider the map

$$H : U \times [0,1] \to U \times [0,1],\ H(p,t) = (f_0(p),t)\ .$$

Then $G := F \circ H^{-1}$ is a diffeotopy between id_U and $f_1 \circ f_0^{-1}$, and $F = G \circ H$. This formal manipulation suggests the following specialization of homotopy. If $G : V \times [0,1] \to V$ is a diffeotopy between $g_0 = \mathrm{id}_V$ and g_1, and $\phi : U \times [0,1] \to V \times [0,1]$ is of the form $\phi(p,t) = (f(p),t)$ for some $f : U \to V$, then $G \circ F$ is called a *diffeotopy* between $f_0 := f$ and $f_1 := g_1 \circ f$; sometimes we also say that f_0 and f_1 are homotopic through an *ambient isotopy*.

Let $f : U \to U$ be a diffeomorphism. The *support of* f is the closure of the subset of U on which $f(x) \neq x$. If F is a diffeotopy between f and id_U, the support of F is the closure of the union of supports of the f_t's.

Homotopy and its relatives define equivalence relations on the pertinent space of maps. Clearly, $F(p,t) = (f(p),t)$ is a homotopy between f and itself. If F is a homotopy between f_0 and f_1, then $\hat{F}(p,t) := F(p,1-t)$ is a homotopy between f_1 and f_0. As for the transitivity, by using the 1-dimensional bump functions, we see that there exists a smooth function $s : [0,1] \to [0,1]$ and $1/3 > \epsilon > 0$ such that $s(t) = 0$ on $[0,\epsilon]$, $s(t) = 1$ on $(1-\epsilon,1]$, and s is strictly increasing on $[\epsilon,1-\epsilon]$. If F is any homotopy between f_0 and f_1, then replace it with $\tilde{F}(p,t) = F(p,s(t))$. If a homotopy $\tilde{F}'$ connects f_0 and f_1, while $\tilde{F}$" connects f_1 and f_2, then set

$$\tilde{F}(p,t) = \tilde{F}'(p,2t),\ t \in [0,1/2]$$

$$\tilde{F}(p,t) = \tilde{F}\text{"}(p,2t-1),\ t \in [1/2,1]\ .$$

It is a *smooth* homotopy between f_0 and f_2. For isotopies and diffeotopies we argue similarly.

1.13. Linearization of diffeomorphisms of $\mathbb{R}^n$ up to isotopy

PROPOSITION 1.15. *Every diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, is diffeotopic to the differential $d_0 f \in \mathrm{GL}(n,\mathbb{R})$, through diffeomorphisms f_t such that $f_t(0) = 0$ for every $t \in \mathbb{R}$.*

Proof : Define $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, by $F(x,t) = f(tx)/t$ if $t \neq 0$, $F(x,0) = d_0 f$. From the very definition of the differential, F is continuous; clearly it is smooth where $t \neq 0$. To check that it is fully smooth we note that, by the division theorem, $F(x,t) = \sum_j g_j(y) x_j$ for $y = tx$, the g_j being smooth maps of y.

■

We can strengthen the above proposition. Let us set $\mathrm{GL}^{\pm} = \mathrm{GL}^{\pm}(n, \mathbb{R})$ the open subsets of $\mathrm{GL}(n, \mathbb{R})$ formed by the matrices A such that either $\det A > 0$ or $\det A < 0$. Take the identity I_n and the matrix $I_{n,1}$ (the notation has been introduced in the proof of Morse's Lemma) as base points of the two sets, respectively.

THEOREM 1.16. *Every diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, such that $d_0 f \in \mathrm{GL}^+$ (resp. $d_0 f \in \mathrm{GL}^-$) is diffeotopic to the linear isomorphism I_n (resp. $I_{n,1}$), through diffeomorphisms f_t such that $f_t(0) = 0$ for every $t \in \mathbb{R}$.*

Proof : If U is a connected open set of some $\mathbb{R}^n$, then it follows easily from the proof of Proposition 1.1 that any two points of U can be connected by a piece-wise smooth path in U. Using bump functions to get a smoothing, we can take a globally smooth path. Then it is enough to prove that both open sets $\mathrm{GL}^{\pm}$ are connected. In fact, it is enough to show that GL^+ is connected. For if $A \in \mathrm{GL}^-$, then $I_{n,1}A$ is in GL^+; if A_t is a path connecting $I_{n,1}A$ with I_n in GL^+, then $I_{n,1}A_t$ is a path connecting A and $I_{n,1}$ in GL^-.

Let us show first that there is a path B_t in GL^+ connecting any given $A = B_0$ with some $B = B_1$ which belongs to

$$\mathrm{SO}(n) := \{P \in \mathrm{GL}(n, \mathbb{R});\ P^{-1} = P^t,\ \det P = 1\}\ .$$

Let $< *, * >$ be the positive definite scalar product on $\mathbb{R}^n$ determined by imposing that the ordered columns of A form an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^n$ for that scalar product. Set

$$(*, *)_t = (1 - t) < *, * > + t(*, *)$$

where $(*, *)$ is the standard Euclidean scalar product, $t \in [0, 1]$. Then $(*, *)_t$ is a path of positive definite scalar products. For every $t \in [0, 1]$, apply the usual Gram-Schmidt orthogonalization algorithm to the basis $\mathcal{B}$ that produces an orthonormal basis $\mathcal{B}_t$ for $(*, *)_t$; by considering the ordered vectors of $\mathcal{B}_t$ as columns of a matrix B_t, we eventually get a path of matrices such that $B_0 = A$ and $B_1 \in \mathrm{SO}(n)$. It remains to show that every $B \in \mathrm{SO}(n)$ can be connected to I_n by a path in $\mathrm{SO}(n)$. Let us consider $B : \mathbb{R}^n \to \mathbb{R}^n$ as a linear isometry with respect to $(*, *)$. By linear algebra, we know that $\mathbb{R}^n$ can be decomposed as the orthogonal direct sum of B-invariant linear subspaces V_i of dimension either 1 or 2. In the first case the restriction of B to V_i is the identity; in the second case B acts on V_i as a rotation. We conclude using the elementary fact that every rotation on $\mathbb{R}^2$ can be connected to I_2 by a path of rotations.

■

1.14. Homogeneity

PROPOSITION 1.17. *Let $p, q \in \mathbb{R}^n$ such that $||p - q|| = d > 0$. Then for every $\epsilon > 0$ there is a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that*

(1) $f(p) = q$

(2) *f is diffeotopic to the identity of $\mathbb{R}^n$ by a diffeotopy of compact support contained in $B^n(p, d+\epsilon)$.*

Proof : In this proof we use some tools that shall be developed in Chapter 7. Without the requirement about the supports, the proof is immediate: set $v = q - p$, then $f_t(x) := x + tv$, $t \in \mathbb{R}$, $f = f_1$ satisfy the thesis. Note that for every $x \in \mathbb{R}^n$, $f_t(x)$ is the integral line defined on the whole real line of the vector field on $\mathbb{R}^n$ constantly equal to v. Now we use a bump function to modify this vector field making it with compact support. Let $d + \epsilon/3 < a < b < d + \epsilon/2$, and consider the bump function $\gamma = \gamma_{p,a,b}$. Take the smooth vector field on $\mathbb{R}^n$ defined by $\gamma(x)v$. For every $x \in \mathbb{R}^n$, there is a unique maximal parametric integral curve denoted again $f_t(x)$ such that $f_0(x) = x$. As the field has compact support, also in this case every $f_t(x)$ is defined on the whole real line; $f_t(x)$ for $t \in [0, 1]$ realizes the required isotopy.

∎

The above proposition is a sort of a local case of the following more general result.

THEOREM 1.18. *Let $U \subset \mathbb{R}^n$ be a connected open set. Then for every $p \neq q \in U$ there is a diffeotopy F of U between $f_0 = \mathrm{id}_U$ and $f = f_1$ such that $f(p) = q$, and F has compact support.*

Proof : The proof is qualitatively similar to the one of Proposition 1.1. Being 'connected' via a diffeotopy with compact support defines an equivalence relation on U. By applying Proposition 1.17 on a chart diffeomorphic to $\mathbb{R}^n$ at each $p \in U$, we realize that every equivalence class is an open set, hence there is only one because U is connected.

∎

CHAPTER 2

The category of embedded smooth manifolds

In this chapter, we introduce a first generalization of the smooth category of open sets in Euclidean spaces. This is rather straightforward as these more general objects are also embedded in some $\mathbb{R}^n$ and their smooth structure is directly derived from the one on the ambient space. As a consequence, for example, the construction of the tangent functor extends directly. In Chapter 4 we will consider the most general category of (abstract) smooth manifolds by taking as *definition* some properties satisfied by the embedded ones. These last will be incorporated in the general theory as being 'submanifolds' of some Euclidean space.

Let us begin by widely extending the notions of smooth map and diffeomorphism to *arbitrary* topological subspaces of some $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be arbitrary subspaces. Then $f : X \to Y$ is $\mathcal{C}^k$, $k \geq 0$, if for every $x \in X$, there exists an open neighbourhood U of x in $\mathbb{R}^n$ and a map $g_U \in \mathcal{C}^k(U, \mathbb{R}^m)$ such that for every $y \in U \cap X$, $f(y) = g_U(y)$. Such a map g_U is called a *local* $\mathcal{C}^k$ *extension of* f *at* $x \in X$.

f is $\mathcal{C}^\infty$ (i.e. *smooth*) if for every $x \in X$ there are smooth local extensions of f at x.

A map $f : X \to Y$ is a *diffeomorphism* if it is a homeomorphism and both f and f^{-1} are smooth maps.

It is easy to verify, by using the results of Chapter 1, that $\mathcal{C}^k$ maps, smooth maps, and diffeomorphisms are stable under the composition of maps. By using this very general notion of diffeomorphism, we can define embedded smooth manifolds.

Definition 2.1. For every $0 \leq k \leq n$, a topological subspace $M \subset \mathbb{R}^n$ is an *embedded smooth* k*-manifold* (k is called the *dimension* of M) if for every $p \in M$, there exists an open neighbourhood W of p in M, an open set U of $\mathbb{R}^k$ and a diffeomorphism $\phi : W \to U$.

Every such a (W, ϕ) is called a *chart* of M; set $\psi = \phi^{-1}$, then (U, ψ) is called a *local parametrization* of M. The family of all charts is called *the atlas* $\mathcal{A} = \mathcal{A}_M$ of M. Hence, by definition, $\mathcal{A}$ incorporates an open covering of M. *An atlas* $\mathcal{U} \subset \mathcal{A}$ of M is any family of charts that incorporates an open covering of M.

The *category of embedded smooth manifolds* has as *objects* the embedded smooth manifolds in some $\mathbb{R}^n$, $n \in \mathbb{N}$; the *morphisms* are the smooth maps between embedded smooth manifolds; the diffeomorphisms are the *equivalences* in the category.

2.0.1. Basic properties and examples. We are going to list a few basic examples or properties that follow immediately from the definitions or are a consequence of the results of Chapter 1.

• A 0-manifold in $\mathbb{R}^n$ is a subset of isolated points. It is compact if and only if it is finite; otherwise it is countable.

• To show that $M \subset \mathbb{R}^m$ is a smooth manifold (sometimes we will omit to say "embedded"), it is enough to exhibit an atlas $\mathcal{U}$. The whole atlas $\mathcal{A}$ is implicitly determined by $\mathcal{U}$. For example, for every chart $(W,\phi) \in \mathcal{U}$, for every open subset $W' \subset W$, the restriction $(W', \phi' := \phi_{|W'})$ is a chart belonging to $\mathcal{A}$.

• Every open set $U \subset \mathbb{R}^n$ is an n-manifold; the inclusion $j : U \to \mathbb{R}^n$ forms an atlas of U with only one chart. More generally, an open set in a k-manifold M is also a k-manifold.

• Let U be an open set in $\mathbb{R}^n$, $f : U \to \mathbb{R}^m$ a smooth map. Then its *graph*

$$G(f) := \{(x,y) \in U \times \mathbb{R}^m;\ y = f(x)\}$$

is an n-smooth manifold embedded in $\mathbb{R}^{n+m}$. In fact $W = G(f) \cap (U \times \mathbb{R}^m) = G(f)$, $\phi : W \to U$, $\phi(x, f(x)) = x$ form an atlas of $G(f)$ with only one chart; the inverse parametrization is $\psi : U \to W$, $\psi(x) = (x, f(x))$.

• Let V be a linear (or affine) k-subspace of $\mathbb{R}^n$. It is a k-manifold and the atlas $\mathcal{A}$ contains any linear (affine) isomorphism $L : V \to \mathbb{R}^k$.

• Let $M \subset \mathbb{R}^m$, $N \subset \mathbb{R}^n$ be embedded smooth manifolds. Then the product $M \times N$ is a smooth manifold embedded in $\mathbb{R}^{n+m}$, and

$$\dim(M \times N) = \dim M + \dim N\ .$$

In fact if (W, ϕ) is a chart of M at p, (W', ϕ') of N at q, then $(W \times W', \phi \times \phi')$ is a chart of $M \times N$ at (p, q).

• If $(W, \phi), (W', \phi') \in \mathcal{A}$ are charts of a k-manifold M, and $W \cap W' \neq \emptyset$, then

$$\beta_{W,W'} := \phi' \circ \psi : \tilde{U} \to \tilde{U}'$$

is a *diffeomorphism between open sets of* $\mathbb{R}^k$ (that is $\tilde{U} = \phi(W \cap W') \subset U$ and $\tilde{U}' = \phi'(W \cap W') \subset U'$). It is called indifferently *change of charts* or *of local parametrizations* or also *of local coordinates.*

• If $f : M \to N$ is a smooth map between embedded smooth manifolds, (W, ϕ) is a chart of M, (W', ϕ') of N such that $f(W) \subset W'$, then

$$f_{U,U'} := \phi' \circ f \circ \psi : U \to U'$$

is a smooth map between open sets of Euclidean spaces called *a representation of f in local coordinates* or shortly *a local representation of f.*

• *The dimension of embedded smooth manifolds is invariant up to diffeomorphism.* This follows immediately from the above items and the "invariance of dimension" already discussed in Chapter 1.

LEMMA 2.2. *(1) An embedded smooth k-manifold* $M \subset \mathbb{R}^n$ *is connected if and only if it is path-connected.*

(2) Every path-connected component of M *is a k-manifold.* M *is the disjoint union of its path-connected (equivalently, connected) components.*

Proof : It is a general topological fact that a path-connected space is connected. For the other implication, we can repeat the argument already used for the open sets in $\mathbb{R}^k$. By using a chart around any point $p \in M$, we can argue that the path-connected component of p is open in M, hence there is only one if M is connected. This proves (1) and also (2) indeed.

■

The definition of embedded smooth manifold $M \subset \mathbb{R}^n$ implies some strong *local* constraint on the relative configuration of the pair $(\mathbb{R}^n, M)$.

LEMMA 2.3. *Let* $M \subset \mathbb{R}^n$ *be an embedded smooth k-manifold;* $p \in M$*. Then there exists a chart* (Ω, β) *of* $\mathbb{R}^n$*,* $p \in \Omega$*, such that* $(\Omega \cap M, \beta_{|\Omega\cap M})$ *is chart of* M *and*

$$\beta(\Omega, \Omega \cap M, p) = (B^n(0,1), B^n(0,1) \cap \mathbb{R}^k, 0)$$

(where $\mathbb{R}^k \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$ *as usual). Such a* β *is called a* relative normal chart *of the pair* $(\mathbb{R}^n, M)$*.*

Proof : It follows immediately from the definition of embedded manifold that there exists an open neighbourhood Ω of p in $\mathbb{R}^n$, an open set U of $\mathbb{R}^k$, and an *injective immersion* $\psi : U \to \Omega$, such that $\psi(U) = \Omega \cap M := W$. By Theorem 1.5 on the local normal form of immersions, possibly by shrinking Ω, there is chart (Ω, β) of $\mathbb{R}^n$ that satisfies the statement of the lemma.

■

LEMMA 2.4. *Let* U *be an open set of* $\mathbb{R}^k$*,* $\psi : U \to \mathbb{R}^n$ *be an injective immersion such that* $\psi : U \to \psi(U)$ *is a homeomorphism. Then* $M = \psi(U)$ *is a smooth manifold embedded in* $\mathbb{R}^n$ *and* $\psi : U \to M$ *is a (global) smooth parametrization of* M*.*

Proof : By using again Theorem 1.5 and the fact that f is a homeomorphism to its image, we readily see that at every $p \in M$, we can find relative normal charts of $(\mathbb{R}^n, M)$, and eventually ψ is a diffeomorphism onto M.

■

The condition that ψ is a homeomorphism to its image is *necessary*, as shown in the following example.

EXAMPLE 2.5. Consider the smooth map

$$E : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2,\ E(x,y) = (\cos(2\pi x), \sin(2\pi x), \cos(2\pi y), \sin(2\pi y))\ .$$

The image of E is contained in $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$. For every $a \in \mathbb{R}$, $a \neq 0$, consider the map

$$f : \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}^2,\ f(x) = E(x, ax)\ .$$

This is an injective immersion, but if a is *not a rational number* then it is not a homeomorphism to its image in $S^1 \times S^1$. In fact, we can verify that $f(\mathbb{R})$ is *dense* in $S^1 \times S^1$ (every nonempty open set of $S^1 \times S^1$ intersects $f(\mathbb{R})$), hence $f(\mathbb{R})$ is not an embedded manifold in $\mathbb{R}^2 \times \mathbb{R}^2$.

2.0.2. Submanifolds. If $Y \subset M$ are embedded smooth manifolds in $\mathbb{R}^n$, we say that Y is a *submanifold* of M . In particular both Y and M are submanifolds of $\mathbb{R}^n$. By extending the argument of Lemma 2.3, we have the following lemma.

LEMMA 2.6. *Let Y be a submanifold of $M \subset \mathbb{R}^n$, with Y and M having dimension k and m, respectively. Let $p \in Y$. Then there exist relative normal charts (for triples)*

$$\beta : (\Omega, \Omega \cap M, \Omega \cap Y, p) \to (B^n(0,1), B^n(0,1) \cap \mathbb{R}^m, B^n(0,1) \cap \mathbb{R}^k, 0)$$

where as usual we consider $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$, $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$.

By using the immersions, we have indicated above a way to get embedded manifolds (endowed with global smooth parametrizations). Now we see how embedded manifolds can be defined *implicitly*.

LEMMA 2.7. *If $f : U \to W$ is a surjective smooth submersion between open sets of Euclidean spaces, $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$, then for every $q \in W$, $M = f^{-1}(q)$ is a submanifold of U and $\dim M = n - m$.*

Proof : Being an embedded manifold is a local property. The lemma is an immediate consequence of Theorem 1.6 on local normal form of submersions or (equivalently) of the implicit function Theorem 1.8.

■

REMARK 2.8. In spite of the existence of relative normal charts at every point of a submanifold, the relative position of two submanifolds of some $\mathbb{R}^n$ can look stranger than one could expect. This is mainly due to the fact that submanifolds are not necessarily closed subsets. Consider, for example, the map $f : (0, +\infty) \to \mathbb{C} \sim \mathbb{R}^2$

$$f(x) = \frac{x}{1+x} e^{ix} \ .$$

This is an immersion and a homeomorphism onto its image N. Then the unitary circle S^1 and N are disjoint 1-submanifolds of $\mathbb{R}^2$. Nevertheless, two points $p \in S^1$ and $q \in N$ cannot be separated by normal charts of S^1 and N at p and q, respectively. In other words, the union $N \cup S^1$ is not an embedded submanifold.

EXAMPLE 2.9. *(Spheres)* Let us show, in several ways, that the unitary sphere $S^n \subset \mathbb{R}^{n+1}$, $n \in \mathbb{N}$, is an embedded smooth n-manifold. This is trivial for $n = 0$, so assume $n \geq 1$. Let $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. Let $W^+ = S^n \setminus \{e_{n+1}\}$, $\phi_+ : W^+ \to \mathbb{R}^n$ be the *stereographic projection* with centre e_{n+1}. It is defined geometrically by $\phi_+(x) = r(x, e_{n+1}) \cap \mathbb{R}^n$, where $r(x, e_{n+1})$ is the straight line passing through the two points. Analytically we have

$$\phi_+(x) = \frac{1}{1 - x_n}(x_1, \ldots, x_{n-1}) \ .$$

This is a diffeomorphism to $\mathbb{R}^n$ with inverse given by

$$\psi_+(y) = \left(\frac{2y}{1 + ||y||^2}, \frac{||y||^2 - 1}{||y||^2 + 1} \right) \ .$$

Then (W^+, ϕ_+) is a chart of S^n at every point different from e_{n+1}. By using the similar projection with centre $-e_{n+1}$, we get a chart (W^-, ϕ_-) which misses only $-e_{n+1}$. Hence $\{(W^\pm, \phi_\pm)\}$ is an atlas of S^n, formed by two charts.

For every $p \in S^n$, let $p^\perp$ be the subspace of $\mathbb{R}^{n+1}$ orthogonal to p. By using the projection of $S^n \setminus \{p\}$ onto $p^\perp$ with centre p (followed by any linear chart of $p^\perp$ to $\mathbb{R}^n$), we obtain other charts of the atlas $\mathcal{A}_{S^n}$.

Further charts are obtained as graphs of functions defined on the unitary open disk of $p^\perp$ with centre p. The basic example for $p = e_{n+1}$ is the function $h : B^n \to \mathbb{R}$,

$$h(x) = \sqrt{1 - \sum_{i=1}^{n-1} x_i^2} \ .$$

$S^n = f^{-1}(1)$, where $f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$, $f(x) = ||x||^2$. As $df_x = (2x_1, \ldots, 2x_{n+1})$, then f is a submersion and this implies again (implicitly) that S^n is an n-manifold by Lemma 2.7.

S^n is a *compact* manifold; it is closed because $S^n = f^{-1}(1)$ as above, and obviously it is bounded.

S^n is connected; given $x \neq y \in S^n$, let P be the 2-plane spanned by these two vectors. Then $P \cap S^n$ is a maximal circle; x, y separate it in two arcs both connecting x and y.

Important examples of embedded smooth manifolds (widely generalizing the spheres) are discussed in Chapter 3.

2.1. The embedded tangent functor

Let us fix a setting we will refer to all through the rest of this Chapter.

- $M \subset \mathbb{R}^h$ is an embedded smooth manifold of dimension m, $p \in M$; $N \subset \mathbb{R}^k$ is an embedded smooth manifold of dimension n, $q \in N$;
- $f : M \to N$ is a smooth map, $f(p) = q$.
- $\phi : W \to U \subset \mathbb{R}^m$ is a chart of M at p, $\phi(p) = a$, with inverse local parametrization $\psi : U \to W \subset M$.
- $f_{U,U'} : U \to U'$ is a representation of f in local coordinates at p. Recall that this is obtained as follows: we take a local chart of M

at p, for simplicity still denoted (W, ϕ), and a local chart (W', ϕ') of N at q, $\phi'(q) = b$, such that $f(W) \subset W'$; then

$$f_{U,U'} = \phi' \circ f \circ \psi : U \to U'$$

(U and U' being open sets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively).

- Possibly by shrinking W, we can also assume that there is an open neighbourhood Ω of p in $\mathbb{R}^h$ such that $\Omega \cap M = W$, a local smooth extension $\Phi : \Omega \to \mathbb{R}^m$ of ϕ, and a local smooth extension $F : \Omega \to \mathbb{R}^k$ of f.

We are going to define the tangent space T_pM to $M \subset \mathbb{R}^h$ at each point $p \in M$ and for every smooth $f : M \to N$, $p \in M$, the differential $d_pf : T_pM \to T_{f(p)}N$. This is done in the following lemma. The facts collected in the lemma are easy consequences of the definitions and of the results of Chapter 1. The reader can justify them in detail by exercise.

Lemma 2.10. *(1) The differential $d_a\psi$ is injective so it is a linear isomorphism to its image $d_a\psi(\mathbb{R}^m)$, $\mathbb{R}^m = T_aU$, which is an m-linear subspace of $\mathbb{R}^h = T_p\mathbb{R}^h$. This image does not depend on the choice of the local parametrization ψ of M at p. Hence*

$$T_pM = d_a\psi(\mathbb{R}^m)$$

is well defined and is called the tangent space to M at the point p .

(2) The restriction of the differential $d_p\Phi$ to T_pM is the inverse isomorphism $(d_a\psi)^{-1}$. Hence it does not depend on the choice of the local extension Φ of ϕ, and

$$d_p\phi := d_p\Phi_{|T_pM}$$

is a well defined linear isomorphism

$$d_p\phi : T_pM \to T_aU \ .$$

(3) The restriction of d_pF to T_pM does not depend on the choice of the local extension of f and takes values in T_qN. Hence it is well defined

$$d_pf := d_pF_{|T_pM}$$

it is a linear map

$$d_pf : T_pM \to T_qN$$

and it is called the differential of f at p. *We have*

$$d_af_{U,U'} = d_q\phi' \circ d_pf \circ d_a\psi : T_aU \to T_bU'$$

and this is the representation in local coordinates *of d_pf.*

(4) If $g \circ f$ is a composition of smooth maps between embedded smooth manifolds, $f(p) = q$, then

$$d_p(g \circ f) = d_qg \circ d_pf \ .$$

If f is a diffeomorphism, then d_pf is a linear isomorphism and $d_qf^{-1} = (d_pf)^{-1}$. If $f = \mathrm{id}$, then $d_pf = \mathrm{id}_{T_pM}$.

(5) If $G(g)$ is the graph of a smooth map $g : U \to \mathbb{R}^s$ defined on an open set $U \subset \mathbb{R}^m$, then

$$T_{(x,g(x))}G(g) = G(d_x g) \ .$$

(6) If $M = g^{-1}(q)$, where $g : \Omega \to \mathbb{R}^s$ is a submersion, $p \in M$, then

$$T_p M = \ker d_p g \ .$$

2.1.1. Interpretations of the tangent spaces. In Section 1.6.1 we have given two intrinsic interpretations of the tangent space T_pU at a point p of an open set $U \subset \mathbb{R}^n$. These extend straightforwardly to the tangent space T_pM at a point p of an arbitrary embedded smooth m-manifold $M \subset \mathbb{R}^h$. In accordance with the first interpretation, $T_pM \subset T_p\mathbb{R}^h = \mathbb{R}^h$ corresponds to the subspace $\mathcal{V}_pM$ of $\mathcal{V}_p\mathbb{R}^h$ of the velocities at time 0 of smooth curves in M, $\gamma : \mathbb{R} \to M$, passing through p ($\gamma(0) = p$). If $f : (M,p) \to (N,q)$ is as above, then the differential $d_pf : T_pM \to T_qN$ corresponds to the map $Vf_p : \mathcal{V}_pM \to \mathcal{V}_qN$ obtained by associating to the class of each smooth curve $\gamma : \mathbb{R} \to M$ passing through the point p, the class of the composite curve $f \circ \gamma : \mathbb{R} \to N$, passing through q. If $\phi : W \to U \subset \mathbb{R}^m$ is a chart of M at the point p, then the map $V\phi_p : \mathcal{V}_pM \to \mathcal{V}_aU = T_aU = \mathbb{R}^m$ is a linear isomorphism. In accordance with the second interpretation, T_pM is identified with the space $\mathrm{Der}(\mathcal{E}_pM)$ of derivations on the space of germs at p of smooth functions $g : M \to \mathbb{R}$. By using the interpretation of T_pM as $\mathcal{V}_pM$, every derivation δ is uniquely expressed as $\delta([h]_p) = d_0(h \circ \gamma)$, for a suitable $[\gamma] \in \mathcal{V}_pM$. A differential $d_pf : T_pM \to T_qN$ as above corresponds to the map $\mathfrak{d}f_p : \mathrm{Der}(\mathcal{E}_pM) \to \mathrm{Der}(\mathcal{E}_qN)$ by associating to every derivation δ on $\mathcal{E}_pM$ the derivation $\mathfrak{d}f_p(\delta)$ on $\mathcal{E}_qN$ such that $\mathfrak{d}f_p(\delta)([g]) = \delta([g \circ f])$. If $\delta([h]) = d_0(h \circ \gamma)$ for some curve γ passing through p, then $\mathfrak{d}f_p(\delta)([g]) = d_0(g \circ f \circ \gamma)$. If $\psi : U \to W$ is a local parametrization of M at p, then $\mathfrak{d}\psi_a : \mathrm{Der}(\mathcal{E}_a) \to \mathrm{Der}(\mathcal{E}_pM)$ is a linear isomorphism.

2.1.2. The tangent bundle. Set

$$T(M) = \{(x,v) \in \mathbb{R}^h \times \mathbb{R}^h;\ x \in M,\ v \in T_xM\} \ .$$

The restriction of the projection of $\mathbb{R}^h \times \mathbb{R}^h$ to the first factor $\mathbb{R}^h$ defines a smooth projection

$$\pi_M : T(M) \to M \ .$$

Example 2.11. The reader can check that

$$T(S^n) = \{(x,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1};\ x \in S^n,\ v \in x^\perp\} \ .$$

As a set, $T(M) = \cup_{x \in M} T_xM$. Note that for every open set $W \subset M$, $T(W)$ coincides with $\pi_M^{-1}(W)$, it is naturally included in $T(M)$ as an open set, and $\pi_W = (\pi_M)_{|T(W)}$.

We are going to show that $T(M)$ *is an embedded smooth manifold of dimension* $2m$*, of a special nature indeed.*

Every chart $\phi : W \to U \subset \mathbb{R}^m$ of M can be enhanced to a chart

$$T\phi : T(W) \to T(U) = U \times \mathbb{R}^m,\ T\phi(x,v) := (\phi(x), d_x\phi(v)) \ .$$

The inverse parametrization is

$$T\psi : U \times \mathbb{R}^m \to T(W),\ T\psi(y,w) = (\psi(y), d_y\psi(w)) \ .$$

If π_U is the natural projection onto U, it is immediate that the following diagram denoted $[\psi, T\psi]$ commutes

$$\begin{array}{ccc} U \times \mathbb{R}^m & \stackrel{T\psi}{\to} & T(W) \\ \downarrow \pi_U & & \downarrow \pi_W \\ U & \stackrel{\psi}{\to} & W \end{array}$$

We say that $\pi_M : T(M) \to M$ is *locally a product* over W and that the above diagram is a *local trivialization.* By varying the chart (W, ϕ) in the atlas $\mathcal{A}$ of M, we get an atlas

$$T\mathcal{A} = \{(T(W), T\phi)\}$$

of $T(M)$. The local coordinates for $T\mathcal{A}$ change in a special way as they are of the form

$$T\beta := T\phi' \circ T\psi : \tilde{U} \times \mathbb{R}^m \to \tilde{U}' \times \mathbb{R}^m$$

$$T\beta(x,v) = (\phi' \circ \psi(x), d_x(\phi' \circ \psi)(v)) = (\beta(x), d_x\beta(v)) \ .$$

Hence, for every x varying in M, it is a linear isomorphism on the second argument which "varies smoothly" with the point x. This means that *the intrinsic linear structure of every fibre* $T_xM = \pi_M^{-1}(x)$ *of the projection* π_M *is respected by the changes of coordinates for the atlas* $T\mathcal{A}$*.*

We can encode the same information by lifting $T\mathcal{A}$ at the level of the open covering $\{W\}$ of M; that is we have the locally trivializing commutative diagrams

$$\begin{array}{ccc} W \times \mathbb{R}^m & \stackrel{\tilde{T}\psi}{\to} & T(W) \\ \downarrow \pi_W & & \downarrow \pi_W \\ W & \stackrel{\mathrm{id}_W}{\to} & W \end{array}$$

where

$$\tilde{T}\psi = T\psi \circ (\phi, \mathrm{id}_{\mathbb{R}^m}) \ .$$

Any change of local trivialization for $\tilde{T}\mathcal{A}$ is of the form

$$\tilde{T}\beta : (W \cap W') \times \mathbb{R}^m \to (W \cap W') \times \mathbb{R}^m,\ (x,v) \to (x, d_x\beta(v)) \ .$$

We summarize all these facts by saying that

$$\pi_M : T(M) \to M$$

is the *tangent vector bundle* of the embedded smooth manifold M and that $T\mathcal{A}$ (actually and equivalently $\tilde{T}\mathcal{A}$) is its *vector bundle atlas*.

In Section 4.3.1, we will formalize these notions in a more general setting.

Now we extend the definition of the *tangent map* . Let $f : M \to N$ be our smooth map between embedded smooth manifolds, then set:

$$Tf : T(M) \to T(N),\ Tf(x,v) = (f(x), d_x f(v))\ .$$

Note that the defining inclusion $T(M) \subset \mathbb{R}^h \times \mathbb{R}^h = T(\mathbb{R}^h)$ is nothing else than Tj, $j : M \to \mathbb{R}^h$ being the inclusion. Clearly, the following diagram denoted $[f, Tf]$ commutes

$$\begin{array}{ccc} T(M) & \overset{Tf}{\to} & T(N) \\ \downarrow\pi_M & & \downarrow\pi_N \\ M & \overset{f}{\to} & N \end{array}$$

that is Tf sends every fibre T_xM linearly to the fibre $T_{f(x)}N$, by a 'smooth field' of linear maps.

If $g \circ f$ is a composition of smooth maps between embedded smooth manifolds, then

$$T(g \circ f) = Tg \circ Tf$$
$$T\mathrm{id_M} = \mathrm{id_{T(M)}}\ .$$

If f is a diffeomorphism, then Tf is a diffeomorphism and

$$Tf^{-1} = (Tf)^{-1}\ .$$

All verifications are local and follow immediately from Lemma 2.10 and the properties of the tangent map in the case of open sets in Euclidean spaces.

We can summarize these considerations as follows:

The tangent category *of the category of embedded smooth manifolds has as* objects *the tangent vector bundles of embedded smooth manifolds and as* morphisms *the tangent maps of smooth maps between embedded smooth manifolds. Then*

$$M \ \Rightarrow\ \pi_M : T(M) \to M$$
$$f : M \to N \ \Rightarrow\ [f, Tf]$$

define a covariant functor *from the category of embedded smooth manifolds to its tangent category.*

2.2. Immersions, submersions, embeddings, Monge charts

The notions of immersion and submersion extend immediately to maps between embedded smooth manifolds: $f : M \to N$ is an *immersion* (resp. *submersion*) if for every $x \in M$, $d_x f$ is *injective* (*surjective*). We say that $f : M \to N$ is an *embedding* if f is a diffeomorphism between M and its image $f(M)$ (in particular, the inclusion $M \subset \mathbb{R}^h$ is an embedding). The proof of the following proposition is of local nature and follows from Lemmas 4.32 and 2.7.

Proposition 2.12. *(1) Let $f : M \to N$ be a surjective submersion; then for every $q \in N$, $Y = f^{-1}(q)$ is a submanifold of M,* $\dim Y = \dim M - \dim N$.

(2) If $f : M \to N$ is an embedding then $f(M)$ is a submanifold of N.

(3) $f : M \to N$ is an embedding if and only if f is an immersion and a homeomorphism to its image.

(4) If $f : M \to N$ is both an immersion and a submersion, then it is a local diffeomorphism.

We have seen in Example 2.9 a distinguished local graph chart of S^n. Here we show that such a kind of chart exists for every embedded smooth m-manifold $M \subset \mathbb{R}^h$ at every point. For every multi-index $J = (j_1, \dots, j_m)$, $|J| = m$, let J', $|J'| = h - m$, be its complementary multi-index. Denote by $\mathbb{R}_J$ the subspace of $\mathbb{R}^h$ generated by $(e_{j_1}, \dots, e_{j_m})$; hence we have the orthogonal direct sum decomposition $\mathbb{R}^h = \mathbb{R}_J \oplus \mathbb{R}_{J'}$ and the orthogonal projection to $\mathbb{R}_J$, $\pi_J(x) = (x_{j_1}, \dots, x_{j_m})$. For every $p \in M$, denote by $\pi_{J,p} : \mathbb{R}^h \to \mathbb{R}_J$ the composition of the translation $x \to x - p$, followed by π_J. Denote by $\phi_{J,p}$ the restriction of $\pi_{J,p}$ to (any suitable subset of) M.

Proposition 2.13. (Monge charts) *For every embedded smooth m- manifold $M \subset \mathbb{R}^h$, for every $p \in M$, there exists J, $|J| = m$, and an open neighbourhood W of p in M such that $(W, \phi_{J,p})$ is a chart of M at p. The inverse local parametrization is of the form $\psi_{J,p} : U \to W$, $U \subset \mathbb{R}_J$, $\psi_{J,p}(y) = (y, f_{J,p}(y))$ (by using the above decomposition $\mathbb{R}^h = \mathbb{R}_J \oplus \mathbb{R}_{J'}$). Hence at every point p, M is locally a graph of a smooth function defined on some $\mathbb{R}_J$.*

Proof : By elementary linear algebra, there exists J such that the restriction of π_J to T_pM is a linear isomorphism to $\mathbb{R}_J$. As $d_p\phi_{J,p}$ coincides with such a restriction, then $\phi_{J,p}$ is a local diffeomorphism.

■

CHAPTER 3

Stiefel and Grassmann manifolds

Let $M \subset \mathbb{R}^h$ be an embedded smooth m-manifold. Denote by $G_{h,m}$ the *set* of m-dimensional linear subspaces of $\mathbb{R}^h$. The definition of the embedded tangent bundle of M includes the map $t : M \to G_{h,m}$, $t(p) = T_pM$. We sense that the tangent space varies 'smoothly' with the point p. The natural way to give substance to intuition is to equip the set $G_{h,m}$ with a natural structure of smooth manifold so that t becomes a smooth map. In this chapter, we introduce these Grassmann and relative Stiefel manifolds. Stiefel manifolds are naturally embedded; we provide embedded models also for Grassmann manifolds. They will play a key role in the study of vector bundles on smooth manifolds, beyond the tangent bundle. Considered by themselves, they form remarkable families of nontrivial embedded smooth manifolds that deserve to be described in detail. However, to a large extent, what we need afterwards is the existence of such smooth structures. At first reading you can skip these details; this chapter can be read independently at a later stage.

3.1. Stiefel manifolds

We introduce first the *Stiefel manifolds*. There are two versions that we call *linear* and *orthogonal*, respectively. For every $n \in \mathbb{N}$ and every $0 \leq k \leq n$, the *linear Stiefel manifold* $L_{n,k}$, as a *set*, is the set of ordered k-tuples $(v_1, \dots, v_k)$ of linearly independent vectors in $\mathbb{R}^n$. By arranging each of them in an $n \times k$ matrix A (so that v_j is the j-column of A), $L_{n,k} \subset M(n,k,\mathbb{R})$. It is *an open subset*: consider the smooth function $\delta : M(n,k,\mathbb{R}) \to \mathbb{R}$ defined in the proof of Proposition 1.10, then $L_{n,k} = M(n,k,\mathbb{R}) \setminus \delta^{-1}(0)$. This specifies how $L_{n,k}$ is an embedded smooth manifold. As a particular case, we have $\mathrm{GL}(n,\mathbb{R}) = L_{n,n}$. For every $P \in \mathrm{GL}(n,\mathbb{R})$, $A \to PA$ defines a diffeomorphism $L_{n,k} \to L_{n,k}$. This action is transitive; in particular, for every $A \in L_{n,k}$, there exists $P \in \mathrm{GL}(n,\mathbb{R})$ such that $PI_{n,k} = A$, where $I_{n,k}$ is the matrix whose columns are $e_1, \dots, e_k$, the first k vectors of the canonical basis of $\mathbb{R}^n$.

Now, let $S_{n,k} \subset L_{n,k}$ be the closed subset defined as $f^{-1}(I_k)$ where

$$f : L_{n,k} \to S(k,\mathbb{R})$$

is the smooth map $f(A) = A^tA$ with values in the space $S(k,\mathbb{R})$ of $k \times k$ *symmetric* matrices, which can be identified with $\mathbb{R}^{\frac{k(k+1)}{2}}$. In other words, we require that the columns of any $A \in S_{n,k}$ form a system of orthonormal vectors. As particular cases, we have $S_{n,1} = S^{n-1}$ and $S_{n,n} = O(n)$, the classical (real) *orthogonal group*. As $M(n,k,\mathbb{R}) = (\mathbb{R}^n)^k$, we see immediately that $S_{n,k} \subset (S^{n-1})^k$, hence $S_{n,k}$ is compact. The above action of $\mathrm{GL}(n,\mathbb{R})$ on $L_{n,k}$ restricts to a transitive action of $O(n)$ on $S_{n,k}$: for every $A \in S_{n,k}$, there exists $P \in O(n)$ such that $PA = I_{n,k}$. It follows that to prove that $S_{n,k}$ is an embedded smooth manifold in $(\mathbb{R}^n)^k$, it is enough to prove that there is a chart (W,ϕ) of $S_{n,k}$ at $J := I_{n,k}$. Hence, it is enough to prove that $d_J f$ is surjective and conclude by applying Theorem 1.6. Let us compute $d_J f$ by the definition of the differential. Then

$$df_J(B) = \lim_{t\to 0} \frac{(J+tB)^t(J+tB) - I_k}{t} =$$

$$\lim_{t\to 0}(J^tB + B^tJ + tB^tB) = J^tB + B^tJ\ .$$

We have to prove that, for every symmetric matrix $C \in S(k,\mathbb{R})$, there exists $B \in M(n,k,\mathbb{R})$ such that $J^tB + B^tJ = C$. Set $B = \frac{1}{2}JC$. Then

$$J^tB + B^tJ = \frac{1}{2}J^tJC + \frac{1}{2}C^tJ^tJ = \frac{1}{2}C + \frac{1}{2}C^t = C$$

because $C = C^t$. Summarizing, $S_{n,k}$ is a compact embedded smooth manifold in $L_{n,k} \subset M(n,k,\mathbb{R}) = (\mathbb{R}^n)^k$, of dimension

$$\dim S_{n,k} = nk - \frac{k(k+1)}{2}\ .$$

$S_{n,k}$ is called an *orthogonal Stiefel manifold* . In particular, the orthogonal group $O(n)$ is a compact embedded smooth submanifold of $(S^{n-1})^n$ of dimension

$$\dim O(n) = n^2 - \frac{n(n+1)}{2}\ .$$

REMARK 3.1. The operation $(A,B) \to AB$, and the map $A \to A^{-1}$ that define the group structure of $\mathrm{GL}(n,\mathbb{R})$ are smooth (concerning A^{-1}, recall the determinant formula based on *Cramer's rule*). These restrict to smooth operations giving the group structure of the manifold $O(n)$. Hence $\mathrm{GL}(n,\mathbb{R})$ and $O(n)$ are basic examples of *Lie groups*. $O(n)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$, in the sense that the first is a submanifold of the second and the smooth operations are compatible.

The *Gram-Schmidt orthonormalization algorithm* applied to the ordered columns of every $A \in L_{n,k}$ defines a smooth map

$$\mathfrak{r}_{n,k} : L_{n,k} \to S_{n,k}$$

which is surjective and such that $\mathfrak{r}_{n,k}(A) = A$ for every $A \in S_{n,k}$. The map $\mathfrak{r}_{n,k}$ is the *canonical retraction* of $L_{n,k}$ to $S_{n,k}$.

3.2. Fibrations of Stiefel manifolds by Stiefel manifolds

In Section 4.3.1 we will formalize the general notion of a locally trivializable fibre bundle. Here we are going to see some concrete and motivating examples of this notion.

For every $0 \leq h < k \leq n$, $L_{n,k}$ is a submanifold (an open set) in the product $L_{n,h} \times L_{n,k-h}$ and denote by

$$l_{k,h} : L_{n,k} \to L_{n,h}$$

the restriction of the natural projection onto the first factor. This map is *equivariant* for the above actions of $\mathrm{GL}(n, \mathbb{R})$ on both Stiefel manifolds (i.e. $l_{k,h}(PA) = Pl_{k,h}(A)$), hence to study local properties such as the smoothness of the map, it is enough to study the restriction of $l_{k,h}$ on $l_{k,h}^{-1}(\Omega)$ where Ω is a neighbourhood of $I_{n,h}$. Clearly, $l_{k,h}(I_{n,k}) = I_{n,h}$. The fibre $F_{k,h} := l_{k,h}^{-1}(I_{n,h})$ over $I_{n,h}$ is made by the 2×2 block matrices of the form

$$Y(S, D) := \begin{pmatrix} I_h & S \\ 0 & D \end{pmatrix}$$

where $(S, D) \in M(h, k-h, \mathbb{R}) \times L_{n-h,k-h}$. If $P \in \mathrm{GL}(n, \mathbb{R})$ is such that $PI_{n,h} = A$, then $P(l_{k,h}^{-1}(I_{n,h})) = l_{k,h}^{-1}(A)$ and all fibres are diffeomorphic to each other. Let Ω be the open neighbourhood of $I_{n,h}$ made by matrices of the form

$$X = \begin{pmatrix} B \\ R \end{pmatrix}$$

where $B \in \mathrm{GL}(h, \mathbb{R})$. We define the smooth map $X \to P(X) \in \mathrm{GL}(n, \mathbb{R})$

$$P(X) = \begin{pmatrix} B & 0 \\ R & I_{n-h} \end{pmatrix}$$

such that $P(X)I_{n,h} = X$. Finally, we have the following commutative diagram of smooth maps

$$\begin{array}{ccc} \Omega \times F_{k,h} & \stackrel{\Psi}{\to} & l_{k,h}^{-1}(\Omega) \\ \downarrow \pi_\Omega & & \downarrow l_{k,h} \\ \Omega & \stackrel{\mathrm{id}_\Omega}{\to} & \Omega \end{array}$$

such that the first row is the diffeomorphism defined by

$$(X, S, D) \to P(X)Y(S, D) \ .$$

The constant section of the product on the left, $X \to (X, 0, I_{n-h,k-h})$ is transformed into the section of $l_{k,h}$ over Ω:

$$s(X) = \begin{pmatrix} B & 0 \\ R & I_{k-h} \end{pmatrix} \ .$$

A similar construction can be performed for the orthogonal Stiefel manifolds. For every $0 \leq h < k \leq n$, $S_{n,k}$ is a submanifold in the product

$S_{n,h} \times S_{n,k-h}$ and denote by

$$h_{k,h} : S_{n,k} \to S_{n,h}$$

the restriction of the natural projection onto the first factor. This map is *equivariant* for the actions of $O(n)$ on both Stiefel manifolds. Clearly $h_{k,h}(I_{n,k}) = I_{n,h}$. The fibre $h^{-1}_{k,h}(I_{n,h})$ over $I_{n,h}$ is made by the 2×2 block matrices of the form

$$Y(D) := \begin{pmatrix} I_h & 0 \\ 0 & D \end{pmatrix}$$

where $D \in S_{n-h,k-h}$. If $P \in O(n)$ is such that $PI_{n,h} = A$, then $P(h^{-1}_{k,h}(I_{n,h})) = h^{-1}_{k,h}(A)$ and all fibres are diffeomorphic to each other. Let Ω be the open neighbourhood of $I_{n,h}$ in $S_{n,h}$ made by matrices of the form

$$X = \begin{pmatrix} B \\ R \end{pmatrix}$$

where $B \in O(h)$. Recall the "Gram-Schmidt" retractions $\mathfrak{r}_{n,k}$ defined above. Then we define the smooth map $X \to P(X) \in O(n)$

$$P(X) = \mathfrak{r}_{n,n}(\begin{pmatrix} B & 0 \\ R & I_{n-h} \end{pmatrix})$$

such that $P(X)I_{n,h} = X$. Finally, we have the following commutative diagram of smooth maps

$$\begin{array}{ccc} \Omega \times S_{n-h,k-h} & \stackrel{\Psi}{\to} & h^{-1}_{k,h}(\Omega) \\ \downarrow \pi_\Omega & & \downarrow h_{k,h} \\ \Omega & \stackrel{\mathrm{id}_\Omega}{\to} & \Omega \end{array}$$

such that the first row is the diffeomorphism defined by

$$(X, D) \to P(X)Y(D) \ .$$

The constant section of the product on the left, $X \to (X, 0, I_{n-h,k-h})$ is transformed into the section of $h_{k,h}$ over Ω:

$$s(X) = \mathfrak{r}_{n,k}(\begin{pmatrix} B & 0 \\ R & I_{k-h} \end{pmatrix}) \ .$$

In summary, all these restrictions of natural projections onto Stiefel manifolds are locally trivializable fibrations with a transitive action of either the group $GL(n, \mathbb{R})$ or $O(n)$, respectively, which sends fibres to fibres. In the case of the orthogonal Stiefel manifolds, the fibre is an orthogonal Stiefel manifold itself.

• A case of particular interest is when $n = k$. In the linear case we have a fibration of the linear group $\mathrm{GL}(n, \mathbb{R})$ over $L_{n,h}$ with fibre the *subgroup* of $\mathrm{GL}(n, \mathbb{R})$ made by the matrices of the form

$$Y(S, D) := \begin{pmatrix} I_h & S \\ 0 & D \end{pmatrix}$$

where $(S, D) \in M(h, n-h, \mathbb{R}) \times \mathrm{GL}(n-h, \mathbb{R})$.

In the orthogonal case we have a fibration of the orthogonal group $O(n)$ over $S_{n,h}$ with fibre the orthogonal group $O(n-h)$. Sometimes this is summarized by writing

$$S_{n,h} = O(n)/O(n-h) \ .$$

• Another useful fibration is $h_{k,1} : S_{n,k} \to S^{n-1}$ with fibre $S_{n-1,k-1}$.

• Recall that $O(n)$ has two connected components and that the component containing I_n is the special orthogonal group $SO(n)$. If $h < n$, also the action of $SO(n)$ on $S_{n,h}$ is transitive, hence we can specialize all the discussion obtaining a fibration

$$sh_{n,h} : SO(n) \to S_{n,h}$$

with fibre $SO(n-h)$, so that

$$S_{n,h} = SO(n)/SO(n-h) \ .$$

In particular, this implies that *for $h < n$, the Stiefel manifold $S_{n,h}$ is connected.*

3.3. Grassmann manifolds

For every (n, k) as above, we are going to define now the *Grassmann manifold* $\mathfrak{G}_{n,k}$.

Denote by $G_{n,k}$ the *set* of linear subspaces of $\mathbb{R}^n$ of dimension k. Let $\mathfrak{G}_{n,k}$ be the closed subset of $S(n, \mathbb{R}) = \mathbb{R}^{\frac{n(n+1)}{2}}$ defined by the polynomial matrix equations

$$A^2 - A = 0, \ \mathrm{trace}(A) = k \ .$$

If $A \in S(n, \mathbb{R})$ satisfies $A^2 - A$, then its spectrum of eigenvalues is $\{0, 1\}$ and, by the spectral theorem for real symmetric matrices, the respective eigenspaces provide an orthogonal direct sum decomposition of $\mathbb{R}^n$; the last condition on the trace is equivalent to the fact that the eigenspace for the eigenvalue $\lambda = 1$ has dimension equal to k and also to the fact that A has rank equal to k.

We fix a bijection $V \to A_V$ from $G_{n,k}$ to $\mathfrak{G}_{n,k}$ as follows. For every $V \in G_{n,k}$ we have the orthogonal direct sum decomposition $\mathbb{R}^n = V \oplus V^\perp$ ($V^\perp$ being the orthogonal space to V for the standard Euclidean scalar product) and the linear map $A_V \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) = M(n, \mathbb{R})$ such that $A_V(v + v') = v$. We readily verify that $A_V \in \mathfrak{G}_{n,k}$. The inverse map $A \to V_A$ is defined by setting V_A equal to the eigenspace of A relative to the eigenvalue $\lambda = 1$.

Next, we prove that $\mathfrak{G}_{n,k}$ is an embedded smooth manifold in $S(n, \mathbb{R})$, of dimension $k(n-k)$. The action by smooth diffeomorphisms of $O(n)$ on $S(n, \mathbb{R})$ given by $(P, A) \to P^t A P$, restricts to an action on $\mathfrak{G}_{n,k}$: for every $A \in \mathfrak{G}_{n,k}$, $(P^t A P)^2 - P^t A P = P^t(A^2 - A)P = 0$; as $P^t = P^{-1}$, then $\mathrm{trace}(PAP^{-1}) = k$ because the trace is invariant up to conjugation. This action corresponds, via the above bijection $V \to A_V$, to the action of $O(n)$ on the set $G_{n,k}$ defined by $(P, V) \to PV$. These actions are transitive, hence for every $A \in \mathfrak{G}_{n,k}$ there exists $P \in O(n)$ such that $P^t A P = H$ where H is the 2×2 block diagonal matrix

$$H = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} .$$

So it is enough to find a chart of $\mathfrak{G}_{n,k}$ at H. First, note that the space of symmetric matrices of rank k (denote by $S(n|k, \mathbb{R})$) is a submanifold of $S(n, \mathbb{R})$ of dimension $\frac{k(k+1)}{2} + k(n-k)$. A local parametrization of $S(n|k, \mathbb{R})$ at H is given by

$$(S(k, \mathbb{R}) \cap \mathrm{GL}(k, \mathbb{R})) \times M(k, n-k, \mathbb{R}) \to \mathcal{W} \subset S(n|k, \mathbb{R}),\ (D, B) \to Z(D, B)$$

where $Z(D, B)$ is the 2×2 block symmetric matrix

$$Z(D, B) = \begin{pmatrix} D & B \\ B^t & B^t D^{-1} B \end{pmatrix} .$$

To see that $Z(D, B)$ is of rank k, consider the nonsingular matrix

$$X(D, B) = \begin{pmatrix} I_k & 0 \\ -B^t D^{-1} & I_{n-k} \end{pmatrix} .$$

Then

$$X(D, B) Z(D, B) = \begin{pmatrix} D & B \\ 0 & 0 \end{pmatrix} .$$

This last matrix has the same rank of $Z(D, B)$ and this is equal to $\mathrm{rank}(D) = k$. The same argument shows that if we change the second block along the diagonal of $Z(D, B)$ to any block other than $B^t D^{-1} B$, then the resulting matrix would have rank $> k$. $Z(I_k, 0) = H$. Hence $\mathcal{W} \cap \mathfrak{G}_{n,k}$ is given by the restriction to $\mathcal{W}$ of the matrix equation $A^2 - A = 0$. The matrix equation carried by the first $k \times k$ block along the diagonal reads

$$BB^t + D^2 - D = 0$$

and by replacing $BB^t = D - D^2$ in the equations carried by the other blocks, a direct computation shows that they are automatically satisfied. We are reduced to study the map

$$h : (S(k, \mathbb{R}) \cap \mathrm{GL}(k, \mathbb{R})) \times M(k, n-k, \mathbb{R}) \to S(k, \mathbb{R}),\ (D, B) \to BB^t + D^2 - D$$

which is a submersion at $(I_k, 0)$; hence, possibly by shrinking $\mathcal{W}$, we conclude that $Z(h^{-1}(0)) = \mathcal{W} \cap \mathfrak{G}_{n,k}$ is an embedded smooth manifold of dimension $k(n-k)$.

An alternative way to get the same conclusion is to provide a local parametrization of $\mathfrak{G}_{n,k}$ at H. Let $\tilde{U}$ be the subset of $G_{n,k}$ formed by the k-linear subspaces V of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that $V \cap \mathbb{R}^{n-k} = \{0\}$. Every $V \in \tilde{U}$ is the graph of a uniquely determined linear map $L_V : \mathbb{R}^k \to \mathbb{R}^{n-k}$. The restriction to V of the projection to $\mathbb{R}^k$ is a linear isomorphism; hence the inverse isomorphism is of the form $x \to (x, L_V(x))$. Then $\tilde{U}$ can be identified with $M(n-k,k,\mathbb{R})$. The restriction to $M(n-k,k,\mathbb{R})$ of the above map $V \to A_V$ can be explicitly computed as follows. For every $L \in M(n-k,k,\mathbb{R})$, let $V = V_L$ be the graph of L. Consider the ordered basis of $\mathbb{R}^n$

$$\mathcal{B}_L = \{(e_1, L(e_1)), \dots, (e_k, L(e_k)), e_{k+1}, \dots, e_n)\}$$

such that the first k-vectors form a basis of V. Apply to $\mathcal{B}_L$ the Gram-Schmidt orthogonalization algorithm which produces an orthonormal basis $\mathcal{D}_L$ of $\mathbb{R}^n$, whose first k vectors are an orthonormal basis of V and the last $n-k$ of $V^\perp$. By organizing as usual $\mathcal{D}_L$ in an $n \times n$ matrix, we get $P_L \in O(n)$. Finally, $A_L = A_V = P_L^t H P_L$. The map $L \to A_L$ is smooth; by a bit of direct computation, we see that it is indeed an immersion. Finally, if Ω is a sufficiently small neighbourhood of H in $S(n,\mathbb{R})$ and $W = \Omega \cap \mathfrak{G}_{n,k}$, then for every $A \in W$, V_A belongs $\tilde{U}$; the restriction to W of $A \to L_V$ is a chart of $\mathfrak{G}_{n,k}$ with values in an open neighbourhood U of $0 \in M(n-k,k,\mathbb{R})$. We have now proved again that $\mathfrak{G}_{n,k}$ is an embedded smooth manifold of dimension $k(n-k)$ in $S(n,\mathbb{R})$.

3.4. Stiefel manifolds as fibre bundles over Grassmann manifolds

There are natural surjective maps

$$\mathfrak{l}_{n,k} : L_{n,k} \to \mathfrak{G}_{n,k}$$

$$s_{n,k} : S_{n,k} \to \mathfrak{G}_{n,k}$$

defined in both cases by $B \to A_{[B]}$ where $[B]$ denotes the linear k-subspace of $\mathbb{R}^n$ generated by the columns of B.

Let us concentrate on the map $s_{n,k}$. Note that $[B] = [C]$ if and only if there exists $Q \in O(k)$ such that $C = BQ$, and that $A_{[B]} = H$ if and only if it is of the form

$$B = \begin{pmatrix} Q \\ 0 \end{pmatrix}, \; Q \in O(k) \; .$$

It follows that every fibre of $s_{n,k}$ is diffeomorphic to $O(k)$ and there is a transitive action (on the *right*) of $O(k)$ itself on each fibre.

The map $s_{n,k}$ is *equivariant* for the actions of $O(n)$: $(P,B) \to PB$ on $S_{n,k}$, $(P,A) \to P^t A P$ on $\mathfrak{G}_{n,k}$, respectively. Recall that 'equivariant' means that for every (P,B), $A_{[PB]} = P^t A_{[B]} P$. Then it is enough to analyze the behaviour of the restriction of the map to the inverse image $\tilde{\Omega} := s_{n,k}^{-1}(\Omega)$ (which is an open neighbourhood of J in $S_{n,h}$) of some open neighbourhood Ω of H in $\mathfrak{G}_{n,h}$. For every $B \in S_{n,k}$, if P is the top $k \times k$ submatrix of B, let us express this by writing $B = (P|D)$. Let $\tilde{\Omega}$ be the open neighbourhood of J in $S_{n,k}$ formed by the matrices $B = (P|D)$ such that P is nonsingular. If $B \in \tilde{\Omega}$ then $[B] \cap \mathbb{R}^{n-k} = \{0\}$, hence its image Ω in $\mathfrak{G}_{n,k}$ is an open set. Moreover, if $[(P|D)] = [(R|S)]$, then there is $Q \in O(k)$ such that $(P|D) = (RQ|SQ)$. If P is nonsingular, then R is also necessarily nonsingular. This means that $\tilde{\Omega} = s_{n,k}^{-1}(\Omega)$ is a *saturated* open set of $S_{n,k}$ for the surjective map

$s_{n,k}$. We can make explicit $s_{n,k}(B)$ on $\tilde{\Omega}$ by applying to every $[B]$ and its orthonormal basis given by B itself the construction already used above to construct a local parametrization of $\mathfrak{G}_{n,k}$ at H. This shows that $s_{n,k}$ is smooth. Moreover, define $\phi : \tilde{\Omega} \to M(k, n-k, \mathbb{R})$ by $\phi((P|D)) = DP^{-1}$. If $(P|D) = (RQ|SQ)$ as above, then $SQQ^{-1}R^{-1} = SR^{-1}$. Then there is an induced smooth map $\Omega \to M(k, n-k, \mathbb{R})$ whose inverse map is

$$\psi : M(k, n-k, \mathbb{R}) \to \Omega, \ \psi(Z) = A_{[\mathfrak{r}_{n,k}(I_k|Z)]}$$

providing once again a local parametrization of $\mathfrak{G}_{n,k}$ at H. We can summarize this discussion by saying that there is a locally trivializing commutative diagram at H

$$\begin{array}{ccc} \Omega \times O(k) & \stackrel{\Psi}{\to} & \tilde{\Omega} \\ \downarrow \pi_\Omega & & \downarrow s_{n,k} \\ \Omega & \stackrel{\mathrm{id}_\Omega}{\to} & \Omega \end{array}$$

where $\Psi(A, Q) = \psi(Z)Q$, $A = \psi(Z)$. Its orbit by the action of $O(n)$ provides a fibred atlas for the submersion $s_{n,k}$. So we have proved

Proposition 3.2. *The map $s_{n,k} : S_{n,k} \to \mathfrak{G}_{n,k}$ is a fiber bundle with fibre $O(k)$. Every change of trivialization*

$$\Phi' \circ \Psi(\Omega \cap \Omega') \times O(k) \to (\Omega \cap \Omega') \times O(k)$$

is of the form

$$(p, P) \to (p, PQ(p))$$

where $p \to Q(p)$ defines a smooth map $\Omega \cap \Omega' \to O(k)$.

We also have the following topological corollaries.

Corollary 3.3. *Every $\mathfrak{G}_{n,k}$ is a compact and connected embedded smooth manifold. As a topological space it has the quotient space topology $S_{n,k}/s_{n,k}$.*

• *Real projective spaces.* A particular case of the above discussion is when $k = 1$. In such a case $\mathfrak{G}_{n,1}$ is also denoted by $\mathbf{P}^{n-1}(\mathbb{R})$ and is called the (real) $(n-1)$-*projective space*. In this case, $S_{n,1} = S^{n-1}$ and the map $s = s_{n,1} : S^{n-1} \to \mathbf{P}^{n-1}(\mathbb{R})$ is a *smooth covering map of degree* 2.

• *Complex Stiefel and Grassmann manifolds.* As a smooth manifold $\mathbb{C}^n = \mathbb{R}^{2n}$, hence $M(n, \mathbb{C})$ is a submanifold of $M(2n, \mathbb{R})$ etc. All along with the above discussion, let us replace:

- $\mathbb{R}^n$ with $\mathbb{C}^n$. The real linear subspaces of $\mathbb{R}^n$ with the *complex* linear subspaces of $\mathbb{C}^n$.
- The standard positive definite scalar product on $\mathbb{R}^n$ with the standard positive definite *Hermitian product* on $\mathbb{C}^n$, $< v, w >= v^t \bar{w}$.
- The (real) orthogonal groups $O(n)$ with the *unitary groups*

$$U(n) := \{A \in \mathrm{GL}(n, \mathbb{C});\ A^{-1} = A^* := \bar{A}^t\}\ .$$

- The spaces of real symmetric matrices $S(n,\mathbb{R})$ with the spaces of *Hermitian matrices*
$$H(n,\mathbb{C}) = \{A \in M(n,\mathbb{C});\ A = A^*\}\ .$$
- The spectral theorem for real symmetric matrices with the *spectral theorem for complex Hermitian matrices.*

Then, by repeating *verbatim* the above constructions, for every (n,k) as above, we realize the (unitary) *complex Stiefel manifold* $S_{n,k}(\mathbb{C})$ as a compact embedded smooth manifold in $M(n,k,\mathbb{C})$, the complex Grassmann manifold $\mathfrak{G}_{n,k}(\mathbb{C})$ as a compact embedded smooth manifold in $H(n,\mathbb{C})$ (defined by the usual equations $A^2 - A = 0$, $\mathrm{trace}(A) = k$), the complex projective spaces $\mathbf{P}^{n-1}(\mathbb{C}) = \mathfrak{G}_{n,1}(\mathbb{C})$, and so on. Although we are dealing with spaces based on the *complex numbers*, we stress that in this way we have realized them as *real* embedded smooth manifolds.

We understand that also the next considerations about Stiefel and Grassmann manifolds would have a counterpart for the complex version.

3.5. A cellular decomposition of the Grassmann manifolds

We describe a natural partition of $\mathfrak{G}_{n,k}$ by a finite number of subsets, each one diffeomorphic to some $\mathbb{R}^h$, $0 \leq h \leq \dim \mathfrak{G}_{n,k}$, (i.e. an *open h-cell*) and such that its closure in $\mathfrak{G}_{n,k}$ is a union of cells of lower dimension. Let $L \in \mathfrak{G}_{n,k}$; that is, L is a k-dimensional linear subspace of $\mathbb{R}^n$ (here we make the abuse of identifying $G_{n,k}$ and $\mathfrak{G}_{n,k}$). For every $i = 0,\dots,n$, denote by
$$p_i : \mathbb{R}^n \to \mathbb{R}^{n-i}$$
the projection to the first $n-i$ coordinates, $p_i((x_1,\dots,x_n)^t) = (x_1,\dots,x_{n-i})^t$. The dimensions of $p_i(L) \subset \mathbb{R}^{n-i}$ decrease from k to 0 in exactly k steps; that is, there are integers
$$1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq n$$
such that for j that decreases from k to 1,
$$\dim p_{\sigma_j+1}(L) - \dim p_{\sigma_j}(L) = 1\ .$$
We call
$$\sigma(L) := (\sigma_1,\dots,\sigma_k)$$
the *Schubert symbol* of L. There is a concrete, elementary way to determine $\sigma(L)$:

- Fix any rank k, $n \times k$ matrix $A \in L_{n,k}$ which projects to $L \in \mathfrak{G}_{n,k}$.
- Apply to A the Gauss algorithm via elementary operations on the columns and get a matrix
$$\hat{A} \in L_{n,k}$$
in *column echelon form* which also projects to L.
- For every $j = 1,\dots,k$, the (σ_j, j) entry of $\hat{A}$ is equal to 1 and is a so-called 'pivot' of $\hat{A}$; the transpose of the σ_jth row of $\hat{A}$ is the σ_jth vector

of the standard basis of $\mathbb{R}^k$. Beyond the pivots, for every $1 \leq j \leq k$, an (s, j) entry of $\hat{A}$ is possibly non-zero only if $\sigma_j < s \leq n$ and s is not the row index of any pivot row. The computation of $\sigma(L)$ through $\hat{A}$ is immediate from the very definition. This means, in particular, that the initial choice of the matrix A is immaterial to this computation of $\sigma(L)$; $\sigma(\hat{A}) := \sigma(L)$ is the symbol of the matrix $\hat{A}$ and two matrices in column echelon form have the same index if and only if they share the same pivot positions.

• Furthermore, we claim that the whole matrix $\hat{A}$ does not depend on the choice of A as it is completely determined by L. Given $\sigma = \sigma(L)$, denote by p_σ the projection of $\mathbb{R}^n$ to the k coordinates $(x_{\sigma_1}, \dots, x_{\sigma_k})$; then the restriction of p_σ to L is a linear isomorphism and the columns of $\hat{A}$ are characterized as the vectors of L which are mapped in the order by p_σ to the vectors $e_{\sigma_1}, \dots, e_{\sigma_k}$ of the standard basis of $\mathbb{R}^k$.

In summary, there are $\binom{n}{k}$ Schubert symbols. For every such a symbol σ, the subset C_σ of $\mathfrak{G}_{n,k}$ formed by the k-subspaces of $\mathbb{R}^n$ which share the symbol σ is in bijection with the subset $\hat{C}_\sigma$ of $L_{n,k}$ formed by the matrices in column echelon forms which also share the symbol σ. $\hat{C}_\sigma$ has a natural base point, that is the matrix J_σ whose non-pivot entries are all zeros; then

$$\hat{C}_\sigma = J_\sigma + \mathbf{V}_\sigma$$

and it is easy to check that $\mathbf{V}_\sigma$ is a linear subspace of $M(n, k, \mathbb{R})$ formed by the matrices with a given pattern of zero entries determined by the symbol σ. The other entries contain free parameters. By counting the free parameters column by column, we readily verify that

$$d_\sigma := \dim \mathbf{V}_\sigma = \sum_{j=1}^{k} (n - \sigma_j - (k - j)) \ .$$

It follows that $C_\sigma \subset \mathfrak{G}_{n,k}$ admits a smooth parametrization

$$\psi_\sigma : \mathbb{R}^{d_\sigma} \to C_\sigma \ .$$

By varying the symbols, we have obtained a partition of $\mathfrak{G}_{n,k}$ by open cells. We claim that:

The closure of every C_σ in $\mathfrak{G}_{n,k}$ is formed by the $C_{\sigma'}$'s such that for every j, $\sigma'_j \geq \sigma_j$.

This claim is not obvious. We omit the proof, however, next item (4) should help the reader to reconstruct a proof.

Remarks and examples. (1) There is one top-dimensional (i.e. of dimension $k(n-k)$) cell of $\mathfrak{G}_{n,k}$ corresponding to the symbol $(1, 2, 3, \dots, k)$. This covers a chart around the image of $I_{n,k}$ in $\mathfrak{G}_{n,k}$. In general, every cell C_σ has a natural base point, that is the image in $\mathfrak{G}_{n,k}$ of the matrix $J_\sigma \in \hat{C}_\sigma$. There is one 0-cell corresponding to the symbol $(n-k+1, n-k, \dots, n)$.

(2) In the case of projective spaces $\mathbf{P}^n(\mathbb{R}) = \mathfrak{G}_{n+1,1}$, there are $n+1$ cells, one cell for each dimension $n, \dots, 0$, corresponding to the symbols $(1), (2), \dots, (n+1)$. The closure of each cell of dimension d is a copy of $\mathbf{P}^d(\mathbb{R})$ linearly embedded in $\mathbf{P}^n(\mathbb{R})$.

(3) For example, $\mathfrak{G}_{4,2}$ has six cells corresponding to the Schubert symbols $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, $(3,4)$, and these cells have dimensions $4,3,2,2,1,0$, respectively.

(4) The cells of $\mathfrak{G}_{n,k}$ can be described also in terms of the orthogonal Stiefel manifold $S_{n,k}$. A matrix $\tilde{A} \in S_{n,k}$ is in *orthogonal* column echelon form of symbol σ if its standard column echelon form $\hat{A}$ is of symbols σ and $\tilde{A}$ may differ from $\hat{A}$ only by: (i) the pivot entries of $\tilde{A}$ are non-zero not necessarily equal to 1; (ii) the entries of a pivot row of $\tilde{A}$ on the left of the pivot are not necessarily equal to 0; (iii) the last non-zero entry of every column is positive. We can verify that, for every $L \in \mathfrak{G}_{n,k}$, there is only one $\tilde{A} \in S_{n,k}$ which projects to L; in fact, if $\hat{A}$ is the unique matrix in standard column echelon form which projects to L, then we can obtain $\tilde{A}$ by applying the Gram-Schmidt algorithm to the columns of $\hat{A}$ considered in the backward order, normalized to also achieve the above condition (iii). The subset $\tilde{C}_\sigma$ of $S_{n,k}$ formed by the matrices in echelon form of symbol σ is diffeomorphic to $\hat{C}_\sigma \subset L_{n,k}$ and maps diffeomorphically to $C_\sigma \subset \mathfrak{G}_{n,k}$. We can prove that the closure of $\tilde{C}_\sigma$ in $S_{n,k}$ is diffeomorphic to a *closed* disk of dimension d_σ which maps to the closure of C_σ in $\mathfrak{G}_{n,k}$.

3.6. Stiefel and Grassmannian manifolds as real algebraic sets

For the notions and basic results of (real) algebraic geometry mentioned in this section, we can refer, for example, to [**BCR**] or to [**BR**].

By definition, a *real algebraic set* $Z \subset \mathbb{R}^m$, for some $m \in \mathbb{N}$, is of the form $Z = F^{-1}(0)$ for some *polynomial* map $F : \mathbb{R}^m \to \mathbb{R}^h$. The orthogonal Stiefel and Grassmann manifolds (even in the complex version) are examples of real algebraic sets. We are going to outline a way to show again that they are embedded smooth manifolds using algebraic geometry, obtaining indeed a stronger result.

For every algebraic set Z as above,

$$I(Z) := \{p(X) \in \mathbb{R}[X_1, \dots, X_m];\ p(x) = 0 \text{ for every } x \in Z\}$$

is called the (defining) *ideal of* Z. By a theorem of Hilbert, $I(Z)$ is *finitely generated*; that is, there exist some polynomials $p_1(X), \dots, p_k(X) \in I(Z)$ such that $I(Z)$ coincides with the set of linear combinations of the $p_j(X)$'s with polynomials coefficients in $\mathbb{R}[X_1, \dots, X_m]$. Consider the polynomial map

$$P : \mathbb{R}^m \to \mathbb{R}^k,\ P(x) = (p_1(x), \dots, p_k(x)) \ .$$

For every $p \in Z$, set

$$r(p) = \text{rank } d_pP \ .$$

It is not too hard to show that $r(p)$ does not depend on the choice of the generators $p_1, \dots, p_k$. So it is well defined

$$r(Z) = \max\{r(p);\ p \in Z\}\ .$$

Assume, for simplicity, that Z is *irreducible*; that is, it cannot be expressed as $Z = Z_1 \cup Z_2$ where Z_1 and Z_2 are algebraic sets both different from Z (you can prove that the connected Stiefel and Grassmann algebraic sets are irreducible, except for the orthogonal groups $O(n)$). Then $p \in Z$ is a *regular point* if $r(p) = r(Z)$. By the definition, the set $R(Z)$ of regular points of Z is *nonempty*. A *Zariski open set* in $\mathbb{R}^m$ is of the form $\mathbb{R}^m \setminus Y$ where Y is an algebraic set in $\mathbb{R}^m$. The following is a nontrivial result.

THEOREM 3.4. *Let $Z \subset \mathbb{R}^m$ be an irreducible algebraic set of rank $r = r(Z)$. Then, for every $p \in R(Z)$, there exists a Zariski open set U of $\mathbb{R}^m$ and a polynomial map $F = (F_1, \dots, F_r) : \mathbb{R}^m \to \mathbb{R}^r$ such that:*

(1) $p \in U$.
(2) $F_j \in I(Z)$, $j = 1, \dots, r$.
(3) $Z \cap U = U \cap F^{-1}(0)$.
(4) For every $x \in U \cap Z$,

$$\text{rank } d_x F = r\ .$$

In particular, $R(Z)$ is an embedded smooth manifold in $\mathbb{R}^m$ of dimension $m - r$.

COROLLARY 3.5. *Let $Z \subset \mathbb{R}^m$ be one of our favourite (Stiefel or Grassmann) algebraic sets. Then $Z = R(Z)$. In particular, Z is an embedded smooth manifold of dimension $m - r(Z)$.*

Proof : We know that $R(Z) \neq \emptyset$. Let $p \in R(Z)$. By using the suitable transitive action on Z of orthogonal (unitary) groups, we realize that, for every $q \in Z$, there is a particularly simple diffeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ such that $\phi(Z) = Z$ and $\phi(p) = q$. Although this is a particular case of a general result on the invariance of $R(Z)$ up to "algebraic isomorphism", these diffeomorphisms are so simple that we can check directly that, since p is regular, then q is also regular. Then $Z = R(Z)$.

■

Note that the linear Stiefel manifolds are Zariski open sets of the pertinent matrix spaces.

REMARK 3.6. The notion of regular point is rather delicate. For example, it can happen that there is some irreducible algebraic set $X \subset \mathbb{R}^m$ which is an embedded smooth manifold, nevertheless $R(X) \neq X$.

CHAPTER 4

The category of smooth manifolds

In this chapter, we introduce the most general category of smooth manifolds considered in this text. *Abstract* smooth manifolds and smooth maps between them will be introduced by taking as *definitions* some properties satisfied by embedded ones. Embedded manifolds are eventually incorporated in the general theory as being submanifolds of some Euclidean space. We will see in Section 4.34 that (abstract) compact manifolds can be embedded in some $\mathbb{R}^n$. As we are mainly interested in compact manifolds, considered up to diffeomorphism, this abstraction would sound a bit superfluous. However, there are good reasons to do so. We will point out natural constructions to build new abstract compact manifolds, starting from given ones, even embedded. It would be artificial to force these constructions in the embedded setting. It is more convenient to use the embedding result *a posteriori*, to exploit the facts that we will establish for compact embedded manifolds in Chapter 6.

DEFINITION 4.1. A topological space M is an *m-smooth manifold* (we will omit the adjective "abstract") if:

- M is Hausdorff and with a countable basis of open sets.
- M admits a *smooth atlas* $\mathcal{U} = \{W_j, \phi_j\}_{j\in J}$ (J being any set of indices); that is
  (i) $\{W_j\}_{j\in J}$ is an open covering of M;
  (ii) every *chart* $\phi_j : W_j \to U_j \subset \mathbb{R}^m$ is a *homeomorphism* to an open set U_j of $\mathbb{R}^m$ ($\psi_j : U_j \to W_j$ denotes the inverse *local parametrization*);
  (iii) for every $i, j \in J$,

$$\phi_j \circ \psi_i : \phi_i(W_i \cap W_j) \to \phi_j(W_i \cap W_j)$$

  is a smooth *diffeomorphism*.

We summarize this second item by saying that M is (smoothly) *locally m-Euclidean*.

REMARKS 4.2. (1) Two smooth atlantes of M are said *compatible* if their union is a smooth atlas. The union of all compatible atlantes is the *maximal smooth atlas* of M. Every smooth atlas $\mathcal{U}$ of M implicitly determines such a unique maximal atlas $\mathcal{A}_M$; this is identified with a specific *smooth structure on M*.

(2) Every embedded smooth manifold is a smooth manifold. In the embedded case, the charts of an atlas were smooth by themselves, referring to the smooth structure of the ambient Euclidean space. In the abstract case, every single chart is only a homeomorphism; the smooth structure is carried by the changes of local coordinates.

(3) Being locally Euclidean does not imply any of the global topological requirements of the first item. For example, consider $M = \mathbb{R}^m \times (\mathbb{R}, \tau_d)$ where the second factor is endowed with the *discrete topology*. Then M is Hausdorff and locally m-Euclidean, but it has no countable basis of open sets. On another hand, consider on $\mathbb{R} \times \{0, 1\}$ (with the product topology) the equivalence relation such that $(x, j) \sim (y, i)$ if and only if either $(x, j) = (y, i)$ or $x = y$ and $x > 0$. Let M be the quotient topological space. Then M is 1-locally Euclidean and has a countable basis of open sets, but it is not Hausdorff. In fact the two points $[(0, 0)] \neq [(0, 1)] \in M$ cannot be separated by disjoint neighbourhoods. $M \times \mathbb{R}^k$ presents the same phenomenon in an arbitrary dimension.

(4) The previous remark poses some principle question when we use manifolds as a model of some physical space or space-time. Local observations can support the idea that phenomena live in a locally Euclidean environment, but it is much more arbitrary to also assume those global topological properties. For example, in some models of space-time, they are not assumed *a priori*, they are derived *a posteriori* as a consequence of certain global "causality assumptions" which look founded on physical (or philosophical) considerations, see for instance [**HE**]. To our aims, we do not hesitate to make these topological assumptions; as the theory is already rich, there are no reasons to renounce, for example, the limit uniqueness or the equivalence between compact and sequentially compact spaces.

(5) A smooth manifold M is connected if and only if it is path-connected. In general, every path-connected component of M is a smooth manifold. We can repeat word by word the proof of the embedded case.

Definition 4.3. Let $f : M \to N$ be a continuous map between smooth manifolds of dimension m and n, respectively. A *representation in local coordinates* of f is of the form

$$\hat{f} = \phi' \circ f \circ \psi : U \to U'$$

where $\phi : W \to U \subset \mathbb{R}^m$ is a chart of $\mathcal{A}_M$, $\phi' : W' \to U' \subset \mathbb{R}^n$ is a chart of $\mathcal{A}_N$, and $f(W) \subset W'$. Then f is *smooth* if for every $p \in M$, there is a local representation of f such that $p \in W$ and $\hat{f}$ is a smooth map between open sets of Euclidean spaces. The map f is a *diffeomorphism* if it is a homeomorphism and both f and f^{-1} are smooth.

The following lemma is an easy consequence of the definitions and of the basic fact that the composition of smooth maps between open sets of Euclidean spaces is smooth.

LEMMA 4.4. *If $f : M \to N$ is a smooth map between smooth manifolds, then every local representation of f in local coordinates is smooth.*

REMARK 4.5. A chart $\phi : W \to U$ of a smooth manifold M and its inverse local parametrization $\psi : U \to W$ are smooth diffeomorphisms as for example, $\mathrm{Id}_U = \phi \circ \psi$ is a representation in local coordinates of ψ. Like the embedded case, the dimension is invariant up to diffeomorphism.

DEFINITION 4.6. A smooth map $f : M \to N$ is a *submersion* (*immersion*) if for every $p \in M$, there is a local representation of f which is a submersion (immersion).

DEFINITION 4.7. Let M be a smooth m-manifold. A subset $Y \subset M$ is a *submanifold* of M of dimension k, $0 \leq k \leq m$, if for every $p \in Y$, there exist relative normal charts of the form

$$\beta : (W, W \cap Y, p) \to (B^m(0,1), B^n(0,1) \cap \mathbb{R}^k, 0)$$

where as usual we consider $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$. The restrictions of these relative charts to Y provide a smooth atlas of Y.

Embedded manifolds as in Chapter 2 coincide with (abstract) submanifolds of some Euclidean space. Smooth maps, diffeomorphisms, submersions and immersions between embedded manifolds fulfill the above definitions. The category of smooth manifolds and smooth maps extends the embedded one.

Let us describe some constructions that naturally produce (abstract) smooth manifolds.

(1) *(Quotient manifolds)* Let $\tilde{M}$ be a smooth manifold. Let G be a subgroup of the group $\mathrm{Aut}(\tilde{M})$ of smooth automorphisms of $\tilde{M}$. Assume that G acts *freely* and *properly discontinuously* on $\tilde{M}$. This means that for every $p \in \tilde{M}$, the identity is the only element of G that fixes p, and that for every compact subset K of $\tilde{M}$, the set of $g \in G$ such that $K \cap g(K) \neq \emptyset$ is *finite*. Let $M := \tilde{M}/G$ be the quotient topological space. We can prove that M is Hausdorff and with countable basis. Moreover, the projection $\pi : \tilde{M} \to M$ is a covering map; for every $p \in M$, there is an open connected neighbourhood W of p such that the restriction of π to every connected component $\tilde{W}$ of $\pi^{-1}(W)$ is a homeomorphism, and $\tilde{W}$ carries a chart $(\tilde{W}, \phi)$ belonging to $\mathcal{A}_{\tilde{M}}$. Then, by varying p in M, $\{(W, \phi \circ \pi^{-1})\}$ is a smooth atlas of M, such that π becomes a smooth locally diffeomorphic map.

(2) *(Grassmann manifolds again)* We have already defined (Chapter 3) the projective spaces $\mathbf{P}^k(\mathbb{R})$ as special instances of (embedded) Grassmann manifolds. There is another classical way to obtain it. Consider $\mathbb{R}^{k+1}$. The multiplicative group $\mathbb{R}^*$ acts on $\mathbb{R}^{k+1}$. Consider the quotient topological space $\mathbb{R}^{k+1}/\mathbb{R}^*$. This is not Hausdorff; the only saturated open set of $\mathbb{R}^{k+1}$ containing 0 is the whole of $\mathbb{R}^{k+1}$ and this intersects any other saturated open set. If we remove 0, and we restrict the action of $\mathbb{R}^*$, things go better.

The orbits, i.e. the equivalence classes, are in bijective correspondence with the set of 1-dimensional linear subspaces of $\mathbb{R}^{k+1}$. Then we easily verify that the quotient topological space $\mathbf{P}^k(\mathbb{R}) := (\mathbb{R}^{k+1} \setminus \{0\})/\mathbb{R}^*$ is now Hausdorff and with a countable basis. We obtain the same quotient space if we restrict the equivalence relation to the unit sphere S^k, then the projection to the quotient, $\pi : S^k \to \mathbf{P}^k(\mathbb{R})$, is a $2:1$ local *homeomorphism*. It is the quotient map by the action on S^k of the group G of order 2 generated by the antipodal map $x \to -x$. Then we can endow $\mathbf{P}^k(\mathbb{R})$ with a smooth manifold structure as a particular case of point (1). We can do it also without restricting to S^k. A finite atlas of $\mathbf{P}^k(\mathbb{R})$ is formed by $\{(W_j, \phi_j)\}_{j=1,\dots,k+1}$, where W_j is the image of the saturated open set $\{x_j \neq 0\}$ of $\mathbb{R}^{k+1} \setminus \{0\}$;

$$\phi_j([x_1, \dots, x_{k+1}]) = (x_1/x_j, \dots, x_{j-1}/x_j, x_{j+1}/x_j, \dots, x_{k+1}/x_j)$$

is a homeomorphism of W_j to $\mathbb{R}^k$. It is immediate to check that the changes of local coordinates are smooth (actually rational). *A posteriori*, we can define, in a natural way, a diffeomorphism of this abstract model of the projective space to the embedded model already constructed.

Every Grassmann manifold could be treated similarly. First, define it as the quotient topological space of the associated linear Stiefel manifold (which is an open set in some Euclidean space); prove that this quotient is Hausdorff and with a countable basis, and finally give it a (abstract) smooth atlas made by the image of suitable saturated open sets of the Stiefel manifold. *A posteriori*, you can construct a diffeomorphism to the already constructed embedded model of Chapter 3.

(3) This example could sound a bit artificial, but it reveals some subtleties. Let M be a smooth manifold (even embedded). Let $f : X \to M$ be any *homeomorphism*. Then

$$\mathcal{U}_f := \{(f^{-1}(W), \phi \circ f)\}_{(W,\phi)\in\mathcal{A}_M}$$

is a *smooth* atlas on X so that f becomes tautologically a diffeomorphism. If $X = M$ (as a topological space), the two smooth structures carried by $\mathcal{U}_f$ and $\mathcal{A}_M$ are diffeomorphic to each other but they are not, in general, the same structure (in other words id_M is not a diffeomorphism). Even if M is embedded, in no natural way is the structure given by $\mathcal{U}_f$ embedded.

Example 4.8. We are going to establish that $SO(3) \sim \mathbf{P}^3(\mathbb{R})$. An elegant way to see it is by using *quaternions*. Let $\mathbf{H}$ be the *quaternion* algebra in its matrix form. That is, $\mathbf{H}$ is the subalgebra of the matrix algebra $M(2, \mathbb{C})$ of the matrices of the form

$$A = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$. Then $\mathbf{H}$ is generated by the matrix

$$A(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ A(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ A(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

which verifies the relations

$$A(i)^2 = A(j)^2 = A(k)^2 = -I$$

$$A(i)A(j) = A(k) = -A(j)A(i),\ A(j)A(k) = A(i) = -A(k)A(j)$$

$$A(k)A(i) = A(j) = -A(j)A(k)\ .$$

By setting

$$A^* := \bar{A}^t$$

we have

$$(AB)^* = A^* + B^*,\ (AB)^* = A^*B^*$$

$$|A|^2 := AA^* = \det A$$

and if $A \neq 0$

$$A^{-1} = \frac{1}{|A|^2} A^*\ .$$

Set

$$\mathbf{H}_1 = \{A \in \mathbf{H};\ |A| = 1\}\ .$$

This is a group for the restriction of the multiplication. This group $\mathbf{H}_1$ is naturally identified with the special unitary group $SU(2)$ that, as a manifold, is naturally identified with the unit sphere S^3 in $\mathbb{R}^4$. Set

$$\mathbf{H}_0 = \{A \in \mathbf{H};\ A^* = -A\}\ ,$$

which is naturally identified with a Euclidean space $\mathbb{R}^3$. For every $A \in \mathbf{H}_1$,

$$\alpha_A : \mathbf{H}_0 \to \mathbf{H}_0,\ X \to AXA^{-1}$$

acts as a rotation on $\mathbf{H}_0 = \mathbb{R}^3$. This gives us a degree 2 covering map

$$SU(2) \to SO(3),\ A \to \alpha_A$$

such that $\alpha_A = \alpha_B$ if and only if $B = \pm A$. Hence finally

$$SO(3) \sim SU(2)_{/\pm I} \sim \mathbf{P}^3(\mathbb{R})$$

as claimed.

Let us consider now for every $(P, Q) \in SU(2) \times SU(2) = \mathbf{H}_1 \times \mathbf{H}_1$, the map

$$\alpha_{P,Q} : \mathbf{H} \to \mathbf{H},\ A \to PAQ^{-1}$$

by identifying $\mathbf{H} \sim \mathbb{R}^4$, $\alpha_{P,Q} \in SO(4)$ and $\alpha_{P,Q} = \alpha_{P',Q'}$ if and only if $(P, Q) = \pm(P', Q')$. Similarly as above, we get that

$$(SU(2) \times SU(2))/\pm 1 \sim SO(4)\ .$$

4.1. Topologies on spaces of smooth maps

Let M, N be smooth manifolds. We define the *weak topology* on every set $\mathcal{C}^r(M,N)$, $r \geq 0$, the topological spaces $\mathcal{E}^r(M,N)$ (the subspaces of $\mathcal{C}^r(M,N)$ formed by the smooth maps), and the space $\mathcal{E}(M,N)$ (that is $\mathcal{C}^\infty(M,N)$ equipped with the union of the $\mathcal{E}^r$ topologies). This extends the special case of open sets treated in Chapter 1. As in that case, we provide a basis of open neighbourhoods of every element in the pertinent map space. For every $f \in \mathcal{C}^r(M,N)$, we consider neighbourhoods of the form

$$\mathcal{U}_r(f, f_{U,U'}, K, \epsilon)$$

where

- $f_{U,U'} : U' \to U$ is a (necessarily $\mathcal{C}^r$) representation of f in local coordinates ($U \subset \mathbb{R}^m$, $U' \subset \mathbb{R}^n$ being open sets);
- $K \subset U$ is a compact set;
- $\epsilon > 0$.

Then $g \in \mathcal{C}^r(M,N)$ belongs to

$$\mathcal{U}_r(f, f_{U,U'}, K, \epsilon)$$

if and only if it admits a local representation (over the same open sets U, U') $g_{U,U'} : U \to U'$ such that $g_{U,U'} \in \mathcal{U}_r(f_{U,U'}, K, \epsilon) \subset \mathcal{C}^k(U,U')$.

If $M \subset \mathbb{R}^h$, $N \subset \mathbb{R}^k$ are embedded manifolds, there is an equivalent way to define these topologies. For every $f \in \mathcal{C}^r(M,N)$ we consider neighbourhoods of the form

$$\mathcal{U}_r(f, \hat{f}, K, \epsilon)$$

where

- $\hat{f} : \Omega \to \mathbb{R}^k$ is a local $\mathcal{C}^r$ extension of $f_{|W} : W \to N$, $W = \Omega \cap M$, $\Omega \subset \mathbb{R}^h$ being open;
- $K \subset W$ is a compact set;
- $\epsilon > 0$.

Then $g \in \mathcal{C}^r(M,N)$ belongs to $\mathcal{U}_r(f, \hat{f}, K, \epsilon)$ if and only if there exists a $\mathcal{C}^r$ extension $\hat{g} : \Omega \to \mathbb{R}^k$ of $g_{|W}$ such that $\hat{g} \in \mathcal{U}_r(\hat{f}, K, \epsilon) \subset \mathcal{C}^r(\Omega, \mathbb{R}^k)$.

4.2. Homotopy, isotopy, diffeotopy, homogeneity

These notions, already introduced in Chapter 1 in the special case of open sets, extend *verbatim* to smooth manifolds. They correspond to continuous paths in suitable map spaces and carry equivalence relations.

The proof of the homogeneity Theorem 1.18 is essentially local and extends straightforwardly.

THEOREM 4.9. *Let N be a connected smooth manifold. Let $p, q \in N$. Then there is a diffeotopy with compact support between $f_0 = \mathrm{id}_N$ and $f = f_1$ such that $f(p) = q$.*

4.3. The (abstract) tangent functor

For embedded manifolds, tangent bundles and maps have been constructed as a direct generalization of the basic case of open sets in Euclidean spaces. For abstract manifolds, they must be somehow "invented", with the constraint that they must be compatible with what is already done in the embedded category. This will lead us in Section 4.4 to the general notion of *fibre bundle in the sense of Steenrod* [**Steen**].

4.3.1. Fibre bundles. The tangent vector bundle of an embedded manifold is the first fundamental example of the general notion of *fibre bundle*. We saw other examples in Chapter 3, dealing with Grassmann and Stiefel manifolds. As we will find many more examples, at this point it is convenient to formalize this notion.

A *smooth fibre bundle* with *base space* X, *total space* E, and *fibre* F, is a surjective submersion $f : E \to X$ between smooth manifolds such that every fibre $f^{-1}(q)$, $q \in X$, is a submanifold of E diffeomorphic to a given manifold F, and which is *locally trivializable* at every point q of X. This means that, for every $q \in X$, there is an open neighbourhood Ω in X and trivializing commutative diagram of the form

$$\begin{array}{ccc} \Omega \times F & \stackrel{\Phi}{\to} & \tilde{\Omega} \\ \downarrow \pi_\Omega & & \downarrow f_| \\ \Omega & \stackrel{\mathrm{id}_\Omega}{\to} & \Omega \end{array}$$

where $\tilde{\Omega} := f^{-1}(\Omega)$ and Φ is a diffeomorphism (with inverse Ψ). If $E = X \times F$ and $f = \pi_X$ is the natural projection, then it is a *trivial* (also-called 'product') fibre bundle. The family of all *local trivializations* form the *maximal fibred atlas* $\mathcal{F}$ of the fibre bundle. A fibred atlas is a subfamily of $\mathcal{F}$ such that the Ω's form an open covering of X, hence the $\tilde{\Omega}$'s of E. Every fibred atlas is contained in a unique maximal one, so it is enough to give a fibred atlas to determine a fibre bundle structure. Every change of local trivialization is of the form

$$\Phi' \circ \Psi : (\Omega \cap \Omega') \times F \to (\Omega \cap \Omega') \times F$$

$$(p, y) \to (p, \rho(p)(y))$$

where $\rho(p)$ belongs to the group $\mathrm{Aut}(F)$ of the smooth automorphisms of the fibre F.

In many cases, the fibre F has an additional structure which is preserved by a subgroup G of $\mathrm{Aut}(F)$ (for example, $F = \mathbb{R}^n$, $G = \mathrm{GL}(n, \mathbb{R})$); if the $\rho(p)$'s as above belong to G then we have a *G-fibre bundle* (*vector bundle*, ...).

A particular case is when $\dim F = 0$. In such a case, a fibration $f : E \to X$ is also-called a *covering map* (of *degree* d if F is compact hence finite, and $d = |F|$). For every local trivialization, the restriction of f to every connected component of $\tilde{\Omega}$ is a diffeomorphism to Ω, provided that Ω is connected.

EXAMPLE 4.10. *(Grassmann manifolds of oriented sub-spaces)* The set $\tilde{G}_{m,n}$ of *oriented* n-subspaces of $\mathbb{R}^m$ can be naturally endowed with a smooth compact manifold structure $\tilde{\mathfrak{G}}_{m,n}$ such that the map

$$p : \tilde{G}_{m,n} \to G_{m,n}$$

that forgets the orientation becomes a degree-2 smooth covering map

$$p : \tilde{\mathfrak{G}}_{m,n} \to \mathfrak{G}_{m,n} \ .$$

As a special case we have the natural covering map $S^n \to \mathbf{P}^n(\mathbb{R})$.

A *fibred map* between fibre bundles is a commutative diagram of smooth maps $[g, \tilde{g}]$ of the form

$$\begin{array}{ccc} E & \stackrel{\tilde{g}}{\to} & E' \\ \downarrow f & & \downarrow f' \\ X & \stackrel{g}{\to} & X' \end{array}$$

so that every fibre $E_x \sim F$ is mapped to the fibre $E'_{g(x)} \sim F'$. It is a *fibred diffeomorphism* if both g and $\tilde{g}$ are diffeomorphisms. In such a case, $F = F'$. The diagrams $[f, Tf]$ of the tangent functor are basic examples of fibred maps.

Fibred equivalences. Consider the set $\mathcal{F}(X, F)$ of fibred bundles over a given base space X, with given fibre F. There are two natural equivalence relations on $\mathcal{F}(X, F)$:

(1) The *full equivalence*, generated by the fibred diffeomeorphisms $[g, \tilde{g}]$ such that g belongs to the group $\mathrm{Aut}(X)$ of smooth automorphisms of X.

(2) The *strict equivalence* (often we will omit to say "strict"), generated by the fibred diffeomorphism of the form $[\mathrm{id}_X, \tilde{g}]$.

This specializes directly to the case of G-fibred bundles.

4.3.2. Tangent spaces. Let M be a smooth m-manifold, $p \in M$. First we define the tangent space T_pM to M at the point p. We do it by extending Sections 1.6.1 and 2.1.1. Following the first interpretation of the tangent spaces in the embedded case, on the set of smooth curves in M, $\gamma : \mathbb{R} \to M$, passing through the point p ($\gamma(0) = p$), we consider the equivalence relation such that $\gamma \sim \tau$ if and only if for every smooth function $h : M \to \mathbb{R}$, $d_0(h \circ \gamma) = d_0(h \circ \tau) : \mathbb{R} \to \mathbb{R}$. We denote by $\mathcal{V}_pM$ the quotient set. For every local parametrization $\psi : U \to M$ of M at p, $\psi(a) = p$, every $[\gamma]$ can be represented in a unique way as $[\psi \circ \bar{\gamma}]$, where $\bar{\gamma} : \mathbb{R} \to U$ represents an element of $\mathcal{V}_aU \sim T_aU = \mathbb{R}^m$. This establishes a linear isomorphism between $\mathbb{R}^m$ and $\mathcal{V}_pM$. Following the second interpretation of the tangent spaces, $\delta_{[\gamma]}([h]) := d_0(\gamma \circ h)$ well defines a linear isomorphism between $\mathcal{V}_pM$ and the space of derivations $\mathrm{Der}(\mathcal{E}_pM)$ on the space of germs at p of smooth functions $h : M \to \mathbb{R}$. Arguing as above we have a natural linear isomorphism between

$\mathrm{Der}(\mathcal{E}_p M)$ and $\mathrm{Der}(\mathcal{E}_a U) = \mathbb{R}^m$. In other words, since the tangent space $T_p M$ is a purely local object, by using local coordinates of M at p, we can carry on M the considerations made before for embedded manifolds. The next more demanding task is to embody the tangent spaces in a bundle structure.

4.3.3. Construction of the tangent bundle. Let M be an m-smooth manifold with its maximal smooth atlas $\mathcal{A} = \{(W_j, \phi_j)\}_{j\in J}$. For every $(i,j) \in J^2$, define the map

$$\mu_{ji} : W_i \cap W_j \to \mathrm{GL}(m,\mathbb{R}),\ \mu_{ji}(x) = d_{\phi_i(x)}(\phi_j \circ \phi_i^{-1}) \ .$$

This family of maps $\{\mu_{ji}\}_{(i,j)\in J^2}$ satisfies the following properties:

(1) Every μ_{ji} is smooth.

(2) For every $j \in J$, for every $x \in W_j \cap W_j = W_j$,
$$\mu_{jj}(x) = I_m \ .$$

(3) For every $(j,i) \in J^2$, for every $x \in W_i \cap W_j = W_j \cap W_i$,
$$\mu_{ji}(x) = \mu_{ij}(x)^{-1} \ .$$

(4) For every $(i,j,k) \in J^3$, for every $x \in W_i \cap W_j \cap W_k$
$$\mu_{ik}(x)\mu_{kj}(x)\mu_{ji}(x) = I_m \ .$$

We summarize these properties by saying that $\{\mu_{j,i}\}$ *is a smooth cocycle on the open covering* $\mathcal{A}$ *with values in the Lie group* $\mathrm{GL}(m,\mathbb{R})$.

As $\mathrm{GL}(m,\mathbb{R})$ is *non-commutative* (if $m > 1$), then the order of the factors in property 4 is not negligible.

Let us consider now the topological product $M \times \mathbb{R}^m \times J$, where J is endowed with the discrete topology. Let $\mathcal{T}$ be the subspace made by the triples (x,v,j) such that $x \in W_j$. Hence $\mathcal{T}$ is the *disjoint* union of the open sets $W_j \times \mathbb{R}^m \times \{j\}$, $j \in J$, each one being canonically homeomorphic to $W_j \times \mathbb{R}^m$. Let us put on $\mathcal{T}$ the relation $(x,v,j) \sim (x',v',k)$ if and only if $x = x'$ and $v' = \mu_{kj}(x)v$. The cocycle properties 2–4 ensure that it is an equivalence relation. We set

$$T(M) := \mathcal{T}/\sim$$

the topological quotient space and denote by $q : \mathcal{T} \to T(M)$ the canonical continuous projection. We have the well defined surjective map

$$\pi_M : T(M) \to M,\ \pi_M([x,v,j]) = x$$

which is continuous. In fact, for every open set A of M, $(\pi_M \circ q)^{-1}(A)$ is the intersection of $\mathcal{T}$ with $A \times \mathbb{R}^m \times J$, hence it is a satured open set, with open image in $T(M)$. It is a topological exercise to show that $T(M)$ *is Hausdorff and with countable basis*; this is left to the reader.

Local trivializations. For every $j \in J$, set

$$\Psi_j : W_j \times \mathbb{R}^m \to T(M),\ (x,v) \to q(x,v,j) = [(x,v,j)] \ .$$

We verify that

(1) Ψ_j is continuous (because q is continuous);
(2) Ψ_j takes values in $\pi_M^{-1}(W_j)$ and $\pi_M \circ \Psi_j = p_j$, where $p_j : W_j \times \mathbb{R}^m \to W_j$ is the projection.
(3) In fact Ψ_j *is a homeomorphism to* $\pi_M^{-1}(W_j)$. For if $b = [x, v, k] \in \pi_M^{-1}(W_j)$, then $b = \Psi_j(x, \mu_{jk}(x)v)$, hence Ψ_j is surjective. If $[x, v, j] = [x', v', j]$, then $x = x'$ and $v = v'$ because $\mu_{jj} = I_m$. Hence Ψ_j is injective. Finally, to show that the inverse of Ψ_j is continuous, it is enough to show that if A is open in $W_j \times \mathbb{R}^m$, the $q^{-1}(\Psi_j(A))$ is open in $\mathcal{T}$. Since the $W_k \times \mathbb{R}^m \times \{k\}$'s form a open covering of $\mathcal{T}$, it is enough to prove that every $q^{-1}(\Psi_j(A)) \cap (W_k \times \mathbb{R}^m \times \{k\})$ is open. This intersection is contained in the open set $(W_j \cap W_k) \times \mathbb{R}^m \times \{k\}$ of $\mathcal{T}$. On this open set $q = \Psi_j \circ r$, where $r(x, v, k) = (x, \mu_{jk}(x)v)$ which is continuous; the thesis follows.

Changes of local trivializations. These are of the form

$$\Psi_j^{-1} \circ \Psi_i(x, v) = (x, \mu_{ji}(x)v)$$

defined on $(W_j \cap W_i) \times \mathbb{R}^m$ to itself. They are smooth, and pointwise linear in the second argument. So we have proved that

$$\pi_M : T(M) \to M$$

is a (abstract) smooth vector bundle over M with fibre $\mathbb{R}^m$, called the *tangent bundle of* M. For every $p \in M$, the fibre $T_pM := \pi_M^{-1}(p)$ is *by definition* the *tangent space of* M *at* p. $T(M)$ is a smooth manifold because it is locally diffeomorphic to spaces of the form $W_j \times \mathbb{R}^m$, W_j being an open set in the smooth manifold M. To be even more concrete, we can exhibit the following special smooth atlas of $T(M)$ made of fibred maps:

$$T\mathcal{A} = \{\pi_M^{-1}(W_j), \Phi_j)\}_{j \in J}$$

where $\Phi_j := (\phi_j, \mathrm{id}) \circ \Psi_j^{-1}$, and

$$(\phi_j, \mathrm{id}) : W_j \times \mathbb{R}^m \to U_j \times \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m,\ (x, v) \to (\phi_j(x), v)\ .$$

Any change of local coordinates is of the form

$$\Phi_j \circ \Phi_i^{-1}(x, v) = (\phi_j \circ \phi_i^{-1}(x), \mu_{ji}(x)v)$$

which *ultimately is nothing else than the tangent map of the change of coordinates on* M.

Tangent map. Let $f : M \to M'$ be a smooth map between smooth manifolds. We want to define now the tangent map

$$Tf : T(M) \to T(M')$$

in such a way that $[f, Tf]$ is a vector bundle fibred map. We have constructed the tangent bundles by patching together the product pieces. We do similarly for Tf. Precisely, let $(\pi_M^{-1}(W), \Phi)$, $(\pi_{M'}^{-1}(W'), \Phi')$ be fibred charts of

$T(M)$ and $T(M')$ which dominate charts (W,ϕ), (W',ϕ') of M and M' respectively. Assume also that this system of charts gives us a representation in local coordinates of f, $\hat{f} = \phi' \circ f \circ \phi^{-1}$. Then we *locally* define

$$Tf_{W,W'} : \pi_M^{-1}(W) \to \pi_{M'}^{-1}(W'),\ Tf_{W,W'} = \Phi' \circ T\hat{f} \circ \Phi^{-1}\ .$$

Recalling the equivalence relation that we have used to build the tangent bundles, we readily check that these locally defined Tf's are in fact *representations in local (fibred) coordinates of a* **globally** *defined fibred map* $Tf : T(M) \to T(M')$. For every $p \in M$, the restriction d_pf of Tf to T_pM is a linear map

$$d_pf : T_pM \to T_{f(p)}M'$$

which by definition is the *differential of f at p*.

Tangent functor. The basic functorial properties of the chain rule globalize so that we have:

The tangent category of the category of smooth manifolds has as objects the tangent vector bundles of smooth manifolds and as morphisms the tangent maps of smooth maps between smooth manifolds. Then

$$M \ \Rightarrow\ \pi_M : T(M) \to M,\ f : M \to M' \ \Rightarrow [f, Tf]$$

define a covariant functor from the category of smooth manifolds to its tangent category. This extends the embedded tangent functor.

Immersions, submersions and embeddings. We can reformulate now the definition of immersions and submersions, already given in Definition 4.6 in terms of representations in local coordinates, in the same way as for embedded manifolds. A smooth map $f : M \to N$ is an immersion (submersion) if for every $p \in M$, d_pf is injective (surjective); it is an embedding if it is an immersion and a homeomorphism to its image. The proof of Proposition 2.12 is of local nature and extends straightforwardly.

PROPOSITION 4.11. *(1) Let $f : M \to N$ be a surjective submersion; then for every $q \in N$, $Y = f^{-1}(q)$ is a submanifold of M and* $\dim Y = \dim M - \dim N$.

(2) If $f : M \to N$ is an embedding then $f(M)$ is a submanifold of N.

(3) If $f : M \to N$ is both an immersion and a submersion, then it is a local diffeomorphism.

4.4. Principal and associated bundles with given structure group

The construction of the tangent bundles lends itself to a wide generalization. Let G be a Lie group (such as $\mathrm{GL}(m,\mathbb{R})$, $O(m)$, $SO(n)$, $U(n)$, ...). Assume that it acts on a smooth manifold F. This means that there is a group homomorphism (also-called a *representation*)

$$\rho : G \to \mathrm{Aut}(F)\ ;$$

the associate action is

$$G \times F \to F,\ (g,x) \to \rho(g)(x)$$

and sometimes we simply write gx instead of $\rho(g)(x)$. We also require that ρ is injective so that G is identified with its image in $\mathrm{Aut}(F)$ and considered as a *group of transformations of* F.

REMARK 4.12. A Lie group G acts on itself by the injective homorphism $g \to L_g$ (i.e. by left multiplication)

$$G \times G \to G,\ (g,h) \to L_g(h) := gh\ .$$

Let M be a smooth manifold and $\mathcal{U} = \{A_s\}_{s\in\mathcal{I}}$ be an open covering of M. A *principal cocycle* on $\mathcal{U}$ with values in the *structure group* G is a family of smooth maps

$$\mathfrak{c} = \{c_{ts} : A_s \cap A_t \to G\}_{(s,t)\in\mathcal{I}^2}$$

such that

(1) For every $s \in \mathcal{I}$, for every $x \in A_s$,
$$c_{ss}(x) = 1 \in G\ .$$
(2) For every $(s,t) \in \mathcal{I}^2$, for every $x \in A_s \cap A_t$,
$$c_{st}(x) = c_{ts}(x)^{-1}\ .$$
(3) For every $(s,t,r) \in \mathcal{I}^3$, for every $x \in A_s \cap A_t \cap A_r$
$$c_{sr}(x)c_{rt}(x)c_{ts}(x) = 1\ .$$

For every representation $\rho : G \to \mathrm{Aut}(F)$ as above, we have an *associated cocycle* with values in $\mathrm{Aut}(F)$

$$\{\rho_{ts} := \rho \circ c_{ts} : A_s \cap A_t \to \mathrm{Aut}(F)\}_{(s,t)\in\mathcal{I}^2}$$

which satisfies the same properties 1-3 (by replacing $1 \in G$ with $1 \in \mathrm{Aut}(F)$). We can repeat word by word the above construction of the tangent bundles and get a *smooth fibre bundle over* M *with structure group* G *and fibre* F. So we have a wide family of bundles that share the basic cocycle $\mathfrak{c}$. When $F = G$ and G acts as above by left multiplication, we get the *principal bundle* of the family; all the other bundles are said to be *associated* to such a principal bundle.

4.4.1. Equivalent cocycles. The strict equivalence of fibre bundles can be rephrased in terms of the defining cocycles. Assume that two cocycles c and c' with values in G are defined on the same open covering $\mathcal{U} = \{A_s\}_{s\in\mathcal{I}}$ of M. Then they define strictly equivalent bundles if and only if there is a family of maps

$$\{\lambda_s : A_s \to G\}_{s\in\mathcal{I}}$$

such that for every (s,t), for every $x \in A_s \cap A_t$,

$$c'_{ts}(x) = \lambda_s(x)c_{ts}(x)\lambda_t(x)^{-1}\ .$$

4.5. Tensor bundles

We are going to show that for every smooth m-manifold M, the tangent vector bundle $\pi_M : T(M) \to M$ is associated with a wide family of further vector bundles over M

$$\{\pi_{p,q} : T^p_q(M) \to M\}_{(p,q)\in\mathbb{N}^2}$$

called *tensor bundles* , such that $T(M) = T^1_0(M)$.

Let us recall first some elementary facts of finite-dimensional *multi-linear algebra*. Every finite-dimensional real vector space V has an infinite family of associated *tensor spaces* $T^p_q(V)$, $p, q \in \mathbb{N}$, also denoted $(V)^{\otimes p} \otimes (V^*)^{\otimes q}$. They consist of the multilinear forms

$$\alpha : \prod_{i=1}^{p} V^* \times \prod_{j=1}^{q} V \to \mathbb{R} \ .$$

Hence the *dual space* $V^* = T^0_1(V)$, while V is "equal" to $T^1_0(V)$ via the canonical identification of V with its *bidual space* $(V^*)^*$. If $\dim V = m$, then

$$\dim T^p_q(V) = m^{pq} \ .$$

Moreover, every basis $\mathcal{B}$ of V is associated in a canonical way to a basis $\mathcal{B}^p_q$ of $T^p_q(V)$; we can say that the basis $\mathcal{B}$ "propagates" to each tensor space. The linear group $\mathrm{GL}(V)$ acts on $T^p_q(V)$ by

$$(g, \alpha) \to g(\alpha)$$

$$g(\alpha)(w^1, \dots, w^p, v_1, \dots, v_q) = \alpha((g^t)^{-1}(w^1), \dots, (g^t)^{-1}(w^p), g(v_1), \dots, g(v_q)) \ .$$

By applying this to $V = \mathbb{R}^m$ (endowed with the canonical basis $\mathcal{C}$) and to $T^p_q(\mathbb{R}^m)$ (with the canonical basis $\mathcal{C}^p_q$), we get a representation

$$\rho_{p,q} : \mathrm{GL}(m, \mathbb{R}) \to \mathrm{GL}(T^p_q(\mathbb{R}^m)) \sim \mathrm{GL}(m^{pq}, \mathbb{R})$$

which is an explicit *regular rational* map. The basic example is

$$\rho_{0,1}(A) = (A^t)^{-1} \ .$$

As another example, $T^0_2(\mathbb{R}^m)$ can be identified with $M(m, \mathbb{R})$ by associating with every matrix B the form

$$(v, w) \to v^t B w \ .$$

Then

$$\rho_{0,2}(P)(B) = P^t B P \ .$$

Sometimes, it is interesting to consider subspaces W of $T^p_q(V)$, $\dim W = w$, which are invariant for the action of $\mathrm{GL}(V)$ and are endowed as well with a basis $\mathcal{B}_W$ canonically associated with $\mathcal{B}$. By applying this to $V = \mathbb{R}^m$, this gives rise to other representations

$$\rho_W : GL(m, \mathbb{R}) \to \mathrm{GL}(W) \sim \mathrm{GL}(w, \mathbb{R}) \ .$$

For example, consider the subspace $W = S^2_0(V) \subset T^0_2(V)$ of *symmetric bilinear form on* $V \times V$ (i.e. the space of *scalar products* on V). In this case, the representation ρ_W is just the "restriction" of $\rho_{0,2}$. Another example is the subspace $\Lambda^0_q(V) \subset T^0_q(V)$ of *alternating multilinear forms.* As a particular case, $\Lambda^0_m(\mathbb{R}^m)$ is 1-dimensional with canonical basis

$$\det : M(m,\mathbb{R}) \to \mathbb{R},\ X \to \det(X)$$

considered as an m-linear function of the columns of X. This gives rise to the representation

$$\delta_m : \mathrm{GL}(m,\mathbb{R}) \to \mathrm{GL}(1,\mathbb{R}),\ \delta_m(P) = \det P\ .$$

We can apply the machinery developed in Section 4.4 to the cocycle which carries the tangent bundle and to each representation $\rho_{p,q}$ (and relatives); this produces the required tensor bundles $\pi_{p,q} : T^p_q(M) \to M$ such that, for every $x \in M$, $\pi^{-1}_{p,q}(x) = T^p_q(T_xM)$, and clearly $T(M) = T^1_0(M)$. The determinant representation δ_m leads to the *determinant bundle* of M. The principal bundle of this family is the *frame bundle* of M, once we have identified the columns of any nonsingular matrix with a basis of $\mathbb{R}^m$.

Remark 4.13. If M is embedded, then the tangent bundle is a very concrete object, "embedded" by construction. This holds for every tensor bundle on an embedded manifold.

4.6. Tensor fields, unitary tensor bundles

We can generalize the content of Section 1.7 of Chapter 1 to smooth manifolds.

Let $\pi : E(M) \to M$ be any tensor vector bundle as above, with fibre E_xM over $x \in M$ of dimension r. A *section* of this bundle is a smooth map

$$\sigma : M \to E(M)$$

such that for every $x \in M$, $\pi(\sigma(x)) = x$. In other words, σ determines a smooth *field of tensors* of a certain type on M. Denote by

$$\Gamma(E(M))$$

the set of these sections. As for every vector bundle, every $\Gamma(E(M))$ includes a canonical *zero section*

$$\sigma_0(x) = (x, 0),\ x \in M\ .$$

In this way, M is canonically included in $E(M)$. Every $\Gamma(E(M))$ is a module over the commutative ring $\mathcal{C}^\infty(M,\mathbb{R})$, hence a real vector space.

- An element of $\Gamma(T(M))$ is called a (tangent) *vector field* on M. Generalizing *verbatim* Section 1.6.1, $\Gamma(T(M))$ is isomorphic to the vector space of *derivations* on $\mathcal{C}^\infty(M,\mathbb{R})$, $\mathrm{Der}(\mathcal{C}^\infty(M,\mathbb{R}))$.

- An element in $\Gamma(T^*(M))$ is called a 1-*differential form* on M. If $f : M \to \mathbb{R}$ is a smooth function, then $df \in \Gamma(T^*(M))$.

- A section $g \in \Gamma(S^0_2(M))$ such that $g(x)$ is positive definite for every $x \in M$ is called a *Riemannian metric* on M. Every embedded manifold

$M \subset \mathbb{R}^h$ admits Riemannian metrics: for every Riemannian metric $\hat{g}$ on $\mathbb{R}^h$ (for instance the standard g_0), the restriction of $\hat{g}_x$ to T_xM for each $x \in M$ defines a Riemannian metric g on M. In fact, every smooth manifold admits Riemannian metrics; this could be demonstrated using more general partitions of unity than the finite ones used by us, but we do not discuss this point.

A map $f : (M, g) \to (N, g')$ is an *isometry* if it is a diffeomorphism and for every $x \in M$, $v, w \in T_xM$, $g_x(v, w) = g'_{f(x)}(d_xf(v), d_xf(w))$.

If (W, ϕ) is a chart of (M, g), with inverse parametrization $\psi : U \to W$, then by imposing that ψ is tautologically an isometry we get a representation g_U of g in local coordinates; g_U is an instance of Riemannian metric on the open set $U \subset \mathbb{R}^m$ as defined in Chapter 1.

• Given a Riemannian metric g on M, for every smooth function $f : M \to \mathbb{R}$ there is a unique vector field $\nabla_g f$ (called the *gradient of f for g*) such that, for every $x \in M$, every $v \in T_xM$,

$$d_xf(v) = g_x(\nabla_g f(x), v) \ .$$

• Let (W, ϕ) and (U, ψ) be a chart and a parametrization of M, respectively, as above. Then for every $X \in \Gamma(T(M)$, every $\omega \in \Gamma(T^*(M))$, by using either ϕ_* or ψ^* we get local representations in the coordinates of U of the type described in Section 1.7. Representations in local coordinates can be straightforwardly developed for every field of tensors of arbitrary type on M.

4.6.1. Unitary tensor bundles. Let (M, g) be endowed with the auxiliary Riemannian metric g. Set

$$UT(M) = \{(x, v) \in T(U); ||v||_{g_x} = 1\}$$

with the restriction

$$u\pi_M : UT(M) \to M$$

of $\pi_M : T(M) \to M$. Then $UT(M)$ is a submanifold of $T(M)$ of dimension $m(m-1)$, and $u\pi_M$ is a surjective submersion with every fibre diffeomorphic to the unitary sphere S^{m-1}. More precisely, the local trivializations of $T(M)$,

$$\begin{array}{ccc} U \times \mathbb{R}^m & \stackrel{T\phi}{\to} & T(W) \\ \downarrow \pi_U & & \downarrow \pi_W \\ U & \stackrel{\phi}{\to} & W \end{array}$$

restrict to "unitary" local trivializations

$$\begin{array}{ccc} U \times S^{m-1} & \stackrel{UT\phi}{\to} & UT(W) \\ \downarrow \pi_U & & \downarrow u\pi_W \\ U & \stackrel{\phi}{\to} & W \end{array} \ .$$

Then $u\pi_M : UT(M) \to M$ is called an *unitary tangent bundle of M*.

Let $\pi : E(M) \to M$ be, as before, any of our tensor bundles. For every $x \in M$, the positive scalar product g_x on every T_xM canonically propagates to a positive definite scalar product g_x^E on the fibre E_xM. This is defined as follows: given one g_x-*orthonormal* basis $\mathcal{B}_x$ of T_xM, g_x^E is determined by imposing that the basis $\mathcal{B}_x^E$ of E_xM canonically associated with $\mathcal{B}_x$ is g_x^E-orthonormal (we verify that this does not depend on the choice of the basis $\mathcal{B}_x$). Then, by the very same procedure, we get the *unitary tensor bundle*

$$u\pi : UE(M) \to M$$

with fibre isometric to the unitary sphere S^{r-1}.

If $f : M \to N$ is a diffeomorphism and we endow N with the Riemannian metric $\tilde{g}$ such that $f : (M, g) \to (N, \tilde{g})$ is tautologically an isometry, then f preserves the corresponding unitary bundles. The total spaces of two unitary bundles, defined using two metrics g_0 and g_1 on M, are canonically diffeomorphic via radial diffeomorphisms fibre by fibre, centred at the origin of each tangent space T_xM. Moreover, by using the path of Riemannian metrics $g_t = (1-t)g_0 + tg_1$, this diffeomorphism is connected to the identity by a smooth path (an *isotopy*) through diffeomorphisms of unitary bundles of the same type. These considerations "propagate" to all tensor bundles. *Unitary tensor bundles are well defined up to isotopy.*

4.7. Parallelizable, combable and orientable manifolds

A smooth manifold M of dimension $m \geq 1$ is said *parallelizable* if there are m sections $\Sigma = (\sigma_1, \dots, \sigma_m) \in \Gamma(T(M))^m$ such that, for every $x \in M$, $\Sigma(x)$ is a basis of T_xM. This property "propagates" to every tensor bundle. For every (p, q), the canonical correspondence $\Sigma(x) \to \Sigma(x)^p_q$ determines

$$\Sigma^p_q \in \Gamma(T^p_q(M))^{m^{pq}}$$

such that, for every $x \in M$, $\Sigma(x)^p_q$ is a basis of $T^p_q(T_xM)$; similarly we have a *nowhere vanishing* section $\det \Sigma$ of the determinant bundle $\delta_M : \det(T(M)) \to M$. In generic notations, denote $\Sigma \in \Gamma(E(M))^r$ such a distinguished field of bases. We can define

$$t_\Sigma : M \times \mathbb{R}^r \to E(M),\ t_\Sigma(x, v) = (x, \sum_j v_j \sigma_j(x)) \ .$$

This is a diffeomorphism and also a vector bundle map in the sense that for every $x \in M$, it induces a *linear isomorphism* $\{x\} \times \mathbb{R}^r \to E_xM$. Moreover, the following diagram obviously commutes:

$$\begin{array}{ccc} M \times \mathbb{R}^r & \stackrel{t_\Sigma}{\to} & E(M) \\ \downarrow p_M & & \downarrow \pi \\ M & \stackrel{\mathrm{id}_M}{\to} & M \end{array} .$$

Then t_Σ is called a *global trivialization of the bundle* $E(M)$. M is parallelizable if and only if its tangent bundle is strictly equivalent to a product bundle, and a *necessary* condition is that the determinant bundle of M has a nowhere vanishing section. Let us say that M is *orientable* if it satisfies such a necessary condition. If M is parallelizable, then it is "*combable*", that is it carries a nowhere vanishing tangent vector field. Every open set of $\mathbb{R}^n$ is parallelizable, hence orientable and combable. The same facts hold *locally* on every manifold M. So we have here a bunch of genuine global questions concerning the structure of a generic smooth manifold M in terms of the existence of suitable patterns of sections of naturally defined vector bundles over it.

Let us explicate now the definition of orientability. It is clear that M is orientable if and only if every connected component of M is orientable, so let us assume that M is connected. Consider the *unitary* determinant bundle. The fibre is $S^0 = \{\pm 1\}$, so we can write it as

$$\mathfrak{p} : \tilde{M} \to M$$

where $\tilde{M}$ is an m-manifold and $\mathfrak{p}$ is a covering map of degree 2 called the *orientation covering of* M . The fibre over every $x \in M$ is $\{(x, \pm 1)\}$. There are two possibilities: either $\tilde{M}$ is connected or it has two connected components $\tilde{M} = \tilde{M}_+ \cup \tilde{M}_-$ where $\tilde{M}_\pm = \{(x, \pm 1); x \in M\}$. The restriction of $\mathfrak{p}$ to $\tilde{M}_\pm$ is a diffeomorphism (basically it is the identity). If $x \to (x, \sigma(x))$ is a nowhere vanishing section of the determinant bundle, as M is connected the sign $\frac{\sigma(x)}{||\sigma(x)||_{g(x)}}$ is constant. So we have proved

Proposition 4.14. *M is orientable if and only if $\tilde{M} = \tilde{M}_+ \cup \tilde{M}_-$ is not connected.*

Example 4.15. Referring to Section 3.4, *even-dimensional* projective spaces are nonorientable and the covering maps $S^n \to \mathbf{P}^n(\mathbb{R})$ are examples of *connected* $\tilde{M} \to M$.

The alternative "M orientable/nonorientable" can be reformulated as follows: a *signature* $\mathfrak{s}$ on an atlas $\mathcal{U}$ of M assigns to every chart a sign $\mathfrak{s}(W, \phi) = \pm 1$. Given such an $\mathfrak{s}$, modify $\mathcal{U}$ to $\mathcal{U}_{\mathfrak{s}}$ by post-composing every chart with negative sign with a linear reflection of $\mathbb{R}^m$ (which has the determinant equal to -1). An atlas $\mathcal{U}$ is *oriented* if all changes of coordinates for $\mathcal{U}$ have the determinant sign constantly equal to 1.

Proposition 4.16. *The following facts are equivalent to each other:*

(1) *M is orientable;*
(2) *There exists an oriented atlas $\mathcal{U}$ of M;*
(3) *For every atlas $\mathcal{U}$ of M, there exists a signature $\mathfrak{s}$ such that $\mathcal{U}_{\mathfrak{s}}$ is oriented.*

We leave the proof as a useful exercise on this pattern of definitions. The condition of point (2) is often given as the *very definition of orientability.* Here are some further remarks on these notions.

• If M is connected and orientable, then every oriented atlas $\mathcal{U}$ is contained in a unique maximal oriented atlas. There are exactly two maximal oriented atlantes $\mathcal{A}^{\pm}$. Any signature $\mathfrak{s}$ on $\mathcal{A}$ such that $\mathcal{A}_{\mathfrak{s}}$ is oriented produces one among $\mathcal{A}^{\pm}$; $\mathfrak{s}$ produces $\mathcal{A}^{+}$ if and only if the opposite signature $-\mathfrak{s}$ produces $\mathcal{A}^{-}$. By definition, $\mathcal{A}^{\pm}$ define two opposite *orientations* of M and make it (in two ways) an *oriented* manifold . If M is oriented, $-M$ denotes M endowed with the opposite orientation. The two components of $\tilde{M}$ are naturally oriented and correspond to the two orientations of M.

• The definition via oriented atlas allows us to recover the elementary notion of orienting $\mathbb{R}^m$ as a vector space. By definition, two bases $\mathcal{B}$ and $\mathcal{D}$ of $\mathbb{R}^m$ are *co-oriented* if the determinant of the change of linear coordinates passing from $\mathcal{B}$ to $\mathcal{D}$ is positive. By the multiplicative properties of the determinant, this defines an *equivalence relation* on $\mathrm{GL}(m, \mathbb{R})$ (considered as the space of bases of $\mathbb{R}^m$); then an *orientation* on $\mathbb{R}^m$ is an equivalence class of bases. Let us call *standard orientation* the class $[\mathcal{C}]$ of the canonical basis $\mathcal{C}$. If U is a (connected) open set of $\mathbb{R}^m$ we get the *standard field of orientations* by giving each $T_xU = \mathbb{R}^m$ the standard orientation. U is an orientable manifold and we can take the maximal oriented atlas $\mathcal{A}^+$ of U which contains the chart $\mathrm{id} : U \to U$. Let $\psi : U' \to W' \subset U$ be the local parametrization associated with a chart of $\mathcal{A}^+$. By taking the standard field of orientations on U', $d\psi$ transforms it to the field of orientations $\{[d_y\phi(\mathcal{C})]\}_{x=\psi(y)}$ on W'. The fact that ψ belongs to $\mathcal{A}^+$ just means that this last field coincides with the standard one on W'. Extending these considerations to an arbitrary manifold M, an orientation on M, if any, can be considered as a “locally coherent” field of orientations on each T_xM.

• Let $f : M \to N$ be a diffeomorphism. If $\mathcal{U} = \{(W, \phi)\}$ is an atlas of M, then

$$f(\mathcal{U}) := \{(f(W), \phi \circ f^{-1})\}$$

is an atlas of N. The proof of the following lemma follows immediately from the definitions.

Lemma 4.17. *Let $f : M \to N$ be a diffeomorphism between connected oriented manifolds with maximal oriented atlas $\mathcal{A}^+_M$ and $\mathcal{A}^+_N$, respectively. The following facts are equivalent to each other.*

(1) $f(\mathcal{A}^+_M) = \mathcal{A}^+_N$.
(2) There exists an oriented atlas $\mathcal{U} \subset \mathcal{A}^+_M$ such that $f(\mathcal{U}) \subset \mathcal{A}^+_N$.
(3) For every representation in local coordinates $f_{U,U'} : U \to U'$ of f relative to charts in $\mathcal{A}^+_M$ and $\mathcal{A}^+_N$ and for every $x \in U$, $\det d_x f_{U,U'} > 0$.

If one (hence all) of the above conditions is satisfied, then we say that f is an oriented diffeomorphism.

• By specializing to oriented manifolds we get a sub-category of our favourite one.

REMARK 4.18. *(Oriented 0-Manifolds)* A connected 0-manifold is just one point. We stipulate that it is orientable and is *oriented* by giving it a sign ± 1.

4.8. On complex manifolds

Another reason to introduce the abstract notion of manifold in terms of atlantes with the changes of coordinates in a certain class of homeomorphism (for instance, smooth diffeomorphisms in our favourite setting) is that it is suited to several interesting implementations. Abstract *complex n-manifolds* have as local models the open sets in $\mathbb{C}^n$ and change of coordinates that are complex analytic (i.e. holomorphic) diffeomorphisms (i.e. biholomorphisms). Holomorphic maps between complex manifolds are defined in terms of holomorphic local representations; and so on, by following and specializing several constructions developed above (complex tangent bundle, complex submanifolds, etc.). On the other hand, by the *maximum principle*, the constant functions $c : M \to \mathbb{C}$ are the only holomorphic functions defined on any *compact* connected complex manifold M. So compact complex manifolds *cannot be embedded in any* $\mathbb{C}^m$, as complex submanifolds. This is an important difference compared with our favourite real smooth theory. Bump functions do not exist in the complex setting, so the many constructions which employ this tool cannot be performed on complex manifolds. Although we have introduced them as examples of embedded smooth manifolds, the *complex Stiefel and Grassmann manifolds* (in particular the complex projective spaces) can be naturally endowed with a compact complex manifold structure. By identifying $\mathbb{C}^n \sim \mathbb{R}^{2n}$ and considering holomorphic maps as a special kind of smooth maps, and by forgetting the complex structure, every complex n-manifold M can be considered as a smooth $2n$-manifold (as we have done for the complex Grassmannian); moreover, the complex structure induces on this $2n$-manifold a natural orientation. Especially in dimension 4, 2-complex manifolds (also-called *complex surfaces*) form an important class of oriented 4-manifolds.

The Riemann sphere. As a basic example, let us consider $\mathbf{P}^1(\mathbb{C})$; let us identify $\mathbb{R}^2 \sim \mathbb{C}$ and consider the two-charts atlas of the 2-sphere S^2 given by the stereographic projections from the two poles. These can be considered as $\mathbb{C}$-valued charts. To make it a complex-manifold atlas, it is enough to compose the second projection with the complex conjugation $z \to \bar{z}$. Moreover, it is immediate to identify such an atlas with the standard two-charts complex atlas of $\mathbf{P}^1(\mathbb{C})$. This shows, in particular, that $\mathbf{P}^1(\mathbb{C})$ is diffeomorphic to S^2; this last, considered as a 1-dimensional complex manifold, is called the *Riemann sphere* .

REMARK 4.19. It has been known for some time [**BoS**] that S^6 is the only other sphere that could carry a complex manifold structure. It is a hard open question whether this happens.

4.9. Manifolds with boundary, proper submanifolds

By definition, a smooth m-manifold M is locally diffeomorphic to open sets of the basic model $\mathbb{R}^m$. Let us change this last by taking instead the *half-space*

$$\mathbf{H}^m = \{x \in \mathbb{R}^m;\ x_m \geq 0\}$$

with the *boundary*

$$\partial\mathbf{H}^m = \{x \in \mathbf{H}^m;\ x_m = 0\}\ .$$

DEFINITION 4.20. A topological space M is an *m-smooth manifold with boundary* if:

- M is Hausdorff, with a countable basis of open sets.
- M admits a *smooth $\mathbf{H}^m$-valued atlas* $\mathcal{U} = \{W_j, \phi_j\}_{j\in J}$ (J being any set of indices); that is
  (i) $\{W_j\}_{j\in J}$ is an open covering of M;
  (ii) every *chart* $\phi_j : W_j \to U_j$ is a *homeomorphism* to an open set U_j of $\mathbf{H}^m$, $\psi_j : U_j \to W_j$ denotes the inverse *local parametrization*;
  (iii) for every $i, j \in J$,
  $$\phi_j \circ \psi_i : \phi_i(W_i \cap W_j) \to \phi_j(W_i \cap W_j)$$
  is a smooth *diffeomorphism* (in the sense established in Chapter 2) between open sets of $\mathbf{H}^m$.

The *boundary* ∂M is the set of points $p \in M$ such that there exists a chart (W, ϕ) at p such that $\phi(p) \in \partial\mathbf{H}^m$.

There is also a natural notion of *embedded* smooth m-manifold with boundary $M \subset \mathbb{R}^h$.

The following lemma provides a basic way to produce manifolds with boundary.

LEMMA 4.21. *Let X be an m-manifold with empty boundary, $f : X \to J$ a surjective submersion, where J is an open interval of $\mathbb{R}$, and $0 \in J$. Then $M = \{x \in X;\ f(x) \geq 0\}$ is an m-manifold with boundary $\partial M = \{f(x) = 0\}$.*

Proof : The question being of local nature, we can reduce it to submersions in normal form for which the result is evident.

■

The following lemma contains an extension of Lemma 2.3 and is similarly an application of the inverse map theorem and its corollaries.

LEMMA 4.22. *Let M be a smooth m-manifold with boundary. Then*

(1) If $p \in \partial M$, then for every chart (W, ϕ) of M at p, $\phi(p) \in \partial\mathbf{H}^m$.

(2) $\mathrm{Int}(M) := M \setminus \partial M$ is an open set in M and a manifold with empty boundary (called the interior *of M).*

(3) For every $p \in \partial M$, there are normal relative charts of $(M, \partial M)$ at p:

$$\phi : (W, W \cap \partial M, p) \to (B^m(0,1) \cap \mathbf{H}^m, B^m(0,1) \cap \partial\mathbf{H}^m, 0)$$

(4) If $\partial M \neq \emptyset$, then it is an $(m-1)$-manifold with empty boundary.

The definition of "smooth manifold with boundary" does not exclude that $\partial M = \emptyset$. We have considered at first such a *boundaryless* case. It is formally convenient to stipulate that the empty set $\emptyset$ is a k-boundaryless manifold for every $k \in \mathbb{N}$. In such a way, for example, point (4) of the last lemma holds even if $\partial M = \emptyset$. By setting $M = (M, \emptyset)$ for every boundaryless manifold, the earlier category of *smooth manifolds* extends to the category of *smooth manifolds with boundary.* Let us briefly retrace within such an extension the main facts developed so far .

• The tangent functor and its relatives extend *verbatim*. If ∂M is nonempty, the inclusion $j : \partial M \to M$ leads to a vector bundle embedding $[j, Tj]$ of $\pi_{\partial M} : T(\partial M) \to \partial M$ in $\pi_M : T(M) \to M$. The total space $T(M)$ is a manifold with boundary equal to the restriction over ∂M of the tangent bundle of M (with the notions that we will introduce in Chapter 5, it is the pull-back $j^*T(M)$ over ∂M). Likewise for the other tensor bundles.

• The *topologies* of spaces of smooth maps between manifolds with boundary extend word by word.

• "Orientability/orientation" extends directly. The boundary ∂M of an *oriented* M is *orientable* and we can fix the following procedure to make it the *oriented boundary* of M:

("First the outgoing normal") Take an oriented atlas $\mathcal{U}$ of M made by normal charts. Post-compose every chart along the boundary ∂M with a transformation $r \in SO(m)$ such that $r(e_1, \dots, e_m) = (-e_m, r(e_1, \dots, e_{m-1}))$. The so obtained atlas, $r\mathcal{U}$, is again an oriented atlas of M and its restriction to ∂M is an oriented atlas which carries an orientation of the boundary. Via the usual convention $M = (M, \emptyset)$, the category of *oriented* boundaryless manifolds extends to the category of *oriented* manifolds with *oriented boundary.*

• *(submanifolds)* Submanifolds Y of a manifold with boundary M might have nonempty boundary. Extending Remark 2.8, because of the presence of the boundaries, there are several qualitatively different ways of being a submanifold; let us list a few examples:

(1) $(Y \subset M) = (\overline{B}^n(0,1) \subset B^n(0,2))$: $\partial Y \neq \emptyset$ and Y is contained in the interior of M.
(2) $(Y \subset M) = (\mathrm{Int}(M) \subset M)$; if $\partial M \neq \emptyset$, then Y is not closed in M.
(3) $(Y \subset M) = (N \subset B^n(0,1))$, where N is defined in Remark 2.8: Y is boundaryless, it is contained in the interior of M, and every point of ∂M is in the closure of Y; again, Y is not closed in M.
(4) $(Y \subset M)$ where $Y = \overline{B}^n(0,1)$, $M = \{x_n \geq -1\}$. Then ∂Y is tangent to ∂M, while the interior of Y is contained in the interior of M.
(5) Let $\gamma := \gamma_{1,2} : \mathbb{R} \to \mathbb{R}$ be the bump function defined in Chapter 1. $(Y \subset M) = (N \subset \mathbf{H}^2)$, where $N = \{(x,y) \in \mathbf{H}^2;\ y \geq \gamma(x)\}$. Then ∂Y is partially contained in the interior of M, and partially in ∂M.

(6) ...

Among this wide typology, there is a particularly clean type which deserves to be pointed out by a definition.

DEFINITION 4.23. Let $Y \subset M$ be a smooth manifolds with boundary. Then Y is a *proper submanifold* of M if

(1) Y is closed in M;
(2) $\partial Y = Y \cap \partial M$;
(3) Y is *transverse* to ∂M. This means that for every $p \in Y \cap \partial M$

$$T_pM = T_pY + T_p\partial M \ .$$

None of the examples above are proper. Every M is a proper submanifold of itself. The properness implies, for example, that every boundaryless component of Y is contained in the interior of M. If $\partial M = \emptyset$, then also $\partial Y = \emptyset$; if $\dim Y = \dim M$, then Y is union of connected components of M.

The following proposition extends (1) of Proposition 2.12 in two ways, to manifolds with boundary and oriented manifolds.

PROPOSITION 4.24. *Let M be a manifold with boundary and N a boundaryless one. Let $f : M \to N$ be a surjective* relative submersion *(that is, both f and $\partial f := f_{|\partial M}$ are submersions). Then:*

(1) For every $q \in N$, $Y = f^{-1}(q)$ is a proper submanifold of M and $\dim Y = \dim M - \dim N$.

(2) If both M and N are oriented, then Y is orientable, and we can fix a procedure to orient it in such a way that the orientation of ∂Y as the oriented boundary of Y coincides with the orientation obtained by applying the procedure to ∂f, provided that ∂M is the oriented boundary of M.

Proof : Assume that $\dim M = m$, $\dim N = n$. If q does not belong to the image of ∂f, then we apply directly Proposition 2.12 so that Y is a closed boundaryless submanifold of the interior of M. Assume now that q belongs to the image of ∂f. The question being of local nature, it is enough to analyze a representation (called f as well) of f in local coordinates which are normal for $(M, \partial M)$:

$$f : (B^m(0,1) \cap \mathbf{H}^m, B^m(0,1) \cap \partial \mathbf{H}^m) \to U \subset \mathbb{R}^n$$

and $q = 0 \in U$. Moreover, we can assume that f is the restriction of a smooth map $g : B^m(0,1) \to U$ defined on the whole of $B^m(0,1)$, which is a surjective submersion. By applying again Proposition 2.12 to g, we have that $\tilde{Y} = g^{-1}(0)$ is a boundaryless submanifold of $B^m(0,1)$ of the correct dimension, such that $Y = f^{-1}(0)$ is $Y = \tilde{Y} \cap \mathbf{H}^m$. As f is a relative submersion, we readily check that $\tilde{Y}$ is transverse to $\partial \mathbf{H}^m$ and that the restriction π to $\tilde{Y}$ of the projection to the x_m coordinate is a submersion to its image and that $Y = \{y \in \tilde{Y};\ \pi(y) \geq 0\}$. We conclude by applying Lemma 4.21.

Let us come to the orientation. First, consider the case $f = \mathrm{id}_M$. Then $Y = \{p\}$ is just a point of M. Let us orient it by giving it the sign $+1$. By applying the rule to ∂f we get the same sign. In the general case, for every $p \in Y$ let

$$\nu(p) = (T_pY)^\perp \cap T_pM \ .$$

Clearly

$$T_p(M) = T_pY \oplus \nu(p)$$

and $\nu(p)$ varies "smoothly" when p varies along Y (by using the contents of the next chapter, this means precisely that $\nu : Y \to \mathfrak{G}_{k,n}$ is a smooth map). In our hypotheses, for every $p \in Y$, the restriction of d_pf to $\nu(p)$ is a linear isomorphism to $T_{f(p)}N$. Let us consider the orientation on N as a field of orientations on the T_yN, $y \in N$ (i.e. a field of equivalence classes of bases of T_yN which is locally coherent). Take an orienting (say "positive") basis $\mathcal{B}_q$ of T_qN. For every $p \in Y$, lift it to a basis $\mathcal{B}_p$ of $\nu(p)$ by means of the restriction of the differential of f. This determines a field of "transverse orientations" $[\mathcal{B}_p]$ along Y. At every p, take a basis $\mathcal{D}_p$ of T_pY such that the basis $\mathcal{D}_p \oplus \mathcal{B}_p$ of T_pM (compatible with the above direct sum decomposition of T_pM) is positive with respect to the given orientation of M. This determines a field $[\mathcal{D}_p]$ of orientations on the T_pY, eventually the desired orientation of Y. We can check that the restriction of this procedure to ∂f is compatible with the last statement of the proposition.

■

4.10. Product, manifolds with corners, smoothing

We know that the product of two (embedded) boundaryless manifolds is a (embedded) boundaryless manifold. The situation is more complicated if we consider nonempty boundaries. The following lemma is immediate.

LEMMA 4.25. *Let M be a (embedded) boundaryless smooth m-manifold, N be a (embedded) n-manifold with $\partial N \neq \emptyset$. Then $M \times N$ is a (embedded) $(m+n)$-manifold with $\partial(M \times N) = M \times \partial N$*

However, if both ∂M and ∂N are nonempty, then $M \times N$ *is no longer* a smooth manifold with boundary. In particular, if M and N are embedded, $M \times N$ is no longer an embedded smooth manifold with boundary.

EXAMPLE 4.26. As a basic example, consider the square

$$Q = D_1 \times D_2 := [-1,1] \times [-1,1] \subset \mathbb{R}^2 \ .$$

Its topological frontier is

$$\partial Q = (\partial D_1 \times D_2) \cup (D_1 \times \partial D_2) \ ;$$

its interior

$$Q \setminus \partial Q = \mathrm{Int}(D_1) \times \mathrm{Int}(D_2)$$

is an open set of $\mathbb{R}^2$, hence a 2-manifold with empty boundary;

$$Q \setminus (\partial D_1 \times \partial D_2)$$

is a 2-manifold with boundary equal to

$$\partial Q \setminus (\partial D_1 \times \partial D_2)\ ,$$

and $\partial D_1 \times \partial D_2$ is a 0-manifold. The points where Q fails to be a manifold with boundary are the "corner" points which form $\partial D_1 \times \partial D_2$.

The behaviour of this very simple example is qualitatively the same as the general one:

PROPOSITION 4.27. *Let $(M, \partial M)$ and $(N, \partial N)$ be an m-manifold and an n-manifold with nonempty boundary, respectively. Then $M \times N$ verifies the following properties:*

- *Set*
$$\partial(M \times N) := (\partial M \times N) \cup (M \times \partial N)\ .$$
Then
$$(M \times N) \setminus \partial(M \times N)$$
is a boundaryless $(m+n)$-manifold;
- *$(M \times N) \setminus (\partial M \times \partial N)$ is an $(m+n)$-manifold with boundary equal to $\partial(M \times N) \setminus (\partial M \times \partial N)$;*
- *$\partial M \times \partial N$ is a boundaryless $(m+n-2)$-manifold.*

Hence $M \times N$ fails to be a manifold with boundary at the "corner locus" $\partial M \times \partial N$. This means that the category of (embedded) smooth manifolds with boundary is *not closed* for the product. This is somehow unpleasant. A way to fix this fact is to enlarge our category by extending the sets of basic models, incorporating the corners. We do it in a minimal way suited to dealing with product manifolds.

DEFINITION 4.28. The *basic m-corner model* is

$$\mathbf{C}^m = \{x \in \mathbb{R}^m;\ x_m \geq 0,\ x_{m-1} \geq 0\}$$

that is the intersection between $\mathbf{H}^m$ and another half-space. Its *boundary* (in fact its topological frontier) is

$$\partial \mathbf{C}^m = \{x \in \mathbf{C}^m;\ x_m = 0\} \cup \{x \in \mathbf{C}^m;\ x_{m-1} = 0\}\ .$$

The set $\mathbf{C}^m \setminus \{x_m = 0, x_{m-1} = 0\}$ is a manifold with boundary and the last set is its *corner locus*.

Then Definition 4.20 can be extended directly, giving us the notion of m-manifold *with corners*, and there is an obvious embedded version of it.

The following properties hold for the basic models and extend easily to every manifold with corners.

(i) Every manifold with corners is naturally *stratified* by the disjoint locally finite union of boundaryless connected smooth manifolds (of varying dimension $m-2 \leq d \leq m$) called the *strata*. The top-dimensional strata are the components of the boundaryless smooth m-manifold $M \setminus \partial M$, the $(m-1)$-strata are the components of ∂M from which we have removed the corner locus, and the $(m-2)$-strata are the components of the corner locus which is a boundaryless manifold of dimension $m-2$ contained in the boundary of M. The closure of each stratum is a union of strata, as well as the maximal smooth manifold with boundary contained in the closure of every stratum.

(ii) The product of two smooth manifolds with boundary is a manifold with corners.

However, manifolds with "codimension 2" corners are not closed under the product (take, for example, the cube $[-1,1]^3$). So we have only shifted the difficulty and we should extend furthermore our category of manifolds. This would bring us a bit far away from our original objects of interest. Fortunately, there is another way that leads manifolds with corners (according with the above restrictive definition) back to ordinary manifolds with boundary, even though considered *up to diffeomorphism.* To introduce such a *"smoothing the corner"* procedure, let us consider again our simplest square example. The function

$$f:\mathbb{R}^2\to\mathbb{R},\ f(x)=(x_1-1)(x_2-1)(x_1+1)(x_2+1)$$

has the property that Q is the closure of a connected component of

$$\mathbb{R}^2\setminus f^{-1}(0)$$

and for every $x\in \text{int}(Q)$, $f(x)>0$. For every $\epsilon>0$, sufficiently small, there is a connected component Q_ϵ of $f(x)\geq\epsilon$ contained in the interior of Q, and which is a smooth manifold with boundary *homeomorphic* to Q. Moreover, we can construct a "piece-wise smooth" radial homeomorphism (centred at 0) $s:Q_\epsilon\to Q$ such that the natural stratification of Q lifts to a stratification by smooth submanifolds of Q_ϵ and the restriction to the maximal manifold with boundary contained in the closure of every stratum is a diffeomorphism to its analogous image in Q. Finally, up to diffeomorphism, the result of such a smoothing does not depend on the specific implementation (in particular on the choice of the small ϵ).

This basic idea can be generalized. By applying it to $\mathbf{C}^m$, by using $M_\epsilon=\{x_m x_{m-1}\geq\epsilon\}\cap\mathbf{C}^m$ and $\epsilon>0$ small enough, we get a nice local smoothing homeomorphism $s:M_\epsilon\to\mathbf{C}^m$ with the same qualitative properties as above. Then we should prove that such local smoothing can be patched to give a global smooth atlas. This could be a bit technically demanding and we are not pushing further in that direction. In Section 7.3, we will reconsider and properly establish such a smoothing procedure in a more flexible "abstract" setting. Regardless, we can now state the following.

PROPOSITION 4.29. *For every m-manifold with corner M, then*

(1) by implementing a certain procedure of "smoothing the corner"*, we get a smooth manifold with boundary $\tilde{M}$ and a piece-wise smooth homeomorphism*

$$\mathfrak{s}:(\tilde{M},\partial\tilde{M})\to(M,\partial M)$$

such that the natural stratification of M lifts to a stratification of $\tilde{M}$ by boundaryless smooth submanifolds, and the restriction of $\mathfrak{s}$ to the maximal smooth manifold with boundary contained in the closure of every stratum of $\tilde{M}$ is a diffeomorphism to its analogous image in M.

(2) $\tilde{M}$ is uniquely determined up to diffeomorphism (i.e. it does not depend on the actual implementation of the procedure).

Coming back to our motivating problem, the product of two smooth manifolds with boundary as a smooth manifold with boundary is well defined *up to diffeomorphism*.

4.11. Embedding compact manifolds

To a large extent, in this text we will focus on compact smooth manifolds. We begin in this section. First, we point out that the weak topologies on the map spaces $\mathcal{C}^\infty(M,N)$ are adequate if the source manifold M is compact. Then, the main result of the section is that every compact manifold can be embedded in some Euclidean space; that is, up to diffeomorphism, we can assume that it is an embedded manifold.

4.11.1. Nice atlas and finite partitions of unity. Let us introduce some useful technical devices. Let M be a smooth m-manifold (possibly with boundary). Recall that a normal chart (W,ϕ) of M is either contained in the interior of M and of the form

$$\phi : W \to B^m(0,1)$$

or it intersects ∂M and is of the relative form

$$\phi(W, W\cap \partial M) \to (B^m(0,1)\cap \mathbf{H}^m, B^m(0,1)\cap \partial\mathbf{H}^m)\ .$$

The bump function (recall Section 1.4)

$$\gamma = \gamma_{1/3,1/2} : B^m(0,1) \to \mathbb{R}$$

lifts to a *global bump function*

$$\gamma_W : M \to \mathbb{R}$$

with compact support

$$S_W = \phi^{-1}(\bar{B}^m(0,1/2)) \subset W\ .$$

Set

$$B_W := \phi^{-1}(B^m(0,1/3)) \subset S_W\ .$$

B_W is a relatively compact open set in M.

Definition 4.30. Let M be a compact smooth manifold.

(1) A *nice atlas* of M is a finite atlas $\mathcal{U} = \{(W_j,\phi_j)\}_{j=1,\dots,s}$ formed by normal charts, such that the family $\{B_j\}$ ($B_j := B_{W_j}$) is an open covering of M.

(2) Set $\gamma_j := \gamma_{W_j}$,

$$\lambda_j := \frac{\gamma_j}{\sum_j \gamma_j}$$

so that

$$\sum_j \lambda_j = 1 \ .$$

Then $\{\lambda_j\}_{j=1,\dots,s}$ is the (finite) *partition of unity* subordinate to the nice atlas $\mathcal{U}$.

REMARK 4.31. The finite partitions of unity of $\mathbb{R}^n$ involving a 'bump function at infinity' used in Section 1.4 are the restriction of partitions of unity subordinate to a nice atlas of S^n, where, as usual, $\mathbb{R}^n \subset \mathbb{R}^n \cup \{\infty\} = S^n$, via a stereographic projection.

Every compact M admits nice atlantes. In fact, we will use nice atlantes *adapted* to certain situations or the solution of certain problems.

4.11.2. Spaces of maps with compact source manifold. We adopt the notations of Section 4.1. The so-called *weak topology* is adequate when the source manifold M is compact as it allows a global control over the whole of M. Let $f \in \mathcal{E}^r(M,N)$; let $\mathcal{U}$ be a nice atlas of M such that every (W_j,ϕ_j) carries a local representation f_j of f. Consider the neighbourhoods of f of the form $\mathcal{U}_r(f,f_j,\bar{B}_j,\epsilon)$. Then every $\cap_j\ \mathcal{U}_r(f,f_j,\bar{B}_j,\epsilon)$ is an open neighbourhood of f and, by varying $\epsilon > 0$, we get a *basis of neighbourhoods* of f because $\cup_j \bar{B}_j = M$.

Let us study now some remarkable subsets of $\mathcal{E}^r(M,N)$, $r \geq 1$ or $\mathcal{E}(M,N)$.

LEMMA 4.32. *Let M be compact. Then $f : M \to N$ is an embedding if and only if it is an injective immersion.*

Proof : One implication is evident. We know that without the compactness assumption the other implication is, in general, false. To prove it, recall that in a compact Hausdorff space, a subset is compact if and only if it is closed and that a continuous map sends compact sets to compact sets; it follows that, since M is compact, f is closed, so that f^{-1} is continuous, and f is a homeomorphism to its image in N.

■

PROPOSITION 4.33. *Let M be a compact smooth manifold. Then the subsets of immersions, submersions, embeddings, diffeomorphisms are (possibly empty) open sets in $\mathcal{E}^r(M,N)$, $r \geq 1$, and in $\mathcal{E}(M,N)$.*

Proof : An immersion or submersion f is characterized by the condition of maximum rank of $d_x f$ at every $x \in M$. If g belongs to a neighbourhood of f in $\mathcal{E}^r(M,N)$, $r \geq 1$, which gives a global control on the whole of M as above (with $\epsilon > 0$ small enough), then g satisfies the same maximum rank condition. As for embeddings, thanks to Lemma 4.32 it is enough to prove that if g is close enough to an injective immersion f then g is also an injective immersion. Assume that this fails. Then there exists a sequence $g_n \in \mathcal{C}^\infty(M, N)$ and sequences of points x_n, y_n in M such that:

(1) Every g_n is an immersion;
(2) $g_n \to f$ and $dg_n \to df$ uniformly on M;
(3) $x_n \to x$, $y_n \to y$ in M, $x_n \neq y_n$ and $g_n(x_n) = g_n(y_n)$ for every n.

Then $g_n(x_n) \to f(x)$, $g_n(y_n) \to f(y)$, hence $x = y$ because f is injective. We can localize the situation in a chart of M at x and conclude (getting a contradiction) by applying the local Proposition 1.10. Finally, if f is a diffeomorphism, in particular, it is an embedding, hence g close to f is an embedding. It is enough to prove that g is surjective. It is not restrictive to assume that M and N are connected. An embedding g is an open map and its image is open in N. On the other hand, the image of g is compact, hence closed, because M is compact. Then the image of g coincides with the whole of N.

∎

4.11.3. The embedding result.

PROPOSITION 4.34. *(1) Let M be a compact smooth manifold. Then there is a diffeomorphism $f : M \to M'$ with an embedded manifold $M' \subset \mathbb{R}^h$, for some h.*

(2) The tangent map Tf establishes a vector bundle equivalence between the respective tangent bundles of M and M', the last bundle being embedded. This equivalence propagates to all tensor bundles and the frame bundle.

Proof : (1) Consider a nice atlas of M with subordinate partition of unity as above. This allows us to define the smooth map

$$\beta = (\beta_1, \dots, \beta_s) : M \to (\mathbb{R}^m \times \mathbb{R})^s$$
$$\beta_j = (\lambda_j \phi_j, \lambda_j)\ .$$

We claim that it is an embedding. It is enough to prove that it is an injective immersion. It is an immersion because every $x \in M$ belongs to some B_j. The restriction of β_j is $(\phi_j, 1)$, which is an injective immersion, so β is an immersion. As for the injectivity, let $x \neq y$. If both belong to some B_j, then they are already separated by ϕ_j. Otherwise, they are separated by some λ_j. Hence β is injective.

Point (2) follows from the fact that the abstract functor extends the embedded one.

∎

CHAPTER 5

Tautological bundles and pull-back

For every embedded smooth m-manifold $M \subset \mathbb{R}^h$, the tangent bundle construction includes a smooth map $t : M \to \mathfrak{G}_{h,m}$; a similar map exists for each tensor bundle on M. We will see that these bundles can be reconstructed through the associated maps and that they belong to a wide category of vector bundles constructed via the *pull-back* of *tautological bundles* over Grassmann manifolds. The basic notions about fibred bundles have been already introduced in Section 4.3.1, and we will use them. We will see that all vector bundles on compact manifolds arise in this way; the main applications at the end of the chapter (a so-called *classification theorem* reducing the vector bundle equivalence to purely a homotopy question) will concern compact manifolds.

5.1. Tautological bundles

We are going to construct so-called *tautological fibre bundles* over the Grassmannian $\mathfrak{G}_{n,k}$.

- *(The tautological vector bundle)* Define

$$\mathcal{V}(\mathfrak{G}_{n,k}) = \{(A, v) \in \mathfrak{G}_{n,k} \times \mathbb{R}^n;\ v \in V_A\}$$

i.e. v belongs to the k-linear subspace V of $\mathbb{R}^n$ such that $A = A_V$, via the usual bijection $G_{n,k} \cong \mathfrak{G}_{n,k}$. The restriction of the projection to the first factor defines the smooth surjective map

$$\tau_{n,k} : \mathcal{V}(\mathfrak{G}_{n,k}) \to \mathfrak{G}_{n,k}\ .$$

It is clear that for every $A \in \mathfrak{G}_{n,k}$, the inverse image $\tau_{n,k}^{-1}(A) = V_A$.

Proposition 5.1. *The map $\tau_{n,k} : \mathcal{V}(\mathfrak{G}_{n,k}) \to \mathfrak{G}_{n,k}$ is a smooth vector bundle with fibre $\mathbb{R}^k$. It is called the* tautological vector bundle *over $\mathfrak{G}_{n,k}$.*

- *(The tautological linear frame bundle)* Define

$$\mathcal{L}(\mathfrak{G}_{n,k}) = \{(A, X) \in \mathfrak{G}_{n,k} \times L_{n,k};\ \mathfrak{l}_{n,k}(X) = A\}$$

i.e. X spans the k-linear subspace V of $\mathbb{R}^n$ such that $A = A_V$. The restriction of the projection to the first factor defines the smooth surjective map

$$l\tau_{n,k} : \mathcal{L}(\mathfrak{G}_{n,k}) \to \mathfrak{G}_{n,k}\ .$$

It is clear that for every $A \in \mathfrak{G}_{n,k}$, the inverse image $l\tau_{n,k}^{-1}(A)$ consists of all *linear frames* of V_A.

PROPOSITION 5.2. *The map* $l\tau_{n,k} : \mathcal{L}(\mathfrak{G}_{n,k}) \to \mathfrak{G}_{n,k}$ *is a smooth fibre bundle with fibre* $GL(k, \mathbb{R})$*. It is called the* tautological linear frame bundle *over* $\mathfrak{G}_{n,k}$*.*

- *(The tautological orthogonal frame bundle)* Define

$$\mathcal{S}(\mathfrak{G}_{n,k}) = \{(A, X) \in \mathfrak{G}_{n,k} \times S_{n,k};\ s_{n,k}(X) = A\}$$

i.e. X spans the k-linear subspace V of $\mathbb{R}^n$ such that $A = A_V$. The restriction of the projection to the first factor defines the smooth surjective map

$$s\tau_{n,k} : \mathcal{S}(\mathfrak{G}_{n,k}) \to \mathfrak{G}_{n,k}\ .$$

It is clear that for every $A \in \mathfrak{G}_{n,k}$, the inverse image $s\tau_{n,k}^{-1}(A)$ consists of all *orthonormal frames* of V_A.

PROPOSITION 5.3. *The map* $s\tau_{n,k} : \mathcal{S}(\mathfrak{G}_{n,k}) \to \mathfrak{G}_{n,k}$ *is a smooth fibre bundle with fibre* $O(k)$*. It is called the* tautological orthogonal frame bundle *over* $\mathfrak{G}_{n,k}$*.*

Proofs: Let us prove Proposition 5.1. Recall that $\mathfrak{G}_{n,k}$ is endowed with an atlas $\{(\Omega_V, \phi_V)\}_{V \in G_{n,k}}$ where

$$\Omega_V = \{A \in \mathfrak{G}_{n,k};\ V_A \cap V^\perp = \{0\}\}\ .$$

Equivalently, V_A is the graph of a uniquely determined linear map $L_A : V \to V^\perp$. Set as usual $\tilde{\Omega}_V = \tau_{n,k}^{-1}(\Omega_V)$. Then a vector bundle atlas of $\tau_{n,k}$ is given by the locally trivializing commutative diagrams (V varying in $G_{n,k}$, $\mathcal{B} = \{v_1, \dots, v_k\}$ varying in the linear frames of V)

$$\begin{array}{ccc} \Omega_V \times \mathbb{R}^k & \stackrel{\Psi_{\mathcal{B}}}{\to} & \tilde{\Omega}_V \\ \downarrow_{\pi_{\Omega_V}} & & \downarrow_{\tau_{n,k}} \\ \Omega_V & \stackrel{\mathrm{id}_{\Omega_V}}{\to} & \Omega_V \end{array}$$

where

$$\Psi_{\mathcal{B}}(A, x) = (A, \sum_{i=1}^k x_i v_i + \sum_{i=1}^k x_i L_A(v_i))\ .$$

It is immediate that for every couple $(V, \mathcal{B})$, $(V', \mathcal{B}')$ there is a smooth map

$$\lambda_{\mathcal{B},\mathcal{B}'} : \Omega_V \cap \Omega_{V'} \to \mathrm{GL}(k, \mathbb{R})$$

such that the corresponding change of local trivialization is of the form

$$(\Omega_V \cap \Omega_{V'}) \times \mathbb{R}^k \to (\Omega_V \cap \Omega_{V'}) \times \mathbb{R}^k$$
$$(A, v) \to (A, \lambda_{\mathcal{B},\mathcal{B}'}(A)v)\ .$$

Remark. *By restricting the* $\mathcal{B}$*'s to orthogonal frames of the* V*'s, we get a sub-fibred atlas such that the change of local trivializations are governed by smooth maps*

$$\lambda_{\mathcal{B},\mathcal{B}'} : \Omega_V \cap \Omega_{V'} \to O(k)\ .$$

The proof of the other two propositions is similar and left to the reader. In the spirit of Section 4.4, the change of local trivializations for the frame bundles are governed by *the same* smooth maps $\lambda_{\mathcal{B},\mathcal{B}'}$ as above, with values in $\mathrm{GL}(k,\mathbb{R})$ for $l\tau_{n,k}$, or in $O(k)$ for $s\tau_{n,k}$, respectively; the groups $\mathrm{GL}(k,\mathbb{R})$ or $O(k)$ act on themselves by left multiplication.

■

5.2. Pull-back

We introduce a fundamental construction on smooth fibred bundles. Here we state it in full generality; later we will apply it to the tautological bundles of Section 5.1.

Let us give a smooth fibre bundle

$$\xi := f : E \to X$$

with fibres E_x diffeomorphic to the manifold F (recall Section 4.3.1).

Let $g \in \mathcal{E}(M, X)$. Then set

$$g^*E = \{(p, y) \in M \times E;\ g(p) = f(y)\}$$
$$g^* : g^*E \to E,\ g^*(p, y) = y$$
$$g^*f : g^*E \to M,\ g^*f(p, y) = p\ .$$

Obviously we have the commutative diagram of smooth maps, denoted by $[g, g^*]$

$$\begin{array}{ccc} g^*E & \stackrel{g^*}{\to} & E \\ \downarrow_{g^*f} & & \downarrow_f \\ M & \stackrel{g}{\to} & X \end{array} \ .$$

Moreover, for every $p \in M$, $x = g(p)$, then $g^*E_p := (g^*f)^{-1}(p)$ is equal to the fibre E_x. Hence, also every g^*E_p is diffeomorphic to F.

Proposition 5.4. *(1) For every fibre bundle $\xi := f : E \to X$ with fibre F, for every $g \in \mathcal{E}(M, X)$,*

$$g^*\xi := g^*f : g^*E \to M$$

is a smooth fibre bundle with fibre F. It is called the pull-back of ξ via g. *Moreover, $[g, g^*]$ is a fibred map between fibred bundles.*

(2) For every $h \in \mathcal{E}(N, M)$, every $g \in \mathcal{E}(M, X)$,

$$(g \circ h)^*\xi = h^*(g^*\xi)\ ,\ (g \circ h)^* = g^* \circ h^*\ .$$

Proof : The second point follows from the very definitions. As for the first, consider a fibre bundle atlas of ξ. This is formed as usual by locally trivializing diagrams

$$\begin{array}{ccc} \Omega \times F & \stackrel{\Psi}{\to} & \tilde{\Omega} \\ \downarrow_{\pi_\Omega} & & \downarrow_f \\ \Omega & \stackrel{\mathrm{id}_\Omega}{\to} & \Omega \end{array}$$

and any change of local trivializations is of the form

$$(\Omega \cap \Omega') \times F \to (\Omega \cap \Omega') \times F$$
$$(x, y) \to (x, \rho(x)(y))$$
$$x \to \rho(x) \in \mathrm{Aut}(F) \ .$$

The Ω's form an open covering of X. Fix an open covering $\{W\}$ of M such that $g(W)$ is contained in some Ω. For every W we have the locally trivializing commutative diagram

$$\begin{array}{ccc} W \times F & \stackrel{\Psi\circ(g,\mathrm{id}_F)}{\to} & \tilde{W} \\ \downarrow{\scriptstyle \pi_W} & & \downarrow{\scriptstyle g^* f} \\ W & \stackrel{\mathrm{id}_W}{\to} & W \end{array} \ .$$

The change of local trivialization is of the form

$$(W \cap W') \times F \to (W \cap W') \times F$$
$$(w, y) \to (w, \rho(g(w))(y))$$
$$w \to \rho(g(w)) \in \mathrm{Aut}(F) \ .$$

■

Remark 5.5. If F has an additional structure preserved by a subgroup $G \subset \mathrm{Aut}(F)$, and $x \to \rho(x)$ as above is a smooth map with values in G (i.e. ξ is a "G-bundle"), then also the pull-back $g^*\xi$ has the same property. For example, if ξ is a vector bundle (with fibre $\mathbb{R}^k$), then $g^*\xi$ is also.

5.3. Categories of vector bundles

Let M be a smooth manifold (possibly with boundary). Let

$$f : M \to \mathfrak{G}_{n,k}$$

be a smooth map. Then we can consider the pull-back vector bundle $f^*\tau_{n,k}$, that is

$$\begin{array}{ccc} f^*\mathcal{V}(\mathfrak{G}_{n,k}) & \stackrel{f^*}{\to} & \mathcal{V}(\mathfrak{G}_{n,k}) \\ \downarrow{\scriptstyle f^*\tau_{n,k}} & & \downarrow{\scriptstyle \tau_{n,k}} \\ M & \stackrel{f}{\to} & \mathfrak{G}_{n,k} \end{array} \ .$$

By the strict definition, the total space of $\mathrm{id}^*_{\mathfrak{G}_{n,k}}\tau_{n,k}$ is a submanifold of $\mathfrak{G}_{n,k} \times (\mathfrak{G}_{n,k} \times \mathbb{R}^n)$; however, the projection to the product of the first and third factors gives a canonical fibred diffeomorphism to the total space of $\tau_{n,k}$. Modulo this *normalized embedding*, we can stipulate that

$$\mathrm{id}^*_{\mathfrak{G}_{n,k}}\tau_{n,k} = \tau_{n,k} \ .$$

Similarly, for every $f : M \to \mathfrak{G}_{n,k}$ as above, the total space of $f^*\tau_{n,k}$ has a canonical embedding in $M \times \mathbb{R}^n$; modulo this normalization we can state that

$$\mathrm{id}^*_M(f^*\tau_{n,k}) = f^*\tau_{n,k} \ .$$

Such a normalization is performed by default. Note also that the composition of f^* with the natural projection of $\mathcal{V}(\mathfrak{G}_{n,k})$ to $\mathbb{R}^n$ gives a map which is linear and injective at every fibre of $f^*(\mathcal{V}(\mathfrak{G}_{n,k}))$, from which we can tautologically reconstruct the map f.

Denote $\mathcal{N} = \{(n,k) \in \mathbb{N}\times\mathbb{N};\ 0 \leq k \leq n\}$. For every $(n,k) \in \mathcal{N}$, set

$$\mathcal{V}_{n,k}(M) := \{f^*\tau_{n,k};\ f \in \mathcal{E}(M, \mathfrak{G}_{n,k})\}$$

and

$$\mathcal{V}(M) = \cup_{(n,k)\in\mathcal{N}}\ \mathcal{V}_{n,k}(M)\ .$$

We see immediately that

$$M \ \Rightarrow\ \mathcal{V}(M)$$

$$g : N \to M \ \Rightarrow\ g^\bullet : \mathcal{V}(M) \to \mathcal{V}(N),\ g^\bullet(f^*\tau_{n,k}) = (f\circ g)^*\tau_{n,k}$$

so that

$$(g\circ h)^\bullet = h^\bullet \circ g^\bullet$$

defines a contravariant functor from the category of smooth manifolds (with boundary) to this category of smooth vector bundles. Moreover, for every f and every g as above there is the natural vector bundle map

$$[g, g^*] : g^\bullet(f^*\tau_{n,k}) \to f^*\tau_{n,k}\ .$$

If $g : N \to M$ is a diffeomorphism, then $g^\bullet : \mathcal{V}(M) \to \mathcal{V}(N)$ is a bijection (with inverse $(g^{-1})^\bullet$), and for every f, $[g, g^*]$ is a vector bundle isomorphism between $g^\bullet(f^*\tau_{n,k})$ and $f^*\tau_{n,k}$.

The tangent bundle of an embedded manifold $M \subset \mathbb{R}^n$, as well as all its tensorial relatives, belongs to $\mathcal{V}(M)$. For example, $\pi_M : T(M) \to M$ is the pull-back of the (tautological) map

$$t_M : M \to \mathfrak{G}_{n,m},\ t_M(p) = T_pM\ .$$

REMARK 5.6. If $M \subset \mathbb{R}^k$ is embedded, then every vector bundle in $\mathcal{V}(M)$ is "embedded" in the same sense that the tangent bundle is so.

EXAMPLE 5.7. Recall the map $p : \tilde{\mathfrak{G}}_{m,n} \to \mathfrak{G}_{m,n}$ of Example 4.10. There is a natural tautological bundle $\tilde{\tau} : \mathcal{V}(\tilde{\mathfrak{G}}_{m,n}) \to \tilde{\mathfrak{G}}_{m,n}$ equal to $p^*(\tau)$. The fibres of $\tilde{\tau}$ are tautologically oriented and this is also the case for every pull-back of $\tilde{\tau}$.

5.3.1. Bundle equivalences. We are going to refine the above constructions by introducing suitable quotient sets of $\mathcal{V}(M)$.

For every $f : M \to \mathfrak{G}_{n,k}$, and every inclusion $j_n : \mathfrak{G}_{n,k} \to \mathfrak{G}_{n+1,k}$ (see Section 5.5), the total space of $(j_n \circ f)^*\tau_{n+1,k}$ is embedded in $M \times \mathbb{R}^n$ and coincides with the total space of $f^*\tau_{n,k}$. This gives us a *canonical identification* between these formally different points of $\mathcal{V}(M)$. A first mild quotient of $\mathcal{V}(M)$ is obtained through such canonical identifications. Let us also call it $\mathcal{V}(M)$. For every equivalence class, there is one representative $f^*\tau_{n,k}$ with *minimum* n.

More substantially, we can restrict to $\mathcal{V}(M)$ the *full equivalence* between vector bundles defined in Section 4.3.1, generated by arbitrary vector bundle isomorphisms of the form $[g,\tilde{g}]$. Denote by $\mathbf{V}(M)$ the quotient set.

EXAMPLE 5.8. If $g \in \mathrm{Aut}(M)$, then for every $f : M \to \mathfrak{G}_{n,k}$, the corresponding $[g, g^*]$ realizes a full equivalence between $f^*\tau_{n,k}$ and $g^\bullet(f^*\tau_{n,k})$. This establishes an action of $\mathrm{Aut}(M)$ on $\mathcal{V}(M)$, so that $\mathbf{V}(M)$ is a quotient set of $\mathcal{V}(M)/\mathrm{Aut}(M)$.

We can restrict to $\mathcal{V}(M)$ the *strict equivalence* between vector bundles defined in Section 4.3.1, generated by isomorphisms of the form $[\mathrm{id}_M, \tilde{g}]$. Denote by $\mathbf{V}_0(M)$ the quotient set. $\mathbf{V}(M)$ is a quotient of $\mathbf{V}_0(M)$.

EXAMPLE 5.9. (1) If $f, g : M \to \mathfrak{G}_{n,k}$ are two different constant maps, then $f^*\tau_{n,k}$ and $g^*\tau_{n,k}$ are different points of $\mathcal{V}(M)$ which obviously are strictly equivalent.

(2) Let $g : M \to N$ be a diffeomorphism; then $[g^{-1}, Tg^{-1}] \circ [g, g^*]$ is a strict equivalence between $T(M)$ and $g^*T(N)$.

(3) By generalizing the above item, let $[g,\tilde{g}]$ realize a full equivalence between bundles in $\mathcal{V}(M)$; then $[g, g^*]$, as in the above example, also realizes such an equivalence. Moreover, $[g^{-1}, \tilde{g}^{-1}] \circ [g, g^*]$ realizes instead a strict equivalence.

Associating to every $f^*\tau_{n,k}$ its class in the preferred quotient set of $\mathcal{V}(M)$, we get variants of the basic pull-back functor defined above.

We will focus on $\mathbf{V}_0(M)$. In particular we pose the following natural question.

QUESTION 5.10. Set

$$\mathcal{E}(M, \mathfrak{G}) := \cup_{(n,k)\in\mathcal{N}}\ \mathcal{E}(M, \mathfrak{G}_{n,k})\ .$$

Consider the obvious surjective map

$$(.)^* : \mathcal{E}(M, \mathfrak{G}) \to \mathbf{V}_0(M),\ \ f \to [f^*\tau_{n,k}]$$

so that tautologically

$$\mathbf{V}_0(M) = \mathcal{E}(M, \mathfrak{G})/(.)^*\ .$$

This relation on $\mathcal{E}(M, \mathfrak{G})$ is only implicitly defined. The question is to *make it explicit.* An answer will be discussed later when M is compact.

5.4. The frame bundles

We can repeat the above scheme using the tautological frame bundles instead. It is enough to replace $\mathcal{V}(M)$ either with

$$\mathcal{L}(M) = \cup_{(n,k)\in\mathcal{N}}\ \mathcal{L}_{n,k}(M)$$

$$\mathcal{L}_{n,k}(M) := \{f^* l\tau_{n,k};\ f \in \mathcal{E}(M, \mathfrak{G}_{n,k})\}$$

or with the similarly defined $\mathcal{S}(M)$ and $\mathcal{S}_{n,k}(M)$, using the tautological bundles $s\tau_{n,k}$. For every $f : M \to \mathfrak{G}_{n,k}$, the vector bundle $f^*\tau_{n,k}$ is *associated* to its *linear frame bundle* $f^*l\tau_{n,k}$, provided that both are considered as $\mathrm{GL}(k,\mathbb{R})$-bundle. By the reduction from $\mathrm{GL}(k,\mathbb{R})$ to $O(k)$, $f^*\tau_{n,k}$ is associated to its *orthogonal frame bundle* $f^*s\tau_{n,k}$, both are considered as $O(k)$-bundles. In particular, by applying this to the tangent bundle $T(M)$ of a manifold, we get the linear or orthogonal *frame bundle* of M, $F_l(M)$ or $F_s(M)$. M is parallelizable if and only if $F_l(M)$ (hence $F_s(M)$) has a section.

5.5. Limit tautological bundles

We will deal with a few concrete instances of the following general topological construction. Let $\{X_n\}_{n\in\mathbb{N}}$ be a countable family of Hausdorff topological spaces each admitting a countable basis of open sets. Assume that, for every n, X_n is strictly contained in X_{n+1} as a closed subset. Then consider the "limit" space

$$X_\infty = \cup_n X_n$$

endowed with the *final topology* with respect to the family of inclusions

$$\{i_n : X_n \to X_\infty\}\ ;$$

this means the *finest* topology such that every i_n is continuous. In other words, A is open in X_∞ if and only if for every n, $A \cap X_n$ is open in X_n.

Lemma 5.11. *If $K \subset X_\infty$ is compact, then there is $n \in \mathbb{N}$ such that $K \subset X_n$.*

Proof : Assume that there is not; then there should be an infinite sequence x_n in K such that $x_n \in X_{n+1} \setminus X_n$. The union of these points of K would be a closed subset of K (hence compact) with induced discrete topology (i.e. it would be a *compact and discrete* space). Such a space is necessarily *finite*, counter to our assumption.

■

Some examples:

• For every n, consider the inclusion $\mathbb{R}^n \subset \mathbb{R}^{n+1}$, $(x) \to (x, 0)$. Then we can define the limit space $\mathbb{R}^\infty$.

• The above inclusions induce "equatorial" inclusions $i_n : S^{n-1} \to S^n$ of unit spheres, so we can define the limit space S^∞.

• The definition of S^∞ can be generalized to arbitrary Stiefel manifolds. The inclusions $M(n,k,\mathbb{R}) \to M(n+1,k,\mathbb{R})$

$$A \to \begin{pmatrix} A \\ 0 \end{pmatrix}$$

induce inclusions of embedded smooth manifolds $i_n : S_{n,k} \to S_{n+1,k}$, and we can define the *Stiefel limit space* $S_{\infty,k}$.

- The inclusions $S(n,\mathbb{R}) \to S(n+1,\mathbb{R})$

$$A \to \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

induce the inclusions $j_n := j_{n,n+1} : \mathfrak{G}_{n,k} \to \mathfrak{G}_{n+1,k}$, and we can define the *limit Grassmannian* $\mathfrak{G}_{\infty,k}$.

REMARK 5.12. The cell decompositions of Section 3.5 respect the inclusions

$$j_n : \mathfrak{G}_{n,k} \to \mathfrak{G}_{n+1,k}$$

in the sense that the cells of $\mathfrak{G}_{n,k}$ are also cells of $\mathfrak{G}_{n+1,k}$; hence, we have also a cell decomposition of the limit infinite Grassmannian $\mathfrak{G}_{\infty,k}$.

- We have the family of commutative diagrams of smooth maps

$$\begin{array}{ccc} S_{n,k} & \stackrel{i_n}{\to} & S_{n+1,k} \\ \downarrow s_{n,k} & & \downarrow s_{n+1,k} \\ \mathfrak{G}_{n,k} & \stackrel{j_n}{\to} & \mathfrak{G}_{n+1,k} \end{array}$$

so that we can eventually define the "limit projection" which is continuous

$$\begin{array}{c} S_{\infty,k} \\ \downarrow s_{\infty,k} \\ \mathfrak{G}_{\infty,k} \end{array} .$$

Similarly, by using the linear frames we have the limit projection

$$\begin{array}{c} L_{\infty,k} \\ \downarrow l_{\infty,k} \\ \mathfrak{G}_{\infty,k} \end{array} .$$

EXAMPLE 5.13. As a particular case we have the projection

$$s_{\infty,1} : S^\infty \to \mathbf{P}^\infty(\mathbb{R}) .$$

We easily realize that s_∞ is a continuous covering map of degree 2, like every $s_{n,1}$. Thanks to Lemma 5.11, for every $p \in \mathbb{N}$, every continuous map $f : S^p \to S^\infty$ is of the form $i_n \circ \tilde{f}$, for some $\tilde{f} : S^p \to S^n$ such that the image of $\tilde{f}$ does not contain e_{n+1}. By considering $S^n = \mathbb{R}^n \cup \{\infty\}$ via the stereographic projection with centre $\infty := e_{n+1}$, then $\tilde{f}$ factorizes through a map with values in $\mathbb{R}^n$ which is contractible. We can conclude that such a map f is *homotopically trivial*. In other words, all homotopy groups $\pi_p(S^\infty)$ are trivial. By a theorem of Whitehead (see [**H**]), it follows that S^∞ is contractible, hence $s_\infty : S^\infty \to \mathbf{P}^\infty(\mathbb{R})$ is a *universal covering map*. By the theory of covering maps, we eventually get that the fundamental group $\pi_1(\mathbf{P}^\infty(\mathbb{R})) \sim \mathbb{Z}/2\mathbb{Z}$, while all other groups $\pi_p(\mathbf{P}^\infty(\mathbb{R}))$, $p > 1$, are trivial. We summarize these facts by saying that $\mathbf{P}^\infty(\mathbb{R})$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$ *space*.

• The same limit procedure applies to the tautological bundles. We have the family of commutative diagrams of smooth maps

$$\begin{array}{ccc} \mathcal{V}(\mathfrak{G}_{n,k}) & \stackrel{\tilde{j}_n}{\to} & \mathcal{V}(\mathfrak{G}_{n+1,k}) \\ \downarrow_{\tau_{n,k}} & & \downarrow_{\tau_{n+1,k}} \\ \mathfrak{G}_{n,k} & \stackrel{j_n}{\to} & \mathfrak{G}_{n+1,k} \end{array}$$

so we eventually define the "limit tautological vector bundle" :

$$\begin{array}{c} \mathcal{V}(\mathfrak{G}_{\infty,k}) \\ \downarrow_{\tau_{\infty,k}} \\ \mathfrak{G}_{\infty,k} \end{array} .$$

Similarly we have the limit bundles

$$\begin{array}{ccc} \mathcal{L}(\mathfrak{G}_{\infty,k}) & & \mathcal{S}(\mathfrak{G}_{\infty,k}) \\ \downarrow_{l\tau_{\infty,k}} & & \downarrow_{s\tau_{\infty,k}} \\ \mathfrak{G}_{\infty,k} & & \mathfrak{G}_{\infty,k} \end{array} .$$

5.6. A classification theorem for compact manifolds

In this section, *we assume that M is compact.* First, we show that up to strict equivalence, every abstract vector bundle on M belongs to $\mathcal{V}(M)$. In a sense this is an extension of the embedding Proposition 4.34.

PROPOSITION 5.14. *Every abstract vector bundle ξ over a compact smooth manifold M is strictly equivalent to a bundle in $\mathcal{V}(M)$.*

Proof : By compactness we can assume that the abstract bundle $p : E \to M$ is determined by a cocycle c_{ts} over a nice atlas $\mathcal{U} = \{(W_j, \phi_j)\}_{j=1,\dots,s}$ of M. Consider the family of local trivializations $\Phi_j : p^{-1}|W_j \to W_j \times \mathbb{R}^n$, and let $\{\lambda_j\}$ be the partition of unity over $\mathcal{U}$ as usual. For every j, denote by $q_j : W_j \times \mathbb{R}^n \to \mathbb{R}^n$ the natural projection. Finally, define

$$h : E \to M \times \mathbb{R}^{ns},\ h(e) = (p(e), \lambda_1(p(e))q_1(e), \dots, \lambda_s(p(e))q_s(e)) .$$

The restriction of h to M, included in E as the zero section, is equal to the identity. Moreover, every fibre of the bundle is linearly embedded to an n-subspace of $\mathbb{R}^{ns}$. This determines a map $f : M \to \mathfrak{G}_{ns,n}$ which recovers the bundle via pull-back.

■

By Lemma 5.11, $f \in \mathcal{C}^0(M, \mathfrak{G}_{\infty,k})$ if and only if there is a minimum n such that it factorizes through a continuous map

$$\hat{f} : M \to \mathfrak{G}_{n,k}$$

followed by the inclusion

$$j_{n,\infty} : \mathfrak{G}_{n,k} \to \mathfrak{G}_{\infty,k} .$$

It makes sense to say that such a map f is *smooth* if $\hat{f}$ is smooth in the usual sense. Moreover, the topologies on the spaces $\mathcal{E}(M, \mathfrak{G}_{n,k})$ pass to the limits, giving us the topological space $\mathcal{E}(M, \mathfrak{G}_{\infty,k})$ of such smooth maps. If $f, \hat{f}$ are as before, we have

$$f^*\tau_{\infty,k} = \hat{f}^*\tau_{n,k}$$

provided that we have incorporated the *canonical identifications* illustrated in Section 5.3.1. Set

$$\mathcal{V}_k(M) := \{f^*\tau_{\infty,k};\ f \in \mathcal{E}(M, \mathfrak{G}_{\infty,k})\}\ .$$

It is clear from the above considerations that the already defined space $\mathcal{V}(M)$ can be described as

$$\mathcal{V}(M) = \cup_{k=0}^{\infty}\ \mathcal{V}_k(M)$$

as well as

$$\mathcal{E}(M, \mathfrak{G}) = \cup_{k=0}^{\infty}\ \mathcal{E}(M, \mathfrak{G}_{\infty,k})\ .$$

Thus we have rephrased, in terms of these limits, the surjective maps

$$(.)^* : \mathcal{E}(M, \mathfrak{G}) \to \mathcal{V}(M)$$

$$[(.)^*] : \mathcal{E}(M, \mathfrak{G}) \to \mathbf{V}_0(M)$$

and we stipulate that the target spaces are endowed with the *quotient topology.*

Given $f_0, f_1 \in \mathcal{E}(M, \mathfrak{G})$, we say that they are *smoothly homotopic* if $f_0, f_1 \in \mathcal{E}(M, \mathfrak{G}_{\infty,k})$ for some k, and are connected by a smooth homotopy $F \in \mathcal{E}(M \times [0,1], \mathfrak{G}_{\infty,k})$, provided that $f_t := F_{|M\times\{t\}}$. As usual, this defines an equivalence relation on $\mathcal{E}(M, \mathfrak{G})$. Denote by $[M, \mathfrak{G}]$ the set of smoothly homotopy classes of maps of $\mathcal{E}(M, \mathfrak{G})$.

PROPOSITION 5.15. *Let M be a compact smooth manifold. If $[f_0^*\tau_{\infty,k}] = [f_1^*\tau_{\infty,k}]$ in $\mathbf{V}_0(M)$, then f_0 and f_1 are homotopic. Hence there is a well defined a surjective map*

$$\mathfrak{v} : \mathbf{V}_0(M) \to [M, \mathfrak{G}],\ [f^*\tau_{\infty,k}] \to [f]\ .$$

Proof : We provide two proofs.

First proof: If $[f_0^*\tau_{\infty,k}] = [f_1^*\tau_{\infty,k}] \in \mathbf{V}_0(M)$, we can assume that they both factorize through maps (for simplicity we keep the same names) $f_0, f_1 : M \to \mathfrak{G}_{n,k}$, for some n big enough. Moreover, sometimes we will confuse here a point $A \in \mathfrak{G}_{n,k}$ with the corresponding subspace $V_A \subset \mathbb{R}^n$. For $j = 0, 1$ and for every $p \in M$, we have the direct sum decomposition $\mathbb{R}^n = f_j(p) \oplus f_j(p)^\perp$. The projections of the canonical basis $\{e_1, \dots, e_n\}$ to $f_j(p)$, when p varies, define n-sections $s_{j,1}, \dots, s_{j,n}$ of $f_j^*\tau_{n,k}$ which span the fibre $f_j(p)$ over every $p \in M$. The map f_j can be reconstructed from these set of sections as follows: for every $p \in M$, the linear evaluation map

$$\mathfrak{e}_{j,p} : \mathbb{R}^n \to f_j(p),\ \mathfrak{e}_{j,p}(X) = \sum_i x_i s_{j,i}(p)$$

is surjective, so that $\ker(\mathfrak{e}_{j,p}) = f_j(p)^\perp$ and, finally, $f_j(p) = \ker(\mathfrak{e}_{j,p})^\perp$. A strict equivalence from $f_0^*\tau_{n,k}$ to $f_1^*\tau_{n,k}$ transports the system of sections $s_{0,1}, \dots, s_{0,n}$ to a system $s'_{1,1}, \dots, s'_{1,n}$ over $f_1^*\tau_{n,k}$ which generate all fibres.

Denote by $\mathfrak{e}'_{1,p}$ the corresponding evaluation maps and apply to it the above procedure in order to produce a map from M with values in $\mathfrak{G}_{n,k}$; we realize that this recovers f_0. For every $p \in M$, $\ker(\mathfrak{e}'_{1,p})$ is a graph of a linear map $L_p : f_1(p)^\perp \to f_1(p)$, while $f_1(p)^\perp$ itself is the graph of the zero map. Finally, we can use the homotopy $L_{p,t} = tL_p,\ t \in [0,1]$, to define a desired homotopy between f_0 and f_1.

Second proof: We know that f_j is determined by a map g_j from

$$f_j^*(\mathcal{V}(\mathfrak{G}_{\infty,k}))$$

to $\mathbb{R}^\infty$ which is linear and injective at each fibre. Moreover, it factorizes through a map with values in some $\mathbb{R}^h$ with h big enough. If $[f_0^*\tau_{\infty,k}] = [f_1^*\tau_{\infty,k}] \in \mathbf{V}_0(M)$, we can transport the map g_1 to a map g'_0 defined on $f_0^*(\mathcal{V}(\mathfrak{G}_{\infty,k}))$ with such a property, and we have to show that g_0 and g'_0 are homotopic through maps that are linear injections on fibers. First compose g_0 with the homotopy $a_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $a_t(x_1, x_2, \dots) = (1-t)(x_1, x_2, \dots) + t(x_1, 0, x_2, 0, \dots)$. This moves the image of g_0 into the odd-numbered coordinates. Similarly, we can move g'_0 into the even-numbered coordinates. By keeping the names of these maps, we eventually define the desired homotopy $h_t = (1-t)g_0 + tg'_0$.

■

We can answer Question 5.10, at least in the compact case.

THEOREM 5.16. (Classification Theorem) *Let M be a* compact *smooth manifold. Then the map*

$$\mathfrak{v} : \mathbf{V}_0(M) \to [M, \mathfrak{G}],\ [f^*\tau_{\infty,k}] \to [f]$$

is bijective. That is, for every $f_0, f_1 \in \mathcal{E}(M, \mathfrak{G})$, $[f_0^*\tau_{\infty,k}] = [f_1^*\tau_{\infty,k}] \in \mathbf{V}_0(M)$ if and only if f_0, f_1 are smoothly homotopic. Hence the map $[(.)^*]$ *induces the inverse map of* $\mathfrak{v}$

$$\mathfrak{c} : [M, \mathfrak{G}] \to \mathbf{V}_0(M),\ \mathfrak{c}([f]) = [f^*\tau_{\infty,k}]$$

whenever $f \in \mathcal{E}(M, \mathfrak{G}_{\infty,k})$.

Proof : Thanks to Proposition 5.15, it is enough to prove that if f_0 and f_1 are homotopic, then $f_0^*\tau_{\infty,k}$ and $f_1^*\tau_{\infty,k}$ are equivalent. We can assume that a homotopy factorizes through $F : M \times [0,1] \to \mathfrak{G}_{n,k}$, for large enough n. Take the pull-back $F^*\tau_{n,k}$. The idea is to use it to connect $f_0^*\tau_{n,k}$ and $f_1^*\tau_{n,k}$ by a path $f_t^*\tau_{n,k}$ of bundles strictly equivalent to each other. For every $t \in [0,1]$, $p \in M$, denote by $V_{t,p}$ the fibre of $f_t^*\tau_{n,k}$ over p.

Claim 1. *There is $\epsilon > 0$ such that, for every $0 \le t \le \epsilon$, for every $p \in M$, $\mathbb{R}^n = V_{0,p} \oplus V_{0,p}^\perp = V_{t,p} \oplus V_{0,p}^\perp$.*

Let us prove it. If such an ϵ does not exist, by compactness there would exist a converging sequence $(p_n, t_n) \to (p_0, 0)$ in $M \times [0,1]$ such that, for every n, $\dim V_{t_n,p_n} \cap V_{0,p_n}^\perp > 0$. But this is impossible because $V_{0,p_0} \cap V_{0,p_0}^\perp = \{0\}$ and this is an open condition.

Claim 2. *There is $\epsilon > 0$ such that, for every $t \leq \epsilon$, $f_0^*\tau_{n,k}$ is strictly equivalent to $f_t^*\tau_{n,k}$.*

To prove it, recall the elementary fact that if $\mathbb{R}^n = V' \oplus V = V'' \oplus V$ (V, V' and V'' being linear subspaces), then $\phi : V' \to V''$, $\phi(v') = v''$ if $v' = v'' + v$, is a *canonical* linear isomorphism between V' and V''. Let $\epsilon > 0$ be as in Claim 1. For every $t \leq \epsilon$, the "field" of canonical isomorphisms

$$\phi_p : V_{t,p} \to V_{0,p}$$

when p varies in M, defines a strict equivalence as required by Claim 2.

Set $\epsilon_0 \in [0,1]$ the supremum of the ϵ's which satisfy Claim 2. We claim that ϵ_0 is a *maximum.* In fact, by applying the same argument as above, we see that there is $\epsilon > 0$ such that $f_{\epsilon_0}^*\tau_{n,k}$ is strictly equivalent to $f_t^*\tau_{n,k}$, for $t \in (\epsilon_0 - \epsilon, \epsilon_0]$. Finally, we claim that $\epsilon_0 = 1$: if $\epsilon_0 < 1$, we apply again the same argument to f_{ϵ_0} and we find $\epsilon_1 = \epsilon_0 + \epsilon$, for some small $\epsilon > 0$, which works as well. This is counter to the fact that ϵ_0 is the maximum.

■

5.7. The rings of stable equivalence classes of vector bundles

For every smooth manifold M, the final aim of this section is to endow a suitable quotient space $\mathbf{K}_0(M)$ of $\mathbf{V}_0(M)$ with a natural *ring* structure. This leads to a contravariant functor from the category of smooth manifolds to the category of commutative rings. If M is compact, this functor has the "homotopy invariance property".

5.7.1. Grassmannian operations. The operations of the ring $\mathbf{K}_0(M)$ will descend from simple 'operations' defined on Grassmann manifolds.

- The inclusion $S(n,\mathbb{R}) \to S(n+m,\mathbb{R})$

$$A \to \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

induces, for every $k \leq n$, a smooth inclusion

$$j_{n,n+m} : \mathfrak{G}_{n,k} \to \mathfrak{G}_{n+m,k} \ .$$

- The inclusion $S(n,\mathbb{R}) \times S(m,\mathbb{R}) \to S(n+m,\mathbb{R})$

$$(A,B) \to \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

induces, for every $k \leq n$, $h \leq m$, a smooth inclusion

$$\oplus_{n,k,m,h} : \mathfrak{G}_{n,k} \times \mathfrak{G}_{m,h} \to \mathfrak{G}_{n+m,k+h} \ .$$

- For every $V \in G_{n,k}$, denote by V^* its dual spaces. This is considered as a subspace of $(\mathbb{R}^n)^* = M(n,1,\mathbb{R})$ as follows. Let $\mathbb{R}^n = V \oplus V^\perp$ be the orthogonal direct sum decomposition, $V^\perp \in G_{n,n-k}$ being the orthogonal complement of V for the standard Euclidean scalar product; extend every $\gamma \in V^*$ to a functional defined on the whole of $\mathbb{R}^n$ by setting $\gamma(u+w) =$

$\gamma(u)$. The matrix space $M(n,1,\mathbb{R})$ is canonically isomorphic to $\mathbb{R}^n$ via the transposition.

Let $(V,W) \in G_{n,k} \times G_{m,h}$. Denote by $V \otimes W$ the space of bilinear forms defined on $V^* \times W^*$. Its dimension is kh. In fact, there is the canonical bilinear map

$$\otimes : V \times W \to V \otimes W,\ v \otimes w(\gamma, \rho) := \gamma(v)\rho(w)$$

and, for every couple of bases $(\mathcal{B}, \mathcal{D})$ of V and W, respectively, $\mathcal{B} \otimes \mathcal{D} = \{v_i \otimes w_j;\ v_i \in \mathcal{B},\ w_j \in \mathcal{D}\}$ is a basis of $V \otimes W$. By using the decomposition

$$\mathbb{R}^n \times \mathbb{R}^m = (V \oplus V^\perp) \times (W \oplus W^\perp)$$

and arguing as above, we can consider $V \otimes W$ as a subspace of $\mathbb{R}^n \otimes \mathbb{R}^m$, hence (via canonical isomorphisms) as an element of $G_{nm,kh}$. In this way, we have defined a map (between *sets*):

$$G_{n,k} \times G_{m,h} \to G_{nm,kh}\ .$$

By the usual bijections $V \to A_V$, it can be transported to a map

$$\otimes_{n,k,m,h} : \mathfrak{G}_{n,k} \times \mathfrak{G}_{m,h} \to \mathfrak{G}_{nm,kh}\ .$$

We can check by direct computation that this is a *smooth map.*

Similarly we can check that the set map

$$G_{n,k} \to G_{n,n-k},\ V \to V^\perp$$

induces a *diffeomorphism*

$$\perp_{n,k}: \mathfrak{G}_{n,k} \to \mathfrak{G}_{n,n-k}$$

with inverse $\perp_{n,n-k}$.

5.7.2. The ring $\mathbf{K}_0(M)$. The Grassmannian operations of Section 5.7.1 induce operations

$$\oplus : \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M),\ f^*\tau_{n,k} \oplus g^*\tau_{r,s} = (\oplus \circ (f,g))^*\tau_{n+r,k+s}$$

$$\otimes : \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M),\ f^*\tau_{n,k} \otimes g^*\tau_{r,s} = (\otimes \circ (f,g))^*\tau_{nr,ks}$$

$$\perp: \mathcal{V}(M) \to \mathcal{V}(M),\ \perp (f^*\tau_{n,k}) = (\perp \circ f)^*\tau_{n,n-k}\ .$$

The operations $\oplus, \otimes, \perp$ descend to each quotient set $\mathcal{V}(M)/\mathrm{Aut(M)}$, $\mathbf{V}(M)$ and $\mathbf{V}_0(M)$.

The Grassmannian operations $\oplus$ and $\otimes$ pass to the limits:

$$\oplus : \mathfrak{G}_{\infty,k} \times \mathfrak{G}_{\infty,h} \to \mathfrak{G}_{\infty,k+h}$$

$$\otimes : \mathfrak{G}_{\infty,k} \times \mathfrak{G}_{\infty,h} \to \mathfrak{G}_{\infty,kh}$$

and are continuous in the limit topology. The operation $\perp$ induces a family of continuous maps

$$\perp_n: \mathfrak{G}_{\infty,k} \to \mathfrak{G}_{\infty,n-k},\ n \geq k\ .$$

For every smooth manifold M, these operations define a *ring* structure on a suitable quotient of $\mathbf{V}_0(M)$ that we are going to describe. Denote by ϵ^k the class in $\mathbf{V}_0(M)$ of the trivial (product) bundle $M \times \mathbb{R}^k \to M$. Clearly

$$\epsilon^k \oplus \epsilon^h = \epsilon^{k+h} \ .$$

DEFINITION 5.17. We say that ξ and η in $\mathbf{V}_0(M)$ are *weakly stably equivalent* if there exist ϵ^k and ϵ^h such that

$$\xi \oplus \epsilon^k = \eta \oplus \epsilon^h \ .$$

This is an equivalence relation. Let us just check the transitivity: if

$$\xi \oplus \epsilon^k = \eta \oplus \epsilon^h, \ \ \eta \oplus \epsilon^r = \beta \oplus \epsilon^s$$

then

$$\xi \oplus \epsilon^{k+r} = \beta \oplus \epsilon^{h+s} \ .$$

EXAMPLE 5.18. (1) Let M be a compact smooth manifold with nonempty boundary ∂M. Let $i : \partial M \to M$ be the inclusion. Then $T(\partial M)$ and $i^*T(M)$ are weakly stably equivalent vector bundles on ∂M. Fix any Riemannian metric g on M. For every $x \in \partial M$, consider $\nu(x) = (T_x\partial M)^{\perp_{g(x)}}$; as $T_xM = T_x\partial M \oplus \nu(x)$, this defines a vector bundle ν on ∂M, with 1-dimensional fibres, such that $i^*T(M) = T(\partial M) \oplus \nu$. The bundle ν has a nowhere vanishing section (for every $x \in \partial M$ take the "outgoing" g-unitary vector in $\nu(x)$). Then $[\nu] = \epsilon^1$. In particular $S^n = \partial B^{n+1}(0,1)$, $T(B^{n+1})$ is trivial as it is the restriction of $T(\mathbb{R}^{n+1})$, hence $[T(S^n)]$ is weakly stably trivial.

Denote by $\mathbf{K}_0(M)$ the quotient of $\mathbf{V}_0(M)$ up to weakly stable equivalence. It is clear that if $M = \{p\}$ is one point, then $\mathbf{K}_0(\{p\}) = 0$.

PROPOSITION 5.19. *The operations $\oplus$, $\otimes$ descend to $\mathbf{K}_0(M)$ and make it an Abelian ring.*

The associativity of $\oplus$ is evident. The weakly stable equivalence class $[\epsilon^1]$ is the zero element; for every $[[\xi]]$, assume that $\xi \in \mathcal{V}_{n,k}(M)$. Then $\xi^\perp \in \mathcal{V}_{n,n-k}(M)$ is such that

$$[\xi \oplus \xi^\perp] = \epsilon^n \ ,$$

hence

$$[[\xi^\perp]] = -[[\xi]] \ .$$

With a bit of more work we can check the whole ring structure. We omit these verifications.

■

In summary,

$$M \ \Rightarrow \ \mathbf{K}_0(M)$$

$$g : N \to M \ \Rightarrow \ g^\bullet : \mathbf{K}_0(M) \to \mathbf{K}_0(N), \ g^\bullet([[f^*\tau_{\infty,k}]]) = [[(f \circ g)^*\tau_{\infty,k}]]$$

define a *contravariant functor from the category of smooth manifolds (with boundary) to the category of Abelian rings.*

If M is compact, the above construction of the ring $\mathbf{K}_0(M)$ from $\mathbf{V}_0(M)$ can be rephrased in terms of $[M, \mathfrak{G}]$. So $[f_0]$, $f_0 : M \to \mathfrak{G}_{\infty,s}$, and $[f_1]$, $f_1 : M \to \mathfrak{G}_{\infty,r}$, are weakly stably equivalent if and only if there are constant maps $c_0 : M \to \mathfrak{G}_{\infty,k}$ and $c_1 : M \to \mathfrak{G}_{\infty,h}$ such that $[\oplus \circ (f_0, c_0)] = [\oplus \circ (f_1, c_1)]$ in $[M, \mathfrak{G}]$. Denote by $[[M, \mathfrak{G}]]_0$ the quotient set.

Proposition 5.20. *Let M be compact. The operations $\oplus$ and $\otimes$ descend to $[[M, \mathfrak{G}]]_0$ and make it an Abelian ring such that the map $\mathfrak{v}$ induces a ring isomorphism*

$$\tilde{\mathfrak{v}} : \mathbf{K}_0(M) \to [[M, \mathfrak{G}]]_0$$

with inverse

$$\tilde{\mathfrak{c}} : [[M, \mathfrak{G}]]_0 \to \mathbf{K}_0(M)$$

induced by the map $\mathfrak{c}$ of the Classification Theorem 5.16.

Corollary 5.21. (Homotopy invariance) *Let M, N be compact smooth manifolds. Then:*

(1) If $g_1, g_2 \in \mathcal{E}(N, M)$ are smoothly homotopic, then $g_1^\bullet = g_2^\bullet$.

(2) If M and N are smoothly homotopically equivalent, then $\mathbf{K}_0(M)$ and $\mathbf{K}_0(N)$ are isomorphic. In particular if M is smoothly contractible, then $\mathbf{K}_0(M) \sim \mathbf{K}_0(\{p\}) = 0$.

Proof : (1) and (2) follow from the Classification Theorem, as $[[*, \mathfrak{G}]]_0$ is manifestly homotopically invariant.

■

We conclude this section with a few scattered remarks.

Remarks 5.22. (1) $\mathbf{K}_0(*)$ is a version, in our smooth framework, of so-called *reduced topological K-theory* [**A**] [**B**]. Taking into account, for simplicity, only the additive structure, the *unreduced* group $\mathbf{K}(M)$ is constructed as follows. First we consider the quotient $\tilde{\mathbf{V}}_0(M)$ of $\mathbf{V}_0(M)$ up to *stable equivalence*; this is defined similarly to the above *weak stable equivalence*, by imposing in the definition that $k = h$. The operation $\oplus$ passes to the quotient, so that $(\tilde{\mathbf{V}}_0(M), \oplus)$ is a *commutative monoid* with (the class of) ϵ^0 as zero element. The monoid $(\tilde{\mathbf{V}}_0(M), \oplus)$ satisfies the "cancellation rule". In fact, if $\xi \oplus \eta = \xi \oplus \alpha$, we know that there exists β such that $\xi \oplus \beta = [\epsilon^n]$ (for some n), hence $[\epsilon^n] \oplus \eta = [\epsilon^n] \oplus \alpha$ and finally $\eta = \alpha$.

Then $\mathbf{K}(M)$ is the *Grothendieck group* of this monoid with cancellation rule. It is a general construction (producing, for example, $(\mathbb{Z}, +)$ from $(\mathbb{N}, +)$) that works as follows. Consider the product $\tilde{\mathbf{V}}_0(M) \times \tilde{\mathbf{V}}_0(M)$; an element (ξ, η) is written as a formal difference $\xi - \eta$. Put on this product the relation such that $\xi - \eta \sim \alpha - \beta$ if and only if $\xi \oplus \beta = \alpha \oplus \eta$. The cancellation rule is used to check that it is an equivalence relation. The addition rule on the quotient $\mathbf{K}(M)$ naturally is

$$(\xi - \eta) \oplus (\alpha - \beta) = \xi \oplus \alpha - \eta \oplus \beta \ ;$$

the zero element is given by

$$[\epsilon^0] - [\epsilon^0] = \xi - \xi, \ \forall \xi \in \tilde{\mathbf{V}}_0(M) \ ;$$

The inverse of $\xi - \eta$ is $\eta - \xi$.

Every element of $\mathbf{K}(M)$ *can be represented by a difference of the form* $\xi - [\epsilon^n]$ *(for some* n*).*

In fact, for every $\alpha - \beta$, let $\beta \oplus \gamma = [\epsilon^n]$; then

$$\alpha - \beta = \alpha \oplus \gamma - \beta \oplus \gamma := \xi - [\epsilon^n] \ .$$

The correspondence $\xi - [\epsilon^n] \to \xi$ induces a canonical surjective homomorphism $\mathbf{K}(M) \to \mathbf{K}_0(M)$. It is well defined because if $\xi - [\epsilon^n] = \xi' - [\epsilon^m]$ in $\mathbf{K}(M)$, then $\xi \oplus [\epsilon^m] = \xi' \oplus [\epsilon^n]$, hence $\xi = \xi'$ in $\mathbf{K}_0(M)$. The kernel consists of the elements of the form $[\epsilon^n] - [\epsilon^m]$; it is isomorphic to $\mathbb{Z}$ so that $\mathbf{K}(M) \sim \mathbf{K}_0(M) \oplus \mathbb{Z}$ (in a non-canonical way).

(2) If M is compact, the construction of $\mathbf{K}(M)$ from $\mathbf{V}_0(M)$ can be rephrased in terms of $[M, \mathfrak{G}]$. This produces a group (indeed, a ring) $[[M, \mathfrak{G}]]$ which is isomorphic to $\mathbf{K}(M)$, via the Classification Theorem (similarly to Proposition 5.20). Hence the functor

$$M \ \Rightarrow \ \mathbf{K}(M)$$
$$\dots \ \Rightarrow \dots$$

satisfies the *homotopy invariance* properties, similarly to Corollary 5.21.

(3) We can develop the very same constructions by using the complex Grassmann manifolds $\mathfrak{G}_{n,k}(\mathbb{C})$ and the complex vector bundles; this leads to the functors

$$M \ \Rightarrow \ \mathbf{K}_0(M, \mathbb{C}), \ \mathbf{K}(M, \mathbb{C})$$
$$\dots \ \Rightarrow \dots \ .$$

(4) *Bott's periodicity theorem* [**B**], [**At**] is a fundamental result in this theory. Let us just recall a few related statements that we can formulate in our setting.

- For every compact M, $\mathbf{K}(M \times S^2, \mathbb{C}) \sim \mathbf{K}(S^2, \mathbb{C}) \otimes \mathbf{K}(M, \mathbb{C})$;
- $\mathbf{K}(S^2, \mathbb{C}) = \mathbb{Z}[X]/(X-1)^2$, where X is the tautological complex bundle over $\mathbf{P}^1(\mathbb{C})$ (recall that $\mathbf{P}^1(\mathbb{C})$ is diffeomorphic to S^2, the "Riemann sphere");
- For every $m \geq 1$, $\mathbf{K}_0(S^{m+8}) \sim \mathbf{K}_0(S^m)$, $\mathbf{K}_0(S^{m+2}, \mathbb{C}) \sim \mathbf{K}_0(S^m, \mathbb{C})$.

CHAPTER 6

Compact embedded smooth manifolds

The hypothesis that a smooth manifold M is *compact* usually simplifies the study of several objects associated with it. This is the case of the weak topology on map spaces with compact source manifold in Section 4.34, or of the classification theorem of Chapter 5. We will develop this theme; the title of the chapter alludes to the fact that we will exploit the existence of an embedding in some Euclidean space.

6.1. Tubular neighbourhoods and collars

Let $M \subset \mathbb{R}^h$ be a compact boundaryless smooth m-manifold. Let $\mathbb{R}^h$ be endowed with the standard Riemannian metric g_0. Let us perform the following construction.

(1) Consider the smooth map

$$\nu : M \to \mathfrak{G}_{h,h-m}$$

where for every $p \in M$, $\nu(p)$ is the (matrix corresponding to the) orthogonal space $(T_pM)^\perp$ (with respect to g_0).

(2) Take the pull-back

$$\nu^*\tau_{h,h-m} : \nu^*(\mathcal{V}(\mathfrak{G}_{h,h-m})) \to M \ .$$

Every fibre $\nu(p)$ of this vector bundle is endowed with the restriction of g_0. We consider $M \subset \nu^*(\mathcal{V}(\mathfrak{G}_{h,h-m}))$ via the canonical "zero section".

(3) Define the smooth map

$$f_\nu : \nu^*(\mathcal{V}(\mathfrak{G}_{h,h-m})) \to \mathbb{R}^h, \ f_\nu(p,v) = p + v \ .$$

For every $\epsilon > 0$, set

$$N_\epsilon(M) = \{(p,v) \in \nu^*(\mathcal{V}(\mathfrak{G}_{h,h-m})); \ ||v||_{g_0} \leq \epsilon\} \ .$$

It is immediately verifiable that

- $f_\nu(p) = f_\nu(p,0) = p$, for very $p \in M$;
- there exists $\epsilon > 0$ small enough such that the restriction of f_ν to $N_\epsilon(M)$ is an immersion. In fact, $\dim \nu^*(\mathcal{V}(\mathfrak{G}_{h,h-m})) = \dim \mathbb{R}^h$, and for every $x = (p,0)$, the image of d_xf_ν is equal to $T_pM \oplus \nu(p) = T_p\mathbb{R}^h = \mathbb{R}^h$, so that f_ν is an immersion at M and the claim follows by the compactness of M.

(4) There exists $\epsilon > 0$ small enough such that the restriction (we keep the name)

$$f_\nu : N_\epsilon(M) \to \mathbb{R}^h$$

is an embedding to a compact h-submanifold of $\mathbb{R}^h$ with boundary, containing M in its interior. We already know that for $\epsilon > 0$ small enough, f_ν is an immersion; it is enough to prove that it is also injective. As it is the identity on M and M is compact, this follows from the same argument used in Section 4.34 to show that the embeddings form an open set.

(5) Set

$$U := f_\nu(N_\epsilon(M)) \subset \mathbb{R}^h$$
$$p : U \to M, \ p := \nu^* \tau_{h,h-m} \circ (f_\nu)^{-1} \ .$$

Let us analyze the arbitrary or inessential choices made to perform this construction.

- Certainly ϵ is not unique.
- The standard metric g_0 has nothing special from a differential topological viewpoint (we made a similar consideration when we discussed the unitary tangent bundles). The construction works as well using an *arbitrary* Riemannian metric g on $\mathbb{R}^h$.
- What we have used of the map ν is that it defines a *transverse distribution of* $(h-m)$*-planes along* M; that is, for every $p \in M$,

  $$\mathbb{R}^h = T_pM \oplus \nu(p) \ .$$

  However, this is a fake generalization; it is not hard to prove (by using, as usual, $\mathbb{R}^h \subset S^h$ and a suitable nice atlas) that, for every such a transverse distribution, there is a Riemannian metric g on $\mathbb{R}^h$ that realizes it.

Summing up, we can vary the metric g and the final choice of $\epsilon > 0$. Let us call a *tubular neighbourhood of* M *in* $\mathbb{R}^h$ any couple (U, p) obtained by any implementation of the construction. We have the following *uniqueness up to isotopy* of these tubular neighbourhoods. Fix an auxiliary *base* tubular neighbourhood (U^*, p^*) constructed by using the standard g_0 and some ϵ_0. We have

Proposition 6.1. *Let $M \subset \mathbb{R}^h$ be a compact boundaryless m-manifold. Let (U, p) be a tubular neighbourhood of M in $\mathbb{R}^h$. Then there is a smooth map*

$$H : U^* \times [0,1] \to \mathbb{R}^h$$

such that for every $t \in [0,1]$,

(1) H_t is an embedding of U^* to $U_t \subset \mathbb{R}^h$;
(2) H_t is equal to id_M on M;
(3) (U_t, p_t) is a tubular neighbourhood of M in $\mathbb{R}^h$ where $p_t := p^* \circ H_t^{-1}$. Moreover,
(4) $H_0 = \mathrm{id}_{U^*}$;
(5) $(U_1, p_1) = (U, p)$.

Proof : If (U, p) differs from (U^*, p^*) only by $\epsilon \neq \epsilon_0$, the statement is clearly true (use a radial isotopy fibre by fibre). Assume that (U, p) has been constructed by using a metric g. Take the path of Riemannian metrics $g_t = (1-t)g_0 + tg$, $t \in [0,1]$. Then there is a "path" of tubular neighbourhoods (U_t, p_t) constructed by using g_t and some $\epsilon_t > 0$. We can also assume that ϵ_t is a smooth function of t, and that $\epsilon_1 = \epsilon$. Hence we have the family of embeddings

$$f_{\nu_t} : N_{\epsilon_t}(M, g_t) \to \mathbb{R}^h \ .$$

There is also a family of strict equivalences $[\mathrm{id}_M, \rho_t]$ between $\nu_0^* \tau_{h,h-m}$ and $\nu_t^* \tau_{h,h-m}$ given for every $t \in [0,1]$ by the "field" of canonical linear isomorphisms

$$\nu_0(p) \to \nu_t(p), \ p \in M$$

associated to the two direct sum decompositions

$$\mathbb{R}^h = T_pM \oplus \nu_0(p) = T_pM \oplus \nu_t(p) \ .$$

We can assume (we are free to change ϵ_0) that for every t,

$$\rho_t(N_{\epsilon_0}(M, g_0)) \subset N_{\epsilon_t}(M, g_t)$$

and we can define the embeddings

$$f_{\nu_t} \circ \beta_t \circ (f_{\nu_0})^{-1} : U^* \to U_t \ .$$

This can be transformed to H_t with the required properties by composing it with radial isotopies fibre by fibre.

■

REMARK 6.2. The above constructions work as well if M is compact with nonempty boundary ∂M. The resulting tubular "neighbourhoods" (U, p) are not really neighbourhoods of M in $\mathbb{R}^h$. Rather, they are submanifolds with corners of $\mathbb{R}^h$, containing $(M, \partial M)$ as a proper submanifold.

6.1.1. Tubular neighbourhoods of submanifolds. Assume now that $Y \subset M \subset \mathbb{R}^h$, $\dim Y = s$, $\dim M = m$, $s < m$ and M and Y compact. Assume also that M and Y are boundaryless. Fix a Riemannian metric g on $\mathbb{R}^h$. As above, we have the associated maps

$$\nu_M : M \to \mathfrak{G}_{h,h-m}$$

$$\nu_Y : Y \to \mathfrak{G}_{h,h-s} \ .$$

Set for every $y \in Y$,

$$\hat{\nu}_Y(y) := \nu_Y(y) \cap T_yM \ .$$

This define a smooth map

$$\hat{\nu}_Y : Y \to \mathfrak{G}_{h,m-s} \ .$$

Define

$$f_{\hat{\nu}_Y} : \hat{\nu}_Y^*(\mathcal{V}(\mathfrak{G}_{h,m-s})) \to \mathbb{R}^h, \ f_{\hat{\nu}_Y}(y, v) = y + v \ .$$

Let (U_M, p_M) be a tubular neighbourhood of M constructed by means of ν_M. There is $\epsilon > 0$ small enough such that the image via $f_{\hat{\nu}_Y}$ of

$$\hat{N}_\epsilon(Y, g) = \{(y, v) \in \hat{\nu}_Y^*(\mathcal{V}(\mathfrak{G}_{h,m-s}));\ ||v||_g \leq \epsilon\}$$

is contained in U_M. Finally, define

$$f_{Y,M} : \hat{N}_\epsilon(Y, g) \to M,\ f_{Y,M} := p_M \circ f_{\hat{\nu}_Y}\ .$$

Using an argument similar to the one used for f_ν,

- $f_{Y,M}(y) = f_{Y,M}(y, 0) = y$, for very $y \in Y$;
- there exists $\epsilon > 0$ small enough such that the restriction of $f_{Y,M}$ to $\hat{N}_\epsilon(Y, g)$ is an immersion.
- In fact, there is $\epsilon > 0$ small enough such that the restriction of $f_{Y,M}$ to $\hat{N}_\epsilon(Y, g)$ is an embedding to a neighbourhood $U_{Y,M}$ of Y in M.

Finally $(U_{Y,M}, p_{Y,M})$, where $p_{Y,M} = \hat{\nu}^*\tau_{h,m-s} \circ (f_{Y,M})^{-1}$, is by definition a *tubular neighbourhood of* Y *in* M.

Similarly as above, varying g and ϵ, we have again the *uniqueness of these tubular neighbourhoods of* Y *in* M *up to isotopy.*

6.1.2. Collars. Consider now $M \subset \mathbb{R}^h$ compact with nonempty boundary ∂M. We would apply the above construction, by considering ∂M as a "mono-lateral" submanifold of M. By keeping the above notations, we know that

$$\hat{\nu}_{\partial M}^*(\mathcal{V}(\mathfrak{G}_{h,1}))$$

is strictly equivalent to the product bundle

$$\partial M \times \mathbb{R} \to \partial M$$

and a section is given by the unitary "positive" v (write "$v > 0$"), that is pointing towards the interior of M. So we can define

$$\hat{N}_\epsilon^+(\partial M, g) = \{(y, v) \in \hat{\nu}_{\partial M}^*(\mathcal{V}(\mathfrak{G}_{h,1}));\ ||v||_g \leq \epsilon,\ \text{“}v \geq 0\text{”}\}\ .$$

By using it, the construction can be repeated and we eventually get (by definition) a *collar* of ∂M in M; that is, an embedding $C : \partial M \times [0, \epsilon] \to M$ which is the identity on ∂M. Again we have the *uniqueness of collars up to isotopy.*

Remark 6.3. In the construction of the collars, it is not necessary that the whole M is compact; it is enough that ∂M is so.

Remark 6.4. Assume that $Y \subset M \in \mathbb{R}^h$ are compact manifolds with boundary such that Y is a *proper* submanifold of M. Then we can apply again the above construction to get tubular neighbourhoods of Y in M *relative* to the boundaries; that is, which restrict to tubular neighbourhoods of ∂Y in ∂M.

Tubular neighbourhoods have several interesting applications. Here is a simple one. Assume that $M \subset \mathbb{R}^h$ is compact. We already know (by using the partitions of unity) that every $f \in \mathcal{E}(M, N)$, $N \subset \mathbb{R}^k$, extends to a smooth map $\hat{f} : U \to \mathbb{R}^k$ defined on a neighbourhood of M in $\mathbb{R}^h$. Let (U, p) be a tubular neighbourhood of M. Then $f \circ p : U \to N$ is a smooth extension of f *with values in* N.

6.2. The "double" of a manifold with boundary

Let $M \subset \mathbb{R}^h$ be a compact smooth manifold with $\partial M \neq \emptyset$. The existence of collars suggests a variant in the definition of nice atlas given in Section 4.34.

Definition 6.5. A *nice atlas with collar* of $(M, \partial M)$ is of the form

$$\{(W_\partial, \phi_\partial)\} \cup \{(W_j, \phi_j)\}_{j=1,\dots,s}$$

where

(1) W_∂ is an open neighbourhood of ∂M and

$$\phi_\partial : W_\partial \to \partial M \times [0, 1)$$

is a diffeomorphism which is equal to the identity on ∂M. Define $B_\partial := \phi_\partial^{-1}([0, 1/3))$.

(2) Every (W_j, ϕ_j) is an normal chart contained in the interior of M, and $B_j \subset W_j$ is defined as for the usual nice atlas.

(3) $\{B_\partial\} \cup \{B_j\}$ is an open covering of M.

The existence of nice atlantes with collar is a direct consequence of the existence of collars. Given such a nice atlas with collar, every W_j carries a global bump function $\gamma_j : M \to \mathbb{R}$ as in Definition 4.30. Define the *collar global bump function*

$$\gamma_\partial : M \to \mathbb{R}$$

such that on W_∂ it is equal to $\gamma \circ p_{[0,1)} \circ \phi_\partial$, where $p_{[0,1)} : \partial M \times [0, 1) \to [0, 1)$ is the projection and γ is the restriction to $[0, 1)$ of the 1-dimensional bump function $\gamma_{1/3,1/2}$; on $M \setminus W_\partial$, γ_∂ is constantly equal to 0. Define

$$\lambda_\partial = \frac{\gamma_\partial}{\gamma_\partial + \sum_{i=1}^s \gamma_i}$$

$$\lambda_j = \frac{\gamma_j}{\gamma_\partial + \sum_{i=1}^s \gamma_i} .$$

Then the family of functions

$$\{\lambda_\partial\} \cup \{\lambda_j\}_{j=1,\dots,s}$$

is the *partition of unity* subordinate to the given nice atlas with collar.

Corollary 6.6. *For every compact manifold M with nonempty boundary there is a smooth function $f : M \to [0, 1]$ such that $\partial M = f^{-1}(0)$ and f is a submersion on a neighbourhood of ∂M.*

Proof : Take a nice atlas with collar. Define locally the following functions

$$f_\partial : W_\partial \to \mathbb{R},\ f_\partial = p_{[0,1)} \circ \phi_\partial \ ;$$
$$f_j : W_j \to \mathbb{R},\ f_j(x) = 1/2,\ \forall x \in W_j \ .$$

Finally, set

$$f = \lambda_\partial f_\partial + \sum_j \lambda_j f_j \ .$$

It is not hard to verify that it is smooth and satisfies the required properties.

■

The following is an easy generalization, we leave the details to the reader.

COROLLARY 6.7. *Let M be a compact manifold with boundary ∂M, equipped with a partition $\partial M = N_0 \cup N_1$, where both N_0 and N_1 are unions of connected components of ∂M. Then there exists a smooth function $f : M \to [0,1]$ such that $f^{-1}(0) = N_0$, $f^{-1}(1) = N_1$, and f is a submersion on a neighbourhood of ∂M.*

■

REMARK 6.8. To get the above corollaries, we can even use a simpler covering of M consisting of $(W_\partial, \phi_\partial)$ as above, together with an open set of the form $U = M \setminus W'$, where $W' \subset W_\partial$ is a smaller compact collar of ∂M contained in B_∂. Hence $W' \subset W" \subset B_\partial$, where $W"$ is another collar of ∂M, so that the compact sets B_∂ and $B'_\partial := \overline{M \setminus W"}$ cover M. By playing with collar bump functions and variants, we get smooth functions γ_∂ and γ'_∂ defined on M where γ_∂ is as above, while γ'_∂ is equal to 1 on B'_∂ and is equal to 0 on W'; λ_∂, λ'_∂ denote the functions of the associated smooth partition of unity. Then to prove, for instance, Corollary 6.6, define f_∂ as above, f_U constantly equal to $1/2$ on U, and finally take $f = \lambda_\partial f_\partial + \lambda'_\partial f_U$.

PROPOSITION 6.9. *Let $M \subset \mathbb{R}^h$ be a compact smooth m-manifold with boundary ∂M. Then there is a diffeomorphism $\beta : M \to M' \subset \mathbb{R}^n$ (some n big enough) such that $(M', \partial M')$ is a proper submanifold of $(\mathbf{H}^n, \partial \mathbf{H}^n)$.*

Proof : Take a nice atlas with collar. Define

$$\beta = (\beta_\partial, \beta_1, \dots, \beta_s) : M \to (\mathbb{R}^h \times \mathbb{R}) \times (\mathbb{R}^m \times \mathbb{R})^s := \mathbb{R}^n$$
$$\beta_\partial = (\lambda_\partial \phi_\partial, \lambda_\partial)$$
$$\beta_j = (\lambda_j \phi_j, \lambda_j) \ .$$

We claim that this β works. To show that it is an embedding it is enough to prove that it is an injective immersion. It is an immersion because every $x \in M$ belongs either to B_∂ or to some B_j. The restriction of either $\lambda_\partial \beta_\partial$ or $\lambda_j \beta_j$ is $(\phi_\partial, 1)$ or $(\phi_j, 1)$. In any case it is an injective immersion, so β is an immersion. As for the injectivity, let $x \neq y$. If both belong to either B_∂ or some B_j, then they are already separated by ϕ_∂ or ϕ_j. Otherwise, they are separated by either λ_∂ or some λ_j. Hence β is injective. Finally, it follows by the construction that the image M' of β is contained in $\mathbf{H}^n$ and that $\partial\mathbf{H}^n$ transversely intersects M' at $\partial M = \partial M'$; in fact $\partial M = \partial M'$ is contained in $\partial\mathbf{H}^n$ and there is a small $\epsilon > 0$ such that

$$M' \cap \{x \in \mathbf{H}^n;\ x_n < \epsilon\} = \partial M \times [0, \epsilon) \ .$$

■

REMARKS 6.10. (1) Corollary 6.6 is also a consequence of Proposition 6.9. The function f given by the composition of β with the projection to the x_n coordinate has the required property with value in some $[0, a)$, $a > 0$, and to get $[0, 1)$ is just a simple question of reparametrization.

(2) A proof of Proposition 6.9 can be obtained by using the open covering with associated partition of unity of Remark 6.8. We can take

$$\beta = (\beta_\partial, \beta_U) : M \to (R^h \times \mathbb{R}) \times (\mathbb{R}^h \times \mathbb{R})$$

where β_∂ is as above, $\beta_U = (\lambda'_\partial j_U, \lambda'_\partial)$ and j_U is the inclusion of M in $\mathbb{R}^h$.

The double of M. Let $M' \subset \mathbb{R}^n$ be obtained from M as in the proof of Proposition 6.9. Let M'' be the image of M' via the reflection

$$(x_1, \dots, x_n) \to (x_1, \dots, -x_n) \ ;$$

$\partial M' = \partial M''$. Also M'' is diffeomorphic to M and is a proper submanifold of $\{x_n \leq 0\}$. Then $D(M) := M' \cup M''$ is a compact smooth *boundaryless* manifold, containing both M' and M'' as submanifolds. The boundary $\partial M'$ is given by the *transverse intersection* of $D(M)$ with $\partial\mathbf{H}^n$. Considered up to diffeomorphism, $D(M)$ only depends on M (also considered up to diffeomorphism). It is called the *double of M*.

6.3. A fibration theorem

PROPOSITION 6.11. (Fibration Theorem) *Let M be a compact boundaryless smooth manifold and $f : M \to N$ a surjective submersion to the connected manifold N. Let $q_0 \in N$, $F = f^{-1}(q_0)$. Then f is a smooth fibre bundle with fibre F.*

Proof : Let $q_0 \in N$ and $F = f^{-1}(q_0)$. We know that F is a submanifold of M. Fix a tubular neighbourhood (U, p) of F in M. Let D be a small open disk in N around q_0 such that $f^{-1}(D) \subset U$. Define $h : f^{-1}(D) \to F \times D$, $h(x) = (p(x), f(x))$. Clearly, $f = p_D \circ h$, where p_D is the projection to D. Moreover, $h(x) = (x, 0)$ for every $x \in F$. As f is a submersion, it is easy to verify that the differential of h is invertible on $f^{-1}(D)$ (possibly shrinking D). As h is essentially the identity on F and the fibres are compact, a usual argument (for instance like in the construction of the tubular neighbourhoods) shows that if D is small enough, h is a diffeomorphism and hence a local trivialization of f. If q is an arbitrary point of N, we can cover a smooth arc joining q_0 and q in N by a "chain" of similar local trivializations over a chain $D = D_0, D_1, \dots, D_k$, with D_k around q, of small disks centred at the arc, $D_j \cap D_{j+1} \neq \emptyset$, so that we eventually deduce that the fibre F' over q is diffeomorphic to F. Finally, we have proved that f is a smooth fibration with fibre F.

■

6.4. Density of smooth maps among $\mathcal{C}^r$-maps

Recall that for every $r \geq 0$, $\mathcal{C}^r(M, N)$ denotes the space of $\mathcal{C}^r$ maps endowed with the weak topology; $\mathcal{E}^r(M, N)$ is the subspace of smooth maps.

PROPOSITION 6.12. *Assume that $M \subset \mathbb{R}^h$, $N \subset \mathbb{R}^k$ are boundaryless compact smooth manifolds. Then for every $r \geq 0$, $\mathcal{E}^r(M, N)$ is dense in $\mathcal{C}^r(M, N)$.*

Proof : Let (U_M, p_M) and (U_N, p_N) be respective tubular neighbourhoods. Let $(U, p) \subset (U_M, p_M)$ be a smaller tubular neighbourhood (it just differs by a smaller "ϵ", so that p is the restriction of p_M). Let $f \in \mathcal{C}^r(M, N)$. Consider the $\mathcal{C}^r$ extension $\hat{f} = f \circ p_M$. Apply the Stone-Weierstrass Theorem 1.9 to get a *polynomial* map $P : U_M \to \mathbb{R}^k$ which uniformly approximates (in the $\mathcal{C}^r$-topology) $\hat{f}$ on U (which is compact); we can also require that $P(U) \subset U_N$. Finally, the restriction to M of $p_N \circ P$ is a *smooth* map from M to N which approximates f in the $\mathcal{C}^r$-topology.

■

By a very similar argument we have

LEMMA 6.13. *Let $M \subset \mathbb{R}^h$, $N \subset \mathbb{R}^k$ be compact boundaryless manifolds. If $f \in \mathcal{E}^r(M, N)$ is close enough to $g \in \mathcal{C}^r(M, N)$, then they are $\mathcal{C}^r$-homotopic. If they are both smooth then they are smoothly homotopic.*

Proof : If f is close enough to g, we can assume that, for every $p \in M$, for every $t \in [0, 1]$, $(1-t)g(p) + tf(p)$ belongs to U_N. Then $H(p, t) = p_N((1-t)g(p) + tf(p))$ is a required homotopy.

■

REMARK 6.14. Using Remark 6.2, Proposition 6.12 and Lemma 6.13 hold true if N is the interior of a compact manifold with boundary $\bar{N}$. Clearly they also hold if N is an open set of $\mathbb{R}^k$

6.5. Smooth homotopy groups - Vector bundles on spheres

The above results have the following important application. Fundamental algebraic-topological invariants, the *homotopy groups* $\pi_n(X)$, $n \geq 1$ (considered up to isomorphism) are defined for every path connected topological space X in terms of *continuous* homotopy classes of *continuous* maps $S^n \to X$. If $X = N \subset \mathbb{R}^k$ is as in Remark 6.14, then Proposition 6.12 and Lemma 6.13 imply that we can equivalently define the homotopy groups of N using *smooth* maps $S^n \to N$ up to *smooth* homotopy. If it is necessary to deal with pointed maps, we can do it using the smooth homogeneity of N.

Let us use these facts to classify (up to strict equivalence) the vector bundles on a unit sphere $S^m \subset \mathbb{R}^{m+1}$, $m \geq 2$. We know from Proposition 5.14 that every vector bundle on S^m is of the form $\xi = f^*\tau_{n,k}$, for some smooth map $f : S^m \to \mathfrak{G}_{n,k}$. Let us fix $1 > \epsilon > 0$. Set $D^+ = S^m \cup \{x_{m+1} \geq -\epsilon\}$, $D^- = S^m \cup \{x_{m+1} \leq \epsilon\}$. Clearly, both $D^\pm$ are diffeomorphic to a closed m-disk, $S^m = D^+ \cup D^-$ and $D^+ \cap D^-$ is a tubular neighbourhood of the equatorial sphere $S^{m-1} \subset S^m$, diffeomorphic to $S^{m-1} \times [-1, 1]$. We know by the Classification Theorem that the pull-back of ξ on $D^\pm$ via the respective inclusion maps is strictly equivalent to the product bundle $D^\pm \times \mathbb{R}^k \to D^\pm$. Fix two respective trivializations. The change of trivialization on $D^+ \cap D^-$ produces a smooth map

$$\rho_\xi : D^+ \cap D^- \to \mathrm{GL}(k, \mathbb{R})$$

and we consider its restriction (we keep the name)

$$\rho_\xi : S^{m-1} \to \mathrm{GL}(k, \mathbb{R}) \ .$$

As $D^+ \cap D^-$ is connected, the image of ρ_ξ is contained in one of the two connected components of $\mathrm{GL}(k, \mathbb{R})$ and up to strict equivalence we can assume that this is the subgroup $\mathrm{GL}^+(k, \mathbb{R})$. The arbitrary choices made to define ρ_ξ are the positive scalar ϵ, the representative ξ in its strict equivalence class, and the two trivializations. It is easy to verify (by using the Classification Theorem) that the homotopy class $[\rho_\xi]$ does not depend on these choices, so we have well defined a map

$$\mathbf{V}_{0,k}(S^m) \to [S^{m-1}, \mathrm{GL}(k, \mathbb{R})], [\xi] \to [\rho_{[\xi]}] \ .$$

If $m - 1 > 1$, the (smooth) $\pi_{m-1}(\mathrm{GL}^+(k, \mathbb{R}))$ is Abelian and the choice of a base point is immaterial, so that $[\rho_{[\xi]}] \in \pi_{m-1}(\mathrm{GL}^+(k, \mathbb{R}))$. If $m = 2$, we have to take into account the base points $p_0 = e_1$ of S^1 and $x_0 = I_k$ of $\mathrm{GL}^+(k, \mathbb{R})$ and work with *pointed* smooth maps. However, this is a minor technical point; we can manage it by using the smooth homogeneity of $\mathrm{GL}^+(k, \mathbb{R})$ (we skip the details), so that we can eventually consider again $[\rho_{[\xi]}] \in \pi_1(\mathrm{GL}^+(k, \mathbb{R}))$. Summing up, for every $m \geq 2$, for every $k \geq 1$, we have defined a map

$$\rho : \mathbf{V}_{0,k}(S^m) \to \pi_{m-1}(\mathrm{GL}^+(k, \mathbb{R})) \ .$$

We claim that *this map is bijective*. In fact, we can exhibit ρ^{-1}. Every map $\rho_\xi : S^{m-1} \to \mathrm{GL}^+(k, \mathbb{R})$ extends to a cocycle $\rho_\xi : D^+ \cap D^- \to \mathrm{GL}^+(k, \mathbb{R})$ on the nice covering of the sphere formed by the two smooth disks D^+, D^-. The inverse map ρ^{-1} is obtained by taking the strict equivalence class of the vector bundle over S^{m-1} constructed using this cocycle.

Remark 6.15. The same construction works as well for complex smooth vector bundles on S^m, by replacing $\mathrm{GL}^+(k, \mathbb{R})$ with $\mathrm{GL}(k, \mathbb{C})$ (which is connected), or also for bundles with “reduced group” like, for instance, $SO(k)$.

6.6. Smooth approximation of compact embedded $\mathcal{C}^r$-manifolds

For every $r \geq 0$, there is a natural category of embedded $\mathcal{C}^r$-manifolds and $\mathcal{C}^r$-maps ($\mathcal{C}^r$-diffeomorphisms) between them. When $r = 0$, we have the category of (embedded) *topological manifolds* and continuous maps (homeomorphisms). This presents phenomena (including "wild" ones) that are beyond the aims and the possibilities of this text. On the other hand, we are going to see that to a large extent (at least in the compact case), for $r \geq 1$, there are, essentially, no new phenomena beyond the smooth category. Basically, this depends on the density of smooth maps already established.

For $r \geq 1$, let $M \subset \mathbb{R}^h$ be a boundaryless compact $\mathcal{C}^r$-manifold. The construction of the tubular neighbourhoods of M in $\mathbb{R}^h$ works verbatim in the $\mathcal{C}^r$-category. It is enough to start with a $\mathcal{C}^r$-map $\nu : M \to \mathfrak{G}_{h,h-m}$ defining a distribution of transverse $(h-m)$-planes along M. If we use, for example, the standard metric g_0 on $\mathbb{R}^h$, we obtain only a $\mathcal{C}^{r-1}$-map. However, by applying the same argument of the proof of Proposition 6.12, we can approximate it by a $\mathcal{C}^r$-map, keeping the transversality. Assume that we have fixed one (U, p). We can summarize this by the following commutative diagram (where, for simplicity, we have written τ instead of $\tau_{h,h-m}$):

$$\begin{array}{ccc} U & \xrightarrow{F} & \mathcal{V}(\mathfrak{G}_{h,h-m}) \\ \downarrow p & & \downarrow \tau \\ M & \xrightarrow{\nu} & \mathfrak{G}_{h,h-m} \end{array}$$

where $F = \nu^* \circ (f_\nu)^{-1}$. The map F is a $\mathcal{C}^r$-map and satisfies the following properties (which are easy to check):

- $M = F^{-1}(\mathfrak{G}_{h,h-m})$, where $\mathfrak{G}_{h,h-m} \subset \mathcal{V}(\mathfrak{G}_{h,h-m})$ is the zero section.
- The image of F is contained in the interior of a compact submanifold with boundary of the form $N_\epsilon(\mathfrak{G}_{h,h-m})$ for some $\epsilon > 0$.
- F is *transverse to* $\mathfrak{G}_{h,h-m}$; that is, for every $p \in M$,

$$T_{F(p)}\mathcal{V}(\mathfrak{G}_{h,h-m}) = T_{F(p)}\mathfrak{G}_{h,h-m} + d_pF(T_pU) \ .$$

This means that $M = F^{-1}(\mathfrak{G}_{h,h-m})$ can be considered as a sort of "global equation" defining M, which localizes in terms of very domestic equations: for every given trivialization $\Phi : \tau^{-1}(W) \to W \times \mathbb{R}^{h-m}$ of the tautological bundle, we can consider the restriction of $\Phi \circ F$, obtaining a map

$$(\Phi \circ F)^{-1}(W \times \mathbb{R}^{h-m}) \to W \times \mathbb{R}^{h-m} \ .$$

Let $\pi : W \times \mathbb{R}^{h-m} \to \mathbb{R}^{h-m}$ be the projection. As F is transverse to $\mathfrak{G}_{h,h-m}$, then $\pi \circ \Phi \circ F$ is a submersion (possibly shrinking U), and

$$(\Phi \circ F)^{-1}(W \times \{0\}) = (\pi \circ \Phi \circ F)^{-1}(0) \ .$$

This confirms that M is a submanifold of U of the correct dimension, thanks to Proposition 2.12.

By the density Theorem 6.12, see also Remark 6.14, we can uniformly approximate F (in the $\mathcal{C}^r$-topology) on a slightly smaller compact tubular neighbourhood $U' \subset U$ with a *smooth* map

$$\tilde{F} : U' \to \mathcal{V}(\mathfrak{G}_{h,h-m}) \ .$$

As the transversality is manifestly a $\mathcal{C}^1$-open condition, if $\tilde{F}$ is close enough to F, then it is transverse to $\mathfrak{G}_{h,h-m}$. By applying to $\tilde{F}$ the above construction and Proposition 2.12 again, we conclude that $M' := \tilde{F}^{-1}(\mathfrak{G}_{h,h-m})$ is a compact submanifold of the interior of U', $\dim M' = \dim M$. Moreover, if $\tilde{F}$ is close enough to F, then the restriction of p to M' defines a $\mathcal{C}^r$-diffeomorphism $\rho : M' \to M$. For, as p is the identity on M, this last claim follows by the very same argument used in the construction of the tubular neighbourhood to show that $f_\nu : N_\epsilon(M) \to U$ is a diffeomorphism. Note that M' can be *arbitrarily* $\mathcal{C}^r$*-close to* M, in the sense that the $\mathcal{C}^r$-diffeomorphism $\rho^{-1} : M \to M'$ composed with the inclusion of M' in U' can be arbitrarily close to the inclusion of M in U'.

In summary, we have the following proposition.

Proposition 6.16. (Smooth approximation theorem) *For every $r \geq 1$, for every embedded compact boundaryless $\mathcal{C}^r$-manifold $M \subset \mathbb{R}^h$, there is a smooth manifold $M' \subset \mathbb{R}^h$ $\mathcal{C}^r$-diffeomorphic to M. Moreover, M' can be chosen arbitrarily* $\mathcal{C}^r$-close *to M (i.e. M' is a* smooth approximation *of M in $\mathbb{R}^h$).*

These smooth structures are *unique up to diffeomorphism.*

Proposition 6.17. (Uniqueness of smooth structure) *If M, N are compact boundaryless embedded smooth manifolds which are $\mathcal{C}^r$-diffeomorphic, for some $r \geq 1$, then they are smoothly diffeomorphic.*

Proof : If $f : M \to N$ is a $\mathcal{C}^r$-diffeomorphism, it can be approximated by a smooth map $\tilde{f}$ which is an injective immersion (because $r \geq 1$), hence it is a diffeomorphism.

■

6.7. Sard-Brown theorem

Let us recall some facts of analysis.

(1) Every open set $U \subset \mathbb{R}^n$ is endowed with the (n-dimensional) *Lebesgue measure* and this defines the class of *measure zero* i.e. *negligible* subsets of U.

(2) If $X \subset U$ is negligible and $f : U \to W$ is a $\mathcal{C}^1$-map between open sets of $\mathbb{R}^n$, then $f(X)$ is negligible in W.

(3) If $U' \subset U$ is an open subset and X is negligible in U, then $X \cap U'$ is negligible in U'.

(4) A countable union of negligible subsets of the open set U is negligible.

(5) If X is negligible in the open set U, then $U \setminus X$ is dense in U.

(6) *(Fubini property)* If $U \subset \mathbb{R}^h \times \mathbb{R}^k$, $X \subset U$ and for every $a \in \mathbb{R}^h$, $X \cap \{a\} \times \mathbb{R}^k$ is negligible in $U \cap \{a\} \times \mathbb{R}^k$, then X is negligible in U.

(7) If M is a smooth m-manifold, we say that $X \subset M$ is *negligible in* M if for every chart $\phi : W \to U \subset \mathbb{R}^m$, $\phi(X \cap W)$ is negligible in U.

Thanks to the above properties of negligible sets, to verify that X is negligible in M it is enough to check it for the intersections of X with the open sets of any countable atlas of M. *We stress that we have not defined any measure on M, we have just defined the class of negligible subsets.*

Let $f : M \to N$ be a smooth map between smooth manifolds of dimension m and n, respectively. By definition, a point $p \in M$ is *critical* for f if rank $d_p f < n = \dim N$. Set $C(f) \subset M$ as the set of critical points of M. By definition,

$$N \setminus f(C(f)) \subset N$$

is the set of *regular values* of f, while $q \in f(C(f))$ is said to be a *critical value* of f. The set $M \setminus C(f)$ is open (possibly empty) in M. If M is compact, $f(C(f))$ is compact, hence closed in N.

Theorem 6.18. (Sard's theorem) *Let $f : M \to N$ be a smooth map between smooth manifolds. Then $f(C(f))$ is negligible in N.*

Sard's theorem is a fundamental result for differential topology; in particular, it is the base of *transversality* theory that we will develop later. In differential topological applications, the following corollary, also known as Brown's theorem, is rather used.

Corollary 6.19. (Brown's theorem) *Let $f : M \to N$ be a smooth map between smooth manifolds. Then $N \setminus f(C(f))$ is dense in N (open and dense if M is compact).*

Easy special cases. A special case of Sard's theorem is when $\dim M < \dim N$. Then $C(f) = M$. In this case the proof is easy: clearly M is negligible in $M \times \mathbb{R}^{n-m}$ and $f(M) = f \circ p_M(M)$, $f \circ p_M : M \times \mathbb{R}^{n-m} \to N$, p_M being the projection to M. Then we can apply the above property (2). A special and immediate case of Brown's theorem is when M is the *finite* union of disjoint submanifolds of N of dimensions strictly less than $\dim N$, and f is the union of the inclusion maps.

A very readable proof of Sard's theorem, which *fully* employs the fact that f is $\mathcal{C}^\infty$, is in [**M1**]. We stress that it is an analytic result in nature and rather delicate. To better appreciate this point, let us recall the following Morse-Sard $\mathcal{C}^r$ generalization.

Theorem 6.20. (Morse-Sard theorem) *Let $f : M \to N$ be a $\mathcal{C}^r$-map between smooth manifolds. If $r > \max\{0, m-n\}$, then $f(C(f))$ is negligible in N.*

The condition which relates the "degree of regularity" of f and the dimensions of the manifolds is sharp. Whitney [**Whit**] has constructed an example of a $\mathcal{C}^1$-function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $C(f)$ contains a subset J homeomorphic to an open interval and f is not constant on J. Hence $f(C(f))$ contains an open interval. A proof of the Morse-Sard theorem can be found in [**H**].

6.8. Morse functions via generic linear projections to lines

Let M be a compact boundaryless smooth m-manifold.

DEFINITION 6.21. A smooth function $f : M \to \mathbb{R}$ is a *Morse function* if it has only non-degenerate critical points.

Using the results established in Chapter 1, the notion of non-degenerate critical point p of a certain *index* λ can be defined on any representation in local coordinates of f at p (as it does not depend on the choice of the local coordinates). By the Morse Lemma, the non-degenerate critical points are isolated; hence, by compactness, every Morse function on M has only a finite number of critical points. At least one of them is a minimum (of index $\lambda = 0$) and at least one is a maximum (of index $\lambda = m$). A Morse function on M is *generic* if distinct critical points take distinct (critical) values. In such a case, we can order the critical points $p_0, p_2, \dots, p_r$ so that $c_j := f(p_j) < f(p_{j+1}) =: c_{j+1}$. Up to a linear reparametrization of the image, sometimes we also assume that $f(M) = [0, 1]$.

We want to prove that Morse functions exist and form an open and dense set in $\mathcal{E}(M, \mathbb{R})$.

LEMMA 6.22. *Let M be a compact boundaryless smooth manifold. The set of Morse functions on M is open in $\mathcal{E}(M, \mathbb{R})$.*

Proof : Let $f : M \to \mathbb{R}$ be a Morse function, with critical points $p_1, \dots, p_k$. Fix a nice atlas of M such that every critical point p_j is contained in a B_j of some normal chart and these B_j's are pairwise disjoint. If g is close enough to f (in the $\mathcal{C}^1$ topology), then it has no critical points on the compact set $M \setminus \cup_j B_j$. Let us analyze the local representation of f, $\hat{f}_j$, defined on the compact set $\bar{U}_j := \phi_j(\bar{B}_j) \subset \mathbb{R}^m$, for every $j = 1, \dots, k$. On $\bar{U}_j$, the positive smooth function

$$a_{\hat{f}_j}(x) := ||d_x \hat{f}_j||^2 + (\det(\frac{\partial^2 \hat{f}_j}{\partial x_i \partial x_j}(x))^2$$

never vanishes, because the first term vanishes only at $0 = \phi(p_j)$, and the second term does not vanish because the critical point is non-degenerate. By compactness, there is $d > 0$ such that, for every $x \in \bar{U}_j$, $a_{\hat{f}_j}(x) > d$. If g is close enough to f in the $\mathcal{C}^2$ topology, then $a_{\hat{g}_j}(x) > d/2$, hence also g has only non-degenerate critical points on $\bar{B}_j$. As there is a finite number of critical points of f, we readily conclude that if g is close enough to f in the $\mathcal{C}^2$ topology, then g is a Morse function.

■

We can assume that the compact manifold M is embedded in some Euclidean space. Let $M \subset \mathbb{R}^h$. For every linear function $L \in (\mathbb{R}^h)^*$,

$$L(x) = a_1 x_1 + \cdots + a_h x_h$$

corresponding to $(a_1, \ldots, a_h) \in M(1, h, \mathbb{R})$), consider the restriction L_M to M.

THEOREM 6.23. *Let $M \subset \mathbb{R}^h$ be a compact boundaryless smooth manifold. Then for every $f \in \mathcal{E}(M, \mathbb{R})$, there is an open dense subset $\mathcal{L}_f$ of $(\mathbb{R}^h)^*$ such that for every $L \in \mathcal{L}_f$, $f + L_M$ is a Morse function.*

COROLLARY 6.24. *Let $M \subset \mathbb{R}^h$ be a compact boundaryless smooth manifold. Then:*

(1) There is an open dense set $\mathcal{L}$ in $(\mathbb{R}^h)^*$ such that for every $L \in \mathcal{L}$, L_M is a Morse function.

(2) The set of generic Morse functions is an open dense set in $\mathcal{E}(M, \mathbb{R})$.

Proof of Corollary 6.24. (1) is a consequence of Theorem 6.23 applied to the constant function $f = 0$. Theorem 6.23 together with Lemma 6.22 implies that the set of Morse functions is open and dense in $\mathcal{E}(M, \mathbb{R})$ (if L is close to zero, then $f + L_M$ is close to f). It is evident that the generic Morse functions form an open set in the set of Morse functions. It remains to show that generic Morse functions are dense. Let $f : M \to \mathbb{R}$ be a Morse function. Assume that there is a critical point p which shares its critical value with other critical points. It is enough to show that, arbitrarily close to f, there is a Morse function g with the same set of critical points of f, such that $g(p) \neq g(p')$ for any other critical point p'. Then we conclude by induction on the number of such critical points like p.

Let (W, ϕ) be a normal chart centred at p, such that W does not contain other critical points of f. Let γ be the global bump functions on M associated to this normal chart. For every $\epsilon \neq 0$, set $g_\epsilon = f + \epsilon\gamma$. Clearly, if $|\epsilon|$ is small enough, then g_ϵ is close to f (because M is compact), hence it is a Morse function. It is also clear that g_ϵ coincides with f outside the compact support of γ (contained in W). A discrepancy between the sets of critical points could only occur on the support of γ. But for every $x \in U$, $d_x\hat{g} = d_x\hat{f} + \epsilon d_x\gamma_{1/3,1/2}$. On $B^m(0, 1/3)$ this reduces to $d_x\hat{f}$, hence p is the only critical point of g_ϵ on $B \subset W$ (with the usual notations about normal charts). The function $\hat{f}$ has no critical points on the compact set $\overline{B^m(0, 1/2) \setminus B^m(0, 1/3)}$, hence if $|\epsilon| > 0$ is small enough the same fact holds for g_ϵ. Finally, by the finiteness of the critical set, it is clear that we can take $|\epsilon|$ small enough so that $g_\epsilon(p)$ differs from any other critical value.

■

Before proving Theorem 6.23, let us state a local Lemma.

LEMMA 6.25. *Let $f : U := B^m(0,1) \to \mathbb{R}$ be a smooth function. Then there is a negligible subset X of $(\mathbb{R}^m)^* \sim M(1,m,\mathbb{R})$ such that, for every $L \in (\mathbb{R}^m)^* \setminus X$, $f + L_U$ is a Morse function.*

Proof : For every L, for every $p \in U$, p is a critical point of $f + L_U$ if and only if $d_p f = -L$. Then $-L$ is regular value of *df* if and only if for every $p \in U$ such that $d_p f = -L$,

$$d_p(d_p f) = (\frac{\partial^2 f}{\partial x_i \partial x_j}(p))_{i,j=1,\dots m} \in M(m,\mathbb{R})$$

is invertible. Hence, $-L$ is a regular value of *df* if and only if all the critical points of $f + L_U$ are non-degenerate; that is, $f + L_U$ is a Morse function. We conclude using the Sard-Brown theorem.

■

Proof of Theorem 6.23. Let $M \subset \mathbb{R}^h$ be a compact smooth m-manifold as above. M is covered by a finite set of Monge charts. Possibly reordering the coordinates of $\mathbb{R}^h$, we can assume that every local Monge parametrization of M is of the form

$$U := B^m(0,1) \to (x, \psi(x)) \in M \subset \mathbb{R}^m \times \mathbb{R}^{h-m}$$

so that the associated local representation of f is the map

$$\hat{f}(x_1, \dots x_m) = f(x_1, \dots, x_m, \psi(x_1, \dots, x_m)) \ .$$

Let us write every $L \in M(1,h)$ in the form

$$L(x) = (a_1 x_1 + \cdots + a_m x_m) + (a_{m+1} x_{m+1} + \cdots + a_h x_h) :=$$

$$\alpha(x_1, \dots, x_m) + \beta(x_{m+1}, \dots, x_h) \ ;$$

then the corresponding local representation of $f + L_M$ is

$$(\hat{f}(x_1, \dots, x_m) + \beta(\psi(x_1, \dots, x_m)) + \alpha(x_1, \dots, x_m) := \hat{f}_\beta + \alpha_U \ .$$

For every fixed $\beta \in M(1, h-m, \mathbb{R})$, let us vary $\alpha \in M(1,m,\mathbb{R})$ and apply Lemma 6.25 to $\hat{f}_\beta$. Then for every β, the subset $C_\beta \subset M(1,m,\mathbb{R})$ of α's such that $\hat{f}_\beta + \alpha_U$ is not a Morse function is negligible. The subset C_f of $M(1,h)$ such that the restriction of $f + L_M$ to the given Monge chart is not Morse is the union of the slices C_β, β varying in $M(1, h-m, \mathbb{R})$; hence it is negligible by the Fubini property (6) recalled at the beginning of Section 6.7. As there is a finite number of Monge charts, there is a finite number of such sets C_f in $M(1,h,\mathbb{R})$. The complement $\mathcal{L}_f$ of their union is dense in $M(1,h,\mathbb{R})$ and for every $L \in \mathcal{L}_f$, $f + L_M$ is a Morse function.

■

6.8.1. Manifolds with boundary. Let M be a compact smooth manifold with boundary ∂M, and let us fix a partition $\partial M = V_0 \cup V_1$ as in Corollary 6.7. By this Corollary we know that the set $\mathcal{E}(M, V_0, V_1; \mathbb{R})$ of smooth functions $f : M \to [0, 1]$ such that $f^{-1}(j) = V_j$, $j = 0, 1$, and they have no critical points near ∂M, is nonempty. We can extend the results obtained in the boundaryless case.

PROPOSITION 6.26. *The generic Morse functions in $\mathcal{E}(M, V_0, V_1; \mathbb{R})$ form an open dense set.*

The only point that needs more comments is the existence of such relative Morse functions. By using the notations of Remark 6.8, via the proper embeddings and the double of M, the results in the boundaryless case tell us that there are arbitrarily small linear projections L which restrict to Morse functions on U. If f belongs to $\mathcal{E}(M, V_0, V_1; \mathbb{R})$ and L is small enough, then $\lambda_\partial f + \lambda'_\partial L$ provides a Morse function close to f.

6.9. Morse functions via distance functions

The use of generic linear projections to lines is a geometrically transparent way to produce Morse functions on a compact (embedded) smooth manifold. Here we outline another natural way based on distance functions. Let $M \subset \mathbb{R}^h$ be compact and boundaryless as usual. For every $q \in \mathbb{R}^h$ consider the smooth (actually polynomial) function

$$\delta_q : \mathbb{R}^h \to \mathbb{R},\ \delta_q(x) := ||x - q||^2 .$$

THEOREM 6.27. *There is an open and dense set $\Omega \subset \mathbb{R}^h$ such that, for every $q \in \Omega$, the restriction of δ_q to M is a Morse function.*

Sketch of proof. Consider $\nu : M \to \mathfrak{G}_{h,h-m}$ corresponding to the distribution of normal $(h-m)$-planes for the standard metric g_0 on $\mathbb{R}^h$. Let

$$f_\nu : \nu^*(\mathcal{V}(\mathfrak{G}_{h,h-m})) \to \mathbb{R}^h,\ f_\nu(p, v) = p + v$$

be the map already used to construct a tubular neighbourhood of M in $\mathbb{R}^h$. We can prove that the restriction of δ_q to M has some degenerate critical point if and only if q is not a regular value of f_ν (all details can be found in [**M2**] Part 1-6). Then we conclude by applying the Sard-Brown theorem.

■

6.9.1. Exhaustive sequences of compact submanifolds of non-compact manifolds. The argument of Theorem 6.27 applies also to any boundaryless non-compact submanifold $N \subset \mathbb{R}^h$ which is also a *closed subset* of $\mathbb{R}^h$ (i.e. N is a proper submanifold). Then by using a generic δ_q, we can find a sequence of increasing regular values c_n, $c_n \to +\infty$, of the restriction of δ_q to N such that every

$$N_n := \{x \in N;\ \delta_q(x) \le c_n\}$$

is a compact submanifold with boundary of N, $N_n \subset N_{n+1}$ and $\cup_n N_n = N$. That is, we have an *exhaustive sequence of nested compact submanifolds with boundary of* N. Every compact subset of N is contained in some N_n. In particular, if $f : M \to N$ is a $\mathcal{C}^r$ or an $\mathcal{E}$-map and M is compact, then there is n such that $f(M) \subset N_n$ and we can extend the density result of $\mathcal{E}(M,N)$ in $\mathcal{C}^r(M,N)$. We can also extend to N the notion of tubular neighbourhood. Fix a sequence a tubular neighbourhoods $\pi_n : U_n \to N_n$ constructed using the standard metric g_0 on $\mathbb{R}^h$ and a suitable decreasing sequence of $\epsilon_n > 0$. For every smooth positive function $\epsilon : N \to \mathbb{R}^+$, $N_\epsilon := \{x \in \mathbb{R}^h;\ d(x,N) < \epsilon(x)\}$. We can find such a function ϵ such that $\epsilon(x) < \epsilon_n$ for every $x \in N_n$ so that the projections π_n match with the projection $\pi : N_\epsilon \to N$ such that $\pi(y) \in N$ is the nearest point to y on N.

6.10. Generic linear projections to hyperplanes

Let $M \subset \mathbb{R}^h$ be a compact boundaryless m-manifold as above. We have seen that generic linear projections of M to 1-dimensional subspaces of $\mathbb{R}^h$ are Morse functions. Here we consider projections to hyperplanes, provided that the *codimension* $h-m$ is big enough. Precisely, let $\mathbb{R}^{h-1} \subset \mathbb{R}^{h-1} \times \mathbb{R}$; for every $v \in S^{h-1} \setminus \mathbb{R}^{h-1}$, let $p_v : \mathbb{R}^h = \mathbb{R}^{h-1} \oplus \text{ span}(v) \to \mathbb{R}^{h-1}$ be the associated projection.

PROPOSITION 6.28. *(1) If $h > 2m$, then there is an open dense subset $I_M \subset S^{h-1}$ such that for every $v \in I_M$, the restriction of p_v to M is an immersion.*

(2) If $h > 2m+1$, then there is an open dense subset $E_M \subset S^{h-1}$ such that, for every $v \in E_M$, the restriction of p_v to M is an embedding.

Proof : (1) Let $UT(M) \subset M \times S^{h-1}$ be the total space of the unitary tangent bundle of M (constructed by using the standard metric g_0 on $\mathbb{R}^h$). Let $t : UT(M) \to S^{h-1}$ be the restriction of the projection $M \times S^{h-1} \to S^{h-1}$. The restriction of p_v to M fails to be an immersion if and only if v belongs to the image of t. Clearly, $\dim UT(M) = 2m-1 < h-1$. Hence $S^{h-1} \setminus t(UT(M))$ is open and dense (by the easy case of Sard's theorem). This achieves point (1).

(2) The diagonal Δ is a closed subset of $M \times M$. Consider the smooth map defined on the complementary open set

$$\beta : M \times M \setminus \Delta \to S^{h-1},\ \beta(x,y) = \frac{x-y}{||x-y||}\ .$$

The restriction of p_v to M is not injective if and only if v or $-v$ belongs to the image of β; $\dim(M \times M \setminus \Delta) = 2m < h-1$. Hence $S^{h-1} \setminus \text{Im}(\beta)$ is a dense subset. Its intersection with the dense open set $S^{h-1} \setminus t(UT(M))$ is also dense. Then we have a dense set of v's such that the restriction of p_v is an injective immersion, hence an embedding of M because it is compact. Finally, this set of v's is also open because the set of embeddings is open.

■

As a corollary, we have the following refinement of the embedding of (abstract) compact smooth manifolds. Remarkably, the immersion/embedding dimension only depends on the dimension of the manifold.

Corollary 6.29. (Weak Whitney immersion/embedding theorem) *Every m-dimensional compact smooth manifold M can be immersed in $\mathbb{R}^{2m}$ and can be embedded in $\mathbb{R}^{2m+1}$.*

Proof : If M is boundaryless, it is an immediate consequence of the existence of an embedding in some Euclidean space and of Proposition 6.28. If M has a boundary, we can reduce to the boundaryless case using the double of M.

■

The Morse projections to lines and the above special cases of projections to hyperplanes are the simplest examples of the general problem of understanding "generic" linear projections of compact embedded smooth manifolds to lower-dimensional subspaces. The interested reader can look at the more advanced paper [**Ma**].

6.10.1. Truncated classifying maps. The classification theorem 5.16 has been formulated in terms of the limit Grassmannians $\mathfrak{G}_{\infty,k}$; however, we know that every classifying map $f : M \to \mathfrak{G}_{\infty,k}$ factorizes through some $\hat{f} : M \to \mathfrak{G}_{n,k}$ (similarly for homotopies between maps defining strictly equivalent vector bundles); but, *a priori*, n might vary with M. Arguing similarly to Proposition 6.28, we show that there is a "uniform truncation", depending only on the dimension of M.

Proposition 6.30. *Let M be a compact embedded m-manifold.*

(1) Every $f : M \to \mathfrak{G}_{\infty,k}$ is homotopic to a map g which factorizes through a map $\hat{g} : M \to \mathfrak{G}_{m+k+1,k}$.

(2) Two homotopic classifying maps with values in $\mathfrak{G}_{m+k+1,k}$ are homotopic via a homotopy which factorizes through a map in $\mathfrak{G}_{m+k+2,k}$.

Proof : Start with $\hat{f} : M \to \mathfrak{G}_{n,k}$, with $n > m + k + 1$. Hence the corresponding bundle is embedded in $M \times \mathbb{R}^n$. Consider linear projections $p_v : \mathbb{R}^n \to \mathbb{R}^{n-1}$, as above, and the maps

$$F_v : M \times \mathbb{R}^n \to M \times \mathbb{R}^{n-1},\ (x, v) \to (x, p_v(v))\ .$$

For a generic v, F_v embeds the vector bundle in $M \times \mathbb{R}^{n-1}$; by the classification theorem, this corresponds to a map $M \to \mathfrak{G}_{n-1,k}$ homotopic to the given one. Similar considerations hold for homotopies.

■

6.10.2. On tubular neighbourhood and collars again. We have claimed at the beginning of this chapter that we exploit the embedding in some auxiliary Euclidean space, but the results we obtain hold for arbitrary compact smooth manifolds thanks to the existence of embeddings. This is evident for most results considered so far. The discussion about tubular neighbourhoods of submanifolds and collars deserves some more comments. We have constructed them and proved that they are unique up to isotopy, using a given embedding $M \subset \mathbb{R}^h$. This could depend on the choice of the embedding. However, this is not the case. Every embedding $M \subset \mathbb{R}^h$ can be "stabilized" to $M \subset \mathbb{R}^h \subset \mathbb{R}^{h+k}$; moreover, arguing as above, if k is big enough, up to isotopy, two embeddings of M in $\mathbb{R}^{h+k}$ have disjoint images and can be extended to an embedding of $M \times [0,1]$, so that they are isotopic to each other.

6.11. Approximation by Nash manifolds

We present here a huge refinement due to J. Nash of the approximation theorem of Section 6.6. We believe that this digression has its conceptual interest. However, it is not necessary for the continuation and can be omitted at first reading.

We use the notations of Section 6.6 and also some notions recalled in Section 3.6. We assume that the reader has a basic knowledge of real analytic maps. For the notions of real (semi)-algebraic geometry we refer to [**BCR**], [**BR**].

Dealing with real algebraic sets, $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, a natural class of maps $f : X \to Y$ consists of so-called *regular rational maps* (shortly "algebraic"); that is, restriction of rational maps $r : \mathbb{R}^n \to \mathbb{R}^m$, whose denominators nowhere vanish on X. Let $X \subset \mathbb{R}^k$ be a compact regular real algebraic set of dimension s. Let us specialize the construction of a tubular neighbourhood in this algebraic situation. If we use the standard metric g_0 on $\mathbb{R}^k$, then the associated map $\nu : X \to \mathfrak{G}_{k,k-s}$ is *algebraic*. The map $f_\nu : N_\epsilon(X) \to \mathbb{R}^k$ is algebraic. The pull-back bundle $\nu^*\tau$ is algebraic. Hence the tubular neighbourhood projection $p : U \to X$ is the composition of algebraic maps and of a map obtained by *inverting an algebraic map*. According to Remarks 6.2 and 6.14, these considerations hold also for the tubular neighbourhoods of a compact *regular "semialgebraic" set with boundary*, that is obtained as in Lemma 4.21, assuming that X is a regular real algebraic set and the function f is algebraic (so that the boundary is also a real algebraic set). Then such a projection p is not any smooth map. A basic example of a function of this type is $y = \sqrt{1+x^2}$ and we note that its graph is a branch of the hyperbola defined by the polynomial equation $y^2 - x^2 - 1 = 0$. We would say that such maps belong to the *smallest class of maps containing the algebraic maps, closed by usual algebraic operations and for which the inverse map theorem and its corollaries hold true.* As an algebraic map is *real analytic*, and the inverse map theorem holds for real analytic maps, then p *is at least real analytic.* But we have more. Recall that, by definition, a *semialgebraic set* Y in some $\mathbb{R}^n$ is definable as the union of a *finite* family of subsets of $\mathbb{R}^n$, each one definable as the solution of a finite system of real polynomial inequalities. This extends the notion of algebraic set. Fixing a few technical issues, by developing these considerations we can define the subcategory of

Nash manifolds and maps of the category of smooth embedded manifolds. A Nash m-manifold is an embedded real analytic m-manifold $M \subset \mathbb{R}^n$, for some n, which is also a semialgebraic set; in particular, this implies that M is contained in a real algebraic set X of the same dimension. A Nash map $f : M \to N$ between Nash manifolds is a real analytic map such that its graph is a semialgebraic set. We say that a Nash manifold $M \subset \mathbb{R}^n$ is *normal* if it is contained in the regular part $R(X)$, X being as above. A normal, compact, boundaryless Nash manifold M is a union of connected components of $R(X)$. Although semialgebraic and analytically smooth, in general, M is not normal but it has a *normalization* up to Nash diffeomorphisms. More precisely, we have the following very concrete description of Nash manifolds and maps (see [**AM**]).

PROPOSITION 6.31. *Let $M \subset \mathbb{R}^n$ be a connected Nash m-manifold and $f : M \to \mathbb{R}^h$ be a Nash map. Then there are:*

1. *An irreducible m-dimensional real algebraic set $X \subset \mathbb{R}^n \times \mathbb{R}^k$, for some k;*
2. *A polynomial map $p : X \to \mathbb{R}^h$;*
3. *A Nash manifold $M' \subset M \times \mathbb{R}^k$, such that $M' \subset R(X)$, and it is the graph a Nash map $g : M \to \mathbb{R}^k$, so that $\sigma(x) = (x, g(x))$ is a Nash diffeomorphism;*
4. *$f = p \circ \sigma$.*

If M and N are Nash manifolds, Nash maps form a subspace $\mathcal{N}^r(M,N)$ of $\mathcal{E}^r(M,N)$, for $r \geq 1$ and $\mathcal{N}(M,N)$ of $\mathcal{E}(M,N)$; thanks to the inverse map theorem which holds for Nash maps, a compact Nash manifold M has Nash tubular neighbourhoods (U,p) (U is a compact Nash manifold with boundary, possibly with corners, and p is a Nash map). With the very same proof of Proposition 6.12 we have the following *density of Nash maps.*

PROPOSITION 6.32. (Density of Nash maps) *Assume that $M \subset \mathbb{R}^h$, $N \subset \mathbb{R}^k$ are Nash manifolds, such that M is compact and boundaryless and N is the interior of a compact $\bar{N}$. Then for every $r \geq 1$, $\mathcal{N}^r(M,N)$ is dense in $\mathcal{E}^r(M,N)$ and $\mathcal{N}(M,N)$ is dense in $\mathcal{E}(M,N)$.*

Let $M \subset \mathbb{R}^h$ be a compact smooth boubdaryless m-manifold and consider again the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{F} & \mathcal{V}(\mathfrak{G}_{h,h-m}) \\ \downarrow p & & \downarrow \tau \\ M & \xrightarrow{\nu} & \mathfrak{G}_{h,h-m} \end{array}$$

already used in Section 6.6. The Grassmannian $\mathfrak{G}_{h,h-m}$ is a regular real algebraic set and $N_\epsilon(\mathfrak{G}_{h,h-m})$ is a compact regular semialgebraic set with boundary contained in the regular real algebraic set $\mathcal{V}(\mathfrak{G}_{h,h-m})$, hence we fix for it a Nash tubular neighbourhood $(U_{\mathfrak{G}}, p_{\mathfrak{G}})$. The approximating map $\tilde{F}$ is of the form

$$p_{\mathfrak{G}} \circ P$$

where P is a polynomial map (by application of Stone-Weierstrass); then $\tilde{F}$ is a Nash map close to F and $M' := \tilde{F}^{-1}(\mathfrak{G}_{h,h-m})$ is a Nash manifold $\mathcal{C}^\infty$-close to M. By adapting the very same construction used to give a compact $\mathcal{C}^r$-manifold a smooth structure, we have the following celebrated result by J. Nash [**Na**]. An earlier approximation theorem in this vein is due to Seifert [**Seif**], concerning the case of manifolds with product tubular neighbourhood.

THEOREM 6.33. *(1)* (Nash approximation theorem) *Let $M \subset \mathbb{R}^h$ be a compact connected smooth boundaryless manifold. Then there is a Nash manifold $M' \subset \mathbb{R}^h$ diffeomorphic to M which can be chosen arbitrarily $\mathcal{C}^\infty$-close to M. Up to stabilize the embedding $M \subset \mathbb{R}^h \subset \mathbb{R}^h \times \mathbb{R}^k$, for some suitable k, we can assume that the Nash approximation $M' \subset \mathbb{R}^{h+k}$ is normal, that is M' is a connected component of $R(X)$, $X \subset \mathbb{R}^{h+k}$ being a real algebraic set of the same dimension.*

(2) (Uniqueness of Nash structures) *If two compact embedded boundaryless Nash manifolds $M \subset \mathbb{R}^h$, $N \subset \mathbb{R}^k$ are smoothly diffeomorphic, then they are Nash diffeomorphic to each other.*

REMARKS 6.34. (1) Let M be compact smooth with nonempty boundary ∂M. We can apply the Nash approximation to a double $D(M)$ of M (realized in $\mathbb{R}^n$ as above) and get a boundaryless Nash manifold $D(M)' \subset \mathbb{R}^n$ close to $D(M)$. Then $M' := D(M)' \cap \mathbf{H}^n$ is a Nash model (with boundary) of M.

(2) In his pioneering paper [**Na**], Nash also stated a few conjectures/ questions towards potential improvements of his result. The most natural conjecture was that M can be approximated by a regular real algebraic set, not only by some normal "analytic sheet" of it. We will come back to this in Section 17.5.3. Another question concerned the existence of *rational* real algebraic models; for that, see Sections 15.5, 19.8.

6.11.1. On Nash vector bundle.

By using the classification theorem 5.16, the density of Nash maps, and Lemma 6.13, we readily have the existence and the uniqueness of Nash structures on smooth vector bundles.

PROPOSITION 6.35. *Let M be a compact embedded Nash manifold. Then*

(1) Every embedded smooth vector bundle on M is strictly equivalent to a Nash vector bundle.

(2) If two Nash vector bundles on M are smoothly strictly equivalent, then they are Nash strictly equivalent to each other.

The interested reader can find much information about Nash manifolds in [**BCR**] and mostly in [**Shi**].

6.11.2. A Sard-Brown theorem in the Nash category. Besides its theoretic interest, the approximation by Nash manifolds and the density of Nash maps can be also of practical utility. Whenever we are interested in the density of smooth maps satisfying a certain property, and we are in the condition to apply Nash approximation and density of Nash maps, it will be enough to show that Nash maps with the given property are dense among Nash maps. The main advantage is that we have a much stronger geometric control on the *image* of Nash maps than of arbitrary smooth maps. As an important manifestation of this fact, we discuss here a Nash version of the Sard-Brown theorem, whose statement and proof are purely geometric.

THEOREM 6.36. (Nash-Sard-Brown) *Let $f : M \to N$ be a Nash map between embedded Nash manifolds. Then $f(C(f))$ is the union of a finite set of Nash submanifolds of N of dimensions strictly less than* $\dim N$*.*

REMARK 6.37. Assume that M and N are embedded compact smooth manifolds that we can approximate by Nash manifolds M' and N', so that $\mathcal{N}(M', N')$ is dense in $\mathcal{E}(M', N')$. It follows from the Nash-Sard-Brown Theorem that the set of smooth maps $f : M \to N$ which satisfy Brown's theorem is dense in $\mathcal{E}(M, N)$. In many applications this suffices.

Outline of proof of Theorem 6.36. Like the statement of the theorem, it is purely geometric. For all details, you can look at [**BCR**]. Let us recall the following basic facts about semialgebraic sets:

(1) We know that every embedded Nash manifold is, in particular, a semialgebraic set.

(2) Every semialgebraic set $X \subset \mathbb{R}^n$ is the union of a *finite* number of disjoint connected Nash embedded manifolds.

(3) If $X \subset M$ is a semialgebraic subset of the embedded Nash manifold M, and $f : M \to N$ is a Nash map between Nash manifolds, then $f(X)$ is a semialgebraic subset of N. This is a formulation, adapted to our situation, of the celebrated *Tarski-Seidenberg theorem* that the projection in $\mathbb{R}^{n-1}$ of a semialgebraic set X in $\mathbb{R}^n$ is a semialgebraic set of $\mathbb{R}^{n-1}$. Moreover, all Nash manifolds making a partition of $f(X)$ as in (2) have dimension less than or equal to $\dim M$.

Let us come to the proof of Theorem 6.36. Let $f : M \to N$ be our Nash map between embedded Nash manifolds. As f is a Nash map, it is not hard to check that $C(f)$ is a semialgebraic subset of M. By applying point (2), we realize that $C(f)$ is the *finite* union of disjoint connected Nash submanifolds, each one, Y, satisfying the following property: there exists $0 \leq k < \dim N$ such that for every $p \in Y$, rank $d_p f_{|Y} = k$. The critical value set $f(C(f)) \subset N$ is the union of the images $f(Y)$, hence it is a semialgebraic subset of N. By point (2) again, it is the disjoint union of a finite number of disjoint connected Nash submanifolds of N. We claim that for every such manifold Z, $\dim Z < \dim N$. For example, if $N = \mathbb{R}$, then the restriction of f on every Y has vanishing differential, hence f is constant on Y, so that

$f(C(f))$ is a *finite* subset of $\mathbb{R}$. In general, we can assume that $Z \subset f(Y)$ for some Y as above, and $\dim Z = \dim N$ would be against the *constant rank theorem* 1.7.

■

The Nash-Sard-Brown theorem is an important example of the application of the stronger geometric control on the images of Nash maps. Merely *continuous* maps (between open sets of some Euclidean space) can have "wild" behaviour (i.e. anti-intuitive for an "ordinary" geometric intuition). Let us recall, for example, so-called *Peano's curves*, i.e. *surjective* continuous maps $g : [0, 1] \to [0, 1]^2$. Wild phenomena make the category of topological manifolds much more delicate to deal with. By Sard's theorem (the easy case suffices) *there are not smooth Peano's curves*. In the Nash situation, even better, the image of any such a Nash map g is a finite union of points or connected Nash 1-manifolds. Smooth maps (and manifolds), although much more "tame" than merely $\mathcal{C}^0$ ones, are suited to topological considerations because they are very "flexible". This is due to the existence of bump functions and the *flatness* phenomenon that they incorporate. On the other hand, this also implies, for example, that subsets of a smooth manifold defined by a finite set of smooth equations or inequalities can be weird: for example, it is known that *every* compact subset of $\mathbb{R}^n$ can be realized as the zero set of a smooth function. In a sense, this means that the formulation of the smooth Sard's theorem in *measure* theoretic terms is the best we can say in general about the image of the critical set. The situation is dramatically simpler in the Nash case. It can be profitable to combine the flexibility of smooth manifolds with the Nash approximation and the density of Nash maps (whenever they can be applied).

In this vein, the proof of Theorem 6.23 is purely geometric in the Nash setting. Adopting the notations of that proof, dimensional considerations about C_f can replace Fubini's property, using the Nash-Sard-Brown theorem.

The treatment of Morse functions via distance functions as well as the exhaustive sequences of compact submanifolds specialize directly in the Nash setting.

We end this section with the statement (without proof) of an interesting application of Nash approximation to discrete dynamical systems, in the spirit of the above considerations. Let M be a compact boundaryless smooth manifold, $f : M \to M$ be a smooth map. For every $n \geq 1$, denote by $N_n(f)$ the number of *isolated* fixed points of f^n. We are interested in studying the rate of growth of $N_n(f)$ as n tends to infinity. Let us say that $N_n(f)$ grows at most exponentially if there is a constant $c = c(f) < +\infty$, such that $N_n(f) \leq c^n$, for every $n \geq 1$.

THEOREM 6.38. ([**AM**]) *For every $r \geq 1$, there is a dense subset S_r of $\mathcal{E}^r(M, M)$ such that for every $f \in S_r$, $N_n(f)$ grows at most exponentially.*

CHAPTER 7

Cut and paste compact manifolds

In this chapter, we deal with compact manifolds, or, more generally, with manifolds which can be embedded in some $\mathbb{R}^n$ and also are a closed subset of $\mathbb{R}^n$. Thus we can exploit the results of Chapters 5 and 6.

7.1. Extension of isotopies to diffeotopies

We recall a few notions. Let N be a smooth boundaryless n-manifold. Let M be a smooth m-manifold and

$$F : M \times [0,1] \to N$$

a smooth map such that f_t is an embedding for every $t \in [0,1]$; then F is an *isotopy* connecting f_0 and f_1.

A *diffeotopy* of N (also-called an *ambient isotopy*) is a smooth map

$$G : N \times [0,1] \to N$$

such that g_t is a diffeomorphism for every $t \in [0,1]$. We will also assume that $g_0 = \mathrm{id}_N$. Hence diffeotopies are special isotopies.

Definition 7.1. We say that an isotopy F as above *extends to an ambient isotopy* if there is a diffeotopy G of N such that $f_t = g_t \circ f_0$ for every $t \in [0,1]$. Note that $\{V_t = f_t(M)\}$ is a one-parameter family of submanifolds of N (each diffeomorphic to M), and $V_t = g_t(V_0)$, for every t.

We are going to see that, under mild compactness assumptions, isotopies extend to diffeotopies. This will be a key result to show that several cut-and-paste procedures below are well-defined. To this aim, it is useful to recast diffeotopies as flows of (suitable) vector fields. In doing it, we will tacitly incorporate basic facts about the existence, uniqueness and regular dependence on the initial conditions of the solutions of *ordinary differential equations* (see for instance [**A**]).

For every isotopy F as above, its *track* is the map defined as

$$\hat{F} : M \times [0,1] \to N \times [0,1],\ \hat{F}(x,t) := (f_t(x), t)\ .$$

The *support* of F is the closure in M of the set

$$\{x \in M |\ \exists t \in [0,1],\ f_t(x) \neq f_0(x)\}\ .$$

Given an ambient isotopy G of N and its track $\hat{G}$ (which is a level-preserving diffeomorphism), consider on $N \times [0,1]$ the constant "vertical" tangent vector field V defined by

$$V(x,t) = (0,1) \in T_x N \times \mathbb{R} \ .$$

The tangent map $T\hat{G}$ transforms this field into another tangent vector field on $N \times [0,1]$ of the form

$$X_G(x,t) = (v_G(x,t), 1) \ .$$

The map $\hat{G}$ transforms every vertical integral line $j_x : [0,1] \to N \times [0,1]$ of V such that $j_x(0) = (x,0)$, into the integral line $\hat{j}_x : [0,1] \to N \times [0,1]$ of the field X_G such that $\hat{j}_x(0) = (x,0)$. By construction

$$\hat{G}(j_x(t)) = \hat{j}_x(t) = (g_t(x), t) \ ;$$

that is, G is the *flow* of X_G, with initial values at $N \times \{0\}$. Hence we can reconstruct the diffeotopy G by the integration of the field X_G.

On the other hand, if $v(x,t)$, $t \in [0,1]$, is any *time depending smooth tangent vector field on* N, let $X(x,t) = (v(x,t),1)$ be the corresponding field on $N \times [0,1]$. We say that it has *complete integral lines* if for every initial point $(x,0) \in N \times [0,1]$, the corresponding integral line of X is defined on the whole interval $[0,1]$. If X has complete integral lines, then it *generates a diffeotopy of* N; that is, there is a unique diffeotopy $G = G_X$ such that $X = X_G$. This establishes a bijection between diffeotopies and tangent vector fields X with complete integral lines. If N is not compact, not every X has complete integral lines; by local existence and uniqueness, in general, for every $(x,0)$, there is a maximal interval $[0,t_x) \subset [0,1]$ on which the corresponding integral line is defined. However, if we assume that $v(x,t)$ has *compact support*, then it is not hard to show that X has complete integral lines, and the generated diffeotopy G_X has compact support. Recall that the support of $v(x,t)$ is defined as the closure in N of the set

$$\{x \in N | \ \exists t \in [0,1], \ v(x,t) \neq 0\} \ .$$

Viceversa, if a diffeotopy G has compact support, then v_G also has compact support. This specializes the above bijection to diffeotopies and tangent vector fields with compact support. This gives us a very flexible way to construct diffeotopies, under mild compactness assumptions. Finally, we can state and prove our extension theorem, sometimes known as "Thom's lemma".

PROPOSITION 7.2. *Let* $F : M \times [0,1] \to N$ *be an isotopy of embeddings of the* compact *boundaryless smooth* m*-manifold* M *into the boundaryless* n*-manifold* N*. Then* F *extends to an ambient isotopy of* N *with compact support.*

Proof : Consider the track $\hat{F}$ of the isotopy F. It is a level-preserving embedding of $M \times [0,1]$ to a compact proper submanifold $\hat{M}$ of $N \times [0,1]$.

Consider the constant vertical tangent vector field on $M \times [0,1]$

$$V_M(x,t) = (0,1) \in T_xM \times \mathbb{R} \ .$$

The tangent map $T\hat{F}$ sends V_M to a vector field X_M of the form

$$X_M(y,t) = (v_M(y,t),1), y = f_t(x)$$

defined along $\hat{M}$. The natural idea is to extend X_M to a tangent vector field X of the form

$$X(y,t) = (v(x,t),1)$$

defined on the whole of $N \times [0,1]$ and such that $v(y,t)$ has compact support. The ambient isotopy G_X generated by the field X will eventually extend the isotopy F. This extension task only concerns the "horizontal" part v_M. Under the assumption made at the beginning of this section, we know from Chapter 6 that there is a proper compact tubular neighbourhood U of $\hat{M}$ in $N \times [0,1]$ (which restricts to a tubular neighbourhood of $f_t(M)$ in $N \times \{t\}$ for every $t \in [0,1]$), and a compact submanifold with boundary W of N such that U is contained in $\mathrm{Int}(W) \times [0,1]$. Using the local product structure of U along $\hat{M}$, we can cover $\hat{M}$ by a finite number of smooth closed $(n+1)$-balls B, each one supporting a smooth extension v_B of the restriction of v_M to $B \cap \hat{M}$ and such that their union is contained in U. Such B's can be incorporated in a nice covering with collar $\mathcal{U}$ of $W \times [0,1]$. Locally extend v_M on any open set of such a covering different from the B's by setting it constantly equal to 0. Using a partition of unity supported by $\mathcal{U}$, we finally get the required smooth extension of v_M to a smooth time depending field v defined on the whole of N, constantly equal to zero on the complement of W, and with compact support contained in W.

■

Remarks 7.3. (1) For the sake of simplicity, we have proved Thom's lemma under the assumption that both the compact manifold M and the manifold N are boundaryless. Mild adaptations of the same construction allow extending the results under more general hypotheses. Assuming that both M and N possibly have a boundary, we can cover the following situations, getting a pertinent version of Thom's lemma (details are left to the readers):

(a) F is an isotopy of embeddings of M either in $N \setminus \partial N$ or in ∂N.

(b) F is an isotopy of proper embeddings of $(M, \partial M)$ in $(N, \partial N)$.

(c) Every boundary component of ∂M is embedded by every f_t either in $N \setminus \partial N$ or in ∂N, being $f_t(M)$ transverse to ∂N along $f_t(M)$; for example, this includes the case when, for every t, f_t parametrizes a collar of a compact boundary component of ∂N.

(d) For every $t \in [0,1]$, f_t parametrizes a relative tubular neighbourhood of a compact proper submanifold $(Y, \partial Y)$ of $(N, \partial N)$.

(2) If M is not compact, in general, an isotopy of embeddings of M in N does not extend to any diffeotopy. For example, take $M = \mathbb{R}$ and $N = \mathbb{R}^2$;

then it is easy to construct an isotopy of embeddings connecting f_0, the natural inclusion $\mathbb{R}_x \subset \mathbb{R}^2_{x,y}$, with f_1 having as image the set $\{(x,y);\ x^2 + (y-1)^2 = 1,\ (x,y) \neq (0,2)\}$. For basic topological reasons, it cannot be extended. On the other hand, what is really important to achieve the proof of Thom's lemma is that the isotopy F has compact support, even if M is possibly noncompact.

As a corollary, we have the following sort of relative extension result.

COROLLARY 7.4. *Let Y be a compact submanifold of the manifold M. Let F be an isotopy of embeddings of Y into the manifold N such that a version of Thom's lemma holds. Assume that f_0 can be extended to an embedding $h_0 : M \to N$. Then also f_1 can be extended to an embedding $h_1 : M \to N$; moreover, we can require that h_0 and h_1 are diffeotopic to each other.*

Proof : By Thom's lemma, F extends to a diffeotopy G of N, hence $h_1 := g_1 \circ h_0$ is an embedding of M in N which extends f_1 and is diffeotopic to h_0 by construction.

■

7.2. Gluing manifolds together along boundary components

Let M_1 and M_2 be m-compact manifolds with boundary, V_1 and V_2 unions of connected components of ∂M_1 and ∂M_2, respectively, and let $\rho : V_1 \to V_2$ be a diffeomorphism. Consider the compact topological quotient space

$$M_1 \amalg_\rho M_2$$

by the equivalence relation on the disjoint union $M_1 \amalg M_2$ which identifies every $x \in V_1$ with $\rho(x) \in V_2$; ρ is called the *gluing map*. Denote the projection to the quotient space by

$$q : M_1 \amalg M_2 \to M_1 \amalg_\rho M_2$$

and the inclusion by

$$i_s : M_s \to M_1 \amalg M_2$$

for $s = 1, 2$; finally, set

$$j_s = q \circ i_s \ .$$

It is clear that j_s is a homeomorphism to its image.

PROPOSITION 7.5. *The quotient space $M_1 \amalg_\rho M_2$ can be endowed with the structure of a smooth m-manifold such that, for $s = 1, 2$, j_s is a smooth embedding, and*

$$\partial(M_1 \amalg_\rho M_2) = (\partial M_1 \amalg \partial M_2) \setminus (V_0 \amalg V_1) \ .$$

Proof : Fix a collar $c_1 : [-1, 0] \times V_1 \to M_1$ of V_1 in M_1 and a collar $c_2 : V_2 \times [0, 1] \to M_2$ of V_2 in M_2. Define $\psi_V : (-1, 1) \times V_1 \to M_1 \amalg_\rho M_2$ by

$$\psi_V(t, x) = j_1(c_1(t, x)) \text{ if } t \in (-1, 0],\ \psi_V(t, x) = j_2(c_2(\rho(x), t)) \text{ if } t \in [0, 1) \ .$$

It is clear that ψ_V is a homeomorphism to an open neighbourhood U of

$$V := j_1(V_1) = j_2(V_2)$$

in $M_1 \amalg_\rho M_2$. By composing the charts of a smooth atlas of $(-1,1) \times V_1$ with $\phi_V = \psi_V^{-1}$, we get a smooth atlas $\mathcal{U}_V$ on U such that ψ_V becomes tautologically a diffeomorphism. Similarly, let $\mathcal{U}_s$ be a smooth atlas on $j_s(M_s \setminus V_s)$ such that the restriction of j_s to $M_s \setminus V_s$ is tautologically a diffeomorphism. It is immediate to check that $\mathcal{U}_V \cup \mathcal{U}_1 \cup \mathcal{U}_2$ is a smooth atlas on $M_1 \amalg_\rho M_2$ that determines a smooth manifold structure with the required properties. An equivalent way to get such a smooth structure on $M_1 \amalg_\rho M_2$ is as follows: take the disjoint union $(M_1 \setminus V_1) \amalg (M_2 \setminus V_2)$ and identify the two open sets $c_1((-1,0) \times V_1)$ and $c_2((0,1) \times V_2)$ by identifying $(t,x) \in (-1,0) \times V_1$ with $(1-t, \rho(x)) \in (0,-1) \times V_2$.

■

We say that a smooth structure on $M_1 \amalg_\rho M_2$ obtained so far is given by *gluing M_1 and M_2 together through the gluing map ρ.* Such a smooth structure depends on the choice of collars entering the construction. However, we have the following *uniqueness up to diffeomorphism.*

PROPOSITION 7.6. *Any two smooth structures given by gluing M_1 and M_2 together via the gluing map ρ are diffeomorphic to each other, via a diffeomorphism which is the identity at the boundary.*

Proof : Assume, for example, that two implementations of the construction differ by the choice of two different collars $c_2, c_2' : V_2 \times [0,1] \to M_2$. Denote by M and M' the respective smooth structures on $M_1 \amalg_\rho M_2$. The isotopy (relative to V_2) of the two collars of V_2 in M_2 extends to a diffeotopy G of M_2. Then the map $h : M \to M'$ such that $h = \mathrm{id}_{j_1(M_1)}$ on $j_1(M_1)$, $h = g_1 \circ (j_2)^{-1}$ on $j_2(M_2)$ provides a required diffeomorphism. The general case is achieved by a similar argument.

■

Hence it makes sense to denote by $M_1 \amalg_\rho M_2$ such a diffeomorphism class of smooth manifolds obtained by gluing M_1 and M_2 together. In fact *we will often abuse the notation by confusing such a class with any representative.*

In some cases we can deduce that $M_1 \amalg_\rho M_2$ and $M_1 \amalg_{\rho'} M_3$ are diffeomorphic, where $\rho : V_1 \to V_2$, $\rho' : V_1 \to V_3$ are the respective gluing maps.

PROPOSITION 7.7. *(1) If the diffeomorphism $\rho' \circ \rho^{-1} : V_2 \to V_3$ extends to a diffeomorphism $h : M_2 \to M_3$, then $M_1 \amalg_\rho M_2$ and $M_1 \amalg_{\rho'} M_3$ are diffeomorphic.*

(2) If two gluing maps $\rho_0, \rho_1 : V_1 \to V_2$ are isotopic, then the manifolds obtained by gluing M_1 and M_2 together by means of ρ_0 and ρ_1, respectively, are diffeomorphic to each other.

Proof : A collar of V_3 in M_3, used to define a smooth structure of $M_1 \amalg_{\rho'} M_3$, can be lifted by h to a collar of V_2 in M_2; this can be used to define a smooth structure of $M_1 \amalg_\rho M_2$ which by construction is diffeomorphic to $M_1 \amalg_{\rho'} M_3$. This achieves (1).

As for (2), $\rho_1 \circ \rho_0^{-1}$ is diffeotopic to the identity of V_2 which obviously extends to the identity of the whole M_2. By Corollary 7.4, $\rho_1 \circ \rho_0^{-1}$ also extends to a diffeomorphism of M_2 and we can apply item (1).

■

Oriented version. Keeping the above setting, assume furthermore that M_s is *oriented* and that V_s is part of the *oriented boundary* ∂M_s. If $\rho : V_1 \to V_2$ is an *orientation reversing* diffeomorphism, then $M_1 \amalg_\rho M_2$ is endowed with the structure of an *oriented* smooth m-manifold such that j_1 and j_2 are *orientation-preserving embeddings.* Up to orientation-preserving diffeomorphism, the *oriented* manifold $M_1 \amalg_\rho M_2$ is well defined; it only depends on the isotopy class of the orientation-reversing attaching diffeomorphism ρ.

7.3. On corner smoothing

Here, we re-examine manifolds with corners, already covered in Section 4.10. Using tubular neighbourhoods and collars as in the previous section, it is not hard to see that every compact smooth m-manifold with corner M verifies the following properties:

- M is a topological m-manifold and contains a boundaryless compact smooth $(m-2)$-manifold L (the corner locus) such that $M \setminus L$ is a smooth m-manifold with boundary.
- There is an open neighbourhood U of L in M and a homeomorphism
$$\phi : U \to L \times [0,1) \times [0,1)$$
such that for every $x \in L$, $\phi(x) = (x, 0, 0)$ and the restriction of ϕ to $U \setminus L$ is a diffeomorphism to $L \times [0,1) \times [0,1) \setminus L \times \{(0,0)\}$.

There is a natural corner smoothing *procedure that gives a smooth structure on M which is compatible with the given smooth structures on L and $M \setminus L$.*

To this end, let us fix a homeomorphism $\tau : [0,1) \times [0,1) \to B^2(0,1) \cap \mathbf{H}^2$ which is a diffeomorphism outside $(0,0)$ (for instance, do it by using polar coordinates). Then set

$$\tau' : L \times [0,1) \times [0,1) \to L \times (B^2(0,1) \cap \mathbf{H}^2), \ \tau'(x,y,z) = (x, \tau(y,z))$$

and take the composition $\tau' \circ \phi : U \to L \times (B^2(0,1) \cap \mathbf{H}^2)$. Take on U the differential structure such that $\tau' \circ \phi$ is tautologically a diffeomorphism. A smooth atlas of this structure together with a smooth atlas of $M \setminus L$ make a smooth atlas on M which, by construction, is compatible with the given smooth structures. The induced smooth structure on ∂M coincides, up to diffeomorphism, with the one obtained by gluing the closure of the components of $\partial M \setminus L$ along the common boundary. Arguing similarly to Proposition 7.6, the corner smoothing produces a unique smooth structure up to diffeomorphism.

7.4. Uniqueness of smooth disks up to diffeotopy

Let M be a smooth boundaryless m-manifold; a smooth embedding

$$\beta : D^m \to M$$

of the closed unitary m-disk is called a *smooth m-disk* in M. If M is oriented, two smooth m-disks in M are *co-oriented* if both preserve or reverse the orientation, provided that D^m inherits the standard orientation of $\mathbb{R}^m$.

PROPOSITION 7.8. *Let M be a connected smooth boundaryless m-manifold. For $r = 0, 1$, let $\beta_r : D^m \to D_r \subset M$ be smooth m-disks in M . Then*

(1) If M is oriented and β_0 and β_1 are co-oriented, then there is a diffeotopy of M which connects β_0 and β_1. In particular, there is an oriented smooth automorphism f of M such that $\beta_2 = f \circ \beta_1$.

(2) If M is not orientable, then there is a diffeotopy of M which connects β_0 and β_1. In particular, there is a smooth automorphism f of M such that $\beta_2 = f \circ \beta_1$.

Proof : In both cases, thanks to the homogeneity of M, possibly by composing β_1 with a diffeotopy, we can assume that $x_0 = \beta_0(0) = \beta_1(0)$. Possibly up to radial isotopies centred at 0, we can assume that the images of both β_0 and β_1 are contained in a chart $\phi : W \to \mathbb{R}^m$ of M such that $\phi(x_0) = 0$. Then we are reduced to the case $M = \mathbb{R}^m$, $\beta_r(0) = 0$. Assume that the two disks are co-oriented. Then we can easily adapt the proof of Proposition 1.16 and conclude that both β_r are isotopic to the same linear embedding of the disk in $\mathbb{R}^m$. By applying Thom's lemma we achieve (1).

If M is not orientable, *a priori* the two disks localized in a chart at x_0 as above might be not co-oriented. However, by the non-orientability of M, we can find a smooth simple loop λ based at x_0 such that, by "sliding" β_1 along λ, we return with the opposite orientation. Then, up to isotopy, we can always reduce to two co-oriented disks in $\mathbb{R}^m$ and conclude as before.

■

7.5. Connected sum, shelling

Let us describe a further cut-and-paste procedure to construct compact manifolds.

• Let M_1 and M_2 be boundaryless connected compact smooth m-manifolds, $m \geq 1$.

• For $s = 1, 2$, let

$$\delta_s : D^m \to D_s \subset M_s$$

be a smooth embedding.

• Consider $\tilde{M}_s = M_s \setminus \mathrm{Int}(D_s)$. Then $\tilde{M}_s$ is a compact smooth manifold with one boundary component V_s diffeomorphic to S^{m-1}.

• Let $\rho : V_1 \to V_2$, $\rho = \rho(\delta_1, \delta_2)$ being the diffeomorphism obtained by the restriction of $\delta_2 \circ \delta_1^{-1} : D_1 \to D_2$. Finally, consider the compact boundaryless manifold

$$W := \tilde{M}_1 \amalg_\rho \tilde{M}_2 \ .$$

Here is an equivalent description of the smooth manifold W. Take the disjoint union

$$(M_1 \setminus \delta_1(0)) \amalg (M_2 \setminus \delta_2(0))$$

and for every $(u,t) \in S^{m-1} \times (0,1)$ identify $\delta_1(tv)$ with $\delta_2((1-t)v)$.

Every W obtained by implementing this procedure is called *a connected sum of* M_1 *and* M_2. There is an *oriented* version, where M_1 and M_2 are oriented and $\delta_2 \circ \delta_1^{-1}$ is orientation reversing. The resulting connected sum is naturally oriented in a compatible way with M_1 and M_2.

Every connected sum depends on the choice of the smooth m-disks δ_j. We are going to analyze to which extent it is uniquely defined up to diffeomorphism.

PROPOSITION 7.9. *Let* M_1 *and* M_2 *be boundaryless connected compact smooth m-manifolds. Then*

(1) If both M_1 *and* M_2 *are* oriented*, then the* oriented connected sum $M_1 \# M_2$ *is well defined up to oriented-preserving diffeomorphism (i.e. it does not depend on the choice of the embeddings* δ_s*, provided that* $\delta_2 \circ \delta_1^{-1}$ *reverses the orientation).*

(2) If at least one among M_1 *and* M_2 *is not orientable, then the connected sum* $M_1 \# M_2$ *is well defined up to diffeomorphism (i.e. it does not depend on the choice of the embeddings* δ_s*).*

Proof : If both manifolds are oriented, possibly by pre-composing the smooth disks with the reflection $(x_1, \dots, x_m) \to (-x_1, \dots, x_m)$, we can assume that the m-disks in M_1 preserve while the m-disks in M_2 reverse the orientation; if at least one is nonorientable, say M_1, while M_2 is orientable, then we can assume that the disks in M_2 are co-oriented. By Proposition 7.8, in all cases, the disks in M_1 or M_2, entering different implementations of the connected sum procedure, are diffeotopic to each other. Then the proposition follows by several applications of Proposition 7.7.

■

REMARKS 7.10. (1) When it is well-defined, strictly speaking, $M_1 \# M_2$ denotes a diffeomorphism class of smooth manifolds. Again, we will often abuse the notation by confusing it with any representative.

(2) In the oriented case, if $-M$ denotes the connected oriented manifold M endowed with the opposite orientation, then it can happen that

$M_1 \# M_2$ is not diffeomorphic to $-M_1 \# M_2$, via any orientation preserving-diffemorphism. They are diffeomorphic if there is an orientation-preserving diffeomorphism between M_1 and $-M_1$.

(3) The discussion about the connected sum works as well for compact manifolds with boundary, provided that the disks are embedded in their interior.

7.5.1. Thick connected sum, shelling. Let us keep the above setting. Assume furthermore that M_s is a boundary component of a compact $(m+1)$-manifold N_s. Then we can consider the topological quotient space

$$N_1 \amalg_{\hat{\rho}} N_2$$

where $\hat{\rho} : D_1 \to D_2$ is equal to $\delta_2 \circ \delta_1^{-1}$. Arguing similarly to Section 7.2, we show that this quotient space carries the natural structure of a smooth $(m+1)$-manifold with corners which, by corner smoothing, leads to a well defined smooth manifold denoted

$$N_1 \hat{\#} N_2$$

that is compatible with the smooth inclusions of N_s; moreover

$$\partial(N_1 \hat{\#} N_2) = (\partial N_1 \setminus M_1) \amalg (\partial N_2 \setminus M_2) \amalg (M_1 \# M_2) \ .$$

Everything is well-defined up to diffeomorphism, possibly in the oriented category.

Definition 7.11. In the above setting, if $N_2 = D^{m+1}$, then we say that $N := N_1$ and $\tilde{N} := N \hat{\#} D^{m+1}$ are *related by a shelling* (at $M := M_1$).

Proposition 7.12. *If N and $\tilde{N}$ are related by a shelling, then they are diffeomorphic, as well as $M \# S^m$ is diffeomorphic to M.*

The proof involves several applications of the extension of isotopies and the disk uniqueness as above. We leave the details to the reader.

■

7.5.2. Weak connected sum, twisted spheres. There is a weak variant of the connected sum procedure; by keeping the notations of the beginning of Section 7.5, at the end we take

$$\tilde{M}_1 \amalg_\beta \tilde{M}_2$$

where

$$\beta : V_1 \to V_2$$

is *any* diffeomorphism; that is, we do *not* require that it is the restriction of a composition of m-disks $\delta_2 \circ \delta_1^{-1}$. In the oriented situation, we require also that β reverses the orientation. The essential difference compared with the original procedure is that β does not necessarily extend to a diffeomorphism $\hat{\beta} : D_1 \to D_2$ between the whole embedded smooth m-disks. If we incorporate this last requirement, then the present weak procedure is equivalent to the previous one. Without such a requirement, in general, it is substantially different. We call any manifold obtained by implementing the weak connected sum procedure starting from $M_1 = M_2 = S^m$ a *smooth twisted m-sphere* . We collect below a few (not exhaustive) important facts about this topic.

PROPOSITION 7.13. *(1) If $1 \leq m \leq 4$, then every diffeomorphism $\beta : S^{m-1} \to S^{m-1}$ extends to a diffeomorphism $\hat{\beta} : D^m \to D^m$; hence every m-weak (oriented) connected sum is an (oriented) m-connected sum.*

(2) For every $m \geq 1$, every smooth twisted sphere is homeomorphic *to S^m. If $1 \leq m \leq 4$ it is diffeomorphic to S^m.*

(3) There are smooth twisted 7-spheres that are not diffeomorphic to S^7.

We will limit the following to a few comments, item by item.

(1) For every m, possibly by composing with a reflection along a hyperplane of $\mathbb{R}^{m+1}$, it is not restrictive to assume that β preserves the orientation of S^m.

The validity (or not) of item (1) is invariant on the isotopy class of β.

For $m = 1$, item (1) is immediate via linear parametrizations of the interval D^1.

For $m = 2$, we prove that β is isotopic to the identity (which trivially extends to the identity of D^2). In fact, up to isotopy it is not restrictive to assume that β is the identity on an open sub-arc J of S^1 (diffeomorphic to $(0, 1)$). Let J' be another open sub-arc of S^1 such that $S^1 = J \cup J'$. We get an isotopy of β with the identity as follows

$$H(x, t) = x \text{ if } x \in J, \ H(x, t) = tx + (1 - t)\beta(x) \text{ if } x \in J' .$$

For $m = 3$, item (1), due to Smale [**S1**], is already nontrivial; as above, it is enough to prove that β is isotopic to the identity. A proof can be built by using special dynamical properties of integration of *planar* tangent vector fields, the so-called *Poincaré-Bendixson theory.* Up to isotopy, we can assume that β is the identity on a hemisphere. Via the stereographic projection, it is enough to prove that a diffeomorphism $g : \mathbb{R}^2 \to \mathbb{R}^2$ which is the identity outside the unitary disk D^2 is isotopic to the identity through diffeomorphisms sharing this property. Again up to isotopy, it is not restrictive to assume that these diffeomorphisms are also equal to the identity on a collar of $S^1 = \partial D^2$ in D^2. Consider the constant unitary vertical tangent field on $\mathbb{R}^2$, $\mathfrak{v}_0 = e_2$, and let $\mathfrak{v}_1$ be its image by means of the differential dg. These fields can be considered as smooth maps $\mathfrak{v}_i : D^2 \to \mathbb{C}^*$ (completed by a constant map outside D^2). Via the universal covering map $\exp : \mathbb{C} \to \mathbb{C}^*$, we can lift them to maps $\tilde{\mathfrak{v}}_i : D^2 \to \mathbb{C}$. By taking the convex combinations $\tilde{\mathfrak{v}}_t := t\tilde{\mathfrak{v}}_1 + (1 - t)\tilde{\mathfrak{v}}_0$, $t \in [0, 1]$ and projecting them back to $\mathbb{C}^*$, we get a homotopy $\mathfrak{v}_t$ between $\mathfrak{v}_0$ and $\mathfrak{v}_1$ through nowhere vanishing tangent vector fields which are constant outside D^2 minus a collar of S^1. Now we would integrate the homotopy $\mathfrak{v}_t$ to a diffeotopy between g and the identity. This is a rather delicate task. A key dynamical property which holds in the present situation is that *no maximal integral curves of* $\mathfrak{v}_t$ *are trapped in (the compact set)* D^2. In particular, an integral line which crosses the upper semicircle of S^1 pointing inside D^2, after a certain time crosses the lower semicircle, pointing outside. By elaborating on this fact, one eventually constructs the desired isotopy of diffeomorphisms (for all details see also Section 6.4. of [**Mart**]).

For $m = 4$, (1) is difficult (see [**Ce**]).

(2) It is easy to extend every β as above to a *homeomorphism* $\hat{\beta} : D^m \to D^m$; we can get such a $\hat{\beta}$ by a radial extension, sending, for every $x \in S^{m-1}$, the interval $[x, 0] \subset D^m$ linearly to the interval $[\beta(x), 0]$ (this is also known as the *Alexander trick*). This is a diffeomorphism on $D^m \setminus \{0\}$, the point 0 being, in general, the only non-smooth point. Using this fact, it is easy to show that every twisted m-sphere is homeomorphic to S^m. For $1 \leq m \leq 4$ it is diffeomorphic to S^m thanks to the item (1).

(3) These are the celebrated *Milnor's exotic* 7*-spheres* [**M4**].

Remark 7.14. Let M be a compact oriented boundaryless smooth m-manifold. Let $Y \subset M$ be a submanifold diffeomorphic to S^{m-1} so that $M \setminus Y = M_1 \amalg M_2$ consists of two connected non-compact manifolds. The closure $\hat{M}_s$ of M_s in M is a compact manifold $\hat{M}_s$ with boundary equal to Y. Let us glue to $\hat{M}_s$ a disk D^m via a diffeomorphism $\rho_s : S^{m-1} \to Y$, obtaining two oriented boundaryless manifolds $\tilde{M}_s$. Then

$$M = \tilde{M}_1 \# \tilde{M}_2 \ .$$

In general, this factorization of M is not unique. For example, the standard S^7 can be expressed as $S^7 \# S^7$ as well as the connected sum of two exotic 7-spheres.

7.6. Attaching handles

This is a very important procedure. We will see in Chapter 9 that every compact manifold admits "handle decompositions"; that is, it can be built (up to diffeomorphism) by iterated applications of this basic attaching procedure.

For every $m \geq 0$, for every $0 \leq q \leq m$,

$$H^q = H^{q,m} = D^q \times D^{m-q}$$

is the *standard q-handle of dimension m*. If clear from the context, we will omit to indicate the dimension; q is called the *index* of the handle. Strictly speaking, such a handle H^q is a manifold with corner, with boundary

$$\partial H^q = (S^{q-1} \times D^{m-q}) \cup (D^q \times S^{m-q-1}) \ ;$$

up to smoothing, it is diffeomorphic to D^m endowed with a certain decomposition by submanifolds of $\partial D^m = S^{m-1}$.

Let us fix some terminology.

• $\Sigma_a := S^{q-1} \times \{0\} \subset \mathcal{T}_a := S^{q-1} \times D^{m-q}$ are called, respectively, the *a-sphere* and the *a-tube* of H^q.

• $\Sigma_b := \{0\} \times S^{m-q-1} \subset \mathcal{T}_b := D^q \times S^{m-q-1}$ are called, respectively, the *b-sphere* and the *b-tube* of H^q.

• $C := D^q \times \{0\}$ is called the *core* of the handle.

• $C^* := \{0\} \times D^{m-q}$ is called the *co-core* of the handle.

Note that the a-sphere is the boundary of the core, the b-sphere is the boundary of the co-core; the core and the co-core intersect transversely only at $(0,0)$. $\mathcal{T}_a$ and $\mathcal{T}_b$ intersect at their boundaries, both equal to $S^{q-1} \times S^{m-q-1}$.

Let N be a compact smooth m-manifold with boundary. Given a q-handle H^q of dimension m, let $h : \mathcal{T}_a \to \partial N$ be a smooth embedding. Then $S_a := h(\Sigma_a)$ is the *embedded (attaching) a-sphere* and $T_a := h(\mathcal{T}_a)$ is a tubular neighbourhood of S_a in ∂N, endowed with a *global trivialization* given by h; T_a is called the *embedded (attaching) a-tube*. Consider the topological quotient space

$$N \amalg_h H^q$$

by the equivalence relation on the disjoint union $N \amalg H^q$ which identifies every $x \in \mathcal{T}_a$ with $h(x) \in T_a$. Then $N \amalg_h H^q$ has the natural structure of a manifold with corner which, by smoothing, leads to a smooth manifold well defined up to diffeomorphism. Considered up to diffeomorphism, we say that $N \amalg_h H^q$ is the smooth manifold obtained by attaching a q-handle to N, via the attaching map h. At this point, it is routine to apply , as above, the extension of isotopies to diffeotopies and get the following proposition.

Proposition 7.15. *Up to diffeomorphism, $N \amalg_h H^q$ only depends on the isotopy class of the attaching embedding h.*

Here are a few further comments about attaching handles.

(1) Up to diffeomorphism, the boundary of $N \amalg_h H^q$ is given by

$$\partial(N \amalg_h H^q) = (\partial N \setminus \mathrm{Int}(T_a)) \amalg_{h_{|\partial \mathcal{T}_b}} \mathcal{T}_b \ ;$$

sometimes we denote it by

$$\chi(\partial N, h)$$

and call it the $(m-1)$-manifold obtained by *surgery* on ∂N with *surgery data h*.

(2) If N is oriented and $q > 1$, then $N \amalg_h H^q$ can also be oriented in a compatible way. In fact, as $q > 1$, the a-tube is connected and we can take the orientation of H^q such that the gluing diffeomorphism $h : \mathcal{T}_a \to T_a$ reverses the orientation. For $q = 1$, $\mathcal{T}_a$ is not connected and it is not always possible to make h orientation-reversing on both components. Attaching 1-handles is the only case which imposes some constraints to perform the construction within the oriented category.

(3) If N is connected and $q > 1$, then also $N \amalg_h H^q$ is connected. In fact, as it is connected, T_a is contained in one connected component of ∂N and

connectedness is preserved when attaching H^q, since H^q is connected. By attaching a 1-handle we can reduce the number of connected components by 1. This happens if the connected components of T_a belong to different components of ∂N.

(4) The a-tube of a 0-handle is empty; attaching a 0-handle to N means to "create" a new connected component diffeomorphic to D^m. The a-tube of an m-handle is the whole boundary of D^m. By attaching an m-handle, we fill a spherical component of ∂N (if any such component exists, otherwise we cannot attach any m-handle).

(5) Up to diffeomorphism, the thick connected sum can be rephrased in terms of attaching a 1-handle to N_1 and N_2 with one component of T_a in ∂N_1 and the other in ∂N_2. Similarly, by suitably attaching a 1-handle to

$$(M_1 \times [0,1]) \amalg (M_2 \times [0,1]) \ ,$$

we get a manifold W such that

$$\partial W = (M_0 \amalg M_1) \amalg (M_1 \# M_2)$$

(possibly in the oriented category).

Remark 7.16. Attaching a handle is an instance of the following more general gluing procedure: for $j = 1, 2$, let Y_j be an $(m-1)$ submanifold, with boundary ∂Y_j, of $\partial M_j \subset M_j$. Let $\rho : Y_1 \to Y_2$ be a diffeomorphism. Then $M_1 \amalg_\rho M_2$ is in a natural way an m-manifold with corners, hence a well defined smooth manifold up to corner smoothing (and up to diffeomorphism).

7.7. Strong embedding theorem, the Whitney Trick

This section aims to provide information about the following theorem; the proof introduces the very important so-called "Whitney trick" [**Whit2**].

Theorem 7.17. *Every compact boundaryless smooth m-manifold M can be embedded in $\mathbb{R}^{2m}$.*

We provide only an outline of the proof, though we stress that it is substantially different from the weak immersion/embedding theorem 6.29. This last is entirely based on so-called "general position arguments" or, equivalently, on *transversality* (concepts that we will develop in Chapter 8). By pushing the general position arguments (see Section 8.2), we can at most refine the weak immersion theorem and get that a "generic immersion", $\pi : M \to \mathbb{R}^{2m}$, of our compact boundaryless m-manifold in $\mathbb{R}^{2m}$, has further properties as follows:

The inverse image of every point in $\pi(M) \subset \mathbb{R}^{2m}$ consists of at most 2 points; if $\pi(p) = \pi(p') = q$, then $\mathbb{R}^{2m} = d_p\pi(T_pM) \oplus d_{p'}\pi(T_{p'}M)$. By compactness of M, the number of such "simple normal crossing points", in the image of π, is finite .

We can start with such a generic immersion. If there are normal crossing points, they persist under any small perturbation of the immersion. To get an embedding, we must operate a robust alteration of π. There are two basic "moves":

(1) To introduce, if necessary, a further crossing point.
(2) To eliminate a couple of double points by applying the so-called *Whitney Trick*.

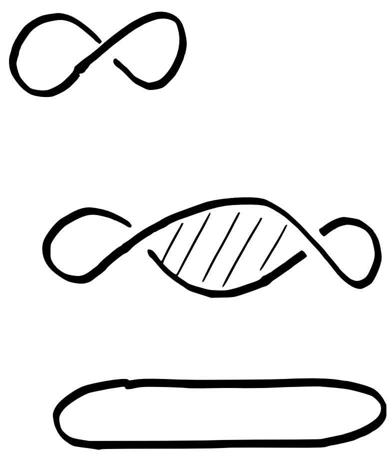

Figure 1. Basic moves for $m = 1$.

In Figure 1 we see a basic application of both moves when $m = 1$. As we are going to see, this scheme works for $m \neq 2$; fortunately, for $m = 2$, the strong embedding theorem holds as a corollary of the *classification of smooth compact surfaces* (see Chapter 15). So we definitively assume here that $m \neq 2$. Moreover, it is not restrictive to assume that M is connected.

• The basic local model for a single self-intersection point of an immersion $\alpha : \mathbb{R}^m \to \mathbb{R}^{2m}$ is as follows:

$$\alpha(t_1, t_2, \cdots, t_m) = \left(t_1 - \frac{2t_1}{u}, t_2, \ldots, t_m, \frac{1}{u}, \frac{t_1 t_2}{u}, \frac{t_1 t_3}{u}, \cdots, \frac{t_1 t_m}{u}\right)$$

where

$$u = (1 + t_1^2)(1 + t_2^2) \cdots (1 + t_m^2) \ .$$

It is an embedding except for the points $(1, 0, \ldots, 0)$ and $(-1, 0, \ldots, 0)$, which are sent to $0 \in \mathbb{R}^{2m}$. Moreover, when $||t|| \to +\infty$, α tends to the usual linear embedding $(t_1, \ldots, t_m) \to (t_1, t_2, \ldots, t_m, 0, \ldots 0)$ of $\mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$. To add such a double point to a given immersion π, we can do it locally in a chart at a point $q \in \pi(M)$ where at $q \sim 0$, $\pi(M)$ looks like the image of the above linear embedding. Then, using two suitable bump functions on $\mathbb{R}^m$, at 0 and at infinity, respectively, and the associated partition of unity, it is not hard to modify π to get one with one more self-intersection point.

REMARK 7.18. Give $\mathbb{R}^m$ and $\mathbb{R}^{2m}$ the standard orientation; then the single self-intersection point has a sign. Its mirror image has the opposite sign.

• The Whitney trick applies at a *Whitney disk* D connecting two crossing points q_1, q_2 in $\pi(M)$. This means that the following pattern is realized:

(1) There is an embedded smooth circle γ in $\pi(M)$ with two corners at q_1 and q_2; these divide γ into two arcs with closures γ_1 and γ_2, respectively; these closed arcs γ_j, $j = 1, 2$, are contained in smooth open m-disks U_j in $\pi(M)$, their union is an open neighbourhood of γ in $\pi(M)$, they intersect transversely at $\{q_1, q_2\}$, and they do not contain other crossing points of $\pi(M)$;

(2) There is:

a 2-disk $\mathcal{D}$ in $\mathbb{R}^2$, with boundary $\partial\mathcal{D}$ and with two corners a_1, a_2, which is contained in the union of two smooth arcs λ_1, λ_2 in $\mathbb{R}^2$ which intersect transversely at $\{a_1, a_2\}$;

an embedding $\psi : U \to \mathbb{R}^{2m}$, where U is an open 2-disk in $\mathbb{R}^2$ containing $\mathcal{D} \cup (\lambda_1 \cup \lambda_2)$, such that

- $\psi(\lambda_j) \subset U_j,\ j = 1, 2;$
- $\psi(\partial\mathcal{D}, \{a_1, a_2\}) = (\gamma, \{q_1, q_2\});$
- for every $x \in \lambda_j$, j=1,2, $d_x\psi(T_xU) \cap T_{\psi(x)}U_j = d_x\psi(T_x\lambda_j);$
- $\psi(\mathrm{Int}(\mathcal{D})) \subset \mathbb{R}^{2m} \setminus \pi(M)$.

We summarize (1) and (2) by saying that the smooth 2-disk with corners $D := \psi(\mathcal{D})$ is *properly embedded* in $(\mathbb{R}^{2m}, \pi(M))$ and *connects the crossing points* q_1, q_2.

Moreover, we require the following

(3) We can extend the embedding ψ to a parametrization of a neighbourhood of D in $\mathbb{R}^{2m}$ by a *standard model*; that is, to an embedding

$$\Psi : U \times \mathbb{R}^{m-1} \times \mathbb{R}^{m-1} \to \mathbb{R}^{2m}$$

such that $\Psi(\lambda_1 \times \mathbb{R}^{m-1} \times \{0\}) = U_1$ and $\Psi(\lambda_2 \times \{0\} \times \mathbb{R}^{m-1}) = U_2$.

Thanks to such a standard model, it is not hard to use a Whitney disk (if any) as a guide to construct a 1-parameter family of immersions, with compact support around D, by "pushing M across D", eventually removing q_1, q_2 without modifying the configuration of the other crossing points.

REMARK 7.19. We can fix local orientations around a Whitney disk. The required properties imply that the two crossing points connected by the disk have *opposite signs* for such orientations.

• To conclude the proof of the embedding theorem, we have to show that for every generic projection, possibly after having inserted a new crossing point (recall Remarks 7.18 and 7.19), there is a pair of crossing points connected by a Whitney disk so that they can be eliminated. For $m = 1$ this follows by somewhat subtle but elementary planar considerations. For

$m > 2$, we will discuss this issue within a larger range of applications of the Whitney trick in Chapter 18 (see Remark 7.20 (2) and Proposition 18.15).

REMARKS 7.20. (1) If $m = 2$, the circle γ can be constructed as well and we could construct a generically immersed disk D in $\mathbb{R}^4$, bounded by γ, but we cannot exclude the existence of crossing points of D itself or of transverse intersection of D with $\pi(M)$ apart from γ.

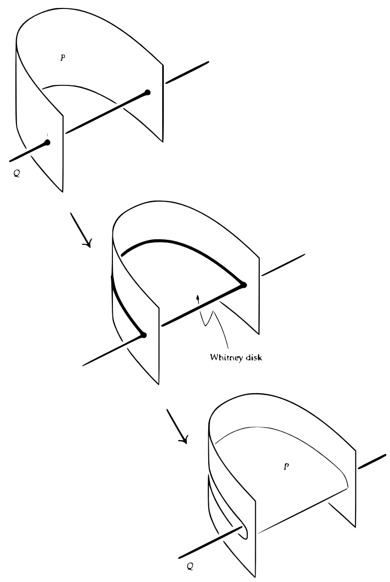

Figure 2. Whitney's trick.

Figure 2 is used with the permission of A. Scorpan ([**Sc**]).

(2) The notion of Whitney disk, and hence the Whitney trick, can be extended to eliminate a couple of transverse intersection points of two submanifolds P, Q of a given manifold M, such that $\dim M = \dim P + \dim Q$ (the boundary loop γ being formed by two arcs in P and Q, respectively). Figure 2 suggests it when $\dim M = 3$. This technique has been of absolute importance in the achievement of fundamental results for smooth manifolds of sufficiently high dimension (see Chapter 18). The fact, noted above, that the scheme does not apply when $\dim M = 4$, is the ultimate reason for special and astonishing phenomena occurring in the realm of 4-manifolds. We will develop these comments much later in the text (see Chapter 20).

7.8. On immersions of n-manifolds in $\mathbb{R}^{2n-1}$

The aim of this section is to provide some information about the following hard immersion theorem [**Whit3**].

THEOREM 7.21. *Every compact boundaryless n-manifold M can be immersed in $\mathbb{R}^{2n-1}$.*

It is not restrictive to assume that M is connected. Similarly to the discussion about the strong embedding theorem, "hard" means that it is not entirely based on general position arguments. These allow (mainly using "jet-and-multi-transversality" - see Section 8.2) to preliminarily determine the *generic maps* $f : M \to \mathbb{R}^{2n-1}$ which, in general, are not immersions. For simplicity, let us give a few details for $n = 2$ (the general case is similar but a bit more complicated). The local models of such a generic map are all realized by

$$g : \mathbb{R}^2_{u,v} \to \mathbb{R}^3_{x,y.z},\ x = u^2,\ y = v,\ z = uv\ .$$

The line $\{v = 0\}$ is the non-injectivity locus of this map and its image is a half-line. The image of every other line $\{v = c\}$ is the parabola $x = (z/c)^2$ in the hyperplane $\{y = c\}$. The point $0 \in \mathbb{R}^2$ is the unique one at which the map g is not an immersion and its image $0 = g(0)$ is called the *Whitney point* in the model. The transverse intersection with the image of g of a small sphere around the Whitney point is a wedge of two circles. The restriction of g to $\mathbb{R}^2 \setminus \{0\}$ is a generic immersion; that is, along the image of $\{v = 0\} \setminus \{0\}$ there are two transverse branches of the image of g.

In general, we can describe qualitatively a generic map $f : M \to \mathbb{R}^{2n-1}$ as follows. Assume first that $n \geq 3$. The image Σ of the non-injectivity locus is a compact 1-dimensional submanifold of $\mathbb{R}^{2n-1}$, possibly with boundary (hence the components of Σ are diffeomorphic to S^1 or to the 1-disk $D^1 = [-1, 1]$ - see Section 9.4); $W = \partial\Sigma$ is formed by the so-called Whitney points of f. The restriction of f to $\tilde{W} := f^{-1}(W)$ is a bijection to its image and f is not an immersion at each point of $\tilde{W}$. $\tilde{\Sigma} := f^{-1}(\Sigma)$ is a smooth compact boundaryless 1-submanifold of M and the restriction of f to $\tilde{\Sigma} \setminus \tilde{W}$ is a double covering map over the interior of Σ. The restriction of f to $M \setminus \tilde{W}$ is a generic immersion, so that locally along every component of the interior of Σ, there are two transverse branches of the image of f.

If $n = 2$ the situation is a bit more complicated. In fact, beyond the Whitney points, Σ has in general also a finite set of three branches crossing points (the "triple points" of the image) at which the local model for the generic immersion of $M \setminus \tilde{W}$ is given by three hyperplanes of $\mathbb{R}^3$ in general position.

These generic maps are *stable* in the sense that their qualitative features are preserved up to small smooth perturbation. Starting from a generic map $f : M \to \mathbb{R}^{2n-1}$, we have to perform a robust alteration of it to get an immersion $\hat{f} : M \to \mathbb{R}^{2n-1}$. The Whitney points are partitioned by pairs of points, the points of each pair being connected by an arc component of Σ. Then we perform a kind of rather subtle "surgery" along each such an arc γ. To give an idea, assume again that $n = 2$ and, for simplicity, the arc γ connecting two Whitney points p_1, p_2 does not include triple points; $\tilde{\gamma} = f^{-1}(\gamma)$ is a smooth circle in M divided in two arcs by $q_j = f^{-1}(p_j) \in \tilde{W}$, $j = 1, 2$. Let us consider a small closed tubular neighbourhood U of $\tilde{\gamma}$ in M. If M is orientable, then U is an annulus (diffeomorphic to $S^1 \times [-1, 1]$);

if M is not orientable, U might be a Möbius band. The restriction f' of f to $M' := M \setminus \mathrm{Int}(U)$ is a generic immersion and an embedding on a collar of $\partial U = \partial M'$ in M'. Moreover, $f'(M')$ can be obtained by removing from $f(M)$ the intersection with a smooth 3-disk $B \subset \mathbb{R}^3$ which retracts to γ and whose boundary transversely intersects $f(M)$ along $f(\partial U)$. Now we want to extend f' to a map $\hat{f} : M \to \mathbb{R}^3$ which is a "proper" immersion on $(U, \partial U)$ in $(B, \partial B)$ (in this way, we eventually eliminate a pair of Whitney points). If $U \sim [-1, 1] \times S^1$ is an annulus, we can assume that it is embedded into the torus $S^1 \times S^1$ or the Klein bottle (both considered as a fibre bundle over S^1 with fibre S^1, the embedding being fibred); then the restriction of $\hat{f}$ on U can be obtained either as the restriction, with values in B, of a standard embedding of the torus or of a standard immersion of the Klein bottle in $\mathbb{R}^3$ without triple points (see Section 19.48). If U is a Möbius band, we can obtain $\hat{f}$ as follows. Consider a (generic) immersion g of the projective plane $\mathbf{P}^2(\mathbb{R})$ in S^3 (it exists, see Section 19.48). Take a small 3-ball $B' \subset S^3$ which intersects transversely $g(\mathbf{P}^2(\mathbb{R}))$ at a 2-disk D in its regular part. The set $B" := S^3 \setminus \mathrm{Int}(B')$ is another smooth 3-disk in S^3, and the restriction g' of g to the Möbius band $\mathcal{M} := \mathbf{P}^2(\mathbb{R}) \setminus g^{-1}(\mathrm{Int}(D))$ is a "proper" immersion of $(\mathcal{M}, \partial\mathcal{M})$ in $(B", \partial B")$. Finally, we identify $(\mathcal{M}, \partial\mathcal{M})$ with $(U, \partial U)$, $(B", \partial B")$ with $(B, \partial B)$, and $(B", g(\partial\mathcal{M}))$ with $(B, f(\partial U))$; we can use g' to extend f' and by doing so along each arc γ we eventually get an immersion $\hat{f} : M \to \mathbb{R}^3$. The maps f and $\hat{f}$ are homotopic; moreover, $\hat{f}$ can be obtained arbitrarily close to the given generic map f in the $\mathcal{C}^0$-topology.

Remark 7.22. The complete proof of Theorem 7.21 in [**Whit3**] is quite demanding. For every n, the local model of a generic map f at a Whitney point is a straightforward generalization of the case $n = 2$. Also, some similar *local* modifications of f at a Whitney point are considered. It is not evident that a suitable pattern of such local modifications can be incorporated into a global immersion $\hat{f}$. This follows from a delicate combinatorial analysis, especially when M is nonorientable.

7.8.1. On Smale-Hirsch immersion theory. Whitney's hard immersion theorem was re-obtained later as a nontrivial application of Hirsch immersion theory [**H2**]. Extending earlier Smale's results concerning the immersions of the spheres, this faces the general questions about the existence of immersions $f : M \to N$, $n = \dim N > \dim M = m$, and the classification of immersions in a given homotopy class of maps from M to N up to regular homotopy.

Definition 7.23. Two immersions $f_0, f_1 : M \to N$ are *regularly homotopic* if they are connected by a smooth homotopy f_t such that for every $t \in [0, 1]$, f_t is an immersion.

Let $M \subset \mathbb{R}^h$, $N \subset \mathbb{R}^k$ be boundaryless smooth properly embedded manifolds; assume that M is compact. If $f : M \to N$ is a smooth map, $[f] \in [M, N]$ denotes its homotopy class.

For every vector bundle ξ on M, we say that *the stable rank of ξ is less or equal to r* and we write

$$\mathrm{rank}_s(\xi) \leq r$$

if there is a rank-r vector bundle η and trivial bundles ϵ^p, ϵ^q on M such that $\xi \oplus \epsilon^p$ is strictly equivalent to $\eta \oplus \epsilon^q$.

We know that

$$\epsilon^h = T(M) \oplus \mathfrak{n}$$

where $\mathfrak{n}$ is the normal bundle on M with classifying map

$$\nu : M \to \mathfrak{G}_{h,h-m},\ \nu(x) = T_xM^\perp$$

for the standard Riemannian metric on $\mathbb{R}^h$.

We have the following existence criterion ([**H2**], [**Tho**]).

THEOREM 7.24. *There exists an immersion $\hat{f} \in [f]$ if and only if*

$$\mathrm{rank}_s(f^*(T(N)) \oplus \mathfrak{n}) \leq \dim N - \dim M$$

The condition is necessary. As M is compact, by the classification theorem, $f^*(T(N)) \sim \hat{f}^*(T(N))$ because f and $\hat{f}$ are homotopic maps. As $\hat{f}$ is an immersion, then

$$\hat{f}^*(T(N)) \sim T(M) \oplus \eta$$

for a certain rank-$(n-m)$ bundle η. Then

$$\hat{f}^*(T(N)) \oplus \mathfrak{n} \sim \eta \oplus (T(M) \oplus \mathfrak{n}) \sim \eta \oplus \epsilon^h\ ,$$

hence $\mathrm{rank}_s(f^*(T(N)) \oplus \mathfrak{n}) \leq n - m$. The other implication is more demanding, being the core of the result.

■

Remarkably, by the classification of vector bundles on compact manifolds, this criterion reduces the question to a question purely in homotopy theory.

When $N = \mathbb{R}^{m+r}$, $r \geq 1$ and for every map f, $f^*(T(N)) = \epsilon^{m+r}$, then there exists an immersion $\hat{f} : M \to \mathbb{R}^{m+r}$ is and only if

$$\mathrm{rank}_s \mathfrak{n} \leq r\ .$$

By the hard Whitney immersion theorem, we know that this condition is satisfied when $r = m - 1$. This can be checked more directly by homotopic considerations, but it is not immediate anyway (see [**H2**]).

The following corollary is immediate.

COROLLARY 7.25. *If M is parallelizable then it can be immersed in $\mathbb{R}^{m+r}$ for every $r \geq 1$.*

For every $m \geq 0$, let $i(m)$ be the minimum $r \geq 1$ such that every compact boundaryless m-manifold M can be immersed in $\mathbb{R}^{m+r}$. We know that $i(m) \leq m-1$. Starting from the reduction of the question to one purely in homotopy theory, after a hard work we eventually know the exact value of $i(m)$; see [**RC**].

THEOREM 7.26. *For every $m \geq 0$, $i(m) = m - \alpha(m)$, where $\alpha(m)$ is the number of* 1 *in the dyadic expansion of m.*

Assume now that $[f] \in [M, N]$ contains immersions. These can be considered up to regular homotopy. Denote by $\mathcal{R}(M, N)_{[f]}$ the set of regular-homotopy classes. If $\hat{f} \in [f]$ is an immersion, $[\hat{f}]_r \in \mathcal{R}(M, N)_{[f]}$ denotes its regular-homotopy class. Denote by $\mathrm{Bun}_{[f]}(T(M), T(N))$ the set of homotopy classes of fibred maps which cover maps homotopic to f and which are injective on each fibre. If $\hat{f} \in [f]$, then the tangent map $T(\hat{f})$ is such a map. One can prove (see [**H2**]) that *the correspondence $[\hat{f}]_r \to [T(\hat{f})]$ well defines a bijection $\mathcal{R}(M, N)_{[f]} \to \mathrm{Bun}_{[f]}(T(M), T(N))$* .

In Section 19.7, we will deal with a materialization of immersion theory of surfaces in 3-manifolds.

7.9. Embedding m-manifolds in $\mathbb{R}^{2m-1}$ up to surgery

By the hard immersion theorem and using multi-transversality (see Section 8.2), we can assume that for every compact boundaryless m-manifold M, there is a *generic* immersion $f : M \to \mathbb{R}^{2m-1}$. So if $m \geq 3$, adopting the above notations, Σ is a compact boundaryless 1-submanifold of $\mathbb{R}^{2m-1}$. For every component C of Σ, locally along C we see two transverse branches of the image of f; $\tilde{C} := f^{-1}(C)$ is a compact boundaryless 1-submanifold of M and the restriction of f to $\tilde{C}$ is a 2-folds covering which might be nontrivial ($\tilde{C}$ connected) or trivial ($\tilde{C}$ with two connected components).

Assume furthermore that M is *orientable*. This section aims to prove the following theorem.

THEOREM 7.27. *Let M be a smooth compact boundaryless orientable m-manifold, $m \geq 3$. Then there exists a compact orientable $(m+1)$-manifold W such that $\partial W = M \amalg \hat{M}$ and $\hat{M}$ can be embedded in $\mathbb{R}^{2m-1}$.*

More precisely, starting from a generic immersion $f : M \to \mathbb{R}^{2m-1}$ as above, we are going to construct W by attaching suitable "round handles" to $M \times [0,1]$ at $M \times \{1\}$, so that $\hat{M}$ is obtained by a kind of "surgery" on M and f can be altered on $\hat{M}$ to get an *embedding* $\hat{f} : \hat{M} \to \mathbb{R}^{2m-1}$. This construction is due to Rohlin (see the translations of his papers in [**GM**]) and it will be used in Chapters 19 and 20.

Let us analyze more closely the properties of such a generic immersion. C has a small tubular neighbourhood $U \sim C \times D^{2(m-1)}$ in $\mathbb{R}^{2m-1}$ which does not intersect the other components of Σ and such that $\tilde{U} := f^{-1}(f(M) \cap U)$ is a tubular neighbourhood of $\tilde{C}$ in M.

As M is orientable, $\tilde{U}$ is a product neighbourhood. There are two models. When the covering $\tilde{C} \to C$ is nontrivial, $\tilde{U}$ can be realized as the mapping cylinder of the map

$$g_0 : \{0,1\} \times D^{m-1} \to \{0,1\} \times D^{m-1},\ g_0(u,x) = (1-u, x)\ ;$$

when the covering $\tilde{C} \to C$ is trivial, $\tilde{U}$ can be realized as the mapping cylinder of the map

$$g_1 : \{0,1\} \times D^{m-1} \to \{0,1\} \times D^{m-1},\ g_1(u,x) = (u,x) \ .$$

In both cases, the mapping cylinder is

$$\{0,1\} \times D^{m-1} \times [0,1]/(u,x,0) \sim (g_j(u,x),1) \ .$$

There are, respectively, two models for $(U, f(M) \cap U)$. Consider $D^{2(m-1)}$ as a subset of $\mathbb{R}^{m-1} \times \mathbb{R}^{m-1}$. Let

$$X := (\{0\} \times D^{m-1}) \cup (D^{m-1} \times \{0\}) \subset D^{2(m-1)}$$
$$h_0 : (D^{2(m-1)}, X) \to (D^{2(m-1)}, X),\ h_0(y,z) = (z,y)$$
$$h_1 : (D^{2(m-1)}, X) \to (D^{2(m-1)}, X),\ h_1(y,z) = (y,z) \ .$$

When the covering is nontrivial, $(U, f(M) \cap U)$ is identified with the mapping cylinder of the map h_0; when the covering is trivial, it is identified with the mapping cylinder of the map h_1. In both cases, the mapping cylinder of the restriction of h_j to X realizes the image $f(\tilde{U})$ in U and the restriction of f to $\tilde{U}$ can be expressed as

$$f(u,x,t) = (ux, (1-u)x, t) \ .$$

As $\mathbb{R}^{2m-1}$ is orientable, then also U must be orientable. This easily implies that the first case does not occur if n is even, because the mapping cylinder would be nonorientable. We have proved

LEMMA 7.28. *If $m = \dim M \geq 3$ is even, then only trivial coverings $\tilde{C} \to C$ can occur.*

We are now going to construct W, $\partial W = M \amalg \hat{M}$, and the embedding $\hat{f} : \hat{M} \to \mathbb{R}^{2m-1}$ with the desired properties. Let C be a component of Σ. Use the above models for $(U, f(M) \cap U)$ and $\tilde{U}$. Consider $\frac{1}{2}\tilde{U} \subset \tilde{U}$ obtained as the mapping cylinder of the restriction of g_j to $\{0,1\} \times \frac{1}{2}D^{m-1}$ and set

$$\tilde{U}' := \tilde{U} \setminus \mathrm{Int}(\frac{1}{2}\tilde{U}) \ .$$

Define the map

$$\hat{f} : \tilde{U}' \to U$$

by

$$\hat{f}(0,x,t) = (\phi(|x|)(-x_1, x_2, \ldots, x_{m-1}), x, t)$$
$$\hat{f}(1,x,t) = (x, \phi(|x|)(-x_1, x_2, \ldots, x_{m-1}), t)$$

where $x = (x_1, \ldots, x_{m-1})$ and

$$\phi : [1/2, 1] \to [0,1]$$

is a smooth strictly decreasing function which coincides with $t \to -t + 3/2$ near $t = 1/2$; $\phi(1) = 0$ and ϕ is flat at 1. For $|x| = 1/2$, the map $\hat{f}$ identifies

$$(u, x_1, x_2, \ldots, x_{m-1}, t) \sim (1-u, -x_1, x_2, \ldots, x_{m-1}, t) \ .$$

For example, when $m = 3$ the image of $\hat{f}$ is defined by the equation

$$yz + \phi(|y|^2 + |z|^2) = 0$$

where we have identified $\mathbb{R}^2 \sim \mathbb{C}$ and $\phi(t) = 3/2 - t$ on $[0, 1/2]$. Case by case, the image of $\hat{f}$ in U is the mapping cylinder of the restriction of h_j to an invariant proper submanifold $\tilde{X}$ of $D^{2(m-1)} \subset \mathbb{R}^{m-1} \times \mathbb{R}^{m-1}$ which coincides with X near the boundary of $D^{2(m-1)}$; $\tilde{X}$ "desingularizes" X. The map $\hat{f}$ extends to the whole of $M \setminus \mathrm{Int}(\frac{1}{2}\tilde{U})$ by taking the restriction of f to $M \setminus \tilde{U}$. Do it for each component of Σ (by using pairwise disjoint tubular neighbourhoods). Thus we have obtained a boundaryless m-submanifold $\tilde{M}$ of $\mathbb{R}^{2m-1}$ which is the image of a smooth map

$$\hat{f} : M_0 \to \mathbb{R}^{2m-1}$$

where M_0 is a submanifold with boundary of M obtained by removing a system of small open tubular neighbourhoods of the $\tilde{C}$'s. It turns out that the quotient

$$\hat{M} := M_0/\hat{f}$$

is, in a natural way, a boundaryless compact manifold and the induced map (we keep the name)

$$\hat{f} : \hat{M} \to \tilde{M}$$

is a diffeomorphism. It remains to describe the "handles" attached to $M \times [0, 1]$ at $M \times \{1\}$, producing an $(m+1)$-manifold W such that $\partial W = M \amalg \hat{M}$. There is one such handle for each component C. If $\tilde{C} \to C$ is the trivial covering, let

$$H = [0, 1] \times D^{m-1} \times [0, 1]/(v, x, 0) \sim (v, x, 1)\ .$$

Then attach H at $M \sim M \times \{1\}$ along $\tilde{U}$, by means of the attaching map which identifies $(0, x, t)$ and $(1, x, t))$ of H with $(0, x, t)$ and $(1, x, t)$ of $\tilde{U}$, respectively. If $\tilde{C} \to C$ is nontrivial (recall that it happens only if m is odd) then we do similarly by using

$$\tilde{H} = [0, 1] \times D^{m-1} \times [0, 1]/(v, x_1, x_2, \ldots, x_{m-1}, 0) \sim (1-v, -x_1, x_2, \ldots, x_{m-1}, 1)\ .$$

This is orientable and the attaching locus is connected. Then W is orientable. This completes the construction and the proof of Theorem 7.27.

■

Remarks 7.29. (1) The constructions and the considerations of this section hold when $f : M \to N$ is any generic immersion of a compact orientable boundaryless m-manifold M in an arbitrary orientable $(2m-1)$-manifold N.

(2) To extend the construction to nonorientable M, we should consider the further case when the covering $\tilde{C} \to C$ is trivial and the componets of $\tilde{U}$ are nonorientable.

7.10. Projectivized vector bundles and blowing up

$\mathbb{R}^n$ can be considered as a vector bundle over the 0-manifold $M = \{0\}$. The projective space $\mathbf{P}^{n-1}(\mathbb{R})$ can be considered as a fibration over M which "projectivizes" the given vector bundle. If

$$\xi := p : E \to M$$

is any vector bundle (for example, the tangent bundle), over a compact m-manifold M with fibre $\mathbb{R}^n$, we can perform the above projectivization fibre by fibre and obtain a fibration

$$\mathbf{p} : \mathbf{P}(E) \to M$$

with fibre $\mathbf{P}^{n-1}(\mathbb{R})$. Every local trivialization $W \times \mathbb{R}^n \sim p^{-1}(W)$ of the vector bundle gives rise to a local trivialization $W \times \mathbf{P}^{n-1}(\mathbb{R}) \sim \mathbf{p}^{-1}(W)$. If (E, p) is defined by means of a cocycle $\{\mu_{i,j} : W_i \cap W_j \to \mathrm{GL}(n, \mathbb{R})\}$, then it induces a cocycle with values in the projectivized linear group $\mathbf{P}\mathrm{GL}(n, \mathbb{R})$ that defines $(\mathbf{P}(E), \mathbf{p})$. The total space $\mathbf{P}(E)$ is a compact manifold of dimension $m+n-1$. A point in $\mathbf{P}(E)$ corresponds to a line l_x in $E_x = p^{-1}(x)$ for some $x \in M$. We can pull-back ξ to $\mathbf{P}(E)$ via the projection $\mathbf{p}$ and obtain the vector bundle $\mathbf{p}^*(\xi)$ over $\mathbf{P}(E)$. We note that the restriction of $\mathbf{p}^*(\xi)$ to every fibre of $\mathbf{p}$ is a product (trivial) bundle. Moreover, $\mathbf{p}^*(\xi)$ has a canonical *tautological* sub-bundle of rank 1 (i.e. a line bundle) λ_ξ : the total space is

$$\Lambda_\xi = \{(l_x, v) \in \mathbf{p}^*(\xi);\ v \in l_x\}$$

with the natural projection to $\mathbf{P}(E)$. Its fibre over l_x is the line contained in the fibre of $\mathbf{p}^*(\xi)$ at l_x, made by the vectors belonging to l_x. By using an auxiliary Riemannian metric on the total space of $\mathbf{p}^*(\xi)$, we realize that, up to strict equivalence, it canonically splits as a direct sum

$$\mathbf{p}^*(\xi) \sim \lambda_\xi \oplus \beta_\xi$$

where the bundle β_ξ is well defined up to strict equivalence. By iterating this construction, starting again using β_ξ, we eventually get the following

PROPOSITION 7.30. *For every vector bundle $\xi : E \to M$ over a compact manifold M, there is a canonical construction (via iterated projectivization of vector bundles) that produces a smooth compact manifold $F(\xi)$ endowed with a surjective smooth map*

$$f_\xi : F(\xi) \to M$$

such that the vector bundle $f_\xi^*(\xi)$ over $F(\xi)$ splits as a direct sum of line bundles. In particular this applies to the tangent bundle of M.

7.10.1. Blowing up along smooth centres. Let us start with the blowing up of $\mathbb{R}^n$, $n \geq 1$, with centre the 0-submanifold $X = \{0\}$. Consider

$$\mathbb{R}^n \times \mathbf{P}^{n-1}(\mathbb{R})$$

where $\mathbb{R}^n$ is endowed with usual coordinates $x = (x_1, \dots, x_n)$, while the projective space is endowed with *homogeneous* coordinates $t = (t_1, \dots, t_n)$. Set

$$\mathbf{B}(\mathbb{R}^n, 0) := \{(x,t) \in \mathbb{R}^n \times \mathbf{P}^{n-1}(\mathbb{R});\ x_i t_j = x_j t_i,\ i,j = 1,\dots,n\}\ ;$$

this is well defined because the equations are homogeneous in the t's. Denote by

$$\rho : \mathbf{B}(\mathbb{R}^n, 0) \to \mathbb{R}^n$$

the restriction of the projection onto $\mathbb{R}^n$. These objects satisfy several interesting properties:

(1) $\mathbf{B}(\mathbb{R}^n, 0)$ *is a smooth n-manifold.* If U_j is the standard chart of the projective space with non-homogeneous coordinates $y_i = t_i/t_j$, $t_j \neq 0$, $i \neq j$, then we readily check that $\mathbf{B}(\mathbb{R}^n, 0) \cap (\mathbb{R}^n \times U_j)$ is given as the graph of the smooth function $x_i = x_j y_i$, $i \neq j$.

(2) *The restriction* $\rho : \mathbf{B}(\mathbb{R}^n, 0) \setminus \rho^{-1}(0) \to \mathbb{R}^n \setminus \{0\}$ *is a diffeomorphism.* Assume that $((a_1, \dots, a_n), (y_1, \dots, y_n)) \in \mathbf{B}(\mathbb{R}^n, 0)$ with some $a_i \neq 0$. Then for every j, $y_j = (a_j/a_i) y_i$ is uniquely determined as a point of $\mathbf{P}^{n-1}(\mathbb{R})$. This also shows that

$$(a_1, \dots, a_n) \to ((a_1, \dots, a_n), (a_1, \dots, a_n)) \in \mathbf{B}(\mathbb{R}^n, 0) \setminus \rho^{-1}(0)$$

defined for $(a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{0\}$ is the inverse diffeomorphism.

(3) *The inverse image* $\rho^{-1}(0) = \{0\} \times \mathbf{P}^{n-1}(\mathbb{R}) \sim \mathbf{P}^{n-1}(\mathbb{R})$ *and it is in natural bijection with the set of lines in* $\mathbb{R}^n$ *passing through* 0*; hence it is the projectivization of* $\mathbb{R}^n$ *considered as a vector bundle over the* 0*-dimensional manifold* $X = \{0\}$. Every such a line L has a parametric equation $x_i = a_i t$, $i = 1, \dots, n$. Consider $L' = \rho^{-1}(L \setminus \{0\})$; L' has parametric equations $x_i = a_i t$, $t_i = a_i t$, $t \neq 0$, $i = 1, \dots, n$. As the t's are homogeneous, equivalently L' is described by $x_i = a_i t$, $y_i = a_i$, $t \neq 0$. These equations extend to define the so-called *strict transform* $\tilde{L}$ of L in $\mathbf{B}(\mathbb{R}^n, 0)$; that is, the closure of L'. Finally, $\tilde{L}$ transversely intersects $\mathbf{P}^{n-1}(\mathbb{R})$ at the point $(a_1, \dots, a_n)$ and

$$L \to \tilde{L} \cap \mathbf{P}^{n-1}(\mathbb{R})$$

defines the required bijection (after all, it corresponds to the bijection between $x \in \mathbf{P}^{n-1}(\mathbb{R})$ and the respective fibre in the tautological bundle over $\mathbf{P}^{n-1}(\mathbb{R})$).

(4) In a more qualitative cut-and-paste fashion, $\mathbf{B}(\mathbb{R}^n, 0)$ is obtained by gluing along the boundary the closure of $\mathbb{R}^n \setminus D^n$ with $\mathbf{B}(D^n, 0)$, and this last can be identified with the *mapping cylinder* of the natural degree two covering map $c : S^{n-1} \to \mathbf{P}^{n-1}(\mathbb{R})$, $S^{n-1} = \partial D^n$.

Consider now $\mathbb{R}^k \subset \mathbb{R}^{k+n} = \mathbb{R}^k \times \mathbb{R}^n$ (defined as usual by the equation $x_i = 0$, $i > k$). The space $\mathbb{R}^{k+n} = \mathbb{R}^k \times \mathbb{R}^n$ can be considered as the total space of the product vector bundle over the manifold $X = \mathbb{R}^k$, with fibre $\mathbb{R}^n$. Then define the *blowing up of* $\mathbb{R}^{k+n}$ *with centre* $X = \mathbb{R}^k$ by

$$\mathbf{B}(\mathbb{R}^{k+n}, \mathbb{R}^k) := \mathbb{R}^k \times \mathbf{B}(\mathbb{R}^n, 0)$$

endowed with the restriction of the natural projection

$$\rho = \rho_{n,k} : \mathbf{B}(\mathbb{R}^{k+n}, \mathbb{R}^k) \to \mathbb{R}^{k+n} \ .$$

The above properties extend directly; set

$$D_{n,k} = \rho^{-1}(\mathbb{R}^k) \ .$$

Then

(1) The restriction $\rho : \mathbf{B}(\mathbb{R}^{k+n}, \mathbb{R}^k) \setminus D_{n,k} \to \mathbb{R}^{k+n} \setminus \mathbb{R}^k$ is a diffeomorphism;

(2) $D_{n,k} = \mathbb{R}^k \times \mathbf{P}^{n-1}(\mathbb{R})$ and it is the total space of the projectivization of the above trivial vector bundle;

(3) $\mathbf{B}(D^n, \mathbb{R}^k)$ is the mapping cylinder of the natural degree two covering map $c : \mathbb{R}^k \times S^{n-1} \to \mathbb{R}^k \times \mathbf{P}^{n-1}(\mathbb{R})$ and $\mathbf{B}(\mathbb{R}^{k+n}, \mathbb{R}^k)$ can be obtained by gluing $\mathbf{B}(D^n, \mathbb{R}^k)$ to the closure of $\mathbb{R}^{k+n} \setminus (\mathbb{R}^k \times D^n)$, along the boundary. Moreover,

(4) If $\mathbb{R}^k \subset \mathbb{R}^{k+h} \subset \mathbb{R}^{k+n}$, $h < n$, then the closure in $\mathbf{B}(\mathbb{R}^{k+n}, \mathbb{R}^k)$ of $\rho^{-1}(\mathbb{R}^{k+h} \setminus \mathbb{R}^k)$ is equal to $\mathbf{B}(\mathbb{R}^{k+h}, \mathbb{R}^k)$, $\rho_{h,k}$ is the restriction of $\rho_{n,k}$, $\mathbf{B}(\mathbb{R}^{k+h}, \mathbb{R}^k)$ transversely intersects $D_{n,k}$ at $D_{h,k}$.

(5) Given $\mathbb{R}^k \times \mathbb{R}^s \times \mathbb{R}^h$, then $\mathbf{B}(\mathbb{R}^{k+s}, \mathbb{R}^k)$ and $\mathbf{B}(\mathbb{R}^{k+h}, \mathbb{R}^k)$ are disjoint submanifolds of $\mathbf{B}(\mathbb{R}^{k+s+h}, \mathbb{R}^k)$. Note that $\mathbb{R}^{k+s} \cap \mathbb{R}^{k+h} = \mathbb{R}^k \subset \mathbb{R}^{k+s+h}$, hence $\mathbb{R}^{k+s} \cup \mathbb{R}^{k+h}$ is 'singular' along $\mathbb{R}^k$. Blowing up with centre $\mathbb{R}^k$ is a way to 'desingularize' it.

Let M be a compact boundaryless smooth $(k+n)$-manifold and $X \subset M$ a proper k-submanifold. We define the *blowing up of M with centre X*

$$\rho = \rho_{M,X} : \mathbf{B}(M, X) \to M$$

as follows: recall that a tubular neighbourhood

$$\pi : U \to X$$

of X in M is, by construction, isomorphic to a neighbourhood fibred by n-disks of the 0-section (identified with X) of a rank k vector sub-bundle

$$p : E \to X$$

of the restriction of $T(M)$ to X, such that

$$\partial\pi : \partial U \to X$$

is isomorphic to the unitary bundle

$$up : UE \to X$$

with fibre S^{n-1}. There is a natural degree 2 covering map

$$c : UE \to \mathbf{P}(E)$$

such that $up = \mathbf{p} \circ c$. Then $\mathbf{B}(M, X)$ is obtained by gluing the mapping cylinder of this map c to the closure of $M \setminus U$, along its boundary. The above $(\mathbf{B}(\mathbb{R}^{k+n}, \mathbb{R}^k), \rho_{n,k})$ provides the local model for $(\mathbf{B}(M, X), \rho_{M,X})$, so that

• $\mathbf{B}(M, X)$ is a smooth compact $(k+n)$-manifold and ρ is a smooth map;

• The restriction

$$\rho : \mathbf{B}(M, X) \setminus D(M, X) \to M \setminus X$$

is a diffeomorphism, where $D(M, X) = \rho^{-1}(X)$.

• The restriction $\rho : D(M, X) \to X$ is isomorphic to the projectivized bundle $\mathbf{p} : \mathbf{P}(E) \to X$.

REMARK 7.31. If X is a hypersurface of M ($\dim X = \dim M - 1$), then $\rho : \mathbf{B}(M, X) \to M$ is a global diffeomorphism.

If Y is a subset of M, the *strict transform* $\tilde{Y}$ of Y in $\mathbf{B}(M, X)$ is, by definition, the closure of $\rho^{-1}(Y \setminus X)$. Then we have the following.

• Let M be as above, $N \subset M$ a proper submanifold of M and $X \subset N$ a proper submanifold of N (whence of M). Then the strict trasform $\tilde{N}$ in $\mathbf{B}(M, X)$ is a proper submanifold diffeomorphic to $\mathbf{B}(N, X)$, moreover, $\tilde{N}$ transversely intersects $D(M, X)$ at $D(N, X)$.

• If N and N' are proper submanifolds of M which intersect transversely at $X = N \cap N' \neq \emptyset$, then the strict transforms $\tilde{N}$ and $\tilde{N}'$ are disjoint in $\mathbf{B}(M, X)$. Note that $N \cup N'$ is not a submanifold of M because it is 'singular' along X. By blowing up the singularity and taking the strict transforms, we can 'desingularize' it.

7.10.2. Blowing-up and connected sum. When $X = \{x_0\} \subset M$ is reduced to one point, blowing up X is related to the connected sum.

PROPOSITION 7.32. *(1) If* $\dim M = m$ *is even, then* $\mathbf{B}(M, x_0) \sim M \# \mathbf{P}^m(\mathbb{R})$ *(recall that* $\mathbf{P}^m(\mathbb{R})$ *is not orientable).*

(2) If M *is oriented and* $\dim M = m$ *is odd, then: (a)* $\mathbf{B}(M, x_0)$ *is oriented in such a way that the restriction*

$$\rho : \mathbf{B}(M, x_0) \setminus D(M, x_0) \to M \setminus \{x_0\}$$

preserves the orientation; (b) Let us stipulate that S^m *is oriented as the boundary of* D^{m+1} *oriented by the standard orientation of* $\mathbb{R}^n$ *and that* $\mathbf{P}^m(\mathbb{R})$ *is oriented in such a way that the standard covering map* $S^m \to \mathbf{P}^m(\mathbb{R})$ *preserves the orientation; then*

$$\mathbf{B}(M, x_0) \sim M \# - \mathbf{P}^m(\mathbb{R})$$

where '$-$' indicates the opposite orietation and we are dealing with the oriented connected sum.

Proof : Forget, for now, the orientation issue. By taking a chart diffeomorphic to $\mathbb{R}^m$ of M at $x_0 \sim 0$, we can assume that D^m is a tubular neighbourhood of x_0. Recall that $\mathbf{B}(D^m, 0) \subset \mathbb{R}^m \times \mathbf{P}^{m-1}(\mathbb{R})$, this last endowed with 'mixed' coordinates (x, t). Let $z = (z_1, \dots, z_{m+1})$ be homogeneous coordinates on $\mathbf{P}^m(\mathbb{R})$ and take the standard affine chart $U = \{t_{m+1} \neq 0\}$;

$U \sim \mathbb{R}^m$, with coordinate $y_1 = (z_1/z_{m+1}, \dots, y_m = z_m/z_{m+1})$. Then it is enough to prove that there is a diffeomorphism

$$\phi : \mathbf{B}(D^m, 0) \to \overline{\mathbf{P}^m(\mathbb{R}) \setminus D^m}$$

which is the identity on ∂D^m. The diffeomorphism ϕ can be defined explicitly as follows:

$$\phi(x_1, \dots, x_m, t_1, \dots, t_m) = (t_1, \dots, t_m, t(\sum_{j=1}^{m} x_j^2)) \in \mathbf{P}^m(\mathbb{R})$$

where $t = t_i/z_i$ if $z_i \neq 0$, $i = 1, \dots, m$. The verification that ϕ is well defined, its image is $\overline{\mathbf{P}^m(\mathbb{R}) \setminus D^m}$, and that it is a diffeomorphism are left to the reader as an exercise. Now we come to the orientation question. If m is even then $\mathbf{P}^m(\mathbb{R})$ is nonorientable, hence the connected sum with it is well defined. In the oriented case, we easily check that $\mathbf{B}(D^m, 0)$ and $\overline{\mathbf{P}^m(\mathbb{R}) \setminus D^m}$ induce opposite orientations on the common boundary ∂D^m. Hence the diffeomorphism ϕ reverses the orientation and (b) follows.

■

7.10.3. On complex blowing up. The (complex) blowing up

$$\mathbf{B}_{\mathbb{C}}(M, X)$$

can be performed in the category of complex manifolds as well. At least, the basic $\mathbf{B}_{\mathbb{C}}(\mathbb{C}^n, 0)$ is defined by the very same formulas of $\mathbf{B}(\mathbb{R}^n, 0)$, in terms of complex coordinates. Hence we can define the blowing up $\mathbf{B}_{\mathbb{C}}(M, x_0)$ of a complex manifold at a point x_0. More generally, if M is an *oriented* $2n$-smooth manifold and $x_0 \in M$, we can define $\mathbf{B}_{\mathbb{C}}(M, x_0)$ (up to oriented diffeomorphism) by taking an oriented chart $\mathbb{R}^{2n} \sim \mathbb{C}^n$ at $x_0 \sim 0$, performing $\mathbf{B}_{\mathbb{C}}(D^{2n}, 0)$ and gluing it to $\overline{M \setminus D^{2n}}$.

PROPOSITION 7.33. *Let M be a compact oriented $2n$-manifold, $x_0 \in M$. Then $\mathbf{B}_{\mathbb{C}}(M, x_0) \sim M \# - \mathbf{P}^n(\mathbb{C})$.*

Proof : As in the proof of Proposition 7.32, the key point is to construct a suitable diffeomorphism

$$\phi_{\mathbb{C}} : \mathbf{B}_{\mathbb{C}}(D^{2n}, 0) \to \overline{\mathbf{P}^n(\mathbb{C}) \setminus D^{2n}}\ .$$

The formula for ϕ defined in that proof works here as well, provided that it is considered in terms of the complex coordinates and we replace each x_j^2 with $|x_j|^2$.

■

REMARK 7.34. Blowing up works in the category of (real or complex) regular algebraic varieties. Algebraic geometry is the first source of this construction and we have just developed a smooth version. In the algebraic setting, $\mathbf{B}(M, X) \setminus D(M, X)$ is a Zariski open set of the regular algebraic variety $\mathbf{B}(M, X)$ as well as $M \setminus X$ is a Zariski open set of the regular algebraic variety M; the restriction of ρ is an algebraic isomorphism between these Zariski open sets, hence (essentially by definition) M and $\mathbf{B}(M, X)$ are *birationally equivalent*. A variety M is said to be *rational* if it is birationally equivalent to the projective space of the same dimension. Blowing up a projective space along regular centres is a basic way to construct rational varieties.

CHAPTER 8

Transversality

We have already used some instances of transversality and related concepts. In this chapter, we will treat this topic more systematically. First, we point out so-called "basic transversality theorems" which, to a large extent, will suffice to our aims. Then we will develop some complements.

8.1. Basic transversality

We consider the following setting.

• M is a smooth m-manifold with (possibly empty) boundary ∂M;

• N is a smooth boundaryless n-manifold and $Z \subset N$ is a proper r-submanifold of N, hence Z is both boundaryless and a closed subset of N;

• $f : M \to N$ is a smooth map. If the boundary is nonempty, then ∂f denotes the restriction of f to ∂M.

DEFINITION 8.1. We say that f is *transverse to* Z (and we write $f \pitchfork Z$) if

(1) For every $x \in M$ such that $y = f(x) \in Z$, we have

$$T_yN = T_yZ + d_xf(T_xM) \ .$$

(2) For every $x \in \partial M$ such that $\partial f(x) \in Z$, we have

$$T_yN = T_yZ + d_x\partial f(T_x\partial M) \ ;$$

(in other words, $\partial f \pitchfork Z$ by itself). If $\partial M = \emptyset$, this second requirement is empty.

We denote by $\pitchfork (M, N; Z)$ the subspace of $\mathcal{E}(M, N)$ formed by the maps transverse to Z. If A is a subset of M, we denote by $\pitchfork_A (M, N; Z)$ the space of maps which satisfy the transversality conditions for every $x \in A$ or $x \in A \cap \partial M$, so that $\pitchfork (M, N; Z) = \pitchfork_M (M, N; Z)$.

Let us consider some particular cases.

• If $f(M) \cap Z = \emptyset$, then $f \pitchfork Z$;

• If $Z = \{y_0\}$ a single point, then $f \pitchfork Z$ if and only if y_0 is a regular value of both f and ∂f.

• If M is a boundaryless submanifold of N and f is the inclusion, then $f \pitchfork Z$ (and we write also $M \pitchfork Z$) if and only if for every $x \in M \cap Z$, $T_xN = T_xM + T_xZ$; if $\dim M + \dim Z = \dim N$, then $T_xN = T_xM \oplus T_xZ$.

• The basic local models for $M \pitchfork Z$ are given by the possible mutual position of two affine subspaces, A and B, in some $\mathbb{R}^n$. If $\dim A + \dim B < n$, then $A \pitchfork B$ if and only if $A \cap B = \emptyset$. If $A \cap B \neq \emptyset$, up to translation we can assume that they are linear subspaces which are transverse if and only if $\mathbb{R}^n = A + B$. Note that $A \cap B$ is also a linear subspace and, by elementary linear algebra, $\dim A \cap B = \dim A + \dim B - n \geq 0$.

There are two kinds of basic transversality theorems; roughly speaking, they respectively claim that transversality implies nice geometric features of the map f and that (at least when M is compact) it is a *generic and stable* property: up to arbitrarily small perturbation every map f becomes transverse, and transversality cannot be destroyed by small perturbations.

THEOREM 8.2. (First basic transversality theorem) *(1) If $f : M \to N$ is transverse to Z, then $(Y, \partial Y) := (f^{-1}(Z), \partial f^{-1}(Z))$ is a proper submanifold of $(M, \partial M)$; moreover $\dim M - \dim Y = \dim N - \dim Z$.*

(2) If $(M, \partial M)$, N and Z are oriented, then Y and ∂Y are orientable and we can fix an orientation procedure in such a way that ∂Y is the oriented boundary of Y.

Proof : When $Z = \{y_0\}$ consists of one point, then the theorem is equivalent to Proposition 4.24. Let us reduce the general to this special case. As Z is a closed subset, then $f^{-1}(Z)$ and $\partial f^{-1}(Z)$ are also closed sets. Now, being a proper submanifold is a local property. For every $z \in Z$ there is a chart of N, $\phi : W \to U \times U' \subset \mathbb{R}^r \times \mathbb{R}^{n-r}$, such that $\phi(z) = 0 \in U \times U'$ and $\phi(W \cap Z) = U \times \{0\}$. Let $p : U \times U' \to U'$ be the projection. The restriction of f to $f^{-1}(W)$ is transverse to Z if and only if $p \circ \phi \circ f$ is transverse to $\{0\}$. This is enough to achieve point (1). As for the orientation, let us orient $\mathbb{R}^{n-r}$ is such a way that the given orientation of $\mathbb{R}^n$ (i.e. of N) is the direct sum of the given orientation of $\mathbb{R}^r$ (i.e. of Z) followed by the selected one on $\mathbb{R}^{n-r}$. Then we can apply to $p \circ \phi \circ f$ the orientation rule of point (2) of Proposition 4.24 to orient the intersection of $(Y, \partial Y)$ with W; by construction, these local orientations are globally coherent.

■

REMARK 8.3. It is useful to make explicit the orientation rule in the case of the transverse intersection $M \pitchfork Z$ of submanifolds of N. For every $x \in M \cap Z$, $T_xN = T_xM + T_xZ$, and by assumption the linear spaces T_xN, T_xM and T_xZ are oriented (in a globally coherent way) and the last two intersect transversely in the first. We have to orient $T_xM \cap T_xZ$. So we have reduced the problem to the basic situation of two transverse oriented linear subspaces (A, ω_A) and (B, ω_B) in $\mathbb{R}^n$ (endowed with the standard orientation ω_n). Given any orientation $\omega_{A \cap B}$ on the intersection, it can be extended in a unique way to A and B so that $\omega_A = \omega_{A \cap B} \oplus \omega'$ and $\omega_B = \omega_{A \cap B} \oplus \omega$". Then $\omega_{A \cap B} \oplus \omega' \oplus \omega$" determines an orientation on the whole $\mathbb{R}^n$. Finally, we select the orientation $\omega_{A \pitchfork B}$ such that the orientation of $\mathbb{R}^n$ obtained so far coincides with the given ω_n. In the nonoriented setting

$M \pitchfork Z = Z \pitchfork M$, but the orientation depends on the order; this can be checked straightforwardly in the linear local model. We get

$$M \pitchfork Z = (-1)^{(\dim N - \dim M)(\dim N - \dim Z)} Z \pitchfork M \ .$$

A very important consequence of Theorem 8.2 is the following *parametric transversality theorem*. It represents the bridge between the two kinds of transversality theorems. Keeping the above setting, consider furthermore a boundaryless "parameter" smooth manifold S, so that $M \times S$ has boundary equal to $\partial M \times S$.

THEOREM 8.4. *Let $F : M \times S \to N$ be transverse to Z. For every $s \in S$, set $f_s : M \to N$ as the restriction of F to $M \sim M \times \{s\}$. Then the set of parameters $s \in S$ such that f_s is not transverse to Z is negligible in S.*

Proof : Let $(Y, \partial Y) = (F^{-1}(Z), \partial F^{-1}(Z))$ be the proper submanifold of $(M \times S, \partial M \times S)$ as in Theorem 8.2. Set $\pi : Y \to S$ as the restriction to Y of the projection $p : M \times S \to S$. We claim that for every regular value s of both π and $\partial\pi$ (i.e. such that $\pi \pitchfork \{s\}$), f_s is transverse to Z. The thesis will then follow from the Sard-Brown theorem, so let us justify the claim. Let $x \in M$ be such that $f_s(x) = F(x, s) = z \in Z$. As $F \pitchfork Z$, for every $w \in T_zN$ there are $(u, v) \in T_xM \times T_sS$ and $t \in T_zZ$ such that

$$w = d_{(x,s)}F(u, v) + t \ .$$

The differential

$$d_{(x,s)}p : T_xM \times T_sS \to T_sS$$

is just the projection to the second factor, and $d_{(x,s)}\pi$ is obtained by restriction. As s is a regular value of π, then there exists a vector of the form $(u', v) \in T_{(x,s)}Y$. By definition of Y, $t' := d_{(x,s)}F(u', v) \in T_zZ$. Finally, we readily verify that

$$w = d_{(x,s)}F(u - u', 0) + d_{(x,s)}F(u', v) + t = d_xf_s(u - u') + (t' - t) \ .$$

This proves that $T_zN = d_xf_s(T_xM) + T_zZ$. Using that s is also a regular value of $\partial\pi$, the very same argument shows that $\partial f_s \pitchfork Z$. This achieves the proof.

■

Before establishing the second transversality theorem, we refine the setting. That is, we assume furthermore that

(1) M is compact;
(2) N can be properly embedded in some $\mathbb{R}^h$, that is, also being a closed subset of $\mathbb{R}^h$.

In many applications, N and Z will also be compact. In any case, these assumptions allow applying the results of Section 6.9.1 to N.

THEOREM 8.5. (Second basic transversality theorem)
(1) The set $\pitchfork (M, N; Z)$ of smooth maps transverse to Z is open and dense in $\mathcal{E}(M, N)$.

(2) Let $f \in \mathcal{E}(M,N)$ *be such that* $\partial f : \partial M \to N$ *is transverse to* Z*. Denote by* $\mathcal{E}(M,N,\partial f)$ *the space of smooth maps that coincide with* ∂f *on* ∂M*, and by* $\pitchfork (M,N,\partial f;Z)$ *the subset of* $\mathcal{E}(M,N,\partial f)$ *of maps that are transverse to* Z*. Then* $\pitchfork (M,N,\partial f;Z)$ *is open and dense in* $\mathcal{E}(M,N,\partial f)$*.*

(3) For every $h \in \mathcal{E}(M,N)$ *(resp.* $h \in \mathcal{E}(M,N,\partial f)$*), there is* $g \in \pitchfork (M,N;Z)$ *(resp.* $g \in \pitchfork (M,N,\partial f;Z)$*) smoothly homotopic to* h*.*

Proof : Let us consider first the property of being open in both items (1) and (2). As M is compact, in earlier chapters we have already achieved it in the case of submersions; this easily implies the validity of the theorem when $Z = \{y_0\}$ consists of one point. Applying the local reduction argument to this case as in the proof of Theorem 8.2, for every $f \in \pitchfork (M,N;Z)$, we can find a finite covering of M by compact sets K such that f reduces to the special case on a neighbourhood of each K in M. For each K, there is an open neighbourhood $\mathcal{U}_K$ of f in $\mathcal{E}(M,N)$ formed by maps which satisfy the transversality conditions at each $x \in K$. The intersection of these finite open sets $\mathcal{U}_K$ is an open neighbourhood of f in $\mathcal{E}(M,N)$ contained in $\pitchfork (M,N;Z)$; hence it is open. The same argument applies to $\pitchfork (M,N,\partial f;Z)$.

Let us come now to the density stated in (1). We consider first the special case when $N = \mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^{n-r}$ and $Z = \mathbb{R}^r = \mathbb{R}^r \times \{0\}$. Let $f \in \mathcal{E}(M,\mathbb{R}^n)$. The map

$$F : M \times \mathbb{R}^n \to \mathbb{R}^n,\ F(x,s) = f(x) + s$$

is transverse to $\mathbb{R}^r$ (in fact, it is a submersion to the whole $\mathbb{R}^n$) and we can apply to it the parametric transversality Theorem 8.4. For every $\epsilon > 0$, there is $s \in \mathbb{R}^n$ such that $||s|| < \epsilon$ and $f_s \pitchfork Z$. As M is compact, by taking ϵ small enough, then $f_s = f + s$ can be arbitrarily close to f in the C^∞-topology.

We are going to apply the same argument in the general case, using a more elaborate construction. Let $f \in \mathcal{E}(M,N)$. For the moment, assume for simplicity that $N \subset \mathbb{R}^h$ is compact and take a tubular neighbourhood $\pi_N : U_N \to N$ of N in $\mathbb{R}^h$ constructed using the standard Riemannian metric g_0 on $\mathbb{R}^h$ and some $\epsilon_0 > 0$. Consider the restriction of the map defined above

$$F : M \times B^h(0,\epsilon) \to \mathbb{R}^h,\ F(x,s) = f(x) + s\ .$$

The parameter space is now restricted to the open ball of radius ϵ; as M and N are compact, if ϵ is smalls enough, the image of F is contained in U_N and we can define

$$\hat{F} : M \times B^h(0,\epsilon) \to N,\ \hat{F}(x,s) = \pi_N(F(x,s))\ .$$

As both F and π_N are submersions, $\hat{F}$ is also a submersion hence $\hat{F} \pitchfork Z$, and we can apply again Theorem 8.4. For s generic and small enough, $\hat{f}_s \pitchfork Z$ and is arbitrarily close to f. If N is not compact, by using the considerations of Section 6.9.1, there is a compact submanifold with boundary $N' \subset N$ such that $f(M) \subset \mathrm{Int}(N')$ and we can repeat the above argument by using a tubular "neighbourhood" $\pi_{N'} : U_{N'} \to N'$. Alternatively, we can use (instead of π_N) the projection $\pi : N_\epsilon \to N$, defined on the ϵ-neighbourhood of N determined by a suitable smooth positive function $\epsilon : N \to \mathbb{R}$, and the modified maps

$$\hat{F} : M \times B^h(0,1) \to N,\ \hat{F}(x,s) = \pi(f(x) + \epsilon(x)s)\ .$$

Let us now address the density stated in (2). We follow the same scheme, modifying the map $\hat{F}$. Let $f \in \mathcal{E}(M,N)$ be such that $\partial f \pitchfork Z$. Using the same considerations developed to prove the openness, it is easy to verify that $f \pitchfork Z$ provided that it is restricted to a small collar C of ∂M. By slightly modifying the construction of a collar bump function, we can construct a smooth function $\gamma : M \to [0,1]$ such that γ is constantly equal to 0 on a smaller closed collar $C' \subset C$, γ is positive on the complement of C', and γ is constantly equal to 1 outside C. Assume again for simplicity that $N \subset \mathbb{R}^h$ is compact and let $\pi_N : U_N \to N$ as above (the discussion when N is noncompact can be repeated word by word). Then define

$$\hat{F} : M \times B^h(0,\epsilon) \to N,\ \hat{F}(x,s) = \pi_N(f(x) + \gamma^2(x)s)\ .$$

We claim that $\hat{F} \pitchfork Z$, then for generic s small enough, $\hat{f}_s = \pi_N \circ (f + \gamma^2 s)$ belongs to $\pitchfork (M,N,\partial f;Z)$ and is arbitrarily close to f. The restriction of $\hat{F}$ to $\{x;\ \gamma^2(x) \neq 0\} \times B^h(0,\epsilon)$ is a submersion because for every fixed x, $s \to \gamma^2(x)s$ is a diffeomorphism to its image, the map $F(x,t) = \pi_N(f(x) + t)$ is a submersion, and $\hat{F}$ is obtained by composition. It follows that if $\hat{F}(x,s) = z \in Z$ and $\gamma^2(x) \neq 0$, then the transversality conditions are verified at (x,s). Assume now that $\hat{F}(x,s) = z \in Z$ and $\gamma^2(x) = 0$; that is, $x \in C'$. We note that $d_x\gamma^2 = 2\gamma(x)d_x\gamma$, hence it vanishes on C'. Using this fact, it is not hard to verify that for every $(u,v) \in T_x(M) \times T_sB^h(0,\epsilon)$,

$$d_{(x,s)}\hat{F}(u,v) = d_xf(u)\ ;$$

hence these differentials have the same image in T_zN. As f restricted to C' is transverse to Z, then the restriction of $\hat{F}$ to C' is also transverse to Z.

Concerning point (3), referring for instance to $f \in \mathcal{E}(M,N)$ and to the above proof of the density, we note that $f = \hat{f}_0$ and is homotopic to $\hat{f}_s \pitchfork Z$ via the path $\hat{f}_{\sigma(t)}$, $\sigma(t) = (1-t)s$, $t \in [0,1]$. On the other hand, we know in general that if g is close enough to f, then they are homotopic (recall Lemma 6.13). The proof is now complete.

■

REMARK 8.6. The proof of the openness does not use that N is embedded. We sketch here a similar proof of the density of (1) and (2) in Theorem 8.5. For simplicity, we consider statement (1) and assume that M is boundaryless. Let $f \in \mathcal{E}(M,N)$. By compactness of M there is a nice atlas $\mathcal{N}$ of M

$$\{\phi_j : W_j \to B^m(0,1)\}_{j=1,\dots,s}$$

and a family $\mathcal{F}$ of charts of (N,Z) of the form

$$\{\alpha_j : (V_j, Z \cap V_j) \to (\mathbb{R}^a \times \mathbb{R}^{n-r}, \mathbb{R}^a)\}$$

such that for every j, $f(W_j) \subset V_j$, so that we have the family

$$\{f_j : U_j \to \mathbb{R}^r \times \mathbb{R}^{n-r}\}$$

of associated representations of f in local coordinates supported by $(\mathcal{N}, \mathcal{F})$. Recall that every $K_j = \overline{B}_j \subset W_j$ is compact and this provides a finite compact covering of M. The subset $A_{\mathcal{N},\mathcal{F}}$ of $\mathcal{E}(M,N)$ formed by the maps admitting local representations supported by $(\mathcal{N}, \mathcal{F})$ is open and nonempty, as it contains f. By applying to every $\mathcal{E}(W_j, V_j)$ the special case of the density considered in the proof of Theorem 8.5 (1) and by using the bump function γ_j in order to extend locally defined maps to maps in $\mathcal{E}(M,N)$, we realize that for every j, $\pitchfork_{K_j} (M,N;Z) \cap A_{\mathcal{N},\mathcal{F}}$ is dense in $A_{\mathcal{N},\mathcal{F}}$. We know that it is also open. Then the intersection of this finite family of open and dense sets is open and dense in $A_{\mathcal{N},\mathcal{F}}$, and is contained in $\pitchfork (M,N;Z)$ since the K_j's cover the whole of M.

Remark 8.7. As already mentioned, *any* compact subset $K \subset B^m(0,1)$ can be realized as $K = f^{-1}(0)$ for some smooth function $f : \overline{B}^m(0,1) \to \mathbb{R}$; compared with the tame behaviour of $f^{-1}(0)$ when $f \pitchfork \{0\}$, this shows that non-transverse situations can be really weird. On the other hand, remarkably, by Theorem 8.5 any weird non-transverse situation can be made stably tame up to arbitrarily small perturbations (at least when M is compact).

8.2. Miscellaneous transversalities

Transversality is a profound, potent and pervasive paradigm beyond the basic results stated in the previous section. Without any pretension of completeness, we collect here a few examples of further applications.

8.2.1. Jet trasversality. First, we perform some constructions within the smooth category of open sets considered in Chapter 1. In particular, we refer to the Taylor polynomials defined in Section 1.2. Recall that a *homogeneous polynomial map of degree* $k \geq 1$

$$\mathfrak{p} : \mathbb{R}^m \to \mathbb{R}^n$$

is of the form $\mathfrak{p}(x) = \phi(x, \dots, x)$, where $\phi : (\mathbb{R}^m)^k \to \mathbb{R}^n$ is a (necessarily unique) *symmetric* k-linear map. The set $\mathcal{P}_k(m,n)$ of these homogeneous polynomial maps has the natural structure of a finite-dimensional real vector space endowed with a standard basis, so that it is identified with $\mathbb{R}^{\dim \mathcal{P}_k(m,n)}$. A *polynomial map of degree* $\leq r$, $p : \mathbb{R}^m \to \mathbb{R}^n$, is of the form

$$p = p_0 + p_1 + \cdots + p_r$$

where $p_0 \in \mathbb{R}^n$ and for $k \geq 1$, p_k is a homogeneous polynomial map of degree k. Denote by $J^r(m,n)$ the set of these polynomial maps. We can use the natural identification

$$J^r(m,n) = \prod_{k=0}^{r} \mathcal{P}^k(m,n)$$

to give it the structure of a finite-dimensional real vector space; $J^r(m,n)$ is identified with $\mathbb{R}^{\dim J^r(m,n)}$.

REMARK 8.8. With some effort we can compute the dimension:

$$\dim J^r(m,n) = n\binom{r+m}{m} \ .$$

Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be nonempty open sets. Define the open set of $\mathbb{R}^m \times J^r(m,n)$

$$J^r(U,V) := \{(x,p) \in U \times J^r(m,n);\ p_0 \in V\} \ .$$

Given a smooth map $f : U \to V$, define the smooth map

$$j^r f : U \to J^r(U,V),\ j^r f(x) = \mathcal{T}_r f(x)$$

sending every point of U to the Taylor polynomial of f at x of degree $\leq r$.

(Composition rule) Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^h$ be nonempty open sets. Set

$$J^r(U,V,W) = \{((y,q),(x,p)) \in J^r(V,W) \times J^r(U,V);\ p_0 = y\} \ .$$

Let $f : U \to V$, $g : V \to W$ be smooth maps. By a suitable extension to higher order derivatives of the chain rule, we can find a unique polynomial map (the explicit expression is called *Faa di Bruno formula*)

$$\mathfrak{P}^r : J^r(U,V,W) \to J^r(U,W)$$

such that

$$j^r(g \circ f)(x) = \mathfrak{P}^r(j^r g(y), j^r f(x)) \ .$$

(Change of coordinates) Let $U, U' \subset \mathbb{R}^m$, $V, V' \subset \mathbb{R}^n$ be nonempty open sets and $\phi : U \to U'$, $\psi : V \to V'$ be diffeomorphisms. Then for every r, there is a unique smooth diffeomorphism

$$j^r_{\psi,\phi} : J^r(U,V) \to J^r(U',V')$$

such that

$$j^r_{\psi,\phi}(j^r f(x)) = j^r f'(x')$$

where

$$x' = \phi(x),\ f' = \psi \circ f \circ \phi^{-1} \ .$$

This is an application of the composition rule.

Now we can globalize the above local considerations, extending what we have done for the (co)-tangent map.

Let M, N be smooth manifolds of dimension m and n, respectively. Define on $M \times \mathcal{C}^\infty(M,N)$ the following relation: $(x,f) \sim_r (x',f')$ if $x = x'$, $f(x) = f'(x)$ and there are *compatible representations in local coordinates* of f and f' at $x = x'$, $y = f(x)$, that are defined on the same charts of M and N, respectively

$$f_{U,V}, f'_{U,V} : U \to V,\ U \subset \mathbb{R}^m,\ V \subset \mathbb{R}^n$$

such that

$$j^r f_{U,V}(x) = j^r f'_{U,V}(x) \ .$$

By using the change of coordinates rule, it is easy to check that this defines an equivalence relation and that if $(x, f) \sim_r (x', f')$, then the above defining property holds *for every* pair of compatible representations in local coordinates. We denote the equivalence class of (x, f), called the r-jet of f at x, by $j^r f(x)$; $J^r(M, N)$ is the *space of* r*-jets* from M to N. For every smooth map $f : M \to N$, the map

$$j^r f : M \to J^r(M, N)$$

is called the r*-jet extension* of f. Clearly, $J^0(M, N) = M \times N$. For every $r \geq 1$, $J^r(M, N)$ has a natural smooth manifold structure of dimension

$$\dim J^r(M, N) = \dim M + \dim J^r(m, n) \ .$$

Local coordinates U and V for M and N carry local coordinates $J^r(U, V)$ for $J^r(M, N)$. This provides a smooth atlas of $J^r(M, N)$ and we have already settled the change of coordinates rules. We also see above the local representations of an extension $j^r f$, which is a smooth map. There is a natural smooth projection

$$\sigma_r : J^r(M, N) \to M$$

and a sequence of smooth "forgetting" maps which factorize σ:

$$M \leftarrow J^1(M, N) \leftarrow \cdots \leftarrow J^r(M, N) \ .$$

The map σ_r is a smooth fibration with fibre *diffeomorphic* to $J^r(m, n)$; note that in spite of the fact that $J^r(m, n)$ is a vector space with a preferred basis, σ_r is *not* a vector bundle for $r > 1$. The atlas of $J^r(M, N)$ is fibred, but the changes of coordinates do not preserve the linear structure of the fibre. Every jet extension $j^r f : M \to J^r(M, N)$ is a section of such a smooth fibre bundle. Also every map $J^s(M, N) \to J^{s-1}(M, N)$ is a smooth fibration with fibre $\mathcal{P}^s(m, n)$.

We are ready to state a version of the so-called *jet transversality theorem*. Let M and N be smooth boundaryless manifolds and Z be a submanifold of $J^r(M, N)$. Denote by

$$\pitchfork j^r(M, N, Z)$$

the set of smooth maps $f \in \mathcal{E}(M, N)$ such that $j^r f \pitchfork Z$.

Theorem 8.9. *Let M be a compact smooth boundaryless manifold and N be a boundaryless proper smooth submanifold of some $\mathbb{R}^h$. Let Z be a proper submanifold of $J^r(M, N)$. Then $\pitchfork j^r(M, N, Z)$ is open and dense in $\mathcal{E}(M, N)$.*

Proof : We provide only a sketch. Note also that $J^r(M, N)$ can be embedded as a proper submanifold of some $\mathbb{R}^k$. When $r = 0$, Theorem 8.9 incorporates (1) of Theorem 8.5 (at least when M is boundaryless). Openness is not hard. As for the density, Theorem 8.5 ensures that every

$j^r f$ can be approximated by a smooth map $g : M \to J^r(M,N)$ transverse to Z, but the statement of Theorem 8.9 requires furthermore that g is the r-jet extension of some map $\tilde{f} : M \to N$. So jet-transversality is not an immediate consequence of standard transversality. Nevertheless, the structure of the proof is essentially the same. A first fundamental case is when $N = \mathbb{R}^n$. In the proof of Theorem 8.5, the key point was the application of parametric transversality to the deformations of a given map $f : M \to \mathbb{R}^n$ of the form $f + s$, $s \in \mathbb{R}^n$. In the present situation, the main difference consists in using polynomial deformations of the form $f + p_0 + p_1 + \cdots + p_r$, where $p = p_0 + \cdots + p_r$ varies among the polynomial maps $p : \mathbb{R}^n \to \mathbb{R}^n$ of degree $\leq r$. Provided this new ingredient, the proof of Theorem 8.5 can be repeated with minor changes.

■

8.2.2. Transversality to stratifications. In several situations, it is convenient to extend the notion of "general position" (i.e. of transversality) to suitable "stratification" either of N for the standard transversality or of $J^r(M,N)$ for the jet-transversality. We do not intend to present here a consistent treatment of stratification theory, merely a few suggestions. At a first sight, a stratification of a smooth manifold X is a partition $\mathcal{S} = \{S_j\}$ through boundaryless, connected *not necessarily proper* smooth submanifolds of X, called the *strata* of the stratification. We usually require more; reasonable requirements are as follows.

- The stratification is locally finite;
- *(Frontier condition)* The frontier $\bar{S}_j \setminus S_j$ of every stratum S_j is a union of strata of strictly lower dimension;
- For every $0 \leq s \leq \dim X$, denote by X^s the *s-skeleton* of the stratification; that is, the union of strata of dimension less or equal to s. Then X^s is a closed subset of X.

For example, if $S \subset X$ is a boundaryless proper submanifold, then $\{X \setminus S, S\}$ is a stratification of X. The open simplices of a smooth triangulation of X (as it is described in Section 14.9) form a stratification; in this case, every stratum of dimension greater or equal to 1 is not a proper submanifold.

Given a stratification $\mathcal{S}$ of N, denote by $\pitchfork(M,N,\mathcal{S})$ the subspace of $\mathcal{E}(M,N)$ formed by the maps $f : M \to N$ which are transverse to each stratum of $\mathcal{S}$ (we write $f \pitchfork \mathcal{S}$). Similarly, for every $r \geq 1$, given a stratification $\mathcal{S}$ of $J^r(M,N)$ we define $\pitchfork j^r(M,N,\mathcal{S})$.

(Nice stratifications) We do not give a formal definition of this notion. The question is to determine further explicit, minimal conditions so that a stratification $\mathcal{S}$ satifies the following properties (assuming that N satisfies the hypotheses of Theorem 8.9).

For every compact boundaryless smooth manifold M, $\pitchfork(M,N,\mathcal{S})$ is open and dense in $\mathcal{E}(M,N)$ and, moreover, for every such a map f transverse to

$\mathcal{S}$, $f^{-1}(\mathcal{S})$ is a stratification of M that inherits the same nice properties of $\mathcal{S}$. Similar properties hold for $\pitchfork j^r(M,N,\mathcal{S})$ and $j^r f^{-1}(\mathcal{S})$.

Roughly speaking, such conditions should imply that the transversality to any stratum S_j forces (at least locally at S_j) the transversality to every stratum S_i such that S_j is in the frontier of S_i. We will not deal with this rather difficult question (see also [**Wall2**]). We will state without justification some results where the occurring stratifications are "nice".

8.2.3. A classification of map singularities. An important field of application of jet-transversality (in the stratified extension) is the study of *singularities of smooth maps* (see [**A2**]). The idea is that, under suitable hypotheses, for a "generic" map $f : M \to N$, the source manifold M carries a nice stratification such that the increasing codimension of the strata corresponds to more and more 'deep' classes of singular points of f determined by a certain specific *lack of transversality.*

(Classification by the differential rank) A first coarse classification is in terms of the rank of differentials. Let M and N be boundaryless manifolds. Let $f : M \to N$ be a smooth map. A point $x \in M$ is said of *class* Σ^i *(with respect to f)* if $\dim \ker d_x f = i$. For every i, denote by $\Sigma^i(f)$ the subset of M of points of class Σ^i. They form a partition of M. If f is arbitrary, this partition might be weird. However, we have:

Proposition 8.10. *Let M and N satisfy the hypotheses of Theorem 8.9. Then there is an open dense set $\mathcal{R}$ in $\mathcal{E}(M,N)$ such that for every $f \in \mathcal{R}$, the connected components of the $\Sigma^i(f)$'s form a nice stratification of M. Moreover, every $\Sigma^i(f)$ is a submanifold of M of dimension given by*

$$\dim M - \dim \Sigma^i(f) = (\dim M - r)(\dim N - r),\ r = \dim M - i\ .$$

In fact, one defines a suitable nice stratification $\mathcal{S}_\Sigma$ of $J^1(M,N)$ and for every generic f, we consider $\mathcal{R} = (j^1 f)^{-1}(\mathcal{S}_\Sigma)$. In local coordinates $J^1(U,V)$, $\mathcal{S}_\Sigma$ corresponds to the stratification of the matrix space $M(n,m,\mathbb{R})$ by the matrix rank.

Example 8.11. (1) If $N = \mathbb{R}$, then the cotangent bundle of M is naturally identified with the restriction of the jet space $J^1(M,\mathbb{R})$ to those jets that have image point 0 in $\mathbb{R}$, so that $df = j^1 f$; f is a Morse function if and only if $j^1 f$ is transverse to the zero section of the bundle. Hence, the result about the Morse functions from Chapter 6 can be obtained again as a special case of jet-transversality (at least when M is compact boundaryless).

(2) *(Whitney cusp)* Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x_1,x_2) = (x_1^3 + x_1 x_2, x_2)\ .$$

The set of singular points is the parabola $S := \{3x_1^2 + x_2 = 0\}$. The nice stratification of the source $\mathbb{R}^2$ is given by $\Sigma^0(f) = \mathbb{R}^2 \setminus S$, $\Sigma^1(f) = S$.

(3) *(Whitney umbrella)* Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(x_1, x_2) = (x_1 x_2, x_2, x_1^2) \ .$$

The point $0 \in \mathbb{R}^2$ is the only one at which f is not an immersion; hence the nice stratification of $\mathbb{R}^2$ is given by $\Sigma^0(f) = \mathbb{R}^2 \setminus \{0\}$, $\Sigma^1(f) = \{0\}$ (see also Section 7.8).

The above examples show that the stratification by the differential rank is, in general, too coarse. In the Whitney cusp, $0 \in \Sigma^1(f) = S$ is clearly special: $\ker d_0 f = T_0 S$ while for other $x \in S$, $\mathbb{R}^2 = \ker d_x f + T_x S$. In the Whitney umbrella, a refinement of the stratification can be obtained by noticing that the line $\{x_2 = 0\}$ is the locus where the map is not injective.

If $f \in \mathcal{R}$, as in Proposition 8.10, a tentative refinement of the stratification $\{\Sigma^i(f)\}$ would be defined by recurrence as follows: assume that for every multi-index of length k, $I = (i_1, \dots, i_k)$ is defined $\Sigma^I(f) \subset M$; then for every multi-index of length $k+1$, $\tilde{I} = (i_1, \dots, i_k, i_{k+1})$, set $\Sigma^{\tilde{I}}(f) := \Sigma^{i_{k+1}}(f|\Sigma^I(f))$. It is not evident that this produces a nice stratification. The correct way to do so (see [**Bo**]) is to extend the above mentioned stratification $\mathcal{S}_\Sigma$ of $J^1(M, N)$, to get a nice stratification $\tilde{\mathcal{S}}_\Sigma$ and extend Proposition 8.10.

8.2.4. Multi-transversality. Assume that $f : M \to N$ is an immersion and that M is connected. Then the above refined nice stratification of M consists of one stratum $\Sigma^0(f) = M$. This does not give any information about the image of f. Clearly, this last might be “nongeneric”. We say that an immersion f is *in general position* if, for every $k \geq 2$, whenever $y = f(x_1) = f(x_2) = \cdots = f(x_k)$ and the points $x_1, \dots, x_k$ are distinct, then

$$T_y N = df_{x_k}(T_{x_k} M) + \cap_{j=1}^{k-1} df_{x_j}(T_{x_j} M) \ .$$

For example, if $\dim N = 2 \dim M$, then the multiple points y are isolated and each is the image of exactly two points of M. The following is a basic example of the multi-transversality result.

PROPOSITION 8.12. *Let M, N satisfy the hypotheses of Theorem 8.9. Assume that the open set $Im(M, N)$ of immersions of M in N is not empty. Then the set of immersions in general position is open and dense in $Im(M, N)$.*

The general concept of multi-jet-transversality was introduced in [**Ma2**]. We consider the products $J^r(M, N)^k$, $k \geq 1$. Then, for every $f \in \mathcal{E}(M, N)$ we have the product map $(j^r f)^k : M^k \to J^r(M, N)^k$. For every submanifold V of $J^r(M, N)^k$ we can consider f such that $(j^r f)^k \pitchfork V$. The submanifolds V of most interest are as follows:

• Given submanifolds V_i of $J^r(M, N)$, $1 \leq i \leq k$, consider the product $\prod V_i \subset J^r(M, N)^k$;

• There is a natural projection $\tau_k : J^r(M,N)^k \to N^k$. Then take

$$V = \tau_k^{-1}(\Delta_k N) \cap \prod V_i$$

where $\Delta_k(N) = \{(y,\ldots,y)\} \subset N^k$ is the (multi) diagonal of N^k.

Multi-transversality to such a manifold V means that the following conditions are satisfied:

• If $f(x_i) = y$ for every $i = 1, \ldots k$, then f is transverse to V_i at x_i with pre-image $X_i = f^{-1}(V_i)$;

• The images B_i of $T_{x_i}X_i$ in T_yN satisfy

$$(\oplus_i B_i) \oplus T_{(y,\ldots,y)}\Delta_k(N) = (T_yN)^k \ .$$

Finally, in the same hypotheses, we get [**Ma2**] a multi-transverse version of Theorem 8.9.

CHAPTER 9

Morse functions and handle decompositions

Let us call *smooth triad* a triple (M, V_0, V_1) where M is a compact smooth m-manifold with (possibly empty) boundary, and V_0 and V_1 are unions of connected components of ∂M in such a way that the boundary is the *disjoint union*

$$\partial M = V_0 \amalg V_1 \ .$$

A boundaryless M corresponds to the triad $(M, \emptyset, \emptyset)$. We stress that different ordered bipartitions of the components of ∂M give rise to different triads. For example, if $\partial M \neq \emptyset$, then $(M, \partial M, \emptyset)$ and $(M, \emptyset, \partial M)$ are different triads. A triad can be considered as a "transition" from V_0 to V_1. We know from Proposition 6.26 that generic Morse functions form a dense open set in $\mathcal{E}(M, V_0, V_1)$, the space of functions $f : M \to [0,1]$ such that $V_j = f^{-1}(j)$, $j = 0, 1$, and without critical points on a neighbourhood of ∂M. Let $f : M \to [0, 1]$ be such a generic Morse function on the triad (M, V_0, V_1). We have a finite set of non-degenerate critical points $p_0, \dots, p_s$ of indices $q_0, \dots, q_s$, and critical values $c_r = f(p_r)$, such that $0 < c_r < c_{r+1} < 1$, $r = 0, \dots, s-1$. For every $X \subset [0, 1]$, denote $V_X := f^{-1}(X)$. For every regular value a of f, V_a is a compact boundaryless submanifold of M of dimension $m - 1$. If $0 \leq a < b \leq 1$ are regular values, then we have the subtriad $(V_{[a,b]}, V_a, V_b)$.

The following lemma ultimately is an instance of a fibration theorem. We give a proof which assumes a few results of analysis about the existence, the uniqueness and the regular dependence on the initial conditions for ordinary differential equations (see [**A**]).

Lemma 9.1. (Cylinder Lemma) *Assume that $[a, b] \subset [0, 1]$ does not contain any critical value of f. Then there is a diffeomorphism*

$$\psi : V_a \times [a, b] \to V_{[a,b]}$$

such that $f \circ \psi(y, t) = t$ for every $y \in V_a$.

Proof : Fix an auxiliary Riemannian metric g on M and let $\nabla_g f$ be the associated gradient field of f, which is non-zero everywhere on $V_{[a,b]}$. We can normalize it by taking, for every $p \in V_{[a,b]}$,

$$\nu(p) = \nabla_g f(p)/||\nabla_g f(p)||_{g(p)} \ .$$

Every integral curve α of ν satisfies $f(\alpha(s)) = s + c$, c being a constant. Possibly by means of the change of parameter $\beta(t) = \alpha(t - c)$, we can assume that $f(\alpha(t)) = t$. Since $V_{[a,b]}$ is compact, every maximal integral curve is defined on the whole $[a,b]$. For every $y \in V_a$, there is a unique maximal integral curve of ν

$$\alpha_y : [a,b] \to V_{[a,b]}$$

such that $\alpha(a) = y$ and $f(\alpha(t)) = t$ for every $t \in [a,b]$. The required diffeomorphism is defined by $\psi(y,t) = \alpha_y(t)$, with inverse $\psi^{-1}(x) = (\alpha_x(a), f(x))$, where α_x is the unique maximal integral curve passing through $x \in V_{[a,b]}$.

■

Remark 9.2. We have developed tubular neighbourhood, collars and other results for compact manifolds by exploiting the existence of embedding in some Euclidean space. However, in several situations, we could provide also an "abstract" treatment. For example, the *existence* of collars of ∂M in M is an immediate consequence of Lemma 9.1, provided that $\mathcal{E}(M, \partial M, \emptyset)$ is nonempty. This last fact can be obtained as follows: fix a nice atlas of M. Define local functions as follows: if (W, ϕ) is an internal chart, then f_j is the constant function equal to $1/2$. If

$$\phi_j : (W_j, W_j \cap \partial M) \to (B^m \cap \mathbf{H}^m, B^m \cap \partial \mathbf{H}^m)$$

is a chart at the boundary, then f_j is the restriction of the projection of B^m to the x_m-axis. By using the partition of unity subordinate to the atlas, define

$$f = \sum_j \lambda_j f_j \ .$$

We can check directly that f has the desired properties. In [**Mu**], one can find an "abstract" proof of the uniqueness up to isotopy of collars.

9.1. Dissections carried by generic Morse functions

We fix a nice atlas with collars on the triad (M, V_0, V_1) *adapted to the given Morse function* $f : M \to [0,1]$. This means the following facts:

- The collars are of the form $V_{[0,\epsilon_0]}$, $V_{[1-\epsilon_0,1]}$, for some $\epsilon_0 > 0$, $\epsilon_0 < c_0 = f(p_0)$, $c_s = f(p_s) < 1 - \epsilon_0$;
- Every critical point p_r of f is contained in a unique internal normal chart (W_r, ϕ_r), in such a way that $B_r \cap B_{r'} = \emptyset$ if $r \neq r'$ (recall that $B_r = \phi_r^{-1}(B^m(0,1/3))$;
- Every (W_r, ϕ_r) is such that $(f \circ \psi_r - c_r) : B^m(0,1/3)) \to \mathbb{R}$ is in normal form according to Morse's Lemma of Section 6.21 at $0 = \phi_r(p_r)$.

Then, we take $\epsilon > 0$ such that

- $\epsilon_0 < c_0 - \epsilon$, $c_0 + \epsilon < c_1 - \epsilon$, $\dots$, $c_{s-1} + \epsilon < c_s - \epsilon$, $c_s + \epsilon < 1 - \epsilon_0$;
- for every $r = 0, \dots, s$, $V_{c_r - \epsilon} \cap B_r \neq \emptyset$ and $V_{c_r+\epsilon} \cap B_r \neq \emptyset$, so that $V_{[c_r-\epsilon, c_r+\epsilon]}$ is the union of $V_{[c_r-\epsilon,c_r+\epsilon]} \cap B_r$ and its complement.

So we have the *dissection* of the triad (M, V_0, V_1) associated to the Morse function f:

$V_{[0,c_0-\epsilon]} \cup V_{[c_0-\epsilon,c_0+\epsilon]} \cup V_{[c_0+\epsilon,c_1-\epsilon]} \cup V_{[c_1-\epsilon,c_1+\epsilon]} \cup \cdots \cup V_{[c_s-\epsilon,c_s+\epsilon]} \cup V_{[c_s+\epsilon,1]}$.

By applying the cylinder and Thom's lemmas, we have that

- $V_{[0,c_0-\epsilon]} \sim V_0 \times [0, c_0 - \epsilon]$, $V_{[c_s+\epsilon,1]} \sim [c_s + \epsilon, 1] \times V_1$;
- For every $r = 0, \dots, s-1$, $V_{[c_r+\epsilon,c_{r+1}-\epsilon]} \sim V_{c_r+\epsilon} \times [c_r + \epsilon, c_{r+1} - \epsilon]$;
- $V_{[0,c_r+\epsilon]} \sim V_{[0,c_{r+1}-\epsilon]}$.

For every $r = 0, \dots, s-1$, $(V_{[c_r-\epsilon,c_r+\epsilon]}, V_{c_r-\epsilon}, V_{c_r+\epsilon})$ is an *elementary triad* in the sense that it carries a Morse function (the restriction of f) with *only one* critical point (p_r of a certain index q_r).

Adapted gradient fields. Using the above nice atlas of (M, V_0, V_1) adapted to f, we can construct an adapted Riemannian metric g on M, so that for every $r = 0, \dots, s$, the gradient field $\nabla f := \nabla_g f$ has the normalized expression in the local coordinates over B_r:

$$2(-x_1, -x_2, \dots, -x_{q_r}, x_{q_r+1}, \dots, x_m) ,$$

while the collars of V_0 and V_1 are obtained by integrating such a (normalized) field as in the proof of Lemma 9.1.

The key point is to understand what happens, up to diffeomorphism, passing from $V_{[0,c_r-\epsilon]}$ to $V_{[0,c_r+\epsilon]}$ (equivalently to $V_{[0,c_{r+1}-\epsilon]}$) through such an elementary triad. It is evident that the choice of the parameters ϵ_0 and ϵ is immaterial. An answer is given by the following proposition. We refer to notions introduced in Chapter 7. The proof is extracted from [**Pa**].

PROPOSITION 9.3. *Let $f : M \to [0, 1]$ be a generic Morse function on the triad (M, V_0, V_1) and consider an associated dissection. Let p be a critical point of f of index q, and c' be the next critical value of f after $c = f(p)$. Then*

(1) $V_{[0,c+\epsilon]}$ is diffeomorphic to $V_{[0,c'-\epsilon]}$.

(2) Up to diffeomorphism, $V_{[0,c+\epsilon]}$ is obtained by attaching a q-handle (of dimension m) to $V_{[0,c-\epsilon]}$ at $V_{c-\epsilon}$.

Proof : As already remarked, (1) follows from the Cylinder and Thom's lemmas.

As for (2), take a nice atlas associated with the given Morse dissection of (M, V_0, V_1). Take the Morse chart $(\psi(B), \phi)$ at p, so that $\phi(p) = 0$ and $\hat{f} = f \circ \psi : B \to \mathbb{R}$ has the normal form

$$\hat{f}(x_1, \dots x_m) = -(x_1^2 + \cdots + x_q^2) + (x_{q+1}^2 + \cdots + x_m^2) + c .$$

According to our usual conventions, B should be $B^m(0, 1/3)$, but up to reparametrization we can normalize the picture as follows. First, we simplify the notations by setting

$$(x_1, \dots, x_q, x_{q+1}, \dots x_m) = (X, Y) \in \mathbb{R}^q \times \mathbb{R}^{m-q} .$$

Then we can assume the following.

- $f : M \to [a_0, a_1]$ for suitable $a_0 < -1$, $1 < a_1$;
- $B = B^m(0, 2)$, $\hat{f}(0) = c = 0$, $\epsilon = 1$;
- $B \cap \phi(W \cap V_{[a_0,-1]}) = \{(X, Y) \in B;\ -||X^2|| + ||Y||^2 \leq -1\}$;
- $B \cap \phi(W \cap V_{[a_0,1]}) = \{(X, Y) \in B;\ -||X^2|| + ||Y||^2 \leq 1\}$.

The standard handle $H^q = D^q \times D^{m-q}$ is contained in

$$B \cap \phi(W \cap V_{[-1,1]}) = \{(X, Y) \in B;\ -1 \leq -||X^2|| + ||Y||^2 \leq 1\}$$

and H^q intersects $\{-||X^2|| + ||Y||^2 = \pm 1\}$ along the union of its a and b-spheres. Moreover, if $H' = (\mathbb{R}^q \times D^{m-q}) \cap \{-1 \leq -||X^2|| + ||Y||^2 \leq 1\}$, then $V_{[a_0,-1]} \cup \psi(H')$ is a submanifold with corners of $V_{[a_0,1]}$ obtained by attaching the q-handle to $V_{[0,-1]}$ at V_{-1}. The idea is to modify the inclusion of H' to an embedding j of H^q (actually an embedded corner smoothing) in such a way that:

(1) $\mathcal{H} := j(H^q) \subset \{(X, Y) \in B;\ -1 \leq -||X^2|| + ||Y||^2 < 1\}$.
(2) $\mathcal{H} \cap \{-||X^2|| + ||Y||^2 = -1\} = j(\mathcal{T}_a)$, the image of the a-tube.
(3) The embedding j is still equal to the identity at the core of the handle.
(4) $\tilde{M} := V_{[a_0,-1]} \cup \psi(\mathcal{H})$ is a smooth submanifold of $V_{[a_0,1]}$ obtained by attaching the q-handle to $V_{[0,-1]}$ at V_{-1}, having the restriction of j to $\mathcal{T}_a$ as attaching map.
(5) $V_{[a_0,1]} \setminus \tilde{M}$ is a collar of V_1 in $V_{[a_0,1]}$.

Take the 1-dimensional bump function $\gamma = \gamma_{1/2,1}$; then define

$$\hat{g} : B \to \mathbb{R};\ \hat{g}(X, Y) = -||X||^2 + ||Y||^2 - \frac{3}{2}\gamma(||Y||^2)\ .$$

Clearly

$$\{\hat{g} \leq -1\} = \{\hat{f} \leq -1\} \cup (\{\hat{f} \geq -1\} \cap \{\hat{g} \leq -1\}) := \{\hat{f} \leq -1\} \cup \mathcal{H}$$

and $\mathcal{H}$ intersects $\{\hat{f} \leq -1\}$ at $\{\hat{f} = -1\}$; $\{\hat{g} \leq -1\}$ is contained in the interior of $\{f \leq 1\}$, and $\{\hat{f} \leq 1\} = \{\hat{g} \leq 1\}$.

Claim. *$\mathcal{H}$ is q-handle attached to $\{\hat{f} \leq -1\}$ at $\{\hat{f} = -1\}$, via a characteristic map $H : D^q \times D^{m-q} \to \mathcal{H}$ which is the identity on the core $D^q \times \{0\}$.*

We are going to write down the explicit formulas establishing the claim. Several verifications are understood; for all details (in a more general setting) we refer to [**Pa**]. The smooth function $\sigma : [0, 1] \to \mathbb{R}$ is uniquely defined by the equation

$$\frac{\gamma(\sigma(s))}{1 + \sigma(s)} = \frac{2}{3}(1 - s)\ .$$

The function σ is strictly increasing, $\sigma(0) = \frac{1}{2}$, $\sigma(1) = 1$ and moreover we have that for every $(X, Y) \in \mathcal{H}$,

$$||Y||^2 < \sigma\Big(\frac{||X||^2}{1 + ||Y||^2}\Big)\ .$$

By using σ and its properties, we can give the explicit characteristic map

$$H : D^q \times D^{m-q} \to \mathcal{H}$$

$$H(X,Y) = (\sqrt{\sigma(||X||^2)||Y||^2+1}\ X, \sqrt{\sigma(||X||^2)}\ Y)$$

which restricts to the attaching map

$$h : S^{q-1} \times D^{m-q} \to \partial\mathcal{H} \subset \{\hat{f} = -1\}$$

$$h(X,Y) = (\sqrt{||Y||^2+1}\ X, Y)\ .$$

Let us consider now

$$M' := [\{f \geq -1\} \cap (M \setminus \psi(B))] \cup \psi(\{(X,Y) \in B;\ \hat{g} \geq -1\})\ .$$

By construction, the functions f and $\hat{g} \circ \phi$ match on M', giving us a global function $g : M' \to \mathbb{R}$, such that

$$\{f \leq 1\} = \{f \leq -1\} \cup \psi(\mathcal{H}) \cup \{p \in M';\ -1 \leq g \leq 1\}\ .$$

The final remark is that $[-1,1]$ does not contain critical values of g. It is enough to verify it for $\hat{g}$ on B. In fact

$$\nabla\hat{g}(X,Y) = 2(-X,Y) - 2(0, \gamma'(||Y||^2)Y)$$

which vanishes only at 0 because $\gamma' \leq 0$ on $(0,+\infty)$. In summary, as

$$\{f \leq -1\} \cup \psi(\mathcal{H})$$

is obtained by attaching a q-handle to $\{f \leq -1\}$ at $\{f = -1\}$, by applying the Cylinder Lemma to g over $[-1,1]$ we conclude that $\{f \leq 1\}$ is also obtained by attaching a q-handle to $\{f \leq -1\}$ along $\{f = -1\}$. Ultimately, by restoring the usual notations, $V_{[0,c+\epsilon]}$ is obtained by attaching a q-handle to $V_{[0,c-\epsilon]}$ at $V_{c-\epsilon}$.

■

Remark 9.4. With the notations of (the proof of) Proposition 9.3, we realize that the core $D^q \times \{0\}$ of the q-handle $\mathcal{H}$ is formed by the integral lines of the adapted gradient field ∇f which start at a point of $V_{c-\epsilon}$ and end in the critical point p. If $c - \epsilon > \delta > 0$ is any value such that $[\delta, c-\epsilon]$ does not contain any critical value of f, then again by the Cylinder Lemma, $V_{[0,c+\epsilon]}$ is also obtained by attaching a q-handle $\mathcal{H}'$ to $V_{[0,\delta]}$ at V_δ. Although the core of $\mathcal{H}$ and the relative attaching map h look "simple and local", the core and the relative attaching map h' of $\mathcal{H}'$ can be far from $V_{c-\epsilon}$ and complicated. In fact, h' is obtained by composing h with the diffeomorphism between $V_{c'-\epsilon}$ and V_δ provided by the Cylinder Lemma; again, the core of $\mathcal{H}'$ is formed by the integral lines of the adapted gradient ∇f (used in the Cylinder Lemma) which start at a point of V_δ and end in p.

9.2. Handle decompositions

Let (M, V_0, V_1) be a triad as before. By definition, a *handle decomposition* of the triad is a sequence of nested triads of the form

$$(M_0, V_0, V_{1,0}) \subset (M_1, V_0, V_{1,1}) \subset (M_2, V_0, V_{1,2}) \subset \cdots \subset (M_k, V_0, V_{1,k})$$

such that

- $V_{1,k} = V_1$, and (M_k, V_0, V_1) is diffeomorphic to (M, V_0, V_1) via a diffeomorphism which is the identity in a neighbourhood of $V_0 \amalg V_1$;
- For every $r = 0, \ldots, k-1$, $(M_{r+1}, V_0, V_{1,r+1})$ is obtained by attaching a q-handle (of dimension m) to $(M_r, V_0, V_{1,r})$ at $V_{1,r}$ (for some q).

Handle decompositions are diffeomorphic if they are related by a diffeomorphism which is the identity near the boundary and respects the sequences of nested triads. We can also *normalize* the form of a given handle decomposition by stipulating that it starts with a "right" collar C_0 of V_0 and ends with a "left" collar C_1 of V_1.

As an immediate Corollary of Proposition 9.3, we have the *existence* of handle decompositions for every triad.

Corollary 9.5. *Every triad (M, V_0, V_1) admits handle decompositions.*

Proof : Take a dissection carried by any generic Morse function on the triad. The sequence of nested submanifolds

$$V_{[0,c_0-\epsilon]} \subset V_{[0,c_1-\epsilon]} \subset V_{[0,c_2-\epsilon]} \subset \cdots \subset V_{[0,1]}$$

leads to a desired handle decomposition.

■

Sometimes, a handle decomposition of (M, V_0, V_1) (in normalized form) is formally indicated as

$$C_0 \cup H_1^{q_1} \cup H_2^{q_2} \cup \cdots \cup H_k^{q_k} \cup C_1$$

where C_0 and C_1 are the respective collars of V_0 and V_1, and for $r = 0, \ldots, k-1$, $M_r = C_0 \cup H_1^{q_1} \cup H_2^{q_2} \cup \cdots \cup H_r^{q_r}$, M_{r+1} is obtained by attaching the q_{r+1}-handle $H_{r+1}^{q_{r+1}}$ to M_r, at $V_{1,r}$. Sometimes, we will omit to indicate the index q_r.

The dual decompositions. Given a triad (M, V_0, V_1), the *dual triad* is by definition (M, V_1, V_2). Given a decomposition $\mathcal{H}$ of the triad (M, V_0, V_1) formally indicated as

$$C_0 \cup H_1^{q_1} \cup H_2^{q_2} \cup \cdots \cup H_k^{q_k} \cup C_1$$

we can consider the dual decomposition $\mathcal{H}^*$ of (M, V_1, V_0) obtained by going from C_1 to C_0 in the opposite direction. Every q-handle H^q of $\mathcal{H}$ is converted into a "dual" $(m-q)$-handle $(H^*)^{m-q}$ of $\mathcal{H}^*$ where the core and the co-core exchange their roles. If $\mathcal{H}$ is associated with a Morse function f, then $\mathcal{H}^*$ is associated with the function $f^* = 1 - f$.

Once we have obtained the existence of handle decompositions, we will develop our discussion in terms of these last, no longer referring to the Morse functions. Morse functions have been rather a tool to produce handle decompositions. On another hand, it is true (but we do not prove it) that for every handle decomposition of a triad, there is a Morse function that recovers it, in such a way that every q-handle corresponds to a critical point of index q. So handle decompositions and Morse functions (with the associated dissections) are equivalent things. This means that any manipulation in terms of handle decompositions should have a counterpart in the realm of Morse functions. We can find such a purely Morse function approach in [**M3**]. However, dealing directly with handle decompositions is often easier and topologically transparent compared with the Morse function counterpart, which sometimes is demanding. Moreover, handle technology works as well for other categories of manifolds (like the *piece-wise-linear* (PL) one, see [**RS**]) where there is not a Morse function counterpart. For these reasons, we will not pursue the equivalence between Morse function and handle approaches, preferring the latter.

9.3. Moves on handle decompositions

There are two basic ways to modify a given handle decomposition of a triad (M, V_0, V_1) (up to equivalence of triads).

Handle sliding. This is synonymous with modifying the attaching map of a handle in the decomposition, say H_r, staying in the same isotopy class. We have already noticed in Chapter 7 that, up to diffeomorphism, this does not modify M_r; then we can continue the decomposition by composing the subsequent attaching maps with such a diffeomorphism. Finally, we obtain a decomposition diffeomorphic to the given one (possibly by attaching a final collar of V_1 to normalize the form).

Before describing the other modification, let us give a definition. Let

$$\cdots \cup H_r^{q_r} \cup H_{r+1}^{q_{r+1}} \cup \dots$$

be a fragment of a handle decomposition of a triad (M, V_0, V_1). Assume that $q_r = q$, $q_{r+1} = q+1$. Both the embedded b-sphere S_b of H_r^q (which is diffeomorphic to S^{m-q-1}) and the embedded a-sphere S_a of H_{r+1}^{q+1} (which is diffeomorphic to S^q) are submanifolds of the $(m-1)$-manifold $V_{1,r}$, and $\dim S_b + \dim S_a = m-1$. By transversality and up to handle sliding, we can assume that S_b and S_a intersect transversely at a finite number of points.

Definition 9.6. The adjacent handles $H_r \cup H_{r+1}$ form *a pair of complementary handles* provided that S_b and S_a intersect transversely in $V_{1,r}$ at *exactly one point.*

Cancelling/inserting pairs of complementary handles. We can state the basic handle cancellation result.

PROPOSITION 9.7. *If*

$$\cdots \cup H_r^q \cup H_{r+1}^{q+1} \cup \ldots$$

is a pair of complementary handles in a handle decomposition of (M, V_0, V_1), *then* $(M_{r-1}, V_0, V_{1,r-1})$ *is diffeomorphic to* $(M_{r+1}, V_0, V_{1,r+1})$. *Hence we can cancel that pair of handles and get a handle decomposition of the form*

$$C_0 \cup H_1^{q_1} \cup \cdots \cup H_{r-1}^{q_{r-1}} \cup H_{r+2}^{q_{r+2}} \cdots \cup H_k^{q_k} \cup C_1 \ .$$

Reciprocally, we can freely insert a pair of complementary handles between any two adjacent handles into a given decomposition.

We postpone the proof until after Proposition 9.10.

A key problem is to study the handle decompositions of a given triad up to the *move-equivalence* relation generated by such basic moves. Using Cerf's theory [**Ce2**] (see [**Kirby**]), we can prove the following nontrivial fact.

THEOREM 9.8. *Any two handle decompositions of a triads* (M, V_0, V_1) *are move-equivalent to each other.*

We will neither prove nor use this rather demanding result. We limit to some remarks and simple applications.

• For every handle decomposition $\mathcal{H}$ of (M, V_0, V_1) set

$$\chi(\mathcal{H}) = \sum_q (-1)^q |\mathcal{H}^q|$$

where $|\mathcal{H}^q|$ denotes the number of q-handles of $\mathcal{H}$. This *characteristic* of $\mathcal{H}$ is move-equivalence *invariant.* It follows from Theorem 9.8 that

$$\chi(M, V_0, V_1) := \chi(\mathcal{H})$$

is a well defined topological characteristic of the triad. Later we will establish this fact in a more elemetary way, at least for the handle decompositions which are move-equivalent to decompositions carried by some Morse function (see Remark 14.2).

• The following is an important application of sliding handle to specialize the handle decompositions.

DEFINITION 9.9. A handle decomposition of (M, V_0, V_1) is said to be *ordered* if

- For every $q = 0, \ldots, m-1$, the $q+1$ handles are attached after the q-handles;
- For every $q = 0, \ldots, m$, the q-handles are attached simultaneously. Precisely, if $\mathcal{H}^q$ denotes the pattern of q-handle, $M_{q-1} = C_0 \cup \mathcal{H}^0 \cup \cdots \cup \mathcal{H}^{q-1}$, then the attaching maps of the handles in $\mathcal{H}^q$ have disjoint images in $V_{1,q-1}$.

PROPOSITION 9.10. (Reordering) *By handle sliding, every handle decomposition of* (M, V_0, V_1) *can be transformed into an ordered decomposition.*

Proof : Let

$$\cdots \cup H_r^{q_r} \cup H_{r+1}^{q_{r+1}} \cup \dots$$

be a fragment of a given handle decomposition $\mathcal{H}$. Set $q_r = p$, $q_{r+1} = q$, and assume that $p \geq q$. The embedded b-sphere S_b of H_r^p is diffeomorphic to S^{m-p-1} while the embedded a-sphere S_a of H_{r+1}^q is diffeomorphic to S^{q-1}. Then $\dim S_b + \dim S_a \leq m-2 < m-1$. Up to handle sliding, we can assume that S_b and S_a are transverse submanifolds of the $(m-1)$-manifold $V_{1,r}$, so that $S_b \cap S_a = \emptyset$. There is a tubular neighbourhood U of S_b contained in the b-tube T_b around S_b, such that $S_a \cap U = \emptyset$; T_b itself is a tubular neighbourhood of S_b. By the uniqueness of the tubular neighbourhood up to isotopy and the extension of isotopy to diffeotopy, there is a diffeotopy of $V_{1,r}$ which keeps S_b fixed and pushes the complement of U in T_b (hence S_a) outside T_b. It follows that, up to handle sliding, the two handles now have disjoint attaching tubes so that we can attach them in the inverse order or even simultaneously. The proposition follows by several applications of this argument.

■

REMARK 9.11. In terms of Morse functions, the last proposition corresponds to the existence of Morse functions such that critical points of the same index share the same critical value, and the critical values strictly increase together with the corresponding indices.

Proof of Proposition 9.7. Let us consider first the simplest case, when $q = 0$. Attaching a 0-handle means "creating" a new disjoint m-ball component

$$H_r^0 = D^m = \{0\} \times D^m \ .$$

The whole boundary S^{m-1} forms the b-sphere. If the 1-handle H_{r+1}^1 is complementary to H_r^0, then its attaching map embeds one component of

$$\partial D^1 \times D^{m-1} = \{-1, 1\} \times D^{m-1}$$

in S^{m-1}, while the other component is embedded in $V_{1,r-1} = V_{1,r} \setminus S^{m-1}$. The partial attachment of $D^1 \times D^{m-1}$ to D^m is a shelling (refer to Section 7.5) of D^m producing another diffeomorphic copy of D^m. Then the remaining component of the attaching map finally produces a shelling of M_{r-1}, hence a diffeomorphic copy of it. The same facts hold in the general case by a more elaborate argument. Assume first that the complementary handles have normalized attaching maps as follows. Let us decompose the b-sphere S_b of H_r^q as $S_b = D_b^+ \cup D_b^-$, where both $D_b^\pm$ are diffeomorphic to D^{m-q-1} and intersect along an equatorial $(m-q-2)$-sphere. Then the b-tube around S_b is given as $T_b = D^q \times (D_b^+ \cup D_b^-)$. Similarly for the a-sphere and the a-tube of H^{q+1}_{r+1}, let $S_a = D^+_a \cup D^-_a$, $D^\pm_a \sim D^q$, $D^+_a \cup D^-_a \sim S^{q-1}$ and $T_a = (D^+_a \cup D^-_a) \times D^{m-q-1}$. Assume that the intersection, say A, between the image of the attaching map of H^{q+1}_{r+1} and T_b is equal to $D^q \times D^+_b$, and that the inverse image of A, say $\hat{A}$, is equal to $D^+_a \times D^{m-q-1}$, so that $\hat{A} \sim A$ and $\hat{A} \cap S_a = D^+_a$ is mapped to $D^q \times \{x_0\}$, x_0 being the 'centre' of D^+_b. In such a normalized situation, we can factorize the attachment of the pattern made by the two complementary handles as follows:

(1) First, glue H^{q+1}_{r+1} to H^q_r by using as attaching map the restriction of the whole attaching map to $\hat{A}$. This is shelling of a disk, so it results in a smooth m-disk with a remaining attaching zone contained in the boundary and diffeomorphic to an $(m-1)$-disk.

(2) Perform the remaining attachment; actually, this is a further shelling over M_{r-1}.

This achieves the result in the normalized situation. In our hypothesis, we have such a normalized situation provided that we replace the whole b-tube T_b with a smaller tubular neighbourhood U of S_b contained in T_b. Now, similarly to the proof of Proposition 9.10, by the uniqueness of the tubular neighbourhood up to isotopy and the extension of isotopy to diffeotopy, there is a diffeotopy which keeps S_b fixed and transforms $U \cup H^{q+1}_{r+1}$ to a pair of complementary handles in a normal situation. This completes the proof.

■

• A measure of the complication of a given handle decomposition is the *total number of handles*. For example, if it is equal to 0, then (M, V_0, V_1) is diffeomorphic to the *product triad* $(V_0 \times [0,1], V_0, V_0)$; in particular, this implies that V_0 and V_1 are diffeomorphic. If a boundaryless M has a decomposition formed by one 0-handle and one m-handle, then M is a twisted sphere. A natural task would be to reduce such a complication using the basic moves. The following is a first, simple but useful, step in this direction.

Proposition 9.12. (Cancellation of 0- and m-handles) *Assume that M is connected. Then the following fact hold.*

(1) For every triad of the form $(M, \emptyset, \emptyset)$ (i.e. M is boundaryless), every handle decomposition $\mathcal{H}$ is move-equivalent to an ordered decomposition $\mathcal{H}'$ with only one 0-handle and only one m-handle.

(2) For every triad of the form $(M, \emptyset, \partial M)$, $\partial M \neq \emptyset$, every handle decomposition $\mathcal{H}$ is move-equivalent to an ordered decomposition $\mathcal{H}'$ with only one 0-handle and without m-handles.

(3) For every triad of the form $(M, \partial M, \emptyset)$, $\partial M \neq \emptyset$, every handle decomposition $\mathcal{H}$ is move-equivalent to an ordered decomposition $\mathcal{H}'$ with only one m-handle and without 0-handles.

(4) For every triad of the form (M, V_0, V_1), both V_0 and V_1 being nonempty, every handle decomposition $\mathcal{H}$ is move-equivalent to an ordered decomposition $\mathcal{H}'$ without both 0- and m-handles.

Proof : By handle sliding, we can assume that the decomposition is ordered. Assume that we have attached a certain number of 0-handles; that is, we have created a set of disjoint components diffeomorphic to D^m. The only way to get the connectedness of M is through the 1-handles. By successive elimination of complementary $H^0 \cup H^1$ or reordering, we eventually reach two possible situations: either we remain with only one 0-handle, and this happens when $V_0 = \emptyset$ (otherwise M would be not connected), or we remain with no 0-handles, which happens when $V_0 \neq \emptyset$ and the 1-handles connect all the components of C_0. To deal with the m-handles, it is enough to apply the same argument to the dual decomposition.

■

REMARK 9.13. In terms of Morse functions, the first case of the above proposition, for example, corresponds to the existence of functions with only one local (hence absolute) minimum and one local (hence absolute) maximum.

9.3.1. The CW complex associated to an ordered decomposition. Let M be boundaryless. Let

$$H^0 \cup \{H^1\} \cup \{H^2\} \cdots \cup \{H^{m-1}\} \cup H^m$$

be an ordered handle decomposition of the triad $(M, \emptyset, \emptyset)$ with one 0-handle and one m-handle; $\{H^j\}$ means a (possibly empty) pattern of i_j j-handles attached simultaneously. Every handle H has a natural retraction

$$r : H \to \text{core}(H) \cup a - \text{tube}(H)$$

which realizes a homotopy equivalence. By using the notations fixed above, we are going to construct inductively homotopy equivalence

$$l_j : W_j \to K_j$$

where K_0 consists of one point and K_j will be obtained by attaching i_j j-cells to K_{j-1}; we eventually get a homotopy equivalence

$$l : M \to K, \ K = K_m \ .$$

By the very definition of this term, K is a *finite CW-complex of dimension* m. Let K_0 be the core of H^0; then $l_0 : M_0 \to K_0$ is an instance of retraction r, as above. Assume we have defined $l_{j-1} : M_{j-1} \to K_{j-1}$. Then

$$M_j = M_{j-1} \cup_{\{h_j\}} \{H^j\}$$

is homotopy equivalent (via the retraction $l_j := l_{j-1} \circ \{r_j\}$) to

$$K_j = K_{j-1} \cup_{\{g_j\}} \{D^j\}$$

where $\{g_j\}$ is the restriction of $l_{j-1} \circ \{h_j\}$.

Assume now that ∂M is not empty and consider the triad $(M, \partial M, \emptyset)$. In such a case, the ordered handle decomposition has no m-handles, hence there is a homotopy equivalence $l : M \to K$, where K *is a finite CW-complex of dimension* $d \leq m - 1$.

9.4. Compact 1-manifolds

We use the handle technology to classify compact 1-manifolds up to diffeomorphism. This is simple and intuitive; nevertheless, it is a fundamental result with many applications (see Chapter 11). It is not restrictive to assume that these manifolds are connected.

PROPOSITION 9.14. *(1) A compact connected boundaryless 1-manifold is diffeomorphic to S^1.*

(2) A compact connected 1-manifold with nonempty boundary is diffeomorphic to the interval D^1.

Proof : In both cases, apply Proposition 9.12. In the second case, there is a handle decomposition of $(M, \emptyset, \partial M)$ formed by one 0-handle (of dimension 1). Hence $(M, \emptyset, \partial M)$ is diffeomorphic to $(D^1, \emptyset, \{\pm 1\})$. In the first case, there is a handle decomposition of $(M, \emptyset, \emptyset)$ formed by one 0-handle and one 1-handle (of dimension 1). Hence M is a twisted 1-sphere and we know from Chapter 7 that it is diffeomorphic to S^1.

∎

CHAPTER 10

Bordism

For every $m \geq 0$, denote by $\mathcal{S}_m$ the class of smooth compact (not necessarily connected) boundaryless m-manifolds. A natural question would be to classify the elements of $\mathcal{S}_m$ up to *diffeomorphism*. We can also specialize the question to the class $\mathcal{O}_m$ of oriented manifolds up to oriented diffeomorphism. Sometimes, we will use $\mathcal{M}_m$ to indicate indifferently either $\mathcal{S}_m$ or $\mathcal{O}_m$. It turns out that, beyond $m \leq 2$, these are very demanding, even hopeless questions. Therefore, it is natural to relax the diffeomorphism to a certain equivalence up to (possibly oriented) *bordism*.

Homotopy groups $\pi_m(X, x_0)$ of any pointed topological space (X, x_0) provide the basic examples of algebraic/topological functors and are constructed by implementing the following idea: to get information about a complicated "unknown" space X, continuously map to it "tame" spaces (the m-sphere) and study the behaviour of these *singular tame* objects in X up to homotopy, which is a basic prototype of bordism between maps. Note that the singular "tame" objects are, in general, not so simple, in spite of the tame source spaces, as the maps and their images in X might be complicated. The same idea can be implemented by considering singular smooth m-manifolds in X; that is, continuous maps $f : M \to X$ where $M \in \mathcal{M}_m$, up to suitable *bordism of maps* (extending the bordism of manifolds mentioned above). This leads in a simple way to further algebraic/topological functors; once the relative theory for topological pairs (X, A) has also been developed, then one easily checks that these functors satisfy the so-called *Eilenberg-Steenrod axioms* which characterize *generalized homology theories*. All this specializes to the case when X itself belongs to $\mathcal{M}_k$, for some k. We will develop this differential/topological specialization in Chapter 11.

10.1. The bordism modules of a topological space

Let X be a topological space. For every $m \geq 0$, a *singular smooth m-manifold in* X is a continuous map $f : M \to X$, where $M \in \mathcal{S}_m$. Denote by

$$\mathcal{S}_m(X)$$

the set of such singular manifolds, to which *we formally add the empty set*.

DEFINITION 10.1. $(M, f) \in \mathcal{S}_m(X)$ is a *singular boundary* if there is a compact smooth $(m+1)$-manifold with boundary $(W, \partial W)$, a diffeomorphism $\rho : M \to \partial W$, and a continuous map $F : W \to X$ such that $F \circ \rho = f$.

Let us put on $\mathcal{S}_m(X)$ the following relation. We say that (M_0, f_0) is *bordant with* (M_1, f_1), and we write $(M_0, f_0) \sim_b (M_1, f_1)$, if the disjoint union $(M_0, f_0) \amalg (M_1, f_1)$ is a singular boundary. It is consistent to state that $(M, f) \sim_b \emptyset$ if and only if (M, f) is a singular boundary.

We claim that this is an *equivalence relation*:

• The cylinder $(M \times [0,1], F)$, $F(x,t) = f(x)$ for every $t \in [0,1]$, establishes that $(M, f) \sim_b (M, f)$, $\rho : M \amalg M \to (M \times \{0\}) \amalg (M \times \{1\})$ being the natural inclusion.

• As the disjoint union is symmetric, then $\sim_b$ is also symmetric.

• Transitivity follows by gluing smooth manifolds along boundary components. Precisely, assume that (W_0, F_0), $\rho_0 : M_0 \amalg M_1 \to \partial W_0$ realize $(M_0, f_0) \sim_b (M_1, f_1)$, while (W_1, F_1), $\rho_1 : M_1 \amalg M_2 \to \partial W_1$ realize $(M_1, f_1) \sim_b (M_2, f_2)$. Then F_0 and F_1 match to define a smooth map F_2 on $W_2 := W_0 \amalg_\psi W_1$, where ψ is the composition of the restriction of ρ_0^{-1} to $\rho_0(M_1)$ with the restriction of ρ_1 to M_1. Finally, (W_2, F_2) together with the disjoint union of ρ_0 restricted to M_0 and ρ_1 restricted to M_2 realize $(M_0, f_0) \sim_b (M_2, f_2)$.

We denote by $\eta_m(X)$ the quotient set $\mathcal{S}_m(X)/\sim_b$ and by $[M, f]$ the equivalence class of (M, f).

The disjoint union is an operation on $\mathcal{S}_m(X)$. It is immediate that it descends to the quotient; that is, $[M, f] + [N, g] := [M \amalg N, f \amalg g]$ is a well defined operation on $\eta_m(X)$.

PROPOSITION 10.2. *The set $\eta_m(X)$ endowed with the operation $+$ is an Abelian group.*

Proof : The operation $+$ is associative and commutative because the disjoint union is associative and commutative. The class $[\emptyset]$ of the singular boundaries is the zero element. For every $\alpha = [M, f]$, $-\alpha = \alpha$; in fact, by using the cylinder as above we see that $[M, f] + [M, f] = 0$.

■

Since for every α, $\alpha = -\alpha$, then $(\eta_m(X), +)$ can be enhanced to be a $\mathbb{Z}/2\mathbb{Z}$-module; that is, a $\mathbb{Z}/2\mathbb{Z}$-*vector space* $(\eta_m(X), +, \cdot)$. We call it the *unoriented m-bordism module* of X.

10.1.1. The oriented bordism $\mathbb{Z}$-modules. We follow the same procedure by using oriented manifolds.

We denote by $\mathcal{O}_m(X)$ the set of *oriented* singular m-manifolds $f : M \to X$, for $M \in \mathcal{O}_m$.

The pair (M, f) is a *singular oriented boundary* if (W, F), $\rho : M \to \partial W$ are as above, $(W, \partial W)$ is oriented and ρ preserves the orientation.

The relation $(M_0, f_0) \sim_{ob} (M_1, f_1)$ on $\mathcal{O}_m(X)$ is defined by requiring that $(M_1, f_1) \amalg (-M_2, f_2)$ is a singular oriented boundary. The verification that it is an equivalence relation is similar:

• The cylinder can be naturally oriented in such a way that its oriented boundary is $M \amalg -M$.

• To get the symmetry, it is enough to replace W with $-W$.

• As for the transitivity, we glue again W_0 and W_1 by taking into account that the gluing diffeomorphism ψ reverses the orientation: in ∂W_0 there is a copy of $-M_1$ while in ∂W_1 there is a copy of M_1. Hence the gluing can be performed in the oriented category.

We denote by $\Omega_m(X)$ the quotient set. Again, the operation $+$ on $\Omega_m(X)$ is induced by the disjoint union on $\mathcal{O}_m(X)$. It results in a commutative group (i.e. a *$\mathbb{Z}$-module*) $(\Omega_m(X), +)$. Again $0 = [\emptyset]$, that is the class of the singular oriented boundaries. By means of the oriented cylinder we see that $-[M, f] = [-M, f]$. This is the *m-oriented bordism module* of the topological space X.

There is a natural group homomorphism

$$\sigma_m : \Omega_m(X) \to \eta_m(X)$$

which maps the class of (M, f) in $\Omega_m(X)$ to its class in $\eta_m(X)$, just by "forgetting the orientation".

As many considerations run formally in the same way for both bordism versions, sometimes we will indifferently indicate by $\mathcal{M}_m(X)$ either $\mathcal{S}_m(X)$ or $\mathcal{O}_m(X)$, and by $\mathcal{B}_m(X) = \mathcal{B}_m(X; R)$, $R = \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}$, the quotient R-module $\eta_m(X)$ or $\Omega_m(X)$.

LEMMA 10.3. *Let $\phi : N \to M$ be a diffeomorphism (preserving the orientation in the oriented setting); $f : M \to X$ and $m = \dim M$. Then $[M, f] = [N, f \circ \phi] \in \mathcal{B}_m(X)$.*

Proof : The cylinder $(M \times [0, 1], f \circ \pi)$ ($\pi : M \times [0, 1] \to M$ being the projection), and $\rho : M \amalg N \to (M \times \{0\}) \amalg (M \times \{1\})$, $\rho = \mathrm{id}_M \amalg \phi$, realize $(M, f) \sim_{\mathcal{B}} (N, f \circ \phi)$.

■

REMARK 10.4. Let (M, f) be a singular boundary in X. Let $((W, \partial W), F)$ and $\rho : M \to \partial W$ realize $(M, f) \sim_{\mathcal{B}} \emptyset$. By applying Lemma 10.3 we have

$$(M, f) \sim_{\mathcal{B}} (\partial W, \partial F)$$

and this is realized by a cylinder; obviously, $((W, \partial W), F)$ and $\mathrm{id}_{\partial W}$ realize

$$(\partial W, \partial F) \sim_{\mathcal{B}} \emptyset \ .$$

By applying to this situation the gluing argument employed to show the transitivity, we can conclude that it is not restrictive to require that $M = \partial W$ and $\rho = \mathrm{id}_M$.

An important special case. When $X = \{x_0\}$ consists of one point, then the maps are immaterial and, by definition, $\mathcal{B}_m := \mathcal{B}_m(\{x_0\})$ is the quotient of $\mathcal{M}_m$ up to *bordism of manifolds*. It follows from Lemma 10.3 that the bordism extends the diffeomorphism equivalence in the category.

10.2. Bordism covariant functors

We have the following proposition. All verifications are straightforward consequences of the definitions.

Proposition 10.5. *For every $m \geq 0$,*

$$X \ \Rightarrow \ \mathcal{B}_m(X)$$

$$g : X \to Y \ \Rightarrow \ g_* : \mathcal{B}_m(X) \to \mathcal{B}_m(Y), \ g_*([M, f]) = [M, g \circ f]$$

is a covariant functor from the category of topological spaces and continuous maps to the category of R-modules and R-linear maps. That is,

$$(g \circ h)_* = g_* \circ h_* \ ,$$

$$(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{B}_m(X)} \ .$$

■

In particular, if $g : X \to Y$ is a homeomorphism, then g_* is an R-linear isomorphism with inverse $(g^{-1})_*$. Considered up to linear isomorphism, $\mathcal{B}_m(X)$ *is an invariant of the topological type of* X. The family introduced above of "forgetting" linear maps

$$\{\sigma_m : \mathcal{B}_m(X; \mathbb{Z}) \to \mathcal{B}_m(X; \mathbb{Z}/2\mathbb{Z})\}$$

is *functorial*; that is, they form commutative squares together with the respective families of g_*'s. We formally express it by saying that "$g_* \circ \sigma = \sigma \circ g_*$".

10.3. Relative bordism of topological pairs

We consider topological pairs (X, A), where A is a subspace of X, and the class $\mathcal{M}^\partial_m$ of compact smooth m-manifolds with boundary $(M, \partial M)$. This incorporates the "absolute situations" by identifying X with the pair $(X, \emptyset)$ and a boundaryless manifold $M \in \mathcal{M}_m$ with $(M, \emptyset)$.

A *relative singular m-manifold in* (X, A) is a continuous map of pairs

$$f : (M, \partial M) \to (X, A)$$

where, by definition, $f(\partial M) \subset A$ and $(M, \partial M) \in \mathcal{M}^\partial_m$. We set $\mathcal{M}_m(X, A)$ as the collection of these relative singular m-manifolds.

Definition 10.6. $f : (M, \partial M) \to (X, A)$ is a *relative singular boundary* if there are continuous pair maps $F : (W, V) \to (X, A)$, $\rho : (M, \partial M) \to (Z, \partial Z)$ such that:

(1) $(W, \partial W) \in \mathcal{M}^\partial_{m+1}$;
(2) $(V, \partial V)$ and $(Z, \partial Z)$ are smooth m-submanifolds of ∂W such that

$$\partial W = V \cup Z, \ V \cap Z = \partial V = \partial Z \ ;$$

(3) $\rho : (M, \partial M) \to (Z, \partial Z)$ is a smooth diffeomorphism (preserving the orientation in the oriented case). In particular if ∂M is empty, then V and Z are also boundaryless, $\partial W = V \amalg Z$ and $F(V) \subset A$.

We put on $\mathcal{M}_m(X, A)$ the equivalence relation

$$(M_0, \partial M_0, f_0) \sim_{\mathcal{B}} (M_1, \partial M_1, f_1)$$

if and only if $(M_0, \partial M_0, f_0) \amalg (-M_1, \partial M_1, f_1)$ is a relative singular boundary (in the unoriented case the sign "$-$" is immaterial). The verification that it is an equivalence relation (in particular the transitivity) incorporates some instances of corner smoothing, in accordance with Remark 7.16.

The disjoint union on $\mathcal{M}_m(X, A)$ descends to an operation $+$ on the quotient set that eventually makes it a R-module $\mathcal{B}_m(X, A) = \mathcal{B}_m(X, A; R)$, called the *realtive m-bordism R-module* of the topological pair (X, A).

Proposition 10.5 extends directly to the following.

PROPOSITION 10.7. *For every $m \geq 0$,*

$$(X, A) \;\Rightarrow\; \mathcal{B}_m(X, A)$$

$$g : (X, A) \to (Y, B) \;\Rightarrow\; g_* : \mathcal{B}_m(X, A) \to \mathcal{B}_m(Y, B)$$

$$g_*([M\partial M, f]) = [M, \partial M, g \circ f]$$

is a covariant functor from the category of pairs of topological spaces and continuous pair maps to the category of R-modules and R-linear maps.

10.4. On Eilenberg-Steenrood axioms

The singular homology (sometimes called "Betti homology") with coefficients in the ring R is a family of functors (indexed by $m \geq 0$), formally of the same kind of Propositions 10.5, 10.7. The (E-S)-axioms are abstractions of some properties satisfied by the singular homology functors and which deserve the name as all models (no matter how they have been produced) that fulfill such axioms are isomorphic to each other, at least if we restrict to pairs of compact CW-complexes (see [**Hatch**]). It turns out that the most critical is the so-called *dimension axiom*; every model which satisfies the other axioms (with the possible exception of "dimension") is called a *generalized homology theory*. We are going to verify that this is the case of bordism. The verifications are geometric/topological in nature and often immediate consequences of the definitions.

The homotopy axiom. *If $g_0, g_1 : (X, A) \to (Y, B)$ are homotopic through pair maps, then $g_{0,*} = g_{1,*}$.*

We have to show that for every $[M, \partial M, f] \in \mathcal{B}_m(X, A)$,

$$[M, \partial M, g_0 \circ f] = [M, \partial M, g_1 \circ f] \text{ in } \mathcal{B}_m(Y, B) .$$

Given a homotopy

$$G : (X \times [0, 1], A \times [0, 1]) \to (Y, B)$$

between g_0 and g_1, then

$$F : (M \times [0, 1], \partial M \times [0, 1]) \to (Y, B), \;\; f_t = g_t \circ f$$

together with the natural inclusion of $(M, \partial M) \amalg (M, \partial M)$ in $\partial(M \times [0,1])$ realize that $(M, \partial M, g_0 \circ f) \sim_{\mathcal{B}} (M, \partial M, g_1 \circ f)$.

■

This implies that if $g : (X, A) \to (Y, B)$ is a relative homotopy equivalence, then g_* is an R-linear isomorphism. Up to isomorphism, *the bordism modules are invariants of the homotopy type rather than the topology type.*

Direct sum over path-connected components. For every topological space X, $\mathcal{B}_m(X)$ is isomorphic to the direct sum of the modules $\mathcal{B}_m(X_c)$, where X_c varies among the path-connected components of X. This follows from the fact that continuous maps send every path-connected component of a manifold M to one path connected component of X. A similar fact holds in the relative version.

Long exact sequence. For every $m \geq 1$ there is the natural well defined R-linear map

$$\partial : \mathcal{B}_m(X, A) \to \mathcal{B}_{m-1}(A), \ \partial([M, \partial M, f]) = [\partial M, \partial f] \ .$$

Denote by $i_* : \mathcal{B}_m(A) \to \mathcal{B}_m(X)$, $j_* : \mathcal{B}_m(X, \emptyset) \to \mathcal{B}_m(X, A)$ the R-linear maps induced by the inclusions. Then we have a *bordism long sequence* of linear maps

$$\cdots \to \mathcal{B}_m(A) \xrightarrow{i_*} \mathcal{B}_m(X) \xrightarrow{j_*} \mathcal{B}_m(X, A) \xrightarrow{\partial} \mathcal{B}_{m-1}(A) \to \cdots$$

which ends on the right with the 0 R-module.

Recall that a sequence of linear maps

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is *exact in* B if $\ker(\beta) = \alpha(A)$. Then we have the following.

(1)The long sequences are functorial: if $g : (X, A) \to (Y, B)$ *then the respective long sequences together with the family of linear maps* $\{g_*\}$ *form commutative squares.*

(2) Every bordism long sequence is exact everywhere.

Fuctoriality is an immediate consequence of the definitions. The verifications of exactness are simple and useful exercises. Let us show, for example, that the above long sequence is exact in $\mathcal{B}_m(X, A)$. If $[N, g] \in \mathcal{B}_m(X)$, then N is boundaryless, so it is clear that $\partial \circ j_*([N, g]) = 0 \in \mathcal{B}_{m-1}(A)$. On the other hand, assume that $(M, \partial M, f)$ is in the kernel of ∂ and $(W, \partial W, F)$ realizes that $(\partial M, \partial f)$ is a boundary. Then, by gluing W and M along ∂M, we get $\tilde{f} : \tilde{M} \to X$, $\tilde{M}$ being boundaryless, where $\tilde{f}$ is obtained by matching f and F, so that $j_*([\tilde{M}, \tilde{f}]) = [M, \partial M, f] \in \mathcal{B}_m(X, A)$.

■

Excision. Let $Z \subset A \subset X$ be a triad of topological spaces. Assume that the closure $\bar{Z}$ of Z in X is contained in the interior $\mathring{A}$ of A.

For every $m \geq 0$*, the linear map induced by the inclusion*

$$i_* : \mathcal{B}_m(X \setminus Z, A \setminus Z) \to \mathcal{B}_m(X, A)$$

is an isomorphism. We say that Z is excisable*.*

Let us prove first that it is surjective. Let $[M, \partial M, f] \in \mathcal{B}_m(X, A)$. The manifold M can be endowed with a distance d compatible with its topology so that (M, d) is a compact metric space; for example, embed M in some $\mathbb{R}^n$ and take the distance induced by the Euclidean distance. The compact set $K := f^{-1}(\bar{Z})$ is contained in the open set $\tilde{A} := f^{-1}(\mathring{A})$. The distance function from K

$$\delta : M \to \mathbb{R}$$

is non-negative, continuous and $K = \{\delta = 0\}$. Then there is a smooth approximation $g : M \to \mathbb{R}$ and a regular value $\epsilon > 0$ of both g and ∂g, sufficiently close to 0, such that $\tilde{M} := \{g \geq \epsilon\}$ is a compact m-submanifold with corners such that $\partial\tilde{M} = \{g = \epsilon\}$ is contained in $\tilde{A}$. Up to smoothing the corners, if $\tilde{f}$ is the restriction of f to $\tilde{M}$, we finally have that $[\tilde{M}, \partial\tilde{M}, \tilde{f}] \in \mathcal{B}_m(X \setminus Z, A \setminus Z)$ and $i_*([\tilde{M}, \partial\tilde{M}, \tilde{f}]) = [M, \partial M, f] \in \mathcal{B}_m(X, A)$. To prove the injectivity we apply the same argument to $(W, \partial W, F)$, which shows that a $(M, \partial M, f) \in \mathcal{M}_m(X \setminus Z, A \setminus Z)$ is a relative singular boundary in (X, A).

■

About the dimension axiom. This axiom for the singular homology (with coefficients in R) determines the homology modules of a singleton. Precisely, the 0-module is isomorphic to R, while the others are all trivial.

For every X, $\mathcal{B}_0(X)$ has a clear topological meaning. In fact, by using the classification of compact 1-manifolds (Proposition 9.14), it is easy to check that it is isomorphic to the direct sum $\oplus_{\pi_0(X)} R$, where $\pi_0(X)$ is the set of path connected components of X. In particular, $\mathcal{B}_0 = R$. On the other hand, we do not know for the moment whether the modules $\mathcal{B}_m$, $m > 1$, are all trivial. We will see in Section 14.8 that *they are not.*

The (E-S)-axioms establish relationships between the modules (in any generalized homology theory) $\mathcal{H}_*(X)$ of a given space and the ones of the presumably simpler pieces of some suitable decomposition of X. If "dimension" also holds, then in many cases they allow computing (up to linear isomorphism) the modules of X. Without "dimension" things are more complicated. The first interesting cases to consider are $X = S^n$ or the pair $(X, A) = (D^n, S^{n-1})$. These are the building blocks of CW-complexes.

• As the n-disk is contractible for every $n \geq 0$, by "homotopy" $\mathcal{H}_m(D^n) \sim \mathcal{H}_m$ for every $m \geq 0$.

• For every $n \geq 1$, we can decompose S^n as the union of the closed northern and southern hemispheres (both diffeomorphic to D^n)

$$S^n = D^+ \cup D^-,\ D^+ \cap D^- = S^{n-1}\ .$$

We claim that the inclusion induces isomorphisms

$$i_* : \mathcal{H}_m(D^+, S^{n-1}) \to \mathcal{H}_m(S^n, D^-)\ .$$

We cannot directly apply "excision" to $Z = \mathring{D}^-$. We can do it by using instead $\tilde{Z} \subset D^-$ equal to the complement of a small collar of S^{n-1} in D^-. Finally, we use "homotopy" and the fact that $(S^n \setminus \tilde{Z}, D^- \setminus \tilde{Z})$ retracts to (D^+, S^{n-1}) to achieve the required isomorphisms.

• Again for $n \geq 1$, we have the exact long sequence of the pair (D^n, S^{n-1})

$$\cdots \to \mathcal{H}_m(S^{n-1}) \xrightarrow{i_*} \mathcal{H}_m(D^n) \xrightarrow{j_*} \mathcal{H}_m(D^n, S^{n-1}) \xrightarrow{\partial} \mathcal{H}_{m-1}(S^{n-1}) \to \cdots$$

and the one of the pair (S^n, D^-)

$$\cdots \to \mathcal{H}_m(D^-) \xrightarrow{i_*} \mathcal{H}_m(S^n) \xrightarrow{j_*} \mathcal{H}_m(S^n, D^-) \xrightarrow{\partial} \mathcal{H}_{m-1}(D^-) \to \cdots \ .$$

• If the theory $\mathcal{H}$ satisfies also "dimension", by simple algebraic considerations we realize that for $n \geq 1$,

- $\partial : \mathcal{H}_m(D^n, S^{n-1}) \to \mathcal{H}_{m-1}(S^{n-1})$ is an isomorphism for $m \geq 2$;
- $\jmath_* : \mathcal{H}_m(S^n) \to \mathcal{H}_m(S^n, D^-)$ is an isomorphism for $m \geq 2$;
- for every $m \geq 1$, $\mathcal{H}_m(S^n) \sim \mathcal{H}_{m-1}(S^{n-1})$ (immediately for $m \geq 2$ and with a little extra work for $m = 1$).

Then by a simple induction, we can finally achieve the computation:

For every $n \geq 1$, $m = 0, n$,

$$\mathcal{H}_m(S^n) \sim \mathcal{H}_m(D^n, S^{n-1}) \sim R \ .$$

For every $n \geq 1$, $m \geq 1$, $m \neq n$,

$$\mathcal{H}_m(S^n) \sim \mathcal{H}_m(D^n, S^{n-1}) = 0 \ .$$

If the theory (like the bordism) does not satisfy "dimension", the considerations based on the other axioms hold as well, but are not immediately conclusive.

10.5. Bordism nontriviality

Combining the axioms with the specific way the bordism has been defined, we will provide a few pieces of evidence that it is not trivial.

• Assume that X is path connected. Consider the long exact sequence of a pair (X, x_0) for some base point in X,

$$\cdots \to \mathcal{B}_m \xrightarrow{i_*} \mathcal{B}_m(X) \xrightarrow{j_*} \mathcal{B}_m(X, x_0) \xrightarrow{\partial} \mathcal{B}_{m-1} \to \cdots \ .$$

It is immediate by the bordism definition that $\partial = 0$ (hence j_* is surjective) and that i_* is injective. Hence every $\mathcal{B}_m(X)$ contains a submodule isomorphic to $\mathcal{B}_m$ which in general is not trivial. Since X is path connected, by "homotopy" this submodule does not depend on the choice of the base point x_0. When $R = \mathbb{Z}/2\mathbb{Z}$ (algebra is simpler in the case of vector spaces) we have $\eta_m(X) \sim \eta_m \oplus \eta_m(X, x_0)$.

• Assume that X is a compact connected boundaryless (possibly oriented) smooth m-manifold. Then by the approximation theorems of continuous maps by smooth maps, *it is not restrictive to assume that all maps entering the bordism treatment are smooth.*

PROPOSITION 10.8. $[X, \mathrm{id}_X] \in \mathcal{B}_m(X)$ *is nontrivial and does not belong to* $\mathcal{B}_m \subset \mathcal{B}_m(X)$*. In particular,* $\dim \eta_m(X) \geq 1 + \dim \eta_m$.

Proof: Assume that it is trivial; then there is a smooth map $F : W \to X$, such that $\partial W = X$ and $F_{|X} = \mathrm{id}_X$. Let $p \in X$. Clearly it is a regular value for ∂F. Apply to F the transversality theorems relative to ∂F. Then we can assume that $F \pitchfork \{p\}$, $Y = F^{-1}(p)$ is a proper 1-submanifold of (W, X) and $p \in Y$. By the classification of compact smooth 1-manifolds, p is contained in an interval component $I \subset Y$. Hence there is another $p' \in \partial I \subset X$ such that $p' \neq p$ and $\partial F(p) = p = \partial F(p') = p'$. This is absurd. This proves that $[X, \mathrm{id}_X] \neq 0$. Let $c : N \to \{p\}$ be a constant map representing some element of $\mathcal{B}_m \subset \mathcal{B}_m(X)$. Let $q \neq p$ so that it is a regular value for both id_X and c. If (W, F) would realize a bordism between (X, id_X) and (N, c), by applying again the relative first transversality theorem to (W, F) we should deduce that $\partial F^{-1}(q) = \{q\}$ is a boundary; again, by the classification of compact 1-manifolds, this is absurd.

■

By a similar argument, we have the following generalization.

PROPOSITION 10.9. *In the setting of Proposition 10.8, let* $[N] \in \mathcal{B}_k$ *be nontrivial, and consider* $(N \times X, \mathrm{id}_X \circ \pi)$*,* π *being the projection to* X*. Then* $[N \times X, \mathrm{id}_X \circ \pi] \in \mathcal{B}_{m+k}(X)$ *is nontrivial.*

The class $[X, \mathrm{id}_X] \in \mathcal{B}_m(X)$ is called the bordism *fundamental class* of the (possibly oriented) manifold X. If X has nonempty boundary, similar facts hold for $[X, \partial X, \mathrm{id}_X] \in \mathcal{B}_m(X, \partial X)$.

• *(On the bordism modules of spheres)* For every $n \geq 1$, consider again S^n and (D^n, S^{n-1}). If $m < n$, by transversality we can assume that every class α in $\mathcal{B}_m(S^n)$ is represented by a smooth and non surjective map $f : M \to S^n$; say that $\infty \notin f(M)$. Then f factorizes through $\mathbb{R}^n \subset \mathbb{R}^n \cup \infty = S^n$, hence it is homotopic to a constant map. By "homotopy" α belongs to $\mathcal{B}_m \subset \mathcal{B}_m(S^n)$, hence if $m < n$, $\mathcal{B}_m(S^n) = \mathcal{B}_m$.

Referring to the long exact sequence for the pair (S^n, D^-), using that D^- is contractible and "homotopy", we have that $\partial = 0$; so j_* is surjective and i_* is injective. In particular, we have

$$\eta_m(S^n) \sim \eta_m \oplus \eta_m(S^n, D^-) \sim \eta_m \oplus \eta_m(D^n, S^{n-1})$$

where for the last isomorphism we have applied "excision" and "homotopy" as above.

Referring to the long exact sequence for the pair (D^n, S^{n-1}), we see that i_* is surjective, hence $j_* = 0$, ∂ is injective. Hence we have, in particular, that

$$\eta_{m-1}(S^{n-1}) \sim \eta_{m-1} \oplus \eta_m(D^n, S^{n-1}) \ ;$$

hence

$$\eta_{m-1}(S^{n-1}) \oplus \eta_m \sim \eta_m(S^n) \oplus \eta_{m-1} \ .$$

By a similar inductive argument already used to compute $\mathcal{H}_*(S^n)$ when the theory $\mathcal{H}$ also satisfies "dimension", we can eventually achieve the determination of $\eta_*(S^n)$.

PROPOSITION 10.10. *(1) For every $m \geq 0$, $\eta_m(S^0) = \eta_m \oplus \eta_m$.*
(2) For every $n \geq 1$, for every $0 \leq m < n$, $\eta_m(S^n) = \eta_m$.
(3) For every $n \geq 1$, $k \geq 1$,

$$\eta_{n+k}(S^n) = \eta_k \oplus \eta_{n+k} \ .$$

Precisely, every class in $\eta_{n+k}(S^n)$ either belongs to η_{n+k} or is of the form $[N \times S^n, \mathrm{id}_{S^n} \circ \pi]$ as in Proposition 10.9

It is already clear from these few remarks that the determination of $\mathcal{B}_m$, for every $m \geq 0$ (that is, of the actual failure of "dimension") is a key point of this story.

10.6. Relation between bordism and homotopy group functors

Here we assume some familiarity with the homotopy group $\pi_m(X, x_0)$, $m \geq 1$, of the *pointed* topological space (X, x_0) (see for instance [**Hatch**]). When $m = 1$ it is called the *fundamental group*. Let us recall a few facts.

• As a set $\pi_m(X, x_0)$ is formed by the classes $< f >$ of pointed continuous maps $f : (S^m, p) \to (X, x_0)$ considered up to pointed homotopy. It is endowed with a natural group operation "$\cdot$" well defined on any given representatives. The 1 element is the class of the constant pointed map. They are Abelian for $m \geq 2$, while the fundamental group is not in general. If X is path-connected, up to group isomorphism they do not depend on the choice of the base point.

• Similarly to the bordism, we have for every $m \geq 1$ a covariant functor

$$(X, x_0) \ \Rightarrow \ \pi_m(X, x_0)$$

$$g : (X, x_0) \to (Y, y_0) \ \Rightarrow \ g_* : \pi_m(X, x_0) \to \pi_m(Y, y_0), \ g_*(< f >) =< g \circ f >$$

from the category of pointed topological spaces and pointed continuous maps to the category of groups (Abelian for $m \geq 2$) and group homomorphisms.

• There is a relative version for pointed pairs (X, A, x_0) $(x_0 \in A)$ of topological spaces. Then the elements of $\pi_m(X, A, x_0)$ are relative homotopy classes $< f >$ of maps $f : (D^m, S^{m-1}, p) \to (X, A, x_0)$. As usual, the "absolute" theory is incorporated by identifying (X, x_0) with (X, x_0, x_0). If $A \neq \{x_0\}$, then $\pi_m(X, A, x_0)$ is Abelian for $m \geq 3$. Similarly to the bordism, for every $m \geq 2$ there is a natural homomorphism

$$\partial : \pi_m(X, A, x_0) \to \pi_{m-1}(A, x_0), \ \partial(< f >) =< \partial f > \ .$$

Together with the homomorphisms

$$i_* : \pi_m(A, x_0) \to \pi_m(X, x_0), \quad j_* : \pi_m(X, x_0) \to \pi_m(X, A, x_0)$$

induced by the inclusions, they give rise to the homotopy long exact sequence of the pointed pair (X, A, x_0)

$$\cdots \to \pi_m(A, x_0) \xrightarrow{i_*} \pi_m(X, x_0) \xrightarrow{j_*} \pi_m(X, A, x_0) \xrightarrow{\partial} \pi_{m-1}(A, x_0) \to \cdots$$

For every $m \geq 1$, the map (in the oriented case we stipulate that D^m inherits the standard orientation of $\mathbb{R}^m$)

$$h_m : \pi_m(X, A, x_0) \to \mathcal{B}_m(X, A), \; h_m(< f >) = [D^m, S^{m-1}, f]$$

obtained by "forgetting the base points" and replacing homotopy by bordism is well defined. It is well defined because homotopy is a special case of bordism where only the cylinders are permitted.

Proposition 10.11. *(1) For every $m \geq 1$, h_m is a group homomorphism.*

(2) The family of homomorphisms $\{h_m\}$ is functorial *("$g_* \circ h = h \circ g_*$") and commutes with the respective long exact sequences.*

Proof : Both the respective morphisms g_* and long exact sequences have the very same definition on representatives. Then (2) follows because the h's are well defined. As for (1), for simplicity, we consider the absolute case $m = 1$, but the argument generalizes without difficulty. Realize an elementary bordism W between $S^1 \coprod S^1$ and S^1 obtained by attaching a 1-handle to $(S^1 \coprod S^1) \times [0,1]$ at $(S^1 \coprod S^1) \times \{1\}$. There is a properly embedded arc $D \sim D^1$ (essentially the core of the handle) which intersects $(S^1 \coprod S^1) \times \{0\}$ at two points belonging to different components and a properly embedded arc D' dual to D (essentially the co-core of the handle) which intersects the other component of ∂W in two points. The set $W \setminus (D \cup D')$ is diffeomorphic to the cylinder $C := ((S^1 \setminus \{p\}) \coprod (S^1 \setminus \{p\})) \times [0,1]$. Let $f_0, f_1 : (S^1, p) \to (X, x_0)$. Up to the natural identification, this induces a map $F : C \to X$, $F(x,t) := f_0 \coprod f_1(x)$ which extends to a continuous map $F : W \to X$ by setting it constantly equal to x_0 on $D \cup D'$. This establishes a bordism between $(S^1, f_0) \coprod (S^1, f_1)$ and a map $g : S^1 \to X$. Recalling the definition of the operation on $\pi_1(X, x_0)$ (see [**Hatch**]), it is immediate that

$$[S^1, g] = h_1(< f_0 > \cdot < f_1 >) \; ;$$

hence

$$h_1(< f_0 > \cdot < f_1 >) = h_1(< f_0 >) + h_1(< f_1 >)$$

as desired.

∎

In general, the study of both $\ker(h_m)$ and its image is a difficult question, even if X is a compact smooth manifold. We can say something more for $m = 1$.

On the 1-bordism. It is evident that the homorphism

$$\sigma_1 : \Omega_1(X) \to \eta_1(X)$$

is surjective: given $[M, f]$ in $\eta_1(X)$, it is enough to arbitrarily orient the components of M (each diffeomorphic to S^1) to get $[\tilde{M}, f]$ in $\Omega_1(X)$ such that $\sigma_1([\tilde{M}, f]) = [M, f]$.

PROPOSITION 10.12. *Assume that X is path connected. Then the homomorphism $h_1 : \pi_1(X, x_0) \to \Omega_1(X)$ is surjective, hence the oriented bordism $\Omega_1(X)$ is a Abelian quotient group of $\pi_1(X, x_0)$. By composition with the surjective homomorphism σ_1, the same fact holds for $\eta_1(X)$.*

Proof : Let $[S^1, f] \in \Omega_1(X)$. Let $p \in S^1$ be the base point and let $q = f(p)$. Up to isotopy, hence up to bordism, we can assume that f is constantly equal to q on a closed interval J such that $p \in J \subset S^1$. Let $J = J_1 \cup J_2$, $J_1 \cap J_2 = \{p\}$. Let $\gamma_i : J_i \to X$ be a continuous path joining q and the base point x_0 and such that $\gamma_i(p) = x_0$. Then define $f' : (S^1, p) \to (X, x_0)$ to be equal to γ_i on J_i and equal to f outside J. Clearly $[S^1, f']$ belongs to the image of h_1. We claim that $[S^1, f] = [S^1, f']$. In fact it is not hard to prove that they are homotopic. For a general $[M, f]$ we can assume that M is the union of a finite number of copies S^1_j of S^1. Consider the corresponding pointed copies (S^1_j, p_j). Let $q_j = f(p_j)$. By applying the above construction for every j, we can assume that $[M, f]$ is the sum of classes each one being the image via h_1 of some $\alpha_j \in \pi_1(X, x_0)$. Finally, by applying inductively on the number of components the same argument used above to show that h_1 is a homomorphism, we conclude that $[M, f]$ is the image of the product of such α_j's.

■

We will complete the analysis of $\Omega_1(X)$ as a quotient of the fundamental group in Chapter 15, Proposition 15.3.

10.7. Bordism categories

There is another important way to organize the bordism matter. As usual, $\mathcal{M}_m$ either denotes $\mathcal{S}_m$ or $\mathcal{O}_m$, $\mathcal{B}$ either denotes η or Ω. For every $m \geq 0$, we define the *bordism category* $\mathbf{CAT}_{\mathcal{B}}(m+1)$.

- $\mathcal{M}_m$ is the class of *objects* (recall that $\emptyset$ is also an object).
- For every couple of objects $M, N \in \mathcal{M}$, a *morphism* ("arrow") $M \mapsto N$ is of the form

$$([\rho_0], [\rho_1], [W, V_0, V_1])$$

where (W, V_0, V_1) is a triad of compact smooth manifolds (recall that V_0 and V_1 are union of components of ∂W, and $\partial W = V_0 \amalg V_1$) considered up to diffeomorphisms which are isotopic to the identity on a neighbourhood of the boundary; $\rho_0 : M \to V_0$ and $\rho_1 : N \to V_1$ are diffeomorphisms (preserving the orientation in the oriented setting) considered up to isotopy.

• Two arrows $f : M \mapsto N$, $g : M' \mapsto N'$ can be composed if $N = M'$. In such a case, if $f = ([\rho_0], [\rho_1], [W, V_0, V_1])$, $g = ([\rho'_0], [\rho'_1], [W', V'_0, V'_1])$, then

$$g \circ f = ([\rho_0], [\rho'_1], [\tilde{W}, V_0, V'_1])$$

where

$$\tilde{W} = W \amalg_\psi W', \ \psi = \rho'_0 \circ \rho_1^{-1} : V_1 \to V'_0 \ .$$

It is consistent because $\tilde{W}$ is defined up to diffeomorphism relatively to the boundary and only depends on the isotopy class of the gluing diffeomorphism. Note again that gluing can be performed in the oriented setting.

• For every object $M \in \mathcal{M}_m$, $M \neq \emptyset$, the *unit arrow* is

$$1_M = ([\mathrm{id}_M], [\mathrm{id}_M], [M \times [0,1], M \times \{0\}, M \times \{1\}]) \ .$$

The discussion made in Chapter 9 about Morse functions on triads, dissections, and handle-decompositions can be rephrased within the bordism category: every arrow is the composition of *elementary arrows* that is supported by triads admitting a handle decomposition with only one handle (of some index).

10.8. A glance at TQFT

An $(m+1)$ *topological quantum field theory* (TQFT) is a kind of nontrivial representation of $\mathbf{CAT}_\mathcal{B}(m+1)$ in the category of vector spaces on some scalar field K. In the last decades, this has emerged as a potent paradigm, the source of plenty of so-called "quantum invariants" for 3-dimensional manifolds and the right conceptual framework for deep 4-dimensional invariants. The actual categorical definition involves many subtleties (see for instance [**Tur**]). Here we merely provide a rough outline of the main features.

First, we note that the objects $\mathcal{M}_m$ of a bordism category are endowed with the disjoint union operation "$\amalg$".

Let K be a field and denote by $\mathcal{V}_K$ the category having as objects the class V_K of *finite-dimensional* K-vector spaces and as morphisms the K-linear maps. The class V_K is also endowed with an operation "$\otimes$" given by the tensor product.

An $(m+1)$ TQFT is a morphism of categories

$$\mathbf{CAT}_\mathcal{B}(m+1) \ \Rightarrow \mathcal{V}_K$$

which satisfies certain conditions:

- To every object $M \in \mathcal{M}_m$ is associated an object $Z(M) \in V_K$.
- To every arrow $f : M \mapsto N$ in $\mathbf{CAT}_\mathcal{B}(m+1)$ is associated a linear map $Z(f) : Z(M) \to Z(N)$, in such a way that the composition is respected:

$$Z(g \circ f) = Z(g) \circ Z(f) \ .$$

- The correspondence $M \Rightarrow Z(M)$ respects the operations on the objects:
$$Z(M \amalg N) = Z(M) \otimes Z(N) \ .$$

Moreover, there are the following 'nontriviality requirements':

- $Z(\emptyset) = K$ (the space of "states" of the "quantum" empty set is nontrivial).
- $Z(1_M) = \mathrm{id}_{Z(M)}$.
- $Z(M)$ is not constantly equal to K and $Z(f)$ is not constantly equal to id_K.

In the oriented setting, on $\mathcal{O}_m$ there is the involution $M \to -M$. On V_K there is the duality "involution" $Z \to Z^*$ (where Z is canonically identified with its bidual space $(Z^*)^*$). Here we also require

- $Z(-M) = Z(M)^*$.

Every TQFT (if any) associates to every $M \in \mathcal{M}_{m+1}$, a scalar $\mu(M)$ which is *an invariant up to (possibly oriented) diffeomorphism.* As M is compact and boundaryless, and $(\emptyset, \emptyset, [M, \emptyset, \emptyset])$ is an arrow $f_{[M]} : \emptyset \mapsto \emptyset$, then $Z(f_{[M]}) : K \to K$ and $\mu([M]) := Z(f_{[M]})(1)$.

We realize quickly that the existence of such a TQFT is not evident at all. A possible approach could be to associate to all nonempty connected $M \in \mathcal{S}_m$ the same vector space $Z(M) = V$ (either V or V^*, in such a way that $Z(-M) = V^*$, in the oriented setting). As every M is the disjoint union of its connected components, $Z(M)$ is the tensor product of some copies of V (of V or V^*). Then we could try to define first the elementary $Z(e)$ associated with the elementary arrows in $\mathbf{CAT}_{\mathcal{B}}(m + 1)$, perhaps in such a way that they depend only on the handle index. A generic $Z(f)$ should be a composition of such elementary morphisms. The key, and hard point, is that the decomposition by elementary arrows in $\mathbf{CAT}_{\mathcal{B}}(m + 1)$ is far from be unique (and also any triad supports infinitely many Morse functions) but the resulting composite $Z(f)$ should not depend on the choice of the decomposition. This means that our elementary $Z(e)$'s must satisfy a huge collection of (*a priori* implicit) relations. If we take $V = K^n$ for some n, $V^* = M(1, n, K)$, and the unknown $Z(e)$'s in matrix form, we should find nontrivial solutions of a huge system of matrix equations. It is not evident that such a solution exists (even if we take $V = K$). We will point out a "baby" (nontrivial) TQFT in Chapter 14.

CHAPTER 11

Smooth cobordism

We specialize the bordism modules $\mathcal{B}_m(X, R)$ introduced in Chapter 10 to X which varies among the boundaryless compact smooth manifolds. More precisely, if X is *not oriented* (even *nonorientable*), then we consider $\eta_m(X) = \mathcal{B}_m(X; \mathbb{Z}/2\mathbb{Z})$; if X is *oriented*, we consider $\Omega_m(X) = \mathcal{B}_m(X; \mathbb{Z})$. A first important fact, already used in Section 10.5, is that by the approximation theorems of continuous maps by smooth maps, we can assume that all maps entering the definition of the bordism modules are smooth; moreover, in dealing with functoriality, we can also assume that the maps $g : X \to Y$ are smooth. Therefore, all discussion will have a differential/topological character. The main issue of this chapter is that using transversality, these "smooth" bordism modules (renamed "cobordism" modules up to a suitable re-indexing) can be embodied into *contravariant functors* and their direct sum can be endowed with a functorial *graded ring* structure. This multiplicative structure is a substantial enrichment of the theory.

11.1. Map transversality

We consider the following variant of the basic transversality setting (Section 8.1):

• All involved smooth manifolds admit an embedding in some $\mathbb{R}^n$, being furthermore a closed subset. This is certainly the case if a manifold is compact.

• All involved smooth maps are *proper* (i.e. the inverse image of a compact set is compact). Of course, this is the case if the source manifold is compact. General topology tells us that proper maps between manifolds are *closed* (i.e. the image of a closed set is closed).

• N and Z are boundaryless smooth manifolds and M is a compact smooth manifold with (possibly empty) boundary ∂M.

• $f : M \to N$, $g : Z \to N$ are smooth maps.

In such a situation, we can define the product map

$$(f \times g) : M \times Z \to N \times N,\ (f \times g)(x, z) = (f(x), g(z))$$

and denote by

$$\Delta_N = \{(y, y) \in N \times N\}$$

the *diagonal submanifold* of $N \times N$, which is obviously diffeomorphic to N by the canonical diffeomorphism $N \to \Delta_N$, $y \to (y, y)$. Recall that $\partial(M \times Z) = \partial M \times Z$.

DEFINITION 11.1. We say that f is *transverse to* g (and we write $f \pitchfork g$) if $(f \times g) \pitchfork \Delta_N$. This incorporates that $\partial f \pitchfork g$.

By using that $T_{(y,y)}\Delta_N = \Delta_{T_y N} \subset T_y N \oplus T_y N$, we readily check that:

LEMMA 11.2. *We have that $f \pitchfork g$ if and only if for every $(x, z) \in M \times Z$ such that $f(x) = g(z) = y$, then $T_y N = d_x f(T_x M) + d_z g(T_z Z)$, and for every $(x, z) \in \partial M \times Z$ such $\partial f(x) = g(z) = y$, then $T_y N = d_x \partial f(T_x \partial M) + d_z g(T_z Z)$.*

We have the following version of the first transversality theorem.

THEOREM 11.3. *In the given setting:*

(1) If $f \pitchfork g$ then

$$(Y, \partial Y) = ((f \times g)^{-1}(\Delta_N), (\partial f \times g)^{-1}(\Delta_N))$$

is a compact proper submanifold of $(M \times Z, \partial M \times Z)$. Moreover,

$$\dim(M \times Z) - \dim(Y) = \dim(N \times N) - \dim(N) = \dim(N) \ .$$

(2) If all involved manifolds are oriented, then Y and ∂Y are orientable and we can fix an orientation procedure such that ∂Y becomes the oriented boundary of Y.

Proof : Except for the compactness of Y, all statements in (1) are a direct consequence of Theorem 8.2 (and they hold also without assuming that g is proper). On the other hand, the compactness of Y follows from the compactness of M and the properness of g. Point (2) is a direct consequence of point (2) of Theorem 8.2, once $N \times N$ is endowed with the product orientation of two copies of the given orientation on N, Δ_N is oriented in such a way that the canonical diffeomorphism is orientation preserving.

■

REMARK 11.4. If $Z \subset N$ is a submanifold and g is the inclusion, then

$$Y = \{(x, z) \in M \times Z;\ f(x) = z\} \ ;$$

that is, the graph of the restriction of f to $f^{-1}(Z)$. If Z is also a closed subset of N, then we are in the setting fixed above, and the projection of Y in M is equal to $f^{-1}(Z)$ and is a proper submanifold of $(M, \partial M)$ recovering the conclusion of Theorem 8.2.

We denote by $\pitchfork (M, N; g)$ the subspace of $\mathcal{E}(M, N)$ formed by the maps transverse to g. If $\partial f \pitchfork g$, then we denote by $\mathcal{E}(M, N, \partial f)$ (resp. $\pitchfork (M, N, \partial f; g)$) the subspace of $\mathcal{E}(M, N)$ ($\pitchfork (M, N; g)$) formed by the maps that coincide with ∂f on ∂M. We have the following version of Theorem 8.5.

THEOREM 11.5. *In the given setting,*

(1) $\pitchfork (M, N; g)$ *is open dense in* $\mathcal{E}(M, N)$.

(2) $\pitchfork (M, N, \partial f; g)$ *is open dense in* $\mathcal{E}(M, N, \partial f)$.

(3) For every $h \in \mathcal{E}(M, N)$ *(resp.* $h \in \mathcal{E}(M, N, \partial f)$*) there is* $\tilde{h} \in \pitchfork (M, N; g)$ *(*$\tilde{h} \in \pitchfork (M, N, \partial f; g)$*) smoothly homotopic to* h.

Proof : The proof is not a direct consequence of the *statement* of Theorem 8.5 but it is a consequence of its proof which can be adapted with minor changes.

■

11.2. Cobordism contravariant functors

Let X be a compact boundaryless smooth manifold. Let $[M, f] \in \mathcal{B}_m(X; R)$ (either $R = \mathbb{Z}/2\mathbb{Z}$ or $R = \mathbb{Z}$ according to the convention fixed at the beginning of this chapter). We say that $[M, f]$ is of *codimension* k *in* X if

$$k = \text{codim}_X[M, f] := \dim(X) - m \ .$$

We can consider the modules $\mathcal{B}_m(X; R)$ indexed by $\mathbb{Z}$ by stipulating that $\mathcal{B}_m(X; R) = 0$ if $m < 0$. If k is the codimension, set

$$\mathcal{B}^k(X; R) := \mathcal{B}_m(X; R)$$

so we have a formal re-idexing by $\mathbb{Z}$ of the family of bordism modules of X in terms of the codimension, so that $\mathcal{B}^k(X; R) = 0$ if $k > \dim X$. To stress it, we say that $\mathcal{B}^k(X; R)$ is the k-cobordism module of X (over R). Formally, for every $k \in \mathbb{Z}$, there are tautological re-indexing isomorphisms

$$d : \mathcal{B}_{\dim(X)-k}(X; R) \to \mathcal{B}^k(X; R), \ D : \mathcal{B}^k(X; R) \to \mathcal{B}_{\dim(X)-k}(X; R)$$

$$d(\alpha) = D(\alpha) = \alpha \ .$$

For every $k \in \mathbb{Z}$, we want to enhance the object correspondence

$$X \ \Rightarrow \ \mathcal{B}^k(X; R)$$

with a correspondence

$$g : X \to Y \ \Rightarrow \ g^* : \mathcal{B}^k(Y; R) \to \mathcal{B}^k(X; R)$$

to build a contravariant functor from the category of compact boundaryless (possibly oriented) smooth manifolds and smooth maps to the category of R-modules and R-linear maps. Hence we want that

$$(g \circ h)^* = h^* \circ g^*$$

whenever the composition makes sense, and

$$\text{id}_X^* = \text{id}_{\mathcal{B}^k(X;R)} \ .$$

We have to define the linear maps g^*. We implement the following procedure, basically it is the same "pull-back" construction that we have used for vector bundles.

- If $k > \dim(Y)$, then $g^* : \{0\} \to \mathcal{B}^k(X;R)$ is uniquely determined.
- Assume that $k \leq \dim(Y)$ and let $\alpha \in \mathcal{B}^k(Y;R)$. Fix a representative

$$\alpha = [M, f] \ .$$

Hence M is compact boundaryless (possibly oriented) of dimension $m = \dim(Y) - k$. By the transversality theorems, up to homotopy, and hence up to bordism, we can assume that $f \pitchfork g$. Then

$$V = (f \times g)^{-1}(\Delta_Y)$$

is a compact boundaryless (possibly oriented) submanifold of $M \times X$ such that $\dim(M \times X) - \dim(V) = \dim(Y)$; that is,

$$\dim(X) - \dim(V) = \dim(Y) - \dim(M) = k \ .$$

Hence $[V, p_X] \in \mathcal{B}^k(X;R)$, where p_X is the restriction of the projection $M \times X \to X$.

Proposition 11.6. *Let $g : X \to Y$ be a smooth map between compact boudaryless (possibly oriented) smooth manifolds. Let $\alpha \in \mathcal{B}^k(Y;R)$. Let $[V, p_X] \in \mathcal{B}^k(X;R)$ obtained by means of any implementation of the above "pull-back" procedure starting from a representative $\alpha = [M, f]$. Then*

(1) The map

$$g^* : \mathcal{B}^k(Y;R) \to \mathcal{B}^k(X;R), \ g^*(\alpha) = [V, p_X]$$

is well defined (it does not depend on the choice of the implementation).

(2) g^* is R-linear.

(3) For every X, $\mathrm{id}_X^* = \mathrm{id}_{\mathcal{B}^k(X;R)}$.

(4) Whenever the composition makes sense, $(g \circ h)^* = h^* \circ g^*$.

(5) If $g_0, g_1 : X \to Y$ are homotopic, then $g_0^* = g_1^*$. This means that the cobordism contravariant functor satisfies the homotopy invariance property*.*

Proof : Assume that g^* is well defined and prove items (2)-(4). The procedure distributes on the addends of a disjoint union, so (2) follows easily.

As for (3) , every $[M, f]$ is transverse to id_X, hence V is the graph of f and clearly $[V, p_X] = [M, f]$.

Concerning (4), if $g^*([M, f]) = [M', f']$, $h^*([M', f']) = [M", f"]$ are the representatives being obtained by iterated application of the pull-back procedure, then $f" \pitchfork (g \circ h)$ and $[M", f"]$ results from an implementation of the procedure applied to $[M, f]$ and $g \circ h$.

Let us show now (1), that g^* is well defined. Let (V, p_X) and (V', p'_X) be obtained by implementing the procedure starting from representatives (M, f) and (M', f'), $f \pitchfork g$, $f' \pitchfork g$; let (W, F) realize a bordism of (M, f) with (M', f'). By applying the transversality theorems, we can assume that $F \pitchfork g$. Then $((F \times g)^{-1}(\Delta_Y), P_X)$ realizes a bordism of (V, p_X) with (V', p'_X).

Finally we show (5). If $F : X \times [0,1] \to Y$ realizes a smooth homotopy of (X, g_0) with (X, g_1), then we can assume that F verifies suitable transversality conditions, so that $((f \times F)^{-1}(\Delta_Y), q \circ P_{X \times [0,1]})$, where $q : X \times [0,1] \to X$

is the natural projection, leads to a bordism of $((f \times g_0)^{-1}(\Delta_Y), p_X)$ with $((f \times g_1)^{-1}(\Delta_Y), p_X)$.

■

11.2.1. Reduction mod(2). When X is oriented, we already known the natural "forgetting" homomorphisms

$$\sigma : \mathcal{B}^k(X; \mathbb{Z}) \to \mathcal{B}^k(X; \mathbb{Z}/2\mathbb{Z}) \ .$$

PROPOSITION 11.7. *For every smooth map $g : X \to Y$ between oriented compact boundaryless manifolds and for every $\alpha \in \mathcal{B}^k(Y; \mathbb{Z})$, $g^*(\sigma(\alpha)) = \sigma(g^*(\alpha))$, where the first g^* refers to the $\mathbb{Z}/2\mathbb{Z}$-cobordism and the second to the $\mathbb{Z}$-cobordism.*

Proof : The construction of $g^*(\sigma(\alpha))$ is obtained by the construction of $g^*(\alpha)$ by forgetting the orientation.

■

11.3. The cobordism cup product

Let X be as above. For every $r, s \in \mathbb{Z}$, we are going to define a bilinear map

$$\sqcup : \mathcal{B}^r(X; R) \times \mathcal{B}^s(X; R) \to \mathcal{B}^{r+s}(X; R) \ .$$

Let us describe the procedure that defines this "cup" product.

• If at least one among r and s is bigger than $\dim(X)$, then $\alpha \sqcup \beta = 0$.

• Let $(\alpha, \beta) \in \mathcal{B}^r(X; R) \times \mathcal{B}^s(X; R)$ and assume that both r and s are $\leq \dim(X)$. Fix representatives $\alpha = [M, f]$ and $\beta = [N, h]$. We claim that

$$[M \times N, f \times h] \in \mathcal{B}^{r+s}(X \times X; R) \ .$$

In fact,

$$2\dim(X) - (\dim(M) + \dim(N)) = 2\dim(X) - (\dim(X) - r + \dim(X) - s) = r + s \ .$$

• Let $\delta_X : X \to X \times X$, $\delta_X(x) = (x, x)$ be the canonical diffeomorphism to the diagonal Δ_X. Finally, take

$$\delta_X^*[M \times N, f \times h] \in \mathcal{B}^{r+s}(X; R) \ .$$

We stress that we are using the contravariant nature of the cobordism functor.

REMARK 11.8. If $f \pitchfork h$ we can explicitly describe representatives of $\delta_X^*[M \times N, f \times h]$. In such a case, $(f \times h) \pitchfork \delta_X$. Then $\delta_X^*[M \times N, f \times h] = [\tilde{V}, p_X]$, where

$$\tilde{V} = \{(x, p, q) \in X \times M \times N;\ f(p) = h(q) = x\} \ .$$

Let

$$V = (f \times h)^{-1}(\Delta_X) = \{(p, q) \in M \times N;\ f(p) = h(q)\} \ .$$

Denote by π_M (π_N) the restriction on V of the projection onto M (N). Then $\tilde{V}$ is the graph of $u := f \circ \pi_M = h \circ \pi_N$, V and $\tilde{V}$ are canonically diffeomorphic, and

$$[\tilde{V}, p_X] = [V, u] \in \mathcal{B}^{r+s}(X; R) \ .$$

In particular, if f and h are the inclusions of two transverse submanifolds M and N of X and j is the inclusion of $M \pitchfork N$, then

$$\delta_X^*[M \times N, f \times h] = [M \pitchfork N, j] \ .$$

Proposition 11.9. *Let X be a compact boundaryless (possibly oriented) smooth manifold. Let $(\alpha, \beta) \in \mathcal{B}^r(X; R) \times \mathcal{B}^s(X; R)$, $\delta_X^*[M \times N, f \times h] \in \mathcal{B}^{r+s}(X; R)$ be obtained by any implementation of the above procedure applied to arbitrary representatives $\alpha = [M, f]$, $\beta = [N, h]$. Then:*

(1) The class $\alpha \times \beta := [M \times N, f \times h]$, whence the class $\alpha \sqcup \beta := \delta_X^*[M \times N, f \times h]$ is well defined (they do not depend on the choice of the implementation).

(2) $\sqcup$ is bilinear.

(3) For every $(\alpha, \beta) \in \mathcal{B}^r(X; R) \times \mathcal{B}^s(X; R)$,

$$\alpha \sqcup \beta = (-1)^{rs} \beta \sqcup \alpha \ .$$

(4) $\sqcup$ is functorial*; that is, for every $g : X \to Y$, for every $(\alpha, \beta) \in \mathcal{B}^r(Y; R) \times \mathcal{B}^s(Y; R)$,*

$$g^*(\alpha) \sqcup g^*(\beta) = g^*(\alpha \sqcup \beta) \ .$$

Proof : Assume that $\sqcup$ is well defined and prove the other items. By the transversality theorems, the assumption allows us to use representatives that satisfy all suitable transversality conditions. The disjoint union distributes to the product of manifolds; (2) follows easily. Item (3) is a local verification and reduces to Remark 8.3. Let (M, f), (N, h) be representatives of α and β such that $f \pitchfork g$, $h \pitchfork g$ and $f \pitchfork h$. It follows that $(g \times g) \circ \delta_X \pitchfork (f \times h)$. By combining the two procedures that define g^* and $\sqcup$ starting from such representatives in general position, we obtain representatives for both terms of the equality of (4) that are evidently bordant to each other (in the same spirit of Remark 11.8). It remains to prove that $\sqcup$ is well defined. As δ_X^* is well defined, it is enough to show that

$$\alpha \times \beta := [M \times N, f \times h] \in \mathcal{B}^{r+s}(X \times X; R)$$

only depends on the class α and β. By symmetry, it is enough to show that it does not depend on the choice of a representative of α. If (W, F) realizes a bordism of (M, f) with (M', f'), then $(W \times N, F \times h)$ realizes a bordism of $(M \times N, f \times h)$ with $(M' \times N, f' \times h)$.

∎

11.3.1. Reduction mod(2).

PROPOSITION 11.10. *For every compact oriented boundaryless manifold X, for every $(\alpha, \beta) \in \mathcal{B}^r(X; \mathbb{Z}) \times \mathcal{B}^s(X; \mathbb{Z})$, $\sigma(\alpha) \sqcup \sigma(\beta) = \sigma(\alpha \sqcup \beta)$, where the first $\sqcup$ refers to the $\mathbb{Z}/2\mathbb{Z}$-cobordism, the second to the $\mathbb{Z}$-cobordism.*

Proof : The construction of $\sigma(\alpha) \sqcup \sigma(\beta)$ is obtained by the construction of $\alpha \sqcup \beta$ just by forgetting the orientation.

■

11.3.2. The cobordism ring. The collection of the above cup products gives a globally defined product

$$\sqcup : \mathcal{B}^\bullet(X; R) \times \mathcal{B}^\bullet(X; R) \to \mathcal{B}^\bullet(X; R)$$

on the direct sum R-module

$$\mathcal{B}^\bullet(X; R) := \oplus_{k \in \mathbb{Z}} \mathcal{B}^k(X; R) \ .$$

The ring $(\mathcal{B}^\bullet(X; R), +, \sqcup)$ is called the *graded R-cobordism ring of X* (it is a graded algebra when $R = \mathbb{Z}/2\mathbb{Z}$).

Similarly, the collection of the above g^*s defines a global graded ring homomorphism

$$g^* : \mathcal{B}^\bullet(Y; R) \to \mathcal{B}^\bullet(X; R) \ .$$

We can summarize the above achievements as follows:

$$X \ \Rightarrow \mathcal{B}^\bullet(X; R)$$

$$g : X \to Y \ \Rightarrow g^* : \mathcal{B}^\bullet(Y; R) \to \mathcal{B}^\bullet(X; R)$$

define a contravariant functor from the category of compact boundaryless (possibly oriented) smooth manifolds and smooth maps to the category of graded rings and graded ring homomorphisms.

REMARK 11.11. A graded ring satisfying the *non*-commutative rule (3) in Proposition 11.9 is sometimes called a "commutative" graded ring.

REMARK 11.12. A particular case of the above constructions is when X is reduced to one point. In this case the product

$$\mathcal{B}^r(R) \times \mathcal{B}^s(R) \to \mathcal{B}^{r+s}(R) \ ,$$

for every couple of indices $r, s \leq 0$, is just defined by the product of representatives

$$[M] \sqcup [N] = [M \times N] \ .$$

REMARK 11.13. *(Non-compact X)* Referring to the setting of the transversality theorems of Section 11.1, we can extend the range of cobordism functors and product to the category of boundaryless, possibly non-compact manifolds X which can be embedded anyway in some $\mathbb{R}^k$, being also a closed subset, and smooth *proper* maps between these manifolds.

11.4. Duality, intersection forms

Assume that X is connected (possibly oriented) and $\dim(X) = n$. Then

$$\mathcal{B}^n(X;R) \sim \mathcal{B}_0(X;R) \sim R\ .$$

If $R = \mathbb{Z}/2\mathbb{Z}$, we have a generator β_X of $\mathcal{B}^n(X;\mathbb{Z}/2\mathbb{Z})$ represented as $\beta_X = [x, i]$, where $x \in X$ and i is the inclusion (it does not depend on the choice of x because X is path connected). If $R = \mathbb{Z}$, we have two generators of the form $[\pm x, i]$. As usual, we encode the point sign by associating to $+x$ the orientation on T_xX carried by the global orientation of X; this selects again one generator β_X. By this choice of generators, we have fixed in both cases an identification of $\mathcal{B}^n(X;R)$ with R.

For every r, s, set $p = n - r$, $q = n - s$. Let r, s be such that $r + s = n$ (hence also $p + q = n$, $p = s$, $q = r$). Then

$$\sqcup : \mathcal{B}^r(X;R) \times \mathcal{B}^s(X;R) \to R\ .$$

Note in particular that

$$d(\alpha_X) \sqcup \beta_X = 1$$

where $\alpha_X = [X, \mathrm{id}_X] \in \mathcal{B}_n(X;R)$ is the bordism fundamental class of X and $d : \mathcal{B}_n(X;R) \to \mathcal{B}^0(X;R)$ is the tautological isomorphism.

Using the tautological isomorphisms, all this can be lifted to a bilinear map

$$\bullet : \mathcal{B}_p(X;R) \times \mathcal{B}_q(X;R) \to R$$

or to a bilinear pairing

$$\sqcap : \mathcal{B}^r(X;R) \times \mathcal{B}_q(X;R) \to R\ .$$

This last induces a linear map $(q = r)$

$$\phi^r : \mathcal{B}^r(X;R) \to \mathrm{Hom}(\mathcal{B}_r(X;R), R),\ \gamma \to \phi_\gamma,\ \phi_\gamma(\sigma) = \gamma \sqcap \sigma\ .$$

Applying the Hom functors, we have a basic way to convert the covariant bordism funtors into contravariant ones

$$X \ \Rightarrow \mathrm{Hom}(\mathcal{B}_m(X;R), R)$$

$$g : X \to Y \ \Rightarrow g_*^t : \mathrm{Hom}(\mathcal{B}_m(Y;R), R) \to \mathrm{Hom}(\mathcal{B}_m(X;R), R)\ ,$$

where $g_*^t(\gamma) = \gamma \circ g_*$. The homomorphisms ϕ^r, g_*^t and g^* are compatible: "$\phi^r \circ g^* = g_*^t \circ \phi^r$".

The map ϕ^r is, in general, not injective nor surjective. A reason is the possible existence of nontrivial submodules of $\mathcal{B}_*(X;R)$ isomorphic to $\mathcal{B}_* = \mathcal{B}_*(\{x_0\};R)$. The image via the tautological isomorphism of such a submodule in $\mathcal{B}^r(X;R)$ is contained in the kernel of ϕ^r. If $R = \mathbb{Z}/2\mathbb{Z}$, so that $\mathcal{B}_r$ can be realized as a direct addend of $\mathcal{B}_r(X;\mathbb{Z}/2\mathbb{Z})$, then any functional γ which holds 1 on $\mathcal{B}_r$ and such that $\mathcal{B}_r(X;\mathbb{Z}/2\mathbb{Z}) = \mathcal{B}_r \oplus \ker\gamma$ does not belong to the image of ϕ^r. If $R = \mathbb{Z}$, then the *torsion submodule* of $\mathcal{B}^r(X;\mathbb{Z})$ is contained in the kernel of ϕ^r. For every r, we set

$$\mathcal{H}^r(X;R) := \mathcal{B}^r(X;R)/\ker(\phi^r)$$

and extending the usual re-indexing set

$$\mathcal{H}_{n-r}(X;R) := \mathcal{H}^r(X;R)$$

where in this last equality only the R-module structure is considered, forgetting the multiplicative structure. Then the above map ϕ^r induces an injective R-linear map

$$\hat{\phi}^r : \mathcal{H}^r(X;R) \to \mathrm{Hom}(\mathcal{H}_r(X;R), R) \ .$$

If X is connected (possibly oriented), then

$$\mathcal{H}^0(X;R) \sim R$$

and is generated by the fundamental class. The map $\sqcap$ can be formally generalized by composing $\sqcup$ with the tautological isomorphisms

$$\sqcap : \mathcal{B}^r(X;R) \times \mathcal{B}_q(X;R) \to \mathcal{B}_{2n-(r+q)}(X;R) \ .$$

In particular,

$$\sqcap : \mathcal{B}^r(X;R) \times \mathcal{B}_n(X;R) \to \mathcal{B}_{n-r}(X;R)$$

and it is a consequence of the definitions that for every $\sigma \in \mathcal{B}^r(X;R)$

$$\sigma \sqcap \alpha_X = D(\sigma) \ .$$

If $\dim(X) = 2m$, we can consider

$$\sqcup : \mathcal{B}^m(X;R) \times \mathcal{B}^m(X;R) \to R \ ,$$

or equivalently

$$\bullet : \mathcal{B}_m(X;R) \times \mathcal{B}_m(X;R) \to R \ .$$

This second is also called the *R-bordism intersection form of X*. These forms are symmetric on $\mathbb{Z}/2\mathbb{Z}$, while on $\mathbb{Z}$ they are symmetric (resp. antisymmetric) if m is even (m is odd). The kernel of ϕ^r coincides in this case with the *radical* of the form, hence the induced form (also called "intersection form")

$$\sqcup : \mathcal{H}^m(X;R) \times \mathcal{H}^m(X;R) \to R$$

determines an inclusion of $\mathcal{H}^m(X;R)$ as a submodule of its dual module

$$\hat{\phi}^m : \mathcal{H}^m(X;R) \to \mathrm{Hom}(\mathcal{H}_m(X;R), R) \ .$$

11.4.1. Cobordism for compact manifolds with boundary. We extend some of the previous constructions to manifolds with nonempty boundary. Let us strengthen first the notion of map between pairs of spaces

$$h : (X, A) \to (Y, B) \ ;$$

we say that it is a *strict* pair map if (as usual) $h(A) \subset B$ and furthermore $h(X \setminus A) \subset Y \setminus B$. Let X be a compact smooth manifold with nonempty boundary ∂X. For example, the inclusion of a *proper* submanifold $(Y, \partial Y)$ in $(X, \partial X)$ is a typical example of a strict map. A strict map $f : (M, \emptyset) \to (X, \partial X)$ sends the boundaryless M in the interior $\mathrm{Int}(X)$ of X. The non-compact manifold $\mathrm{Int}(X)$ satisfies the conditions of Remark 11.13; for example, if $X \subset \mathbb{R}^k$ for some k (this is possible because X is compact) and $h : \mathbb{R}^k \to \mathbb{R}$ is a non-negative smooth function such that $\partial X = h^{-1}(0)$, then the restriction to $X \setminus \partial X$ of $\mathbb{R}^k \setminus \partial X \to \mathbb{R}^{k+1}$, $x \to (x, 1/h(x))$ is an embedding of $\mathrm{Int}(X)$ to a closed subset of $\mathbb{R}^{k+1}$.

The usual definitions of the relative bordism modules $\mathcal{B}_m(X, \partial X; R)$ can be enhanced by stipulating that all involved maps are smooth and strict. Using the approximation theorem of continuous maps by smooth maps and the boundary collars to push into the interior what is necessary to make strict any given "singular" smooth manifold in $(X, \partial X)$, it is not hard to check that these enhanced modules are isomorphic to the original ones. Moreover, $\mathcal{B}_m(X; R)$ is naturally isomorphic to $\mathcal{B}_m(\mathrm{Int}(X); R)$. The re-indexing $\mathcal{B}^k(X; R) = \mathcal{B}_m(X; R)$ or $\mathcal{B}^k(X, \partial X; R) = \mathcal{B}_m(X, \partial X; R)$, $k = \dim(X) - m$, is defined as usual in terms of the codimension in X.

If M is compact boundaryless and $g : X \to M$ is a smooth map, then adapting the above construction, for every $k \in \mathbb{Z}$, we define

$$g^* : \mathcal{B}^k(M; R) \to \mathcal{B}^k(X, \partial X; R) .$$

If N is compact boundaryless and $h : N \to X$ is smooth, up to isotopy it is not restrictive to assume that $h(N) \subset \mathrm{Int}(X)$, and we define

$$h^* : \mathcal{B}^k(X, \partial X; R) \to \mathcal{B}^k(N; R) .$$

In such a situation, we have

$$(g \circ h)^* = h^* \circ g^* .$$

Now, formally using the very same definition given when X is boundaryless, we (partially) extend the cup product as follows:

$$\sqcup : \mathcal{B}^r(X, \partial X; R) \times \mathcal{B}^s(X; R) \to \mathcal{B}^{r+s}(X; R)$$
$$\sqcup : \mathcal{B}^r(X; R) \times \mathcal{B}^s(X; R) \to \mathcal{B}^{r+s}(X; R) .$$

Then we have a linear map

$$\phi^r : \mathcal{B}^r(X, \partial X; R) \to \mathrm{Hom}(\mathcal{B}_r(X; R), R)$$

which restricts to (we keep the same name)

$$\phi^r : \mathcal{B}^r(X; R) \to \mathrm{Hom}(\mathcal{B}_r(X; R), R) .$$

Finally, we have the induced injective map

$$\hat{\phi}^r : \mathcal{H}^r(X, \partial X; R) \to \mathrm{Hom}(\mathcal{H}_r(X; R), R) .$$

CHAPTER 12

Applications of cobordism rings

In this chapter, we will see several, sometimes very classical, applications of the cobordism theory, especially of its multiplicative structure.

12.1. Fundamental class revised, Brouwer's fixed point theorem

Here, we recover Proposition 10.8 in terms of cobordism. Let X be a compact boundaryless connected (possibly oriented) smooth n-manifold. Let $[X, \mathrm{id}_X] \in \mathcal{B}^0(X;R)$ (often we will simply write $[X]$). Let $\beta_X \in \mathcal{B}^n(X;R)$ be the generator given in Section 11.4 in order to fix an identification $\mathcal{B}^n(X;R) = R$. We have already remarked that

$$[X] \sqcup \beta_X = 1 \in R \ ;$$

hence, in particular, $[X] \neq 0$. On the other hand, if γ belongs to the image via the tautological isomorphism $d : \mathcal{B}_n(X;R) \to \mathcal{B}^0(X;R)$ of the natural submodule isomorphic to $\mathcal{B}_n$, then

$$\gamma \sqcup \beta_X = 0 \ ;$$

hence $[X] \neq \gamma$. If X has nonempty boundary ∂X, we can consider

$$[X, \partial X] \in \mathcal{B}^0(X, \partial X; R)$$

and we have again

$$[X, \partial X] \cup \beta_X = 1 \in R \ .$$

The following is a very classical topological application of such a fundamental class.

Theorem 12.1. (Brouwer fixed point theorem) *For every continuous map $f : D^n \to D^n$, there is $x \in D^n$ such that $f(x) = x$.*

Proof : The case $n = 0$ is trivial. For $n > 0$, assume that there is such an f without any fixed point. Define $F : D^n \to S^{n-1}$ by setting $F(x)$ equal to the unique point of intersection between $S^{n-1} = \partial D^n$ and the ray emanating from $f(x)$ and passing through x. As f is continuous, it is easy to verify that F is also continuous and that $\partial F = \mathrm{id}_{S^{n-1}}$. Hence $[S^{n-1}]$ should be trivial in $\mathcal{B}_{n-1}(S^{n-1})$ contrary to Proposition 10.8.

■

12.2. A separation theorem

It is evident that an equatorial $S^{n-1} \subset S^n$ divides this last in two connected components. If $n \geq 2$, every connected hypersurface in S^n shares the same behaviour.

PROPOSITION 12.2. *(1) Let $M \subset S^n$ be a compact boundaryless connected submanifold,* $\dim(M) = n-1$*,* $n \geq 2$*. Then $S^n \setminus M$ has exactly two connected components, W and W', and the closures are compact submanifolds with boundary such that $\partial \bar{W} = \partial \bar{W}' = M$.*

(2) Let $M \subset \mathbb{R}^n$ be a compact boundaryless connected submanifold, $\dim(M) = n-1$*,* $n \geq 2$*. Then $\mathbb{R}^n \setminus M$ has two connected components; one, say W, has compact closure and $\partial \bar{W} = M$.*

Proof : The item (2) follows from (1) by considering $\mathbb{R}^n \subset \mathbb{R}^n \cup \infty = S^n$, such that ∞ does not belong to M. As for (1), we know by Section 10.5 that $[M] := [M, i_M] \in \mathcal{B}_{n-1}(S^n; \mathbb{Z}/2\mathbb{Z}) \sim \mathcal{B}^1(S^n; \mathbb{Z}/2\mathbb{Z})$ (i_M being the inclusion) belongs to the submodule isomorphic to $\mathcal{B}_{n-1}$. Hence we know that $[M]$ belongs to the kernel of the map $\phi : \mathcal{B}^1(S^n; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\mathcal{B}_1(S^n; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$. Assume that $S^n \setminus M$ is connected. Take a small simple arc γ intersecting transversely M at one point. The endpoints of γ belong to $S^n \setminus M$, hence γ can be completed to a smooth simple curve $\hat{\gamma}$ in S^n that transversely intersects M at one point. It follows that $\phi_{[M]}([\hat{\gamma}, i_{\hat{\gamma}}]) = 1$, and this is a contradiction. Hence $S^n \setminus M$ is not connected. A tubular neighbourhood U of M in S^n is diffeomorphic to $M \times (-1, 1)$, in fact $M \times [0, 1)$ can be identified with a collar of M in $\bar{W}$, where W is a component of $S^n \setminus M$. Since $U \setminus M$ has two connected components, then $S^n \setminus M$ has at most two components and this achieves the proof.

■

12.3. Intersection numbers

Let X be a compact connected (possibly oriented) boundaryless smooth n-manifold. Let M and N be compact boundaryless (possibly oriented) submanifolds of X such that $\dim M = p$, $\dim N = q$. Assume that $p + q = n$. Then

$$[M] \bullet [N] \in R$$

is the *R-intersection number* of the two submanifolds. Obviously, it is invariant up to isotopy of M or N in X (isotopy is a particular instance of bordism). Hence if $[M] \bullet [N] \neq 0$, then there is no isotopy that separates M and N. In particular, if $M = N$ (hence $n = 2m$), then $M \bullet M$ is called the *self-intersection number* of the submanifold M.

12.3.1. Lefschetz's number and fixed point theorem. Let X be as above a connected compact boundaryless n-manifold. Let $f : X \to X$ be a smooth map. Consider the submanifolds Δ_X and $G(f)$ of $X \times X$, $G(f)$

being the graph of f. If $n = \dim(X)$, then

$$L_2(f) := [\Delta_X] \sqcup [G(f)] \in \mathcal{B}^{2n}(X \times X; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

is called *the Lefschetz number of* f mod(2). This is invariant if f is considered up to homotopy. As usual, this allows us to define this number also when f is merely a continuous map. It is clear that if $\Delta_X \cap G(f) = \emptyset$ (that is, if f has no fixed points), then $L_2(f) = 0$. Equivalently, we have the following "fixed point theorem":

If $L_2(f) \neq 0$, then f has a fixed point.

If M is oriented, we can define the Lefschetz number in the oriented setting,

$$L(f) \in \mathcal{B}^{2n}(X \times X; \mathbb{Z}) = \mathbb{Z},\ L(f) = L_2(f) \ \mathrm{mod}(2)$$

and repeat *verbatim* the above considerations.

12.4. Linking numbers

Let X be as in Section 12.3, $n \geq 3$. Let $(M, \partial M)$ be an $(n-k)$-compact submanifold (possibly oriented) of X with nonempty boundary, $n - k \geq 1$. M is called a *R-Seifert surface of $T = \partial M$ in X*. Let U be a "small" tubular neighbourhood of T in X, such that $\partial U \pitchfork M$. The closure $(Y, \partial Y)$ $(\partial Y = \partial U)$ of $X \setminus U$ is a compact n-manifold with nonempty boundary; the closure $(N, \partial N)$ of $M \setminus U$ is a proper $(n-k)$-submanifold of $(Y, \partial Y)$. Then $[N, \partial N] \in \mathcal{B}^k(Y, \partial Y; R)$ (we omit to indicate the inclusion map). Let Z be a compact boundaryless (possibly oriented) proper k-submanifold of $(Y, \partial Y)$. Hence $[Z] \in \mathcal{B}^{n-k}(Y; R)$. Then

$$lk_M(T, Z) := [N, \partial N] \sqcup [Z] \in R$$

is called the *R-linking number of Z with T with respect to the Seifert surface M*. By the uniqueness of tubular neighbourhoods up to isotopy, it is well defined. Moreover, it is invariant up to isotopy of Z in $\mathrm{Int}(Y)$. In some cases, the linking number does not depend on the choice of the Seifert surface.

Proposition 12.3. *In the above setting, assume that $X = S^n$. Then*

$$lk(T, Z) := lk_M(T, Z) \in R$$

is well defined; that is, it does not depend on the choice of a Seifert surface of T in S^n.

Proof : Let $T = \partial M = \partial M'$. By (abstractly) gluing M and M' along T and taking the union of the inclusions, we get $[W, f] \in \mathcal{B}^k(S^n; R)$. Let us consider $[Z] \in \mathcal{B}^{n-k}(S^n)$. We have already noticed that

$$[W, f] \sqcup [Z] = 0 \in R\ .$$

On the other hand, it follows from the very geometric definition of the cobordism cup product that

$$[W, f] \cup [Z] = lk_W(T, Z) - lk_{W'}(T, Z)$$

and the proposition follows.

■

REMARK 12.4. A classical example of linking number is the case where $X = S^3$ and T, Z are oriented disjoint *knots* in S^3 (that is, disjoint submanifolds diffeomorphic to S^1). Every oriented knot T in S^3 admits oriented Seifert surfaces; this is a particular case of Proposition 13.7. However, there is an elementary way to construct a Seifert surface of T (or Z) and to compute the linking number $lk(T,Z)$ from any generic projection of $T \amalg Z$ in $\mathbb{R}^2$ (see [**Rolf**]). We eventually have

$$lk(T,Z) = lk(Z,T) \in R\ .$$

Another classical situation is when $X = S^n$, $T \sim S^p$, $Z \sim S^q$ and these last are *unknotted spheres* in S^n; that is, they are the boundary of embedded $(p+1)$ or $(q+1)$ smooth disks respectively.

12.5. Degree

Let X and Y be compact connected boundaryless (possibly oriented) smooth n-manifolds and let $g : X \to Y$ be a continuous map. Let us fix generators β_X of $\mathcal{B}^n(X;R) = R$ and β_Y of $\mathcal{B}^n(Y;R) = R$ as in Section 11.4. Consider

$$g^* : \mathcal{B}^n(Y;R) \to \mathcal{B}^n(X;R)\ ;$$

then define the *R-degree* of g by

$$\deg_R(g) := g^*(\beta_Y) \in R\ .$$

Although we have already given an operative definition of g^* in full generality, it is convenient to spell it again in the present situation: fix $y_0 \in Y$; up to homotopy make g smooth and transverse to y_0 (equivalently move y_0 a little to make it a regular value of g). Then $g^{-1}(y_0) = \{x_1, \dots x_r\}$ is a finite set of points. In the oriented setting they are oriented, that is, endowed with signs ϵ_j, $j = 1, \dots, r$. On $R = \mathbb{Z}/2\mathbb{Z}$ the degree is equal to $r \bmod(2)$; on $\mathbb{Z}$ the degree is the sum of the signs ϵ_j.

Now we list a few properties of the degree.

- If g is not surjective, then $\deg_R g = 0$.
- If $g : X \to Y$ is a diffeomorphism, then $\deg_R(g) = \pm 1$.
- If $h \circ g$ and the degrees of all involved maps make sense, then

$$\deg_R(h \circ g) = \deg_R(h) \deg_R(g)\ ;$$

that is, the degree is multiplicative under composition. This follows immediately from functoriality.

- If g and h are homotopic, then

$$\deg_R(g) = \deg_R(h)\ .$$

This follows from (5) of Proposition 11.6.

• (*Invariance up to bordism*) To define the degree of a map $g : X \to Y$, it is not strictly necessary that X is connected. In fact we can define

$$\deg_R(g) = \sum_{X_c} \deg_R(g_{|X_c})$$

where X_c varies among the connected components of X. We can rephrase the above procedure to compute the degree in such a way it also holds when X is not connected. Consider $[X, g] \in \mathcal{B}^0(Y; R)$. Given a point p_0, define $j_{y_0} : \{p_0\} \to Y$, $j_{y_0}(p_0) = y_0$. Then

$$\deg_R(g) = j^*_{y_0}([X, g]) \in \mathcal{B}^0(p_0; R) = R \ .$$

Note that it does not depend on the choice of $y_0 \in Y$ because the maps j_* are homotopic to each other as Y is connected (apply again (5) of Proposition 11.6). Then we eventually extend the above homotopy invariance to the *invariance of the degree up to bordism.*

PROPOSITION 12.5. *If* $[X_0, g_0] = [X_1, g_1] \in \mathcal{B}_n(Y; R)$, *then* $\deg_R(g_0) = \deg_R(g_1)$.

■

• *For every oriented connected* X *as above,* $n \geq 1$*, for every* $r \in \mathbb{Z}$ *there is* $g : X \to S^n$ *such that* $\deg_{\mathbb{Z}}(g) = r$*.*
First we prove it when $X = S^n$, by induction on $n \geq 1$. Consider S^1 as the unitary circle of $\mathbb{C}$. The restriction of $z \to \bar{z}$ to S^1 has $\mathbb{Z}$-degree equal to -1. For every $r \geq 1$, the restriction of $z \to z^r$ has $\mathbb{Z}$-degree equal to r. As the degree is multiplicative under composition this achieves the result for $n = 1$. For a given $r \in \mathbb{Z}$, let $g : S^n \to S^n$ be of degree equal to r; we have to construct $\hat{g} : S^{n+1} \to S^{n+1}$ having the same degree. Take $\hat{g}$ which fixes the northern and southern poles and holds $\hat{g}(x) = (1 - t^2)g(\frac{x}{1-t^2})$ on $S^{n+1} \cap \{x_{n+2} = t\}$, for every $t \in (-1, 1)$. We check that its $\mathbb{Z}$-degree is equal to r as well. To finish, it is enough to construct $g : X \to S^n$ of $\mathbb{Z}$-degree equal to ± 1. Fix a smooth D^n contained in a chart of X. By using a tubular neighbourhood U of ∂D^n in X, it is not hard to construct a smooth map $g : X \to S^n$ such that the restriction of g to D^n is a diffeomorphism to $D^- = \{x \in S^n |\ x_{n+1} \leq 0\}$, and holds constantly the northern pole of S^n on the complement of $D^n \cup U$ in X. Such a g has the required property.

REMARK 12.6. For arbitrary oriented X and Y as above, it is in general a hard question to determine the set of $r \in \mathbb{Z}$ which can be realized as the $\mathbb{Z}$-degree of some $g : X \to Y$.

• Consider again the case $X = Y = S^n$, $n \geq 1$. If $\rho : S^n \to S^n$ is the restriction of a reflection of $\mathbb{R}^{n+1}$ along a linear hyperplane, then $\deg_{\mathbb{Z}}(\rho) = -1$. Denote by $a_n : S^n \to S^n$, $a_n(x) = -x$ the *antipodal map*; a_n is the composition of the restriction of $n + 1$ reflections (e.g. the reflections along the hyperplanes $\{x_j = 0\}$, $j = 1, \dots, n + 1$). Then we have

$$\deg_{\mathbb{Z}}(a_n) = (-1)^{n+1} \ .$$

• (*Degree and linking number*) In the setting of Remark 12.4, let $S^n = \mathbb{R}^n \cup \infty$, $\infty \in S^n \setminus (T \cup Z)$. Define

$$L : T \times Z \to S^{n-1},\ L(t,z) = \frac{t-z}{||t-z||}\ .$$

Then we can prove that

$$\deg_Z(L) = \pm lk(T,Z)\ .$$

It is not so easy to prove it in general. In the particular case of knots in S^3, this can be easily checked using the Seifert surfaces associated to generic projections in $\mathbb{R}^2$ (see [**Rolf**]) .

12.5.1. A proof of the fundamental theorem of algebra. The fundamental theorem of algebra states that every non-constant complex polynomial $p(Z) \in \mathbb{C}[Z]$ has a complex root a, $p(a) = 0$. There are several proofs; here is a differential topological one based on the degree.

Let $p(Z)$ be a polynomial of degree $m \geq 1$. It is not restrictive to assume that

$$p(Z) = Z^m + \sum_{j=1}^m a_j Z^{m-j}$$

is monic. Define the homotopy through polynomial maps:

$$p_t(z) = tp(z) + (1-t)z^m = z^m + t(\sum_{j=1}^m a_j z^{m-j}),\ t \in [0,1]\ .$$

By the compactness of $[0,1]$, the ratios $p_t(z)/z^m$ tend uniformly to 1 when $|z| \to +\infty$. Hence there is R big enough such that, for every $t \in [0,1]$, the roots of $p_t(Z)$ are in the open ball $B_R = \{|z| < R\}$, with boundary $S_R \sim S^1$. Hence

$$p_t/|p_t| : S_R \to S^1$$

is a well defined smooth map for every t, so that $p_1/|p_1|(z) = p(z)/|p(z)|$ and $p_0/|p_0|(z) = z^m/R^m$ are homotopic to each other. It is immediate that

$$\deg_{\mathbb{Z}}(p_0/|p_0|) = m\ ,$$

hence also $\deg_{\mathbb{Z}}(p/|p|) = m$. On the other hand, if $p(Z)$ has no roots, then $p/|p|$ can be extended to the whole closed ball $\bar{B}_R$; it would be homotopically trivial, hence $\deg_{\mathbb{Z}}(p/|p|) = 0$, a contradiction.

■

12.6. The Euler class of a vector bundle

Let

$$\xi := \pi : E \to X$$

be a vector bundle of *rank* k (that is, k is the dimension of the fibre) over a compact boundaryless smooth n-manifold X. This manifold X is considered as a submanifold of E via the canonical *zero section* $s_0 : X \to E$. Then

$$[X] \in \mathcal{B}^k(E; \mathbb{Z}/2\mathbb{Z})$$

and set

$$w^k(\xi) := s_0^*([X]) \in \mathcal{B}^k(X; \mathbb{Z}/2\mathbb{Z}) \ .$$

This is called the *Euler class of the vector bundle* ξ. Let us describe how to get nice representatives of this class.

LEMMA 12.7. *(1) The subset* $\pitchfork \Gamma(\xi, X)$ *made by the sections* $s : X \to E$ *of* ξ *such that* $s \pitchfork X$ *is open and dense in* $\Gamma(\xi)$*.*

(2) Two sections transverse to X *are homotopic to each other through sections of* ξ*.*

Proof : As X is compact, the openness is now a routine fact. Let us show the density. Let $s : X \to E$ be any section. By transversality theorems, there is a map $z : X \to E$ close to s, $z \pitchfork X$, z not necessarily a section. If z is close enough to s, then $h = \pi \circ z$ is a diffeomorphism to $X \subset E$. Then $z \circ h^{-1} : X \to E$ is a section close to s and transverse to X. Every section is homotopic to s_0 via a natural fibre-wise radial homotopy.

■

Let $s : X \to E$ be any section of ξ transverse to X. Then its zero set

$$Z_s = \{x \in X |\ s(x) = 0\}$$

is a proper submanifold of X of dimension $n - k$.

LEMMA 12.8. *For any section* $s : X \to E$*,* $s \pitchfork X$*, we have*

$$w^k(\xi) = [Z_s] \in \mathcal{B}^k(X; \mathbb{Z}/2\mathbb{Z}) \ .$$

This follows immediately from the definition of s_0^*.

PROPOSITION 12.9. *For every couple* ξ*,* ρ *of vector bundles on* X *of rank* r *and* s*, respectively, then*

$$w^{r+s}(\xi \oplus \rho) = w^r(\xi) \sqcup w^s(\rho) \ .$$

Proof : By using sections s and s' of ξ and ρ transverse to X in $E(\xi)$ and $E(\rho)$, respectively, and such that $s \oplus s'$ is transverse to X in $E(\xi \oplus \rho)$, then

$$Z_{s \oplus s'} = Z_s \pitchfork Z_{s'} \ .$$

We conclude by means of Lemma 12.8.

■

It is evident that if there exists s such that $Z_s = \emptyset$, then $w^k(\xi) = 0$.

The nonvanishing of the Euler class $w^k(\xi)$ is a basic obstruction to the existence of a nowhere vanishing section of the vector bundle ξ.

If $k > n = \dim X$, then for every s as above $Z_s = \emptyset$ and this fits with $\mathcal{B}^k(X;\mathbb{Z}/2\mathbb{Z}) = 0$. It follows that ξ of rank $k > n$ is strictly isomorphic to $\eta \oplus \epsilon^{n-k}$, η being of rank n; in other words, *every vector bundle over X is stably equivalent to a vector bundle of rank $\leq \dim(X)$.*

PROPOSITION 12.10. *Let $g : X \to Y$ be a smooth map between compact boundaryless smooth manifolds. Let ξ be a rank k vector bundle over Y. Then*

$$w^k(g^*(\xi)) = g^*(w^k(\xi)) \in \mathcal{B}^k(X;\mathbb{Z}/2\mathbb{Z}) \ .$$

Proof : We stress that the first g^* refers to the vector bundle pull-back while the second refers to the cobordism pull-back. The two pull-back procedures are formally very similar and equality is a direct consequence.

■

Universal Euler classes. If $g : X \to \mathfrak{G}_{h,k}$ is any classifying map of ξ, so that ξ is strictly equivalent to $g^*(\tau_{h,k})$, then $w^k(\xi) = g^*(w^k(\tau_{h,k}))$, $w^k(\tau_{h,k}) \in \mathcal{B}^k(\mathfrak{G}_{h,k};\mathbb{Z}/2\mathbb{Z})$. These last can be considered as the *universal Euler classes of vector bundles.*

The total cobordism characteristic classes of projective spaces. Consider the particular case of the *real projective space* $\mathbf{P}^n(\mathbb{R}) = \mathfrak{G}_{n+1,1}$ with the tautological line bundle $\tau_{n+1,1}$. Then

$$\gamma^1 := w^1(\tau_{n+1,1}) = [Z^1] \in \mathcal{B}^1(\mathbf{P}^n;\mathbb{Z}/2\mathbb{Z})$$

where $Z^1 \sim \mathbf{P}^{n-1}(\mathbb{R})$ is any projective hyperplane in $\mathbf{P}^n(\mathbb{R})$. For every $s \geq 1$,

$$\gamma^s := \sqcup_{j=1}^s \gamma^1 = [Z^s]$$

where $Z^s \sim \mathbf{P}^{n-s}(\mathbb{R})$ is any codimension s projective subspace of $\mathbf{P}^n(\mathbb{R})$. Set $\gamma^0 := [Z^0] = [\mathbf{P}^n(\mathbb{R})]$ the $\mathbb{Z}/2\mathbb{Z}$-fundamental class. Clearly, if $s \leq n$,

$$\gamma^s \sqcup \gamma^{n-s} = 1 \ ;$$

hence they do not belong to $\ker(\phi^s)$ and $\ker(\phi^{n-s})$ respectively. If $s > n$, $\gamma^s = 0$. By definition

$$\sum_{s=0}^n \gamma^s \in \mathcal{B}^\bullet(\mathbf{P}^n(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$$

is the *total $\mathbb{Z}/2\mathbb{Z}$-cobordism characteristic class of* $\mathbf{P}^n(\mathbb{R})$. If necessary, we write $\gamma^s = \gamma^s_n$ to stress that it refers to $\mathbf{P}^n(\mathbb{R})$. If we consider any linear inclusion $j : \mathbf{P}^k(\mathbb{R}) \to \mathbf{P}^n(\mathbb{R})$, $k \leq n$, $\mathbf{P}^k(\mathbb{R}) = Z^{n-k}$ as above, then for every $m \geq 0$,

$$\gamma^m_k = j^*(\gamma^m_n) \ .$$

12.6.1. Oriented vector bundles. A rank r vector bundle ξ over X is oriented if it is defined by a maximal fibred atlas with $\mathrm{GL}^+(k,\mathbb{R})$ cocycle. If the base manifold is also oriented, then the total space manifold is naturally oriented itself. If X is compact boundaryless, then we can repeat the above constructions in the oriented setting. This defines the *oriented Euler class*

$$e^r(\xi) := j^*([X]) \in \mathcal{B}^r(X;\mathbb{Z}) \ .$$

The class $\omega^r(\xi)$ is the image of $e^r(\xi)$ via the natural forgetting map $\mathcal{B}^r(X;\mathbb{Z}) \to \mathcal{B}^r(X;\mathbb{Z}/2\mathbb{Z})$. For every pair of oriented bundles over X of rank r and s, respectively,

$$e^{r+s}(\xi \oplus \rho) = e^r(\xi) \sqcup e^s(\rho) \in \mathcal{B}^{r+s}(X;\mathbb{Z}) \ ;$$

for every smooth map $f : X \to Y$ between oriented compact boundaryless manifolds, for every oriented rank r vector bundle ξ bundle over Y,

$$g^*(e^r(\xi)) = e^r(g^*(\xi)) \in \mathcal{B}^r(X;\mathbb{Z}) \ .$$

■

A case of main interest is the tangent bundle of X; then

$$w^n(X) := w^n(T(X)) \in \mathcal{B}^n(X;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

provides a basic obstruction to the existence of nowhere vanishing tangent vector fields on X.

If we consider the rank 1 determinant bundle of X, then

$$w^1(X) := w^1(\det T(X)) \in \mathcal{B}^1(X;\mathbb{Z}/2\mathbb{Z})$$

provides a basic obstruction to X being orientable. We will see in Corollary 13.4 that it is a *complete* obstruction.

We will develop the case of (real and complex) rank 1 bundles (also called *line bundles*) in Chapter 13. We will develop the study of the Euler class of the tangent bundle of X in Chapter 14.

12.7. Borsuk-Ulam theorem

By definition, a map $f : S^n \to S^m$ is *antipodal preserving* if for every $x \in S^n$,

$$f(-x) = -f(x) \ .$$

PROPOSITION 12.11. *For every $n \geq 1$, there does not exist any continuous antipodal preserving map $f : S^n \to S^{n-1}$.*

The following corollary is known as the *Borsuk-Ulam theorem* (BUT).

COROLLARY 12.12. *For every $n \geq 1$, for every continuous map $f : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$.*

For example, assuming that the surface of the earth is a round sphere and that temperature and pressure vary continuously on it in space and time, then at every instant there is a couple of antipodal points at which we have the same couple of temperature and pressure values.

Proof of BUT. By contradiction, if a given f does not satisfy the conclusion of the Corollary, then

$$g : S^n \to \mathbb{R}^n,\ g(x) = f(x) - f(-x)$$

is continuous, nowhere vanishing, and for every $x \in S^n$,

$$g(-x) = f(-x) - f(x) = -g(x)\ .$$

Then

$$\hat{g} : S^n \to S^{n-1},\ \hat{g}(x) = g(x)/||g(x||$$

is continuous and would be antipodal preserving, contrary to Proposition 12.11

■

Proof of Proposition 12.11. To lighten the notations, in this proof we will use $\eta_k(*)$ instead $\mathcal{B}_k(*;\mathbb{Z}/2\mathbb{Z})$, and write $\mathbf{P}^m$ instead of $\mathbf{P}^m(\mathbb{R})$.

The case when $n = 1$ is evident, because S^1 is connected while $S^0 = \{\pm 1\}$ is not.

For $n = 2$, we use some basic facts about the fundamental group of a manifold and its action on a universal covering space. Assume that there is such a continuous antipodal preserving map $f : S^2 \to S^1$. It induces a map $\hat{f} : \mathbf{P}^2 \to \mathbf{P}^1 \sim S^1$ such that the following diagram commutes, the vertical maps being the natural degree 2 covering maps:

$$\begin{array}{ccc} S^2 & \stackrel{f}{\to} & S^1 \\ \downarrow p_2 & & \downarrow p_1 \\ \mathbf{P}^2 & \stackrel{\hat{f}}{\to} & \mathbf{P}^1 \end{array} .$$

We know that $\pi_1(\mathbf{P}^2, x_0) \sim \mathbb{Z}/2\mathbb{Z}$, generated by the class of a projective line passing through the base point, while $\pi_1(\mathbf{P}^1, \hat{f}(x_0)) \sim \mathbb{Z}$, generated by the the identity loop. Hence the induced homomorphism $\hat{f}_* : \pi_1(\mathbf{P}^2, x_0) \to \pi_1(\mathbf{P}^1, \hat{f}(x_0))$ is necessarily trivial. On the other hand, take the two antipodal points $x, -x \in S^2$ over x_0 and an arc σ in S^2 that joins them. Then $p_2(\sigma)$ represents a nontrivial element of $\pi_1(\mathbf{P}^2, x_0)$ because it acts nontrivially on S^2, which is the universal covering of the projective plane. The class $\hat{f}_*(< p_2(\sigma) >)$ is represented by $p_1 \circ f \circ \sigma$ and again it is nontrivial because it acts nontrivially on the universal covering space of $\mathbf{P}^1$ that dominates the covering p_1. This is contrary to the fact that $\hat{f}_* = 0$.

If $n > 2$, we have a similar commutative diagram

$$\begin{array}{ccc} S^n & \xrightarrow{f} & S^{n-1} \\ \downarrow p_n & & \downarrow p_{n-1} \\ \mathbf{P}^n & \xrightarrow{\hat{f}} & \mathbf{P}^{n-1} \end{array}$$

where both vertical maps are now universal covering maps. Both fundamental groups are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the very same argument used above shows that

$$\hat{f}_* : \pi_1(\mathbf{P}^n, x_0) \to \pi_1(\mathbf{P}^{n-1}, \hat{f}(x_0))$$

is an isomorphism. Any surjective homomorphism $g : \mathbb{Z}/2\mathbb{Z} \to G$ either is an isomorphism or $G = 0$ and g is trivial. For every $m > 1$, the surjective homorphism

$$\hat{h} := \sigma_1 \circ h_1 : \pi_1(\mathbf{P}^m, x_0) \to \eta_1(\mathbf{P}^m)$$

is nontrivial (the class of a projective line Z^{m-1} passing through the base point is sent by $\hat{h}$ to the nontrivial class $[Z^{m-1}] \in \eta_1(\mathbf{P}^m)$, for via the tautological isomorphism $[Z^{m-1}] = \gamma_m^{m-1} \in \eta^{m-1}(\mathbf{P}^m)$, and we know that $\gamma_m^{m-1} \sqcup \gamma^1 = 1$). Hence $\hat{h}$ is an isomorphism and $\hat{f}$ induces an isomorphism (we keep the notation)

$$\hat{f}_* : \eta_1(\mathbf{P}^n) \to \eta_1(\mathbf{P}^{n-1}) \ .$$

For every $m > 1$, $\mathrm{Hom}(\eta_1(\mathbf{P}^m), \mathbb{Z}/2\mathbb{Z}) \sim \mathbb{Z}/2\mathbb{Z}$. Then in our situation

$$f_*^t : \mathrm{Hom}(\eta_1(\mathbf{P}^{n-1}), \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\eta_1(\mathbf{P}^n), \mathbb{Z}/2\mathbb{Z})$$

is also an isomorphism. For every $m > 1$,

$$\hat{\phi} : \eta^1(\mathbf{P}^m)/\ker(\phi) \to \mathrm{Hom}(\eta_1(\mathbf{P}^m), \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism and $\eta^1(\mathbf{P}^m)/\ker(\phi)$ is generated by γ_m^1. Then, on one hand we would have

$$\hat{f}^*(\gamma_{m-1}^1) = \gamma_m^1, \ \hat{f}_*(\hat{\phi}(\gamma_{m-1}^1)) = \hat{\phi}(\gamma_m^1) \ ;$$

on other hand,

$$0 = \hat{f}^*(0) = \hat{f}^*(\sqcup_{s=1}^m \gamma_{m-1}^1) = \sqcup_{s=1}^m \gamma_m^1 = 1$$

and this is a contradiction.

■

CHAPTER 13

Line bundles, hypersurfaces, and cobordism

In this chapter, X denotes a compact boundaryless smooth manifold and we also assume that X is connected (in general we can apply the next arguments to each connected component). We will use indifferently the notations $\eta_j(X)$ or $\mathcal{B}_j(X;\mathbb{Z}/2\mathbb{Z})$ (resp. $\Omega_j(X)$ or $\mathcal{B}_j(X;\mathbb{Z})$) and so on. Also recall $\mathcal{H}^r(X;R) := \mathcal{B}^r(X;R)/\ker(\phi^r)$ defined in Section 11.4. Using the Euler classes of line bundles over X, we can achieve a good understanding of $\eta^1(X) = \mathcal{B}^1(X;\mathbb{Z}/2\mathbb{Z})$. If X is oriented, we will get information about $\Omega^1(X)$ and, by using *complex* line bundles, also about $\Omega^2(X)$.

13.1. Real line bundles and hypersurfaces

Let X be as above. Denote by $\mathcal{V}_1(X)$ the set of rank 1 real vector bundles on X (also called (real) *line bundles*) considered up to strict equivalence. We know from Chapter 5 that

$$\mathcal{V}_1(X) \sim [X, \mathbf{P}^\infty(\mathbb{R})]$$

where this last is the set of homotopy classes of *classifying maps* $f \in \mathcal{E}(X, \mathbf{P}^\infty(\mathbb{R}))$, and the bijective correspondence is given via the pull-back of the tautological line bundle:

$$[X, \mathbf{P}^\infty(\mathbb{R})] \to \mathcal{V}_1(X),\ [f] \to [f^*(\tau_{\infty,1})]\ .$$

Moreover, by Section 6.10.1 we know that we can "truncate" the classifying maps so that eventually

$$\mathcal{V}_1(X) \sim [X, \mathbf{P}^{m(n)}(\mathbb{R})]$$

where $m = m(n)$ is big enough and $n = \dim X$. We will often confuse a class with a given representative (we write f instead of $[f]$, ξ instead of $[\xi]$, and so on). Recall that the *tensor product* defines an operation

$$\otimes : \mathcal{V}_1(X) \times \mathcal{V}_1(X) \to \mathcal{V}_1(X),\ (\xi, \beta) \to \xi \otimes \beta\ .$$

In Section 12.6, we have defined a map

$$w^1 : \mathcal{V}_1(X) \to \eta^1(X),\ \xi \to w^1(\xi)$$

which associates to every line bundle its Euler class. Precisely, $w^1(\xi)$ can be represented as

$$w^1(\xi) = [Z]$$

where Z is a smooth compact *hypersurface* in X given as the zero set $Z = Z_s$ of any section $s \in \Gamma(\xi)$ transverse to X in $E(\xi)$, where X is canonically embedded in the total space of ξ by the zero section s_0. Moreover, if Z_0 and Z_1 are two such zero sets, then we can realize the equality of their bordism classes $[Z_0] = [Z_1] \in \eta^1(X)$ by means of *embedded bordisms*; that is we have the following.

There exists a proper hypersurface $(Y, \partial Y)$ *of* $(X\times[0,1], (X\times\{0\})\amalg(X\times\{1\}))$ *such that* $\partial Y = Z_0 \amalg Z_1$, $Z_i \subset X \times \{i\}$. *The map which interpolates the two inclusions* $j_i : Z_i \to X$ *is the projection to* X.

We denote by

$$\eta^1_{\mathrm{Emb}}(X)$$

the *set* of proper smooth hypersurfaces of X considered up to embedded bordism. There is a natural projection

$$\mathfrak{p} : \eta^1_{\mathrm{Emb}}(X) \to \eta^1(X)$$

so that the above map w^1 factorizes as

$$w^1 = \mathfrak{p} \circ \hat{w}^1$$

through a well defined map

$$\hat{w}^1 : \mathcal{V}_1(X) \to \eta^1_{\mathrm{Emb}}(X) \ .$$

PROPOSITION 13.1. *(1) The map* $\hat{w}^1 : \mathcal{V}_1(X) \to \eta^1_{\mathrm{Emb}}(X)$ *is bijective.*

(2) For every couple $(\xi, \beta) \in \mathcal{V}_1^2$,

$$w^1(\xi \otimes \beta) = w^1(\xi) + w^1(\beta) \ .$$

(3) The projection $\mathfrak{p}$ *maps* $\eta^1_{\mathrm{Emb}}(X)$ *to a* $\mathbb{Z}/2\mathbb{Z}$*-submodule* $\mathbf{H}^1(X;R)$ *of* $\mathcal{B}^1(X;\ Z/2\mathbb{Z})$*, the one made by the (unoriented) cobordism classes that can be represented by embedded hypersurfaces.*

Proof : Let us describe the inverse map of $\hat{w}^1$. For every proper hypersurface Z of X, we have to construct a line bundle ξ_Z on X such that $Z = Z_s$ for some $s \in \Gamma(\xi_Z)$, $s \pitchfork X$. We can find a finite nice atlas of (X, Z), $\{(W_j, \phi_j)\}$ such that, for every j, there is a submersion $f_j : W_j \to \mathbb{R}$, such that $W_j \cap Z = \{f_j = 0\}$. On $W_i \cap W_j$, the ratio f_i/f_j is defined *a priori* outside the zero set of f_j; however, by Remark 1.3, it extends to a well defined, smooth and nowhere vanishing function

$$g_{i,j} : W_i \cap W_j \to \mathbb{R},\ g_{i,j}(x) = f_i(x)/f_j(x) \ .$$

Hence

$$\{g_{i,j} : W_i \cap W_j \to \mathbb{R}^*\}$$

defines a cocycle of a line bundle ξ_Z on X which has the desired properties by construction.

As for (2), we can assume that ξ and β are defined by means of cocycles $\{\mu_{i,j}\}$ and $\{\nu_{i,j}\}$, respectively, over a same nice atlas of X. Then $\{\mu_{i,j}\nu_{i,j}\}$ is a cocycle for $\xi \otimes \beta$. Then if $\{s_i\}$ and $\{s'_i\}$ are representations in local coordinates of sections s and s' of ξ and β, respectively, such that $s \pitchfork X$ and $s' \pitchfork X$, $s \pitchfork s'$, then $\{s_i s'_i\}$ determines a section ss' of $\xi \otimes \beta$ such that $[Z_s] = w^1(\xi)$, $[Z_{s'}] = w^1(\beta)$; by perturbing ss' to get s" $\pitchfork X$, eventually

$Z_{s''}$ represents $w^1(\xi \otimes \beta)$ and $[Z_{s''}] = [(Z_s, i) \amalg (Z_{s'}, i')]$. In fact $Z_{s''}$ can be considered as an embedded desingularization in X of $Z_s \cup Z_{s'}$, which is singular along the codimension 2 submanifold $Y = Z_s \pitchfork Z_{s'}$.

Item (3) is a consequence of (1) and (2).

∎

13.2. Real line bundles and Rep($\pi_1, \mathbb{Z}/2\mathbb{Z}$)

Recall that we are assuming that X is connected. We denote by

$$\mathrm{Rep}(\pi_1(X), \mathbb{Z}/2\mathbb{Z})$$

the set of group homomorphisms (the base point of X is understood). Recall the linear map

$$\phi : \eta^1(X) \to \mathrm{Hom}(\eta_1, \mathbb{Z}/2\mathbb{Z}),\ \ \phi_\gamma(\sigma) = \gamma \sqcap \sigma\ .$$

Recall the surjective homomorphism

$$\hat{h} : \pi_1(X) \to \eta_1(X)\ .$$

Then we define the map

$$\kappa : \mathcal{V}_1(X) \to \mathrm{Rep}(\pi_1(X), \mathbb{Z}/2\mathbb{Z}),\ \ \kappa(\xi) = \phi_{w^1(\xi)} \circ \hat{h}\ .$$

Here is a concrete way to describe $\kappa(\xi)$. As $\pi_1(\mathbf{P}^\infty(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$, then $\mathcal{V}_1(S^1)$ consists of two line bundles: the trivial and the nontrivial one which has the total space diffeomorphic to an open Möbius band. If

$$\sigma = < f : S^1 \to X > \in \pi_1(X)$$

then $\kappa(\xi)(\sigma) = 1$ if and only if $f^*\xi$ is nontrivial.

PROPOSITION 13.2. *The map* $\kappa : \mathcal{V}_1(X) \to \mathrm{Hom}(\pi_1(X), \mathbb{Z}/2\mathbb{Z})$ *is bijective.*

Proof : We have already remarked in Example 5.13 that $\mathbf{P}^\infty(\mathbb{R})$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$ space. It is a fundamental property of such a space that for every

$$\sigma \in \mathrm{Rep}(\pi_1(X), \mathbb{Z}/2\mathbb{Z})$$

there is a unique

$$f \in [X, \mathbf{P}^\infty(\mathbb{R})]$$

such that

$$\sigma = f_* : \pi_1(X) \to \pi_1(\mathbf{P}^\infty(\mathbb{R}))\ .$$

Then

$$\sigma \to \xi_\sigma := f^*(\tau_{\infty,1})$$

defines the inverse map of κ. Equivalently, we can describe κ^{-1} in terms of degree 2 covering maps. It is known that there is a bijection between the degree 2 covering maps over X (up to strict equivalence) and $\mathrm{Rep}(\pi_1(X), \mathbb{Z}/2\mathbb{Z})$. For every line bundle ξ, $\kappa(\xi)$ corresponds to the double covering of X given by the unitary bundle with fibre S^0 associated with ξ. *Viceversa*, every degree 2 covering of X can be considered as a fibre bundle defined by a cocycle over a finite open covering of X with values in the multiplicative subgroup $\{\pm 1\}$ of $\mathbb{R}^*$. So it can be considered as the unitary bundle associated to the line bundle determined by the same cocycle.

■

Referring to Proposition 13.1, we have the following immediate corollaries.

Corollary 13.3. *(1) The map* $\mathfrak{p} : \eta^1_{\rm Emb}(X) \to \mathbf{H}^1(X;\mathbb{Z}/2\mathbb{Z}) \subset \mathcal{B}^1(X;\mathbb{Z}/2\mathbb{Z})$ *is bijective.*

(2) $\mathbf{H}^1(X;\mathbb{Z}/2\mathbb{Z}) \sim \mathcal{H}^1(X;\mathbb{Z}/2\mathbb{Z}) \sim \mathrm{Hom}(\eta_1(X),\mathbb{Z}/2\mathbb{Z})$.

(3) $\mathcal{V}_1(X) \sim \mathcal{H}^1(X;\mathbb{Z}/2\mathbb{Z})$.

Another consequence of the above discussion is that $\mathcal{H}^1(X;\mathbb{Z}/2\mathbb{Z})$ *is finite-dimensional.* For, as X is compact, $\pi_1(X)$ is finitely generated, hence $\eta_1(X) = \hat{h}(\pi_1(X))$ is a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space as well as $\mathcal{H}^1(X;\mathbb{Z}/2\mathbb{Z})$.

Corollary 13.4. *A compact connected boundaryless smooth manifold* X *is orientable if and only if* $w^1(X) = 0 \in \mathcal{H}^1(X;\mathbb{Z}/2\mathbb{Z})$.

13.3. Oriented hypersurfaces and Ω^1

Assume that X is oriented. Then we have the $\mathbb{Z}$-linear map

$$\phi : \Omega^1(X) \to \mathrm{Hom}(\Omega_1(X),\mathbb{Z})$$

and via the homomorphism

$$h : \pi_1(X) \to \Omega_1(X)$$

we define a map

$$\kappa : \Omega^1(X)/\ker(\phi) \to \mathrm{Rep}(\pi_1(X);\mathbb{Z}) \ .$$

As usual, $[X,S^1]$ is the set of homotopy classes in $\mathcal{E}(X,S^1)$. Denote by β_{S^1} the usual generator of $\Omega^1(S^1)$ which fixes the identification $\Omega^1(S^1)=\mathbb{Z}$. We have the $\mathbb{Z}$-linear map

$$\mathfrak{w} : [X,S^1] \to \Omega^1(X),\ f \to f^*(\beta_{S^1}) \ .$$

In fact, $f^*(\beta_{S^1}) = [Z]$ where Z is an oriented proper hypersurface of X of the form $Z = f^{-1}(s_0)$, s_0 being any regular value of f. We denote by $\Omega^1_{\rm Emb}(X)$ the set of proper oriented hypersurfaces of X considered up to *oriented embedded bordism* (this notion is the natural enhancement of the unoriented one given above). Then we have the projection $\mathfrak{p} : \Omega^1_{\rm Emb}(X) \to \Omega^1(X)$ such that $\mathfrak{w}$ factorizes as $\hat{\mathfrak{w}} \circ \mathfrak{p}$ for a well defined map

$$\hat{\mathfrak{w}} : [X,S^1] \to \Omega^1_{\rm Emb}(X) \ .$$

Finally, we have the map

$$\mathfrak{r} : [X,S^1] \to \mathrm{Rep}(\pi_1(X);\mathbb{Z}),\ f \to f_* : \pi_1(X) \to \pi_1(S^1) = \mathbb{Z} \ .$$

PROPOSITION 13.5. *(1) The map* $\hat{\mathfrak{w}} : [X, S^1] \to \Omega^1_{\mathrm{Emb}}(X)$ *is bijective.*

(2) The map $\mathfrak{r} : [X, S^1] \to \mathrm{Rep}(\pi_1(X); \mathbb{Z})$ *is bijective.*

(3) The map $\kappa : \mathcal{H}^1(X; \mathbb{Z}) \to \mathrm{Rep}(\pi_1(X); \mathbb{Z})$ *is bijective.*

(4) The projection $\mathfrak{p} : \Omega^1_{\mathrm{Emb}}(X) \to \Omega^1(X)$ *is injective to a* $\mathbb{Z}$*-submodule*

$$\mathbf{H}^1(X; \mathbb{Z}) \subset \mathcal{B}^1(X; \mathbb{Z}) \ .$$

(5) $\mathbf{H}^1(X; \mathbb{Z}) \sim \mathcal{H}^1(X; \mathbb{Z}) \sim \mathrm{Hom}(\Omega_1, \mathbb{Z})$.

(6) $\mathcal{H}^1(X; \mathbb{Z})$ *is finitely generated.*

Proof : Let us define the inverse map of $\hat{\mathfrak{w}}$. This is a sample of a general construction that we will study with all details in Chapter 17. We limit here to indicate the main points. Let Z be a proper oriented hypersurface of X. As both X and Z are oriented, we can fix a global trivialization $t : Z \times (-1, 1) \to U$ of a tubular neighbourhood of Z in X. Let s_- be the southern pole of S^1, $s_+ := \infty$ the northern one. Let $D \sim (-1, 1)$ be an open interval in S^1 centred at s_-. Then the composition of t^{-1} with the projection to $(-1, 1)$ defines a local submersion $f : U \to D \subset S^1$. By using a suitable partition of unity as usual, we can globally define $f_Z : X \to S^1$ such that f_Z is constantly equal to ∞ on the complement of U, equals f on $t((-1/2, 1/2))$ and $f^{-1}(s_-) = Z$. We verify that the homotopy class of such a map f_Z is invariant up to oriented embedded bordism of hypersurfaces, so $[Z] \to [f_Z]$ eventually defines the inverse map of $\hat{\mathfrak{w}}$. This achieves (1).

As for (2), it is well known that S^1 is a $K(\mathbb{Z}, 1)$ space. Hence for every $\sigma \in \mathrm{Rep}(\pi_1(X); \mathbb{Z})$, there is $f^\sigma : X \to S^1$, uniquely defined up to homotopy, such that $\sigma = f^\sigma_* : \pi_1(X) \to \pi_1(S^1) = \mathbb{Z}$. This defines the inverse map of κ.

The item (3) follows from (1) and (2) by readily noticing that if $[Z] = \hat{\mathfrak{w}}([f])$, then $f_* = \phi([Z])$. Items (4) and (5) are a rephrasing of the previous ones; (6) follows again from the fact that X is compact, hence $\pi_1(X)$ is finitely generated and the homomorphism h is surjective.

■

13.4. Complex line bundles and Ω^2

Assume again that X is oriented. Denote by

$$\mathcal{V}_1(X, \mathbb{C})$$

the set of *complex* line bundles over X considered up to strict equivalence. Similarly to the real case,

$$\mathcal{V}_1(X, \mathbb{C}) \sim [X, \mathbf{P}^\infty(\mathbb{C})]$$

where this last is the space of homotopy classes of *classifying maps* $f \in \mathcal{E}(X, \mathbf{P}^\infty(\mathbb{C}))$, and the bijective correspondence is given via the pull-back of the tautological complex line bundle:

$$[X, \mathbf{P}^\infty(\mathbb{C})] \to \mathcal{V}_1(X, \mathbb{C}),\ [f] \to [f^*(\tau^{\mathbb{C}}_{\infty,1})] \ .$$

Moreover, we can "truncate" the classifying maps so that eventually

$$\mathcal{V}_1(X,\mathbb{C}) \sim [X, \mathbf{P}^{m(n)}(\mathbb{C})]$$

where $m = m(n)$ is big enough, $n = \dim(X)$. Every complex line bundle ξ underlies a rank 2 *oriented* real bundle $\xi_{\mathbb{R}}$. *Viceversa*, every rank 2 oriented real bundle can be endowed with the structure of a complex line bundle by reducing the structural group to $SO(2)$ and by identifying the rotation by $\pi/2$ to the product by $\sqrt{-1}$. Then we can define

$$e^2 : \mathcal{V}_1(X;\mathbb{C}) \to \Omega^2(X),\ \xi \to e^2(\xi_{\mathbb{R}})$$

which associates to every ξ the oriented Euler class of its "realification". Precisely, $e^2(\xi)$ can be represented as

$$e^2(\xi) = [Z]$$

where Z is a proper codimension 2 oriented smooth submanifold of X given as the oriented zero set $Z = Z_s$ of any section $s \in \Gamma(\xi_{\mathbb{R}})$ transverse to X in $E(\xi_{\mathbb{R}})$. If Z_0 and Z_1 are two such zero sets, then we can realize the equality of their bordism classes $[Z_0] = [Z_1] \in \Omega^2(X)$ by means of *oriented embedded bordisms* via proper oriented codimension 2 submanifold $(Y, \partial Y)$ of $(X \times [0,1], (X \times \{0\}) \amalg (X \times \{1\}))$. Similarly as above, denote by $\Omega^2_{\mathrm{Emb}}(X)$ the set of codimension 2 oriented proper submanifolds of X considered up to embedded oriented bordism, and

$$\mathfrak{p} : \Omega^2_{\mathrm{Emb}}(X) \to \Omega^2(X)$$

the natural projection. The map e^2 factorizes as $\mathfrak{p} \circ \hat{e}^2$, where

$$\hat{e}^2 : \mathcal{V}_1(\mathbb{C}) \to \Omega^2_{\mathrm{Emb}}(X)\ .$$

Recall the $\mathbb{Z}$-linear map

$$\phi^2 : \Omega^2 \to \mathrm{Hom}(\Omega_2(X), \mathbb{Z})$$

which, when composed with e^2 and the homomorphism

$$h : \pi_2(X) \to \Omega_2(X)\ ,$$

leads to the map

$$\kappa : \mathcal{V}_1(\mathbb{C}) \to \mathrm{Rep}(\pi_2(X), \mathbb{Z})\ .$$

Finally, analogously to the real case, $\mathbf{P}^\infty(\mathbb{C})$ is a $K(\mathbb{Z},2)$-space ([**Hatch**]); hence the map

$$\mathfrak{r} : [X, \mathbf{P}^\infty(\mathbb{C})] \to \mathrm{Rep}(\pi_2(X), \mathbb{Z}),\ f \to f_* : \pi_2(X) \to \pi_2(\mathbf{P}^\infty(\mathbb{C})) = \mathbb{Z}$$

is defined and bijective.

PROPOSITION 13.6. *(1) The map $\hat{e}^2 : \mathcal{V}_1(\mathbb{C}) \to \Omega^2_{\mathrm{Emb}}(X)$ is bijective.*

(2) For every $(\xi, \beta) \in \mathcal{V}_1^2(\mathbb{C})$, $e^2(\xi \otimes_{\mathbb{C}} \beta) = e^2(\xi) + e^2(\beta)$.

(3) The map $\kappa : \mathcal{V}_1(\mathbb{C}) \to \mathrm{Rep}(\pi_2(X), \mathbb{Z})$ is bijective.

(4) The projection $\mathfrak{p}$ is injective and maps $\Omega^2_{\mathrm{Emb}}(X)$ to a $\mathbb{Z}$-submodule $\mathbf{H}^2(X;\mathbb{Z})$ of $\mathcal{B}^2(X;\mathbb{Z})$.

(5) $\mathbf{H}^2(X;\mathbb{Z}) \sim \mathcal{H}^2(X;\mathbb{Z}) \sim \mathrm{Hom}(\Omega_2(X)/\phi^2, \mathbb{Z})$.

13.4.1. Relative case. If $(X, \partial X)$ is compact with nonempty boundary, possibly oriented, this is part of the setting of Section 11.4.1. We can elaborate on a relative version of the previous results. We limit to state the existence of isomorphisms

$$\mathcal{H}^1(X, \partial X; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\mathcal{H}_1(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$$

$$\mathcal{H}^1(X, \partial X; \mathbb{Z}) \to \mathrm{Hom}(\mathcal{H}_1(X; \mathbb{Z}), \mathbb{Z})$$

$$\mathcal{H}^2(X, \partial X; \mathbb{Z}) \to \mathrm{Hom}(\mathcal{H}_2(X; \mathbb{Z}), \mathbb{Z}) \ .$$

13.5. Seifert's surfaces

Let X be a compact oriented boundaryless manifold. By applying similar arguments about complex line bundles or rank 2 oriented real bundles, we want to prove the following proposition.

PROPOSITION 13.7. *Let $Y \subset X$ be a proper oriented codimension 2 submanifold of X. Assume that $[Y] \in \ker(\phi)$; that is, $[Y] = 0 \in \mathcal{H}^2(X; \mathbb{Z})$. Let $\pi : U \to Y$ be a tubular neighbourhood of Y in X. Let $W = X \setminus \mathrm{Int}(U)$ with boundary $\partial W = \partial U$. Then there exists a compact oriented hypersurface with boundary $\tilde{Z}$ of X such that $\partial \tilde{Z} = Y$. Such a $\tilde{Z}$ is called a* Seifert surface *of Y. Precisely, $\tilde{Z}$ is transverse to ∂W, $(Z, \partial Z) := (\tilde{Z} \cap W, \tilde{Z} \cap \partial W)$ is a proper oriented hypersurface in $(W, \partial W)$ and $U \cap \tilde{Z}$ is a collar of Y in $\tilde{Z}$.*

Proof : Let $i : Y \to X$ be the inclusion. Any tubular neighbourhood $p : U \to Y$ of Y in X can be associated to a direct sum decomposition of the form

$$i^*(T(X)) = T(Y) \oplus \xi_{\mathbb{R}}$$

where $\xi_{\mathbb{R}}$ is the "realification" of a complex line bundle on Y. As $[Y] \in \ker(\phi)$, then $e^2(\xi) = 0$, hence ξ is trivial so that U admits global trivializations which induce trivializations of ∂W. Let us fix one $h_0 : \partial W \to Y \times S^1$. Take one oriented fibre $D \sim D^2$ of π with oriented boundary $S \sim S^1$. We claim that $[S]$ is of infinite order in $\Omega_1(W)$. By contradiction, let us assume that $p \neq 0$ parallel copies of S are the boundary of a singular manifold $g : (V, \partial V) \to (W, \partial W)$. Then by gluing V and p parallel copies of D along the boundary, we would get an "absolute" singular 2-manifold $(\tilde{V}, \tilde{g})$ in X such that $[Y] \sqcup [\tilde{V}, \tilde{g}] = p$, contrary to the fact that $[Y] \in \ker(\phi)$. There exists $\psi \in \mathrm{Hom}(\Omega_1(W), \mathbb{Z})$ such that $\psi([S]) = 1$. We know that ψ is realized by a map $f_\psi : (W, \partial W) \to S^1$ transverse to a given point $q \in S^1$. Denote by $j : \partial W \to W$ and $r : S \to \partial W$ the two inclusions. Then $\gamma := j^t(\psi)$ is realized by the restriction f_γ of f_ψ to ∂W, while the restriction of f_γ to S realizes $(j \circ r)^t(\phi)$ and is homotopic to the identity. Up to modiyfing the given trivialization h_0 by a suitable one h, f_γ factorizes as $p \circ h$, where $h : \partial W \to Y \times S^1$ and $p : Y \times S^1 \to S^1$ is the projection to the second factor. Then $(Z, \partial Z) = (f_\phi^{-1}(q), f_\gamma^{-1}(q))$ and $\tilde{Z}$, obtained by gluing along ∂Z the mapping cylinder of the restriction of π to it, achieve the proof.

■

From the last step of the above proof, we have the following corollary.

Corollary 13.8. *Let X be an oriented compact n-manifold with boundary ∂X. Let Z be a proper oriented submanifold of dimension $n-2$ of ∂X. Assume that $[Z] = 0$ in $\mathcal{H}^2(X;\mathbb{Z})$. Then there is a proper oriented hypersurface $(W, \partial W)$ such that $Z = \partial W$.*

We also have the following version of Corollary 13.8 when Z is of codimension 2 in ∂X.

Proposition 13.9. *Let X be an oriented compact n-manifold with boundary ∂X. Let Z be e proper submanifold of dimension $n-3$ of ∂X. Assume that $[Z] = 0$ in $\mathcal{H}^3(X;\mathbb{Z})$. Then there is a proper codimension-2 oriented submanifold $(W, \partial W)$ of $(X, \partial X)$ such that $Z = \partial W$.*

Proof : The hypotheses put us in a situation analogous to the last step in the proof of Proposition 13.7, that is, to Corollary 13.8. Here S^1 is replaced by $\mathbf{P}^n(\mathbb{C})$ (n big enough) in the sense that both carry special instances of the *Pontryagin-Thom's construction*, which will be considered in Chapter 17 in full generality. Let $f_0 : Z \to \mathbf{P}^{n-1}(\mathbb{C})$ be a classifying map of the oriented normal rank-2 bundle of Z in ∂X. Note that $\mathbf{P}^n(\mathbb{C}) \setminus \{x_0\}$, $x_0 \in \mathbf{P}^n(\mathbb{C}) \setminus \mathbf{P}^{n-1}(\mathbb{C})$, is diffeomorphic to the total space of the tautological vector bundle on $\mathbf{P}^{n-1}(\mathbb{C})$. Hence f_0 extends to a map $f : \partial X \to \mathbf{P}^n(\mathbb{C})$ such that $f \pitchfork \mathbf{P}^{n-1}(\mathbb{C})$ and $Z = f^{-1}(\mathbf{P}^{n-1}(\mathbb{C}))$. As $[Z] = 0$ in $\mathcal{H}^3(X;\mathbb{Z})$, if n is big enough then f can be extended to a map $F : X \to \mathbf{P}^n(\mathbb{C})$ which we can assume is transverse to $\mathbf{P}^{n-1}(\mathbb{C})$. Finally, $W = F^{-1}(\mathbf{P}^{n-1}(\mathbb{C}))$ satisfies the required property.

■

As a corollary, we have a weak version of Proposition 13.7 when Y has codimension 3.

Corollary 13.10. *Let $Y \subset X$ be a proper oriented codimension 3 submanifold of X. Assume that the normal bundle of Y in X has a nonvanishing section s and let $Y' = s(Y)$ be a copy of Y in the boundary ∂U of a tubular neighbourhood of Y in X. Assume that $[Y'] = 0$ in $\mathcal{H}^3(X \setminus \mathrm{Int}(U);\mathbb{Z})$. Then there is a proper oriented codimension-2 submanifold $(W, \partial W)$ of $(X \setminus \mathrm{Int}(U), \partial U)$ such that $\partial W = Y'$.*

Remark 13.11. *(Nonorientable Seifert surfaces)* In the statement of Proposition 13.7, do not assume that X and Y are orientable and use the space $\mathcal{H}^2(X;\mathbb{Z}/2\mathbb{Z})$ instead. It is natural to inquire about the existence of possibly nonorientable Seifert surfaces. We see an immediate obstruction: if a Seifert surface exists and $i^*(T(X)) = T(Y) \oplus \xi$ is as above (where ξ is now not necessarily trivial or orientable), then ξ has a nowhere vanishing section. The above proof can be adapted to show that this is the only obstruction.

CHAPTER 14

Euler-Poincaré characteristic

X will denote a compact connected oriented boundaryless smooth n-manifold. Then the tangent bundle $\pi : T(X) \to X$ is tautologically an *oriented* rank n vector bundle on X: the orientation of X determines, in a coherent way, an orientation on every fibre T_pX of $T(X)$. Then we can consider the oriented Euler class

$$e^n(X) \in \Omega^n(X) = \mathcal{B}^n(X;\mathbb{Z}) = \mathbb{Z} \ .$$

By a traditional change of notation

$$\chi(X) := e^n(X) \in \mathbb{Z}$$

is called the *Euler-Poincaré characteristic of* X . If X is not connected, $\chi(X)$ is defined as the sum of the characteristics of its connected components.

Recall that $\chi(X)$ is computed by means of any section s of $T(X)$ transverse to X. In other words, $\chi(X)$ is the *self-intersection number* of X in $T(X)$. Such a section $s \pitchfork X$ is a tangent vector field on X with only *non-degenerate zeros*: s can be expressed in local coordinates at every such a zero $p \sim 0$ in the form

$$s(x) = (x, f_p(x)) \ ,$$

where $f_p : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a diffeomorphism. The sign $\epsilon(p) = \pm 1$ of the zero p is readily computed as

$$\epsilon(p) = \mathrm{sign}(\det d_0 f_p)$$

so that

$$\chi(X) = \sum_{p; s(p)=0} \epsilon(p) \ .$$

14.1. E-P characteristic via Morse functions

Let $f : X \to \mathbb{R}$ be a Morse function with critical points $p_1, \dots, p_r$ of index $q_1, \dots, q_r$. Let $\nabla_g f$ be an adapted gradient field of f as in Section 9.1. Then $p_1, \dots, p_r$ are also the zeros of this field. It is easy to check by using the Morse local coordinates that they are non-degenerate zeros and their sign is given by

$$\epsilon(p_j) = (-1)^{q_j} \ .$$

Hence we have

$$\chi(X) = \sum_{j=1}^{r} (-1)^{q_j} \ .$$

This has the following interesting corollary.

COROLLARY 14.1. *If* $\dim(X) = n$ *is odd, then* $\chi(X) = 0$.

Proof : Consider the Morse function $1 - f$; f and $1 - f$ have the same critical points $p_1, \dots, p_r$, of index q_j and $n - q_j$, $j = 1, \dots, r$, respectively. Then

$$\chi(X) = \sum_{j=1}^{r}(-1)^{q_j} = \sum_{j=1}^{r}(-1)^{n-q_j} \ ;$$

as n is odd, this implies that $\chi(X) = -\chi(X)$.

■

REMARK 14.2. If we consider the handle decomposition of X, $\mathcal{H}$, associated to a Morse function f, the above expression of $\chi(X)$ can be rewritten in terms of handle indices; that is, $\chi(X) = \chi(\mathcal{H})$ (see Section 9.3). The characteristic $\chi(\mathcal{H})$ is defined for every handle decomposition, not necessarily associated to any Morse function. We know that it is invariant for the handle move-equivalence. Hence, at least for the handle decompositions which are move-equivalent to one carried by a Morse function, $\chi(\mathcal{H})$ has an intrinsic meaning.

14.2. The index of an isolated zero of a tangent vector field

We are going to reformulate the sign $\epsilon(p)$ of a non-degenerate zero of a tangent vector field on X in a way that will make sense also for any *isolated* zero (not necessarily non-degenerate). Let p be an isolated zero of a vector field s. Let us implement the following procedure:

(1) Take local coordinates of X at $p \sim 0$, so that s is of the form
$$s(x) = (x, f_p(x))$$
where
$$f_p : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$
is a smooth map such that $f_p^{-1}(0) = \{0\}$.

(2) It is well defined the smooth map
$$f_p/||f_p|| : S^{n-1} \to S^{n-1} \ .$$

(3) We can assume that the standard orientation of $\mathbb{R}^n$ associated to the standard basis is coherent with the global orientation of X, so that S^{n-1} is oriented as the boundary of the oriented disk $D^n \subset \mathbb{R}^n$.

(4) Finally set
$$i_p = \deg(f_p/||f_p||) \in \mathbb{Z} \ .$$

LEMMA 14.3. *(1)* $i_p(s) := i_p = \deg(f_p/||f_p||) \in \mathbb{Z}$ *is well defined (i.e. it does not depend on the specific implementation of the procedure) and it is called* the index of the isolated zero p *of the tangent vector field* s.

(2) If p *is a non-degenerate zero of* s*, then (with the notations fixed above)*
$$i_p(s) = \epsilon(p) = \text{sign}(\det d_0 f_p) \ .$$

Proof : Let $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a change of coordinates relating two different implementations. Then $\mathcal{D}^n := \phi^{-1}(D^n)$ is a smooth oriented n-disk around 0, with oriented boundary Σ diffeomorphic to S^{n-1}. Let $s(x) = (x, f_p(x))$ be the expression of s in the source local coordinates. Set

$$g := f_p/||f_p|| : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \ .$$

It is clear that i_p computed with respect to the target local coordinates is equal to the degree of the restriction of g to Σ. So we have to prove that this degree equals i_p computed for the source local coordinates. There is $1 > \epsilon > 0$ small enough such that the closed n-disk ϵD^n (with boundary ϵS^{n-1}) is contained in the interior of $\mathcal{D}^n$. Then the restriction of g to $D^n \setminus \mathrm{Int}(\epsilon D^n)$ establishes an oriented bordism of $g_{|S^{n-1}}$ with $g_{|\epsilon S^{n-1}}$; similarly, the restriction of g to $\mathcal{D}^n \setminus \mathrm{Int}(\epsilon D^n)$ establishes a bordism of $g_{|\Sigma}$ with $g_{|\epsilon S^{n-1}}$. We can conclude by applying twice the invariance of the degree up to bordism, as in Proposition 12.5. This achieves (1).

As for (2), assume that f_p is a diffeomorphism. The result is immediate if f_p is a linear isomorphism. Then we can conclude using the results of Section 1.13 and the invariance properties of the degree again.

■

14.3. Index theorem

Let s be a tangent vector field on X with only isolated zeros, say $p_1, \dots, p_r$ (there is a finite number because X is compact). Then we can set

$$\chi(X, s) := \sum_{j=1}^{r} i_{p_j}(s) \ ;$$

if $s \pitchfork X$ (that is, all zeros are non-degenerate) then we know that

$$\chi(X, s) = \chi(X)$$

has an intrinsic meaning which does not depend on the choice of the field s. The next theorem extends this fact to an arbitrary field as above.

THEOREM 14.4. *For every tangent vector field s on X with only isolated zeros, we have*

$$\chi(X, s) = \chi(X) \ .$$

Proof : For every zero p_j of s, fix an implementation of the procedure that computes $i_{p_j}(s)$. Hence $i_{p_j}(s) = \deg(g_j : S^{n-1}_j \to S^{n-1})$. We can also assume that these charts are pairwise disjoint. Let $\tilde{s}$ be a section of $T(X)$, $\tilde{s} \pitchfork X$, very close to s. Then the non-degenerate zeros of $\tilde{s}$ distribute in bunches $z_{j,1}, \dots, z_{j,r_j}$, contained in the interior of the n-disk D^n_j, $j = 1, \dots, r$. Fix one of these zeros $p = p_j$ and consider the corresponding $z_1, \dots, z_{r_j} \in D^n = D^n_j$. We can take a system of pairwise disjoint small n-disks D^n_i centred at z_i, contained in the interior of D^n. As s and $\tilde{s}$ are homotopic along $S^{n-1} = \partial D^n$, then we can use $\tilde{s}$ instead of s in order to compute $i_{p_j}(s)$ via the degree. On the other hand, we can use the restriction of $\tilde{s}$ to ∂D^n_i to compute the index of the non-degenerate zero z_i of $\tilde{s}$. The normalized field is defined on $D^n \setminus (\cup_i \mathrm{Int}(D^n_i))$ and this establishes a bordism between the restriction on the boundary components. By the invariance of the degree up to bordism, we realize that

$$i_p(s) = \sum_i i_{z_i}(\tilde{s}) \ .$$

By taking the sum over all zeros of s, we eventually get

$$\chi(X, s) = \sum_{j,i} i_{z_{j,i}}(\tilde{s}) = \chi(X) \ .$$

■

14.4. E-P characteristic for nonoriented manifolds

Let us fix first the behaviour of χ when the orientation changes. So let X be as above and $-X$ denote X endowed with the opposite orientation.

Lemma 14.5. $\chi(X) = \chi(-X)$.

Proof : Use the same given tangent field s with isolated zeros to compute both characteristic numbers. As $T(X)$ is tautologically oriented in agreement with the orientation of X, it is immediate that the index of every zero of s does not depend on the choice of this orientation.

■

The computation of the index of an isolated zero p of s is purely local task: *we do not need a* global *orientation of X to compute it; a local orientation of X at p suffices and the same argument of the above lemma shows that it does not depend on the choice of the local orientation.*

This suggests that the procedure to compute $\chi(X)$ can be extended to every X not oriented and even nonorientable. In the computation of the indices, it is enough to replace a global orientation of X (if any) with an arbitrary system of local orientations at the zeros of a given tangent field s with isolated zeros. Then we have defined in general $\chi(X, s)$, that might depend on the choice of s. In fact, it does not. If X is orientable we have already achieved this result. Assume that X is connected and nonorientable. Let $p : \tilde{X} \to X$ be the degree 2 orientation covering of X, where $\tilde{X}$ is the connected orientable total space of the unitary determinant bundle of X. Every field s on X as above lifts to a field $\tilde{s}$ on $\tilde{X}$ so that every isolated zero p of s lifts to a couple $p_\pm$ of isolated zeros of $\tilde{s}$. It follows from the very definition that

$$i_p(s) = i_{p_\pm}(\tilde{s}) \ ,$$

so eventually

$$\chi(X, s) = \frac{1}{2}\chi(\tilde{X}, \tilde{s}) = \frac{1}{2}\chi(\tilde{X}) \ .$$

If X is orientable, then $\tilde{X}$ consists of two copies of X, so that also in this case

$$\chi(X) = \frac{1}{2}\chi(\tilde{X}) \ .$$

Summing up,

$$\chi(X) := \frac{1}{2}\chi(\tilde{X}) \in \mathbb{Z}$$

is always a well defined characteristic number of X, and in every case (X being orientable or not) can be computed as the sum of indices of any tangent vector field s on X with isolated zeros.

Recall that we also have the nonoriented cobordism Euler class

$$w^n(X) \in \eta^n(X) = \mathcal{B}^n(X; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \ .$$

Clearly

$$w^n(X) = \chi(X) \ \mathrm{mod}(2) \ ,$$

and sometimes we write

$$\chi_{(2)}(X) := w^n(X) \ .$$

14.5. Examples and properties of χ

• The unit sphere S^n admits a Morse function with just one minimum and one maximum (for example, take the restriction of the projection to the x_{n+1}-axis), then $\chi(S^n) = 1 + (-1)^n$ and it is zero when n is odd (as it must be), while $\chi(S^n) = 2$ if n is even. This implies that an even-dimensional sphere does not admit any nowhere vanishing tangent vector field.

PROPOSITION 14.6. *S^n admits a nowhere vanishing tangent vector field if and only if n is odd.*

Proof : We have to exhibit such a tangent vector field on S^n when n is odd. For $n = 1$, let $S^1 \subset \mathbb{R}^2$ be the unit circle. For every $p = (x, y) \in S^1$, set $s(p) = (-y, x)$. In general, for every $p = (x_1, y_1, \dots, x_{n+1}, y_{n+1}) \in S^n \subset \mathbb{R}^{n+1}$, set $s(p) = (-y_1, x_1, \dots, -y_{n+1}, x_{n+1})$.

■

• If $\pi : \tilde{X} \to X$ is a degree d covering map, then

$$\chi(\tilde{X}) = d\chi(X) \ .$$

We can argue as above for the degree 2 covering maps, by lifting to $\tilde{X}$ any tangent vector field s with isolated zeros on X; every zero p of s lifts to d isolated zeros of $\tilde{s}$ sharing the same index $i_p(s)$. In particular, by considering the natural degree 2 covering map $\pi : S^n \to \mathbf{P}^n(\mathbb{R})$, we have $\chi(\mathbf{P}^n(\mathbb{R})) = 0$ if n is odd, while $\chi(\mathbf{P}^n(\mathbb{R})) = 1$ if n is even.

• Consider the complex projective space $\mathbf{P}^n(\mathbb{C})$ as the quotient space of the unitary sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. We can verify (the reader can do it as an exercise by using the standard atlas of $\mathbf{P}^n(\mathbb{C})$ with $n+1$ complex affine charts) that

$$f([z_0, z_1, \dots, z_n]) = \sum_{j=0}^{n} (j+1)|z_j|^2$$

defines a Morse function on $\mathbf{P}^n(\mathbb{C})$ with exactly $n+1$ critical points

$$p_0 = [1, 0, \dots, 0], \dots, p_n = [0, \dots, 0, 1]$$

and every even index between 0 and $2n$ occurs exactly once. Hence

$$\chi(\mathbf{P}^n(\mathbb{C})) = n + 1 \ .$$

See Remark 14.12 for another way to get this result.

• The characteristic χ is *multiplicative for the product of manifolds.* That is, if X and X' are compact boundaryless manifolds as above, then

$$\chi(X \times X') = \chi(X)\chi(X') \ .$$

In fact if s (s') is a tangent field on X (on X') with non-degenerate zeros $p_1, \dots, p_r$ ($p'_1, \dots, p'_{r'}$), then $s \times s'$ defines a field on $X \times X'$ with rr' non-degenerate zeros (p_j, p'_i), $j = 1, \dots, r$, $i = 1, \dots, r'$, each one having index

$$i_{(p_j, p'_i)}(s \times s') = i_{p_j}(s) i_{p'_i}(s') \ .$$

For example,

$$\chi(X \times S^1) = 0$$

for every X (in fact, in this case, we can explicitly define a nowhere vanishing tangent vector field on $X \times S^1$ which restricts to the standard field considered above on every S^1 fibre). Whenever both n and m are even, then

$$\chi(\mathbf{P}^n(\mathbb{R}) \times \mathbf{P}^m(\mathbb{R})) = 1 \ .$$

14.6. The relative E-P characteristic of a triad, χ-additivity

Here we adopt the setting of Chapter 9. A *relative tangent vector field on a triad* (W, V_0, V_1) is defined by the property that it looks like the adapted gradient of a Morse function $f : W \to [0, 1]$ at the boundary $\partial W = V_0 \amalg V_1$ of W. Hence it is ingoing W along V_0 and outgoing along V_1. We can develop, with minor changes, a notion of *relative E-P characteristic for triads.* If s is such a relative field with isolated zeros, we define as usual $\chi(s)$ as the sum of the indices of its zeros. Using the double of W, we can reduce to the boundaryless case the verification that

$$\chi(W, V_0) := \chi(s)$$

is well defined. Let s' be another relative field. By gluing each s and s' with $-s$ along the boundary, we get two fields z and z', respectively, with isolated zeros on the double $D(W)$. Use them to compute $\chi(D(W))$. We have $\chi(D(W)) = \chi(s) + \chi(-s) = \chi(s') + \chi(-s)$ hence $\chi(s) = \chi(s')$ as desired.

Every W with nonempty boundary gives rise to several triads (W, V_0, V_1); among these are $(W, \emptyset, \partial W)$ and $(W, \partial W, \emptyset)$. The notation

$$\chi(W) := \chi(W, \emptyset, \partial W)$$

is compatible with

$$\chi(W) = \chi(W, \emptyset, \emptyset)$$

when W is boundaryless.

If $f : W \to [0,1]$ is a Morse function on the triad (W, V_0, V_1), then $\hat{f} = 1 - f$ is a Morse function on (W, V_1, V_0).

Lemma 14.7.

$$\chi(W, V_0) = (-1)^{\dim(W)} \chi(W, V_1) \ .$$

The lemma follows by using two adapted gradient fields to compute the relative characteristics.

Note that $\chi(D^n) = 1$: use a Morse function on $(D^n, \emptyset, S^{n-1})$ with just one minimum.

If X is boundaryless and Y is with boundary, then the very same argument used when Y is boundaryless allows to extend the multiplicative property.

Lemma 14.8.

$$\chi(X \times Y) = \chi(X)\chi(Y) \ .$$

In particular,

$$\chi(X \times D^n) = \chi(X) \ .$$

14.6.1. Additive property of χ. If (W, V_0, V_1), (W', V'_0, V'_1) are triads and $\phi : V_1 \to V'_0$ is a diffeomorphism, we get a new composite triad $(W", V_0, V'_1)$, where $W" = W \amalg_\phi W'$. Any couple of relative fields v and v' with isolated zeros on the given two triads, respectively, can be glued together to produce a relative field $v"$ having as zeros the union of the zeros of v and v', each one keeping its index. Then we have

$$\chi(W", V_0, V'_1) = \chi(W, V_0, V_1) + \chi(W', V'_0, V'_1) \ .$$

This additive property of χ has remarkable consequences.

14.6.2. A baby TQFT. In Section 10.8, we have roughly outlined the axioms of a so-called TQFT and posed the question about the existence of any "nontrivial" one. We use χ to provide a first nontrivial example. Consider $\mathbf{CAT}_\eta(n+1)$. Associate to every object M the vector space $Z(M) = \mathbb{C}$. To every arrow f, carried by a triad (W, M_0, M_1), associate the unitary $\mathbb{C}$-linear map

$$Z(f) : Z(M_0) \to Z(M_1), \ z \to e^{i\chi(W, M_0)} z \ .$$

By the additive property of χ, it is easy to check that all axioms are satisfied. This shows, at least, that there are not logical contradictions within the given pattern of axioms.

14.7. E-P characteristic of tubular neighbourhoods and the Gauss map

The identity $\chi(X \times D^n) = \chi(X)$ is a special case of the following.

PROPOSITION 14.9. *Let $p : U \to X$ be a closed tubular neighbourhood of a submanifold X of some Y. Then $\chi(U) = \chi(X)$.*

Proof : It is enough to show the equality for an ϵ-neighbourhood $N_\epsilon(X)$ of the zero section X of a vector bundle $\pi : E \to X$ endowed with a field of positive definite scalar products on every fibre. Let v be a tangent field on X with non-degenerate zeros. Define the field on $N_\epsilon(X)$,

$$w(z) = (z - p(z)) + v(p(z)) \ .$$

We check that w is a field on the triad $(N_\epsilon(X), \emptyset, \partial N_\epsilon(X))$ and that the zeros of w coincide with the zeros of v, are non-degenerate and keep the sign. The proposition follows.

■

In the special case $X \subset \mathbb{R}^k$, assume that U has been constructed using the standard metric on $\mathbb{R}^k$. By removing from the interior of U a system of pairwise disjoint small open disks $\mathcal{D}_p$ around each zero of w, we get a manifold W with boundary $(\amalg_p S^{k-1}_p) \amalg \partial U$ on which the normalized field $\mathfrak{w} := w/||w||$ is well defined, as well as the map $\mathfrak{w} : W \to S^{k-1}$. The restriction of $\mathfrak{w}$ to ∂U is a field of unitary vectors pointing out from U (in fact normal to the boundary). This restriction, say $g_{\partial U}$ is called the *Gauss map of the hypersurface ∂U.*

COROLLARY 14.10. *Let $p : U \to X$ be a tubular neighbourhood of X in $\mathbb{R}^k$. Then*

$$\chi(X) = \deg(g_{\partial U})$$

where $g_{\partial U}$ is the Gauss map of the hypersurface ∂U.

The corollary follows by computing the characteristic as the sum of zero indices and using the bordism invariance of the degree.

If X itself is an oriented hypersurface in $\mathbb{R}^k$, $k \geq 2$, we can define its Gauss map $g_X : X \to S^{k-1}$, where this field of oriented unitary normal vectors along X is determined by the fact that, for every $x \in X$, $g_X(x) \oplus T_x X = T_x \mathbb{R}^k$ as oriented vector spaces, where $\mathbb{R}^k$ is endowed with the standard orientation. In this case, ∂U consists of two parallel copies of X with opposite orientations. the last corollary specializes to

COROLLARY 14.11. *Let X be an oriented hypersurface of $\mathbb{R}^k$.*
(1) If k is even, then $\chi(X) = 0$;
(2) If k is odd, then

$$\chi(X) = 2 \deg(g_X)$$

where g_X is the Gauss map of X.

The corollary follows by using that the degree of the antipodal map $a_{k-1} : S^{k-1} \to S^{k-1}$ is equal to $(-1)^k$.

REMARK 14.12. We can compute inductively the characteristic of real and complex projective spaces by decomposing $\mathbf{P}^n(K)$ as the union of a tubular neighbourhood of $\mathbf{P}^{n-1}(K) \subset \mathbf{P}^n(K)$ and its complement, and applying Proposition 14.9 together with the additivity of χ.

14.8. Nontriviality of $\eta_\bullet$ and $\Omega_\bullet$

The integer E-P characteristic is *not* invariant up to bordism. For example, $[S^2] = [S^1 \times S^1] = 0 \in \eta_2$, but $\chi(S^2) = 2 \neq 0 = \chi(S^1 \times S^1)$. On the other hand, the E-P characteristic mod(2) is bordism invariant.

PROPOSITION 14.13. *Let* $[X] = 0 \in \eta_n$. *Then* $\chi_{(2)}(X) = 0 \in \mathbb{Z}/2\mathbb{Z}$.

Proof : If n is odd, we know in general that $\chi(X) = 0$. Assume that n is even. Let $X = \partial W$, W being a compact manifold with boundary. Take the double $D(W)$. The double $D(W)$ can be presented as the composition of the triad

$$(X \times D^1, \emptyset, (X \times \{-1\} \amalg (X \times \{1\})))$$

followed by two copies of the triad

$$(W, X, \emptyset)$$

glued to $X \times D^1$ along $X \times \{\pm 1\}$, respectively. By the additive property

$$\chi(D(W)) = \chi(X \times D^1) + 2\chi(W, X, \emptyset) \ .$$

By Lemmas 14.8 and 14.7, and the fact that $n+1$ is odd and the double is boundaryless, we have

$$\chi(X) = \chi(D(W)) - 2\chi(W, X, \emptyset) = 2\chi(W) \in \mathbb{Z}$$

so that

$$\chi_{(2)}(X) = 0 \in \mathbb{Z}/2\mathbb{Z} \ .$$

∎

As an immediate corollary, we have the *nontriviality of* η_{2n} *and* Ω_{4n}.

COROLLARY 14.14. *For every even* $n \geq 1$, $\eta_{2n} \neq 0$ *and* $\Omega_{4n} \neq 0$.

Proof : We know that $\chi(\mathbf{P}^{2n}(\mathbb{R})) = 1$, hence $[\mathbf{P}^{2n}(\mathbb{R})] \neq 0 \in \eta_{2n}$. Similarly $\chi_{(2)}(\mathbf{P}^{2n}(\mathbb{C})) = 1$, hence $[\mathbf{P}^{2n}(\mathbb{C})] \neq 0 \in \Omega_{4n}$.

∎

By using the multiplicative property of χ and the obvious fact that it is additive under disjoint union, we also have the following.

COROLLARY 14.15. $\chi_{(2)} : \eta^\bullet \to \mathbb{Z}/2\mathbb{Z}$ *is a well defined nontrivial* ring *homomorphism.*

Every $\mathbf{P}^a(\mathbb{R}) \times \mathbf{P}^b(\mathbb{R})$, where a and b are even, is nontrivial in η_{a+b}. For example, in η_4 we have the nontrivial elements $[\mathbf{P}^4(\mathbb{R})]$, $[\mathbf{P}^2(\mathbb{R}) \times \mathbf{P}^2(\mathbb{R})]$, $[\mathbf{P}^2(\mathbb{C})]$. At present, we are not able to decide whether they are independent or not. Similarly, we have that $\mathbf{P}^a(\mathbb{C}) \times \mathbf{P}^b(\mathbb{C})$, where a and b are even, is nontrivial in $\Omega_{2(a+b)}$.

14.9. Combinatorial E-P characteristic

We have treated the E-P characteristic of smooth manifolds in purely differential topological terms. However, the reader is probably aware that the name E-P characteristic is used in other different contexts. Probably, she/he has at least encountered a combinatorial formula producing the value $2 = \chi(S^2)$ for every polyhedral realization of the sphere as the boundary of a convex polytope in $\mathbb{R}^3$. In this very sketchy section, we would outline a few bridges between such different ways to recover the E-P characteristic.

14.9.1. Piece-wise smooth triangulations and the combinatorial characteristic. Recall that an m-simplex σ in some Euclidean space $\mathbb{R}^h$, $h \geq m$, is the convex hull of $m+1$ affinely independent points (that is, they span an m-dimensional affine subspace of $\mathbb{R}^h$). These are called the *vertices* of σ. By removing one vertex, say p, we determine an $(m-1)$ simplex σ_p which is the $(m-1)$ face of σ opposite to the vertex p. By iterating the face operation we get the iterated k-faces of σ, $0 \leq k \leq m$, where the vertices are the 0-faces and σ itself is the unique m-face. By definition, a *finite simplicial complex* is a finite family $\mathcal{K}$ of simplexes in some $\mathbb{R}^h$ such that

- $\mathcal{K}$ is closed for the iterated faces.
- Two simplexes of $\mathcal{K}$ may intersect each other only at a common iterated face.

The union $|\mathcal{K}|$ of the simplexes of $\mathcal{K}$ is a topological subspace of $\mathbb{R}^h$ called the *geometric support* of the complex $\mathcal{K}$.

Let X be a compact boundaryless smooth manifold. A *piece-wise smooth triangulation of* X is given by a homeomorphism

$$\tau : |\mathcal{K}| \to X$$

where $\mathcal{K}$ is a finite simplicial complex in some $\mathbb{R}^h$ and the restriction of τ to every n-symplex of $\mathcal{K}$ is a smooth embedding in X. If $\partial X \neq \emptyset$, we require furthermore that $\tau_{|\tau^{-1}(\partial X)}$ is a triangulation of ∂X.

PROPOSITION 14.16. *Let $\tau : |\mathcal{K}| \to X$ be a piece-wise smooth triangulation of the compact boundaryless smooth n-manifold X. Then there is a tangent, the so-called* Whitney *vector field v_τ on X, whose zero set coincides with the set of images of the barycenters $\hat{\sigma}$ of the simplexes σ of $\mathcal{K}$ and every zero has index equal to* $(-1)^{\dim(\sigma)}$.

COROLLARY 14.17. *For every piece-wise triangulation $\tau : |\mathcal{K}| \to X$ as above,*

$$\chi(X) = \sum_{j=0}^{n} (-1)^j c_j := \chi(\mathcal{K})$$

where c_j is the number of j-dimensional simplexes of $\mathcal{K}$. In particular, the combinatorial characteristic $\chi(\mathcal{K})$ *does not depend on the choice of the triangulation of X.*

A few comments about the proof. The Whitney field v_τ can be explicitly defined using the barycentric coordinates on the simplexes of $\mathcal{K}$, see for instance [**HT**]. Every barycenter of an n-simplex of $\mathcal{K}$ corresponds to a source of v_τ and every vertex of $\mathcal{K}$ corresponds to a pit; in general, the barycenter of a j-simplex corresponds to a saddle point with a j-dimensional space of ingoing directions tangent to the simplex and an $(n-j)$-dimensional space of outgoing directions transverse to the simplex.

For the *existence* and a suitable form of *uniqueness* up to subdivision of piece-wise smooth triangulations, see [**Mu**].

14.9.2. Homological characteristic. Here we want to recover the combinatorial characteristic in an algebraic/topological setting.

Fix any field F (for example $F = \mathbb{Z}/2\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$).

Given a triangulation of X as above, enhance $\mathcal{K}$ by fixing an orientation on each simplex σ of $\mathcal{K}$, induced by the choice of orientation on the affine subspace of $\mathbb{R}^h$ spanned by the vertices of σ. We can define the *simplicial homology of the oriented complex $\mathcal{K}$ with coefficients in F* as follows.

• For every $0 \leq j \leq n$, set $C_j(\mathcal{K}; F)$ as the finite-dimensional F-vector space having as a basis the oriented j-simplexes of $\mathcal{K}$ (note that $-\sigma$, considered as the simplex endowed with the opposite orientation, is confused with -1σ, i.e. the product of σ with the scalar $-1 \in F$). Hence , $\dim C_j(\mathcal{K}; F) = c_j$.

• Every $(j-1)$-face σ' of the oriented j-simplex σ of $\mathcal{K}$ inherits a boundary orientation accordingly with our usual convention. Hence σ' has two orientations, the one fixed above and the boundary orientation. Give it the sign $\epsilon(\sigma', \sigma) = 1$ if these orientations agree and the sign -1 otherwise. Then define the unique F-linear map

$$\partial_j : C_j(\mathcal{K}; F) \to C_{j-1}(\mathcal{K}, F)$$

which on every oriented j-simplex σ holds

$$\partial_j(\sigma) = \sum_{\sigma'} \epsilon(\sigma', \sigma)\sigma' \ ,$$

where σ' varies among the $(j-1)$ faces of σ. It is not hard to verify that

$$\partial_{j-1} \circ \partial_j = 0 \ ,$$

basically because two $(j-1)$ faces of the oriented j-simplex σ both endowed with the boundary orientation induce opposite boundary orientations on their common $(j-2)$ face of σ. Hence we can define the quotient F-vector spaces

$$H_j(\mathcal{K};F) = \ker(\partial_j)/\mathrm{Im}(\partial_{j+1}) \ ,$$

and these are the desired simplicial F-homology spaces of the oriented complex $\mathcal{K}$. By using the elementary dimension formula for any finite-dimensional linear map $f : V \to W$:

$$\dim(V) = \dim(\ker(f)) + \dim(\mathrm{Im}(f)) \ ,$$

it is not hard to check that the F-*homological characteristic*

$$\chi(H_\bullet(\mathcal{K};F)) := \sum_{j=0}^{n} (-1)^j \dim H_j(\mathcal{K};F) = \sum_{j=0}^{n} (-1)^j \dim C_j(\mathcal{K};F)$$

hence it equals the combinatorial characteristic, so that

$$\chi(H_\bullet(\mathcal{K};F)) = \chi(X) \ .$$

Remarkably, it does not depend on the choice of the oriented triangulation of X and not even of the field F. It is a fundamental result of algebraic topology (see [**Hatch**], [**Mu2**]) that even the single dimensions (also-called the F-*Betti numbers* of X)

$$\dim H_j(X;F) := \dim H_j(\mathcal{K};F)$$

do not depend on the choice of the triangulation with oriented simplexes, although they depend on the field F.

CHAPTER 15

Surfaces

We apply several tools developed in the previous chapters to classify the compact surfaces (i.e. smooth 2-manifolds) and also to determine both bordisms η_2 and Ω_2.

Let M be a compact connected boundaryless surface. We know from Chapter 9 that M *admits a 'reduced' ordered handle decomposition with one* 0*-handle, followed by* κ *disjoint* 1*-handles and one final* 2*-handle, where* $\kappa := \kappa(M)$ *is intrinsically determined by* $\kappa(M) = 2 - \chi(M)$.

Recall that for any handle decomposition $\mathcal{H}$ of M, its characteristic

$$\chi(\mathcal{H}) := \sum_{j=0}^{2} (-1)^j b(j) \ ,$$

$b(j)$ being the number of index-j handles, is preserved by the basic moves on handle decompositions; if $\mathcal{H}$ is associated to a Morse function on M, then $\chi(\mathcal{H}) = \chi(M)$. Finally, we can get a reduced ordered decomposition of M by performing some basic moves on any given decomposition.

Remark 15.1. We want to argue that for *any* ordered handle decomposition $\mathcal{H}$ of M with one 0-handle, one 2-handle, and a few disjoint 1-handles, the number of 1-handles is always equal to $\kappa(M)$. It is not hard to triangulate M in the following way: take a vertex internal to every handles; take as further vertices on the boundary of the 0-handle the union of the boundaries of the attaching 1-disks of the 1-handles. They also provide a triangulation of the boundary of every 1-handle. Triangulate both the one 0-handle and every 1-handle by the cones on the boundary with centre at the respective internal vertex. These triangulations match and give a triangulation of the union of the 0-handle with the 1-handles. The resulting surface has as boundary a triangulated circle. Finally, complete it to a triangulation of the whole of M by means again of the cones with centre at the internal vertex of the 2-handle. By using the combinatorial computation of $\chi(M)$ applied to such a triangulation, we can easily check that the number of 1-handles is always equal to $\kappa(M)$, as desired.

• Recall from Section 7.5.2 that, in dimension 2, connected sum and weak connected sum are equivalent to each other; moreover, every twisted 2-sphere is diffeomorphic to the standard sphere S^2.

So let $\gamma \subset M$ be any dividing connected simple curve γ; that is,

$$M \setminus \gamma = N_1 \amalg N_2$$

where N_j is a nonempty connected open set of M and the closure $\bar{N}_j$ is a compact smooth manifold with boundary $\partial \bar{N}_j = \gamma$. Let M_j be the boundaryless surface obtained from $\bar{N}_j$ by filling $\partial \bar{N}_j$ with a 2-disk glued along the boundary; then (up to diffeomorphism)

$$M \sim M_1 \# M_2 \ .$$

To be more precise, the result is uniquely determined if at least one among M_1 and M_2 is nonorientable, or both are orientable and admit orientation reversing diffeomorphisms. For the moment, let us say that M is "a" connected sum of M_1 and M_2; however, this precision will eventually be immaterial. By the additive property of χ, we have

$$\kappa(M) = \kappa(M_1) + \kappa(M_2) \ .$$

• Let us consider $\eta_1(M) = \mathcal{B}_1(M; \mathbb{Z}/2\mathbb{Z})$.

LEMMA 15.2. *$\eta_1(M)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space of finite dimension $\leq \kappa(M)$.*

Proof : By Section 10.6 there is a *surjective* homomorphism (a base point being understood)

$$\pi_1(M) \to \eta_1(M) \ .$$

By using a reduced ordered handle decomposition of M as above and applying (an elementary version of) the Van Kampen theorem, we see that $\pi_1(M)$ has a presentation with κ generators and one relation; for the union of the 0-handle with the κ 1-handles has the homotopy type of a wedge of κ copies of S^1 whose fundamental group is a free group with κ generators; the defining relation between them is given by the attaching map of the 2-handle. The lemma follows.

■

LEMMA 15.3. *Every $\alpha \in \eta_1(M)$ can be represented by a* connected *simple smooth curve C traced on M.*

Proof : We already know from general results in Chapter 13 that a codimension-1 class can be represented by hypersurfaces. In the present 2-dimensional situation, we can get an elementary direct proof of this fact as follows. Certainly, $\alpha = [f : \tilde{C} \to M]$ where $\tilde{C}$ is a finite union of copies of S^1. By a standard 'general position' argument (see Section 8.2) we can assume that up to homotopy, hence up to bordism, $f : \tilde{C} \to M$ is a generic immersion possibly having simple double points in its image $f(\tilde{C}) \subset M$. In local coordinates, every crossing of $f(\tilde{C})$ is of the form $\{xy = 0\}$ and has two local 'simplifications' of the form $\{xy \pm \phi(x,y)\epsilon = 0\}$, where $\epsilon > 0$ is small enough and ϕ is a suitable bump function with support in a small disk centred at 0. By locally simplifying each crossing of $f(\tilde{C})$ (choose arbitrarily one way), we get a 1-submanifold C' of M. It is not hard to verify that

$\alpha = [f : \tilde{C} \to M] = [C'] \in \eta_1(M)$. The manifold C' is not necessarily connected. In order to modify C' to get a connected representative C of α, first we can remove all dividing components of C' (keeping the name); if C' is not connected then apply the following argument that decreases the number of components by 1. We can find two components C_1 and C_2 of C' which can be connected by a smooth arc I whose internal part is embedded in $M \setminus C'$, one endpoint x_j is on C_j, $j = 1, 2$, and is transverse to $C_1 \cup C_2$. I can be thickened to an embedded 1-handle $H \sim I \times [-1, 1]$ which intersects C_j at $\{x_j\} \times [-1, 1]$ and is contained in $M \setminus C'$ elsewhere. Then consider

$$C" := (C' \setminus (C_1 \cup C_2)) \cup C^* \ ,$$

where

$$C^* = ((C_1 \cup C_2) \setminus H) \cup (I \times \{\pm 1\})$$

up to corner smoothing. Hence $C_1 \cup C_2$ has been replaced with the connected curve C^*. Again, it is not hard to show that $[C'] = [C"] \in \eta_1(M)$. By iterating the procedure, we eventually get a required connected representative C of α.

■

• Consider now the symmetric intersection form (Section 11.4)

$$\bullet : \eta_1(M) \times \eta_1(M) \to \mathbb{Z}/2\mathbb{Z} \ .$$

LEMMA 15.4. *The intersection form on $\eta_1(M)$ is non-degenerate.*

Proof : We have to show that if $\alpha \neq 0$ in $\eta_1(M)$, then there is $\beta \in \eta_1(M)$ such that $\alpha \bullet \beta = 1 \in \mathbb{Z}/2\mathbb{Z}$. Let $C \subset M$ be a connected smooth representative of α as in Lemma 15.3. As $\alpha \neq 0$, then $M \setminus C$ is connected (otherwise C would be the boundary of the closure of a component of $M \setminus C$, so that $[C] = 0$). Take a fibre I, necessarily transverse to C at one point, of a tubular neighbourhood of C in M. Also $M \setminus (C \cup I)$ is connected, so that the endpoints of the interval I can be connected by a smooth simple arc γ whose internal part is contained in $M \setminus (C \cup I)$. Then (possibly by corner smoothing) $C' := I \cup \gamma$ is a smooth boundaryless curve in M which intersects C transversely at one point, hence $[C] \bullet [C'] = 1$.

■

The next lemma follows from Chapter 13 .

LEMMA 15.5. *Let $C \subset M$ be a connected smooth boundaryless curve. Then there are two possibilities: either $[C] \bullet [C] = 1$, and this happens if and only if C has tubular neighbourhood in M diffeomorphic to a Möbius band, or $[C] \bullet [C] = 0$, and this happens if and only if C has a product tubular neighbourhood in M.*

The following lemma is obvious.

LEMMA 15.6. *If $f : M \to M'$ is a surface diffeomorphism, then*

$$f_* : (\eta_1(M), \bullet_M) \to (\eta_1(M'), \bullet_{M'})$$

is an isometry; that is, f_* is a $\mathbb{Z}/2\mathbb{Z}$-linear isomorphism and for every $\alpha, \beta \in \eta_1(M)$,

$$\alpha \bullet_M \beta = f_*(\alpha) \bullet_{M'} f_*(\beta) \ .$$

Hence *the isometry class of the non-degenerate symmetric intersection form on $\eta_1(*)$ is invariant up to diffeomorphism.*

In what follows, we will abuse the notation by confusing a form with its isometry class. If (V, ρ) and (V', ρ') are finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector spaces endowed with non-degenerate symmetric forms, we can define the *orthogonal direct sum* $(V, \rho) \perp (V', \rho')$ which denotes the non-degenerate symmetric form $\rho \perp \rho'$ on $V \oplus V'$ that restricts to ρ (resp. ρ') on V (V') and such that V and V' are orthogonal to each other. We have

LEMMA 15.7. *If the surface M is a connected sum*

$$M \sim M_1 \# M_2 \ ,$$

then (up to isometry)

$$(\eta_1(M), \bullet_M) = (\eta_1(M_1), \bullet_{M_1}) \perp (\eta_1(M_2), \bullet_{M_2}) \ .$$

Proof : We adopt the notations fixed at the beginning of the section. We can assume that the connected sum has been realized by a connected dividing curve γ in M. It is easy to see that the linear map $i_* : \eta_1(N_j) \to \eta_1(M_j)$ induced by the inclusion is an isomorphism, $j = 1, 2$. Denote by V_j the image of $\eta_1(N_j)$ in $\eta_1(M)$ by the inclusion. It is evident that V_1 and V_2 are orthogonal to each other with respect to $\bullet_M$. It is enough to show that $\eta_1(M) = V_1 + V_2$, whence $\eta_1(M) = V_1 \perp V_2$ because $\bullet_M$ is non-degenerate, and that V_j is actually isomorphic to $\eta_1(N_j)$, $j = 1, 2$. Let $\alpha \in \eta_1(M)$ and $C \subset M$ be a smooth representative as above. By transversality, we can assume that $C \pitchfork \gamma$. As $[\gamma] = 0$ in $\eta_1(M)$, then $C \cap \gamma$ consists of an even number of points $\{p_1, \dots, p_{2d}\}$. We can assume that they are the endpoints of a family $\{I_1, \dots, I_d\}$ of pairwise disjoint intervals embedded into γ. Take a 'small' tubular neighbourhood $U \sim \gamma \times [-1, 1]$ of γ in M. Then $M \setminus U$ consists of two connected components W_1 and W_2 such that $W_j \subset N_j$. The boundary of W_j is a parallel copy γ_j of γ. Denote by $I_{i,j}$, $j = 1, 2$, $i = 1, \dots d$, the parallel copy in γ_j of the interval I_i. Finally, for $j = 1, 2$, set

$$C_j = (C \cap W_j) \cup (\bigcup_{i=1}^{d} I_{i,j}) \ .$$

Up to corner smoothing, C_j is a smooth curve (not necessarily connected) in N_j and it is easy to see that

$$[C_1 \amalg C_2] = [C] \in \eta_1(M) \ ;$$

this shows that $\eta_1(M) = V_1 + V_2$. Finally, let $\alpha \in \eta_1(N_1) \sim \eta_1(M_1)$ and denote by α' its image in $\eta_1(M)$ by the inclusion. If α is not zero, as $\bullet_{M_1}$ is non-degenerate, then there is $\beta \in \eta_1(N_1)$ such that $\alpha \bullet_{M_1} \beta = 1$; due to the geometric way one computes the intersection forms, it also follows that $\alpha' \bullet_M \beta' = 1$, whence α' is non-zero.

■

We are going to see that the isometry class of the intersection form contains all relevant information about the diffeomorphism type.

15.1. Classification of symmetric bilinear forms on $\mathbb{Z}/2\mathbb{Z}$

Here we classify non-degenerate symmetric bilinear forms on $\mathbb{Z}/2\mathbb{Z}$-vector spaces of finite dimension, up to isometry. We denote by $\mathbf{U}$ the unique 1-dimensional isometry class; by $\mathbf{H}$ the isometry class of *hyperbolic planes*, i.e. 2-dimensional spaces endowed with a non-degenerate symmetric form admitting a basis made by isotropic vectors (recall that a vector v is *isotropic* for a form β if $\beta(v,v) = 0$). Although $\mathbf{H}$ is non-degenerate, it is *totally isotropic* (every vector is so). This depends on the fact that the characteristic of the field $\mathbb{Z}/2\mathbb{Z}$ is equal to 2; in characteristic $\neq 2$ the zero-form is the only totally isotropic by the so-called 'polarization formula'. For every $n \geq 1$, denote by $n\mathbf{U}$ (resp. $n\mathbf{H}$) the orthogonal direct sum of n copies of $\mathbf{U}$ (resp. of $\mathbf{H}$).

Proposition 15.8. *Let (V,β) be a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space endowed with a non-degenerate symmetric bilinear form, $\dim V > 0$. Then we have one of the following exclusive occurrences:*

(1) (V,β) admits an orthogonal basis so that it is isometric to $n\mathbf{U}$, $n = \dim V$, and this happens if and only if it is not totally isotropic.

(2) $\dim V = 2n$, (V,β) is isometric to $n\mathbf{H}$, and this happens if and only if it is totally isotropic.

Proof : Assume first that (V,β) is totally isotropic. Let $\mathcal{B} = \{v_1, \dots v_k\}$ be a basis of V, $\mathcal{B}^* = \{v_1^*, \dots, v_k^*\}$ its dual basis, w_1 the vector which represents the functional v_1^* by means of the non-degenerate form β. Then the subspaces spanned by $\{v_1, w_1\}$ endowed with the restriction of β is a hyperbolic plane $\mathbf{H}$. As this last is non-degenerate, then (up to isometry)

$$(V,\beta) = \mathbf{H} \perp \mathbf{H}^\perp \ ,$$

all spaces being endowed with the restriction of β. The restriction to $\mathbf{H}^\perp$ is non-degenerate and totally isotropic, and $\dim \mathbf{H}^\perp = \dim V - 2$. So we can achieve the item (2) by induction on the dimension. Assume now that $v \in (V,\beta)$ is not isotropic. Then the subspace spanned by v endowed with the restriction of β represents $\mathbf{U}$ and (up to isometry)

$$(V,\beta) = \mathbf{U} \perp \mathbf{U}^\perp \ .$$

By iterating the argument, we get that either

$$(V,\beta) = n\mathbf{U}, \ n = \dim V \ ,$$

or

$$(V,\beta) = k\mathbf{U} \perp T$$

for some $k \geq 1$, where T is totally isotropic, $\dim T > 0$. If we are not in the first case, we apply (2) to T, and get

$$(V, \beta) = k\mathbf{U} \perp h\mathbf{H}$$

for some $k, h \geq 1$. Finally, item (1) is achieved by means of the following lemma. By the way, it also shows that $\perp$ does not satisfy the 'cancellation property'.

LEMMA 15.9. *Up to isometry,* $\mathbf{U} \perp \mathbf{H} = 3\mathbf{U}$.

Proof : Let $\mathcal{D} = \{u, w, t\}$ be a basis for $\mathbf{U} \perp \mathbf{H}$ adapted to the decomposition so that $\{w, t\}$ is a basis of the hyperbolic plane. Let N be the subspace spanned by $\{u + w, u + t\}$. We readily verify that this last is an orthogonal basis of N so that $N = 2\mathbf{U}$. Then $\mathbf{U} \perp \mathbf{H} = N \perp N^{\perp}$ and the last space is 1-dimensional and non-degenerate, so eventually $\mathbf{U} \perp \mathbf{H} = 3\mathbf{U}$.

■

The proof of Proposition 15.8 is now complete.

■

15.2. Classification of compact surfaces

We are going to prove the following topological classification theorem.

THEOREM 15.10. *(0) Let M be a compact connected boundaryless surface. Then the following facts are equivalent to each other:*

- *M is diffeomorphic to S^2;*
- *$\kappa(M) = 0$;*
- $\dim \eta_1(M) = 0$.

(1) For every $n \geq 1$, the isometry class $n\mathbf{U}$ is realized by the intersection form of $\eta_1(n\mathbf{P}^2(\mathbb{R}))$, where $n\mathbf{P}^2(\mathbb{R})$ denotes the connected sum of n copies of the real projective plane.

(2) For every $n \geq 1$, the isometry class $n\mathbf{H}$ is realized by the intersection form of $\eta_1(n(S^1 \times S^1))$, where $n(S^1 \times S^1)$ denotes the connected sum of n copies of the torus.

(3) Two compact connected boundaryless surfaces M and M' are diffeomorphic if and only if the intersection forms on $\eta_1(M)$ and $\eta_1(M')$, respectively, are isometric to each other.

This theorem has several interesting corollaries.

COROLLARY 15.11. *In the hypotheses of Theorem 15.2:*

(1) $\dim \eta_1(M) = \kappa(M) = 2 - \chi(M)$. *If M is orientable then $\kappa(M) = 2g(M)$ is even ($g(M)$ is called the* genus *of M).*

(2) Two surfaces M and M' are diffeomorphic if and only if $\chi(M) = \chi(M')$, and either they are both orientable or both nonorientable.

(3) Every orientable surface M admits orientation reversing diffeomorphisms. Hence the connected sum of two surfaces $M_1 \# M_2$ is always uniquely defined up to diffeomorphism.

(4) Every M can be embedded in $\mathbb{R}^4$. If M is orientable, then it can be embedded in $\mathbb{R}^3$.

Proofs. First, item (0) of Theorem 15.10; that is, the characterization of the 2-sphere up to diffeomorphism. If $\kappa(M) = 0$, then M has a handle decomposition with only one 0-handle and one 2-handle. So it is a twisted 2-sphere, whence it is diffeomorphic to S^2. Then M is simply connected, hence $\dim \eta_1(M) = 0$. Let us show now that if $\kappa(M) > 0$ then $\dim \eta_1(M) > 0$. Take a reduced ordered handle decomposition with $\kappa(M)$ 1-handles. The core of every 1-handle can be completed with a simple arc embedded in the 0-handle to get a connected simple smooth curve C in M. There are two possibilities: either $[C] \bullet_M [C] = 1$ or there are two such curves C and C' such that $[C] \bullet [C'] = 1$ (here we use that the boundary of the union of the 0-handle with the disjoint 1-handles must be connected). In any case, $\dim \eta_1(M) > 0$. The other implications of item (0) are evident.

Let us show now that $\mathbf{U}$ and $\mathbf{H}$ can be realized. $\mathbf{P}^2(\mathbb{R})$ can be obtained by gluing a 2-disk along the boundary of a Möbius band. By the Van Kampen theorem, we realize that $\pi_1(\mathbf{P}^2(\mathbb{R})) \sim \mathbb{Z}/2\mathbb{Z}$ and it is generated by the core C of the Möbius band. Another way to check this fact is by using the orientation covering $S^2 \to \mathbf{P}^2(\mathbb{R})$. Then also $\eta_1(\mathbf{P}^2(\mathbb{R})) \sim \mathbb{Z}/2\mathbb{Z}$, generated by $[C]$ and $[C] \bullet [C] = 1$. The above Möbius band can be realized by attaching one 1-handle to an initial 0-handle, and we get $\mathbf{P}^2(\mathbb{R})$ by adding one final 2-handle; this provides a reduced ordered handle decomposition with $\kappa(\mathbf{P}^2(\mathbb{R})) = 1$ 1-handle. By the way, we realize also that if $\kappa(M) = 1$ then M is diffeomorphic to $\mathbf{P}^2(\mathbb{R})$.

The fundamental group $\pi_1(S^1 \times S^1) \sim \mathbb{Z} \oplus \mathbb{Z}$ and is generated by the simple loops $C_1 = S^1 \times \{b_0\}$, $C_2 = \{a_0\} \times S^1$ with base point (a_0, b_0). It is immediate that $[C_1] \bullet [C_2] = 1$ in $\eta_1(S^1 \times S^1)$, while $[C_j] \bullet [C_j] = 0$, $j = 1, 2$. Hence $[C_1]$ and $[C_2]$ are non-zero and linearly independent, $\dim \eta_1(S^1 \times S^1) = 2$ and the intersection form realizes $\mathbf{H}$. The union B of a tubular neighbourhood U_1 of C_1 with a tubular neighbourhood U_2 of C_2 can be realized by attaching two disjoint 1-handles to one initial 0-handle, and we get $S^1 \times S^1$ by adding one final 2-handle; this provides a reduced ordered handle decomposition with $\kappa(S^1 \times S^1) = 2$ 1-handles.

Now items (1) and (2) of Theorem 15.10 follow from Lemma 15.7. Note that every $n\mathbf{P}^2(\mathbb{R})$ is not orientable (because it contains a connected curve C such that $[C] \bullet [C] = 1$) while every $n(S^1 \times S^1)$ is orientable, and that all items of Corollary 15.11 hold at least if we limit to consider surfaces M, M' belonging to the families of $n\mathbf{P}^2(\mathbb{R})$'s or $n(S^1 \times S^1)$'s.

It remains to prove item (3) of Theorem 15.10. This is the main point. Thanks to the above characterization of the 2-sphere, we can assume that $\dim \eta_1(M) > 0$. We will follow the proof of the algebraic classification

Theorem 15.8, pointing out, step by step, a topological counterpart. We have already obtained the counterpart of $n\mathbf{U}$ and $n\mathbf{H}$. Assume first that $(\eta_1(M), \bullet_M)$ is totally isotropic. Then every connected smooth simple curve $C \subset M$ has a product tubular neighbourhood, as $[C] \bullet_M [C] = 0$. Take such a curve C such that $[C] \neq 0$. By the proof of Lemma 15.3, there is another connected curve $C' \subset M$ which transversely intersects C at one point (so that $[C] \bullet [C'] = 1$, $[C'] \neq 0$, and $[C'] \bullet_M [C'] = 0$). We check straightforwardly that the union $\tilde{B}$ of a tubular neighbourhood U of C with a tubular neighbourhood U' of C' is diffeomorphic to the union B of tubular neighbourhoods of the geometric generators of $\pi_1(S^1 \times S^1)$ considered above. Hence the boundary of $\tilde{B}$ is a connected dividing curve in M and this gives rise to a connected sum decomposition

$$M \sim (S^1 \times S^1)\# M' \ ,$$

and we know that

$$\kappa(M') = \kappa(M) - 2 \ .$$

Again by Lemma 15.7, $(\eta_1(M'), \bullet_{M'})$ is also totally isotropic. Then we can conclude by induction on the dimension that in the totally isotropic case

$$M \sim n(S^1 \times S^1),\ 2n = \kappa(M) = 2 - \chi(M) \ .$$

Assume now that there is $\alpha \in \eta_1(M)$ such that $\alpha \bullet_M \alpha = 1$. Let $C \subset M$ be a connected simple smooth representative of α. Then a tubular neighbourhood U of C is a Möbius band, its boundary is a dividing curve, we have a connected sum decomposition

$$M \sim \mathbf{P}^2(\mathbb{R})\# M'$$

and we know that

$$\kappa(M') = \kappa(M) - 1 \ .$$

By iterating the argument, either we get

$$M \sim \kappa(M)\mathbf{P}^2(\mathbb{R})$$

or

$$M \sim k\mathbf{P}^2(\mathbb{R})\# M'$$

for some $k \geq 1$, where $\dim \eta_1(M') > 0$ and $\bullet_{M'}$ is totally isotropic. By applying the above result in this case, we eventually get

$$M \sim k\mathbf{P}^2(\mathbb{R})\# h(S^1 \times S^1) \ ,$$

$$\kappa(M) = k + 2h \ ,$$

for some $k, h \geq 1$. We conclude by applying the following final lemma. By the way, it shows also that # does not satisfy the 'cancellation property'.

Lemma 15.12. $\mathbf{P}^2(\mathbb{R})\#(S^1 \times S^1) \sim 3\mathbf{P}^2(\mathbb{R})$.

Proof : First, we outline a bare hands proof. After, we will outline another (equivalent) proof based on a transparent geometric construction, using the blowing up of Section 7.10.1.

First proof. Consider $S^1 \times S^1$ with the geometric generators C_1 and C_2 of $\pi_1(S^1 \times S^1)$ transversely intersecting at the base point (a_0, b_0) as above. Remove an open 2-disk D centred at (a_0, b_0) and glue a Möbius band $\mathcal{M}$ along the boundary to get $(S^1 \times S^1)\#\mathbf{P}^2(\mathbb{R})$. Then $(C_1 \cup C_2) \setminus D$ can be completed by means of two fibres of the natural fibration of $\mathcal{M}$ over its core getting two disjoint simple curves $\tilde{C}_1$ and $\tilde{C}_2$ in $(S^1 \times S^1)\#\mathbf{P}^2(\mathbb{R})$, each transversely intersecting the core of $\mathcal{M}$ at one point. We check that these curves have disjoint Möbius band tubular neighbourhoods U_1 and U_2, respectively, which can be filled to give two copies of $\mathbf{P}^2(\mathbb{R})$; moreover, $(S^1 \times S^1)\#\mathbf{P}^2(\mathbb{R}) \setminus (U_1 \cup U_2)$ is connected. By filling each boundary component with a 2-disk, we get a connected boundaryless surface Z such that

$$(S^1 \times S^1)\#\mathbf{P}^2(\mathbb{R}) \sim \mathbf{P}^2(\mathbb{R})\#Z\#\mathbf{P}^2(\mathbb{R})$$

and $\kappa(Z) = 1$, so that eventually $Z \sim \mathbf{P}^2(\mathbb{R})$.

Second proof. Consider the product $\mathbf{P}^1(\mathbb{R}) \times \mathbf{P}^1(\mathbb{R}) \sim S^1 \times S^1$, endowed with a couple of homogeneous coordinates $(t, s) = ((t_1, t_2), (s_1, s_2))$. Let $\mathbf{P}^3(\mathbb{R})$ with homogeneous coordinates $x = (x_1, x_2, x_3, x_4)$. Define

$$\psi : \mathbf{P}^1(\mathbb{R}) \times \mathbf{P}^1(\mathbb{R}) \to \mathbf{P}^3(\mathbb{R})$$

$$\psi(t, s) = (t_1 s_1, t_1 s_2, t_2 s_1, t_2 s_2)\ .$$

We verify that ψ is a well defined smooth embedding onto the quadric $Q \subset \mathbf{P}^3(\mathbb{R})$ defined by the homogeneous equation $x_1 x_4 = x_2 x_3$. Let $p_0 = (1, 0, 0, 0) \in Q$ and consider the "stereographic projection"

$$\phi : V \setminus \{p_0\} \to P$$

where $P \sim \mathbf{P}^2(\mathbb{R})$ is the projective plane $P \subset \mathbf{P}^3(\mathbb{R})$ defined by the equation $x_1 = 0$. Denote by T the plane tangent to Q at p_0. It is defined by the equation $x_4 = 0$. The intersection $T \cap Q$ consists of the union of the two lines passing through p_0, $l_1 = \{x_4 = x_2 = 0\}$ and $l_2 = \{x_4 = x_3 = 0\}$. The intersection $T \cap P$ is the line $l_0 = \{x_1 = x_4 = 0\}$. We verify that the restriction of ϕ is a diffeomorphism

$$\phi : Q \setminus (l_1 \cup l_2) \to P \setminus l_0\ .$$

Let us blow up $\mathbf{P}^3(\mathbb{R})$ at the point p_0 and take the strict transform $\tilde{Q}$. We know from the results of Section 7.10.1 that $\tilde{Q} \sim (S^1 \times S^1)\#\mathbf{P}^2(\mathbb{R})$. Blow up $\mathbf{P}^3(\mathbb{R})$ at the two points $p_1 = l_1 \cap P = (0, 1, 0, 0)$ and $p_2 = l_2 \cap P = (0, 0, 1, 0)$. Take the strict transform $\tilde{P} \sim 3\mathbf{P}^2(\mathbb{R})$. Finally, one verifies that ϕ extends to a diffeomorphism

$$\tilde{\phi} : \tilde{Q} \to \tilde{P}\ .$$

■

The proof of Theorem 15.10 and of Corollary 15.11 is now complete.

■

The above classification extends to *compact connected surfaces with boundary*. We limit to a few indications. Details are left to the reader.

• Let M be a compact connected smooth surface with $r \geq 1$ boundary components. Denote by $\hat{M}$ the boundaryless surface obtained by filling each boundary component with a 2-disk. *Viceversa*, M is obtained from $\hat{M}$ by removing the interior of r disjoint closed 2-disks. By the uniqueness of the disks up to isotopy, M is determined up to diffeomorphism by r and the diffeomorphism type of $\hat{M}$.

• The radical $\mathrm{Rad}(\bullet_M) \subset \eta_1(M)$ of the intersection form $\bullet_M$ is of dimension $r-1$ and is generated by the boundary components of M. The nondegenerate form $\hat{\bullet}_M$, uniquely induced up to isometry by $\bullet_M$ on $\eta_1(M)/\mathrm{Rad}(\bullet_M)$, is isometric to $\bullet_{\hat{M}}$. Hence M is determined up to diffeomorphism by the isometry class of the intersection form $\bullet_M$; that is, by $\dim \mathrm{Rad}(\bullet_M)$ and the isometry class of $\bullet_{\hat{M}}$.

• Two compact connected smooth surfaces with boundary M and M' are diffeomorphic if and only if they have the same number of boundary components, $\chi(M) = \chi(M')$, and either they are both orientable or both nonorientable.

15.3. $\Omega_1(X)$ as the Abelianization of the fundamental group

Recall that in Proposition 10.12 we established a natural epimorphism

$$h_1 : \pi_1(X, x_0) \to \Omega_1(X) \ ,$$

X being a path-connected topological space. Now we are able to determine the kernel of this epimorphism.

Proposition 15.13. *The kernel* $\ker h_1$ *coincides with the commutator subgroup of* $\pi_1(X, x_0)$*, hence* $\Omega_1(X)$ *is the Abelianization of the fundamental group.*

Proof : Let $\gamma : (S^1, p) \to (X, x_0)$ be a homotopically non-trivial loop which represents $0 \in \Omega_1(X)$. Then γ can be extended to a map $h : \Sigma \to X$ where Σ is a compact orientable surface with boundary $\partial\Sigma = S^1$ such that, by attaching a 2-disk along $\partial\Sigma$, we get a boundaryless compact orientable surface $\tilde{\Sigma}$ of a certain genus $g \geq 1$. By using the concrete models for such a surface provided by the classification theorem, we see that there is embedded in $\tilde{\Sigma}$ a wedge of $2g$-simple loops based at p, not intersecting $D^2 \setminus \{p\}$, such that by cutting the surface along these loops we get a $4g$-gon, and γ retracts to that wedge within Σ. Finally, we realize that these loops can be distributed into two families, $a_1, \dots a_g$, $b_1, \dots, b_g$, in such a way that the above retraction realizes a homotopy between γ and the composite loop

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \ .$$

The proposition follows.

■

The above proposition means that every homomorphism

$$\phi : \pi_1(X, x_0) \to G \ ,$$

G being Abelian, factorizes as $\phi = \hat{\phi} \circ h_1$, $\hat{\phi} : \Omega_1(X) \to G$.

15.4. Ω_2 and η_2

THEOREM 15.14. *(1)* $\Omega_2 = 0$*;*
(2) $\eta_2 \sim \mathbb{Z}/2\mathbb{Z}$ *and is generated by* $[\mathbf{P}^2(\mathbb{R})]$*.*
(3) $\psi : \eta_2 \to \mathbb{Z}/2\mathbb{Z}$, $\phi([M]) := \chi_{(2)}(M)$ *is a well defined isomorphism.*

Proof : Recall that

$$[M_1 \# M_2] = [M_1] + [M_2] \in \eta_2$$

(resp. $\in \Omega_2$ in the oriented setting). It follows from the classification that every compact connected oriented surface is the boundary of an oriented 3-manifold (in fact $n(S^1 \times S^1)$ can be embedded in $S^3 = \mathbb{R}^3 \cup \infty$ and divides it). Hence $\Omega_2 = 0$.

On the other hand, for every compact connected surface M,

$$[M \# 2\mathbf{P}^2(\mathbb{R})] = [M] \in \eta_2$$

and

$$M \# 2\mathbf{P}^2(\mathbb{R}) \sim (\kappa(M) + 2)\mathbf{P}^2(\mathbb{R})$$

by the classification. Hence

$$[M] = \chi_{(2)}(M)[\mathbf{P}^2(\mathbb{R})] \in \eta_2 \ .$$

As $[\mathbf{P}^2(\mathbb{R})] \neq 0$, then items (2) and (3) follow.

■

15.4.1. The group η_2 as a Witt group. Apparently, Theorem 15.14 is exhaustive. However, the topological classification of surfaces runs parallel to the algebraic classification of $\mathbb{Z}/2\mathbb{Z}$-symmetric bilinear forms up to isometry. We would like to recast the content of Theorem 15.14 within this vein.

Denote by $I(\mathbb{Z}/2\mathbb{Z})$ the set of isometry classes of non-degenerate symmetric bilinear forms defined on $\mathbb{Z}/2\mathbb{Z}$-vector spaces of arbitrary finite dimension. The set $I(\mathbb{Z}/2\mathbb{Z})$ is a semigroup provided it is endowed with the operation $\perp$. An element $S \in I(\mathbb{Z}/2\mathbb{Z})$ is said *neutral* if $\dim S = 2m$ is even and there is a subspace $Z \subset S$, $\dim Z = m$, such that $Z = Z^\perp$. It follows from Theorem 15.8 that S is neutral if and only if either $S = 2m\mathbf{U}$ or $S = m\mathbf{H}$, for some m. Put on $I(\mathbb{Z}/2\mathbb{Z})$ the equivalence relation $X \sim X'$ if and only if there are neutral spaces S, S' such that

$$X \perp S = X' \perp S' \ .$$

Denote by $W(\mathbb{Z}/2\mathbb{Z})$ the quotient set. For every $X \in I(\mathbb{Z}/2\mathbb{Z})$, $X \perp X$ is neutral, hence $\perp$ descends to $W(\mathbb{Z}/2\mathbb{Z})$ and makes it an Abelian group called the *Witt group* of the field $\mathbb{Z}/2\mathbb{Z}$; $0 \in W(\mathbb{Z}/2\mathbb{Z})$ is the class of neutral spaces, and for every $[X] \in W(\mathbb{Z}/2\mathbb{Z})$, $-[X] = [X]$. It follows from Theorem 15.8 that

$$r_{(2)} : W(\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z},\ r_{(2)}([X]) := \dim X \bmod(2)$$

is a well defined isomorphism of groups. Finally, Theorem 15.14 can be reformulated as follows.

THEOREM 15.15.

$$\mathfrak{w} : \eta_2 \to W(\mathbb{Z}/2\mathbb{Z}),\ \mathfrak{w}([M]) = [\bullet_M]$$

is a well defined isomorphism; moreover,

$$r_{(2)} \circ \mathfrak{w} = \chi_{(2)}\ .$$

■

15.4.2. A direct derivation of Ω_2 and η_2. Theorem 15.14 has been derived as a corollary of the classification. Here, we outline a direct derivation; the mechanism is interesting. For example, if M is orientable, starting from a 2-dimensional handle decomposition of M, it produces a 3-dimensional handle decomposition of a certain orientable 3-manifold such that the given surface is its boundary; in a sense, M *builds its 'simplest bulk'*.

For every compact connected surface M as usual, take a reduced ordered handle decomposition with $\kappa = \kappa(M)$ 1-handles. Starting from $(M_0, \partial M_0) = (D^2, S^1)$, we have a sequence $(M_i, \partial M_i)$, $i = 1, \dots, \kappa$, obtained by attaching one 1-handle to $(M_{i-1}, \partial M_{i-1})$; finally, M is obtained by attaching a 2-handle to $(M_\kappa, \partial M_\kappa)$. Consider the product 3-manifold $W = M \times [0,1]$. On the copy $M' = M \times \{1\}$ of M, consider the family of pairwise disjoint not necessarily connected curves ∂M_i, $i = 0, \dots, \kappa$. There is a system of pairwise disjoint tubular neighbourhoods $U_i \sim \partial M_i \times [-1,1]$ of these curves in M'. Let us attach to W at M' a family of κ disjoint three-dimensional 2-handles, each one attached to U_i, $i = 0, \dots, \kappa$. In this way, we get a 3-manifold W' such that

$$\partial W' = (M \times \{0\}) \amalg M"$$

where $M"$ has $\kappa + 2$ connected components, each one associated with one of the handles of the original decomposition of M. It is not hard to see that a component of $M"$ corresponding either to the 0-handle or the 2-handle of M is diffeomorphic to S^2. For a component associated with a 1-handle, there are two possibilities:

(1) Starting from an annulus $A \sim S^1 \times [0,1]$ we attach the 1-handle to $S^1 \times \{1\}$ in such a way that the resulting surface is orientable; then this surface is a 'pant' P and the corresponding component of $M"$

is obtained by filling every component of ∂P with a 2-disk so that it is diffeomorphic to S^2.

(2) Starting from an annulus $A \sim S^1 \times [0,1]$ we attach the 1-handle to $S^1 \times \{1\}$ in such a way that the resulting surface is nonorientable; then this surface is a Möbius band $\mathcal{M}$ and the corresponding component of M" is obtained by filling $\partial\mathcal{M}$ with a 2-disk so that it is diffeomorphic to $\mathbf{P}^2(\mathbb{R})$.

It follows that M is bordant with the disjoint union of k copies of $\mathbf{P}^2(\mathbb{R})$ for some $k \geq 0$. This is enough to conclude that $\eta_2 \sim \mathbb{Z}/2\mathbb{Z}$ and is generated by $[\mathbf{P}^2(\mathbb{R})]$.

Assume now that M is orientable. Hence W is orientable, and also W' is orientable because attaching a 2-handle does not destroy the orientability. Also $\partial W'$ is orientable so that M" is a disjoint union of 2-spheres. This is enough to conclude that $\Omega_2 = 0$. But we can say more. Let W" be obtained from W' by filling every component of M" with a 3-disk. By construction, W" is obtained from W by attaching a few disjoint 2-handles followed by a few 3-handles. By considering the dual decomposition, we see that W is obtained starting from a few 0-handles followed by a few disjoint 1-handles. By cancellation of 0-handles, we can assume that there is only one 0-handle. By sliding handles, we realize that up to diffeomorphism W" $:= \mathcal{H}_h$ is uniquely determined by the number h of 1-handles, it is called a *handlebody* of *genus* h, and $M = \partial\mathcal{H}_h$. By some elementary considerations about the Euler-Poincaré characteristic, we finally realize that $\kappa(M) = 2h$; in this way we have re-obtained a classification up to diffeomorphism, at least in the orientable case.

15.5. Stable equivalence

The classification of surfaces up to diffeomorphism contains a coarse classification up to *stabilization*: let us say that two (compact connected boundaryless, as usual) surfaces M and M' are *stably equivalent* if there are $n, m \in \mathbb{N}$ such that

$$M\#n\mathbf{P}^2(\mathbb{R}) \sim M'\#m\mathbf{P}^2(\mathbb{R}) \ .$$

As an immediate corollary of the full classification, we have that *every surface is stably equivalent to each other*. In the orientable setting, we have a similar result up to stabilization by some $n(S^1 \times S^1)$.

This coarse classification deserves to be pointed out because it is a sort of toy model of phenomena occurring, for example, in dimension 4 (even though a full classification is not known in such a case), and also because of the different flavour it acquires once we interpret $\#\mathbf{P}^2(\mathbb{R})$ as the *blowing up at a point*, according to Section 7.10.1. Then a stable equivalence between M and M' is realized by an $\tilde{M}$ which dominates both, being obtained by blowing up some points of each respectively; equivalently, we can say that M' is obtained from M by firstly blowing up some points of M and then performing a certain *blowing down* to M'.

Recall (Remark 7.34) that a compact real algebraic set X is *rational* if it is *birationally equivalent* to the projective space of the same dimension; that is, X contains a nonempty Zariski open set which is algebraically isomorphic to a Zariski open set in the projective space of the same dimension. If $X = B(\mathbf{P}^n(\mathbb{R}), Y)$ is obtained from $\mathbf{P}^n(\mathbb{R})$ by blowing up along a regular algebraic centre (in particular a finite set of points), then X is a rational regular algebraic set. A so-called "Nash's conjecture" stated in [**Na**] asked if every compact smooth manifold admits any rational regular real algebraic model, up to diffeomorphism. We have a rather complete answer in the case of surfaces:

• Every nonorientable surface $M \sim \mathbf{P}^2(\mathbb{R})\#n\mathbf{P}^2(\mathbb{R}) \sim B(\mathbf{P}^2(\mathbb{R}), Y)$, $\kappa(M) = n+1$, Y consisting of n points, has a rational model;

• If M is orientable, $M\#\mathbf{P}^2(\mathbb{R})$ admits a rational model $B(\mathbf{P}^2(\mathbb{R}), Y)$, where Y consists of $2n = \kappa(M)$ points. We can ask if Y can be chosen in such a way that a blowing down that returns M can be done in the algebraic setting, providing a rational model for M itself. For example, in the second proof of Lemma 15.12, we see such a mechanism which produces $\mathbf{P}^1(\mathbb{R}) \times \mathbf{P}^1(\mathbb{R}) \sim S^1 \times S^1$ by blowing down $B(\mathbf{P}^2(\mathbb{R}), \{p_1, p_2\})$, collapsing to a point p_0 the strict transform of the line of $\mathbf{P}^2(\mathbb{R})$ passing through the points p_1 and p_2. We can prove in general that if $Y = \{p_1, \dots, p_{2n}\}$ is contained in a projective line $l \subset \mathbf{P}^2(\mathbb{R})$, then by blowing down to a point p_0 the strict transform $\tilde{l}$ of l in $B(\mathbf{P}^2(\mathbb{R}), Y)$ we get a rational algebraic set X, which is *homeomorphic* to M via an algebraic homeomorphism which restricts to an algebraic isomorphism between regular Zariski open sets

$$B(\mathbf{P}^2(\mathbb{R}), Y) \setminus \tilde{l} \to X \setminus \{p_0\} \ .$$

However, *if $n > 1$, X is not regular* as it has one isolated singularity at p_0. These rational models with one isolated singularity are the best we can do because it is known since Comessati [**COM**] that $S^1 \times S^1$ is the only *orientable* surface admitting a *regular* rational model.

15.6. Quadratic enhancement of surface intersection forms

Let $(\eta_1(M), \bullet_M)$ be as above, where M is a compact connected boundaryless surface. In several situations, one is interested in the immersions of M in a given higher-dimensional manifold, considered up to suitable equivalence relations which often enhance the abstract surface bordism. In such situations, so-called *quadratic enhancements* of the intersection form naturally arise. In this section, we will develop a few aspects of the *abstract theory* of such structures. Many proofs are simple exercises and we will omit them. Later in the text, we will see concrete applications (see Sections 17.4.3, 19.7.1, 19.8, 20.6).

Let (V, β) be a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space endowed with a non-degenerate symmetric bilinear form β.

(Totally isotropic case) Assume first that β is totally isotropic, so that (V,β) is isometric to $g\mathbf{H}$, $\dim V = 2g$.

DEFINITION 15.16. A map $q : V \to \mathbb{Z}/2\mathbb{Z}$ is a *quadratic enhancement* of (V,β) (sometimes we simply say "of β") if for every $x, y \in V$,

$$q(x+y) = q(x) + q(y) + \beta(x,y)\ .$$

We can enhance the equivalence relation "up to isometry" to the set of such triples:

$$f : (V_1, \beta_1, q_1) \to (V_2, \beta_2, q_2)$$

is an *isometry* if and only if

$$f : (V_1, \beta_1) \to (V_2, \beta_2)$$

is an isometry in the usual sense and, moreover, for every $x \in V_1$, $q_1(x) = q_2(f(x))$. We denote by

$$I_q^{\mathbf{H}}(\mathbb{Z}/2\mathbb{Z})$$

the set of isometry classes of these triples. The operation "$\perp$" gives it a semigroup structure.

It is rather easy to enhance the results of Section 15.1 (in the totally isotropic case); as usual, sometimes we will confuse representatives with their isometry classes.

Up to isometry there are exactly two quadratic enhancement of $\mathbf{H}$ (endowed with the standard hyperbolic basis $\{e_0, e_1\}$):

- $q_0(e_0) = q_0(e_1) = 0$, $q_0(e_0 + e_1) = 1$; denote by $\mathbf{H}^{0,0}$ the corresponding equipped space;
- $q_1(e_0) = q_1(e_1) = q_1(e_0 + e_1) = 1$; denote by $\mathbf{H}^{1,1}$ the corresponding equipped space.

Then every triple (V, β, q) is isometric to

$$m\mathbf{H}^{0,0} \perp n\mathbf{H}^{1,1}$$

for some $m, n \in \mathbb{N}$ such that $2(m+n) = 2g = \dim V$. Such integers m and n are not unique; in fact we have the following.

LEMMA 15.17. $\mathbf{H}^{0,0} \perp \mathbf{H}^{0,0} = \mathbf{H}^{1,1} \perp \mathbf{H}^{1,1}$.

PROPOSITION 15.18. *(1)*

$$\mathrm{Arf} : (I_q^{\mathbf{H}}(\mathbb{Z}/2\mathbb{Z}), \perp) \to (\mathbb{Z}/2\mathbb{Z}, +),\ \mathrm{Arf}([V, \beta, q]) = n \bmod(2)\ ,$$

provided that $[V, \beta, q] = m\mathbf{H}^{0,0} \perp n\mathbf{H}^{1,1}$ *for some* $(m, n) \in \mathbb{N}^2$*, is a well defined surjective semigroup homomorphism.*

(2) $\mathrm{Arf}([V, \beta, q]) = 1$ *if and only if* $|q^{-1}(1)| > |q^{-1}(0)|$*;* $\mathrm{Arf}([V, \beta, q]) = 0$ *if and only if* $|q^{-1}(1)| < |q^{-1}(0)|$.

(3) If $[V,\beta] = g\mathbf{H}$ *and the* j*-copy of* $\mathbf{H}$ *is endowed with its standard hyperbolic basis* $\{e_0^j, e_1^j\}$, $j = 1, \dots, g$, *then*

$$\mathrm{Arf}([V,\beta,q]) = \sum_j q(e_0^j)q(e_1^j)\ .$$

Arf is called the *Arf invariant.*

We can define the *Witt group* associated to the semigroup

$$(I_q^{\mathbf{H}}(\mathbb{Z}/2\mathbb{Z}), \perp)\ .$$

The class $[V,\beta,q] \in I_q^{\mathbf{H}}(\mathbb{Z}/2\mathbb{Z})$, $\dim V = 2g$, is said *neutral* if there is a subspace $Z \subset V$ such that $\dim Z = g$, $Z = Z^\perp$ and q vanishes on Z. Put on $I_q^{\mathbf{H}}(\mathbb{Z}/2\mathbb{Z})$ the equivalence relation $X \sim X'$ if and only if there are neutral spaces S, S' such that

$$X \perp S = X' \perp S'\ .$$

Denote by $W_q^{\mathbf{H}}(\mathbb{Z}/2\mathbb{Z})$ the quotient set. The operation $\perp$ descends to $W_q^{\mathbf{H}}(\mathbb{Z}/2Z)$ and makes it an Abelian group.

PROPOSITION 15.19. *The Arf homomorphism passes to the quotient*

$$\mathrm{Arf} : W_q^{\mathbf{H}}(\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$$

and is a group isomorphism. The Witt group is generated by $\mathbf{H}^{1,1}$.

We know that $(I^{\mathbf{H}}(\mathbb{Z}/2\mathbb{Z}), \perp)$ is isomorphic to the semigroup of *orientable* compact connected boundaryless surfaces (considered up to diffeomorphism) endowed with the "#" operation. The isomorphism is given by associating to every surface M the class of $(\eta_1(M), \bullet_M)$. So the above algebraic discussion can be interpreted in such a topological setting. In particular, the bases evoked in item (3) of Proposition 15.18 can be realized geometrically: if M is a surface of genus g then we can find two families of g smooth circles $\{A_1, \dots, A_g\}$ and $\{B_1, \dots, B_g\}$ such that

- $A_i \cap A_j = B_i \cap B_j = \emptyset$ if $i \neq j$;
- A_i and B_j intersect transversely at one point if and only if $i = j$, otherwise $A_i \cap B_j = \emptyset$.

These $2g$ circles form a basis of $\eta_1(M)$; if q is a quadratic enhancement of $\bullet_M$, then

$$\mathrm{Arf}(q) = \sum_j q([A_j])q([B_j])\ .$$

The set $W_q^{\mathbf{H}}(\mathbb{Z}/2\mathbb{Z})$ can be considered as a formal nontrivial refinement of $\Omega_2 = 0$.

(General case) Now we consider arbitrary non-degenerate spaces (V,β). In this generality, the notion of quadratic enhancement is subtler, due to the presence of nonisotropic elements. The key point is to consider $\mathbb{Z}/4\mathbb{Z}$ instead of $\mathbb{Z}/2\mathbb{Z}$-valued forms q.

DEFINITION 15.20. A map

$$q : V \to \mathbb{Z}/4Z$$

is a quadratic enhancement of β if for every $x, y \in V$,

$$q(x+y) = q(x) + q(y) + 2\beta(x,y) \ ,$$

where $a \to 2a$ is the unique nontrivial homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$.

REMARK 15.21. Assume that (V, β) is totally isotropic. If $\bar{q} : V \to \mathbb{Z}/2\mathbb{Z}$ is a quadratic enhancement of β in the earlier sense, then $q = 2\bar{q}$ is a quadratic enhancement in the new sense. On the other hand, if $q : V \to \mathbb{Z}/4\mathbb{Z}$ is as in Definition 15.20, then it takes only even values and there is a unique $\bar{q} : V \to \mathbb{Z}/2\mathbb{Z}$ such that $q = 2\bar{q}$. So if we restrict to totally isotropic spaces we recover the previous setting.

The set of quadratic enhancements of (V, β) has the structure of an *affine space over* V.

LEMMA 15.22. *There are* $2^{\dim V}$ mod *(4) quadratic enhancements of* (V, β)*; if* q *is one, the others are of the form*

$$q'(x) = q(x) + 2\beta(u, x)$$

for a unique $u \in V$.

Proof : The map $l(x) := 2^{-1}(q'(x) - q(x))$ is linear, hence represented by a unique $u \in V$ by means of the non-degenerate form β.

■

The notion of isometry of triples extends *verbatim* and we denote by

$$(I_q(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}), \perp)$$

the semigroup of isometry classes.

• Up to isometry, on $\mathbf{H}$ there are two $\mathbb{Z}/4\mathbb{Z}$-valued quadratic enhancenents, that is $\mathfrak{q}_j = 2q_j$, $j = 0, 1$, where $q_j : \mathbf{H} \to \mathbb{Z}/2\mathbb{Z}$ have already been defined above. We keep the notations $\mathbf{H}^{j,j}$ for the associated equipped spaces.

• Up to isometry, on $\mathbf{U}$ there are two quadratic enhancements $q^{\pm} : \mathbf{U} \to \mathbb{Z}/4\mathbb{Z}$, $q^{\pm}(1) = \pm 1$. Denote by $\mathbf{U}^{\pm}$ the corresponding equipped spaces.

Hence for every (V, β) totally isotropic, we still have

$$[V, \beta, q] = m\mathbf{H}^{0,0} \perp n\mathbf{H}^{1,1}, \ 2(m+n) = \dim V \ .$$

If (V, β) is not totally isotropic, then

$$[V, \beta, q] = a\mathbf{U}^{-} \perp b\mathbf{U}^{+}$$

for some $(a, b) \in \mathbb{N}^2$, $a + b = \dim V$. As above, we are not claiming that (a, b) is unique.

In any case, we say that $[V,\beta,q]$ is *neutral* if there exists a subspace $Z \subset V$ such that $Z = Z^\perp$ (so that $\dim V = 2h$ is even and $\dim Z = h$) and q vanishes on Z. As above, we can define the *Witt group*, denoted by

$$W_q(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \ ,$$

as a quotient of the semigroup $(I_q(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}), \perp)$.

For every $[V,\beta,q] \in I_q(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$, for every $x \in V$, define

$$\psi_{[V,\beta,q]}(x) := \exp(\frac{i\pi}{2} q(x)) = i^{q(x)} \ .$$

Finally, set

$$\gamma([V,\beta,q]) := (\frac{1}{\sqrt{2}})^{\dim V} \sum_{x\in V} \psi_{[V,\beta,q]}(x) \ .$$

This is called the multiplicative *Brown invariant* of $[V,\beta,q]$.

For every $k \geq 2$, denote by U_k the multiplicative subgroup of $U(1)$ formed by the kth-roots of 1. Denote by

$$\alpha_k : (\mathbb{Z}/k\mathbb{Z}, +) \to U_k$$

the natural isomorphism of groups.

LEMMA 15.23. *If (V,β) is totally isotropic so that $q = 2\bar{q}$ for a unique*

$$\bar{q} : V \to \mathbb{Z}/2\mathbb{Z}$$

then

$$\gamma([V,\beta,q]) = \alpha_2(\mathrm{Arf}([V,\beta,\bar{q}]) \ .$$

Hence the Brown invariant extends the Arf one.

For every $X = [V,\beta,q]$, set $-X := [V,\beta,-q]$.

LEMMA 15.24. *Let $X, Y \in I_q(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$. Then:*

(1) $\gamma(X \perp Y) = \gamma(X)\gamma(Y)$;

(2) If X is neutral, then $\gamma(X) = 1$;

(3) $4X = 4(-X)$.

Proof : Item (1) follows from the very definition.

As for (2), let $X = [V,\beta,q]$, $Z \subset V$, $\dim V = 2n$, $\dim Z = n$, $Z = Z^\perp$, q vanishing on Z. For simplicity we omit the index X in denoting ψ. Let $V = Z \oplus L$ be any direct sum decomposition. Then

$$\gamma(q) = (\frac{1}{\sqrt{2}})^{2n} \sum_{z\in Z, l\in L} \psi(z+l) = (\frac{1}{\sqrt{2}})^{2n} \sum_{z\in Z, l\in L} \psi(l)(-1)^{\beta(l,z)} =$$

$$(\frac{1}{\sqrt{2}})^{2n}[\sum_{l\in L\setminus\{0\}} (\sum_{z\in Z}(-1)^{\beta(l,z)}\psi(l)) + |Z|)] =$$

$$(\frac{1}{\sqrt{2}})^{2n}|Z| = 1 \ ,$$

where the fourth equality depends on the fact that for every $l \neq 0$, $z \to \beta(l,z)$ defines a linear form ϕ on Z, and $\dim \ker(\phi) = \dim Z - 1$ as β is non-degenerate.

As for (3), it is enough to show that $4\mathbf{U}^+ = 4\mathbf{U}^-$. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $4\mathbb{C} \sim \mathbb{C}^4$. Let $\rho_j : \mathbb{C} \to \mathbb{C}^4$ be the linear embedding such that $\rho_j(1) = e_j$. Then we verify that the linear isomorphism

$$\rho = (\rho_1, \dots, \rho_4) : \mathbb{C}^4 \to \mathbb{C}^4$$

induces a required isomorphism

$$\rho : 4\mathbf{U}^+ \to 4\mathbf{U}^- \ .$$

■

Finally, we can state the main result of this matter.

THEOREM 15.25. *The Brown semigroup morphism γ passes to the quotient, and in fact it determines, a group isomorphism*

$$\tilde{\gamma} := \alpha_8^{-1} \circ \gamma : W_q(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \to \mathbb{Z}/8\mathbb{Z} \ .$$

The Witt group is generated by $\mathbf{U}^+$.

Proof : The space $\mathbf{U}^+ \perp \mathbf{U}^-$ is neutral, then the Witt group is cyclic generated by $\mathbf{U}^+$. By the previous lemma, $8\mathbf{U}^+$ is neutral, hence the order of $\mathbf{U}^+$ divides 8. Finally, by direct computation, $\gamma(\mathbf{U}^+) = \exp(\frac{i\pi}{4})$; that is, it is a primitive eighth root of 1.

■

The following corollary is easy.

COROLLARY 15.26. *The Brown invariant of q, the dimension of V and the fact that β is, or is not, totally isotropic form a complete set of invariants which classifies $[V, \beta, q] \in I_q(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$.*

By rephrasing everything in the topological 2-dimensional setting, we can say that the Witt group $W_q(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \sim \mathbb{Z}/8\mathbb{Z}$ is a formal enhancement of the Witt group $W(\mathbb{Z}/2\mathbb{Z}) \sim \eta_2 \sim \mathbb{Z}/2\mathbb{Z}$.

We conclude this section by outlining a constructive way to build quadratic enhancements of $(M, \bullet_M)$ for a given compact boundaryless surface M (see [**KT**], Lemma 3.4). It is enough to define a function q which associates an element in $\mathbb{Z}/4\mathbb{Z}$ to every disjoint union of smooth circles on M (considered up to ambient isotopy) provided that the following conditions are satisfied:

(1) The function q is *additive on disjoint unions*: if $L_1 \amalg L_2$ is again a disjoint union of circles, then $q(L_1 \amalg L_2) = q(L_1) + q(L_2)$;
(2) If K_1 and K_2 are two circles that cross transversely at r points, then by resolving (in one of the two possible ways) each crossing we get a disjoint union L of embedded circles. Then $q(L) = q(K_1) + q(K_2) + 2r \mod(4)$.
(3) If K is a smooth circle that bounds a 2-disk in M, then $q(K) = 0$.

In such a situation, a quadratic enhancement of $(\eta_1(M), \bullet_M)$ is defined by setting $q(\alpha) = q(C)$, where C is any smooth circle representing α.

CHAPTER 16

Bordism characteristic numbers

The Euler-Poincaré characteristic mod(2) is a first example of characteristic number for the nonoriented bordism modules η_m; that is, for every $m \geq 0$, it defines a homomorphism $\chi_{(2)} : \eta_m \to \mathbb{Z}/2\mathbb{Z}$ which is nontrivial for each even m. Pontryagin remarked that a huge family of characteristic numbers can be produced using the cohomology ring with $\mathbb{Z}/2\mathbb{Z}$-coefficients of the infinite Grassmann manifolds $\mathfrak{G}_{\infty,\bullet}$ and the classifying map $M \to \mathfrak{G}_{\infty,m+1}$ of the stable tangent bundle $T(M) \oplus \epsilon^1$ of each compact boundaryless m-manifold M. These are called *SW-characteristic numbers* as they are incorporated in the theory of (cohomological) Stiefel-Whitney *characteristic classes*. We do not dispose of cohomology, but it is easy to reformulate the definition using the cobordism rings, which we have defined from scratch, instead of the cohomology rings. We call (stable) η-*characteristic numbers* the ones obtained in this way. In [**T**], Thom determined the ring $\eta^\bullet$, using the Pontryagin-Thom construction (that we treat in Chapter 17) and combining geometric tools and homotopy theory. A byproduct of Thom's work is the *completeness* of the SW- characteristic numbers: *$\beta \in \eta_m$ is equal to zero if and only if every SW- characteristic number vanishes on β.* Later, the authors obtained in [**BH**] a nice geometric proof of this remarkable result, ultimately based on transversality and simple cohomological computations. We show that this proof can be entirely performed using the cobordism rings and eventually get the completeness of the η-characteristic numbers. The ultimate reason for this is another theorem of Thom in [**T**] (the solution of so called Steenrod's representation problem; the paper [**BH**] provides a geometric proof of this result too). Combined with the Poincaré duality and the "compatibility" between the cohomological (with $\mathbb{Z}/2\mathbb{Z}$-coefficients) and the η cup products, it implies that SW and η characteristic numbers are basically the same thing. However, we do not use these facts. We directly get the completeness of the η-characteristic numbers adapting the proof of [**BH**]. At the end of the chapter, we extend η to Ω-characteristic numbers and we briefly discuss why they are not sufficient to completely detect the oriented boundaries. Cohomology cannot be avoided for the oriented bordism. However, any characteristic number for $\Omega_\bullet$, whatever it is defined, should vanish on $[M]$ if the m-manifold M is *parallelizable*. At least in this special case we prove that $[M] = 0 \in \Omega_m$ for any choice of the orientation of M, using similar geometric tools.

16.1. η-numbers

Let us give a definition of an η-characteristic number modeled on the Euler-Poincaré characteristic mod(2), $\chi_{(2)}$. As usual, denote by $\mathcal{S}_n$ the class of compact boundaryless smooth n-manifolds. For every $X \in \mathcal{S}_n$, let

$$t_X : X \to \mathfrak{G}_{m,n}$$

be a "truncated" classifying map of the tangent bundle $T(X)$, where $m = m(n)$ big enough only depends on n. An *η-characteristic number* is a function

$$\mathfrak{c} : \mathcal{S}_n \to \mathbb{Z}/2\mathbb{Z}$$

such that

(1) It is of the form

$$\mathfrak{c}(X) = \mathfrak{c}_\alpha(X) := \sum_j t_X^*(\alpha) \sqcap [X_j]$$

for some $\alpha \in \eta^n(\mathfrak{G}_{m,n})$, where X_j varies among the connected components of X. Such a $\mathfrak{c}(X)$ is a diffeomorphism invariant.

(2) If $[X] = 0 \in \eta_n$, then $\mathfrak{c}(X) = 0$. It follows that $\mathfrak{c}$ induces a $\mathbb{Z}/2\mathbb{Z}$-linear map

$$\mathfrak{c} : \eta_n \to \mathbb{Z}/2\mathbb{Z}\ .$$

Here is another characteristic η-number besides $\chi_{(2)}$. For every X, consider the nth-power (with respect to the $\sqcup$ product)

$$w^1(X)^n$$

of the Euler class of the determinant line bundle of X.

PROPOSITION 16.1. *$\mathfrak{c}_{w^1(X)^n}$ is an η-characteristic number, different from $\chi_{(2)}$.*

Proof : To see that it is characteristic, it is enough to show that if $X = \partial W$ is a boundary, then $\mathfrak{c}_{w^1(X)^n}(X) = 0$. Note that

$$w^1(X) = j^* w^1(W) \in \eta^1(W, \partial W)\ ,$$

where $j : \partial W \to W$ is the inclusion. Then $w^1(X)^n = (j^*(w^1(W)))^n$, and $w^1(X)^n$ is represented by the boundary of the proper 1-dimensional submanifold of $(W, \partial W)$ which represents $w^1(W)^n \in \eta^n(W, \partial W)$, hence it consists of an even number of points. To see that it is different from $\chi_{(2)}$, consider for example $w^1(\mathbf{P}^4(\mathbb{R}))^4 = 1$, while we can show (do it by exercise) that $w^1(\mathbf{P}^2(\mathbb{R}) \times \mathbf{P}^2(\mathbb{R}))^4 = 0$. We know that both E-P characteristics mod(2) are equal to 1. Hence $[\mathbf{P}^4(\mathbb{R})]$ and $[\mathbf{P}^2(\mathbb{R}) \times \mathbf{P}^2(\mathbb{R})]$ are nontrivial independent elements of η_4. Similarly, $w^1(*)^4$ distinguishes $[\mathbf{P}^4(\mathbb{R})]$ from $[\mathbf{P}^2(\mathbb{C})]$. The same argument extends to any couple $[\mathbf{P}^{a+b}(\mathbb{R})]$, $[\mathbf{P}^a(\mathbb{R}) \times \mathbf{P}^b(\mathbb{R})]$ in η_{a+b}, where both a and b are even.

∎

16.2. Stable η-numbers

It is not so easy to check directly if a function of the form $\mathfrak{c}_\alpha$ as above is a characteristic number or not (that is, if it vanishes on boundaries). On the other hand, this becomes almost immediate if we consider so-called "stable classes" in the Grassmannian cobordism. Consider the "stabilized tautological bundle"

$$\tau_{m,n} \oplus \epsilon^1 ;$$

it corresponds to an evident classifying map

$$s_n : \mathfrak{G}_{m,n} \to \mathfrak{G}_{m+1,n+1} .$$

Then $\alpha \in \eta^k(\mathfrak{G}_{m,n})$ (not necessarily $k = n$) is by definition a *stable class* if

$$\alpha = s_n^*(\tilde{\alpha})$$

for some $\tilde{\alpha} \in \eta^k(\mathfrak{G}_{m+1,n+1})$. The sum and the product of stable classes are stable. A class of the form $\alpha = (s_{n+j} \circ \cdots \circ s_n)^*(\tilde{\alpha})$ is stable for every $j \geq 0$.

For every $X \in \mathcal{S}_n$, the classifying map of the *stable tangent bundle*

$$T(X) \oplus \epsilon^1$$

is the composite map

$$s_X := s_n \circ t_X .$$

PROPOSITION 16.2. *For every $n \geq 0$, if $\alpha \in \eta^n(\mathfrak{G}_{m,n})$ is a stable class, then $\mathfrak{c}_\alpha$ is a (*stable *by definition) η-characteristic number defined on $\mathcal{S}_n$.*

Proof : Assume that $X = \partial W$; then

$$j^*(T(W)) = T(X) \oplus \epsilon^1 ,$$

so that $s_X = t_W \circ j$, $t_X^*(\alpha) = (t_W \circ j)^*(\tilde{\alpha})$, where j is the inclusion. Let us apply to t_W and $\tilde{\alpha}$ the geometric procedure that defines g^* as in Proposition 11.6. Then we find a smooth map $\pi : (Y, \partial Y) \to (W, X)$, where Y is a compact 1-manifold and $[\partial Y, \partial \pi] = t_X^*(\alpha)$; hence $\mathfrak{c}_\alpha$ vanishes on X because ∂Y consists of an even number of points. In other words, $j_*[X] = 0 \in \eta_n(W)$ and $t_X^*(\alpha) \sqcap [X] = t_W^*(\tilde{\alpha}) \sqcap j_*[X] = 0$.

■

Note that it is not evident whether $\chi_{(2)}$ is a *stable* characteristic number (see the end of this chapter).

16.3. Completeness of stable η-numbers

The above definition of stable η-characteristic numbers is quite implicit, as it is based on the cobordism of Grassmann manifolds which we do not know. Nevertheless, we are going to see that it is enough to show that they are "complete". This "completeness" refers to the fact that the necessary condition to be a boundary established in Proposition 16.2 (the vanishing of all stable η-characteristic numbers) is also sufficient.

THEOREM 16.3. *$[X] = 0 \in \eta_n$ if and only if every stable η-characteristic number vanishes on X.*

We will propose a geometric proof extracted from [**BH**], provided that it is entirely performed using the cobordism rings instead of the cohomology rings with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

It is enough to show the "if" implication. This will be an immediate consequence of the next two lemmas.

By the classification of compact 1-manifolds, if $n = 0$ then X is a boundary if and only if it consists of an *even* number of points, thus it is easy to check that Theorem 16.3 holds true for $n = 0$. If $\dim X > 0$, there is a *special case* such that the stable characteristic numbers clearly vanish, when (X, s_X) is bordant with a constant map (N, c); in other words, $[X, s_X]$ belongs to a copy of η_n embedded in $\eta_n(\mathfrak{G}_{m+1,n+1})$. Let us prove first that X is a boundary under such a stronger hypothesis.

Lemma 16.4. *Let* $\dim X > 0$ *and* $F : Q \to \mathfrak{G}_{m+1,n+1}$ *realize a bordism of* (X, s_X) *with a constant map* $c : N \to \mathfrak{G}_{m+1,n+1}$. *Then* N *(hence* X*) is a boundary.*

Proof : The map F pulls back the tautological bundle over the Grassmannian to a rank $(n+1)$ vector bundle ξ on Q which restricts to $\tau_X := T(X) \oplus \epsilon^1$ on X and to a trivial bundle ϵ^{n+1} on N. Denote by $D(\xi)$, $S(\xi) = \partial D(\xi)$ the total spaces of the unitary $(n+1)$-disk and n-sphere bundles of ξ, respectively. Similarly, denote the restrictions $D(\tau_X)$, $S(\tau_X)$ and $D(\epsilon^{n+1})$, $S(\epsilon^{n+1})$. Let ι be the fibrewise antipodal involution on ξ. Then $S(\xi)$ is a compact $(2n+1)$-manifold with boundary

$$\partial S(\xi) = S(\tau_X) \amalg S(\epsilon^{n+1})$$

equipped with the involution ι_S (the restriction of ι). Consider the $(2n+1)$-manifold with boundary

$$Y = X \times X \times [-1, 1]$$

equipped with the involution

$$\sigma(x, y, t) = (y, x, -t)$$

so that ∂Y is an invariant set of σ. The fixed point set of σ is given by

$$\mathcal{X} = \Delta_X \times \{0\} = \{(x, x, 0)\} \subset Y$$

which can be naturally identified with X itself. We can find a tubular neighbourhood U of $\mathcal{X}$ in Y such that, removing the interior of U from Y, we get a compact $(2n+1)$-manifold Z, with boundary

$$\partial Z = \partial U \amalg \partial Y$$

such that $(Z, \partial Z)$ is invariant for σ and the restriction of σ to ∂U can be identified with the restriction of ι_S to $S(\tau_X)$. Then we can glue Z and $S(\xi)$ along $\partial U \sim S(\tau_X)$ and get a compact $(2n+1)$-manifold W with boundary

$$\partial W = \partial Y \amalg S(\epsilon^{n+1})$$

equipped with a smooth fixed point free involution σ_W, which coincides with $\sigma \amalg \iota_S$ on ∂W. The quotient space $\mathcal{W} := W/\sigma_W$ is a smooth manifold with boundary such that the quotient map

$$q : W \to \mathcal{W}$$

is a degree 2 smooth covering map. The restriction of q to ∂Y is a trivial covering, while

$$S(\epsilon^{n+1})/\sigma_W \sim N \times \mathbf{P}^n(\mathbb{R})$$

and the restriction of q to $S(\epsilon^{n+1}) \sim N \times S^n$ can be identified with the map $\mathrm{Id}_N \times s$, $s : S^n \to \mathbf{P}^n(\mathbb{R})$, being the standard double covering. The associated real line bundle on $\mathcal{W}$ (see Chapter 13) is the pull-back by a classifying map

$$\phi : \mathcal{W} \to \mathbf{P}^a(\mathbb{R})$$

for some a big enough, considered up to homotopy. By the above remark about the restriction of the covering to $\partial\mathcal{W}$, we can assume that $\phi_{|\partial Y}$ is a constant map, while $\phi_{|N\times\mathbf{P}^n(\mathbb{R})}$ is the composition of the projection $N \times \mathbf{P}^n(\mathbb{R}) \to \mathbf{P}^n(\mathbb{R})$ followed by the inclusion $\mathbf{P}^n(\mathbb{R}) \subset \mathbf{P}^a(\mathbb{R})$. Let $\mathbf{P}^{a-n}(\mathbb{R})$ be a projective subspace of $\mathbf{P}^a(\mathbb{R})$ which intersects $\mathbf{P}^n(\mathbb{R})$ transversely at one point x_0. We can also assume that $\phi(\partial Y) \cap \mathbf{P}^{a-n}(\mathbb{R}) = \emptyset$, so that $\phi_{|\partial\mathcal{W}} \pitchfork \mathbf{P}^{a-n}(\mathbb{R})$ and

$$\phi^{-1}_{|\partial Y}(\mathbf{P}^{a-n}(\mathbb{R})) = N \times \{x_0\} \sim N\ .$$

Finally, using usual transversality theorems, we can also assume that the whole map ϕ is transverse to $\mathbf{P}^{a-n}(\mathbb{R})$ so that the proper $(n+1)$-submanifold $(R, \partial R)$ of $(\mathcal{W}, \partial\mathcal{W})$ given by $R = \phi^{-1}(\mathbf{P}^{a-n}(\mathbb{R}))$ is such that $N\times\{x_0\} = \partial R$. This achieves the proof of Lemma 16.4.

∎

As Theorem 16.3 holds true for $n = 0$, we will argue by induction on the dimension $n \geq 0$. The inductive step is provided by the following lemma combined with Lemma 16.4.

LEMMA 16.5. *Let* $\dim X = n > 0$*. Assume that all stable* η*-characteristic numbers of* X *vanish, and that Theorem 16.3 holds true for all dimensions* m *smaller than* n*. Then* (X, s_X) *is bordant with a constant map* $c : N \to \mathfrak{G}_{m+1,n+1}$.

Proof : This proof is not completely self-contained within the content of this text. In fact, we consider a triangulation $\mathcal{K}$ of $\mathfrak{G}_{m+1,n+1}$ made by smoothly embedded simplices, whose existence has been evoked in Section 14.9.1 (without a proof). The interior of every such an h-simplex is a submanifold of $\mathfrak{G}_{m+1,n+1}$ diffeomorphic to $\mathbb{R}^h$ and called an (open) h-cell of $\mathcal{K}$. Alternatively, we can use the open cells of the natural cellular decomposition of the Grassmannian depicted in Section 3.5 (the geometric properties of this decomposition had been only sketched as well). For every $0 \leq h \leq \dim \mathfrak{G}_{m+1,n+1}$, the union of the cells of dimension less or equal to

h is called the *h-skeleton* $\mathcal{K}_h$ of $\mathcal{K}$. Fix a base point x_e in each open cell e and call it the "centre" of the cell. For every h as above, by removing the centre from every h-cell, we get a subspace $\tilde{\mathcal{K}}_h$ of $\mathcal{K}_h$ which retracts to $\mathcal{K}_{h-1}$. By basic transversality, we can assume that the smooth map s_X misses the centre of every cell of dimension greater than $n = \dim X$; hence, up to (continuous) homotopy, we can assume that

$$s_X : X \to \mathfrak{G}_{m+1,n+1}$$

is continuous with values in the n-skeleton $\mathcal{K}_n$, it is smooth on $s_X^{-1}(\mathcal{K}_n \setminus \mathcal{K}_{n-1})$, and is transverse to the centre x_e of every n-cell e.

We claim that for every n-cell e, the 0-submanifold $Y := s_X^{-1}(x_e)$ of X consists of an *even* number of points; that is, it is a 0-dimensional boundary. In fact, by collapsing $\mathcal{K}_n \setminus \{e\}$ to one point, we get a projection

$$p_e : \mathcal{K}_n \to S^n$$

which restricts to a smooth embedding of the n-cell e to $\mathbb{R}^n \subset \mathbb{R}^n \cup \infty = S^n$, so that we will confuse x_e with $p_e(x_e)$. Then

$$Y = (p_e \circ s_X)^{-1}(x_e)$$

and we easily realize that

$$[Y] = s_X^*(p_e^*([x_e]) \in \eta^n(X) \ ,$$

which vanishes as it is a stable η-characteristic number of X. Fix a small n-ball D around x_e in e. Then

$$s_X^{-1}(D) = (\tilde{D}_1 \cup \tilde{D}_2) \cup \cdots \cup (\tilde{D}_s \cup \tilde{D}_{s+1})$$

and the restriction of s_X to every $\tilde{D}_j$ is a diffeomorphism to D. Remove from X the interior of every $\tilde{D}_j$ and pairwise glue together the boundary components $\partial\tilde{D}_{2j-1}$ and $\partial\tilde{D}_{2j}$, $j = 1, 2, \ldots, (s+1)/2$, by means of the above identifications with ∂D. Do it simultaneously at the centre of every n-cell. Then we get a boundaryless n-manifold N_1 such that the map s_X descends to a stable classifying map

$$s_1 = s_{N_1} : N_1 \to \mathcal{K}_n \subset \mathfrak{G}_{m+1,n+1} \ ,$$

which misses the centres of every n-cell; hence up to homotopy we may assume that s_1 *takes values in $\mathcal{K}_{n-1}$, it is smooth on $s_1^{-1}(\mathcal{K}_{n-1} \setminus \mathcal{K}_{n-2})$, and is transverse to the centre of every $(n-1)$-cell. Moreover, it is not hard to check that, by construction (X, s_X), is bordant with (N_1, s_1), so that also all stable η-characteristic numbers of N_1 vanish.*

Now we proceed by induction on the codimension of the skeleton to eventually reach (N_n, s_n), which takes values in $\mathcal{K}_0$ and is bordant with (N_{n-1}, s_{n-1}) (hence with the initial (X, s_X)). As the Grassmannian is connected, (N_n, s_n) will be homotopic to a required constant map (N, c), $N = N_n$ (this last step is not necessary if we use the natural cellular decomposition which has only one 0-cell).

So let us assume inductively that for some $h \geq 1$ we have obtained

$$s_h = s_{N_h} : N_h \to \mathcal{K}_{n-h} \subset \mathfrak{G}_{m+1,n+1}$$

bordant with (X, s_X), which is smooth on $s_h^{-1}(\mathcal{K}_{n-h} \setminus \mathcal{K}_{n-h-1})$, transverse to the centre x_e of every $(n-h)$-cell e, and such that the stable η-characteristic numbers of N_h vanish.

By a similar argument as above, for every such a cell e, there is a collapsing projection

$$p_e : \mathcal{K}_{n-h} \to S^{n-h}$$

which restricts to a smooth embedding of the cell e to $\mathbb{R}^{n-h} \subset \mathbb{R}^{n-h} \cup \infty = S^{n-h}$; by confusing x_e with $p_e(x_e)$, set $Y = (p_e \circ s_h)^{-1}(x_e)$. Using the terminology that we will define in Chapter 17, Y is *framed*; that is, it has a trivialized tubular neighbourhood $U \sim Y \times D^{n-h}$ in N_h such that the restriction of s_h to U can be identified with the projection $Y \times D^{n-h} \to D^{n-h}$, where D^{n-h} is a small disk in e around x_e. We claim that this h-submanifold Y of N_h is a boundary. Let us conclude assuming this fact. Then Y is a boundary of a manifold W. We make a surgery on N_h by replacing the above product neighbourhood $U \sim Y \times D^{n-h}$ with $W \times \partial D^{n-h}$; do it simultaneously at every $(n-h)$-cell. We get a manifold N_{h+1}; the map s_h descends to $s_{h+1} : N_{h+1} \to \mathfrak{G}_{m+1,n+1}$, which can be identified with the projection $W \times \partial D^{n-h} \to \partial D^{n-h}$ at every $(n-h)$-cell. By construction (N_{h+1}, s_{h+1}) is bordant with (N_h, s_h) and this eventually achieves the inductive step. Finally, let us prove now that Y is a boundary. By the inductive assumption of Lemma 16.5, it is enough to show that every stable η-characteristic number of Y vanishes. As Y is framed in N_h, this implies that a stable classifying map s_Y for Y is given by $s_h \circ j$, where $j : Y \to N_h$ is the inclusion. We have to show that for every $\alpha \in \eta^h(\mathfrak{G}_{m+1,n+1})$,

$$s_Y^*(\alpha) \sqcap [Y] = 0 \in \mathbb{Z}/2\mathbb{Z} \ .$$

As $s_Y = s_h \circ j$, the geometric definition of the cobordism products implies that

$$s_Y^*(\alpha) \sqcap [Y] = s_h^*(\alpha) \sqcap j_*([Y]) = s_h^*(p_e^*[x_e] \sqcup \alpha) \sqcap [N_h] \in \mathbb{Z}/2\mathbb{Z} \ ,$$

the last term vanishes, being a stable η-characteristic number of N_h.

The proofs of Lemma 16.5 and of Theorem 16.3 are now complete.

■

16.4. On parallelizable manifolds

Recall that a compact boundaryless n-manifold X is parallelizable if the tangent bundle admits a global trivialization so that its total space is diffeomorphic to $X \times \mathbb{R}^n$; in such a case, X is orientable. If X is parallelizable, then any classifying map $t_X : X \to \mathfrak{G}_{m,n}$ of $T(X)$ is homotopic to a constant map as well as any stable classifying map $s_X : X \to \mathfrak{G}_{m+1,n+1}$. If X is parallelizable and $\dim X = n > 0$, certainly it satisfies the hypothesis of Lemma 16.4, hence $[X] = 0 \in \eta_n$. We can strengthen this result.

PROPOSITION 16.6. *Let X be a parallelizable and oriented compact boundaryless n-manifold, $n > 0$. Then $[X] = 0 \in \Omega_n$.*

Proof : It is enough to prove the statement when X is connected. We will use and refine the proof of Lemma 16.4. If $\dim X = n$ is *even*, we can apply such a proof starting from a homotopy $F : X \times [0,1] \to \mathfrak{G}_{m+1,n+1}$ between s_X and a constant map. Clearly $X \times [0,1]$ is orientable. At the end of the proof, we may assume that both $\mathbf{P}^a(\mathbb{R})$ and $\mathbf{P}^{a-n}(\mathbb{R})$ are odd-dimensional, hence they are both orientable. We conclude using the oriented version of the transversality theorems.

If $\dim X = n$ is *odd*, we modify the construction as follows: we consider

$$Y = X \times X$$

endowed with the involution $\sigma(x, y) = (y, x)$. The fixed point set consists of the diagonal Δ_X which is naturally identified with X itself. A tubular neighbourhood U of Δ_X can be identified with the unitary disk bundle of $T(X)$, hence with the product $X \times D^n$. By removing the interior of U from Y, we get a compact $2n$-manifold W with boundary $\partial W = X \times S^{n-1}$; σ restricts to a fixed point free involution on W, and it can be identified with the fibrewise antipodal map on ∂W, that is the trivial unitary sphere bundle of $T(X)$. Then the proof runs similarly to the one of Lemma 16.4. In the end, we can assume that both $\mathbf{P}^a(\mathbb{R})$ and $\mathbf{P}^{a-n+1}(\mathbb{R})$ are orientable and conclude again by oriented transversality.

■

16.5. On Ω-characteristic numbers

Via the forgetting projection $\Omega^\bullet \to \eta^\bullet$, every stable η-characteristic number lifts to a stable Ω-characteristic number, with the obvious meaning of the term.

If the manifold X is oriented we can consider also the *complexification* $T_{\mathbb{C}}(X)$ of the tangent bundle: every real vector bundle ξ can be complexified to $\xi_{\mathbb{C}}$ via the inclusion $\mathbb{R} \subset \mathbb{C}$, so that every real cocycle defining ξ can be considered as a cocycle defining $\xi_{\mathbb{C}}$. Then $T_{\mathbb{C}}(X)$ corresponds to a classifying map

$$t_{X,\mathbb{C}} : X \to \mathfrak{G}_{m,n}(\mathbb{C}) \ .$$

We can repeat almost *verbatim* the above definition of stable characteristic numbers in the complexified setting. This gives rise to further stable Ω-characteristic numbers with values in $\mathbb{Z}$ instead of $\mathbb{Z}/2\mathbb{Z}$. We call generically *stable Ω-characteristic numbers* any one belonging to the union of such two families. One would wonder that an Ω-analogous of Theorem 16.3 holds as well in terms of these Ω-characteristic numbers; unfortunately, the situation is more complicated. The classical treatment of the stable characteristic numbers (see [**MS**], [**BT**]) is developed using the *singular cohomology ring* of real or complex Grassmannians with $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$ coefficients (instead of cobordism). They are called *Stiefel-Whitney* and *Pontryagin* numbers, respectively. All this is part of the theory of *Stiefel-Whitney* or *Pontryagin* classes of vector bundles, which we do not develop in this text. Above, we have just 'lifted' some facts of this theory in terms of the cobordism rings that we have introduced from scratch. In the case of η, "to lift" is a quite appropriate term because, by the considerations in the introduction of the chapter, we can prove that, for every compact boundaryless n-manifold X, there is a *ring epimorphism* from the η-cobordism to the $\mathbb{Z}/2\mathbb{Z}$-cohomology of X. Hence, stable η-characteristic numbers and SW-numbers essentially are the same thing. Incidentally, using the properties of the SW classes, one can prove that $\chi_{(2)}$ is a *stable* characteristic number.

In the oriented case, the Ω-characteristic numbers do not recover all Pontryagin numbers. We cannot avoid cohomology in the oriented setting. By using Stiefel-Whitney and Pontryagin numbers, we have the following oriented version of Theorem 16.3. The proof [**Wall**] is quite complicated. Parallelizable manifolds as in Proposition 16.6 represent the basic instance for this theorem.

THEOREM 16.7. *Let X be a compact oriented boundaryless n-manifold. Then $[X] = 0 \in \Omega_n$ if and only if all Stiefel-Whitney and Pontryagin numbers of X vanish.*

CHAPTER 17

The Pontryagin-Thom construction

The original Pontryagin construction was invented to rephrase the homotopy groups of spheres in terms of a certain, more geometric bordism theory, presumably more accessible at that time (about 1938). *Viceversa*, later Thom's extension of Pontryagin construction was mainly intended as a way to rephrase the bordism ring $\eta^\bullet$ (or $\Omega^\bullet$) in terms of the homotopy groups of certain so-called Thom's spaces, more accessible at that time after the impressive progress in homotopy theory since Serre's Thesis ([**Se**] about 1954).

Let us start by describing the Pontryagin construction (see the later exposition in [**Pont**] and also [**M1**], [**DNF**] Chapter 5 of the second part). We are primarily interested here in the determination of the homotopy groups

$$\pi_m(S^n, p)$$

for $m, n \geq 1$. We know that we can manage with them in a purely differential topological way. We know that

$$\pi_1(S^1, p) \sim \mathbb{Z},\ \pi_m(S^1, p) = 1 \text{ for } m > 1,$$

$$\pi_m(S^n, p) = 1 \text{ for } n \geq 2,\ 1 \leq m < n\ .$$

Hence we will assume that $m \geq n > 1$. In such a case, $\pi_m(S^n, p)$ is Abelian, the base point is immaterial and the group can be identified with $[S^m, S^n]$, the set of smooth homotopy classes of maps $f : S^m \to S^n$. Moreover, it is convenient to extend the discussion to $[M, S^n]$ where M is any *compact, connected boundaryless* smooth m-manifold, $m \geq n \geq 1$.

17.1. Embedded and framed bordism

In Chapter 13, we have already encountered instances of embedded bordism within a given manifold. Let us state it in general.

Definition 17.1. Let M be a compact connected boundaryless m-manifold. Let $0 \leq k < m$. Given compact boundaryless smooth k-submanifolds V_0, V_1 of M, we say that V_0 *is bordant with* V_1 *within* M (and we write $V_0 \sim_{b,M} V_1$) if there is a smooth triad (W, V_0, V_1), properly embedded in $M \times [a_0, a_1]$, for some $a_0 < a_1$, such that for $j = 0, 1$,

$$\partial W \cap (M \times \{a_j\}) = V_j\ .$$

The relation "$\sim_{b,M}$" is an equivalence relation on the set of compact boundaryless k-submanifolds of M: every such a V is in relation with itself because the cylinder $V \times [a_0, a_1]$ properly embeds in $M \times [a_0, a_1]$; the relation is obviously symmetric; as for the transitivity, up to isotopy we can normalize the proper embeddings of the triads (W, V_0, V_1) in such a way that they are locally cylinder-like near the boundary. Given properly embedded triads (W, V_0, V_1) in $M \times [a_0, a_1]$ and (W', V_0', V_1') in $M \times [a_0', a_1']$, such that $V_1 = V_0'$, then we can construct (W'', V_0, V_1') in $M \times [a_0, a_1 + a_1' - a_0']$ just by stacking $M \times [a_0', a_1']$ over $M \times [a_0, a_1]$.

We denote by

$$\eta_k^{\mathrm{emb}}(M)$$

the quotient set.

By restriction to *oriented* k-submanifolds of M, we can get an oriented version of the above definition leading to a quotient set

$$\Omega_k^{\mathrm{emb}}(M) \ .$$

Note that we are not assuming here that M is oriented.

Let M be as above.

Definition 17.2. A compact boundaryless k-submanifold $V \subset M$ is *framed* if it is endowed with a *framing*. This last is of the form

$$\mathfrak{f} = (s_1, \dots, s_{m-k}) \ ,$$

where

(1) Every s_j is a nowhere vanishing section of the bundle $i_V^* T(M)$,
$$i_V : V \to M$$
being the inclusion;
(2) For every $x \in V$, the vectors $s_1(x), \dots, s_{m-k}(x)$ are linearly independent in $T_x M$;
(3) For every $x \in V$, $T_x M = T_x V \oplus F_x$, where
$$F_x := \mathrm{Span}\{s_1(x), \dots, s_{m-k}(x)\} \ .$$

Hence $x \to F_x$ defines a smooth field of transverse $(m-k)$-planes along V tangent to M and the framing provides a global trivialization of the associated normal bundle F, so that

$$i_V^* T(M) \sim T(V) \oplus F \sim T(V) \oplus \epsilon^{m-k} \ ,$$

thereby providing a global trivialization of every tubular neighbourhood of V in M constructed using such a field. This means, in particular, that a necessary (and sufficient) condition so that V admits a framing is that it has globally trivializable tubular neighbourhoods in M.

We are going to specialize and enhance the embedded bordism to framed submanifolds. First, let us extend the definition of framing to properly embedded triads. Let (W, V_0, V_1) be a properly embedded $(k+1)$-triad in

$M \times [a_0, a_1]$; from now on we will assume by default that the embedding is normalized (i.e. cylinder-like near the boundary as above). A framing of the triad in $M \times [a_0, a_1]$ is of the form

$$\mathfrak{f} = (s_1, \dots, s_{m-k}) \ ,$$

where these are point-wise linearly independent sections of the bundle

$$i_W^* T(M \times [a_0, a_1]) \ .$$

They induce a smooth field of transverse $(m-k)$-planes along W tangent to $M \times [a_0, a_1]$, and we require furthermore that the restriction of $\mathfrak{f}$ to the boundary defines a framing of V_j in M, $j = 0, 1$.

DEFINITION 17.3. Let $(V_0, \mathfrak{f}_0)$ and $(V_1, \mathfrak{f}_1)$ be framed k-submanifolds of M. We say that $(V_0, \mathfrak{f}_0)$ *is framed bordant with* $(V_1, \mathfrak{f}_1)$ *within* M , and we write

$$(V_0, \mathfrak{f}_0) \sim_{fb} (V_1, \mathfrak{f}_1)$$

if there is a properly embedded framed triad $((W, V_0, V_1), \mathfrak{f}_W)$ in some $M \times [a_0, a_1]$ such that the restriction of the framing $\mathfrak{f}_W$ to the boundary coincides with the union of the framings $\mathfrak{f}_0$ and $\mathfrak{f}_1$.

As above, we check that this defines an equivalence relation on the set of framed k-submanifolds of M, and we denote by

$$\eta_k^{\mathcal{F}}(M)$$

the quotient set.

REMARK 17.4. If M is *oriented*, then every framed submanifold $(V, \mathfrak{f})$ is naturally oriented itself using the orientation procedure stated in Theorem 8.2 (every T_xV is endowed with the orientation such that when followed by the orientation of F_x determined by $\mathfrak{f}(x)$ it recovers the orientation of T_xM). Similarly any framed triad $((W, V_0, V_1), \mathfrak{f}_W)$ properly embedded in some $M \times [a_0, a_1]$ dominates an embedded *oriented* bordism of V_0 with V_1. Hence, if M is oriented, we can write $\Omega_k^{\mathcal{F}}(M)$ instead of $\eta_k^{\mathcal{F}}(M)$.

17.2. The Pontryagin map

Let us keep the above setting. We establish the following procedure.

• Fix $x_0 \in S^n$. For every $\alpha \in [M, S^n]$, thanks to transversality take $f : M \to S^n$ belonging to α and such that $f \pitchfork \{x_0\}$.

• $V := f^{-1}(x_0)$ is submanifold of M of dimension $\dim V = k := m - n$. Fix a positive basis $\mathcal{B}$ of $T_{x_0}S^n$ (as usual, the unitary sphere is the oriented boundary of the unit disk D^{n+1} of $\mathbb{R}^{n+1}$ endowed with the standard orientation). For every $x \in V$, set

$$\mathfrak{f}(x) = (d_x f)^{-1}(\mathcal{B}) \ ;$$

by the very definition of transversality, this defines a framing $\mathfrak{f}$ of V in M. Hence we have constructed a framed k-submanifold $(V, \mathfrak{f})$ of M. We denote by $[V, \mathfrak{f}]$ its class in $\eta_k^{\mathcal{F}}(M)$.

PROPOSITION 17.5. *Let M be a connected boundaryless compact smooth m-manifold M, $m \geq n$, $k = m - n$. Let us associate to every $\alpha \in [M, S^n]$ a class $\mathfrak{p}(\alpha) = [V, \mathfrak{f}] \in \eta_k^{\mathcal{F}}(M)$ by means of an arbitrary implementation of the procedure stated above. Then this well defines the* Pontryagin map

$$\mathfrak{p} : [M, S^n] \to \eta_k^{\mathcal{F}}(M) \ .$$

Proof : Every implementation of the procedure involves a few arbitrary choices. We have to check that they are immaterial with respect to the framed bordism class of the resulting framed manifold $(V, \mathfrak{f})$. Given $\alpha \in [M, S^n]$, let us assume first that two implementations just differ by the choice of the maps f_0 and f_1 in α and transverse to $x_0 \in S^n$. By the basic transversality theorems, we can assume that a homotopy $F : M \times [0, 1] \to S^n$ which connects f_0 to f_1 is also transverse to $x_0 \in S^n$; hence $W = F^{-1}(x_0)$ endowed with the framing $x \to (d_x F)^{-1}(\mathcal{B})$ gives rise to a framed cobordism between $(V_0, \mathfrak{f}_0)$ and $(V_1, \mathfrak{f}_1)$ constructed by means of f_0 and f_1 respectively. Assume now that the two implementations just differ by the choice of the positive bases $\mathcal{B}_0$ and $\mathcal{B}_1$ of $T_{x_0} S^n$. Then the resulting framed manifolds $(V, \mathfrak{f}_0)$ and $(V, \mathfrak{f}_1)$ just differ by the framing. As $\mathrm{GL}^+(n, \mathbb{R})$ is connected, there is a smooth path $\mathcal{B}_t$, $t \in [0, 1]$, of such bases connecting $\mathcal{B}_0$ and $\mathcal{B}_1$. Clearly this gives rise to a 1-family of framed manifolds of the form $(V, \mathfrak{f}_t)$, and eventually to a framing of $V \times [0, 1]$ properly embedded in $M \times [0, 1]$ which realizes a framed bordism between $(V, \mathfrak{f}_0)$ and $(V, \mathfrak{f}_1)$. Finally, let us assume that we deal with two different points $x_0, x_1 \in S^n$. By the homogeneity of S^n, there is a diffeotopy h_t, $t \in [0, 1]$, of S^n such that $h_0 = \mathrm{Id}_{S^n}$, $h_1(x_0) = x_1$. Given $f_0 \in \alpha$, $f_0 \pitchfork \{x_0\}$, clearly also $f_1 := h_1 \circ f_0$ belongs to α and $f_1 \pitchfork \{x_1\}$. Thanks to the above results, it is enough to show that the framed manifold $(V_0, \mathfrak{f}_0)$ constructed by using $x_0, \mathcal{B}, f_0$ and the framed manifold $(V_1, \mathfrak{f}_1)$ constructed by means of $x_1, \mathcal{B}_1 := d_{x_0} h_1(\mathcal{B}), f_1$ belong to the same framed cobordism class. This is easy to achieve by using the 1-parameter family of framed manifolds $(V_t, \mathfrak{f}_t)$ constructed by means of $x_t := h_t(x_0), \mathcal{B}_t := d_{x_0} h_t(\mathcal{B}), f_t := f_0 \circ h_t$. The proposition is proved.

■

We can state the main result about this Pontryagin construction.

THEOREM 17.6. *Let M be a compact, connected and boundaryless smooth m-manifold, $m \geq n \geq 1$, $k = m - n$. Then the Pontryagin map*

$$\mathfrak{p} : [M, S^n] \to \eta_k^{\mathcal{F}}(M)$$

is bijective.

REMARK 17.7. Following Remark 17.4, if M is oriented then the Pontryagin map takes values in $\Omega_k^{\mathcal{F}}(M)$ and is bijective.

Before giving a proof, let us state immediately an interesting corollary, earlier due to Hopf.

COROLLARY 17.8. *Assume that* $\dim M = \dim S^n \geq 1$.

1) If M is oriented, then $f_0, f_1 : M \to S^n$ are homotopic to each other if and only if

$$\deg_{\mathbb{Z}}(f_0) = \deg_{\mathbb{Z}}(f_1) \ .$$

2) If M is nonorientable, then $f_0, f_1 : M \to S^n$ are homotopic to each other if and only if

$$\deg_{\mathbb{Z}/2\mathbb{Z}}(f_0) = \deg_{\mathbb{Z}/2\mathbb{Z}}(f_1) \ .$$

Proof : As M and the sphere have the same dimension, the respective framed manifolds $(V_0, \mathfrak{f}_0)$ and $(V_1, \mathfrak{f}_1)$ constructed by means of f_0 or f_1 consist of a finite number of (possibly oriented) points. It follows from the very definition of $\deg_R$, $R = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$, that they are framed bordant (possibly in the oriented setting) if and only if the two maps have the same degree. The result follows by Theorem 17.6.

■

COROLLARY 17.9. *If M is oriented, the map* $\deg_{\mathbb{Z}} : [M, S^n] \to \mathbb{Z}$ *is bijective.*

Proof : We already remarked in Section 12.5 that it is surjective. By the above corollary, it is also injective.

■

Proof of Theorem 17.6: Let us show first that the Pontryagin map is surjective. Let $(V, \mathfrak{f})$ be a framed k-submanifold of M. It is enough to prove that there is a map $f : M \to S^n$ such that $[(V, \mathfrak{f})]$ is produced by some implementation of the procedure used to define the Pontryagin map, starting from the map f. As usual, let us decompose the sphere as $S^n = D^+ \cup D^-$, such that $D^+ \cap D^- = S^{n-1}$. By the stereographic projection from the northern pole, we can identify D^- with the unit disk D^n; take $x_0 = 0 \in D^n \subset S^n$. By using the framing $\mathfrak{f}$, we can define a global trivialization

$$\tau : V \times D^n \to U$$

of a tubular neighbourhood of V in M, such that the restriction of τ to $V \times \{0\}$ is the identity. Then we can define the map

$$\tilde{f} : U \to D^n, \ \tilde{f}(u) := \pi \circ \tau^{-1} \ ,$$

π being the projection $V \times D^n \to D^n$. By construction,

- $\tilde{f} \pitchfork \{0\}$.
- $\tilde{f}^{-1}(0) = V$.
- Up to framed bordism (use again that $\mathrm{GL}^+(n, \mathbb{R})$ is connected), the framing $\mathfrak{f}$ can be recovered by the usual construction applied to 0, $\tilde{f}$ and a basis $\mathcal{B}$ of $T_0 D^n$.

By using a collar of ∂U in M and a collar bump function, it is not hard to extend $\tilde{f}$ to a smooth map

$$f : M \to S^n$$

such that

- $f = \tilde{f}$ on U;
- The map f sends the complement of U in D^+ and is constantly equal to the northern pole of S^n, say ∞, on the complement of a slightly bigger tubular neighbourhood of V in M;
- $f^{-1}(0) = \tilde{f}^{-1}(0) = V$.

By construction, the map f has the desired property. So we have proved that the Pontryagin map is surjective.

Let us prove now that it is injective. Let us say that a map $f : M \to S^n$ is *in standard form* if it has the qualitative properties of the map f constructed above to prove the surjectivity. Let us prove first the result for the homotopy classes that admit representatives in standard form.

LEMMA 17.10. *Assume that $f_0, f_1 : M \to S^n$ are in standard form, let α_0 and α_1 be the respective homotopy classes, and assume that $\mathfrak{p}(\alpha_0) = \mathfrak{p}(\alpha_1)$. Then $\alpha_0 = \alpha_1$.*

Proof : Let $(V_0, \mathfrak{f}_0)$ and $(V_1, \mathfrak{f}_1)$ be framed manifolds obtained by implementing the procedure with respect to 0, $\mathcal{B}$ and f_0 or f_1. By hypothesis there is a properly embedded framed triad $((W, V_0, V_1), \mathfrak{f}_W)$ in $M \times [0,1]$ which realizes a framed bordism between them. Let us apply to the triad the construction used above to define $\tilde{f}$. This produces a suitable map

$$\tilde{F} : U_W \to D^n ,$$

where U_W is a properly embedded relative tubular neighbourhood of W in $M \times [0,1]$ which restricts to tubular neighbourhoods U_j of V_j in M, $j = 0, 1$. As we have extended above $\tilde{f}$ to $f : M \to S^n$ (in normal form), we can extend $\tilde{F}$ to

$$F : M \times [0,1] \to S^n$$

in relative normal form with respect to U_W. As f_0 and f_1 are themselves in normal form by hypothesis, up to diffeotopy we can assume that the restriction of F to the boundary recovers the given maps f_0 and f_1. Then F establishes a required homotopy between them.

■

To achieve the proof of the main theorem, it is enough to prove that the assumptions in the above lemma are not restrictive. Let $g : M \to S^n$. It is not restrictive to assume that $g \pitchfork 0$. Let $(V, \mathfrak{f})$ be obtained by implementing the usual procedure using 0, $\mathcal{B}$ and g. Let $f : M \to S^n$ be a map in normal form obtained from $(V, \mathfrak{f})$ as in the proof of surjectivity. Up to diffeotopy, we can assume that the tubular neighbourhood U of V which supports $\tilde{f}$ coincides with $g^{-1}(D^-)$ and that eventually g and f coincide on U. Both f and g send the complement of U in D^+ which retracts to ∞. Using these facts, it is an exercise to show that f and g are homotopic. This completes the proof of the main Theorem 17.6.

■

17.3. Characterization of combable manifolds

Recall that a manifold is *combable* if it carries a nowhere vanishing tangent vector field. We are now able to characterize this property.

THEOREM 17.11. *Let M be a compact connected boundaryless smooth manifold. Then M is combable if and only if $\chi(M) = 0$. In particular, if $m = \dim M$ is odd, then M is combable.*

Proof : We already know that $\chi(M) = 0$ is a necessary condition. Let us prove the other implication. Let $\mathfrak{v}$ be any tangent vector field on M with isolated zeros. By using the homogeneity of M, up to a diffeotopy we can assume that there is a chart $\phi : W \to \mathbb{R}^m$ such that the zeros $x_1, \dots, x_k$ of $\mathfrak{v}$ are contained in W and their images are contained in the unitary disk $D^m \subset \mathbb{R}^m$. For simplicity, let us keep the name $\mathfrak{v}$ for its expression in such local coordinates, and x_j for the images of the zero sets in D^m. We can fix an auxiliary Riemannian metric g on M, which restricts as the standard Euclidean metric g_0 on a neighbourhood of D^m. Fix a system of small pairwise disjoint disks $D_j \subset D^m$, centred at the x_j, $j = 1, \dots, k$. The field $\hat{\mathfrak{v}} := \mathfrak{v}/||\mathfrak{v}||_g$ is well defined on $M \setminus \cup_j \text{Int}(D_j)$ and homotopic to the restriction of $\mathfrak{v}$. The restriction of $\hat{\mathfrak{v}}$ to $D^m \setminus \cup_j \text{Int}(D_j)$ defines a map

$$\rho : D^m \setminus \cup_j \text{Int}(D_j) \to S^{m-1} \ .$$

Assume first that M is oriented. By the bordism invariance of the degree, we have

$$\deg_{\mathbb{Z}}(\rho_{|\partial D^m}) = \sum_j \deg_{\mathbb{Z}}(\rho_{|\partial D_j})$$

and the second term is equal to $\chi(M) = 0$. By Corollary 17.8, $\rho_{|\partial D^m}$ is homotopically trivial, hence can be extended to a map $\hat{\rho} : D^m \to S^{m-1}$. By matching this last map with the restriction of $\hat{\mathfrak{v}}$ to $M \setminus \text{Int}(D^m)$, we eventually get a nowhere vanishing vector field on M. If M is not orientable, arguing similarly to the proof of Proposition 7.8, we can assume that the local picture at D^m agrees with the one in the oriented case and we can conclude as well.

■

The above result extends to triads with very similar proof.

PROPOSITION 17.12. *A smooth triad (W, V_0, V_1) carries a nowhere vanishing triad tangent vector field if and only if the relative characteristic $\chi(W, V_0) = 0$.*

17.4. On (stable) homotopy groups of spheres

In accordance with the basic motivation of the Pontryagin construction, let us manage with

$$\pi_m(S^n) \sim [S^m, S^n] \sim \Omega^{\mathcal{F}}_{m-n}(S^m)$$

for $m \geq n > 1$, in terms of framed bordism. The first step is to transport on $\Omega^{\mathcal{F}}_{m-n}(S^m)$ the group operation of $\pi_m(S^n)$. Recall that the operation of the ordinary bordism modules is induced by the disjoint union of representatives; moreover, disjoint union and connected sum belong to the same bordism class. This implies that every ordinary bordism class can be represented by *connected* manifolds. The operation of the framed bordism of the spheres is an embedded version of the disjoint union, again with the help of connected sum. Let $(V_1, \mathfrak{f}_1)$ and $(V_2, \mathfrak{f}_2)$ be oriented framed $(m-n)$-submanifolds of S^m; then the operation on $\Omega^{\mathcal{F}}_{m-n}(S^m)$ is defined by

$$[V_1, \mathfrak{f}_1] + [V_2, \mathfrak{f}_2] = [(V_1, \mathfrak{f}_1) \amalg (V_2, \mathfrak{f}_2)]$$

where we assume at first that the given framed manifolds are embedded in two disjoint copies of S^m, and the disjoint union $(V_1, \mathfrak{f}_1) \amalg (V_2, \mathfrak{f}_2)$ means the framed submanifold of

$$S^m = S^m \# S^m \ ,$$

understanding that the connected sum is performed at disks which are respectively disjoint from the two given framed submanifolds. It is not hard to verify that this operation is well defined and recovers (via the Pontryagin map) the usual operation of the homotopy group $\pi_m(S^n)$.

Remark 17.13. In the ordinary setting, we have noticed that every class has connected representatives. Through embedded connected sums performed by attaching embedded 1-handles, we can obtain that also every class in $\Omega^{\mathcal{F}}_k(S^m)$, $k > 0$, has a representative $[V, \mathfrak{f}]$ with connected V. This is easy if we forget the framing, but more complicated taking it into account (see [**Pont**] or [**DNF**] second part, Chapter 5, Section 23).

Proposition 17.14. *For every $m \geq 2$, $\deg : \pi_m(S^m) \to \mathbb{Z}$ is an isomorphism of $\mathbb{Z}$-modules, and $[S^m, \mathrm{id}_{S^m}]$ is a generator of $\pi_m(S^m)$.*

The same result was already known for $m = 1$.

17.4.1. The J-homomorphism. For every $m, n \geq 1$, there is an important homomorphism (earlier defined by Whitehead)

$$J : \pi_m(SO(n)) \to \pi_{m+n}(S^n)$$

which can be naturally expressed in terms of

$$J : \pi_m(SO(n)) \to \Omega^{\mathcal{F}}_m(S^{m+n}) \ .$$

In fact, by taking the usual equatorial embedding $S^m \subset S^{m+n}$, every $a \in \alpha \in \pi_m(SO(n))$ can be considered as a framing $\mathfrak{f}_a$ of S^m in S^{m+n}; hence $J(\alpha) = [S^m, \mathfrak{f}_a]$ is well defined.

17.4.2. Freudenthal's homomorphism, stable homotopy groups. Let $S^m \subset S^{m+1}$ be the usual equatorial embedding. Set $m = k+n$, so that $m+1 = k+(n+1)$. If $(V, \mathfrak{f})$ is an oriented framed k-submanifold of S^m, then we can consider the framed k-submanifold of S^{m+1}, $(V, \mathfrak{s}\mathfrak{f})$, where the framing $\mathfrak{s}\mathfrak{f}$ is obtained by completing $\mathfrak{f}$ with the unitary normal vectors along S^m which point toward the northern pole of S^{m+1}. It is an easy consequence of the definition of the operation, that this induces a $\mathbb{Z}$-modules homomorphism

$$\mathfrak{s} : \Omega_k^{\mathcal{F}}(S^m) \to \Omega_k^{\mathcal{F}}(S^{m+1}) \ ;$$

that is, via the Pontryagin map, a homomorphism

$$\mathfrak{s} : \pi_{n+k}(S^n) \to \pi_{n+1+k}(S^{n+1}) \ ,$$

called *Freudenthal suspension homomorphism.* Using the same "general position argument" employed for the weak Whitney embedding theorem (Corollary 6.29), we have:

PROPOSITION 17.15. *For every $k \geq 1$,*
1) If $n \geq k+1$ then

$$\mathfrak{s} : \pi_{n+k}(S^n) \to \pi_{n+1+k}(S^{n+1})$$

is surjective;
2) If $n \geq k+2$ then

$$\mathfrak{s} : \pi_{n+k}(S^n) \to \pi_{n+1+k}(S^{n+1})$$

is an isomorphism.

We say that, for every $k \geq 0$, the homotopy groups $\pi_{n+k}(S^n)$ *stabilize* for $n \geq k+2$, being all isomorphic to the (by definition) *stable homotopy group* denoted by π_k^∞.

By keeping the above notations, it is convenient to organize the groups

$$\pi_{n+k}(S^n) \sim \Omega_k^{\mathcal{F}}(S^{n+k})$$

as being indexed by the couples of integers (k, n), $k \geq 0$, $n \geq 2$, endowed with the lexicographic order. So for every k, by increasing n we encounter a few groups in the "unstable regime", until we reach

$$\pi_k^\infty \sim \pi_{2+2k}(S^{k+2}) \sim \Omega_k^{\mathcal{F}}(S^{2+2k}) \ .$$

17.4.3. Homotopy groups of spheres for small k. The descriptor "small" here will mean $k \leq 3$. Pontryagin himself succeeded to compute the cases $k \leq 2$. We will give some information about these cases. Certain key constructions will be described with some details, but we will omit many verifications, especially when $k = 2$. Our presentation is similar to the one in Chapter 5 of the second part of [**DNF**]. The reader could try to fill the missed facts or refer to the exposition [**Pont**] which contains detailed proofs.

($k = 0$) In agreement with Proposition 17.14, the situation stabilizes immediately:

$$\pi_0^\infty \sim \pi_2(S^2) \sim \mathbb{Z} .$$

($k = 1$) The group in the unstable regime is

$$\pi_3(S^2) \sim \Omega_1^{\mathcal{F}}(S^3)$$

while

$$\pi_1^\infty \sim \Omega_1^{\mathcal{F}}(S^4) \sim \pi_4(S^3) .$$

Let us analyze the first one. Every finite family of embedded r smooth circles in S^3 can be transformed into the boundary of r pairwise disjoint embedded smooth 2-disks by means of a generic homotopy which is an embedding for every $t \in [0,1]$, except for a finite number of t at which two branches of two circles (possibly the same one) cross each other with distinct tangents. Such a generic homotopy induces an embedded framed bordism. So $\Omega_1^{\mathcal{F}}(S^3)$ is generated by classes of the form $[S^1, \mathfrak{f}]$, where S^1 is the standard $S^1 \subset S^2 \subset S^3$ via equatorial embeddings; such representatives only differ by the framings. We can take as reference framing $\mathfrak{f}_0$, the one having as first component a transverse field along a collar of S^1 in the standard 2-disk $D^+ \subset S^2$. In fact, $[S^1, \mathfrak{f}_0]$ corresponds to $1 \in \pi_3(S^2)$. In this way every framing is of the form $\mathfrak{f} = h_\mathfrak{f}\mathfrak{f}_0$ for a map

$$h_\mathfrak{f} : S^1 \to SO(2) .$$

As $SO(2) \sim S^1$, the class $\alpha_\mathfrak{f}$ of $h_\mathfrak{f}$ belongs to $\mathbb{Z} \sim \pi_1(SO(2))$. We claim that $[S^1, \mathfrak{f}_1] = [S^1, \mathfrak{f}_2] \in \Omega_1^{\mathcal{F}}(S^3)$ if and only if $\alpha_{\mathfrak{f}_1} = \alpha_{\mathfrak{f}_2}$. In fact, if $f : S^3 \to S^2$ corresponds to $(S^1, \mathfrak{f})$ via the Pontryagin construction, then we realize that $\alpha_\mathfrak{f}$ coincides with the linking number of two generic fibres of f over two distinct regular values (this is called the *Hopf number* of f). By the above consideration we have a well defined and surjective map

$$\pi_3(S^2) \to \Omega_1^{\mathcal{F}}(S^3) \sim \mathbb{Z} ;$$

using the interpretation of the linking number as the degree of the map defined at the end of Section 12.5 and the bordism invariance of the degree, we check directly that the above map is also injective. Eventually we have that

$$\pi_3(S^2) \sim \Omega_1^{\mathcal{F}}(S^3) \sim \mathbb{Z} .$$

We can also exhibit a very interesting geometric generator. This is the so-called *Hopf map*: let S^3 be realized as the unitary sphere in $\mathbb{C}^2$ and recall that

$$\mathbf{P}^1(\mathbb{C}) \sim S^2$$

the *Riemann sphere*. Then the mentioned map is

$$\mathfrak{h} : S^3 \to S^2$$

given by the restriction of the natural projection $\mathbb{C}^2 \setminus \{0\} \to \mathbf{P}^1(\mathbb{C})$. We can see that $\mathfrak{h}$ is a fibre bundle map with fibre S^1; the union of two distinct fibres is the so-called (oriented) *Hopf link* formed by two simply linked unknotted knots in S^3 with linking number equal to 1.

With similar and easier considerations (now every embedding of S^1 is "standard" by dimensional reasons), we see that $\Omega_1^{\mathcal{F}}(S^4)$ is generated by classes of the form $[S^1, \mathfrak{f}]$, and every framing induces a classifying map $\alpha_{\mathfrak{f}} \in \pi_1(S0(3))$; we know that $SO(3) \sim \mathbf{P}^3(\mathbb{R})$ (see Example 4.8), so that $\pi_1(SO(3)) \sim \mathbb{Z}/2\mathbb{Z}$, and eventually

$$\pi_1^\infty \sim \Omega_1^{\mathcal{F}}(S^4) \sim \pi_4(S^3) \sim \mathbb{Z}/2\mathbb{Z} .$$

Again we can exhibit geometric generators. We have

$$\mathfrak{s}^{n-2} : \pi_3(S^2) \to \pi_{n+1}(S^n) ;$$

then

$$\mathfrak{s}^{n-2}([\mathfrak{h}]) = [\mathfrak{h}_n]$$

for a suitable "suspended Hopf map"

$$\mathfrak{h}_n : S^{n+1} \to S^n$$

eventually generates $\pi_{n+1}(S^n)$ for $n \geq 3$.

($k = 2$) We have $\pi_4(S^2)$ and $\pi_5(S^3)$ in the unstable range, while $\pi_2^\infty \sim \pi_6(S^4)$. It turns out that they are all isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Again we can exhibit geometric generators. In fact, the class of the map

$$\mathfrak{g} := \mathfrak{h} \circ \mathfrak{h}_3 : S^4 \to S^2$$

generates $\pi_4(S^2)$, while

$$\mathfrak{s}^{n-2}([\mathfrak{g}]) := [\mathfrak{g}_n]$$

generates $\pi_{n+2}(S^n)$ for $n \geq 2$.

This is subtler to establish than the case $k = 1$. It is achieved via the following steps.

(a) The map

$$\pi_4(S^3) \to \pi_4(S^2),\ [\alpha : S^4 \to S^3] \to [\mathfrak{h} \circ \alpha]$$

is an isomorphism. Assuming it, $\pi_4(S^2) \sim \pi_4(S^3) \sim \mathbb{Z}/2\mathbb{Z}$ by the case $k = 1$.

(b) One constructs geometrically an explicit isomorphism

$$\delta : \pi_6(S^4) \to \mathbb{Z}/2\mathbb{Z} .$$

This determines the stable group π_2^∞.

(c) One shows that

$$\mathfrak{s} : \pi_4(S^2) \to \pi_5(S^3)$$

is surjective.

Assuming (a), (b) and (c), and recalling that $\mathfrak{s} : \pi_5(S^3) \to \pi_6(S^4)$ is surjective by Proposition 17.15, it follows that also $\pi_5(S^3) \sim \mathbb{Z}/2\mathbb{Z}$.

We limit to outline a proof of steps (a) and (b). Step (b) is where the Pontryagin construction is applied. Step (a) follows from some general facts in homotopy theory. Even omitting several verifications, we believe that some key ideas are rather transparent.

(a) A fundamental tool in homotopy theory is the so-called *homotopy long exact sequence of a fibre bundle* (see for instance [**Hu**], [**Hatch**]). We apply it to the Hopf fibration $\mathfrak{h} : S^3 \to S^2$ with fibre S^1; extract from the exact sequence the strings

$$\cdots \to \pi_n(S^1) \to \pi_n(S^3) \to \pi_n(S^2) \to \pi_{n-1}(S^1) \to \dots$$

where the middle homomorphism is $\mathfrak{h}_*$ induced by $\mathfrak{h}$. As $\pi_m(S^1) = 1$ for $m \geq 2$, we get that for $n \geq 3$,

$$\pi_n(S^3) \sim \pi_n(S^2) \ ;$$

in particular,

$$\pi_4(S^3) \sim \pi_4(S^2)$$

as desired. Note that this also proves again that $\pi_2(S^3) = \pi_2(S^2) \sim \mathbb{Z}$.

(b) This is the most interesting step. To construct the isomorphism δ we will use several facts about surfaces discussed in Chapter 15. Let $(V, \mathfrak{f})$ be a framed surface in S^6, representing a class in $\Omega_2^{\mathcal{F}}(S^6)$. We can assume that V is connected (recall Remark 17.13), then it is orientable of a certain genus $g \geq 0$. By dimensional reasons, up to diffeotopy V is embedded in a standard way in $S^3 \subset S^6$. So only the framing contribution is relevant. Let C be a compact oriented smooth circle on V. The restriction of the framing $\mathfrak{f} = (s_1, \dots, s_4)$ to C can be completed by adding s_5, that is a normal field along C tangent to V which, together with an oriented field tangent to C, gives the orientation of T_xV at every $x \in C$. In this way we have constructed a framed circle $(C, \mathfrak{f}_C)$ representing an element of $\Omega_1^{\mathcal{F}}(S^6) \sim \mathbb{Z}/2\mathbb{Z}$. Hence we can associate to $(C, \mathfrak{f}_C)$ the corresponding value $q(C) := q([C, \mathfrak{f}_C]) \in \mathbb{Z}/2\mathbb{Z}$. Such a value does not depend on the orientation of C. If $L = \amalg_j C_j$ is a disjoint union of smooth circles on V, set

$$q(L) := \sum_j q(C_j) \in \mathbb{Z}/2\mathbb{Z} \ .$$

LEMMA 17.16. *The map*

$$q_{(V,\mathfrak{f})} : \eta_1(V) \to \mathbb{Z}/2\mathbb{Z}, \ q_{(V,\mathfrak{f})}(\alpha) = q(C) \ ,$$

provided that C is any smooth circle on V which represents α, is a well defined quadratic enhancement of $(\eta_1(V), \bullet)$

To prove the lemma, one verifies that the function q defined so far satisfies the conditions stated at the end of Chapter 15.

We can associate to $(V, \mathfrak{f})$, the Arf invariant $\mathrm{Arf}(q_{(V,\mathfrak{f})}) \in \mathbb{Z}/2\mathbb{Z}$.

PROPOSITION 17.17.

$$\delta : \Omega_2^{\mathcal{F}}(S^6) \to \mathbb{Z}/2\mathbb{Z}, \ \delta(\alpha) = \mathrm{Arf}(q_{(V,\mathfrak{f})})$$

is a well defined isomorphism, provided that $(V,\mathfrak{f})$ *represents* α *and* V *is connected.*

Thus $\Omega_2^{\mathcal{F}}(S^6)$ is isomorphic to the Witt group $W_q^{\mathbf{H}}(\mathbb{Z}/2\mathbb{Z})$ and realizes in a geometric way the formal nontrivial enhancement of $\Omega_2 = 0$ mentioned in Section 15.6. The group $\Omega_2^{\mathcal{F}}(S^6)$ is generated by a framed torus $S^1 \times S^1$ embedded in the standard way in $S^3 \subset S^6$, such that the framing realizes $\mathbf{H}^{1,1}$. Let us outline now the key step in the proof of Proposition 17.17. Let $(V,\mathfrak{f})$ be as above. Let C be smooth circle traced on V, and assume that $q([C,\mathfrak{f}_C]) = 0$. Abstractly, we can attach a 2-handle to $V \times [0,1]$ at $V \times \{1\}$ in such a way that the embedded attaching tube is a tubular neighbourhood of C in V. In this way, we have constructed a triad (W,V,V') such that $g(V') = g(V) - 1$. By easy dimensional reasons, we can extend the embedding $V \subset S^6$ to a proper embedding of the triad (W,V,V') in $S^6 \times [0,1]$. Then we realize that the condition $q([C,\mathfrak{f}_C]) = 0$ is sufficient (and necessary) so that this can be enhanced to a framed bordism between $(V,\mathfrak{f})$ and $(V',\mathfrak{f}')$ for some framing $\mathfrak{f}'$. Moreover, $\mathrm{Arf}(q_{(V,\mathfrak{f})}) = \mathrm{Arf}(q_{(V',\mathfrak{f}')})$. If $g(V) \geq 2$, then there exists C such that $q([C,\mathfrak{f}_C]) = 0$. Repeatedly applying the above argument, we eventually reach either a framed sphere which represents the null class or a generating framed torus.

($k = 3$) This remarkably more complicated case was settled (using the Pontryagin construction) by Rohlin in a series of four papers in 1951-52 of great historical importance, mostly for the relation with the theory of 4-manifolds. We refer to [**GM**] for the translation (in french) of these papers and deep commentaries. Here, we limit to state the final results. We will come back to it in Chapter 20, Section 20.6.

There is a quaternionic version of the Hopf map (recall Example 4.8)

$$\mathfrak{h}^{\mathbf{H}} : S^7 \to S^4$$

obtained in the following way. Let us identify $\mathbb{R}^4$ with $\mathbf{H}^2$, with quaternionic coordinates (q_0, q_1). The unitary sphere S^7 is defined by the equation $|q_0|^2 + |q_1|^2 = 1$. The group of unitary quaternion ($|q| = 1$) $SU(2)$ acts on S^7 by left multiplication. The quotient space is diffeomorphic to S^4, and $\mathfrak{h}^{\mathbf{H}}$ is just the quotient projection. It is a fibre bundle map with fibre S^3. Then we have the following.

- $\pi_6(S^3) \sim \mathbb{Z}/12\mathbb{Z}$;
- $\pi_7(S^4) \sim \mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$, where the first free factor is generated by $[\mathfrak{h}^{\mathbf{H}}]$, the finite factor is generated by the suspension of a generator of $\pi_6(S^3)$;
- $\pi_{n+3}(S^n) \sim \mathbb{Z}/24\mathbb{Z}$, and is generated by $\mathfrak{s}^{n-2}([\mathfrak{h}^{\mathbf{H}}])$, for every $n \geq 5$.

This geometric way of dealing with the homotopy groups of spheres has been worked out only for $k \leq 3$. Presumably, the difficulty would increase too much with k. On the other hand, especially from the viewpoint of *low-dimensional* differential topology, the main interest of such a direct method consists of the method itself. Since Serre's thesis ([**Se**]), powerful tools (including the use of so-called *spectral sequences*) have been developed in homotopy theory; being interested only in the final result, the above cases $k \leq 3$ become first "trivial" applications of these potent methods. Moreover, we can get some general structural information; for example, we have the following Serre's result.

PROPOSITION 17.18. *For every $k \geq 0$ and $n > 1$, the homotopy group $\pi_{n+k}(S^n)$ is finite with the following exceptions:*

- *$k = 0$, as $\pi_n(S^n) \sim \mathbb{Z}$;*
- *$k = 2h - 1$, $n = 2h$, $h > 0$, where $\pi_{n+k}(S^n) \sim \mathbb{Z} \oplus F$, F being a finite group.*

■

Nevertheless, in spite of such powerful tools, the groups $\pi_{n+k}(S^n)$ (even the stable groups π_k^∞) are largely unknown; their behaviour for increasing k is quite irregular (see [**To**]).

17.5. Thom's spaces

Here the purpose is to reduce the determination of the bordism $\mathbb{Z}/2\mathbb{Z}$-vector spaces η_k to the homotopy groups of certain so-called *Thom's spaces*, $\mathbf{T}_k^\eta$. Having the Pontryagin construction as an ideal model, S^n would be the "Thom space" for the framed bordism $\Omega_k^{\mathcal{F}}(S^{n+k})$.

To reach a setting closer to the Pontryagin construction, let us recover first the "absolute" bordism in the embedded one into spheres. For every sphere S^m, $m > k$, consider the sets $\eta_k^{emb}(S^m)$ defined in Section 17.1. Employing the embedded disjoint union already used above to define the operation on $\Omega_k^{\mathcal{F}}(S^m)$, we can endow $\eta_k^{emb}(S^m)$ with a $\mathbb{Z}/2\mathbb{Z}$-vector space structure, so that the natural map obtained by forgetting the embedding is a $\mathbb{Z}/2\mathbb{Z}$-linear map:

$$\phi_{k,m} : \eta_k^{emb}(S^m) \to \eta_k \ .$$

Via the usual equatorial embedding $S^m \subset S^{m+1}$, we get linear maps

$$\mathfrak{s}_{k,m} : \eta_k^{emb}(S^m) \to \eta_k^{emb}(S^{m+1}) \ .$$

By means of general position considerations as in the weak Whitney embedding theorem, and dealing also with proper embeddings in $S^m \times [0,1]$, we easily have the following.

LEMMA 17.19. *1) If $m \geq 2k+1$, then $\phi_{k,m}$ is surjective;*

2) If $m \geq 2k+2$, then $\phi_{k,m}$ is a isomorphism; moreover $\phi_{k,m} = \phi_{k,m+1} \circ \mathfrak{s}_{k,m}$.

From now on, we stipulate that, for every $k > 0$, we will take $m \geq 2k+2$ and set $h = m - k$.

Let M be an $(r+h)$-manifold which is the interior of a (possibly boundaryless) compact smooth manifold with boundary; let $Y \subset M$ be a boundaryless compact r-submanifold. The following facts are now well-known.

If $f : S^m \to M$ is transverse to Y, then $V_f = f^{-1}(Y)$ is a compact boundaryless k-submanifold of S^m; if f_0 and f_1 are homotopic and both transverse to Y, then $[V_{f_0}] = [V_{f_1}] \in \eta_k^{emb}(S^m)$. Then, using the transversality theorems, we well define the map

$$[S^m, M] \to \eta_k^{emb}(S^m),\ \alpha = [f : S^m \to M] \to [f^{-1}(Y)]$$

provided that f is any representative of α transverse to Y. Recall that in our situation

$$[S^m, M] \sim \pi_m(M)\ .$$

This would suggest looking for such a pair (M, Y) (if any) such that the map defined so far is bijective. The pair $(S^n, \{x_0\})$ has played this role for the framed bordism $\Omega_k^{\mathcal{F}}(S^{n+k})$.

With this perspective in mind, let us recall a construction already employed in Section 6.6. For every (k, m) as above, $h = m - k$, take the tautological vector bundle

$$\tau : \mathcal{V}(\mathfrak{G}_{m,h}) \to \mathfrak{G}_{m,h}\ ,$$

the Grassmannian $\mathfrak{G}_{m,h}$ being identified with the zero section of this bundle. As usual, present the sphere as $S^m = \mathbb{R}^m \cup \infty$. Up to diffeotopy, every compact boundaryless k-submanifold V of S^m misses ∞; that is, $V \subset \mathbb{R}^m \subset S^m$. Let

$$\nu : V \to \mathfrak{G}_{m,h},\ \nu(x) = (T_x V)^\perp$$

be the orthogonal distribution of h-planes along V with respect to a Riemannian metric on $\mathbb{R}^m$, for instance the standard one g_0. We can use ν to build a tubular neighbourhood $p : U \to V$ of V in $\mathbb{R}^m$ and this can be incorporated into a commutative diagram of maps

$$\begin{array}{ccc} U & \xrightarrow{\tilde f} & \mathcal{V}(\mathfrak{G}_{m,h}) \\ \downarrow p & & \downarrow \tau \\ V & \xrightarrow{\nu} & \mathfrak{G}_{m,h} \end{array}$$

where the image of $\tilde f$ is contained in a tubular neighbourhood of the zero section in $\mathcal{V}(\mathfrak{G}_{m,h})$, $\tilde f$ is a fibred map, it is transverse to $\mathfrak{G}_{m,h}$, and $\tilde f^{-1}(\mathfrak{G}_{m,h}) = V$. It would be tempting to take $(M, Y) = (\mathcal{V}(\mathfrak{G}_{m,h}), \mathfrak{G}_{m,h})$, but we immediately realize that there is no reason to believe that $\tilde f$ can be extended to the whole of S^m. The situation is very similar to the step in the proof of the surjectivity of the Pontryagin map, when we have constructed the map also called $\tilde f : U \to \mathbb{R}^n$, where $(\mathbb{R}^n, \{0\})$ played the role of $(\mathcal{V}(\mathfrak{G}_{m,h}), \mathfrak{G}_{m,h})$. The key fact to extend that map $\tilde f$ to a map $f : S^m \to S^n$ was that the complement of the image of $\tilde f$ retracts to the northern pole of S^n; note that $S^n = \mathbb{R}^n \cup \infty$ can be considered as the one-point compactification of $\mathbb{R}^n$. This suggests a very simple way to compactify $\mathcal{V}(\mathfrak{G}_{m,h})$ and make the extension of the present map $\tilde{f}$ possible. Set

$$\mathbf{T}^{\eta}_{m,h} := \mathcal{V}(\mathfrak{G}_{m,h}) \cup \infty \ ;$$

that is, the one-point compactification. This space has some remarkable features:

- It is no longer a manifold; however, the only non-manifold point is just the added point at infinity;
- This point ∞ has a fundamental system of conical neighbourhoods centred at it and with base diffeomorphic to the total space of the unitary bundle of the tautological bundle τ;
- The one-point compactification (which is isomorphic to the sphere S^h) of every fibre of τ is embedded in $\mathbf{T}_{m,h}$, which can be considered as the wedge of such infinite family of h-spheres, based at ∞;
- $\mathbf{T}^{\eta}_{m,h} \setminus \mathfrak{G}_{m,h}$ retracts to ∞.

Although it is not a manifold, $\mathbf{T}^{\eta}_{m,n}$ is a rather tame path connected compact space; in particular, it has the structure of a finite CW complex whose homotopy groups lend themselves to being treated by the powerful tools mentioned above.

Arguing similarly to the Pontryagin construction, we can extend the above map

$$\tilde{f} : U \to \mathcal{V}(\mathfrak{G}_{m,h})$$

to a map

$$f : S^m \to \mathbf{T}^{\eta}_{m,h}$$

such that the complement of U is mapped in the complement of the image of U in $\mathbf{T}^{\eta}_{m,h}$, f is constantly equal to ∞ on the complement of a slightly bigger tubular neighbourhood U' of V in S^m, and f is smooth on U'. Let us say that a map sharing these properties of f is *in standard form.*

Lemma 17.20. *Every $\alpha \in [S^m, \mathbf{T}^{\eta}_{m,h}]$ has representatives in standard form.*

Proof : Let $\alpha = [g : S^m \to \mathbf{T}^{\eta}_{m,h}]$. Up to a first homotopy we can assume that g is smooth on $D^- \subset S^m$ (as usual $D^- \sim D^m$), $g^{-1}(\infty) \cap D^- = \emptyset$ and $g_{|D^-}$ is transverse to $\mathfrak{G}_{m,h}$. Then we can construct $f : S^m \to \mathbf{T}^{\eta}_{m,h}$ in normal form, which coincides with g on the tubular neighbourhood U of $V = g^{-1}(\mathfrak{G}_{m,h})$ involved in the construction of $\tilde{f}$, and therefore of f itself. Set $A = g(U) = f(U)$. As $\mathbf{T}^{\eta}_{m,h} \setminus A$ is contractible to ∞, we can conclude that g and f are homotopic.

∎

We summarize the above discussion in the following main result of the present section. Thanks to Lemma 17.20, the proof runs parallel to the one of Theorem 17.6; details are omitted.

THEOREM 17.21. *For every $k > 0$, $m \geq 2k+2$, $h = m-k$, the map*

$$\mathfrak{t}_{m,h} : [S^m, \mathbf{T}^{\eta}_{m,h}] \to \eta^{emb}_k(S^m),\ \mathfrak{t}_{m,h}(\alpha) = [f^{-1}(\mathfrak{G}_{m,h})] ,$$

provided that $f : S^m \to \mathbf{T}^{\eta}_{m,h}$ is any representative in normal form of α, is well defined and eventually establishes group isomorphisms

$$\pi_m(\mathbf{T}^{\eta}_{m,h}) \sim \eta^{emb}_k(S^m) \sim \eta_k \ .$$

Every such a $\mathbf{T}^{\eta}_{m,h}$ is called a Thom spaces for η_k. Sometimes, we prefer to write them as $\mathbf{T}^{\eta}_{k+h,h}$; the homotopy groups $\pi_{k+h}(\mathbf{T}^{\eta}_{k+h,h})$ stabilize when $h \geq k+2$.

17.5.1. On Thom's spaces for Ω_k. First, we identify Ω_k with $\Omega^{emb}_k(S^m)$, $m \geq 2k+2$. Then we replace the tautological bundle τ with the tautological bundle of the Grassmannian of *oriented* h-planes in $\mathbb{R}^m$ (see Chapter 4)

$$\tilde{\tau} : \mathcal{V}(\tilde{\mathfrak{G}}_{m,h}) \to \tilde{\mathfrak{G}}_{m,h} \ .$$

The fibres of this bundle are tautologically oriented. Set $\mathbf{T}^{\Omega}_{m,h}$ as the one-point compactification of $\mathcal{V}(\tilde{\mathfrak{G}}_{m,h})$. For every $[V] \in \Omega_k(S^m)$, in a very similar way as above, we can construct

$$\tilde{f} : U \to \mathcal{V}(\tilde{\mathfrak{G}}_{m,h})$$

which extends to a map in standard form

$$f : S^m \to \mathbf{T}^{\Omega}_{m,h} \ .$$

The given orientation of V coincides with the one obtained by the usual rule, already employed in the Pontryagin construction.

THEOREM 17.22. *For every $k > 0$, $m \geq 2k+2$, $h = m-k$, the map*

$$\tilde{\mathfrak{t}}_{m,h} : [S^m, \mathbf{T}^{\Omega}_{m,h}] \to \Omega^{emb}_k(S^m),\ \tilde{\mathfrak{t}}_{m,h}(\alpha) = [f^{-1}(\tilde{\mathfrak{G}}_{m,h})] ,$$

provided that $f : S^m \to \mathbf{T}^{\Omega}_{m,h}$ is any representative in normal form of α, is well defined and eventually establishes group isomorphisms

$$\pi_m(\mathbf{T}^{\Omega}_{m,h}) \sim \Omega^{emb}_k(S^m) \sim \Omega_k \ .$$

Every such a $\mathbf{T}^{\Omega}_{m,h} = \mathbf{T}^{\Omega}_{k+h,h}$ is called a Thom spaces for Ω_k; again, the homotopy groups $\pi_{k+h}(\mathbf{T}^{\Omega}_{k+h,h})$ stabilize when $h \geq k+2$.

17.5.2. On the determination of $\eta_\bullet$. The homotopy groups $\pi_m(\mathbf{T}^{\eta}_{m,h})$ look qualitatively simpler than those of spheres; for example, we know that they are $\mathbb{Z}/2\mathbb{Z}$-vector spaces. They can be computed by advanced homotopy theory methods ([**Se**]), providing the full determination of

$$\eta_\bullet = \oplus_k \eta_k \ .$$

Recall that $\eta_\bullet$ has furthermore a $\mathbb{Z}/2\mathbb{Z}$-graded algebra structure, where the product is induced by the Cartesian product of manifolds (Remark 11.12):

$$[V] \cdot [W] = [V \times W] \ .$$

In [**T**], this algebra has been eventually determined.

THEOREM 17.23. *The $\mathbb{Z}/2\mathbb{Z}$-graded algebra $\eta_\bullet$ is isomorphic to the polynomial algebra*

$$\mathbb{Z}/2\mathbb{Z}[X_i;\ i \in J]$$

where

$$J = \mathbb{N} \setminus \{2^j - 1;\ j \in \mathbb{N}\}\ .$$

We can give explicit geometric generators (see [**M5**]). For every $m \leq n$, let $H_{m,n}$ denote the regular real algebraic hypersurface in the product of projective spaces $\mathbf{P}^m(\mathbb{R}) \times \mathbf{P}^n(\mathbb{R})$, defined in terms of the respective homogeneous coordinates $(w_0, \dots, w_m)$ and $(z_0, \dots, , z_n)$, as the locus

$$H_{m,n} = \{w_0 z_0 + w_1 z_1 + \cdots + w_m z_m = 0\}\ .$$

Then take

$$\{X_{2j} := [\mathbf{P}^{2j}(\mathbb{R})],\ j > 1\}\ ,$$
$$\{X_{2^{k+1}+1} := [H_{2^k, 2+2^k}],\ k > 1\}\ .$$

As a remarkable qualitative consequence, we have the following.

COROLLARY 17.24. *For every $k \geq 0$, every $\alpha \in \eta_k$ can be represented by regular real algebraic projective sets.*

The determination of $\Omega_\bullet$ can be performed in the same vein, however the proof, even the statement of the result, is more complicated (see [**Wall**]).

17.5.3. On Nash-Tognoli theorem. We have discussed in the last section of Chapter 6 how every compact boundaryless m-submanifold M of $\mathbb{R}^n$ can be approximated by a Nash manifold M' (normal if the embedding dimension is big enough). As already said, in his paper [**Na**], Nash also stated a few conjectures/questions towards potential improvements of this result (see also Sections 15.5, 19.8). The most natural conjecture was that M can be approximated by a regular real algebraic set (not only by some "analytic sheet" of it). A first step was accomplished in [**Wa2**] by proving the conjecture under the restrictive hypotheses that the embedding dimension is big enough (as for normality), and $[M] = 0 \in \eta_m$ (i.e. it is a boundary $M = \partial W$). Roughly, one realizes the double $D(W) \subset \mathbb{R}^n$ in such a way that M is the transverse intersection of $D(W)$ with a hyperplane P. Then one shows that $D(W)$ can be approximated by a normal Nash manifold N made by regular components of a real algebraic set X such that $X \setminus N$ is far from the hyperplane. Finally, $M' = P \pitchfork X$ is a required regular real algebraic approximation of M. Corollary 17.24 can be expressed by saying that the conjecture holds *up to bordism*. By using this fact, the actual conjecture has been proved in general [**Tog**], assuming again that the embedding dimension is big enough. By Corollary 17.24, there is a regular m-dimensional real algebraic set Σ such that $M \amalg \Sigma = \partial W$. A suitable *relative approximation theorem* allows us to refine the above construction in such a way that

$$P \pitchfork X = M' \amalg \Sigma\ ;$$

as both $M' \amalg \Sigma$ and Σ are regular algebraic sets, it is not hard to conclude that M' is also regular algebraic, hence it is a required approximation of M. In [**Ki**], Nash-Tognoli's theorem is refined in the projective setting and it is proved that $M \subset \mathbf{P}^n(\mathbb{R})$ can be approximated by regular algebraic subsets of the projective space.

We know that Nash maps are dense in $\mathcal{E}(M, N)$, provided that M and N are compact embedded Nash manifolds. Moreover, the Nash structure on M is unique up to Nash diffeomorphism. After the Nash-Tognoli theorem, we can pose analogous, much subtler questions in the real algebraic setting. We recall some results from [**BTo**] and [**BD**]. It is not hard to see that, in general, a regular real algebraic model $\hat{M}$ of a compact smooth manifold M is not unique up to algebraic isomorphism. Given algebraic models, the algebraic maps are, in general, not dense in $\mathcal{E}(\hat{M}, \hat{N})$. Given a smooth map $f : M \to \hat{N}$, there is an algebraic model $\hat{M}$ and an algebraic map $\hat{f} : \hat{M} \to \hat{N}$ that approximates f, if and only if f is bordant with an algebraic map $g : \Sigma \to \hat{N}$. This holds, for example, if $\hat{N} = \mathfrak{G}_{n,k}$ is any Grassmann algebraic set. Elaborating on this fact, we can prove that every M admits an algebraic model $\hat{M}$ such that the ring $\mathbf{K}(\hat{M})$, as in Section 5.7.2, can be entirely realized by real algebraic vector bundles. On the negative side, there are compact smooth manifolds M and N and smooth maps $f : M \to N$ such that, for any algebraic model $\hat{N}$ of N, f is not bordant with any algebraic map $g : \Sigma \to \hat{N}$. For more details about this matter see also [**BCR**].

CHAPTER 18

High-dimensional manifolds

In this context, "high" means of dimension greater or equal to 6. The reason for this specific distinction, "low dimensions less or equal to 5" *vs* "high dimensions greater or equal to 6", mainly depends on the fact that in high dimension Smale's [**S2**] *h-cobordism theorem* holds, and, moreover, we have a "stable" proof; that is, working in the same way for every high dimension. Such proof does not work for low dimensions. In dimension 5, the h-cobordism theorem fails and this reflects specific phenomena of a persistent *geometric* intersection between surfaces embedded in boundaryless compact simply connected 4-manifolds, even though they have vanishing *algebraic* intersection number. In dimension 4, the proof does not apply because of specific geometric linking phenomena between knots in S^3 with vanishing (algebraic) linking number; the validity of the 4-dimensional h-cobordism theorem still is an *open question*. The 3-dimensional h-cobordism theorem is equivalent to the celebrated *Poincaré conjecture*; this last has been eventually proved using deep 3-dimensional methods of *geometric analysis*. In a sense, dimension 5 is really in the border; as already said, it is influenced by the behaviours of 4-dimensional manifolds; but on the other hand, with some specific additional care, it shares some remarkable behaviours with higher dimensions.

We will not provide a proof of the h-cobordism theorem (see [**M3**] for a proof in terms of Morse functions, see [**RS**] for a proof in terms of handle decompositions which actually works also for PL manifolds); rather we will focus on a key point where the high-dimensional assumption is crucial.

Together with Chapter 15, Chapters 19 and 20 will be devoted to some aspects of low-dimensional theory.

18.1. On the h-cobordism theorem

Let us start with a definition.

DEFINITION 18.1. Let (W, V_0, V_1) be a smooth m-dimensional triad ($m = \dim W$). It is an *h-cobordism* if both inclusions $j_i : V_i \to W$, $i = 0, 1$, are homotopy equivalences (i.e. they have an inverse up to smooth homotopy $r_i : W \to V_i$ such that (by definition) $r_i \circ j_i$ is homotopic to id_{V_i} and $j_i \circ r_i$ is homotopic to id_W).

The basic example of h-cobordism is a cylinder $(V \times [0,1], V, V)$. The general, vague question is under which minimal hypotheses the cylinders are the unique instance of h-cobordism up to diffeomorphism of triads. We can formulate the following more specific question.

QUESTION 18.2. *(Simply connected m-dimensional h-cobordism question)* Let (W, V_0, V_1) be an h-cobordism, $\dim W = m$; assume that W (whence both V_0 and V_1) is *simply connected*. Is it true that the triad is diffeomorphic to the cylinder $(V_0 \times [0,1], V_0, V_1)$, so that, in particular, V_0 is diffeomorphic to V_1?

Note that the question is empty for $m = 2$. Assuming the positive answer, let us derive some important consequences.

PROPOSITION 18.3. *Assume that m-dimensional simply connected h-cobordisms are diffeomorphic to cylinders. Then the following facts hold.*

(1) (Characterization of the m-disk) *Every contractible compact m-manifold M with simply connected boundary is diffeomorphic to the closed disk D^m.*

(2) (Generalized Poincaré conjecture) *If Σ is a compact m-manifold which is homotopically equivalent to S^m (i.e. it is a* homotopy sphere*), then it is homeomorphic to S^m.*

(3) (Smooth Schoenfliess property) *If Σ is a smooth embedded $(m-1)$-sphere in S^m, then there is a diffeotopy of S^m that sends Σ to the standard equator $S^{m-1} \subset S^m$.*

Sketch of proof. Some of the facts claimed below are not so evident; to prove them one should dispose of more advanced algebraic/topological tools; we limit to an outline.

(1) Remove from M a standard m-disk D embedded in a chart of M. Set $W = M \setminus \text{Int}(D)$. The triad $(W, \partial D, \partial M)$ is a simply connected h-cobordism, hence it is diffeomorphic to the cylinder $(S^{m-1} \times [0,1], S^{m-1}, S^{m-1})$ and M is diffeomorphic to the manifold obtained by gluing D to this cylinder by a diffeomorphism $\phi : \partial D \to S^{m-1} \times \{1\}$; it is not hard to conclude that M is diffeomorphic to D^m.

(2) Remove from Σ a standard m-disk D in a chart as above. The manifold $M = \Sigma \setminus \text{Int}(D)$ verifies the hypothesis of item (1), then it is diffeomorphic to a disk, Σ is eventually a twisted sphere (see Section 7.5.2) and we know that it is homeomorphic (not necessarily diffeomorphic) to S^m.

(3) By the separation theorem of Section 12.2, $S^m \setminus \Sigma$ has two connected components; the closure of each one of these components verifies the hypothesis of item (1), hence it is an embedded smooth m-disk in S^m and we conclude through the uniqueness of disks up to diffeotopy.

■

REMARK 18.4. The above proposition shows that the h-cobordism question is strictly related to (in fact, motivated by) fundamental questions about the topology of smooth manifolds. For example, for $m = 3$, if (W, V_0, V_1) is a simply connected h-cobordism, then $V_0 \sim V_1 \sim S^2$ by the classification of surfaces. As a 3-dimensional twisted sphere is a true sphere, it follows that a positive answer to question 18.2 for $m = 3$ is equivalent to the validity of the original celebrated *Poincaré conjecture*, with the further refinement that, for $m = 3$, every smooth homotopy sphere Σ is *diffeomorphic* to S^3. Probably, the reader is aware that this has been proved by G. Perelman at the beginning of the new century, by achieving the program based on the Ricci flows of Riemannian metrics on 3-manifolds, earlier introduced by R. Hamilton. We stress that this 3-dimensional geometric/analytic approach is very far from the differential/topological methods discussed in this text. As the 3-dimensional Poincaré conjecture is true, then if (W, V_0, V_1) is a simply connected 4-dimensional h-cobordism, then $V_0 \sim V_1 \sim S^3$. Thus, as a twisted 4-sphere is a true sphere, a positive answer to question 18.2 for $m = 4$ is equivalent to the fact that every smooth 4-dimensional homotopy sphere is actually *diffeomorphic* to S^4. This still is an *open question*, as well as the validity of the 4-dimensional smooth Schoenfliess property. On the other hand, we recall that the *purely topological* 4-dimensional Poincaré conjecture (even dealing with topological, not necessarily smooth, 4-manifolds) has been proved in 1982 by M.H. Freedman [**Fr**].

Now we can state the *high-dimensional simply connected h-cobordism theorem.*

THEOREM 18.5. *Let (W, V_0, V_1) be a simply connected h-cobordism and assume that* $\dim W \geq 6$. *Then it is diffeomorphic to the cylinder* $(V_0 \times [0, 1], V_0, V_0)$.

Hence, all consequences stated in Proposition 18.3 hold for $m \geq 6$. We have mentioned before that the h-cobordism theorem fails for $m = 5$; nevertheless, this dimension shares some behaviour with higher dimensions. Referring to the statement of Proposition 18.3, we recall for example (without proof) that:

(1) The characterization of the 5-disk holds under the *stronger* hypothesis that the boundary of the contractible 5-manifold M is *diffeomorphic* to S^4;

(2) The 5-dimensional generalized Poincaré conjecture holds;

(3) The 5-dimensional smooth Schoenfliess property holds.

18.1.1. On the proof of the high-dimensional h-cobordism theorem. The strategy to prove the h-cobordism theorem is based on handle decompositions (refer to Chapter 9). Given a simply connected h-cobordism (W, V_0, V_1), $\dim W = m$, we can start with an ordered handle decomposition

$$C_0 \cup H_1^{q_1} \cup \dots H_k^{q_k} \cup C_1$$

without 0- and m-handles (Proposition 9.12). If necessary, we can also assume that the handles of a given index $q < m$ are attached simultaneously at pairwise disjoint attaching tubes. Note also that, in the hypothesis of the theorem, all involved manifolds (W and all submanifolds W_r of W obtained by attaching till the rth-handle) are orientable. We dispose of two basic handle moves to possibly make it simpler and simpler. If we succeed, eventually reaching a decomposition without handles of any index, then the theorem will be proved. *A priori*, the only way to reduce the number of handles is the cancellation of pairs of complementary handles. The core of the proof is a much more flexible *cancellation theorem* which applies in the setting of the theorem. Consider a fragment of a given handle decomposition of the form

$$\cdots \cup H^q_r \cup H^{q+1}_{r+1} \cup \dots \quad .$$

Then both the (embedded) b-sphere S_b of H^q_r and the a-sphere S_a of H^{q+1}_{r+1} are submanifolds of ∂W_r and $\dim S_b + \dim S_a = \dim \partial W_r = m - 1$. Fixing auxiliary orientations, we can compute their intersection number in ∂W_r, $[S_b] \bullet [S_a] \in \mathbb{Z}$.

Definition 18.6. In the situation depicted above, we say that $H^q_r \cup H^{q+1}_{r+1}$ is a pair of *algebraically* complementary handles if $[S_b] \bullet [S_a] = \pm 1$.

This extends the notion of complementary handles. Now we can state a *stronger cancellation theorem.*

Theorem 18.7. *Let (U, Z_0, Z_1) be a smooth triad of dimension m which admits a handle decomposition*

$$C_0 \cup H^q \cup H^{q+1} \cup C_1$$

made by two algebraically complementary handles. Assume that both Z_0 and Z_1 are simply connected, and that

$$m \geq 6, \;\; q \geq 2, \;\; m - q \geq 4 \; .$$

Then the given triad is diffeomorphic to the cylinder $(Z_0 \times [0,1], Z_0, Z_0)$.

By transversality and handle sliding, we can assume that $S_b \pitchfork S_a$ in ∂M, $M := C_0 \cup H^q$ and that the intersection consists of an odd number of signed points, such that the sum of the signs is equal to ± 1. Therefore, using handle sliding, we would progressively cancel pairs of intersection points of *opposite sign*, so that, at the end, we reach a decomposition made by two genuine complementary handles that can be cancelled. In the discussion about the strong Whitney embedding theorem (Section 7.7) of compact n-manifolds in $\mathbb{R}^{2n}$, for $n \geq 3$, we have already mentioned the so-called "*Whitney trick*" as a tool to cancel pairs of crossing points. The hypotheses of the stronger cancellation theorem allow applying it. This will be discussed with some care in the next section.

18.2. Whitney trick and unlinking spheres

First, we state a lemma under the hypotheses of the stronger cancellation theorem.

LEMMA 18.8. *In the hypotheses of Theorem 18.7, denote by* $\partial(C_0 \cup H^q) = Z_0 \amalg M$*, so that the* b*-sphere* S_b *of* H^q *and the* a*-sphere* S_a *of* H^{q+1} *are transverse submanifolds of* M*. Then* $M \setminus (S_b \cup S_a)$ *is simply connected.*

Proof : Set $m = n+1$. Denote by S'_a the a-sphere of H^q. Its codimension is $\dim Z_0 - \dim S'_a = n - (q-1) \geq 4$. By transversality, also $Z_0 \setminus S'_a$ is simply connected; as both $Z_0 \setminus S'_a$ and $M \setminus S_b$ retract to $Z_0 \setminus \mathrm{Int}(T'_a)$, it follows that also $M \setminus S_b$ is simply connected. The codimension of S_a is $\dim M - q = n - q \geq 3$. Therefore, by the same transversality argument, we have that $(M \setminus S_b) \setminus S_a = M \setminus (S_b \cup S_a)$ is simply connected.

■

Referring to the last lemma, we can abstractly formalize some features of the situation occurring on the manifold M.

By *a situation* $(M, R, S, \pm x)$ *of type* $(n, r) \in \mathbb{N}^2$ we mean:

- M is a connected oriented boundaryless smooth manifold of dimension n;
- R and S are boundaryless compact connected oriented submanifolds of M such that $\dim R = r$, $\dim S = s$, $n > s \geq r > 0$, $r + s = n$, $R \pitchfork S$.
- $M \setminus (S \cup R)$ is simply connected;
- $\pm x \in R \cap S$ are intersection points of *opposite sign.*

REMARKS 18.9. (1) In a situation of type (n, r) as above, if both codimensions of S and R are greater or equal to 3, then by a usual transversality argument, $M \setminus (S \cup R)$ is simply connected if and only if M is simply connected.

(2) In situations arising under the hypotheses of Theorem 18.7, we have furthermore that $n \geq 5$ and $r \geq 2$.

(Whitney disk) Let $(M, R, S, \pm x)$ be a situation of type (n, r). By a *Whitney disk* D for $(M, R, S, \pm x)$ we mean the realization of the following pattern (recall Section 7.7)

(1) There is an embedded smooth circle γ in $R \cup S$ with two corners at $\pm x$. These divide γ in two arcs with closures γ_R and γ_S, respectively; γ_R (resp. γ_S) is contained into an smooth open r-disk (s-disk) $U_R \subset R$ ($U_S \subset S$). The open set $U_R \cup U_S$ is a neighbourhood of γ in $R \cup S$; $U_R \pitchfork U_S = \{\pm x\}$ and $U_R \cup U_S$ does not contain other points of $R \cap S$.

(2) There are:

- A 2-disk $\mathcal{D}$ in $\mathbb{R}^2$, with boundary $\partial \mathcal{D}$ and with two corners a_1, a_2, which is contained in the union of two smooth arcs λ_R, λ_S in $\mathbb{R}^2$ which intersect transversely at $\{a_1, a_2\}$;

• An embedding $\psi : U \to M$ where U is a closed 2-disk in $\mathbb{R}^2$ containing $\mathcal{D} \cup (\lambda_R \cup \lambda_S)$, such that

- $\psi(\lambda_*) \subset U_*, \ * = R, \ S$;
- $\psi(\partial\mathcal{D}, \{a_1, a_2\}) = (\gamma, \{q_1, q_2\})$;
- for every $x \in \lambda_*$, $d_x\psi(T_xU) \cap T_{\psi(x)}U_* = d_x\psi(T_x\lambda_*)$;
- $\psi(\mathrm{Int}(\mathcal{D})) \subset M \setminus (R \cup S)$.

We summarize (1) and (2) by saying that the smooth 2-disk with corners $D := \psi(\mathcal{D})$ is *properly embedded* in $(M, R \cup S)$ and *connects the crossing points* $\pm x$. Moreover, we require that

(3) We can extend the embedding ψ to a parametrization of a neighbourhood of D in M by a *standard model*; that is, to an embedding

$$\Psi : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \to M$$

such that $\Psi(\lambda_R \times \mathbb{R}^{r-1} \times \{0\}) = U_R$ and $\Psi(\lambda_S \times \{0\} \times \mathbb{R}^{s-1}) = U_S$.

Remark 18.10. We stress that the existence of a Whitney disk (in particular condition (3)) for a situation (M, R, S, x_0, x_1) implies that the two points are of opposite sign.

(Whitney trick) The Whitney trick applies to $(M, R, S, \pm x)$ at a Whitney disk connecting $\pm x$: thanks to the standard model, such a Whitney disk can be easily used as a guide to construct an isotopy of R in M with support not intersecting the other points of $R \cap S$ and carrying R to $R' \pitchfork S$ such that $R' \cap S = R \cap S \setminus \{\pm x\}$ (recall Figure 2 of Chapter 7, by renaming $R = P$, $S = Q$).

Definition 18.11. For every type (n, r) as above, we say that $\mathbf{WT}(n, r)$ holds if every situation $(M, R, S, \pm x)$ of type (n, r) admits a Whitney disk.

We are going to relate the validity of $\mathbf{WT}(n, r)$ with a certain *unlinking property* of *unknotted spheres* in a sphere.

A smooth p-sphere $\Sigma \subset S^k$, $k > p \geq 1$, is *unknotted* if it is the boundary of a smooth $(p+1)$-disk embedded in S^k. The following lemma is easy, by using the uniqueness of disks up to diffeotopy.

Lemma 18.12. *Let $\Sigma \subset S^k$ be unknotted. Let D be a smooth k-disk in S^k disjoint from Σ. Then Σ is the boundary of a smooth $(p+1)$-disk embedded in $S^k \setminus D$.*

A *link of unknotted spheres* (S^k, Σ, Σ') *of type* $(k, p) \in \mathbb{N}^2$ consists of two *disjoint* unknotted smooth spheres $\Sigma, \Sigma' \subset S^k$ such that

$$p = \dim \Sigma, \ q = \dim \Sigma', \ p \leq q, \ k = p + q + 1 \ .$$

Such a link (S^k, Σ, Σ') is (geometrically) *unlinked* if the two spheres are the boundary of disjoint $(p+1)$- and $(q+1)$-disks, respectively. By using again the unicity of disks up to diffeotopy, we have the following.

LEMMA 18.13. *Up to diffeotopy, there is a unique unlinked link of type* (k,p).

For every link (S^k, Σ, Σ'), give the spheres auxiliary orientations; then we can define the *linking number* (recall Section 12.4 and Remarks 12.4)

$$lk(\Sigma, \Sigma') \in \mathbb{Z} \ .$$

A link is *algebraically unlinked* if

$$lk(\Sigma, \Sigma') = 0 \ .$$

We know (see the end of Section 12.5) that the choice of auxiliary orientations is immaterial with respect to the vanishing of the linking number; moreover this property is symmetric: $lk(\Sigma, \Sigma') = 0$ if and only if $lk(\Sigma', \Sigma) = 0$. Geometrically unlinked links are algebraically unlinked.

DEFINITION 18.14. For every link type $(k,p) \in \mathbb{N}^2$, we say that the *unlinking property* $\mathbf{U}(k,p)$ holds, if every link (of unknotted spheres) (S^k, Σ, Σ') of type (k,p) which is algebraically unlinked is, in fact, geometrically unlinked.

It follows from the above discussion that Theorem 18.7 will be a corollary of item (1) in the next proposition.

PROPOSITION 18.15. *(1) For every type* (n,r) *such that* $n \geq 5$ *and* $r \geq 2$, $\mathbf{WT}(n,r)$ *holds.*

(2) For every link type (k,p) *such that* $k \geq 4$, $\mathbf{U}(k,p)$ *holds.*

Proof : First, let us prove that $\mathbf{U}(k,1)$ holds for every $k \geq 4$; for consider an algebraically unlinked link (S^k, Σ, Σ'), $\dim \Sigma = 1$, $\dim \Sigma' = q = k-2 \geq 2$. Then $S^k \setminus \Sigma'$ is homotopically equivalent to the standard $S^1 \subset S^k$ and the embedding of Σ in $S^k \setminus \Sigma'$ is homotopically trivial; as $k > 2\dim \Sigma + 1 = 3$, then Σ is isotopic in $S^k \setminus \Sigma'$ to a geometrically unlinked circle.

Next, we prove the following claim.

Claim 1. *For every* $n \geq 5$, *if* $\mathbf{WT}(n,r)$ *holds, then* $\mathbf{U}(n, \min(r,q))$, $q = n - r - 1$, *holds.*

Proof of the claim: Consider an algebraically unlinked link (S^n, Σ, Σ'), $\dim \Sigma = r$, $\dim \Sigma' = q$. Assume for simplicity that $r \leq q$. Let $D \subset S^n$ be a $(q+1)$-disk such that $\partial D = \Sigma'$. Then the intersection number $[\Sigma] \bullet [D]$ in $S^n \setminus \Sigma'$ is equal to $0 \in \mathbb{Z}$. Then as $\mathbf{WT}(n,r)$ holds, Σ is isotopic to Σ" such that $\Sigma" \cap D = \emptyset$. We can assume that Σ" is embedded in $S^n \setminus B$, where $B \sim D^n$ is an n-disk of S^n which thickens D. Then we conclude by means of Lemma 18.12.

Next, we propose two ways to conclude. The first way consists of a direct proof of item (1); then item (2) will follow as a corollary of Claim 1 and the case $\mathbf{U}(k,1)$ already achieved. By the second way, both statements will be proved simultaneously by implementing the *concatenated inductive scheme* obtained by combining Claim 1 with the following Claim 2 (the case $\mathbf{U}(k,1)$ is the initial step of this induction):

Claim 2. *For every $k \geq 4$, if $\mathbf{U}(k,p)$ holds, then $\mathbf{WT}(k+1,p+1)$ holds.*

The second way makes fully manifest the strict relationship between **WT** and **U**. The presentation of this second way is very close to Chapter 5 of [**RS**].

Proof of item (1): As $n \geq 5$, by general position we can assume that points (1) and (2) in the definition of a Whitney disk for $(M,R,S,\pm x)$ are fulfilled. It remains to achieve point (3). This is rephrased in terms of a suitable configuration of sub-bundles of $T(M)$ over $(D,\partial D)$. We can assume that an auxiliary Riemannian metric g on M is fixed in such a way that R and S are orthogonal at their intersection points, and the normal bundles and the associated tubular neighbourhoods are constructed using g. We use the notation $\nu_X Y$ to mean the normal bundle of Y in X. The tangent bundle $T(R)$ splits over γ_R as

$$T(R)|\gamma_R = T(\gamma_R) \oplus E_R$$

where E_R is a rank-$(r-1)$ sub-bundle of $(\nu_M D)|\gamma_R$. Thus E_R is tangent to R and normal to D. The normal bundle $\nu_M S$ splits over γ_S as

$$(\nu_M S)|\gamma_S = \nu_D \gamma_S \oplus E_S$$

where E_S is a rank-$(r-1)$ sub-bundle of $(\nu_M D)|\gamma_S$. Thus E_S is normal to both S and D. The bundles E_R and E_S match at the intersection points $\pm x$, so that we have a rank-$(r-1)$ bundle E defined over the whole ∂D. By construction E is tangent to R and normal to S. We claim that E can be extended to a sub-bundle of the whole $\nu_M D$. Using a trivialization of $\nu_M D$ we can encode E as a map $E : \partial D \to \mathfrak{G}_{n-2,r-1}$. Then E extends if and only if it is homotopically trivial. It is known that under our dimensional hypotheses (see for instance [**Steen**])

$$\pi_1(\mathfrak{G}_{n-2,r-1}) = \mathbb{Z}/2\mathbb{Z}$$

and that E as above is homotopically trivial if and only if the corresponding rank-$(r-1)$ bundle is *orientable*. This is the case because the intersection points have opposite signs. At this point, it is not hard to build compatible trivializations of the bundles considered so far and achieve point (3) in the definition of a Whitney disk.

A sketch of proof of Claim 2: Let $(M,R,S,\pm x)$ of type $(k+1,p+1)$, $k \geq 4$. Argue as in the above proof of item (1), so that we can assume that points (1) and (2) in the definition of a Whitney disk for $(M,R,S,\pm x)$ are fulfilled. Again, it remains to achieve point (3). Assume that it holds. We analyze the standard model and then we transport the conclusions in M around the disk D using the embedding Ψ. Up to corner smoothing,

$B := U \times D^p \times D^{k-p-1}$ is a $(k+1)$-disk, and

$$(\partial B, \partial(\lambda_R \times D^p \times \{0\}), \partial(\lambda_S \times \{0\} \times D^{k-p-1}))$$

is diffeomorphic to an unlinked link of type (k,p). Moreover, the whole B can be reconstructed from such an unlinked link. Assume now that, *a priori*, only (1) and (2) are verified. We can nevertheless find a smooth $(k+1)$-disk B in M around D, which retracts to D, such that

$$\partial B \pitchfork U_R := \Sigma_R,\ \partial B \pitchfork U_S := \Sigma_S$$

are smooth spheres in the sphere $\partial B \sim S^k$ forming a link of type (k,p). To incorporate it in a standard model, by Lemma 18.13 it is enough to prove that it is unlinked. As $\pm x$ have opposite signs, it follows that the link is algebraically unlinked, and we can conclude because $\mathbf{U}(k,p)$ holds by the hypothesis of Claim 2.

The proposition is proved.

∎

REMARK 18.16. As for low dimensions, we note that:

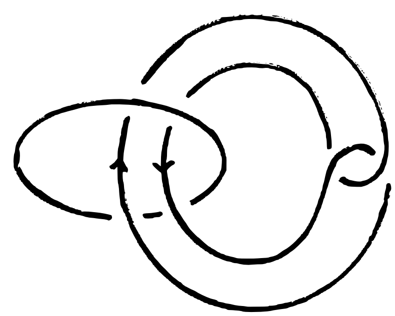

Figure 1. Whitehead's link.

• $\mathbf{U}(3,1)$ fails. The simplest counterexample is the so-called *Whitehead link*; several classical knot invariants show that it is geometrically linked in spite of the fact that it is algebraically unlinked (see [**Rolf**]).

• Trying to perform the construction to approach $\mathbf{WT}(4,2)$, it is not hard to realize item (1) in the definition of Whitney disk; however (2) and even more (3) are very problematic; we will see in Chapter 20 that there are actual obstructions.

CHAPTER 19

On 3-manifolds

In this chapter, we deal with 3-manifolds. In no way we will touch Thurston's *geometrization* approach that has dominated the study of 3-manifolds in the last decades. We will not even touch fundamental facts of 3-dimensional geometric topology such as the decomposition in prime manifolds or the so-called JSJ-decomposition. This chapter aims to show how different tools developed in this text combine to obtain, in a very concrete way, some primary results about 3-manifolds. An important amount of the chapter will be devoted to several proofs of "$\Omega_3 = 0$" and of the equivalent Lickorish-Wallace theorem about surgery equivalence. We also provide a few elementary proofs that compact orientable boundaryless 3-manifolds are parallelizable and we study combings of arbitrary 3-manifolds. Each proof will illuminate different facets of the matter. The last two sections are more advanced. We will study immersions of surfaces in a given 3-manifold M, including the determination of the bordism group $\mathcal{I}_2(M)$ of immersed surfaces. An emerging theme will be the quadratic enhancements of the intersection form of a surface associated with such immersions in an orientable 3-manifold. In the last section, we consider the classification of 3-manifolds up to certain equivalence relations defined in terms of blowing up along smooth centres.

19.1. Heegaard splitting

We begin with a classical way to realize all compact orientable boundaryless 3-manifolds. It is an easy application of handle theory.

Let M be a connected, *orientable*, boundaryless compact 3-manifold. We know that there is an ordered handle decomposition $\mathcal{H}$ of M with only one 0-handle, only one 3-handle, and such that both 1- and 2-handles, respectively, are attached simultaneously at disjoint attaching tubes. Denote by M_1 the submanifolds with boundary of M obtained by attaching the 1-handles at the boundary of the unique 0-handle. As M is orientable, then also M_1 is orientable; by the uniqueness of disks up to diffeotopy applied to the attaching tubes of 1-handles and handle sliding, M_1 only depends, up to diffeomorphism, on the number $g \geq 0$ of 1-handles and is called a *handlebody of genus* g, denoted by $\mathfrak{H}_g$. Its boundary $\Sigma = \partial M_1$ is a surface of genus g, that is diffeomorphic to the connected sum of g copies of the torus $S^1 \times S^1$. If $g = 1$, $\mathfrak{H}_1 = D^2 \times S^1$ is also-called a *solid torus*. Consider the dual handle decomposition $\tilde{\mathcal{H}}$, so that the 2-handles of $\mathcal{H}$ become the 1-handles of $\tilde{\mathcal{H}}$.

Apply the above discussion to $\tilde{M}_1$. Then $\partial M_1 = \partial \tilde{M}_1 = \Sigma$, and also $\tilde{M}_1$ is a handlebody of genus g. Then

$$M = M_1 \cup \tilde{M}_1$$

is called a *Heegaard splitting* of M of genus g and the separating surface Σ is the corresponding *Heegaard surface*.

Therefore, every such an M admits a Heegaard splitting of some genus, and we define the *Heegaard genus* $g_H(M)$ of M as the minimum g such that M has a splitting of genus g. As it often happens, such an invariant is easy to define but in general, it is hard to compute or even to estimate.

Up to diffeomorphism, a Heegaard splitting of M can be described equivalently as follows: fix a standard model $\mathfrak{H}_g$ of genus g handlebody (for instance embedded in $\mathbb{R}^3$ and endowed with the standard induced orientation); let $\Sigma_g = \partial \mathfrak{H}_g$ with the boundary orientation. Fix an auxiliary smooth automorphism γ of Σ_g which reverses the orientation. There is an orientation-preserving (we say "positive") smooth automorphism $\phi \in \mathrm{Diff}^+(\Sigma_g)$ such that

$$M = M_1 \cup \tilde{M}_1 \sim \mathfrak{H}_g \amalg_{\gamma\circ\phi} \mathfrak{H}_g \ .$$

Moreover, we know that up to diffeomorphism, the last term only depends on the isotopy class of ϕ; in other words, define

$$\mathrm{Mod}(\Sigma_g) := \mathrm{Diff}^+(\Sigma_g)/\mathrm{Diff}^0(\Sigma_g)$$

that is the quotient group mod the normal subgroup of automorphisms isotopic to the identity. This is called the *mapping class group* of Σ_g (also called its *modular group*) and it is an object of main importance and interest. Then every splitting is of the form

$$M \sim \mathfrak{H}_g \amalg_{[\phi]} \mathfrak{H}_g \sim \mathfrak{H} \amalg_{\gamma\circ\phi} \mathfrak{H}_g, \ [\phi] \in \mathrm{Mod}(\Sigma_g) \ .$$

EXAMPLE 19.1. (1) If $g_H(M) = 0$ then M is a twisted, hence a true smooth 3-sphere.

(2) The 3-manifolds such that $g_H(M) = 1$ are classified and called *lens spaces* [**Brod**]. Let us recall the main facts. Realize the torus as the quotient manifold $\mathbb{R}^2/\mathbb{Z}^2$. The matrix group $SL(2,\mathbb{Z})$ acts linearly on $\mathbb{R}^2$, preserving the lattice $\mathbb{Z}^2$. Then the action descends to the quotient. We can prove that

$$\mathrm{Mod}(\Sigma_1) \sim SL(2,\mathbb{Z}) \ .$$

Fix an identification of the torus as the boundary Σ_1 of $\mathfrak{H}_1$ in such a way that the circle image in $\mathbb{R}^2/\mathbb{Z}^2$ of the x-axis of $\mathbb{R}^2$ becomes a *meridian* m, meaning it bounds a 2-disk properly embedded in $(\mathfrak{H}_1, \Sigma_1)$, while the image of the y-axis is a longitude l which transversely intersects m at one point; m, l form a basis of $\Omega_1(\Sigma_1) \sim \mathbb{Z}^2$. Let $A \in SL(2,\mathbb{Z})$, so that $A(m) = pm + ql$, $gcd(p,q) = 1$. Denote by $L_{p,q}$ the resulting lens space obtained by using A as gluing map. It is not hard to check via Van Kampen theorem that $\pi_1(L_{p,q}) \sim \mathbb{Z}/p\mathbb{Z}$. Then $L(p,q)$ is diffeomorphic to $L(p,q')$ if and only if

$$\pm q' = q^{\pm 1} \ \mathrm{mod}(p) \ .$$

For higher genus, the situation is much more complicated.

19.1.1. Heegaard diagrams and a diagrammatic "calculus".

Heegaard splittings can be encoded by means of suitable Heegaard diagrams.

DEFINITION 19.2. A *genus g Heegaard diagram* consists of a triple (Σ, C^-, C^+) where

(1) Σ is a surface of genus g;
(2) $C^\pm = \{c_1^\pm, \dots, c_g^\pm\}$ is a family of g disjoint simple smooth circles on Σ whose union does not divide Σ, meaning that by removing from Σ the interiors of small pairwise disjoint annular neighbourhoods of these circles we get a 2-sphere with $2g$ holes;
(3) $C^- \pitchfork C^+$, meaning the union of the c_j^-'s is transverse to the union of the c_j^+'s.

Given a Heegaard diagram, we can construct a 3-manifold M endowed with a Heegaard splitting as follows. Take the product $\Sigma \times [-1, 1]$ and stipulate that the circle of $C^\pm$ are traced on

$$\Sigma \times \{\pm 1\} := \Sigma^\pm \ .$$

The surface Σ is identified with the separating surface $\Sigma \times \{0\}$. Then take a system of pairwise disjoint annular neighbourhoods $T_j^\pm$ for $C^\pm$ on $\Sigma^\pm$. Consider the T_j^+ as a system of attaching tubes of disjoint 2-handle attached to $\Sigma \times [0, 1]$ at Σ^+. Thanks to the properties of the circles in C^+ this produces a 3-manifold with boundary diffeomorphic to $\Sigma \amalg S^2$. By filling the spherical component with a 3-handle we get the piece $\tilde{M}_1$ of the desired handle decomposition of M. Doing similarly on the other side $\Sigma \times [-1, 0]$ we get the piece M_1 and eventually the splitting

$$M \sim M_1 \cup \tilde{M}_1$$

with Heegaard surface Σ.

REMARK 19.3. The fact that the resulting 3-manifold is *unique up to diffeomorphim* follows from Smale theorem recalled in Proposition 7.13, (1), $m = 3$.

On the other hand, every Heegaard splitting with Heegaard surface Σ gives rise to an encoding Heegaard diagram, possibly by handle sliding to reach the transversality requirement of the definition.

(Heegaard diagram moves) The elementary handle moves induce elementary moves on Heegaard diagrams which keep the resulting manifold M fixed up to diffeomorphism.

- Handle sliding produces the following diagram moves (called *H-diagram sliding*):

  1) Of course we can modify $C^\pm$ up to ambient isotopy (keeping that $C^+ \pitchfork C^-$);

2) More substantially we have the following. Let $T_j^{\pm}$ and $T_i^{\pm}$ be disjoint annular neighbourhoods of two circles of $C^{\pm}$ as above. Connect these annuli by attaching an embedded 1-handle H at $\partial(T_j^{\pm} \amalg T_i^{\pm})$ in such a way that, apart from the attaching segments, H is contained in $\Sigma \setminus \cup_{s=1}^{g} T_s^{\pm}$. The boundary of $T_j^{\pm} \cup T_i^{\pm} \cup H$ contains a component c'_j which is the embedded connected sum of a parallel copy of $c_j^{\pm}$ with a parallel copy of $c_i^{\pm}$. Then get a new $C^{\pm}$ by replacing $c_j^{\pm}$ with c'_j.

- Cancellation/introduction of a pair of complementary handles produces the following diagram move. Consider the diagram
$$(S^1 \times S^1, c^- = S^1 \times \{y_0\}, c^+ = \{x_0\} \times S^1)\ .$$
Given any diagram (Σ, C^-, C^+) of genus g, replace Σ with $\Sigma \# (S^1 \times S^1)$ provided that the sum is performed at 2-disks disjoint from $C^- \cup C^+$ and $c^- \cup c^+$ respectively; then add to $C^{\pm}$ the circle $c^{\pm}$ to get the new diagram of genus $g+1$. In terms of the resulting 3-manifolds, we replace M with $M \# S^3 \sim M$. This move is called *elementary stabilization.*

The stabilization shows, by the way, that for every $g \geq g_H(M)$, M admits Heegaard splitting of genus g. In particular, S^3 has splittings of any genus.

THEOREM 19.4. ([**Sing**]) *Two Heegaard diagrams encode Heegaard splittings of a same* 3*-manifold M (considered up to diffeomorphism) if and only if they become equal up to finite sequences of H-diagram sliding or stabilizations.*

REMARK 19.5. Once the existence of Heegaard splitting has been easily established, several nontrivial questions naturally arise.

• For a given M, estimate in effective terms its genus $g_H(M)$;

• For every $g \geq g_H(M)$, study the Heegaard splittings of M of genus g up to ambient isotopy.

Concerning the second question, a complete answer is known for the 3-sphere and lens spaces defined above; that is, for manifolds such that $g_H \leq 1$, we have the following.

For every $g \geq 1$, S^3 and every lens space have, up to diffeotopy, a unique Heegaard splitting of genus g.

On the other hand, for $g \geq 2$, there are manifolds with nonisotopic splittings of genus g.

We refer to the body and the references of [**BO**] for more information about this question.

19.1.2. From Heegaard diagrams to spines and Δ-complexes. This section aims to briefly show other ways to present all orientable 3-manifolds derived from Heegaard splittings. We will use some of these facts in Section 19.6. However, this section is of a purely technical nature and may be skipped at a first reading.

Let (Σ, C^-, C^+) be a Heegaard diagram of M as above. Up to H-sliding, we can assume that not only $C^- \pitchfork C^+$, also that every component (called a *region*) of $\Sigma \setminus (C^- \cup C^+)$ is an open 2-disk. By following the reconstruction of the Heegaard splitting

$$M \sim M_1 \cup \tilde{M}_1$$

of M encoded by the diagram, we see that the core of every 2-handle attached to a circle c of $C^+ \times \{1\}$ can be extended by means of the annulus $c \times [0,1]$ and we get an embedded 2-disk in $\tilde{M}_1$ which transversely intersects $\Sigma = \Sigma \times \{0\}$ at c. Do it for each c in $\mathbb{C}^+$ and similarly for each c in $\mathbb{C}^-$, getting a disk in M_1. Denote by $\mathbf{P}$ the union of Σ with all such disks. This $\mathbf{P}$ is a kind of singular surface embedded in M with the following properties:

(1) $S(\mathbf{P}) := (C^- \cup C^+) \subset \Sigma$ is the singular locus of $\mathbf{P}$;
(2) $V(\mathbf{P}) := C^- \cap C^+$ is the singular locus of $S(P)$ and its points are the *vertices* of $\mathbf{P}$. The components, each diffeomorphic to the open 1-disk $(-1, 1)$, of $S(\mathbf{P}) \setminus V(\mathbf{P})$ are the *edges* of $\mathbf{P}$; at every vertex there are four edge germs.
(3) The components, each diffeomorphic to an open 2-disk, of $\mathbf{P} \setminus S(\mathbf{P})$ are the *regions* of $\mathbf{P}$. Along every edge there are three region germs. At every vertex there are six region germs.
(4) If B^+ and B^- are the 0 and 3-handles of the splitting, then $\mathbf{P}$ is a deformation retract of

$$\hat{M} := M \setminus (\mathrm{Int}(B^-) \cup \mathrm{Int}(B^+)\ .$$

In fact, there is a *normal retraction* $r : \hat{M} \to \mathbf{P}$ such that: the fibre over a region point is diffeomorphic to $[-1, 1]$; the fibre over an edge point is a *tripod* that is the wedge of three segments $[0, 1]$ with common endpoint 0; the fibre over a vertex is a wedge of four such segments $[0, 1]$; $\hat{M}$ can be reconstructed as being the mapping cylinder of such a normal retraction.

We summarize all this by saying that $\mathbf{P}$ is a *standard spine* of $\hat{M}$. By using the language of CW-complexes, $\mathbf{P}$ is the 2-skeleton of such a complex over M which is obtained by attaching two 3-cells to it.

Now we can give $\mathbf{P}$ a suitable system of region orientations as follows. Give Σ, hence every region of $\mathbf{P}$ contained in Σ, an orientation; give every circle c in $C^- \cup C^+$ an orientation, hence give the region of $\mathbf{P}$ bounded by c the orientation with the prescribed boundary orientation. In this way, $S(\mathbf{P})$ is a union of oriented circles crossing transversely on Σ at some vertices; every region of $\mathbf{P}$ is oriented in such a way that there is a *prevailing orientation* induced on every edge of $\mathbf{P}$ and this agrees with the one of the circle c in $S(\mathbf{P})$ which contains the edge. At every vertex of $\mathbf{P}$ the four configurations at the edge germs automatically match. We call such a system of region orientations a *branching* $\mathbf{b}$ of $\mathbf{P}$.

We can modify any branched standard spine $(\mathbf{P}, \mathbf{b})$ of $\hat{M}$ obtained so far to become a branched standard spine $(\mathbf{P}_0, \mathbf{b}_0)$ of M_0, where M_0 is of the form

$$M_0 = M \setminus \mathrm{Int}(B) \ ,$$

B being some smooth 3-ball in M. Then $\mathbf{P}_0$ will be the 2-skeleton of a CW-complex over M with a unique 3-cell. Take a point p on an edge of $\mathbf{P}$ and locally insert an embedded triangle t such that

(1) The interior of t is contained in $M \setminus \mathbf{P}$;
(2) The point p is a vertex of t, t transversely intersects $\mathbf{P}$ at the two edges of t having p as common vertex, so that t has a "free edge" l.

Take an embedded 1-handle with core parallel to l, intersecting transversely t along its b-tube at l, with attaching tube on $\mathbf{P}$. Then $\mathbf{P}_0$ results from $\mathbf{P}$ by the surgery which removes the interior of the attaching tube and replaces it with the union of t and the b-tube. It is easy to see that the handle has fused the two components of $\partial\hat{M}$ into one spherical boundary component of a manifold M_0 of the desired form. By construction, $\mathbf{P}_0$ is a standard spine of M_0 and it carries a branching $\mathbf{b}_0$ which agrees with $\mathbf{b}$ on the regions that have not been affected by the surgery. The branched spines $(\mathbf{P}, \mathbf{b})$ or $(\mathbf{P}_0, \mathbf{b}_0)$ can be considered as the 2-skeleton of the *dual cell decomposition* to a Δ-*complex structure* over M in the sense of [**Hatch**]. This is a kind of *triangulation* of M having respectively two vertices or one vertex, obtained by assembling "abstract" tetrahedra with ordered vertices by gluing their abstract 2-facets in pairs in such a way that no face remains unglued and the vertex orders match.

19.1.3. Nonorientable Heegaard splitting. If M is compact connected boundaryless and nonorientable, then using a nice handle decomposition as above, we see that

$$M \sim M_1 \cup \tilde{M}_1 \ ,$$

where M_1 is nonorientable and is obtained by attaching $h+1$ disjoint 2-handles to the unique 0-handle at the boundary $\partial D^3 = S^2$ (and similarly $\tilde{M}_1$ for the dual decomposition). Up to handle sliding, we can assume that only one of these 2-handles destroyed the orientability and that $M_1 \sim \tilde{M}_1$ are completely determined up to diffeomorphism by the handle number $h+1$. Let us call it a *nonorientable handlebody of genus h*. The separating (nonorientable) Heegaard surface is now diffeomorphic to

$$\tilde{\Sigma}_h := (\mathbf{P}^2(\mathbb{R})\#\mathbf{P}^2(\mathbb{R}))\#h(S^1 \times S^1) \ .$$

The reader would imagine how to develop a nonorientable version of Heegaard diagrams and diagram moves supported by such surfaces. Stabilization extends verbatim; a bit of care is necessary for the sliding diagram moves.

19.2. Surgery equivalence

We define a "surgery" equivalence relation on compact connected boundaryless 3-manifolds in terms of certain special 4-dimensional triads; then we characterize the 3-dimensional oriented boundaries as the manifolds that are surgery equivalent to the sphere S^3.

DEFINITION 19.6. Let M_0 and M_1 be compact connected boundaryless nonempty 3-manifolds. We say that M_1 can be obtained by *(longitudinal) surgery (along a framed link) of* M_0 (and we write $M_1 \sim_\sigma M_0$) if there exists a 4-dimensional triad (W, M_0, M_1) which admits a handle decomposition $\mathcal{H}$ consisting only of 2-handles attached simultaneously at disjoint attaching tubes.

To justify the terminology let us analyze the situation of the above definition. The decomposition is of the form

$$C_0 \cup (\cup_{j=1}^d H_j^2) \cup C_1$$

where $C_0 = M_0 \times [0,1]$ and $C_1 = [-1,0] \times M_1$ are respective collars of M_0 and M_1 in W. The union of the embedded attaching spheres of the 2-handles

$$L = \cup_{j=1}^d K_s$$

is a so-called *link* in $M_0 \sim M_0 \times \{1\}$. Every component K_s is a *knot* in M_0. Moreover, we have a family of disjoint attaching tubes T_s, each one equipped with a trivialization (also called a "framing") by $S^1 \times D^2$, so that $K_s \sim S^1 \times \{0\}$. The manifold M_1 is obtained from M_0 by removing the interior of these attaching tubes and attaching back a copy of $D^2 \times S^1$ to every boundary component ∂T_s, in such a way that a meridian $S^1 \times \{x_0\}$ of $D^2 \times S^1$ is mapped to a longitude $l_s \sim S^1 \times \{y_0\}$, $y_0 \in \partial D^2$ of K_s determined by the framing (such a longitude is unique up to isotopy).

This defines an equivalence relation; in particular, $M_1 \sim_\sigma M_0$ implies $M_0 \sim_\sigma M_1$ because the dual decomposition of such an $\mathcal{H}$ also consists of 2-handles only. If $M_0 \sim_\sigma M_1$, then M_0 is orientable if and only if M_1 is orientable, and in such a case any special triad connecting them is necessarily orientable.

Let us consider for a while the orientable case. We have the following (see [**Wa**]).

PROPOSITION 19.7. *Let M_0, M_1 be compact connected orientable boundaryless 3-manifolds. Then $M_1 \sim_\sigma M_0$ if and only if there is an orientable 4-dimensional triad (W, M_0, M_1); that is, for suitable orientations, $[M_0] = [M_1] \in \Omega_3$.*

COROLLARY 19.8. *$M \sim_\sigma S^3$ if and only if for every orientation of M, $[M] = 0 \in \Omega_3$.*

Proofs: Let us prove the corollary, assuming the proposition. If $M_1 \sim_\sigma S^3$, then by completing with one 4-handle attached at S^3 the dual $\mathcal{H}^*$ of a special decomposition $\mathcal{H}$ of a given triad (W, S^3, M), we get a triad $(V, M, \emptyset)$ so that $M = \partial V$. On the other way round, assume that $M = \partial V$ for some orientable connected 4-manifold V. Then the triad $(V, \emptyset, M)$ admits an ordered handle decomposition with one 0-handle, and no 4-handles. By removing the 0-handle we get an orientable triad (W, S^3, M) and we conclude by applying to it the proposition.

Let us prove now the proposition. One implication is trivial. For the other, let us start with any orientable triad (W, M_0, M_1). It has an ordered handle decomposition without both 0 and 4-handles. Moreover, we can assume that all handles of a given index are attached simultaneously at disjoint attaching tubes. The idea is to *trade* first every 1-handle for a 2-handle in such a way that the 4-manifold W could be modified, but its boundary is kept fixed. Every 1-handle does not destroy the orientability. Moreover, by the uniqueness of disks up to diffeotopy, we can assume that all attaching tubes of the 1-handles are contained in a smooth 3-disk D in $M_0 \sim M_0 \times \{1\}$; then after having attached the 1-handles to $C_0 = M_0 \times [0, 1]$ at $M_0 \sim M_0 \times \{1\}$, we get a 4-manifold W_1 such that ∂W_1 is the disjoint union of M_0 with the connected sum of M_0 with some copies of $S^2 \times S^1$. A 4-manifold V_1 with the same boundary can be obtained by surgery along a link L in M_0 formed by d (d being equal to the number of the above 1-handles) unknotted and unlinked components contained in the above disk D, such that each component K_s is endowed with the framing associated with the distinguished longitude carried by a collar in a 2-disk D_s in D such that $\partial D_s = K_s$. In fact, such a surgery performed on S^3 at one knot component realizes the standard genus-1 Heegaard splitting of $S^2 \times S^1$ with Heegaard surface $S^1 \times S^1 \subset S^2 \times S^1$, the first S^1 factor being the equator of S^2. The rest of the handle decomposition is unchanged and we get a 4-dimensional triad (W', M_0, M_1) having an ordered handle decomposition $\mathcal{H}'$ without 0, 1 and 4-handles. To also trade the 3-handles for some 2-handles, we manage similarly by using the dual decomposition of $\mathcal{H}'$. Finally, we get a triad $(W", M_1, M_0)$ with a handle decomposition $\mathcal{H}"$ consisting only of 2-handles. The proposition is proved.

■

Now we state two main theorems of this chapter.

THEOREM 19.9. (Lickorish-Wallace) *Every orientable connected compact boundaryless 3-manifold M is surgery equivalent to S^3 ($M \sim_\sigma S^3$).*

THEOREM 19.10. $\Omega_3 = 0$.

By Corollary 19.8, they can be, and have been, considered as a corollary of each other. For example, Lickorish proved Theorem 19.9 as an application of his main results about the generators of the mapping class groups of surfaces and, in doing so, he got a new proof that $\Omega_3 = 0$. On the other hand, Wallace obtained Theorem 19.9 via Corollary 19.8, as it was already known (by several different proofs) that $\Omega_3 = 0$. We will develop in great detail this theme.

19.2.1. Nonorientable surgery. There is a nonorientable version of Corollary 19.8. Denote by $\mathfrak{M}$ the nonorientable 3-manifold which is the boundary of the nonorientable 4-manifold $\mathfrak{V}$ (unique up to diffeomorphism) with a handle decomposition consisting of one 0-handle and one 1-handle. $\mathfrak{M}$ is the nonorientable total space of a fibration over S^1 with fibre S^2.

PROPOSITION 19.11. *Let M be a compact connected boundaryless nonorientable 3-manifold. Then $M \sim_\sigma \mathfrak{M}$ if and only if $[M] = 0 \in \eta_3$.*

The proof is similar to the orientable case.

19.3. Proofs of $\Omega_3 = 0$

In this section we discuss a few "direct" proofs of Theorem 19.10, so that Theorem 19.9 will result as a corollary.

- *(Via immersions in $\mathbb{R}^5$ and Seifert's surfaces)* This is the first proof of $\Omega_3 = 0$ (Rohlin 1950, see his papers translated in [**GM**]). If a compact connected orientable boundaryless 3-manifold $\hat{M}$ is *embedded* in $\mathbb{R}^5$, then by Proposition 13.7 it admits an orientable Seifert's surface W; in particular, $\hat{M} = \partial W$. To prove the theorem, it is enough to show that for every orientable M there is an orientable triad $(V, M, \hat{M})$ such that $\hat{M}$ is embedded in $\mathbb{R}^5$. This is a particular case of Theorem 7.27.
- *(Via vanishing of characteristic numbers)* We will see in Section 19.6 that a compact orientable 3-manifold M is parallelizable (without using that $\Omega_3 = 0$). Then M is an oriented boundary, this being a particular case of Proposition 16.6. In a sense this is the most "modern" proof, being a special case of the general determination of bordism groups based on Thom's spaces and characteristic numbers.

19.4. Proofs of Lickorish-Wallace theorem

In this section we discuss a few "direct" proofs of Theorem 19.9, so that Theorem 19.10 will result as a corollary.

These proofs are based on Heegaard splittings.

(*Via Dehn twists)* The main Lickorish result establishes a distinguished set of generators of the mapping class group $\text{Mod}(\Sigma_g)$. Let C be a smooth circle on the surface Σ_g. Assume that C is *essential*; that is, it is not the boundary of a smooth disk embedded in Σ_g. Fix an auxiliary trivialization

$$\psi : S^1 \times [-1, 2] \to U$$

of a tubular neighbourhood of C. Give $S^1 \times [-1, 2]$ the coordinates $(e^{i\theta}, t)$, $\theta \in [0, 2\pi]$. Let $\rho : [-1, 2] \to [0, 1]$ be a smooth function which is constantly equal to 0 on $[-1, 0]$ and constantly equal to 1 on $[1, 2]$, while it is strictly increasing on $[0, 1]$. Then define the diffeomorphism

$$\tau_C : \Sigma_g \to \Sigma_g$$

which is the identity outside U, and is defined on U as $\psi \circ h \circ \psi^{-1}$, where

$$h(e^{i\theta}, t) = (e^{i(\theta + 2\pi\rho(t))}, t) \ .$$

The maps τ_C and τ_C^{-1} are called Dehn's twists along C. Their classes in $\mathrm{Mod}(\Sigma_g)$ do not depend on the arbitrary choices we made, including the fact that C is considered up to ambient isotopy. We also refer to these classes as Dehn's twists.

THEOREM 19.12. [**Lick**] $\mathrm{Mod}(\Sigma_g)$ *is generated by the Dehn twists along essential smooth circles.*

We assume this theorem and show how to deduce that $M \sim_\sigma S^3$.

LEMMA 19.13. *Let* $[\psi] = [\tau_k] \circ \cdots \circ [\tau_1]$ *be an element of* $\mathrm{Mod}(\Sigma_g)$ *expressed as a composition of* k *Dehn's twists. Then there exist two systems of* k *disjoint solid tori,* $V_1, \ldots, V_k$ *and* $V'_1, \ldots, V'_k$*, in the interior of the handlebody* $\mathfrak{H}_g$ *such that* ψ *extends to a diffeomorphism*

$$\bar{\psi} : \mathfrak{H}_g \setminus \cup_j \mathrm{Int}(V_j) \to \mathfrak{H}_g \setminus \cup_j \mathrm{Int}(V'_j) \ .$$

Proof : If $k = 0$, then ψ is isotopic to the identity and the statement is trivially verified. Assume that $k = 1$, $\psi = \tau = \tau_C^{\pm 1}$. Consider a collar $C(\Sigma_g) \sim \Sigma_g \times [0,1]$ of $\Sigma_g = \partial \mathfrak{H}_g$ in $\mathfrak{H}_g$. Set $V \sim U(C) \times [1/2, 1] \subset C(\Sigma)$ (up to corner smoothing) where $U(C)$ is a annular neighbourhood of C in Σ_g. Set $V' = V$. Then an extension of τ is obtained by taking a parallel copy of τ on every leaf $U(C) \times \{s\}$, $0 \leq s \leq 1/2$, and setting $\bar{\tau}$ equal to the identity on the remaining part of $\mathfrak{H}_1 \setminus \mathrm{Int}(V)$. If $k = 2$ we can extend τ_2 along C_2 by the same method, provided that the "tunnel" V_2 is more deeply in the interior of $\mathfrak{H}_g$, so that $V_1 \cap V_2 = \emptyset$ and $\bar{\tau}_1 = \mathrm{id}$ along V_2. Then set $V'_2 = V_2$, $V'_1 = \bar{\tau}_2(V_1)$, so that $\bar{\tau}_2 \circ \bar{\tau}_1$ is a desired extension of ψ. Iterating the same method, by induction we get the result for every $k \geq 0$.

■

Consider any genus g Heegaard splitting presented in the form

$$M \sim \mathfrak{H}_g \amalg_{[\phi]} \mathfrak{H}_g, \quad [\phi] \in \mathrm{Mod}(\Sigma_g)$$

as in Section 19.1; recall that this includes the choice of an auxiliary diffeomorphism γ which reverses the orientation. We know that also S^3 admits a genus g splitting

$$S^3 = \mathfrak{H}_g \amalg_{[\phi']} \mathfrak{H}_g \ .$$

Set $\psi = \phi^{-1} \circ \phi' = (\phi^{-1} \circ \gamma^{-1}) \circ (\gamma \circ \phi')$. Apply the above lemma to ψ. Then we get an extension

$$\bar{\psi} : \mathfrak{H}_g \setminus \cup_j \mathrm{Int}(V_j) \to \mathfrak{H}_g \setminus \cup_j \mathrm{Int}(V'_j)$$

which by construction extends to a diffeomorphism

$$\bar{\psi} : S^3 \setminus \cup_j \mathrm{Int}(V_j) \to M \setminus \cup_j \mathrm{Int}(V'_j) \ ,$$

and this readily shows that $M \sim_\sigma S^3$.

■

(By induction on a Heegaard diagram complexity) Last but not least, we present the clever proof of [**Rourke**]. Let us fix an orientation of M; it is understood that all manifolds produced by the following construction are oriented and that the orientations are compatible. We are going to realize that $S^3 \sim_\sigma M$.

LEMMA 19.14. *If* $M = M_1 \# M_2$ *and* $S^3 \sim_\sigma M_j$, $j = 1, 2$, *then* $S^3 \sim_\sigma M$.

Proof : As $S^3 = S^3 \# S^3$, the lemma follows immediately.

■

We write
$$M = M(x, y)$$
to mean that M is encoded by a genus g Heegaard diagram (Σ, x, y) where $x = \{x_1, \dots, x_g\}$ and $y = \{y_1, \dots, y_g\}$ are the two non-dividing families of simple smooth circles on the surface Σ, earlier denoted by C^- and C^+, respectively. Recall that $x \pitchfork y$.

Let $z = \{z_1, \dots, z_g\}$ be another family of g smooth circles on Σ whose union does not divide the surface. Assume that $z \pitchfork x$ and $z \pitchfork y$. Recalling the reconstruction of $M = M(x, y)$ from the diagram, we can assume that z is traced on the Heegaard surface $\Sigma \sim \Sigma \times \{0\}$. Give an orientation to each z_j, fix a system of disjoint tubular neighbourhoods U_j of each z_j in M such that $\partial U_j \pitchfork \Sigma$ along a pair of curves parallel to z_j, and select the longitude $l_j \subset \partial U_j$ given by the component of $\partial U_j \cap \Sigma$ whose orientation is parallel to the one of z_j. For every j, up to isotopy there is a unique framing $\rho_j : S^1 \times D^2 \to U_j$ so that the longitude l_j is carried by ρ_j; thus we have determined a framed link $L := \cup_j (z_j, l_j)$ in $M = M(x, y)$. These trivializations are used as attaching maps of disjoint 2-handles so that we have constructed a special triad
$$(W, M, \tilde{M}), \ \tilde{M} \sim_\sigma M \ .$$
The following simple lemma, which is, in fact, the core of the proof, establishes a key relationship between surgery equivalence and Heegaard splitting.

LEMMA 19.15. $\tilde{M} \sim M(x, z) \# M(z, y)$.

Proof : Denote by $M_0(x, z)$ the manifold with spherical boundary obtained by removing from $M(x, z)$ the interior of a smooth embedded 3-disk. Similarly, $M_0(z, y)$ is obtained from $M(z, y)$. It follows straightforwardly by comparing the reconstruction of $M(x, z)$ and $M(z, y)$ from the diagrams and the construction of $\tilde{M}$ by surgery on M along the framed link $L := \cup_j (z_j, l_j)$ that, up to diffeomorphism, $\tilde{M}$ is obtained by gluing $M_0(x, z)$ and $M_0(z, y)$ by a diffeomorphism between the boundaries. With the terminology of Section 7.5.2, $\tilde{M}$ is a weak connected sum of $M(x, z)$ and $M(z, y)$. Then by Smale's theorem (Proposition 7.13, (1), $m = 3$) it is a true connected sum.

■

The last ingredient is a suitable *measure of the complexity of the Heegaard diagrams.* Let (Σ, x, y) be such a diagram of genus g. Recall that every

$x_i \cap y_j$ is a finite set and denote by $|x_i \cap y_j|$ the number of elements (we stress that it is the "geometric" number, no algebraic intersection numbers are involved). Then set

$$c(\Sigma, x, y) := (g, r := \min_{i,j} |x_i \cap y_j|) \in \mathbb{N}^2$$

where $\mathbb{N}^2$ is endowed with the lexicographic order. We will achieve the result by (double) induction on the complexity c of a given splitting of M.

The initial step is when $g = 0$; in such a case, by the very definition, M is a twisted 3-sphere, so it is a true smooth sphere again by Smale theorem (Proposition 7.13, (2), $m = 3$); the empty surgery does the job.

Let $M = M(x, y)$ be of complexity $c = (g, r)$ and assume that $S^3 \sim_\sigma M'$ for every M' admitting an encoding diagram of complexity $c' = (g', r') < c = (g, r)$.

If $c = (g, 1)$, then the given diagram is a stabilization of a diagram of genus $g - 1$, hence $S^3 \sim_\sigma M$ by the inductive hypothesis.

If $c = (g, 0)$, it is not restrictive to assume that $x_1 \cap y_1 = \emptyset$.

Claim 1. *There exists a non-separating circle z_1 on Σ which intersects each of x_1 and y_1 transversely at a single point.*

Assuming this fact, extend z_1 to a non-dividing family z of g circles on Σ, $z \pitchfork x$ and $z \pitchfork y$. Then both $M(x, z)$ and $M(z, y)$ have encoding diagrams with $r = 1$ and we conclude by applying the previous case and Lemmas 19.14, 19.15.

Assume that $r > 1$. It is not restrictive to assume that $r = |x_1 \cap y_1|$.

Claim 2. *There exists a non-separating circle z_1 on Σ which intersects each of x_1 and y_1 transversely at a number of points strictly less than r.*

Assuming this fact, extend z_1 to a non-dividing family z of g circles on Σ, $z \pitchfork x$ and $z \pitchfork y$. Then both $M(x, z)$ and $M(z, y)$ have encoding diagrams of the same genus g but with strictly smaller complexity. Then by the inductive hypothesis, S^3 is surgery equivalent to both and again we can conclude by applying Lemmas 19.14 and 19.15.

It remains to prove the two claims. As for Claim 1, there are two possibilities; $\Sigma' := \Sigma \setminus (x_1 \cup y_1)$ is either connected or nonconnected. Take a small segment γ in Σ transverse to x_1 at one point, with endpoints p_0, p_1; similarly let γ' be transverse to y_1 at one point, with endpoints p'_0, p'_1. If Σ' is not connected, up to reordering, we can assume that the couples of endpoints p_0, p'_0 and p_1, p'_1 belong to different connected components. Then in both cases a smooth circle z_1 in Σ with the required properties can be obtained of the form

$$z_1 = \gamma \cup \alpha \cup \gamma' \cup \alpha' \ ,$$

where α is a smooth arc which connects p_0 and p'_0 and α' is such an arc connecting p_1 and p'_1.

As for Claim 2, let A and B be two points of $x_1 \cap y_1$ which are adjacent in x_1. Then there is an arc α in x_1 which intersects y_1 only at its endpoints

A and B. These points also divide y_1 into two arcs, β and γ. As y_1 does not separate Σ, there is at least one of these arcs, say β, such that $\alpha \cup \beta$ does not separate Σ. Then we can construct z_1 using a parallel copy α' of α which near A is in the direction of β, completed by a segment β' close to β. One realizes that z_1 intersects x_1 in at most $r-1$ points and intersects y_1 in at most one point. So z_1 has the desired properties. This proof of Theorem 19.9 is now complete.

∎

19.4.1. On Kirby's calculus. We have proved that for every orientable compact connected, boundaryless 3-manifold M, there is a special triad (W, S^3, M) which realizes the surgery equivalence $S^3 \sim_\sigma M$, so that W admits an ordered handle decomposition consisting only of 2-handles. Every such a handle decomposition with k 2-handles is encoded by a framed link L in S^3 with k constituent knots K_j, $j = 1, \dots, k$. For every K_j, its framing is encoded by a parallel longitude l_j; fixing auxiliary parallel orientations of both K_j and l_j, this last is encoded by the linking number $L(K_j, l_j)$; that is, equivalently, by the intersection number of l_j with any oriented Seifert surface of K_j in S^3. The natural question is how two framed links representing a given manifold are related to each other. A handle decomposition can be modified by handle sliding and this can be translated in terms of the corresponding framed links. Moreover, we must consider the possibility of modifying a special triad without changing its boundary. A distinguished way to do it consists of attaching a 2-handle with attaching circle contained and unknotted in a 3-ball disjoint from the other link components, and with framing equal to ± 1. One realizes that this does not modify the boundary, while we pass from W to $W \# \pm \mathbf{P}^2(\mathbb{C})$. This is called an *elementary blow-up move*. We can consider also the inverse (negative) move of removing such a handle. An important Kirby's result [**Kirby2**] can be formulated, somewhat qualitatively, as follows.

THEOREM 19.16. *Two framed links L_1 and L_2 in S^3 carry two realizations of the surgery equivalence $S^3 \sim_\sigma M$ if and only if they are related to each other by a finite sequence of modifications which either translate* 2*-handle sliding or are positive/negative elementary blow-up moves.*

The proof is rather difficult, based on Cerf's theory [**Ce2**]. After such a qualitative statement, successive efforts have been devoted to convert it into an efficient diagrammatic calculus on framed links in S^3. Kirby himself found a generator (called "band move") for the handle sliding; this is *not a 'local' move* and resembles a move described above on Heegaard diagrams. Later, in [**FR**], the authors point out an *infinite* family of *local* moves (qualitatively, "one" move depending on an integer parameter) generating the whole calculus. Finally, in [**Mart2**], we can find a generating *finite* family of *local* moves.

19.5. On $\eta_3 = 0$.

Referring to Section 19.2.1, the following two theorems can be obtained as a corollary of each other.

THEOREM 19.17. *Every nonorientable compact connected boundaryless 3-manifold M is surgery equivalent to $\mathfrak{M}$ ($M \sim_\sigma \mathfrak{M}$).*

THEOREM 19.18. $\eta_3 = 0$.

19.5.1. On some proofs that $\eta_3 = 0$. Certainly, it is contained in the general statement of Thom's Theorem 17.23 and, in a sense, this is its first proof. However, Rohlin claimed, without further explanation (see [**GM**]), that the method he had used to prove $\Omega_3 = 0$ also allows proving that $\eta_3 = 0$. This is not so immediate. Starting from a generic immersion of M (possibly nonorientable) in $\mathbb{R}^5$, the "embedding up to bordism" of Theorem 7.27 can be extended; to eliminate the double points of a generic immersion, we have to consider another case where a circle of double point C has trivial double covering $\tilde{C} \to C$, and the components of $\tilde{C}$ have nonorientable tubular neighbourhoods in M. Let us assume that M is embedded in $\mathbb{R}^5$. If a tubular neighbourhood U of M in $\mathbb{R}^5$ is associated with a splitting $T(M) \oplus \xi$ of the restriction of $T(\mathbb{R}^5)$ to M, we cannot assume in general that ξ has a nowhere vanishing section and hence we cannot assume that there is a possibly nonorientable Seifert surface (recall Remark 13.11). To conclude, it would be enough to find M' embedded in some 5-manifold X such that $[M] = [M'] \in \eta_3$, $[M'] = 0 \in \mathcal{H}^2(X, \mathbb{Z}/2\mathbb{Z})$, and there is a splitting $T(M) \oplus \xi'$ of the restriction of $T(X)$ to M' such that ξ' has a nowhere vanishing section. This can be achieved as follows (see also the suggestion at pag. 91 of [**GM**]). Let M be embedded in $\mathbb{R}^5$ as above. Consider the Euler class of ξ belonging to $\eta_1(M)$. This is represented by a smooth circle C on M. Take the blow up X of $\mathbb{R}^5$ along C (see Section 7.10); let M' be the blow up of M along C which is embedded in X as the strict transform of M. One can check that $M' \subset X$ satisfy the required properties. In particular $[M'] = [M] + [S^1 \times \mathbf{P}^2(\mathbb{R})] = [M] \in \eta_3$.

19.5.2. On some proofs that $M \sim_\sigma \mathfrak{M}$. In [**Lick2**], Lickorish extended to nonorientable surfaces his main result on the generators of the mapping class group. This allows him to also extend the corollary about the surgery equivalence to the nonorientable case. In [**AG**], the simpler clever proof of [**Rourke**] has been extended to the nonorientable case.

19.6. Combing and framing

A main result of this section will be the following.

Every compact connected orientable boundaryless 3-manifold M is parallelizable.

Current modern proofs of this primary result in 3-dimensional differential topology (originally attributed to Stiefel [**Sti**]) use either a mixture of *spin structures* and *Stiefel Whitney classes* theories (see for instance [**Ge**], Section 4.2), or a refinement due to Kaplan [**Ka**] of the Lickorish-Wallace theorem, using the Kirby calculus (see also [**FM**], Section 9.4.). We do not dispose of these tools. Nevertheless, by following [**BL**], we will provide two self-contained elementary proofs, revealing different aspects of the question. The first proof uses some ideas of the last-mentioned approach but *it avoids the use of both Lickorish-Wallace Theorem and Kirby calculus.* Recall that one of our proofs that $\Omega_3 = 0$, hence of Lickorish-Wallace theorem, is based on the fact that M is parallelizable, so we are not introducing any loop in our discussion. The second proof will result from a study of combings of a 3-manifold, associated with othogonal 2-planes distributions.

From now on, M will denote a compact connected orientable boundaryless 3-manifold. Like every odd-dimensional manifold, M is combable; that is, it carries nowhere vanishing tangent vector fields v. These are considered up to smooth homotopy through such fields, and a homotopy class is called a *combing* of M. We will often confuse a combing with suitable representatives determined case by case. If necessary, we use the term 'field' instead of combing, to stress that we are dealing with a certain representative. A *framing* $\mathcal{F}$ of $T(M)$ is a triple (v, w, z) of pointwise linearly independent tangent vector fields. Framings are considered up to homotopy as well. Fixing any auxiliary Riemannian metric g on M, we can assume that a given combing is represented by a unitary field for g, and every framing is represented by pointwise orthonormal fields. A framing determines an orientation of M so that orientability of M is a necessary condition. If M is *oriented* and parallelizable, then some framings induce the given orientation. From now on, we will assume that M is *oriented*, by fixing an auxiliary orientation.

19.6.1. Framing via even surgery. The first remark is that it is enough to prove that M is almost-parallelizable. A *quasi-framing* of M is a framing of $T(M)$ over a submanifold of the form

$$M_0 := M \setminus \mathrm{Int}(B) \ ,$$

where B is a smooth 3-disk in M. We say that M is *almost-parallelizable* if it admits a quasi-framing. In such a case, by the uniqueness of the disk up to ambient isotopy, we see that the choice of the disk B is immaterial.

Lemma 19.19. *M is parallelizable if and only if it is almost-parallelizable.*

Proof : One implication is trivial. As for the other implication, we can assume that B is contained in a chart of M and looks standard therein as well as the auxiliary metric. Then the restriction of a quasi-framing $\mathcal{F}'$ to $\partial B = S^2$ is encoded by a map

$$\rho : S^2 \to SO(3) \ .$$

We know that $SO(3) \sim \mathbf{P}^3(\mathbb{R})$ (Example 4.8), with S^3 as universal covering space, hence $\pi_2(SO(3)) \sim \pi_2(S^3) = 0$. It follows that ρ extends to

$$\hat{\beta} : B \to SO(3) \ ,$$

and that $\mathcal{F}'$ extends to a framing $\mathcal{F}$ of the whole $T(M)$.

■

Let M be obtained by longitudinal surgery along a framed link L in S^3; we write

$$M = \chi(S^3, L) \ ;$$

M is the final boundary of a triad $(W, \emptyset, \chi(S^3, L))$, where W is obtained by attaching disjoint 2-handles to D^4 at $S^3 = \partial D^4$. Every 2-handle $D^2 \times D^2$ determines a constituent knot K of L, so that $\partial D^2 \times D^2 \sim N(K)$, $N(K)$ being a tubular neighbourhood of K in S^3, $\partial D^2 \times \{0\}$ being identified with a longitude l_K on $\partial N(K)$ along K. The framing of every component K of L is encoded by the linking number $n_K \in \mathbb{Z}$ between K and the longitude l_K, where K and l_K are co-oriented in such a way that the projection of L_K to K is of degree 1. We say that the surgery is *even* if for every constituent knot K of L, $n_K \in 2\mathbb{Z}$.

PROPOSITION 19.20. *Let $(W, \emptyset, M)$ be the triad associated to an even surgery $M = \chi(S^3, L)$. Then W is parallelizable.*

Proof : To simplify the notation, we give the proof for a one-component link, but this generalizes straightforwardly. So let $L = (K, n)$, $n \in 2\mathbb{Z}$. Both D^4 and $D^2 \times D^2$ are parallelizable, so we have to show that they carry some framings which match on $N(K)$. Fix a reference framing $\mathcal{F}_0$ on D^4; the restriction to $N(K)$ of any framing $\mathcal{F}$ on the 2-handle is encoded by a map $\rho : N(K) \to SO(4)$. Viewing S^3 as the group of unit quaternions, we can construct a 2-fold covering map $S^3 \times S^3 \to SO(4)$ showing that $\pi_1(SO(4)) = \mathbb{Z}/2\mathbb{Z}$ (see Example 4.8). As the solid torus $N(K)$ retracts to $K \sim S^1$, ρ determines an element of $\mathbb{Z}/2\mathbb{Z}$, and the two framings coincide on $N(K)$ if and only if this is equal to 0. It can be readily seen that this element is equal to the number n mod (2).

■

COROLLARY 19.21. *Let $M = \chi(S^3, L)$ be an even surgery. Then M is stably-parallelizable (i.e. $T(M) \oplus \epsilon^1$ is a product bundle).*

Proof : Let $(W, \emptyset, \chi(S^3, L))$ be as above. Then $T(W)_M = T(M) \oplus \nu$, where ν is a trivial normal line bundle of $M = \partial W$ in W. We know by the proposition that $T(W)$ is a product bundle.

■

LEMMA 19.22. *If M is stably parallelizable then it is almost-parallelizable.*

Proof : As $T(M) \oplus \epsilon^1 = M \times \mathbb{R}^4$, every T_xM is an oriented 3-plane in $\mathbb{R}^4$. So we have a smooth classifying map $\rho : M \to S^3$, where the sphere is considered as the space of oriented 3-planes in $\mathbb{R}^4$, and $T(M)$ is the pull-back of the corresponding tautological bundle (see Chapter 4). Now we know that M_0 retracts to a 2-dimensional spine $\mathbf{P}_0$ as in Section 19.1.2. Hence the restriction of ρ to $\mathbf{P}_0$ is not surjective; then it is homotopic to a constant map. It follows that the restriction of $T(M)$ to $\mathbf{P}_0$, hence to M_0, is a product bundle.

■

REMARK 19.23. Lemma 19.22 holds in every dimension n; the key point is that $M \setminus \mathrm{Int}(B^n)$ has the homotopy type of a CW-complex of dimension less than or equal $n-1$ (see Section 9.3.1).

Recall the notion of weak connected sum given in Section 7.5.2. We know by Smale's theorem that 3-dimensional weak connected sums are veritable connected sums, but we do not need this fact in the present discussion. The following lemma is trivial.

LEMMA 19.24. *If there exists M' such that a weak connected sum of M and M' is parallelizable, then M is almost parallelizable (hence, parallelizable).*

Recall that a Heegaard splitting (of some genus g) of M can be encoded by a non-dividing family L of g smooth circles on the boundary $\partial\mathfrak{H}_g$ of a handlebody $\mathfrak{H}_g$. We can assume that $\mathfrak{H}_g$ is embedded in a standard way in S^3 so that $\mathfrak{H}'_g := \overline{S^3 \setminus \mathfrak{H}_g}$ is also a handlebody of genus g, and we have a Heegaard splitting of S^3. Give every component K of L the framing carried by a tubular neighbourhood of K in $\partial\mathfrak{H}_g$. Then we have a framed link L in S^3. By applying the proof of Lemma 19.15, we readily prove the following.

LEMMA 19.25. *$\chi(S^3, L)$ is a weak connected sum of M and M', for some M'.*

By combining the above lemmas, to show that M is almost parallelizable (hence parallelizable) it is enough to show that we can implement the above construction in such a way that the surgery $\chi(S^3, L)$ is even. Fix any embedding $L \subset \partial\mathfrak{H}_g \subset \mathfrak{H}_g \subset S^3$ as above. Fix a system $\mu = \{m_1, \dots, m_g\}$ of g meridians on $\partial\mathfrak{H}_g$ (which bound 2-disks properly embedded in $\mathfrak{H}_g$) and a dual system of g meridians $\lambda = \{l_1, \dots, l_g\}$ for the complementary handlebody $\mathfrak{H}'_g$. A Dehn twist on $\partial\mathfrak{H}_g$ along a curve m_i extends to a diffeomorphism of the whole $\mathfrak{H}_g$. Hence we can modify the family L by applying any finite sequence of such Dehn twists, keeping the fact that $\chi(S^3, L)$ is a weak connected sum of M and M', for some M'. We are reduced to prove the following lemma.

LEMMA 19.26. *Up to a suitable finite sequence of Dehn twists along the meridians in μ, $\chi(S^3, L)$ is an even surgery.*

Proof : The question can be reduced to $\mathbb{Z}/2\mathbb{Z}$-linear algebra on $\eta_1(\partial\mathfrak{H}_g)$. Start with any surgery $\chi(S^3, L)$ which realizes a weak connected sum of M and a certain M' as above. The union of curves in the families μ and λ form a symplectic basis of $\eta_1(\partial\mathfrak{H}_g)$ with respect to the intersection form. So, by confusing classes mod (2) and representatives and setting $L = \{K_1, \dots, K_g\}$, we have the $\mathbb{Z}/2\mathbb{Z}$-linear combinations:

$$K_j = \sum_{i=1}^{g} (a_i^j m_i + b_i^j l_i) \ .$$

The framing mod (2) of K_j is given by

$$n_j = \sum_i a_i^j b_i^j \in \mathbb{Z}/2\mathbb{Z} \ .$$

A Dehn twist T_j along m_i acts on $\eta_1(\partial \mathfrak{H}_g)$ so that

$$T_i(l_i) = l_i + m_i \ ,$$

while it is the identity on the other $2g - 1$ elements of the given basis. All intersection numbers mod (2) of the curves of L vanish; that is,

$$K_r \bullet K_s = 0, \ r, s = 0, \dots g \ .$$

This means that the coefficients of the above linear combinations satisfy the system of conditions:

$$\sum_{i=1}^{g} (a_i^r b_i^s + a_i^s b_i^r) = 0, \ r, s = 0, \dots g \ . \tag{19.1}$$

We allow ourselves to apply twist combinations of the form $T_1^{x_1} \dots T_g^{x_g}$. Then we want to show that the $\mathbb{Z}/2\mathbb{Z}$-linear non homogeneous system

$$\sum_{i=1}^{g} (x_i + b_i^r) a_i^r = 0, \ r = 1, \dots g \tag{19.2}$$

admits a solution in $(\mathbb{Z}/2\mathbb{Z})^g$. Note that we tacitly use several times that $z = z^2$ for every $z \in \mathbb{Z}/2\mathbb{Z}$. If for every r, all $a_i^r = 0$, then every $(x_1, \dots, x_g)$ is a solution. Otherwise we can assume that $a_1^1 = 1$. Then the solution of the equation

$$\sum_{i=1}^{g} (x_i + b_i^1) a_i^1 = 0$$

is of the form $x_1 = \sum_{j=2}^{g} c_j x_j$. By replacing in the other equations and using the relations 19.1, we are reduced to solve a system in $x_2, \dots, x_g$ of the same form

$$\sum_{i=2}^{g} (x_i + \tilde{b}_i^r) \tilde{a}_i^r = 0, \ r = 2, \dots, g \ ,$$

with

$$\tilde{a}_i^r = a_1^r a_i^1 + a_i^r, \ \tilde{b}_i^r = a_1^r b_i^1 + b_1^r \ .$$

One verifies directly that these new coefficients formally satisfy the corresponding conditions 19.1. So we can conclude by recurrence.

∎

Our first proof that M is parallelizable is now complete.

REMARK 19.27. In [**Ka**] (see also [**FM**]) the general result is proved that, *for every* M as above, there is an even surgery $M = \chi(S^3, L)$. Starting from any surgery presentation of M with associated triad $(W, \emptyset, M)$, which exists by the Lickorish-Wallace theorem, the proof consists of an algorithm that progressively reduces the number of “odd” link components using the moves of the Kirby calculus. This proof does *not* use the harder fact that the Kirby calculus connects any two surgery presentations of M [**Kirby2**].

Next, we will elaborate on the second proof.

19.6.2. On the cobordism ring of an orientable 3-manifold. We now specialize the results of Chapters 13. In the present situation, the relevant co-bordism modules are

$$\mathcal{H}^j(M;\mathbb{Z}/2\mathbb{Z}),\ \mathcal{H}^j(M;\mathbb{Z}),\ j = 0, 1, 2, 3\ .$$

We summarize here some properties which we will use.

- $\mathcal{H}^3(M;\mathbb{Z}/2\mathbb{Z}) \sim \mathcal{H}^0(M;\mathbb{Z}/2\mathbb{Z}) \sim \mathbb{Z}/2\mathbb{Z}$ by the isomorphism which associates the usual generator of $\mathcal{H}^3(M;\mathbb{Z}/2\mathbb{Z})$ to the fundamental class mod (2) $[M]$; similarly over $\mathbb{Z}$.
- $\mathcal{H}^2(M;\mathbb{Z}/2\mathbb{Z}) = \eta^2(M) = \eta_1(M)$
- $\mathcal{H}^2(M;\mathbb{Z}/2\mathbb{Z}) \sim \mathcal{H}^1(M;\mathbb{Z}/2\mathbb{Z})$ in a natural way: if

$$\alpha = [F] \in \mathcal{H}^1(M;\mathbb{Z}/2\mathbb{Z})$$

we can assume that the embedded surface $F \subset M$ is connected and does not divide M if $\alpha \neq 0$. If γ is a smooth simple arc in M transverse to F at one point, it can be completed to a smooth circle c by means of an arc γ' contained in $M \setminus F$, so that $[F] \sqcup [c] = 1$. *Viceversa*, if $[c] \neq 0 \in \mathcal{H}^2(M;\mathbb{Z}/2\mathbb{Z})$, then it is part of a basis $\mathcal{B}$ of $\mathcal{H}^2(M;\mathbb{Z}/2\mathbb{Z})$ which is finite-dimensional. The functional $[c]^*$, belonging to the dual basis composed with the natural homomorphism $\pi_1(M) \to \eta_1(M)$, defines a $\mathbb{Z}/2\mathbb{Z}$-valued representation of the fundamental group that can be realized by a connected hypersurface F, so that in particular $[F] \sqcup [c] = 1$. Moreover, we can assume that F transversely intersects c at one point: if F intersects c at an odd number of points, we can reduce them to one by attaching suitable embedded 1-handles along arcs of c and performing surgeries of F.

- If c is a connected oriented smooth circle in M such that $[c] = 0 \in \mathcal{H}^2(M;\mathbb{Z})$, then there is an oriented Seifert surface for c in M; if $[c] = 0 \in \mathcal{H}^2(M;\mathbb{Z}/2\mathbb{Z})$ then there is a possibly nonorientable Seifert surface for c in M;
- Consider the natural forgetting morphism $\mathcal{H}^2(M;\mathbb{Z}) \to \mathcal{H}^2(M;\mathbb{Z}/2\mathbb{Z})$.

LEMMA 19.28. *A class $\alpha \in \mathcal{H}^2(M;\mathbb{Z})$ belongs to the kernel of the forgetting morphism $\mathcal{H}^2(M;\mathbb{Z}) \to \mathcal{H}^2(M;\mathbb{Z}/2\mathbb{Z})$ if and only if α is an* even class*; that is, there is $\beta \in \mathcal{H}^2(M;\mathbb{Z})$ such that $\alpha = 2\beta$.*

Proof : We can assume that α is represented by a connected oriented smooth circle c. By hypothesis, c is the boundary of a possibly nonorientable connected compact surface F embedded in M. If F is orientable, then $\alpha = 0$ and we are done. If F is not orientable, it follows from the classification of surfaces that there is a smooth 1-submanifold C on $\text{Int}(F)$ such that a tubular neighbourhood $U(C)$ of C in F is a union of Möbius strips and $F \setminus C$ is orientable. Then orient $F \setminus C$ in such a way the oriented c inherits the boundary orientation, and orient consequently $C' := \partial U(C) \subset F \setminus \text{Int}(U(C))$. Then $[c] = [C'] \in \mathcal{H}^2(M; Z)$ and $[C'] = 2[C"]$, where $C"$ is the union of the cores of $U(C)$ oriented in such a way that the restriction of the projection of C' to every core is of positive degree.

■

19.6.3. Combings and orthogonal plane distributions. Let v be a field representing a combing of M. Fix an auxiliary metric as above. We have the distribution of orthogonal tangent 2-planes

$$\{P_x := \text{span}(v(x))^{\perp}\}_{x \in M} \ .$$

These planes P_x are oriented by the unique orientation which, added to $v(x)$, agrees with the given orientation on T_xM. This defines an oriented rank-2 vector bundle ξ_v on M whose strict equivalence class does not depend on the choice of the combing representative nor of the auxiliary metric. We consider the oriented Euler class

$$e^2(\xi_v) \in \Omega^2(M) = \Omega_1(M) \ .$$

In fact, $e^2(\xi_v) \in \mathcal{H}^2(M; \mathbb{Z})$. If ξ_v has a non-vanishing unitary section w orthogonal to v, then (v, w) extends to the unique orthonormal framing $\mathcal{F} = (v, w, z)$ of $T(M)$ such that the orientations are compatible. So ξ_v is trivial if and only if it admits a nowhere vanishing section w as above.

LEMMA 19.29. *The bundle ξ_v has a non-vanishing section if and only if the Euler class $e^2(\xi_v)$ vanishes.*

This is known from section 13.4.

As usual, $\omega^2(\xi_v) \in \mathcal{H}^2(M; \mathbb{Z}/2\mathbb{Z})$ is the image of $e^2(\xi_v)$ via the natural forgetting map.

The comparison class. We can associate to an ordered pair of unitary fields (v, v'), representing combings of M, a smooth section $v \times v'$ of ξ_v as follows. At a point $x \in M$ where $v(x) \neq \pm v'(x)$, $v \times v'(x) \in P_v(x) \subset T_xM$ is the "vector product" of $v(x)$ and $v'(x)$, i.e. the only tangent vector such that

- $\|v \times v'(x)\|^2_{g(x)} = 1 - g(v, v')^2$;
- $v \times v'(x)$ is $g(x)$-orthogonal to $v(x)$ and $v'(x)$;
- $(v(x), v'(x), v \times v'(x))$ is an oriented basis of T_xM.

At a point $x \in M$ where $v(x) = \pm v'(x)$, we set $v \times v'(x) = 0$.

If the two unitary fields v and v' are generic in the respective combings, the section $v \times v'$ of F_v is transverse to the zero section and the zero locus

$$C := \{x \in M \mid v \times v'(x) = 0\} \subset M$$

is a disjoint collection of simple closed curves. Moreover, $C = C_+ \cup C_-$, where

$$C_+ = \{x \in M \mid v(x) = v'(x)\} \quad \text{and} \quad C_- = \{x \in M \mid v(x) = -v'(x)\}.$$

By the very definition of $e^2(\xi_v)$, C can be oriented to represent the Euler class of ξ_v. Indeed, let $E(\xi_v)$ denote the total space of ξ_v, $M_0 \subset E(\xi_v)$ the zero-section and $M_1 = v \times v'(M) \subset E(\xi_v)$. Under the natural identification of M with M_0, the submanifold C is identified with $M_0 \cap M_1$. By transversality, for each $x \in M_0 \cap M_1$ the natural projection $p_x : T_x E(\xi_v) \to P_v(x)$ isomorphically maps the image under $(v \times v)'_*$ of the fiber $N_x(C)$ of the normal bundle of $TC \subset TM|_C$ to $P_v(x)$. Therefore, the given orientation on $\xi_v(x)$ can be pulled-back to $N_x(C)$ and, together with the orientation of T_xM, it induces an orientation on T_xC in a standard way.

Definition 19.30. For every ordered pair of generic unitary fields (v, v') on M as above, we define the *comparison class* $\alpha(v, v') \in \Omega^2(M)$ as the class $[C_-]$ carried by the collection of curves C_- oriented as part of the oriented zero locus of $v \times v' : M \to \xi_v$ representing $e^2(\xi_v)$.

Lemma 19.31. *Let (v, v') be a generic pair of unitary fields on M. Then,*

$$\alpha(v, v') = -\alpha(v', v) \quad \textit{and} \quad \alpha(v, -v') = \alpha(v', -v).$$

Proof : For each $x \in C$ the equality $\xi_v(x) = \xi_{v'}(x)$ holds, with the orientations of $\xi_v(x)$ and $\xi_{v'}(x)$ being the same or different according to, respectively, whether $x \in C_+$ or $x \in C_-$. We may choose a tubular neighborhood $U = U(C)$ such that the restrictions of the tangent plane fields $P_v|_U$ and $P_{v'}|_U$ are so close that there is a vector bundle isomorphism $\varphi : \xi_v|_U \overset{\cong}{\to} \xi_{v'}|_U$ which is the identity map on the intersections $P_v(x) \cap P_{v'}(x)$, $x \in U$, and which is orientation-preserving near $C_+ = \{x \in M \mid v(x) = v'(x)\}$ and orientation-reversing near $C_- = \{x \in M \mid v(x) = -v'(x)\}$. Since $\varphi \circ (v \times v') = v \times v' = -v' \times v$ and $-v' \times v$ is obtained by composing the section $v' \times v$ with the orientation-preserving automorphism of $F_{v'}$ given by minus the identity on each fiber, the orientation on C_- as part of the zero locus of $v \times v' : M \to \xi_v$ is the opposite of its orientation as part of the zero locus of $v' \times v = -v \times v' : M \to \xi_{v'}$. This implies $\alpha(v, v') = -\alpha(v', v)$. Similarly, the orientation on C_+ as part of the zero locus of $v \times (-v') : M \to \xi_v$ coincides with its orientation as part of the zero locus of $(-v') \times v = v' \times (-v) : M \to \xi_{v'}$, which implies $\alpha(v, -v') = \alpha(v', -v)$.

∎

Lemma 19.32. *Let* (v, v') *be a generic pair of unitary fields on* M. *Then,*

$$e^2(\xi_v) - e^2(\xi_{v'}) = 2\alpha(v, v').$$

Proof : According to the definitions, we have

$$e^2(\xi_v) = \alpha(v, v') + \alpha(v, -v') \quad \text{and} \quad e^2(\xi_{v'}) = \alpha(v', v) + \alpha(v', -v).$$

The statement follows applying Lemma 19.31, after taking the difference of the two equations.

■

Pontryagin surgery. Let v be a unitary field on M and $C \subset M$ an oriented, simple closed curve such that the positive unit tangent field along C is equal to $v|_C$ and there is a trivialization

$$j : D^2 \times S^1 \overset{\cong}{\to} U(C)$$

of a tubular neighborhood of C in M such that

$$v \circ j = j_*(\partial/\partial\phi),$$

where ϕ is a periodic coordinate on the S^1-factor of $D^2 \times S^1$. Let (ρ, θ) be polar coordinates on the D^2-factor. We say that a unitary field v' is obtained from v by a *Pontryagin surgery* along C if, up to homotopy, v' coincides with v on $M \setminus U(C)$ and

$$v' \circ j = j_* \left(-\cos(\pi\rho)\frac{\partial}{\partial\phi} - \sin(\pi\rho)\frac{\partial}{\partial\rho} \right)$$

on $U(C)$.

The combing represented by such a field v' is well defined (it does not depend on the choices made to define v') and we also say that it is obtained by a Pontryagin surgery on the combing represented by v.

Lemma 19.33. *For every combing on* M *and every* $\beta \in \mathcal{H}^2(M; \mathbf{Z})$, *there is a unitary generic pair* (v, v') *of unitary fields on* M *such that* v *represents the combing and* $\alpha(v, v') = \beta$.

Proof : Let $C \subset M$ be an oriented simple closed curve representing the Poincaré dual of β and let $j : D^2 \times S^1 \to U(C)$ be a trivialization of a neighborhood of C. Without loss of generality, we may assume that the pull-back $j^*(g)$ of the auxiliary metric g on M is the standard product metric on $D^2 \times S^1$. We can also assume that a representative v of the combing is suited to perform a Pontryagin surgery along C. Consider a normal disc $D_{\phi_0} = j(D^2 \times \{\phi_0\})$ and let $p = D_{\phi_0} \cap C$. Then $T_pD_{\phi_0}$ coincides, as an oriented 2-plane, with $P_v(p)$ as well as with the $g(p)$-orthogonal subspace of T_pC inside T_pM. Let v' be a unitary combing obtained from v by first performing a Pontryagin surgery on $U(C)$ and then applying a small generic perturbation supported on a small neighborhood of $M \setminus U(C)$. Then (v, v') is a generic pair of unitary combings and $C = \{x \in M \mid v(x) = -v'(x)\}$. By the definition of $\alpha(v, v')$, to prove the statement it suffices to show that the given orientation of C coincides with its orientation as part of the zero set of $v \times v' : M \to \xi_v$. Near C, we have

$$(v \times v') \circ j = j_* \left(-\sin(\pi\rho) \frac{\partial}{\partial\theta} \right) = j_* \left(\frac{\sin(\pi\rho)}{\rho} \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \right),$$

where $x = \rho \cos\theta$ and $y = \rho \sin\theta$ are rectangular coordinates on the D^2-factor. Observe that j_* sends the pair $(\partial/\partial x, \partial/\partial y)$ to an oriented framing of ξ_v. Using the resulting trivialization of ξ_v, we can write locally the restriction of $v \times v'$ to the disc D_{ϕ_0}, followed by projection to ξ_v as follows:

$$v \times v'|_{D_{\phi_0}} : (x, y) \mapsto \frac{\sin(\pi\rho)}{\rho}(y, -x) = \pi(y, -x) \ + \ \text{higher order terms.}$$

It is easy to compute that $(v \times v')_* \circ j_*$ sends $\partial/\partial x$ to $-\pi\partial/\partial y$ and $\partial/\partial y$ to $\pi\partial/\partial x$, and since the matrix $\left(\begin{smallmatrix} 0 & \pi \\ -\pi & 0 \end{smallmatrix}\right)$ has determinant $\pi^2 > 0$ this shows that the restriction of $(v \times v')_*$ to the normal bundle to C composed with the projection to ξ_v is orientation-preserving along C, concluding the proof.

■

We are ready to state the main theorem of this section.

THEOREM 19.34. *Let M be a compact connected oriented boundaryless 3-manifold. The following facts are equivalent to each other and all true.*

(1) M is parallelizable.

(2) There exists a combing v of M such that $e^2(\xi_v) = 0$.

(3) There exists a combing v of M such that $e^2(\xi_v)$ is an even class; that is, of the form $e^2(\xi_v) = 2\beta$ for some $\beta \in \mathcal{H}^2(M; \mathbb{Z})$.

(4) For every combing v of M, $e^2(\xi_v)$ is an even class.

(5) For every combing v of M, $\omega^2(\xi_v) = 0 \in \mathcal{H}^2(M; \mathbb{Z}/2\mathbb{Z})$.

Proof : Let us prove first the equivalence of the five statements. We will prove $(j) \Leftrightarrow (j+1)$ for $j = 1, \dots, 4$.

(1) $\Rightarrow$ (2): If $\mathcal{F} = (v, w, z)$ is a framing of M, then $e^2(\xi_v) = 0$.

(1) $\Leftarrow$ (2): We have already remarked above that if $e^2(\xi_v) = 0$, then v can be extended to a global framing $\mathcal{F} = (v, w, z)$.

(2) $\Rightarrow$ (3): This is trivial.

(2) $\Leftarrow$ (3): If $e^2(\xi_v) = 2\beta$, then by applying the Pontryagin surgery to v and the class $-\beta$, we get v' such that

$$e^2(v') = -2\beta + e^2(v) = 0 \ .$$

(3) $\Rightarrow$ (4): If $e^2(\xi_v) = 2\beta$ and v' is another combing, then by Lemma 19.32

$$e^2(v') = 2(\alpha(v, v') - \beta) \ .$$

(3) $\Leftarrow$ (4): This is trivial.

(4) $\Rightarrow$ (5): This is trivial.

(4) $\Leftarrow$ (5): This follows from Lemma 19.28.

The equivalence of the five statements is achieved. Now it is enough to show that at least one among them is true. We are going to prove it for statement (5).

PROPOSITION 19.35. *For every combing v of M, $\omega^2(\xi_v) = 0 \in \mathcal{H}^2(M;\mathbb{Z}/2\mathbb{Z})$.*

Equivalently, we have to show that for every compact closed surface F embedded in M, possibly F nonorientable, then

$$\omega^2(\xi_v) \sqcup [F] = 0 \in \mathbb{Z}/2\mathbb{Z} ;$$

that is,

$$\omega^2(i^*\xi_v) \sqcup [F] = 0$$

where $i : F \to M$ is the inclusion, and it is not restrictive to assume that F is connected.

Consider the restriction $i^*T(M)$ of the tangent bundle of M to F. Similarly, consider $i^*\xi_v$. Then we have the following two splittings as direct sum:

$$i^*T(M) = i^*\xi_v \oplus \epsilon^1 = T(F) \oplus \nu ,$$

where ν denotes the orthogonal line bundle along F, and ϵ^1 is the restriction to F of the trivial line bundle which has v as a nowhere vanishing section. Here is the key lemma.

LEMMA 19.36. *For every combing v of M and every compact closed embedded surface F we have*

$$\omega^2(i^*\xi_v) \sqcup [F] = \omega^2(T(F)) \sqcup [F] + (\omega^1(\det T(F)) \cup \omega^1(\nu)) \sqcup [F] .$$

Claim. *Lemma 19.36 $\Rightarrow$ Proposition 19.35:*

Proof of the Claim: If F is *orientable*, then the identity of Lemma 19.36 reduces to

$$\omega^2(i^*\xi_v) \sqcup [F] = \omega^2(T(F)) \sqcup [F] = \chi_2(F)$$

and we conclude because $\chi(F)$ is even. If F is *nonorientable*, then $F \sim h\mathbf{P}^2(\mathbb{R})$; that is, the connected sum of h copies of the projective plane. As M is orientable, then ν is isomorphic to the determinant line bundle $\det T(F)$, hence also in this case

$$\omega^2(i^*\xi_v) \sqcup [F] = \chi_2(F) + (\omega^1(F) \cup \omega^1(F)) \sqcup [F] = 2 - h + h = 0 \text{ mod}(2) .$$

∎

Proof of Lemma 19.36: Consider again the two splittings

$$i^*T(M) = i^*\xi_v \oplus \epsilon^1 = T(F) \oplus \nu$$

realized geometrically by a field of splittings

$$T_xM = P_x \oplus l(x) = T_xF \oplus \nu(x),\ x \in F$$

where $l(x)$ is the (oriented) line spanned by $v(x)$, while $\nu(x)$ is the (unoriented) line orthogonal to T_xF. Let s be a generic section of $i^*(\xi_v)$; that is, a field of vectors $s = \{s(x) \in P_x\}_{x \in F}$. For every $x \in F$, the direct sum $T_xF \oplus \nu(x)$ induces the decompositions

$$s(x) = s_F(x) + s_\nu(x),\ v(x) = v_F(x) + v_\nu(x)\ .$$

By transversality, we can assume the following.

(1) $\{s = 0\}$ is a finite number of points representing $\omega^2(i^*\xi_v)$.
(2) $s_\nu = \{s_\nu(x)\}$ and $v_\nu = \{v_\nu(x)\}$ are generic sections of ν, so that both are smooth curves on F representing $\omega^1(\nu)$ and moreover are transverse to each other in F, so that their intersection represents $\omega^1(\eta) \cup \omega^1(\eta) = \omega^1(\det T(F)) \cup \omega^1(\nu)$.
(3) $\{s = 0\} \cap \{v_\nu = 0\} = \emptyset$.
(4) $s_F = \{s_f(x)\}$ is a generic section of $T(F)$, so that $\{s_F = 0\}$ is a finite number of points representing $\omega^2(T(F))$.

For every finite set X, let $\#X$ denote the number of its elements mod (2). Then we have

$$\omega^2(i^*\xi_v) \sqcup [F] = \#\{s = 0\},\ \omega^2(T(F)) \sqcup [F] = \#\{s_F = 0\}\ ,$$

$$(\omega^1(\det T(F)) \cup \omega^1(\nu)) \sqcup [F] = \#(\{v_\nu = 0\} \cap \{s_\nu = 0\})\ .$$

So we have to prove that

$$\#\{s = 0\} = \#\{s_F = 0\} + \#(\{v_\nu = 0\} \cap \{s_\nu = 0\})\ .$$

On the other hand, obviously

$$\{s_F = 0\} = (\{v_\nu = 0\} \cap \{s_F = 0\}) \amalg (\{v_\nu \neq 0\} \cap \{s_F = 0\})\ .$$

We claim that

$$\{v_\nu \neq 0\} \cap \{s_F = 0\} = \{s = 0\}\ ;$$

in fact, by item (3) above

$$\{s = 0\} = \{v_\nu \neq 0\} \cap \{s = 0\}\ .$$

Clearly

$$\{v_\nu \neq 0\} \cap \{s = 0\} \subset \{v_\nu \neq 0\} \cap \{s_F = 0\}\ ;$$

on the other hand if $s(x) \neq 0$, then $s_F(x) \neq 0$, because the projection $P_x \to T_xF$ is an isomorphism being $v_\nu(x) \neq 0$. It remains to check that

$$\#(\{v_\nu = 0\} \cap \{s_F = 0\}) = \#(\{v_\nu = 0\} \cap \{s_\nu = 0\})\ .$$

Set $C = \{v_\nu = 0\}$ and $j : C \to F$ as the inclusion of this smooth curve; for every $x \in C$, the line $\nu(x)$ is contained in P_x and we have the splitting as direct sum

$$P_x = (P_x \cap T_xF) \oplus \nu(x)\ .$$

Hence we have a splitting as the direct sum of line bundles,

$$j^*\xi_v = \lambda \oplus j^*\nu\ .$$

These two lines bundles are isomorphic to each other; in fact, along every component of C, $j^*\xi_v$ is trivial because it is oriented. Then the two line bundles are both trivial or both nontrivial; eventually

$$\omega^1(\lambda) \sqcup [C] = \omega^1(j^*\nu) \sqcup [C] \ .$$

We conclude by noticing that the restriction of s_F and s_ν are, respectively generic sections of these line bundles.

■

The proof of Proposition 19.35, hence of the main Theorem 19.34 (including that M is parallelizable), is now complete.

■

REMARK 19.37. Lemma 19.32 shows, in particular, that the class $2\alpha(v, v')$ does not depend on the choice of the generic pair of fields v and v' representing the respective combings. If $\mathcal{F} = (v, w, z)$ is a framing of $T(M)$, and v' is any other generic field, then $e^2(\xi_{v'}) = 2\alpha(v', v)$. Thanks to the framing, v' is encoded by a map $s : M \to S^2$ and it is not hard to verify (do it by exercise) that $\alpha(v', v) = s^*(u) \in \Omega^2(M)$, where u is the usual standard generator of $\Omega^2(S^2) \sim \mathbb{Z}$. More generally, if $\tilde{v}$ is another combing encoded by the map $\tilde{s} : M \to S^2$, then $\alpha(\tilde{v}, v') = \tilde{s}^*(u) - s^*(u)$; this shows that the comparison class itself only depends on the combings, not on the choice of the generic representatives.

19.6.4. What is the simplest proof that $\Omega_3 = 0$? We have earlier discussed several proofs that $\Omega_3 = 0$ and of the equivalent Lickorish-Wallace theorem on surgery equivalence. We have finally established that orientable 3-manifolds are parallelizable, thus all those proofs are now complete. We can ask about the "simplest one"; that is, more precisely, the one with a minimal mathematical background. Rohlin's first proof certainly uses nontrivial facts about immersions of 3-manifolds in $\mathbb{R}^5$. Lickorish's proof arises as a corollary of an important result on the surface mapping class group which, nevertheless, is rather expensive if one is just interested in the corollary. The proof in [**Rourke**] is certainly very simple and self-contained, provided one assumes Smale's theorem. Then the most basic proof would be by combining one with a minimal background of parallelizability of 3-manifolds (in particular, we have noticed that Smale's theorem is not necessary) and the specialization to the 3-dimensional case of Proposition 16.6.

19.6.5. Classification of framings. We provide a classification of the framings on M with respect to a given reference framing $\mathcal{F}_0$. Any other framing $\mathcal{F}$ is encoded by a map

$$\rho_{\mathcal{F}} : M \to SO(3)$$

considered up to homotopy. The set $[M, SO(3)]$ can be endowed with a group structure by pointwise multiplication. As $SO(3) \sim \mathbf{P}^3(\mathbb{R})$ there is a natural homomorphism (see Section 13.1)

$$\psi : [M, SO(3)] \to \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z}),\ [h] \to h^*([\mathbf{P}^2(\mathbb{R})])\ .$$

Denote by $p : S^3 \to SO(3) \sim \mathbf{P}^3(\mathbb{R})$ the universal covering. Recall that by Corollary 17.8

$$[M, S^3] \sim \Omega_0^{\mathcal{F}}(M) \sim \mathbb{Z}$$

every homotopy class being classified by the common $\mathbb{Z}$-degree of its representative maps. There is a natural homomorphism

$$\phi : [M, S^3] \to [M, \mathbf{P}^3(\mathbb{R})],\ [f] \to [p \circ f]\ .$$

Finally we can state

PROPOSITION 19.38. *The homomorphism sequence*

$$0 \to \mathbb{Z} \xrightarrow{\phi} [M, \mathbf{P}^3(\mathbb{R})] \xrightarrow{\psi} \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z}) \to 0$$

is exact.

Proof : If $p \circ f$ is homotopic to a constant map, then the homotopy can be lifted to S^3, hence f is homotopically trivial and ϕ is injective.

Given $g : M \to SO(3)$, $\psi([g]) = 0$ if and only if g lifts to S^3, hence the kernel of ψ is the image of ϕ.

We are left to prove that ψ is surjective. We use a spine $\mathbf{P}_0$ of M_0 constructed in Section 19.1.2. First, one proves that every homomorphism $\alpha : \pi_1(\mathbf{P}_0) \to \mathbb{Z}/2\mathbb{Z}$ is induced by a map $j : \mathbf{P}_0 \to \mathbf{P}^2(\mathbb{R})$. Let $a : (S^1, e) \to (\mathbf{P}^1(\mathbb{R}), x_0)$, $\mathbf{P}^1(\mathbb{R}) \subset \mathbf{P}^2(\mathbb{R})$, be a loop which generates $\pi_1(\mathbf{P}^2(\mathbb{R})) \sim \mathbb{Z}/2\mathbb{Z}$. We choose a maximal tree T in the singular set of the spine $\mathbf{P}_0$, and define $j : \mathrm{Sing}(\mathbf{P}_0) \to \mathbf{P}^2(\mathbb{R})$ by setting it constantly equal to x_0 on T, while on every other edge of the singular set it equals either the constant map or the map a, according to the value of α on the loop determined by such an edge. On the boundary of every region of $\mathbf{P}_0$ there is an even number of edges at which j is not constant, hence the map j extends to the whole of $\mathbf{P}_0$. Now we consider $\mathbf{P}^2(\mathbb{R}) \subset \mathbf{P}^3(\mathbb{R}) \sim SO(3)$. The map j extends to M_0, and finally to the whole of M because $\pi_2(SO(3)) = 0$.

■

19.6.6. Classification of combings. Fix a reference framing $\mathcal{F}_0$ of M as above. The set of combings of M can be identified with $[M, S^2] \sim \Omega_1^{\mathcal{F}}(M)$ by the Pontryagin construction of Chapter 17. We want to make it explicit. There is a natural forgetting projection

$$\pi : \Omega_1^{\mathcal{F}}(M) \to \Omega_1(M)\ .$$

In fact, $\pi(v) = v^*(u) \in \Omega^2(M) = \Omega_1(M)$, where $u = [y_0]$ is a standard generator of $\Omega^2(S^2)$. We have already remarked that

$$e^2(\xi_v) = 2\pi(v)\ .$$

The projection π is surjective. So we have to understand the fibre $\pi^{-1}(x)$ of every $x \in \Omega_1(M)$. If we consider the comparison class $\alpha(v, v')$ as the first obstruction in order that the combings coincide, to distinguish the combings in a same fibre we have to point out a secondary comparison invariant. Given an oriented framed knot $(K, \mathfrak{f})$ in M which projects to x, we can modify the framing to $(K, n\mathfrak{f})$ by adding n twists to the given framing. This gives a transitive action of $\mathbb{Z}$ on such a fibre. We have to understand when $(K, \mathfrak{f})$ and $(K, n\mathfrak{f})$ represent the same element of $\Omega_1^{\mathcal{F}}(M)$. Assume this is the case, realized by a framed surface S in $M \times I$. By taking the double of $M \times I$, diffeomorphic to $M \times S^1$, the double Σ of S embedded therein is an oriented boundaryless surface in $M \times S^1$ such that $[\Sigma] \bullet [\Sigma] = n \in \Omega_0(M \times S^1)$. We have

$$([\Sigma] - \lambda) \bullet [M \times \{1\}] = 0 \ ,$$

where $\lambda = [K \times S^1]$. Then

$$([\Sigma] - \lambda) \bullet ([\Sigma - \lambda]) = [\lambda] \bullet [\lambda] = 0$$

$$n = 2([\Sigma] - \lambda) \bullet \lambda = [\Sigma - \lambda] \bullet e^2(\xi_v) \ .$$

Then there are two cases:

(1) $\pi(v)$ is a torsion element; then also $e^2(\xi_v)$ is so, and then $n = 0$.

(2) $e^2(\xi_v)$ is not a torsion element; if d is the biggest integer such that $\pi(v) = d\beta$ for some β, then

$$n = 0 \text{ mod } (2d) \ .$$

In summary, we have the following.

Proposition 19.39. *(1) Every framing $\mathcal{F}_0$ on M determines a surjective map*

$$\pi : \Omega_1^{\mathcal{F}}(M) \to \Omega^2(M)$$

such that for every combing $v \in \Omega_1^{\mathcal{F}}(M)$, $2\pi(v) = e^2(\xi_v)$.

(2) If $e^2(\xi_v)$ is a torsion element, set $d = 0$; then for every $v, v_0 \in \pi^{-1}(\pi(v_0))$, it is defined a secondary comparison invariant $h(v, v_0) \in \mathbb{Z}/2d\mathbb{Z} = \mathbb{Z}$ such that $v = v_0$ iff and only if $h(v, v_0) = 0$.

(3) If $e^2(\xi_v) = 2\pi(v)$ is not a torsion element, let d be the maximum integer such that $\pi(v) = d\beta$ for some β. Then it is defined a secondary comparison invariant $h(v, v_0) \in \mathbb{Z}/2d\mathbb{Z}$ such that $v = v_0$ iff and only if $h(v, v_0) = 0$.

Remark 19.40. If $\Omega_1(M)$ has no nontrivial elements of order 2, then the map π does not depend on the choice of the framing $\mathcal{F}_0$. In general, this is not true. Let $M = \mathbf{P}^3(\mathbb{R})$. Fix a trivialization $b : U\mathbf{P}^3(\mathbb{R}) \to \mathbf{P}^3(\mathbb{R}) \times S^2$ of its unitary tangent bundle (associated to a framing $\mathcal{F}_0$). Identify $\mathbf{P}^3(\mathbb{R})$ with $SO(3)$. Consider a new trivialization c defined by $c(b^{-1}(p, y)) = (p, py)$ Let v be a combing encoded by a constant map with respect to b. Then $\pi_b(v) = 0$. On the other hand, $\pi_c(v)$ is represented by the loop in $SO(3)$ given by the rotation in a certain plane, hence it is not trivial.

Finally, we want to outline that Pontryagin surgery acts transitively.

Proposition 19.41. *Let v, v_0 be combings of M. Then they are connected by a finite sequence of combing Pontryagin surgeries.*

Proof : Up to Pontryagin surgery we can assume that the first comparison obstruction vanishes: $\alpha(v, v_0) = 0$. Fix a reference framing $\mathcal{F}_0$ as above. Then combings are encoded by $[M, S^2] \sim \Omega_1^{\mathcal{F}}(M)$, and we can assume that v, v_0 belong to the same fibre of $\pi : \Omega_1^{\mathcal{F}}(M) \to \Omega^2(M)$. It remains to be shown that, up to further Pontryagin surgeries on v_0 which preserve the fibre, the second comparison invariant $h(v, v_0)$ also vanishes. As $\alpha(v, v_0) = 0$, we can assume that v and v_0 coincide on $M_0 = M \setminus \mathrm{Int}(B)$, where B is a standard 3-disk in a chart of M diffeomorphic to $\mathbb{R}^3$, and they are constantly equal to a base point $s_0 \in S^2$ on $\partial B \sim S^2$. As $B/\partial B \sim S^3$ and is endowed with the base point $p_0 = [\partial B]$, then v and v_0 determine two elements $\bar{v}, \bar{v}_0 \in \pi_3(S^2)$. We know that this last is isomorphic to $\mathbb{Z}$ and is generated by the Hopf map $\mathfrak{h}$; then $\bar{v} = n\mathfrak{h}$, $\bar{v}_0 = n_0\mathfrak{h}$. It is not hard to verify that (with the notations of Proposition 19.39)

$$h(v, v_0) = n - n_0 \bmod (2d) \ ,$$

where d only depends on the given fibre of π. Then we are essentially reduced to prove that, starting from the map $c_0 : S^3 \to S^2$, $c_0(x) = s_0$, for every $n \in \mathbb{Z}$, we can realize a map $f : S^3 \to S^2$ such that $[f] = [n\mathfrak{h}]$ by means of a finite sequence of Pontryagin surgeries. Assume that $B \subset \mathbb{R}^3$ is a suitably big radius; consider the following loops in $\mathbb{R}^3$:

$$\gamma_\pm : [0, 2\pi] \ni \phi \to 3(0, \cos(\phi), \pm\sin(\phi)) \in \mathbb{R}^3 \ .$$

Parametrize a tubular neighbourhood of $\gamma_\pm$ as

$$j_\pm : [0, 2] \times [0, 2\pi] \times [0, 2\pi] \ni (\rho, \theta, \phi) \to$$
$$(3 + \rho\cos(\theta))(0, \cos(\phi), \pm\sin(\phi)) + (\rho\sin(\theta), 0, 0) \in \mathbb{R}^3 \ .$$

Now, by taking convex combinations in S^2 on the region $1 \le \rho \le 2$, we can construct a homotopy between the constant field s_0 and the field

$$e_\pm^{(0)}(j_\pm(\rho, \theta, \phi)) = (0, -\sin(\phi), \pm\cos(\phi)) = \dot{\gamma}_\pm(\phi)/3 \ .$$

Up to rescaling the field, we can apply the Pontryagin surgery along the tube $\{\rho \le 1\}$. This produces another field $e^{(1)}$ which coincides with $e^{(0)}$ outside the tube and is given there by:

$$e_\pm^{(1)}(j_\pm(\rho, \theta, \phi)) =$$
$$-\cos(\pi\rho)(0, -\sin(\phi), \pm\cos(\phi)) - \sin(\pi\rho)(\sin(\theta), \cos(\theta)\cos(\phi), \pm\cos(\theta)\sin(\phi)) \ .$$

The value $(-1, 0, 0)$ is regular and the inverse image is the curve

$$\delta_\pm : [0, 2\pi] \ni \phi \to j_\pm(1/2, \pi/2, \phi) = (1/2, 3\cos(\phi), \pm 3\sin(\phi)) \ .$$

By direct computation, one checks that the framing on $\delta_\pm$ is given by the normal field

$$\nu_\pm(\phi) = -\frac{\sin(\phi)}{\pi}(1, 0, 0) - \frac{\cos(\phi)}{2}(0, \cos(\phi), \pm\sin(\phi))$$

so that one finally checks that

$$lk(\delta_\pm, \delta_\pm + \nu_\pm) = \mp 1 \ .$$

We can therefore conclude that starting from the constant field c_0, the element of $\pi_3(S^2)$ which corresponds to the integer n can be realized by $|n|$ Pontryagin surgeries.

∎

19.7. The bordism group of immersed surfaces in a 3-manifold

Let S be a compact boundaryless surface and M be a connected boundaryless 3-manifold. As usual, $[S, M]$ denotes the set of homotopy classes of maps $f : S \to M$. By Section 7.8, we know that every class $\alpha \in [S, M]$ contains generic immersions whose local models are the same as for immersions in $\mathbb{R}^3$ described therein. Generic immersions in a given homotopy class can be considered up to the finer relation of *regular homotopy*. This is a particular case of Smale-Hirsch theory, but the resulting classification is a bit implicit; several efforts have been made to make it more transparent. We can also consider generic immersions of compact boundaryless surfaces in a given 3-manifold up to a notion of bordism which extends the one of embedded bordism. In this section we mainly refer to [**HH**], [**Pi**], [**BS**]. We will refer to these papers for details of some proofs. Nevertheless, we hope to provide a substantial report.

Let us recall first the notion of *regular homotopy*.

DEFINITION 19.42. Let $\alpha \in [S, M]$; we say that two generic immersions $f_0, f_1 : S \to M$ belonging to α are *regularly homotopic* if they are connected by a smooth homotopy f_t, $t \in [0, 1]$, such that f_t is an immersion for every t. We denote by $\mathcal{R}[S, M]_\alpha$ the set of regular homotopy classes of immersions in α, and by $[f]_r$ the class of a generic immersion belonging to α.

Let us define now the $\mathfrak{i}$-bordism.

DEFINITION 19.43. Let $f_j : S_j \to M$, $j = 0, 1$, be generic immersions of surfaces in the 3-manifold M. Then f_0 is $\mathfrak{i}$-*bordant* with f_1 if there is a 3-dimensional triad (W, S_0, S_1) and an immersion $F : W \to M \times [0, 1]$ such that $F \pitchfork M \times \{0, 1\}$ and $f_j \times \{j\} = F_{|S_j}$, $j = 0, 1$.

First, we have a few remarks.

- As usual, $\mathfrak{i}$-bordism is an equivalence relation. Denote by $[f]_{\mathfrak{i}}$ the equivalence class of a generic immersion f.
- If $\phi : S \to S$ is a smooth diffeomorphism, then for every generic immersion $f : S \to M$, f is $\mathfrak{i}$-bordant with $f \circ \phi$: the bordism relation incorporates the reparametrizations of surfaces. For every immersion f, the intrinsic object of interest is rather its image $f(S) \subset M$ which is a kind of singular surface in M, called an *immersed surface*.

• If $f_0, f_1 : S \to M$ are connected by a regular homotopy $F : S \times [0,1] \to M$, then

$$F \times \mathrm{id} : S \times [0,1] \to M \times [0,1]$$

realizes an $\mathfrak{i}$-bordism of f_0 with f_1. Hence, in a sense, $\mathfrak{i}$-bordism embodies regular homotopy, but we stress that reparametrization is not included in the definition of regular homotopy. By enhancing the regular homotopy of immersions with the reparametrization of S, we have an intermediate relation between bordism and regular homotopy of immersions of surfaces which can be interpreted as the *regular homotopy on immersed surfaces* with source S.

• Denote by $\mathcal{I}_2(M)$ the set of $\mathfrak{i}$-bordism classes. The disjoint union defines an Abelian *semigroup* structure $(\mathcal{I}_2(M), +)$ with the class of the empty immersion as 0:

$$[S_1, f_1]_{\mathfrak{i}} + [S_2, f_2]_{\mathfrak{i}} = [S_1 \amalg S_2, f_1 \amalg f_2]_{\mathfrak{i}} \ .$$

A priori, it is not evident that it is a group; that is, it is not clear how to define the inverses $-[f]_{\mathfrak{i}}$.

• By using 1-handles embedded in M, we can define a *connected sum* between immersions $f_1 \# f_2 : S_1 \# S_2 \to M$ such that

$$[S_1 \# S_2, f_1 \# f_2]_{\mathfrak{i}} = [S_1, f_1]_{\mathfrak{i}} + [S_2, f_2]_{\mathfrak{i}} \in \mathcal{I}_2(M) \ ;$$

it follows that every class in $\mathcal{I}_2(M)$ can be represented as $[S, f]_{\mathfrak{i}}$ where S is connected, and the operation $+$ is induced by $\#$ as well.

We will be mainly concerned with compact 3-manifolds M, and we distinguish two cases depending on M being orientable or nonorientable. When M is orientable, an important ingredient of the discussion will be a certain quadratic enhancement of the intersection form of a surface S associated with each immersion of S in M. We will discuss in detail the orientable case following [**HH**], [**Pi**], [**BS**]. Later we will give a few indications about the nonorientable one.

An important special case is $M = S^3$ [**Pi**]. In this case, for every surface S there is only one homotopy class of maps $f : S \to S^3$, and via the usual inclusion $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \infty = S^3$, we easily see by transversality that $\mathcal{R}[S, S^3] = \mathcal{R}[S, \mathbb{R}^3]$ and $\mathcal{I}_2(S^3) = \mathcal{I}_2(\mathbb{R}^3)$.

19.7.1. From immersions in 3-manifolds to quadratic enhancements of the intersection forms. Let us recall the current setting:

• M is an *orientable* connected compact boundaryless 3-manifold;

• S is a compact and boundaryless surface, not necessarily orientable. For a while, we will assume also that S is connected.

• $f : S \to M$ is a generic immersion.

We know that M is parallelizable, so let us fix an auxiliary *framing* $\mathcal{F}$ of M. This includes also the choice of the orientation of M. The framing $\mathcal{F}$ can be equivalently identified with an ordered triple $\mathcal{F} = (v, w, z)$ of pointwise linearly independent tangent vector fields on M. By taking an auxiliary Riemannian metric g on M, we can also assume that these fields are pointwise orthonormal.

Let K be a smooth knot in M ($K \sim S^1$). Give K an auxiliary orientation. The restriction of v along K can be considered as a map $v : K \to S^2$; then, up to homotopy of framings, we can assume that v coincides along K with the positive unitary tangent field on K. Thus, along K, $\mathfrak{n}_{\mathcal{F}} := (w, z)$ is an ordered couple of pointwise orthonormal vectors normal to K; i.e. it is a *normal framing* and it determines a tubular neighbourhood $N(K)$ of K in M equipped with a trivialization. If $\mathfrak{n} = (w_1, z_1)$ is any other normal framing along K, then by using $\mathfrak{n}_{\mathcal{F}}$ as a reference, we encode $\mathfrak{n}$ by a map $\rho : K \to SO(2) \sim S^1$ and we associate to $\mathfrak{n}$ the degree $\phi(\mathfrak{n}) := \deg_{\mathbb{Z}}(\rho) \in \mathbb{Z}$, so that, obviously, $\phi(\mathfrak{n}_{\mathcal{F}}) = 0$. This number can be equivalently obtained as follows. The framing $\mathfrak{n}_{\mathcal{F}}$, that is its first component w, determines a longitude $l_{\mathcal{F}}$ on $\partial N(K)$ oriented in such a way that the projection to K is of degree 1. Another framing $\mathfrak{n}$ also determines a longitude $l_{\mathfrak{n}}$. Then

$$\phi(\mathfrak{n}) = [l_{\mathfrak{n}}] \bullet [l_{\mathcal{F}}] \in \Omega_0(\partial N(K)) \sim \mathbb{Z} ,$$

where $\partial N(K)$ is endowed with the boundary orientation. We say that $\mathfrak{n}$ differs from $\mathfrak{n}_{\mathcal{F}}$ by $\phi(\mathfrak{n})$ positive or negative twists along K. We can modify $\mathfrak{n}$ by adding an arbitrary number of twists. We stipulate that $\mathfrak{n}_{\mathcal{F}}$ is the basic *odd* normal framing of K determined by $\mathcal{F}$ and that a normal framing is odd if it differs from $\mathfrak{n}_{\mathcal{F}}$ by an even number of twists. Otherwise, a framing is *even*. So we have distributed the normal framings to K in two classes; these classes of odd/even framings do not depend on the choice of the auxiliary orientation on K. If we apply this construction to $S^1 = \partial D^2 \subset \mathbb{R}^2 \subset \mathbb{R}^3$ with respect to the standard constant framing of $\mathbb{R}^3$, we realize that even normal framings along S^1 are characterized by the property that they cannot be extended to a framing of the restriction of $T(\mathbb{R}^3)$ to the spanning 2-disk D^2. On the other hand, odd normal framings along S^1 can be extended. The typical even framing along S^1 has as field w the ingoing normals to S^1, tangent to D^2; the associated longitude is determined by a collar of S^1 in D^2.

Consider now a smooth circle C on the surface S. By transversality, we can assume that the restriction $f_{|C}$ of the immersion is an embedding of C to a knot $K \subset f(S) \subset M$ which extends to an embedding of a tubular neighbourhood $U(C)$ of C in S to a band $B(K)$ in $f(S)$, with core K. We can assume that $B(K)$ is the transverse intersection with $f(S)$ of a neighbourhood $N(K)$ of K in M as above. We can apply to this knot K the above considerations. Give C, hence K, an auxiliary orientation. Let us orient $\partial B(K)$ in such a way that the natural projection to its core K is a degree-2 covering. Fix an *even* normal framing $\mathcal{F}_e$ along K, with associated longitude $l_{\mathcal{F}_e}$. For every normal framing $\mathfrak{n}$, define as above $\phi_e(\mathfrak{n}) \in \mathbb{Z}$ with respect to $\mathcal{F}_e$. We can consider the integer

$$[\partial B(K)] \bullet [l_{\mathcal{F}_e}] \in \Omega_0(\partial N(K)) \sim \mathbb{Z} .$$

Then set

$$q_f(C) := [\partial B(K)] \bullet [l_{\mathcal{F}_e}] \mod(4) \ .$$

If $U(C)$ is annular, then a normal framing, say u, of C in S gives rise to a normal framing $\mathfrak{n}_f = (w, z)$ of K in M, provided that w is the image of u by the differential of f, and (v, w, z) agrees with the given orientation of $T_x M$ along K, where v is tangent to K as above. Then

$$[\partial B(K)] \bullet [l_{\mathcal{F}_e}] = 2\phi_e(\mathfrak{n}_f) \ .$$

We can say that $q_f(C)$ counts the number mod(4) of *half-twists* that the band $B(K)$ makes along its core K. The same interpretation also makes sense when $U(C)$ is a Möbius strip. In this case $[\partial B(K)] \bullet [l_{\mathcal{F}_e}]$ is odd.

REMARK 19.44. If $M = \mathbb{R}^3$, $q_f(C)$ is the linking number mod(4) between $\partial B(K)$ and the core K of the band (co-oriented as before).

If $L = \amalg_j C_j$ is the finite disjoint union of smooth circles on S, set

$$q_f(L) = \sum_j q_f(C_j) \ .$$

LEMMA 19.45. *(1) The procedure described above well defines a function q_f which associates to every finite disjoint union of smooth circles on the surface S, considered up to ambient isotopy, an element $q_f(C) \in \mathbb{Z}/4\mathbb{Z}$.*

(2) The function q_f satisfies the conditions stated at the end of Chapter 15; hence by setting for every $\alpha \in \eta_1(S)$, $q_f(\alpha) := q_f(C)$, where C is any smooth circle on S representing α, we well define a quadratic enhancement of $(\eta_1(S), \bullet_S)$.

As for item (1), it is a bit complicated to show that $q_f(C)$ in invariant up to ambient isotopy. A generic isotopy between two copies of C, which embed in M by the restriction of f, might pass through non-injective immersions and we have to check that these accidents are immaterial for the value of q_f. As for item (2), we are reduced to a local analysis at a single crossing point (the choice of the simplification of the crossing also turns out to be immaterial); this is not very hard. We leave the details to the reader as an exercise.

REMARKS 19.46. 1) The choice of the framing $\mathcal{F}$ is not immaterial, in the sense that the quadratic form q_f might depend on such a choice. However, it will be immaterial for the statement of the main Theorems 19.47 and 19.51. In the case of $S^3 = \mathbb{R}^3 \cup \infty$, we will deal with the unique framing (up to homotopy) of $\mathbb{R}^3$.

2) The above construction would be placed in a more conceptual framework in terms of *spin structures* on M and induced *pin*$^-$ on S. In such a framework (the interested reader may look to [**KT**]), given $f : S \to M$ as above, as M is oriented, $f^*T(M) = T(S) \oplus \Lambda(S)$, where this last is the determinant bundle of S. For every spin structure Θ on M, we have the pull-back spin structure $f^*(\Theta)$ on $f^*T(M)$, and there is a natural bijection

between the spin structures on $T(S) \oplus \Lambda(S)$ and the pin$^-$ structures on S; moreover these last are in natural bijection with the quadratic enhancements of the intersection form of S. Rather than the framing $\mathcal{F}$ itself, above we have used the spin structure carried by it. In this framework, the statement of the last lemma becomes conceptually clear and even simpler to prove. However, to our present aims, we have preferred the above direct operative presentation, without introducing the general theory.

3) The constructions of the present section work as well if M is any framed 3-manifold, not necessarily compact.

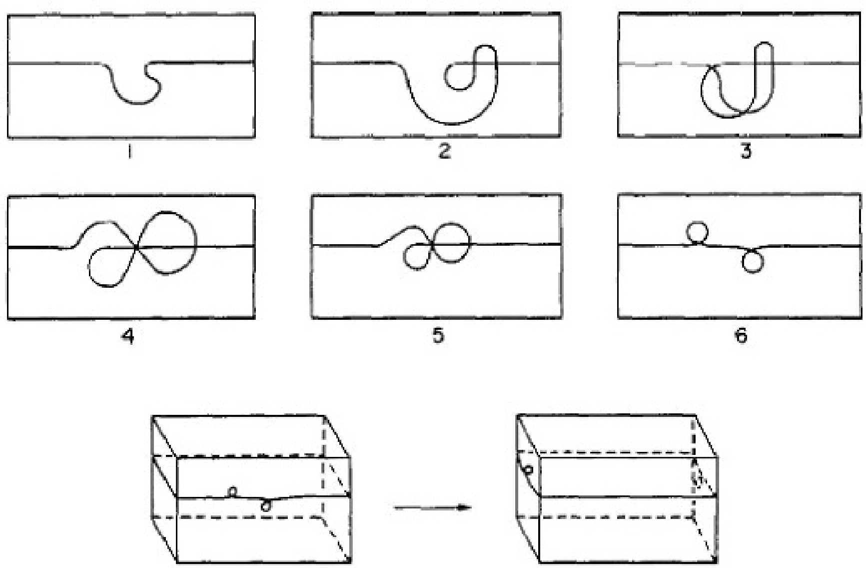

Figure 1. A kink box.

Figure 1 is reprinted from [**HH**], with permission from Elsevier.

19.7.2. Adding kinks. Let $f : S \to M$ be a generic immersion, S connected. Let C be a smooth circle on S such that f restricts to an embedding of a small tubular neighbourhood $U(C)$ of C in S. We are going to modify the immersion f by *adding a kink along* C. This nice and crucial construction has been introduced in [**HH**]. Denote by $K = f(C)$, $B(K) = f(U(C))$. The neighbourhood $U(C)$ is either an annulus or a Möbius strip. As M is orientable, then any tubular neighbourhood $N(K)$ of K in M is diffeomorphic to the product $S^1 \times D^2$. As usual we can assume that $\partial N(K)$ is transverse to $f(S)$ and that $B(K) = N(K) \cap f(S)$. We have two possible models for the pair $(N(K), B(K))$, depending on $U(C)$ being orientable or not. Consider (D^2, X) where $X = \{(x_1, x_2) \in D^2; x_1 x_2 = 0\}$; $X = X_1 \cup X_2$, where $X_1 = \{x_2 = 0\}$, $X_2 = \{x_1 = 0\}$.

• If $U(C)$ is an annulus, then the model for $(N(K), B(K))$ is the mapping cylinder of $\mathrm{id} : (D^2, X_1) \to (D^2, X_1)$.

• If $U(C)$ is a Möbius strip, then the model for $(N(K), B(K))$ is the mapping cylinder of $-\mathrm{id} : (D^2, X_1) \to (D^2, X_1)$.

Accordingly, there are two models for adding a kink along C. Let $\tilde{X}_1$ be the image of an immersion $\alpha : [-1, 1] \to D^2$ such that $\tilde{X}_1$ is contained in $x_2 \geq 0$, is symmetric with respect to the x_2-axis, has one double point and coincides with the inclusion of X_1 near the end-points. Denote by $-\tilde{X}_1$ its image by $-\mathrm{id}$.

If $U(C)$ is an annulus the kink model is very simple: take the mapping cylinder of $\mathrm{id} : (D^2, \tilde{X}_1) \to (D^2, \tilde{X}_1)$.

If $U(C)$ is a Möbius strip, then the kink model is more complicated (see [**HH**] pages 104-105); one constructs a so-called "kink box"; that is, a certain immersion of the 2-disk in D^3 with one triple point. A way to visualize this immersion is given in Figure 1. First, we consider the immersion of D^2 in $D^3 = D^2 \times D^1$ described by the movie in the first two rows; it results in the bottom left-hand picture. Then we apply an isotopy to it and reach the eventual kink box of the bottom right-hand picture. We can consider it as an immersion $X_1 \times [-1, 1]$ in $D^2 \times [-1, 1]$ such that the following items hold for some $\epsilon > 0$:

1. The image of $X_1 \times [-1, -1 + \epsilon]$ coincides with the embedding of $\tilde{X}_1 \times [-1, -1 + \epsilon]$;
2. The image of $X_1 \times [1 - \epsilon, 1]$ coincides with the embedding of $-\tilde{X}_1 \times [1 - \epsilon, 1]$;
3. The image along the boundary of $D^2 \times [-1, 1]$ coincides with the inclusion of $X_1 \times [-1, 1]$;
4. There is one triple point in the middle.

Denote by Z the image of this immersion.

The kink model is obtained by taking

$$(D^2 \times [0, 1], Z)/(x_1, x_2, 0) \sim (-x_1, -x_2, 1) .$$

Then Z projects to a new immersion of $U(C)$ which agrees with $B(K)$ along the boundary.

By using these models we can modify the given immersion $f : S \to M$ just along $U(C)$ and get $f_C : S \to M$. It is clear by the construction that f_C is homotopic to f.

19.7.3. Determination of $\mathcal{R}[S, M]_\alpha$. We give here a first remarkable application of adding kinks. Let $f : S \to M$ be a generic immersion as above and let q_f be the associated quadratic enhancement of $(\eta_1(S), \bullet_S)$. We know by Lemma 15.22 that every other enhancement is *abstractly* of the form

$$q'(x) = q_f(x) + 2x \bullet u$$

for a unique $u \in \eta_1(S)$. Adding kinks is a natural way to realize it geometrically, by keeping the homotopy class α of f fixed. Assume that $u = [C]$,

C being a smooth circle on S to which we can apply the kink construction. Let C' be another smooth circle on S which transversely intersects C at one point. Denote by $U(C')$ a small tubular neighbourhood of C' in S. Then it is immediate that $f(U(C'))$ and $f_C(U(C'))$ differ by one full twist. Recalling the geometric definition of q_f in terms of counting half twists mod (4), one easily realizes that

$$q_{f_C}([C']) = q_f([C]) + 2[C'] \bullet [C] \text{ mod}(4)$$

as desired.

THEOREM 19.47. *Let S be a compact connected boundaryless surface, $\alpha \in [S, M]$. Denote by $Q(S)$ the set of quadratic enhancements of $(\eta_1(S), \bullet_S)$. Then the map*

$$\mathfrak{q} : \mathcal{R}[S, M]_\alpha \to Q(S), \ \ \mathfrak{q}([f]_r) = q_f$$

is well defined and bijective.

Proof : We provide a sketch of the proof. It is not hard to check that it is well defined. We already know that the map $\mathfrak{q}$ is surjective. The proof that it is injective is nontrivial and consists of rephrasing Smale-Hirsch immersion theory in terms of the quadratic enhancement. This theory provides a simply transitive action of $\eta_1(S)$ on $\mathcal{R}[S, M]_\alpha$; a key result of [**HH**] is that this action can be realized by adding kinks as well as the one on $Q(S)$. So eventually $\mathfrak{q}$ is an equivariant bijection.

■

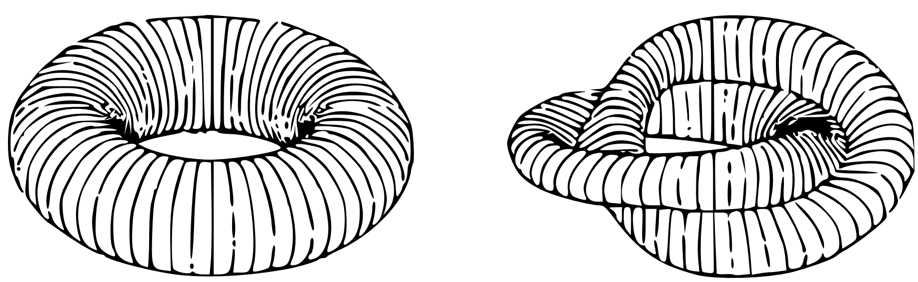

Figure 2. Immersed tori.

Figure 2 is reprinted from [**Pi**], with permission from Elsevier.

REMARKS 19.48. *(Basic immersed surfaces in $\mathbb{R}^3$)* We refer to [**Pi**].

1) By Theorem 19.47, $\mathcal{R}[S^2, \mathbb{R}^3]$ is trivial (i.e. it is reduced to one point). A regular homotopy connecting the standard inclusion i of S^2 in $\mathbb{R}^3$ with $-i$ is called a *sphere eversion*, whose surprising existence was discovered by S. Smale [**S0**].

2) The elementary surface bricks, besides the sphere, are the torus $S^1 \times S^1$ and the projective plane $\mathbf{P}^2(\mathbb{R})$. We denote by T the standard embedding of the torus in $\mathbb{R}^3$ bounding a solid torus. We denote by $\tilde{T}$ the immersion obtained by adding a kink along a meridian of T and then along

the privileged longitude of T which bounds a 2-disk in the complement of the solid torus. These realize the two *isometry classes* of quadratic enhancements of $(\eta_1(S^1 \times S^1), \bullet)$; T and $\tilde{T}$ are illustrated in Figure 2. Recall that there are four regular homotopy classes of *immersions* of the torus in $\mathbb{R}^3$. These can be realized by adding suitable kinks to T or $\tilde{T}$. We can say that T and $\tilde{T}$ represent the two regular homotopy classes of *immersed surfaces* associated to the torus immersions.

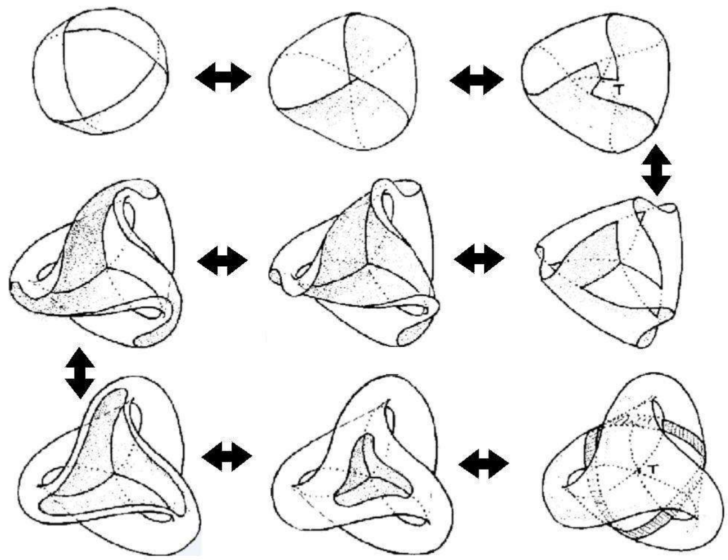

Figure 3. Boy's surface.

Figure 3 is used with the permission of Jean-Pierre Petit.

There is a famous immersion of the projective plane with one triple point called *Boy's surface* (see for instance the body and the references of

[**Ap**]). Figure 3 suggests how to construct it, starting from an embedded Möbius band. Such an immersion, denoted by B, and its mirror $\bar{B}$ (that is B composed with a reflection at a hyperplane of $\mathbb{R}^3$) realize the two quadratic enhancements of $(\eta_1(\mathbf{P}^2(\mathbb{R}), \bullet)$.

19.7.4. Determination of $(\mathcal{I}_2(M), +)$. First we will point out a few invariants up to $\mathfrak{i}$-bordism.

The Arf-Brown invariant. Let $f : S \to M$ be a generic immersion, S connected, with the associated q_f. Accordingly with Section 15.6, we can consider the Arf-Brown multiplicative invariant

$$\gamma(f) := \gamma(q_f) \in U_8$$

where, for simplicity, we have written $\gamma(q_f)$ instead of $\gamma(S, \bullet_S, q_f)$. If $f : S \to M$, where $S = \amalg_j S_j$ is a union of several connected components, then set

$$\gamma(f) := \prod_j \gamma(f_j) \ ,$$

where $f_j = f_{|S_j}$.

LEMMA 19.49. *Let $f_j : S_j \to M$ be generic immersions, $j = 0, 1$. If $[f_0]_\mathfrak{i} = [f_1]_\mathfrak{i}$, then $\gamma(q_{f_0}) = \gamma(q_{f_1})$.*

Proof : Let (W, S_0, S_1), $F : W \to M \times [0, 1]$ be as in Definition 19.43, and let $t : M \times [0, 1] \to [0, 1]$ be the projection. Without loss of generality, we can assume that $t \circ F$ is a Morse function on the triad. Then consider the possible accidents when passing through a critical point of $t \circ F$. Modifications occur locally in a chart of M at the critical point. We use the notations of Remark 19.48. At local minima/maxima, a new spherical component appears/disappears. For the other kinds of critical point, there are three possibilities:

• One performs the immersed connected sum of two components of the surface;

• One performs the connected sum with either a standard torus T or a Klein bottle immersion $B \# \bar{B}$.

In every case, the value of γ does not change (for all details one can see [**Pi**], pp. 432-433).

■

We have detected a first important U_8-valued invariant $\gamma([f]_\mathfrak{i})$ defined on $\mathcal{I}_2(M)$.

From now on we will use the standard isomorphism $U_8 \sim (\mathbb{Z}/8\mathbb{Z}, +)$ and hence adopt the additive notation.

Other invariants. Let $f : S \to M$ be a generic immersion (S not necessarily connected). It is obvious, just by forgetting part of the structure of $[f]_\mathfrak{i}$, that $[f] = [S, f] \in \eta_2(M)$ is invariant under $\mathfrak{i}$-bordism. Recall the quotient module $\mathcal{H}^1(M, \mathbb{Z}/2\mathbb{Z})$ of $\eta^1(M) = \eta_2(M)$ defined in Corollary 13.3;

recall also that the cup product $\sqcup$ descends to this quotient with values in $\eta^2(M) = \eta_1(M)$. Keep the notation $[f]$ for its image in $\mathcal{H}^1(M, \mathbb{Z}/2\mathbb{Z})$.

Denote by $\Sigma \subset S$ the non-injectivity locus of f. The image $\Sigma_f := f(\Sigma)$ determines an element $[\Sigma_f] \in \eta_1(M)$ as follows. The components of $f^{-1}(\Sigma_f)$ are of two kinds:

1) They belong to a couple $\tilde{C} = C \amalg C'$ such that $f(C) = f(C')$ and f is generically $1-1$ on such C, or C'.

2) Components $\tilde{C}$ such that $\tilde{C} = f^{-1}(f(\tilde{C}))$ and in such a case f is generically 2-1 on $\tilde{C}$.

Then select one component C in every couple $\tilde{C}$ of the first kind; for the second kind one finds a quotient C of $\tilde{C}$ such that f induces a map (we keep the name) $f : C \to M$, such that $f(C) = f(\tilde{C})$ and f is generically 1-1. Then set

$$[\Sigma_f] := \sum_{\tilde{C}} [C, f] \in \eta_1(M) \ .$$

The triple points of $f(S)$ determine a class $t_f \in \eta_0(M) \sim \mathbb{Z}/2\mathbb{Z}$.

Lemma 19.50. *If* $[f_0]_{\mathrm{i}} = [f_1]_{\mathrm{i}}$, *then* $[\Sigma_{f_0}] = [\Sigma_{f_1}] \in \mathcal{H}^1(M, \mathbb{Z})$ *and* $t_{f_0} = t_{f_1} \in \eta_0(M)$.

Proof : Let (W, S_0, S_1), $F : W \to M \times [0,1]$ be as in Definition 19.43. We can also assume that F is generic. Then $F(\Sigma_F)$ is a kind of singular surface properly embedded in $M \times [0,1]$ such that $F(\Sigma_F) \cap (M \times \{0,1\}) = f_0(\Sigma_0) \amalg f_1(\Sigma_1)$; by using the regular surface $F^{-1}(\Sigma_F))$ we can explicitly define a triad which connects the sum of the components that form $[\Sigma_{f_0}]$ and $[\Sigma_{f_1}]$ respectively. We deal similarly with the triple points.

■

Consider the product

$$\Gamma(M) = \eta_1(M) \times \mathcal{H}^1(M, \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/8\mathbb{Z}$$

endowed with the *twisted* group structure defined by the operation

$$(\delta, h, a) + (\delta', h', a') := (\delta + \delta' + h \sqcup h', h + h', a + a') \ .$$

We can state now the main result of this section.

Theorem 19.51. *The map* $\psi : \mathcal{I}_2(M) \to \Gamma(M)$, *well defined by*

$$[f]_{\mathrm{i}} \to ([\Sigma_f], [f], \gamma(f)) \ ,$$

is a semigroup isomorphism. In particular the semigroup $(\mathcal{I}_2(M), +)$ *is a group. Moreover, the invariant* $t_{[f]_{\mathrm{i}}}$ *is determined by the others.*

The rest of this section is occupied by the proof of Theorem 19.51. Immediately, ϕ is a semigroup homomorphism.

The 3-sphere.

Theorem 19.52. *The map* $\phi : \mathcal{I}_2(S^3) \to \mathbb{Z}/8\mathbb{Z}$, $\phi([f]_{\mathrm{i}}) = \gamma(f)$ *is a group isomorphism.*

Proof : This is the main result of [**Pi**] to which we refer for all details. We can use $\mathbb{R}^3$ instead of S^3. Note that in this case, we know *a priori* that $\mathcal{I}_2(\mathbb{R}^3)$ is a group : inverses are obtained by mirror image along a hyperplane. By using connected sums (or disjoint unions) of the basic immersed surfaces of Remark 19.48, it is easy to prove that ϕ is surjective. Recall that, at the beginning of this section, we have defined the regular homotopy of immersed surfaces by adding to the regular homotopy of immersions the reparametrization of the source surface. By Proposition 19.47 (and the classification of surfaces), one realizes that every immersed surface in $\mathbb{R}^3$ is regularly homotopic to a connected sum of several copies of the standard embedding T and one among the following eight independent immersed surfaces:

$$B,\ \bar{B},\ K_0,\ K_+,\ K_-,\ K_+\#B,\ K_0\#\tilde{T},\ K_-\#\bar{B}\ ,$$

where $K_0 = B\#\bar{B}$, $K_+ = B\#B$, $K_- = \bar{B}\#\bar{B}$. Up to i-bordism, the T-components are immaterial and we can conclude that those eight generators suffice and this achieves the desired bijection with $\mathbb{Z}/8\mathbb{Z}$.

■

The map ψ is surjective. Let us prove now in general that the map ψ is surjective.

LEMMA 19.53. *The map $\psi : \mathcal{I}_2(M) \to \Gamma(M)$ is surjective.*

Proof : As $M = M\#S^3$, we see that $\mathcal{I}_2(M)$ contains the subgroup $\mathcal{I}_2(S^3)$; it consists of the classes with a representative contained in a 3-disk of M.

The semigroup $\mathcal{I}_2(M)$ contains also the *subset* $E(M)$ given by the classes which are represented by *embedded* surfaces. By the description of $\mathcal{H}^1(M,\mathbb{Z}/2\mathbb{Z})$ as the embedded surfaces in M up to embedded bordism, we see that $E(M)$ is, in fact, the image of $\mathcal{H}^1(M,\mathbb{Z}/2\mathbb{Z})$ in $\mathcal{I}_2(M)$ by a natural quotient map.

Let $(\delta, h, a) \in \Gamma(M)$. Represent h by an embedding $e : S \to M$. Represent δ by a knot K in M. Consider the boundary $\mathcal{T} \sim S^1 \times S^1$ of a tubular neighbourhood of K in M. Add a kink along a longitude K' of K on $\mathcal{T}$ and get a generic immersion $j : \mathcal{T} \to M$. By construction, $\delta = [\Sigma_j]$, while $[j] = 0 \in \mathcal{H}^1(M,\mathbb{Z}/2\mathbb{Z})$, hence $[j] \sqcup [e] = 0$. By the elementary fact that γ is surjective in the case of S^3, there is $s : S \to S^3$ such that $\gamma(s) = a - \gamma(e) + \gamma(j)$. Clearly $[s] = 0$ and $[\Sigma_s] = 0$. Finally,

$$\psi([j]_{\mathfrak{i}} + [e]_{\mathfrak{i}} + [s]_{\mathfrak{i}}) = (\delta, h, a)\ .$$

■

A normal decomposition of i-bordism classes. Now the idea is that every $[f]_{\mathfrak{i}}$ admits a certain *normal decomposition* modelled on the classes used to prove that ψ is surjective. Precisely, we have the following key proposition.

PROPOSITION 19.54. *Every* $[f]_{\mathrm{i}}$ *can be represented by a sum*

$$[f]_{\mathrm{i}} = [j]_{\mathrm{i}} + [e]_{\mathrm{i}} + [s]_{\mathrm{i}} \ ,$$

where $j : \mathcal{T} \to M$ *is obtained by adding a kink along a longitude* K' *on the boundary* $\mathcal{T}$ *of a tubular neighbourhood of a knot* K *in* M*,* $[e]_{\mathrm{i}} \in E(M)$*,* $[s]_{\mathrm{i}} \in \mathcal{I}_2(\mathbb{R}^3)$ *where* $\mathbb{R}^3$ *is a chart of* M*. Moreover, we can choose the decomposition in such a way that* $q_j(K') = 0$ *hence so that* $\gamma(j) = 0$*.*

Proof : We will proceed in several steps. We adopt the notations of Remark 19.48; in particular, B and $\bar{B}$ are the two versions of Boy's surface.

Step 1. *We have that* $[f]_{\mathrm{i}} = [f']_{\mathrm{i}} + [s]_{\mathrm{i}}$*, where* f' *has no triple points and* $[s]_{\mathrm{i}} \in \mathcal{I}_2(\mathbb{R}^3)$*.*

Notice that $K_0 = B \# \bar{B}$ is regularly homotopic to the usual immersion of the Klein bottle in $\mathbb{R}^3$ without triple points (and a plane of symmetry) and recall that $[K_0]_{\mathrm{i}} = 0$. Similarly, if x_0 is a triple point of f, either $f \# B$ or $f \# \bar{B}$ is regularly homotopic to $\tilde{f}$ with one triple point less than f, and either $[f]_{\mathrm{i}} = [\tilde{f}]_{\mathrm{i}} + [\bar{B}]_{\mathrm{i}}$ or $[f]_{\mathrm{i}} = [\tilde{f}]_{\mathrm{i}} + [B]_{\mathrm{i}}$. So the step is achieved by induction on the number of triple points.

The double line locus $\Sigma_{f'}$ consists of the disjoint union of a finite number of embedded circles in M. If K is such a circle, then it has a neighbourhood in $f(S)$ which is a bundle over K, sub-bundle of a tubular neighbourhood of K in M, with fibre isomorphic to $X = \{(x_1, x_2) \in D^2;\ x_1 x_2 = 0\}$. We can count the number mod(4) of *quarter turns* this configuration does when moving along K . Denote it by $l(K) \in \mathbb{Z}/4\mathbb{Z}$; it characterizes the bundle. The cases $l(K) = 0, 2$ correspond to the situation where $f' : (f')^{-1}(K) \to K$ is a trivial double covering; if $l(K) = 0$ then the two components of this inverse image have annular tubular neighbourhoods in S', and if $l(K) = 2$, both have Möbius strip neighbourhoods. The cases $l(K) = 1, 3$ correspond to a nontrivial double covering.

Step 2. *We have that* $[f]_{\mathrm{i}} = [f']_{\mathrm{i}} + [s]_{\mathrm{i}}$ *as in Step 1 and moreover, we can require that* $\Sigma_{f'}$ *is connected.*

If $\Sigma_{f'}$ is not connected, there are two components K and K' and points $p \in K$, $p' \in K'$ belonging to the closure of a single connected component of $M \setminus \mathrm{Im}(f')$. So there is a smooth simple arc σ in M, connecting p and p' and without any further intersection with $\mathrm{Im}(f')$. Locally in a chart of M at p, the image of f' looks like two transverse planes P_1 and P_2. The image appears similarly at p', with planes P'_1 and P'_2. Remove from the image of f' the intersection, B_p, of the interior of a small 3-ball centred at p, with transverse boundary spheres. The closure of B_p is the union $D_1 \cup D_2$ of two 2-disks, $D_j \subset P_j$, $j = 1, 2$, which intersect transversely at a segment of K. Do similarly at p'. Possibly up to reordering the planes, we can attach two embedded 1-handles H_j along the arc σ, $j = 1, 2$, with attaching tube $T_{a,j} = D_j \cup D'_j$, and transverse b-tubes such that $T_{b,1} \pitchfork T_{b,2}$ consists of two disjoint double arcs having as endpoints the four points of

$(D_1 \cap D_2) \cup (D'_1 \cap D'_2)$. Ultimately, (up to corner smoothing) we get the immersed surface

$$\mathrm{Im}(\tilde{f}) := (\mathrm{Im}(f') \setminus (B_p \cup B_{p'})) \cup (T_{b,1} \cup T_{b,2})$$

which by construction is $\mathfrak{i}$-bordant with f', and like f' has no triple points; the two knots K and K' of $\Sigma_{f'}$ have fused into one knot K'' of $\Sigma_{\tilde{f}}$, so that this last has one component less. The step is achieved by induction on the number of components of $\Sigma_{f'}$. We stress that by the above construction, we have furthermore that

$$l(K'') = l(K) + l(K') \; .$$

Step 3. *Let $[f]_{\mathfrak{i}} = [f']_{\mathfrak{i}} + [s]_{\mathfrak{i}}$ be as in Step 2 (i.e. with $\Sigma_{f'} = K$ connected) and assume that $l(K) = 0, 2$. Then it is not restrictive to assume that $l(K) = 0$.*

By using the results about the group $\mathcal{I}_2(\mathbb{R}^3)$, we see that there is an immersion s_0 of the Klein bottle in a chart of M, without triple points and having connected $\Sigma_{s_0} = K_0$ such that $l(K_0) = 2$. Take

$$[f' \# s_0]_{\mathfrak{i}} + [s]_{\mathfrak{i}} - [s_0]_{\mathfrak{i}} = [f]_{\mathfrak{i}}$$

and apply Step 2 to $f' \# s_0$. This achieves the step.

Let $[f]_{\mathfrak{i}} = [f']_{\mathfrak{i}} + [s]_{\mathfrak{i}}$ be as in Step 3, so that $l(K) = 0$. Set $q_f(K) := q_f(C)$, where C is a component of $(f')^{-1}(K)$. It is well defined, and either $q_f(K) = 0$ or $q_f(K) = 2$.

Step 4. *Let $[f]_{\mathfrak{i}} = [f']_{\mathfrak{i}} + [s]_{\mathfrak{i}}$ be as in Step 3, so that $l(K) = 0$. Then it is not restrictive to assume that $q_f(K) = 0$*

There is an immersion s_1 of the torus in a chart of M, without triple points and with connected $\Sigma_{s_1} = K_1$ such that $l(K_1) = 0$ and $q_{s_0}(K_1) = 2$. If $q_f(K) = 2$, take

$$[f' \# s_0]_{\mathfrak{i}} + [s]_{\mathfrak{i}} - [s_0]_{\mathfrak{i}} = [f]_{\mathfrak{i}}$$

and apply Step 2 to $f' \# s_0$. This achieves the step.

Step 5. *Let $[f]_{\mathfrak{i}} = [f']_{\mathfrak{i}} + [s]_{\mathfrak{i}}$ be as in Step 2 (i.e. with $\Sigma_{f'} = K$ connected) and assume that $l(K) = 0, 2$. Then Proposition 19.54 holds in this case.*

By Steps 3 and 4 we can assume that $l(K) = 0$ and $q_f(K) = 0$. Perform a *Rohlin surgery* along K (recall Section 7.9). This splits f' into two disjoint immersed surfaces: an embedding e and the immersion j of torus having as image a product sub-bundle of (the interior of) a tubular neighbourhood $N(K)$ of K in M with fibre a lemniscate; the germ of j along K equals the germ of f'. It is easy to see that j is obtained by adding a kink along a longitude C on the boundary $\mathcal{T}$ of a smaller tubular neighbourhood $N'(K) \subset N(K)$, and that $q_{\mathcal{T}}(C) = q_f(K) = 0$. By construction, $[f]_{\mathfrak{i}} = [j]_{\mathfrak{i}} + [e]_{\mathfrak{i}} + [s]_{\mathfrak{i}}$. The Proposition is proved under such restrictive hypotheses.

To proceed, we need the following lemma.

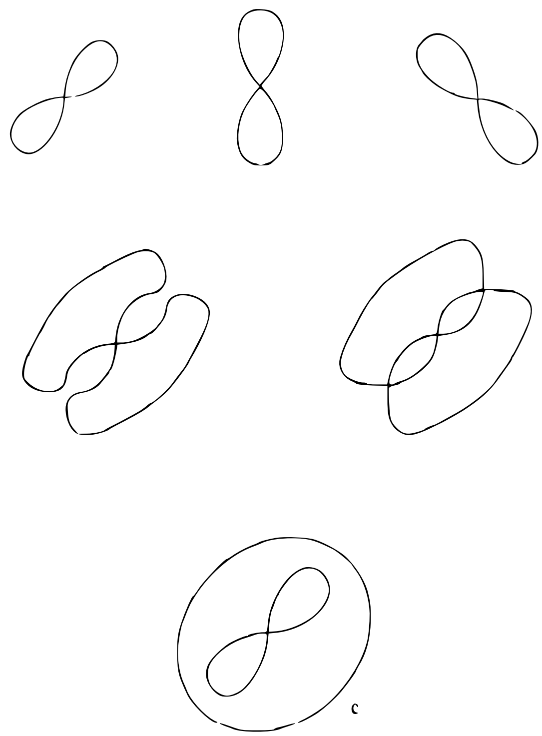

Figure 4. An auxiliary immersed surface.

LEMMA 19.55. *There is an immersion s_2 in $\mathbb{R}^3$ of a surface F of Euler-Poincaré characteristic $\chi(F) = -1$ such that the following points hold.*

1) s_2 has one triple point;

2) Σ_{s_2} consists of the union of a smooth circle K_2 endowed with an X-bundle neighbourhood in the image of s_2 such that $l(K_2) = 1$, and a lemniscate in a 2-disk D contained in the image of s_2, intersecting K at the triple point; D is transverse to K and the germ of the lemniscate at the triple point is a fibre of the X-bundle along K.

Proof : First, we construct an immersion of a surface G with boundary in $D^2 \times D^1$. This is given by the movie of Figure 4. Note that at the initial time $t = -1$ and at the final time $t = 1$ of the movie we see two copies of the same lemniscate L; in the final configuration, L is encircled by a smooth circle c. Finally we complete G by filling the curve c by a 2-disk, and identifying, by the identity of $\mathbb{R}^2$, the two copies of L over -1 and 1. One readily check that this is the image of an immersion of a surface F with the required properties.

∎

We denote by $\bar{s}_2$ the mirror image of the immersion s_2 as above.

Step 6. *Proposition 19.54 holds in full generality.*

It remains to prove it when $[f]_{\mathfrak{i}} = [f']_{\mathfrak{i}} + [s]_{\mathfrak{i}}$ is again as in Step 2, but we assume now that $l(K) = 1, 3$. Let $l(K) = 1$. By realizing s_2 in a chart of M, take

$$[f' \# \bar{s}_2]_{\mathfrak{i}} + [s]_{\mathfrak{i}} + [s_2]_{\mathfrak{i}} = [f]_{\mathfrak{i}}$$

and apply Step 2 to $f' \# \bar{s}_2$. In this way we reach a decomposition $[f]_{\mathfrak{i}} = [f"]_{\mathfrak{i}} + [s']_{\mathfrak{i}}$, where $\Sigma_{f"}$ is qualitatively similar to the one of s_2; that is, it consists of the union of a smooth circle $K"$ endowed with an X-bundle neighbourhood in the image of $f"$ and a lemniscate in a 2-disk D contained in the image of $f"$, intersecting $K"$ at one triple point. The 2-disk D is transverse to $K"$ and the germ of the lemniscate at the triple point is a fibre of the X-bundle along $K"$. Moreover, $l(K") = 0$. By applying Step 4, we can also assume that $q_{f"}(K") = 0$. Now, although there is a triple point, we can apply Step 5 along $K"$. This produces a decomposition of the form $[f]_{\mathfrak{i}} = [j]_{\mathfrak{i}} + [g]_{\mathfrak{i}} + [s']_{\mathfrak{i}}$, where $[j]_{\mathfrak{i}}$ has the required final properties, while Σ_g is contained in D and consists of the union of a lemniscate fibre of j and two further simple double circles. We can eliminate such circle by applying again Steps 4 and 5; eventually we get the required decomposition

$$[f]_{\mathfrak{i}} = [j]_{\mathfrak{i}} + [e]_{\mathfrak{i}} + [s"]_{\mathfrak{i}} \ .$$

If at the beginning $l(K) = 3$, we manage similarly by exchanging the roles of s_2 and $\bar{s}_2$. This achieves Step 6.

REMARK 19.56. *We stress that when $l(K) = 0, 2$, the images of j and e in the normal decomposition obtained above are disjoint. When $l(K) = 1, 3$, they intersect producing one triple point. In the first case $[\Sigma_f] \bullet [f] = 0 \in \eta_0(M) \sim \mathbb{Z}/2\mathbb{Z}$, in the second $[\Sigma_f] \bullet [f] = 1$.*

The proof of Proposition 19.54 is now complete.

■

The map ψ is injective.

LEMMA 19.57. *The map $\psi : \mathcal{I}_2(M) \to \Gamma(M)$ is injective.*

Proof : We can use normal decompositions of $\mathfrak{i}$-bordism classes. Assume that

$$\psi([j]_{\mathfrak{i}} + [e]_{\mathfrak{i}} + [s]_{\mathfrak{i}}) = \psi([j']_{\mathfrak{i}} + [e']_{\mathfrak{i}} + [s']_{\mathfrak{i}}) \ .$$

As $[e] = [e'] \in \mathcal{H}^1(M, \mathbb{Z}/2\mathbb{Z})$, then they are bordant by means of an embedded bordism, hence $\gamma(e) = \gamma(e')$. As $\gamma(j) = \gamma(j') = 0$, then $\gamma(s) = \gamma(s')$ and by Theorem 19.52, we have $[s]_{\mathfrak{i}} = [s']_{\mathfrak{i}}$. It remains to prove that $[j]_{\mathfrak{i}} = [j']_{\mathfrak{i}}$. Now $[j]_{\mathfrak{i}} + [j']_{\mathfrak{i}} = [j \# j']_{\mathfrak{i}}$, and this last can be obtained from the embedding $\mathcal{T} \# \mathcal{T}'$ by adding kinks along two disjoint circles K', $K"$ at which the quadratic enhancement vanishes. Let C be a smooth circle on $\mathcal{T} \# \mathcal{T}'$ such that $[K'] + [K"] = [C] \in \eta_1(\mathcal{T} \# \mathcal{T}')$. Then, up to regular homotopy, $j \# j'$ can be obtained by adding a kink to $\mathcal{T} \# \mathcal{T}'$ along C. It follows from the hypotheses that $[C] = 0 \in \eta_1(M)$ and that the quadratic enhancement of $\mathcal{T} \# \mathcal{T}'$ vanishes on C. We claim that in such a situation $[j \# j']_{\mathfrak{i}} = 0$. As the same considerations hold for $[j\#j]_{\mathfrak{i}}$, we will eventually conclude $[j]_{\mathfrak{i}} = -[j]_{\mathfrak{i}}$, and hence that $[j]_{\mathfrak{i}} = [j']_{\mathfrak{i}}$ as desired.

We need the following lemma.

LEMMA 19.58. *Let F be a compact surface with connected boundary embedded in a framed 3-manifold N (F might be nonorientable and N noncompact). Then the normal framing of $C = \partial F$ determined by a collar in F is even compared with the ambient framing.*

Proof : We can extend the embedding of F to a generic immersion of the double $D(F)$ of F in N. If F is orientable, up to corner smoothing, we can take the boundary of a tubular neighbourhood of F in N; if F is not orientable, we can take an immersion which looks like the orientable case along the boundary and has double lines in the interior of F. We use the ambient framing to define a quadratic enhancement $q_{D(F)}$ of the intersection form of the double. As $[C] = 0 \in \eta_1(D(F))$, then $q_{D(F)}(C) = 0$. This means that the collar normal framing is even.

■

To simplify the notations, denote by $f : S \to M$ the embedding $\mathcal{T}\#\mathcal{T}'$, so that $q_f(C) = 0$. As $[C] = 0 \in \eta_1(M)$, then there is a (possibly nonorientable) embedded Seifert surface $F \subset M$ such that $\partial F = C$. Apply Lemma 19.58 to F. As $q_f(C) = 0$, then both the normal framings of C, determined by a tubular neighbourhood in S and by a collar in F, respectively, differ to each other by an even number of twists. It follows that we can "roll up" F in a tubular neighbourhood U of C in M, in such a way that F is transverse to S along C, and transversely intersects S outside U.

Assume first that $F = D$ is a 2-disk. Let τ be a Dehn twist on S along C. For every $\alpha \in \eta_1(S)$,

$$\tau_*(\alpha) = \alpha + ([C] \bullet \alpha)[C] \ .$$

As $q_f(C) = 0$, by recalling the geometric definition of q_f, we readily see that

$$q_{f_C} = q_{f \circ \tau} \ .$$

We claim that f_C and $f \circ \tau$ are homotopic (equivalently, f and $f \circ \tau$ are homotopic). To prove the last statement, let U denote now a tubular neighbourhood of C in S; there is a natural map $h : U \to D$ which realizes a homotopy to a point of $f_{|C}$. Then f and $f \circ \tau$ are homotopic to maps f' and f" such that:

- They coincide outside U;
- f'_U and $f"_U$ factor though h.

Since D is contractible, they are homotopic relative to $S \setminus U$. By Theorem 19.47, $[f_C]_r = [f \circ \tau]_r$, hence $[f_C]_i = [f]_i$.

It remains to reduce to such a special case $F = D$. To this aim, consider a generic Morse function

$$r : F \to [0, 1]$$

such that $r^{-1}(0) = C$ and r has no minima and only one maximum. Then we can find a non-critical value $\lambda \in [0,1)$ such that $D = r^{-1}([\lambda, 1])$ is a 2-disk embedded in M with boundary denoted by $\hat{C}$. By following the level lines of r between 0 and λ, we can modify the configuration (S, f, C, F) into a configuration $(\hat{S}, \hat{f}, \hat{C}, D)$ such that $[f]_{\mathrm{i}} = [\hat{f}]_{\mathrm{i}}$, $[f_C]_{\mathrm{i}} = [\hat{f}_{\hat{C}}]_{\mathrm{i}}$, $q_{\hat{f}}(\hat{C}) = 0$. This provides the required reduction to the special case $F = D$. In fact, between two consecutive critical values, we can extend the isotopy between level lines to a diffeotopy of M. At a critical point, the analysis is local in a chart of M: the critical level of r containing a crossing point x_0 is contained in a "critical" surface $\tilde{S}$ with one isolated singular point at x_0, locally isomorphic to a cone centred at x_0 and bases at two disjoint circles; F and $\tilde{S}$ intersect along such a critical level, transversely outside x_0. After having passed such a critical level, we get an intermediate configuration (S', f', C', F') satisfying the required properties.

■

The proof of the main Theorem 19.51 is now complete.

■

19.7.5. More quasi-framing. Now we give another proof of the existence of a quasi-framing on M based on some constructions established in Section 19.7.1.

By contradiction, assume that there is v such that

$$\beta := \omega^2(\xi_v) \neq 0 \ .$$

Let K be an oriented knot in M which represents $e^2(\xi_v)$. By forgetting the orientation, K represents $\omega^2(\xi_v)$. Then it follows from the hypotheses that (see Section 19.6.2):

(1) There is a framing $\mathcal{F}'$ of $T(M)$ over $M \setminus K$.
(2) There is a (possibly nonorientable) compact boundaryless surface F embedded in M such that $F \pitchfork K$ at exactly one point.

Let $N(K) \sim S^1 \times D^2$ be a tubular neighbourhood of K in M transverse to F. By removing the interior of $N(K)$ from F, we can assume to get a surface F_0 properly embedded in

$$M' := M \setminus \mathrm{Int}(N(K))$$

such that $C := \partial F_0$ is a meridian of $\partial N(K)$ bounding a fibre D of $N(K)$. As in Section 19.7.1, we can use the framing $\mathcal{F}'$ to construct a quadratic enhancement of the intersection form of every surface immersed in M'. By Lemma 19.58, we see that the normal framing of C determined by a collar of C in F_0 (equivalently, by a collar of C in the meridian disk D) is *even* with respect to $\mathcal{F}'$, and it is also *even* with respect to a framing of a 3-ball containing D. Then the normal framing determined by $\mathcal{F}'$ is odd within the 3-ball. Consequently, $\mathcal{F}'$ can be extended over a neighbourhood $U \sim D \times [-1, 1]$ of D in $N(K)$; as the closure of $N(K) \setminus U$ is a closed 3-ball, we have obtained an almost-framing of M. By Lemma 19.19 and (1) $\Rightarrow$ (5) of Theorem 19.34, we get that $\omega^2(\xi_v) = 0$, contrary to the assumption that $\omega^2(\xi_v) \neq 0$. This is a contradiction.

■

19.7.6. On $\mathcal{I}_2(M)$ when M is nonorientable. If M is nonorientable, the structure of $\mathcal{I}_2(M)$ is eventually simpler. Consider the product

$$\Gamma_0(M) = \eta_1(M) \times \mathcal{H}^1(M, \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$$

with the twisted group structure given by the operation

$$(\delta, h, a) + (\delta', h', a') := (\delta + \delta' + h \sqcup h', h + h', a + a') \ .$$

Then we have the following (see [**G**]).

THEOREM 19.59. *Let M be a nonorientable compact connected boundaryless 3-manifold. The map*

$$\psi_0 : \mathcal{I}_2(M) \to \Gamma_0(M),\ \ \psi_0([S, f]_{\mathrm{i}}) = ([\Sigma_f], [f], \chi_{(2)}(S))$$

is a well defined semigroup isomorphism (hence $\mathcal{I}_2(M)$ is eventually a group).

To a large extent, the proof is an adaptation of the one when M is orientable, but we have to face several differences (the existence of knots in M with solid-Klein-bottle tubular neighbourhoods, the absence of framing of M, etc.). The basic reason for the final simpler form of $\mathcal{I}_2(M)$ is that the subgroup of the immersed surfaces in a 3-ball of M is a quotient isomorphic to $\mathbb{Z}/2\mathbb{Z}$ of $\mathcal{I}_2(\mathbb{R}^3) \sim \mathbb{Z}/8\mathbb{Z}$. For $\mathcal{I}_2(\mathbb{R}^3)$ is generated by the Boy surface B and, as M is nonorientable, there is a diffeotopy of M which sends a 3-disk of M containing a copy of B into itself, reversing the orientation; hence $[B]_{\mathrm{i}} = [\bar{B}]_{\mathrm{i}} = -[B]_{\mathrm{i}}$.

19.8. Tear and smooth-rational equivalences

Blowing up a manifold along a smooth centre has been defined in Section 7.10.1. In Section 15.5 we have interpreted the stable equivalence between surfaces in terms of blowing up of points which are the only possible smooth centres in such a case. If M is now a compact boundaryless 3-manifold, then besides the points we also have any link of knots in M as a possible smooth centre. In this section, referring to [**BM**], we widely study some equivalence relations generated by blowing up 3-manifolds along smooth centres (and diffeomorphisms). We will also discuss applications of this study to the so-called 3-*dimensional Nash's rationality conjecture.*

19.8.1. 3-dimensional blowing-up-or-down. We denote by $\mathcal{M}_3$ the class of all compact connected boundaryless 3-manifolds. Let M be such a manifold. A possible smooth centre X of a blow-up

$$\pi : B(M, X) \to M$$

is either a finite set of points or a link of a finite number of pairwise disjoint knots in M, $L = K_1 \cup \cdots \cup K_s$. We know that $D_X := \pi^{-1}(X)$ is a hypersurface of $B(M, X)$ called the *exceptional hypersurface*. We also say that M is obtained by *blowing down* $\tilde{M} := B(M, X)$ along the hypersurface D_X.

For simplicity, let us analyze connected centres. A connected smooth centre in M is either a point or a knot K. We know that the effect of blowing up one point consists (up to diffeomorphism) of performing a connected sum $M\#\mathbf{P}^3(\mathbb{R})$, the exceptional hypersurface being a one-side projective plane $\mathbf{P}^2(\mathbb{R})$; that is, a projective plane with an oriented tubular neighbourhood.

As M is not necessarily orientable, then a knot K either preserves the orientation (that is, it has a solid-torus tubular neighbourhood in M), or it reverses the orientation (that is, it has a solid-Klein-bottle tubular neighbourhood in M). In the first case, the exceptional hypersurface D_K in $B(M, K)$ is a one-side torus. In the second, it is a one-side Klein-bottle.

PROPOSITION 19.60. *Let $\tilde{M}$ be in $\mathcal{M}_3$ and Y be a hypersurface of $\tilde{M}$ which is either a projective plane with an oriented tubular neighbourhood, a one-side torus or a one-side Klein bottle. Then there exists M in $\mathcal{M}_3$ and a smooth centre $X \subset M$ such that $\tilde{M} = B(M, X)$ and $Y = D_X$.*

Proof : If $Y \sim \mathbf{P}^2(\mathbb{R})$ with orientable tubular neighbourhood $N(K)$, then $N(K) \sim \mathbf{P}^3(\mathbb{R}) \setminus \mathrm{Int}(B)$, where B is a 3-ball. Then $\tilde{M} = M\#\mathbf{P}^3(\mathbb{R})$ for some M, so that $\tilde{M}$ is the blow-up of M at a point.

The standard model of a tubular neighbourhood of a one-side torus is obtained by taking the blow up

$$\pi : N := B(D^2 \times S^1, \{0\} \times S^1) \to D^2 \times S^1 \ .$$

Denote by $p : D^2 \times S^1 \to S^1$ the natural projection, $D^2_x = p^{-1}(x)$. The manifold N is diffeomorphic to $\mathcal{M} \times S^1$, $\mathcal{M}$ being a Möbius strip, with natural projection $\tilde{p} : \mathcal{M} \times S^1 \to S^1$ such that $\tilde{p} = p \circ \pi$; for every $x \in S^1$, $\mathcal{M}_x = \tilde{p}^{-1}(x) = B(D^2_x, \{0\} \times \{x\})$. On the torus $\partial N \sim \partial D^2 \times S^1$, it is defined the involution τ which restricts to the antipodal one on every ∂D^2_x. The manifold N (and coherently every $\mathcal{M}_x$) can be identified with the mapping cylinder of τ. The exceptional hypersurface is the torus $D = s_0 \times S^1$, where $s_0 = \pi^{-1}(\{0\} \times \{x_0\})$ and x_0 is a base point on S^1. The mapping cylinder structure realizes also N as being a tubular neighbourhood of D, endowed with its projection $q : N \to D$. The restriction of q to ∂N is a fibred double covering of D.

If $Y \subset \tilde{M}$ is a one-side torus, there are in fact *several ways* to fix a parametrization

$$\phi : (N, D) \to (N(Y), Y)$$

so that the blowing down $\pi : N \to D^2 \times S^1$ gives rise to a blowing down $\pi : \tilde{M} \to M$, for some M in $\mathcal{M}_3$, where $(N(Y), Y)$ is mapped to $(N(K), K)$, K is a knot in M which preserves the orientation and $N(K)$ is a tubular neighbourhood of K in M. To do it, assume that $N(Y)$ is constructed using a normal line bundle ξ on Y in $\tilde{M}$. By hypothesis, the Euler class $\omega^1(\xi) \neq 0$. Fix any fibration $\mathcal{F}_s$ of Y by smooth circles parallel to a circle s such that $\omega^2(\xi) \sqcup [s] \neq 0$. This means that the restriction of the line bundle ξ to s is not trivial. Then there is a diffeomorphism $\phi : (N, D) \to (N(Y), Y)$ such that the fibration $\mathcal{F}_{s_0}$ of D by the circles parallel to s_0 is mapped to the fibration $\mathcal{F}_s$. To see it, we can transfer the question to the above standard model. The fibration $\mathcal{F}_{s_0}$ of D lifts, by the projection q, to the fibration by meridians of $\partial N \sim \partial D^2 \times S^1$; set $m_0 = \partial D^2 \times \{x_0\}$ and denote by $\tilde{\mathcal{F}}_{m_0}$ such fibration. Fix on D another fibration $\mathcal{F}_s$ parallel to an s with the properties fixed above. This lifts by the projection q to a fibration $\tilde{\mathcal{F}}_{\tilde{s}}$ of ∂N by circles parallel to a $\tilde{s}$ such that $[\tilde{s}] = [m_0] \in \eta_1(\partial N)$. Moreover, by construction, $\tilde{\mathcal{F}}_{\tilde{s}}$ is invariant by the involution τ. We claim that, possibly up to isotopy of s, there is a diffeomorphism h of the torus ∂N which sends $\tilde{\mathcal{F}}_{m_0}$ to $\tilde{\mathcal{F}}_{\tilde{s}}$ and extends to a diffeomorphism of (N, D), sending the fibration $\mathcal{F}_{s_0}$ of D to $\mathcal{F}_s$. In such a case, it is easy to see that the topological space obtained by collapsing each fibre of $\mathcal{F}_s$ to one point results from another blow down of (N, D) obtained by the *flip* $\mathcal{F}_{s_0} \to \mathcal{F}_s$ of fibrations of the exceptional hypersurface D. To justify the claim, let us identify ∂N with $\mathbb{R}^2/\mathbb{Z}^2$, endowed with "linear" coordinates such that the line $\{y = 0\}$ is mapped to $l_0 = \{p_0\} \times S^1$, while the line $\{x = 0\}$ is mapped to m_0 and the involution can be expressed as $\tau(x, y) = (x, y + 1/2)$; up to isotopy, a generic diffeomorphism is of the form $h(x, y) = (ax + by, cx + dy)$, with the coefficients belonging to a matrix in $GL(2, \mathbb{Z})$. Under our hypotheses, $h(0, y) = (by, dy)$, where b is even and d is odd, so that clearly $h \circ \tau = \tau \circ h$ and this is enough to conclude.

The discussion for the one-side Klein bottle is similar (however, see Remark 19.63).

■

19.8.2. Tears and Dehn surgery. The possibility of *flippig the fibrations of an exceptional hypersurface*, hence modifying the corresponding blowing down (sometimes this modification is called a *flop*), suggests a way to modify the topology of 3-manifolds.

Definition 19.61. Let M be in $\mathcal{M}_3$ and $L = K_1 \cup \cdots \cup K_s$ be a link in M whose constituent knots preserve the orientation. We say that M' in $\mathcal{M}_3$ is obtained from M by a *tear along* L, if up to diffeomorphism there is a blow-down flop

$$M \leftarrow B(M, L) \to M'$$

associated to a system of flips of fibrations of the exceptional hypersurfaces D_{K_i} as in the proof of Proposition 19.60. In other words, $(B(M, L), D_L) =$

$(B(M', L'), D_{L'})$ for some link $L' = K'_1 \cup \cdots \cup K'_s$ in M' whose constituent knots preserve the orientations.

LEMMA 19.62. *Tears define an equivalence relation called* tear equivalence*, and we write* $M \sim_t M'$.

Proof : If we move a centre by an ambient isotopy, the result of a blowing up does not change up to diffeomorphism relative to the exceptional hypersurfaces. Given a tear from M to M' (with associated links L in M and L'_1 in M') and a tear from M' to M" (with associated links L'_2 in M' and L" in M"), by transversality we can assume that $L'_1 \cap Ł'_2 = \emptyset$, hence there is a copy of L'_2 in M and a copy of L'_1 in M" so that $L \cup L'_2$ and $L" \cup L'_1$ are links in M and M", respectively, supporting a tear from M to M". This proves that the relation is transitive. It is trivially reflexive and symmetric.

∎

REMARK 19.63. *A priori*, one should consider also tears along knots which reverse the orientation. However, for any such a tear $M \leftarrow \tilde{M} \rightarrow M'$, M and M' are diffeomorphic; this happens because on a Klein bottle there is only one isotopy class of smooth circles with annular tubular neighbourhood. So we consider only tears along knots preserving the orientation.

It is convenient to rephrase tears in terms of more usual modifications performed on 3-manifolds. As above, let M be in $\mathcal{M}_3$, $L = K_1 \cup \cdots \cup K_s$ be a link in M with constituent knots preserving the orientation. Let $N(L) = N(K_1) \amalg \cdots \amalg N(K_s)$ be a tubular neighbourhood of L in M. Consider the manifold with s toric boundary components

$$N := M \setminus \mathrm{Int}N(L) \ .$$

We say that M' is obtained by a *Dehn surgery* of M along L if, up to diffeomorphism, it is obtained by gluing back every $N(K_i)$ to N along the torus $\partial N(K_i)$ by means of a diffeomorphism $h_i : \partial N(K_i) \rightarrow \partial N(K_i)$, $i = 1, \ldots, s$. The link $L \subset N(L)$ determines a link $L' = K'_1 \cup \cdots \cup K'_s$ in M', and the identity map of N extends to a diffeomorphism $\psi : M \setminus L \rightarrow M' \setminus L'$. If m_i is a meridian of $\partial N(K_i)$, then $h_i(m_i) = s_i$ is a smooth circle on $\partial N(K_i)$. The fibration of $\partial N(K_i)$ by meridians parallel to m_i is mapped by h_i to a fibration by circles parallel to s_i. These are meridians of a tubular neighbourhood of L' in M'. If every s_i is a *longitude* of $\partial N(K_i)$, then M' is obtained from M by an ordinary surgery already considered above; that is, Dehn surgery generalizes the ordinary surgery associated to special 4-dimensional triads. The diffeomorphism ϕ extends to a diffeomorphism $\phi : M \rightarrow M'$ if and only if every s_i is a meridian of $\partial N(K_i)$.

Now, up to diffeomorphism, $B(M', L')$ is obtained from $B(M, L)$ by gluing back every $B(N(K_i), K_i)$ to N along the torus $\partial N(K_i)$ by means of the same diffeomorphism $h_i : \partial N(K_i) \rightarrow \partial N(K_i)$, $i = 1, \ldots, s$, as before.

DEFINITION 19.64. We say that a Dehn surgery *lifts to a tear* if the diffeomorphism $\tilde{\phi} : B(M, L) \setminus D_L \to B(M', L') \setminus D_{L'}$, which lifts $\phi : M \setminus L \to M' \setminus L'$, extends to a diffeomorphism $\tilde{\phi} : B(M, L) \to B(M', L')$, preserving the exceptional hypersurface.

PROPOSITION 19.65. *A Dehn surgery from M to M' lifts to a tear if and only if, for every $i = 1, \dots, s$, $[s_i] = [m_i] \in \eta(\partial N(K_i)) = \mathcal{H}^1(\partial N(K_i); \mathbb{Z}/2\mathbb{Z})$.*

Proof : The condition is necessary because the meridians generate the kernel of the unoriented bordism morphism induced by the inclusions $\partial N(K_i) \to N(K_i)$. The other implication rephrases the proof of Proposition 19.60.

■

Concerning ordinary surgery, we have the following immediate corollary.

COROLLARY 19.66. *Let M', $M"$ be obtained by ordinary (longitudinal) surgery on M along a same link $L = \cup_i K_i$ with different normal framings $\{\mathfrak{f}'_i\}$ and $\{\mathfrak{f}"_i\}$, respectively. Let $L' \subset M'$ and $L" \subset M"$ be the links corresponding to L. Then $M"$ is obtained (up to diffeomorphism) from M' by a tear of the form*

$$M' \leftarrow B(M', L') = B(M", L") \to M"$$

if and only if every $\mathfrak{f}'_i$ differs from $\mathfrak{f}"_i$ by an even number of twists.

Hence the tear equivalence can be considered as a specialization of the equivalence relation generated by Dehn surgery. As this last extends ordinary surgery and preserves orientability, then we already know that *being, or not being, orientable is a complete invariant for the Dehn surgery equivalence*. We are going to see that this is no longer true for the tear equivalence. We refine the 'orientable/nonorientable' partition $\mathcal{M}_3 = \mathcal{M}_3^+ \amalg \mathcal{M}_3^-$, introducing three *types* completely determined by the behaviour of $\omega^1(*)$.

(1) $\omega^1(M) = 0 \in \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z})$; that is, M is *orientable*;

(2) $\omega^1(M) \neq 0$ and $\omega^1(M)^2 := \omega^1(M) \sqcup \omega^1(M) = 0$. In this case we say that M is *weakly nonorientable* and we write $M \in \mathcal{M}_3^w$;

(3) $\omega^1(M) \neq 0$ and $\omega^1(M)^2 := \omega^1(M) \sqcup \omega^1(M) \neq 0$; in this case we say that M is *strongly nonorientable* and we write $M \in \mathcal{M}_3^s$.

Characteristic surfaces: If M is nonorientable, every hypersurface F which represents $\omega^1(M)$ is called a *characteristic surface* of M. We can assume that F is connected and it is necessarily orientable: the boundary $\partial N(F)$ of a tubular neighbourhood is connected and orientable, as it is the boundary of the orientable manifold $M \setminus \mathrm{Int} N(F)$; the projection of $\partial N(F)$ to F is $2:1$ and every orientation on $\partial N(F)$ descends to F.

PROPOSITION 19.67. *Let $M \sim_t M'$ be realized by a tear*

$$M \xleftarrow{\pi} B(M, L) = \tilde{M} = B(M', L') \xrightarrow{\pi'} M',\ L = \cup_{i=1}^s K_i \ .$$

1) For every $j = 0, \dots, 3$, $\pi^* : \mathcal{H}^j(M; \mathbb{Z}/2\mathbb{Z}) \to \mathcal{H}^j(\tilde{M}; \mathbb{Z}/2\mathbb{Z})$ is an injective homomorphism and the similar fact holds for π'.

2) $\mathcal{H}^1(\tilde{M};\mathbb{Z}/2\mathbb{Z}) \sim \mathcal{H}^1(M;\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^s$*, where the last factor is generated by the components* D_{K_i} *of* D_L *and* $\mathcal{H}^2(\tilde{M};\mathbb{Z}/2\mathbb{Z}) \sim \mathcal{H}^2(M;\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^s$*, where the last factor is generated by the fibres* $\mathcal{M}_i$ *of the fibrations* $\mathcal{M} \times K_i \to K_i$ *of* D_{K_i} *; similarly for* π'*.*

3) For every $j = 0, \dots, 3$*, there is a linear isomorphism*

$$h_j : \mathcal{H}^j(M;\mathbb{Z}/2\mathbb{Z}) \to \mathcal{H}^j(M';\mathbb{Z}/2\mathbb{Z})$$

such that $(\pi')^* \circ h^j = \pi^*$*. Moreover,* $h_1(\omega^1(M)) = \omega^1(M')$ *and for every* $\alpha \in \mathcal{H}^1(M;\mathbb{Z}/2\mathbb{Z})$*,* $h^2(\alpha \sqcup \omega^1(M)) = h^1(\alpha) \sqcup \omega^1(M')$*.*

4) M *and* M' *are of the same type.*

Proof : Let us justify $(1)-(3)$. For every j, every class in $\mathcal{H}^j(M;\mathbb{Z}/2\mathbb{Z})$ can be represented by an embedded proper $(3-j)$-submanifold S transverse to the link L. The corresponding class in $\mathcal{H}^j(\tilde{M};\mathbb{Z}/2\mathbb{Z})$ is represented by the strict transform $\tilde{S}$ of S via the blow-up. If $j = 2,3$, $\tilde{S}$ is mapped diffeomorphically to S by π. If $j = 0$, $\tilde{S} = \tilde{M}$. If $j = 1$, then $\tilde{S} = B(S, S \pitchfork L)$. As for (2), notice that $\mathcal{M}_i \bullet D_{K_j} = \delta_{i,j}$. As for (3), consider the diffeomorphism

$$\phi : M \setminus L \to M' \setminus L' \ .$$

If $j = 2,3$, then h_j is determined by the diffeomorphism $S \sim \phi(S)$. If $j = 0$, then $h_0([M]) = [M']$, and notice that $[\tilde{M}, \pi] = [M]$, $[\tilde{M}, \pi'] = [M']$. If $j = 1$, then S is a hypersurface transverse to L. Then $S \setminus \text{Int } N(L)$ is sent diffeomorphically to $\bar{S}'$ properly embedded in $M' \setminus \text{Int}N(L')$; as ϕ preserves the class of meridians mod (2), then $\bar{S}'$ can be completed to a boundaryless hypersurface S' transverse to L'. This geometric correspondence $S \leftrightarrow S'$ induces h_1. If S is a characteristic surface of M, then as the constituent knots of L preserve the orientation, we can assume that $S \cap L = \emptyset$, so that the diffeomorphic surface $S' = \phi(S)$ does not intersect L' and is a characteristic surface of M'. The last statements of (3) follow. Item (4) is an easy corollary of the other items.

■

Whenever S' is obtained from S as in the above proof, we will say that S' is obtained by *darning* S.

Remark 19.68. One would wonder about a graded ring isomorphism in statement (3), but this is not true. For example, $S^1 \times S^2$ and $\mathbf{P}^3(\mathbb{R})$ can be obtained by ordinary surgery along an unknot $K \subset \mathbb{R}^3 \subset S^3$ with the standard even normal framing $\mathfrak{f}_0$ and the framing which differs from it by two twists, respectively. By Corollary 19.66, they are connected by a tear, but their $\mathbb{Z}/2\mathbb{Z}$-cobordism rings are different.

19.8.3. ***rs*-equivalence.** We define now a coarser equivalence relation generated by blowing-up-or-down.

DEFINITION 19.69. Let M, M' be in $\mathcal{M}_3$. We say that, *up to diffeomorphism, M' is obtained from M by a finite chain of blowing-up-or-down* if there is a finite chain of the form

$$M \to M_0 \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow \cdots \leftrightarrow M_n \leftarrow M' \ ,$$

where

(1) Every M_i is in $\mathcal{M}_3$;
(2) The right and left arrows are diffeomorphisms;
(3) For every $i \neq n$, $M_i \leftrightarrow M_{i+1}$ is either a blow-up along a smooth centre

$$M_i \leftarrow M_{i+1} = B(M_i, C_i)$$

or a blow-up

$$M_i = B(M_{i+1}, Z_{i+1}) \to M_{i+1}$$

so that M_{i+1} is obtained by a *blow-down* of M_i.

This defines another equivalence relation called *smooth-rational equivalence*; it extends the diffeomorphism and also the tear equivalence. We write $M \sim_{sr} M'$. Note that no one of the tear invariants pointed out in Proposition 19.67 persists for the sr-equivalence.

We want to determine the quotient set of $\mathcal{M}_3$ mod $\sim_{sr}$ or mod $\sim_t$. Tear equivalence preserves the type, so we can split the study of $\mathcal{M}_3$ mod $\sim_t$, type by type.

The results for $\mathcal{M}_3^+$ mod $\sim_t$ and for $\mathcal{M}_3$ mod $\sim_{sr}$ are easy to state.

THEOREM 19.70. *For every M, M' in $\mathcal{M}_3^+$, $M \sim_t M'$ if and only if $\dim \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z}) = \dim \mathcal{H}^1(M'; \mathbb{Z}/2\mathbb{Z})$. If $\dim \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z}) = h$, then*

$$M \sim_t S^3 \# h\mathbf{P}^3(\mathbb{R}) \ .$$

PROPOSITION 19.71. *For every M in $\mathcal{M}_3^-$ there exists $M' \in \mathcal{M}_3^+$ such that $M \sim_{sr} M'$.*

THEOREM 19.72. *For every M in $\mathcal{M}_3$, $M \sim_{sr} S^3$.*

Proof: If M is in $\mathcal{M}_3^+$, then the result follows immediately from Theorem 19.70, as $S^3 \# h\mathbf{P}^3(\mathbb{R})$ is obtained by blowing-up S^3 at h points. If $M \in \mathcal{M}_3^-$, Proposition 19.71 reduces it to the orientable case.

■

The structure of $\mathcal{M}_3^-$ mod $\sim_t$ is more complicated; we will face it later. First, we will develop a proof of Theorem 19.70 and of Proposition 19.71.

19.8.4. Disorientated surfaces and weakly trivial knots. Let N be a compact 3-manifold with possibly nonempty boundary ∂N. A connected, properly embedded surface F in N is said to be *disorientated* if it is nonorientable and has an orientable neighbourhood in N.

Let M be in $\mathcal{M}_3$, and let $K \subset M$ be a knot that preserves the orientation, endowed with a tubular neighbourhood $N(K)$. The knot K is said to be *weakly trivial* if there exists a longitude l on $\partial N(K)$ which bounds a disorientated surface F properly embedded in $M \setminus \mathrm{Int}N(K)$.

The notion of tears makes sense also for a manifold with boundary N, provided that the supporting link is contained in the interior of N. The following proposition shows the tear's power to simplify disorientated hypersurfaces and eventually the topology of 3-manifolds.

PROPOSITION 19.73. *Let $S \subset N$ be a disorientated hypersurface. Assume that S has at most two boundary components. Then we have:*

- *A link $L \subset \mathrm{Int}(S) \subset \mathrm{Int}(N)$ with constituent knots preserving the orientation;*
- *A tear $N \leftarrow B(N,L) = \tilde{N} = B(N',L') \rightarrow N'$;*
- *A surface $S' \subset N'$ obtained by darning S (over the tear) such that:*

  (1) *If S is boundaryless, then S' is a disorientated projective plane.*
  (2) *If ∂S is connected, then S' is a disk properly embedded in N'.*
  (3) *If ∂S has two components, then S' is a two-sides annulus properly embedded in N'.*

Proof : The surface S is diffeomorphic to the connected sum of s copies of $\mathbf{P}^2(\mathbb{R})$, $s \geq 1$, from which we have removed k disjoint open 2-disks, either $k = 0, 1, 2$. Let $L = K_1 \cup \cdots \cup K_s$ be formed by the cores of s pairwise disjoint Möbius strips $\mathcal{M}_i$ embedded in S. Each K_j reverses the orientation of S and preserves the orientation of N (because S has an orientable neighbourhood). Then $[\partial \mathcal{M}_i]$ is a meridian of $\partial N(K_i)$ mod (2), and we can consider the corresponding tear $N \leftarrow B(N,L) = B(N',L') \rightarrow N'$. Then every K_i collapses to one point in a darning surface S' properly embedded in N' with orientable neighbourhood. If $k = 0$ then S' is a 2-sphere; in order to get a disorientated $\mathbf{P}^2(\mathbb{R})$, it is enough to remove from L one constituent knot. In the other two cases we get either a disk or an annulus.

■

COROLLARY 19.74. *For every $M \in \mathcal{M}_3$ there is a chain of the form*

$$M \rightarrow M_0 \leftrightarrow M_1 \leftrightarrow \cdots \leftrightarrow M_n \leftarrow M'$$

such that:

(1) *Every M_i is in $\mathcal{M}_3$, the right and left arrows being diffeomorphisms;*
(2) *$\mathcal{H}^1(M'; \mathbb{Z}/2\mathbb{Z})$ is generated by $\omega^1(M')$;*
(3) *For every $i \neq n$, $M_i \leftrightarrow M_{i+1}$ either is:*
   - *A tear;*

• *A blow-up* $M_i = B(M_{i+1}, x_0) \to M_{i+1}$ *at a point of* M_{i+1}*;*

• *A blow-up* $M_i = B(M_{i+1}, K) \to M_{i+1}$ *along a smooth knot of* M_{i+1} *which preserves the orientation.*

Proof : If M already satisfies (2), then take $M' = M$. Otherwise, there is a hypersurface S, such that $[S] \neq 0 \in \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z})$ and S is not a characteristic surface of M. We can assume that S is connected and that there is a characteristic surface F such that either:

• $S \cap F = \emptyset$, meaning $S \subset M \setminus N(F)$ for a small tubular neighbourhood of F;

• $S \pitchfork F$ along a knot $K \subset S$ which does not divide it.

• In both cases, $S \setminus \mathrm{Int}N(F)$ is properly embedded in $M \setminus \mathrm{Int}N(F)$, has oriented neighbourhood therein, and there is a smooth circle $C \subset M \setminus \mathrm{Int}N(F)$ with nontrivial intersection number mod (2) with $S \setminus \mathrm{Int}N(F)$. By adding an embedded 1-handle along a suitable arc of C, we can also assume that $S \setminus \mathrm{Int}N(K)$ is disorientated. Now, if S is disjoint from F, by Proposition 19.73 there is a tear which converts S into a disorientated projective plane; this can be considered as the exceptional hypersurface of a blow-up of a point. In the other case there is a tear converting $S \setminus \mathrm{Int}N(F)$ into an annulus; together with $S \cap N(F)$, they form a one-side torus which can be considered as the exceptional hypersurface of a blow-up along a knot.

■

COROLLARY 19.75. *If* M *is orientable and* $\dim \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z}) = h$*, then*

$$M \sim_t \tilde{M} \ ,$$

where

$$\tilde{M} = h\mathbf{P}^3(\mathbb{R}) \# M'$$

and $\mathcal{H}^1(M'; \mathbb{Z}/2\mathbb{Z}) = 0$.

Proof : As M is orientable, $\omega^1(M) = 0$; hence the statement and the proof of Corollary 19.74 tell us that $\mathcal{H}^1(M'; \mathbb{Z}/2\mathbb{Z}) = 0$ and that only blow-up of points occur. Up to isotopy, a point misses any possible already present exceptional hypersurface, tears and blowing up of points commute and the corollary follows.

■

COROLLARY 19.76. *Let* M *and* M' *in* $\mathcal{M}_3$ *be such that*

$$\mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z}) = \mathcal{H}^1(M'; \mathbb{Z}/2\mathbb{Z}) = 0 \ .$$

Assume that M' *is obtained from* M *by an ordinary longitudinal surgery of* M *along a weakly trivial knot* $K \subset M$*. Then* $M \sim_t M'$*.*

Proof : By Proposition 19.73, there is a tear from M to M_1 converting K to a genuine trivial knot $K_1 \subset M_1$. Up to tear equivalence, we can assume that M' is obtained from M by an ordinary surgery along a trivial knot K. As they both have vanishing $\mathcal{H}^1$, the normal framing $\mathfrak{f}$ of this surgery must be odd for the framing $\mathfrak{f}_0$ determined by a collar of K in a spanning 2-disk. On the other hand, M is diffeomorphic to the manifold obtained by using the framing $\mathfrak{f}_1$ which differs from $\mathfrak{f}_0$ by one twist. Hence, by Corollary 19.66, there is a tear from M to M'.

■

As a further corollary, we can prove Proposition 19.71, which we state again

For every M in $\mathcal{M}_3^-$ there exists $M' \in \mathcal{M}_3^+$ such that $M \sim_{sr} M'$.

Proof : Assume that M has a connected characteristic surface F of genus $g+1>1$. We are going to show that $M \sim_{sr} M'$ such that M' either has a characteristic surface F' of genus g, if $g>0$, or it is orientable. This will achieve the result by induction on g. First, we can assume that F is one-side in M. Let $K \subset F$ be a smooth circle which does not divide F. Then the strict transform $\tilde{F}$ of F in $B(M,K)$ is a one-side characteristic surface of the same genus. If F is a one-side torus, then it is the exceptional hypersurface of a blow-down to an orientable M' and we are done. If $g>1$, there is a smooth circle C on F which divides it by a one-side torus T_0 with one hole and a bilateral surface S_0 of genus $g-1$ with one hole. By adding an embedded 1-handle as in the proof of Corollary 19.74, we can modify S_0 far from C and make it disorientated. By Proposition 19.73, there is a tear from M to M_1 which converts S_0 to a 2-disk so that C becomes a trivial knot in M_1. The manifold M_2 obtained by ordinary surgery along C, with normal framing given by a tubular neighbourhood of C in F, is tear equivalent to $M_1 \# \mathbf{P}^3(\mathbb{R})$, hence it is sr-equivalent to M_1 and therefore to M. We conclude by noticing that a characteristic surface of M_2 is given by the disjoint union of a surface of genus g and a one-side torus, which again can be considered as the exceptional hypersurface of a blow-down.

■

19.8.5. $\mathcal{M}_3^+$ mod $\sim_t$ and $\mathcal{M}_3$ mod $\sim_{sr}$. We are ready to prove Theorems 19.70 and 19.72. Thanks to Corollary 19.76 and Proposition 19.71, it is enough to prove the following lemma.

LEMMA 19.77. *For every M in $\mathcal{M}_3$ such that $\mathcal{H}^1(M;\mathbb{Z}/2\mathbb{Z})=0$, there exists a sequence $S^3=M_0,M_1,\dots,M_n \sim M$, such that:*

(1) For every M_i, $\mathcal{H}^1(M_i;\mathbb{Z}/2\mathbb{Z})=0$;
(2) M_{i+1} is obtained from M_i by an ordinary surgery along a weakly trivial knot $K_{i+1} \subset M_i$.

Proof : We use some notions that we will develop in Chapter 20, Section 20.2.1. Here we outline the main points. We know that $S^3 \sim_\sigma M$; that is, there is a triad (W, S^3, M) with a handle decomposition made by 2-handles only, so that M is obtained by longitudinal surgery along a framed link $L = \cup_i K_i$ in S^3. The framing $\mathfrak{f}_i$ is encoded by an integer which expresses the number of twists compared with the framing given by the collar of K_i in a Seifert surface. The intersection form of $\mathcal{H}^2(W; \mathbb{Z}/2\mathbb{Z})$ is represented by the linking matrix mod (2) of this framed link L, so that along the diagonal we have the reduction mod (2) of such integers. As $\mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z}) = 0$, the intersection form is non-degenerate. Possibly performing an elementary blow-up move (Section 19.4.1), we can also assume that the form is not totally isotropic, hence it has an orthogonal basis (see Section 15.1). By realizing such a change of basis by handle sliding, we get that every K_i is the boundary of a surface S_i disjoint from the rest of the link, and the new normal framings are odd. So the knot K_{i+1} is weakly trivial in the manifold M_i obtained by the surgery along the partial framed link $K_1 \cup \cdots \cup K_i$.

■

19.8.6. $\mathcal{M}_3^-$ mod $\sim_t$. As already said, this is more complicated. We will give exhaustive statements. For detailed proofs, a curious reader may refer to [**BM**].

We can manage type by type. For $\mathcal{M}_3^s$ the statement is simpler; like the orientable case, the necessary conditions of Proposition 19.67 are also sufficient.

THEOREM 19.78. *Let M, M' be strongly nonorientable. Then $M \sim_t M'$ if and only if for every $j = 0, \dots, 3$, there is a linear isomorphism*

$$h_j : \mathcal{H}^j(M; \mathbb{Z}/2\mathbb{Z}) \to \mathcal{H}^j(M'; \mathbb{Z}/2\mathbb{Z})$$

such that $h_1(\omega^1(M)) = \omega^1(M')$ and for every $\alpha \in \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z})$,

$$h^2(\alpha \sqcup \omega^1(M)) = h^1(\alpha) \sqcup \omega^1(M') \ .$$

For weakly nonorientable manifolds, another tear invariant comes up.

We begin with a construction that makes sense for every orientable compact boundaryless surface S embedded in any M in $\mathcal{M}_3$. Consider the subspace of $\eta_1(S)$ formed by the 1-*boundary* in M; that is,

$$\mathcal{B}(S, M) = \ker i_*$$

where $i : S \to M$ is the inclusion. Let $\alpha \in \mathcal{B}(S, M)$. Then $\alpha = [c]$ for some smooth circle c on S. By hypothesis, c bounds a *membrane* $\mathfrak{M} \subset M$: $\mathfrak{M}$ is a compact surface embedded in M, such that $c = \partial\mathfrak{M}$, and moreover $\mathfrak{M}$ is in "general position" with respect to S; this means that $S \pitchfork \mathrm{Int}(\mathfrak{M})$ and $S \cap \mathfrak{M}$ is the union $c \cup d$ where d is a smooth curve properly embedded in S (i.e. $\partial d = \cap \partial\mathfrak{M}$). Tubular neighbourhoods of d, $N(d, S)$ and $N(d, \mathfrak{M})$ in S and $\mathfrak{M}$ respectively, coincide at ∂d along a tubular neighbourhood of $\partial d = d \cap c$ in c. Then along the abstract double $D(d) = d_+ \cup d_-$ of d, we can define a band $N(D(d))$ equal to $N(d, S)$ on d_+, equal to $N(d, \mathfrak{M})$ on d_-, glued by the identity on $\partial d_+ = \partial d_-$. Using the self-intersection of $D(d)$ in $N(D(d))$, we can define

$$\rho_{\mathfrak{M}}(c) = D(d) \bullet D(d) \in \mathbb{Z}/2\mathbb{Z} \ .$$

The question is under which hypotheses this construction *well defines* a map

$$\rho_S : \mathcal{B}(S, M) \to \mathbb{Z}/2\mathbb{Z}, \ \rho(\alpha) = \rho_{\mathfrak{M}}(c), \ \alpha = [c] \ .$$

This is widely discussed in [**BM**]. Here we are concerned with the application of this construction to a characteristic surface F of M in $\mathcal{M}_3^-$.

PROPOSITION 19.79. *Let F be a characteristic surface of the nonorientable 3-manifold M. Then $\rho_F : \mathcal{B}(F, M) \to \mathbb{Z}/2\mathbb{Z}$ is well defined if and only if M is weakly nonorientable ($M \in \mathcal{M}_3^w$).*

A first point where the vanishing of $\omega^1(M) \sqcup \omega^1(M)$ is relevant is in showing that the value of $\rho_{\mathfrak{M}}(c)$ does not depend on the choice of the membrane $\mathfrak{M}$. One verifies that:

(i) $\sigma \sqcup \sigma \sqcup \omega^1(M) + \sigma \sqcup \omega^1(M) \sqcup \omega^1(M) = 0$ for every $\sigma \in \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z})$ if and only if $\omega^1(M) \sqcup \omega^1(M) = 0$;

(ii) given two membranes $\mathfrak{M}$ and $\mathfrak{M}'$ of c, $\tau = \mathfrak{M}' \cup \mathfrak{M}$ defines a cycle mod (2) in M and ones verifies that

$$\rho_{\mathfrak{M}'} - \rho_{\mathfrak{M}} = [\tau] \sqcup [\tau] \sqcup \omega^1(M) + [\tau] \sqcup \omega^1(M) \sqcup \omega^1(M) \ .$$

This is the first step to show that $\rho(c)$ only depends on the class $[c] \in \eta_1(F)$.

Let $M \in \mathcal{M}_3^w$, F, ρ_F, β be as in the above proposition. In general, β is degenerate; that is, its radical $\mathcal{B}(F, M)^\perp \neq \{0\}$. Set

$$\hat{\mathcal{B}}(F, M) = \mathcal{B}(F, M)/\mathcal{B}(F, M)^\perp \ .$$

Then β induces on this quotient space a non-degenerate form $\hat{\beta}$. There are two possibilities:

(1) $\rho_F \neq 0$ on $\mathcal{B}(F, M)^\perp$.

(2) $\rho_F = 0$ on $\mathcal{B}(F, M)^\perp$.

PROPOSITION 19.80. *If $\rho_F = 0$ on $\mathcal{B}(F, M)^\perp$, then ρ_F descends to a map*

$$\hat{\rho}_F : \hat{\mathcal{B}}(F, M) \to \mathbb{Z}/2\mathbb{Z}$$

which is a quadratic enhancement of the non-degenerate form $\hat{\beta}$.

We can associate with the characteristic surface F the symbol

$$\tau_F \in \{\emptyset\} \cup \mathbb{Z}/2\mathbb{Z}$$

where $\tau_F = \emptyset$ if $\rho_F \neq 0$ on $\mathcal{B}(F, M)^\perp$, $\tau_F = \delta_F$ if $\rho_F = 0$ on $\mathcal{B}(F, M)^\perp$, where $\delta_F := \delta(\hat{\rho}_F) \in \mathbb{Z}/2\mathbb{Z}$ is the *Arf invariant* (see Section 15.6).

PROPOSITION 19.81. *Let $M \in \mathcal{M}_3^w$, then $\tau_M := \tau_F$ is well defined, that is it does not depend on the choice of the characteristic surface F.*

We have refined the type of weakly nonorientable manifolds according to the value of τ_M. Finally, we can complete the classification up to tear equivalence.

THEOREM 19.82. *Let M, M' be weakly nonorientable. Then $M \sim_t M'$ if and only if, for every $j = 0, \dots, 3$, there is a linear isomorphism*

$$h_j : \mathcal{H}^j(M; \mathbb{Z}/2\mathbb{Z}) \to \mathcal{H}^j(M'; \mathbb{Z}/2\mathbb{Z})$$

such that $h_1(\omega^1(M)) = \omega^1(M')$, for every $\alpha \in \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z})$, $h^2(\alpha \sqcup \omega^1(M)) = h^1(\alpha) \sqcup \omega^1(M')$ and, moreover, $\tau_M = \tau_{M'}$.

The eventual result contains more information. We have explicitly given normal representatives for every orientable tear class α, that is $h\mathbf{P}^3(\mathbb{R})$, $h = \dim \mathcal{H}^1(\alpha; \mathbb{Z}/2\mathbb{Z}) := \mathcal{H}^1(M; \mathbb{Z}/2\mathbb{Z})$, $\alpha = [M]_t$. This can be done for every tear equivalence class. Here, we limit to state some qualitative features of these normal representatives. Let us say that M is *smooth-rational elementary* if it is obtained by means of a tower of blowing up along smooth centres over the standard 3-sphere S^3:

$$S^3 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_k = M \ .$$

PROPOSITION 19.83. *With the exception of the weakly nonorientable class α_0 such that $\dim \mathcal{H}^1(\alpha_0; \mathbb{Z}/2\mathbb{Z}) = 1$ and $\tau_{\alpha_0} = 1$, the normal representative of every tear class α is smooth-rational elementary. In the exceptional case, α_0 cannot be represented by any smooth-rational elementary manifold, and for the normal representative M_{α_0} there is a smooth-rational elementary $\tilde{M}_{\alpha_0}$ and a blow-up $\tilde{M}_{\alpha_0} = B(M_{\alpha_0}, x_0) \to M_{\alpha_0}$, where x_0 is a point.*

19.8.7. On 3-dimensional Nash's rationality conjecture. Using the classification up to tear equivalence, in [**BM**] we get an answer to the so-called rationality Nash's conjecture in three dimensions.

Let us say that a regular 3-dimensional real algebraic set X is *rational elementary* if it is obtained by a tower of blowing up along regular real algebraic centres over the standard sphere S^3.

First, one proves that every tear equivalence class has an explicitly given rational model which is elementary with one exception. Referring to Proposition 19.83, using variations on Nash-Tognoli theorem (see Section 17.5.3) we have the following.

PROPOSITION 19.84. *With the exception of the weakly nonorientable class α_0 such that $\dim \mathcal{H}^1(\alpha_0; \mathbb{Z}/2\mathbb{Z}) = 1$ and $\tau_{\alpha_0} = 1$, the normal representative of every tear class α is a regular rational elementary real algebraic set Y_α. In the exceptional case, the normal representative Y_{α_0} is a rational algebraic set with one singular point y_0; moreover, there is an 'algebraic resolution of singularity' $\psi : \hat{Y}_{\alpha_0} \to Y_{\alpha_0}$ such that $\hat{Y}_{\alpha_0}$ is regular rational elementary and $\psi : \hat{Y}_{\alpha_0} \setminus \psi^{-1}(y_0) \to Y_{\alpha_0} \setminus \{y_0\}$ is an algebraic isomorphism.*

THEOREM 19.85. *For every tear equivalence class* $\alpha \neq \alpha_0$*, for every* $M \in \alpha$*, there is a tear from* M *to* Y_α *of the form*

$$M \xleftarrow{\sigma} Y_M \xleftarrow{\mathfrak{p}} B(Y_\alpha, L_M) \xrightarrow{\pi} Y_\alpha$$

where:

- π *is a blowing up of* Y_α *along a regular real algebraic link* $L_M \subset Y_\alpha$*; hence* $B(Y_\alpha, L_M)$ *is regular rational elementary.*
- Y_M *is a rational real algebraic set with* 1*-dimensional singular set* $\mathrm{Sing}(Y_M) = \mathfrak{p}(D_{L_M})$ *consisting of a union of regular real algebraic circles;*
- *The surjective algebraic map* $\mathfrak{p}$ *is a 'resolution of singularity'; that is,*

$$\mathfrak{p} : B(Y_\alpha, L_M) \setminus D_{L_M} \to Y_M \setminus \mathrm{Sing}(Y_M)$$

is an algebraic isomorphism between regular Zariski open sets;

- σ *is a homeomorphism which restricts to a diffeomorphism on* $Y_M \setminus \mathrm{Sing}(Y_M)$ *and on* $\mathrm{Sing}(Y_M)$*;*

As for $M \in \alpha_0$*, we have a similar realization of the form*

$$M \xleftarrow{\sigma} Y_M \xleftarrow{\mathfrak{p}} B(Y_0, L_M) \xrightarrow{\pi} Y_{\alpha_0}$$

where $L_M \subset R(Y_{\alpha_0})$*, and eventually the rational model* Y_M *of* M *has a further isolated singular point and admits an algebraic resolution of singularity by means of the rational elementary* $B(\hat{Y}_{\alpha_0}, \hat{L}_M)$*,* $\hat{L}_M = \psi^{-1}(L_M)$*.*

The theorem shows that every M in $\mathcal{M}_3$ has a *singular* rational algebraic model Y_M with a mild controlled singular set. The situation is very similar to what we have done in the case of surfaces (Section 15.5). In the case of surfaces, Comessati tells us that for a genus greater than 1, the presence of one singular point in a rational model of an orientable surface is not only an accident of the construction, it is intrinsically unavoidable. The same question has been faced for threefolds (see [**Ko**]); roughly summarizing, one realizes that also in dimension 3, orientable manifolds admitting a regular rational model are very special. On the other hand, we have the following interesting fact (see [**Ko2**]).

For every α*, for every* $M \in \alpha$*, there are* nonsingular *rational models, provided that one deals with a category of "abstract" algebraic-like varieties (also called* Moishezon varieties*) which are only locally but not globally isomorphic to ordinary algebraic sets in some* $\mathbb{R}^n$*.*

In this larger setting also the singular blowing down $\mathfrak{p}$ as in Theorem 19.85 can be realized as the inverse of an algebraic blow-up along a nonsingular centre.

CHAPTER 20

On 4-manifolds

In this chapter, we will apply several results established so far to compact 4-manifolds. Similarly to the attitude of Chapter 19 concerning the geometrization of 3-manifolds, we will develop a few classical differential topological themes; in no way (except for a final informative and discursive section) we will touch the study of 4-manifolds through *gauge theory* that has dominated in the last decades. For a more up to date treatment of 4-manifolds theory, one can refer, for example, to [**Sc**]. In particular, we will determine Ω_4, we present some instances of "classification of simply connected 4-manifolds up to stabilization" and Rohlin's theorem about the signature mod (16) of 4-manifold intersection forms. The intersection form will be the principal player.

We will deal with *oriented* 4-manifolds. M will denote a compact, connected, oriented, boundaryless smooth 4-manifold. By using the notations and the results of Sections 11.4, 13.4 and 13.5, we have that the *intersection form*

$$\sqcup : \mathcal{H}^2(M;\mathbb{Z}) \times \mathcal{H}^2(M;\mathbb{Z}) \to \mathbb{Z} \ ;$$

equivalently,

$$\bullet : \mathcal{H}_2(M;\mathbb{Z}) \times \mathcal{H}_2(M;\mathbb{Z}) \to \mathbb{Z}$$

is symmetric and induces a $\mathbb{Z}$-linear isomorphism

$$\hat{\phi} : \mathcal{H}^2(M;\mathbb{Z}) \to \mathrm{Hom}(\mathcal{H}_2(M;\mathbb{Z}),\mathbb{Z}) \ .$$

Then the free $\mathbb{Z}$-module $\mathcal{H}^2(M;\mathbb{Z}) = \mathcal{H}_2(M;\mathbb{Z})$ is of finite rank n, and the intersection form is *unimodular*: for any basis of $\mathcal{H}^2(M;\mathbb{Z})$, the representing matrix A belongs to $GL(n,\mathbb{Z})$ (i.e. $|\det A| = 1$). Every class $\alpha \in \mathcal{H}^2(M;\mathbb{Z})$ can be represented by an oriented 2-dimensional proper submanifold F; $\alpha = [F] = 0$ if and only if F is the boundary of an embedded Seifert hypersurface. The isometry class of the intersection form is an invariant up to orientation-preserving diffeomorphism. We are in a situation formally similar to the case of compact boundaryless surfaces endowed with their intersection forms. In the case of surfaces, we have seen in Chapter 15 that this intersection form contains all relevant information; moreover, there is perfect parallelism between the abstract algebraic theory of symmetric $\mathbb{Z}/2\mathbb{Z}$-bilinear forms and the 2-dimensional differential/topological realization. We would try to pursue this analogy as far a possible, obtaining, in fact, only very partial results.

20.1. Symmetric unimodular $\mathbb{Z}$-bilinear forms

In analogy to Section 15.1, we face here the question of the classification of finite rank, symmetric, unimodular $\mathbb{Z}$-bilinear forms up to isometry. It turns out that this abstract classification is complete only for the class of *indefinite forms*, while the *definite* case is a wide, largely unknown territory. This is an important difference with respect to the $\mathbb{Z}/2\mathbb{Z}$-case. For more information and detailed proofs, we refer the reader to [**MH**].

We consider free $\mathbb{Z}$-modules V of finite rank, endowed with a symmetric unimodular $\mathbb{Z}$-bilinear form ρ. This means that the $\mathbb{Z}$-linear map

$$V \to \mathrm{Hom}(V,\mathbb{Z}),\ v \to f_v,\ f_v : V \to \mathbb{Z},\ f_v(w) = \rho(v,w)$$

is an isomorphism. Equivalently, the symmetric matrix A representing ρ with respect to any basis of V belongs to $GL(n,\mathbb{Z})$, $n = \mathrm{rank}V$; that is, $|\det A| = 1$. Isometry is defined in the usual way. Sometimes, we will abuse the notation to confuse a form with its isometry class. Given (V,ρ) and (V',ρ'), we can define the *orthogonal direct sum*

$$(V,\rho) \perp (V',\rho') \ ;$$

that is, the symmetric unimodular form $\rho \perp \rho'$ on $V \oplus V'$ that restricts to ρ (resp. ρ') on V (V') and such that V and V' are orthogonal to each other.

20.1.1. Some invariants. We point out some isometry invariants besides the rank.

(Signature) By extension of the coefficients $\mathbb{Z} \subset \mathbb{R}$, V becomes a lattice in a $\mathbb{R}$-vector space $V_\mathbb{R}$ so that $\dim V_\mathbb{R} = \mathrm{rank}V = n$, and ρ extends to a $\mathbb{R}$-bilinear non-degenerate form $\rho_\mathbb{R}$. We know by Sylvester's theorem that a complete isometry invariant of $\rho_\mathbb{R}$ is given by the pair of *positivity* and *negativity indices* $(i_+(\rho_\mathbb{R}), i_-(\rho_\mathbb{R}))$, where $i_\pm(\rho_\mathbb{R})$ is the maximum of dimensions of $\mathbb{R}$-linear subspaces of $V_\mathbb{R}$ such that the restriction of $\rho_\mathbb{R}$ to them is either positive or negative definite. This pair of indices is an isometry invariant for the $\mathbb{Z}$-bilinear form ρ. We set

$$\sigma(\rho) = i_+(\rho_\mathbb{R}) - i_-(\rho_\mathbb{R}) \ ,$$

which is called the *signature* of ρ (some authors call it the *index* of ρ). As $i_+ + i_- = n$, then $\sigma \equiv n \bmod (2)$ and

$$(i_+, i_-) = (\frac{n+\sigma}{2}, \frac{n-\sigma}{2}) \ .$$

The signature is *additive with respect to orthogonal direct sum*:

$$\sigma(\rho \perp \rho') = \sigma(\rho) + \sigma(\rho') \ .$$

We can distribute the unimodular $\mathbb{Z}$-forms in the following classes, which are invariant up to isometry.

(Definite/indefinite) The form (V, ρ) is *definite*, either *positive* or *negative*, if either for every $v \in V$, $v \neq 0$, $\rho(v, v) > 0$ or $\rho(v, v) < 0$. Otherwise, ρ is *indefinite*.

(Parity) The form (V, ρ) is *even* if for every $v \in V$, $\rho(v, v) \in 2\mathbb{Z}$ is even. If ρ is not even, then it is said to be *odd*. (V, ρ) is even if and only if there is a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V such that for every $j = 1, \dots, n$, $\rho(v_j, v_j) \in 2\mathbb{Z}$; in such a case, this happens for every basis of V.

We have the combination sub-classes "definite/indefinite and even", "definite/indefinite and odd"; the study up to isometry can be made sub-class by sub-class.

20.1.2. Some basic forms. We denote by $\mathbf{U}_+$, $\mathbf{U}_-$ the unique (up to isometry) rank-1 forms. They are both definite (of opposite sign) and odd,

$$\sigma(\mathbf{U}_\pm) = \pm 1 \ .$$

We denote by $\mathbf{H}$ the (isometry class of the) form defined on $\mathbb{Z}^2$ by

$$(x, y) \to x^t H y \ ,$$

where

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

The form $\mathbf{H}$ is indefinite and even; $\sigma(\mathbf{H}) = 0$.

Let us denote by $\mathbf{E}_8$ the (isometry class of the) form defined on $\mathbb{Z}^8$ by

$$(x, y) \to x^t E y \ ,$$

where $E = (e_{i,j})$ is the symmetric matrix 8×8 such that:

- For every i, $e_{i,i} = 2$;
- For $i = 1, \dots, 6$, $e_{i,i+1} = 1$;
- $e_{5,8} = 1$;
- $e_{i,j} = 0$ otherwise.

One verifies by direct computation that $\mathbf{E}_8$ is unimodular, even and positive definite; hence $\sigma(\mathbf{E}_8) = 8$. The form $-\mathbf{E}_8$ (that is the isometry class of $(\mathbb{Z}^8, -E)$) is even, negative definite with $\sigma(-\mathbf{E}_8) = -8$. Being even, $\pm\mathbf{E}_8$ is *not diagonalizable*; that is, it is not isometric to $8\mathbf{U}_\pm$.

20.1.3. Full classification up to rank 4. We have

PROPOSITION 20.1. *Isometry classes of symmetric unimodular $\mathbb{Z}$-bilinear forms of rank n up to 4 are either diagonalizable (i.e. they admit an orthonormal basis) or even with zero signature. The normal representatives are respectively:*

(1) **(Diagonalizable)** *The normal representative is*

$$|\sigma|\mathbf{U}_\epsilon \perp \frac{n-|\sigma|}{2}(\mathbf{U}_+ \perp \mathbf{U}_-)\ ,$$

where ϵ is the sign of the signature σ.

(2) **(Even)** *The normal representatives are either $\mathbf{H}$ or $2\mathbf{H}$.*

The key geometric fact to get this result is that for every (V,ρ) such that $\mathrm{rank}(V) \leq 4$, there is $v \neq 0$ in V such that $|\rho(v,v)| < 2$; this is an application of a theorem of Minkowski on the volume of lattice in Euclidean spaces.

20.1.4. Classification of indefinite forms. This is summarized in the following theorem.

THEOREM 20.2. *(1) The triple*

$$(\text{rank},\ \text{signature},\ \text{parity})$$

is a complete invariant for the indefinite forms considered up to isometry.

(2) For every indefinite isometry class, we have the following distinguished representative, depending on the parity:

(Indefinite and odd normal representatives) *For every rank n and signature σ, this is*

$$|\sigma|\mathbf{U}_\epsilon \perp \frac{n-|\sigma|}{2}(\mathbf{U}_+ \perp \mathbf{U}_-)$$

where ϵ is the sign of σ. Hence indefinite odd forms are diagonalizable; that is, they admit orthonormal basis.

(Indefinite and even normal representatives) *For every rank n and signature σ, $\sigma \equiv 0$ mod (8), $n-|\sigma|$ is even and non-zero and the normal representative is*

$$\frac{\sigma}{8}\mathbf{E}_8 \perp \frac{n-|\sigma|}{2}\mathbf{H}$$

where we mean $a\mathbf{E}_8 = -a(-\mathbf{E}_8)$ if $a < 0$.

The key fact for the indefinite classification is the number-theoretic *Meyer theorem* which states that for every indefinite (V,ρ), there is $v \neq 0$ in V such that $\rho(v,v) = 0$. If $n \leq 4$ this follows from the above full classification. If $n \geq 5$, via the extension of coefficients $\mathbb{Z} \subset \mathbb{Q}$, one is reduced to prove that, alike for $\mathbb{R}$-spaces, a scalar product on a $\mathbb{Q}$-vector space of dimension $n \geq 5$ is definite if and only if for every non-zero vector v, $\rho(v,v) \neq 0$. Note that the last statement fails for $n = 4$. The proof is based on the *Hasse-Minkowski Theorem.* Then the indefinite odd case follows by a rather easy inductive argument. An important relation to achieve the odd case is

$$\mathbf{H} \perp \mathbf{U}_\pm = \mathbf{U}_\mp \perp 2\mathbf{U}_\pm\ .$$

The classification in the indefinite and even case is more delicate, employs the already achieved odd classification, and involves in the very statement certain congruence mod (8). We limit to clarify this last point.

20.1.5. Characteristic elements and congruences mod (8)**.** Let (V,ρ) be as above. An element $u \in V$ is by definition *characteristic* if for every $v \in V$, $\rho(v,v) \equiv \rho(u,v)$ mod(2). We have the following so-called *van der Blij* lemma.

LEMMA 20.3. *(1) Every (V,ρ) admits characteristic elements.*
(2) For every characteristic element u, $\sigma \equiv \rho(u,u)$ mod(8).
(3) If ρ is even, then $\sigma \equiv 0$ mod(8).

Proof : For (1), fix a basis of V so that $V \sim \mathbb{Z}^n$ and let the $n \times n$ symmetric matrix A represent the form ρ. By reducing mod (2), we have the $\mathbb{Z}/2\mathbb{Z}$-linear function $(\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$, $y \to y^tAy$. As $\det A = 1$ mod(2), there is a unique representing vector $\bar{u} \in (\mathbb{Z}/2\mathbb{Z})^n$ such that for every y, $y^tAy = \bar{u}^tAy$. Every $u \in \mathbb{Z}^n$ whose reduction mod (2) is equal to $\bar{u}$ is a characteristic element of ρ.

As for (2), if u and u' are characteristic elements, so that $u' = u + 2x$ for some $x \in V$, then $\rho(u',u') = \rho(u,u) + 4(\rho(u,x) + \rho(x,x)) \equiv \rho(u,u)$ mod(8). So $\rho(u,u)$ is invariant mod (8). It is additive with respect to the orthogonal direct sum and it holds ± 1 on $\mathbf{U}_\pm$. Then item (2) holds for indefinite and odd forms thanks to the classification in this case. Furthermore, $\rho \perp \mathbf{U}_+ \perp \mathbf{U}_-$ has the same signature of ρ and is indefinite and odd; so (2) holds in general.

Item (3) is an immediate corollary of (2).

■

20.1.6. Indefinite stabilizations. Given any form ρ, there are simple ways to transform it into an indefinite one. The first is called *elementary odd stabilizations*:

$$\rho \to \rho \perp \mathbf{U}_\epsilon$$

for a suitable $\epsilon = \pm$; the resulting form is indefinite and odd. The signature changes by $\sigma \to \sigma \pm 1$. The form $\rho \perp (\mathbf{U}_+ \perp \mathbf{U}_-)$ is always indefinite odd and the signature does not change.

The *elementary even stabilization* is

$$\rho \to \rho \perp \mathbf{H} \ ;$$

the resulting form is indefinite and is even if and only if ρ is even. The signature does not change.

Then the classification of indefinite odd forms induces a classification of *all* forms up to such odd stabilizations. Similarly, the classification of indefinite even forms induces a classification of all *even* forms up to even stabilization. In particular, we have the following.

For every pair of forms ρ and ρ', there are $m_1, m_2, m_1', m_2', m \in \mathbb{N}$ such that

$$\rho \perp m_1\mathbf{U}_+ \perp m_2\mathbf{U}_- = \rho' \perp m_1'\mathbf{U}_+ \perp m_2'\mathbf{U}_- = m(\mathbf{U}_+ + \mathbf{U}_-) \ .$$

20.1.7. Neutral forms and the Witt group. Similarly to Section 15.4.1, denote by $I(\mathbb{Z})$ the set of isometry classes of unimodular symmetric $\mathbb{Z}$-bilinear forms defined on free $\mathbb{Z}$-modules of arbitrary finite rank. The operation $\perp$ makes it a semigroup. A class $S \in I(\mathbb{Z})$ is said to be *neutral* if rank $S = 2m$ is even and there is a submodule $Z \subset S$, rank $Z = m$ such that $Z = Z^\perp$. The following lemma is an immediate consequence of Theorem 20.2.

LEMMA 20.4. *An indefinite odd class is neutral if and only if it is of the form $m(\mathbf{U}_+ \perp \mathbf{U}_-)$ for some $m \geq 1$. An indefinite even class is neutral if and only if it is of the form $m\mathbf{H}$ for some $m \geq 1$.*

Put on $I(\mathbb{Z})$ the equivalence relation $X \sim X'$ if and only if there are neutral spaces S, S' such that

$$X \perp S = X' \perp S' \ .$$

Denote by $W(\mathbb{Z})$ the quotient set. The operation descends to $W(\mathbb{Z})$ and makes it an Abelian group called the *Witt group* of the ring $\mathbb{Z}$. All this can be restricted to the set $I_0(\mathbb{Z})$ of even classes and gives rise to the restricted Witt group $W_0(\mathbb{Z})$. The following proposition is again an easy consequence of Theorem 20.2.

PROPOSITION 20.5. *Both following maps are well defined group isomorphisms:*

$$\sigma : W(\mathbb{Z}) \to (\mathbb{Z}, +), \ \frac{\sigma}{8} : W_0(\mathbb{Z}) \to (\mathbb{Z}, +) \ .$$

Moreover, $W(\mathbb{Z})$ is generated by $\mathbf{U}_+$, while $W_0(\mathbb{Z})$ is generated by $\mathbf{E}_8$.

20.2. Some 4-manifold counterparts

In analogy with the surface case, one would like to determine 4-manifold counterparts of the above abstract theory, at least for indefinite forms where the arithmetic classification is complete. In particular one would wonder that every indefinite normal representative is realized as the intersection form $\bullet_M$ of some 4-dimensional smooth manifold M as above. Unfortunately, this is too optimistic.

Notation: We set $\sigma_{\bullet_M} = \sigma(M)$.

First, we establish a topological counterpart of the operation $\perp$.

LEMMA 20.6. *Let $(M_1, \bullet_{M_1})$ and $(M_2, \bullet_{M_2})$ be 4-manifolds equipped with the respective intersection forms and set $M = M_1 \# M_2$. Then, up to isometry,*

$$\bullet_M = \bullet_{M_1} \perp \bullet_{M_2} \ .$$

Proof : This lemma is analogous to Lemma 15.7 in the case of surfaces.. Let $\alpha = [F] \in \mathcal{H}_2(M;\mathbb{Z})$, where F is a proper oriented surface embedded in M. Up to isotopy we can assume that $F \pitchfork S$, where S is a smooth 3-sphere in M which realizes the connected sum splitting of M. The intersection $L = F \cap S$ is a link in $S \sim S^3$. Then M is obtained by gluing $M'_j = M_j \setminus \mathrm{Int}(D^4)$, $j = 1, 2$, along the two boundary components of a tubular neighbourhood $N(S) \sim S^3 \times [-1, 1]$ of S in M. The intersection $F_j = F \cap \hat{M}'_j$ is a proper submanifold of M'_j with boundary L; F_j can be capped by means of a Seifert surface of L in S^3. So we get boundaryless surfaces $\hat{F}_j$ in M_j which, up to isotopy, can be embedded in M'_j. Hence, via the isomorphism induced by the inclusions and a slight abuse of notation, we have $[F] = [\hat{F}_1] + [\hat{F}_2]$. Doing in a similar way for another class $\alpha' = [F']$, we get $\alpha \bullet \alpha' = [\hat{F}_1] \bullet [\hat{F}'_1] + [\hat{F}_2] \bullet [\hat{F}'_2]$.

■

REMARK 20.7. We stress that we are **not** claiming that every direct sum decomposition of an intersection form $\bullet_M$ corresponds to a connected sum decomposition of the manifold M (see Example 20.11).

It is easy to realize $\mathbf{U}_\pm$ and $\mathbf{H}$. In fact, $\mathbf{U}_\pm$ is the intersection form of $\pm\mathbf{P}^2(\mathbb{C})$, where $\mathbf{P}^2(\mathbb{C})$ is endowed with the natural orientation as a complex manifold. $\mathcal{H}_2(\mathbf{P}^2(\mathbb{C});\mathbb{Z})$ is generated by $[\mathbf{P}^1(\mathbb{C})]$ that is represented by any complex line embedded in $\mathbf{P}^2(\mathbb{C})$. Hence *every indefinite and odd normal representative can be realized*.

Notation: To simplify the notation, set $\mathcal{P} = \mathbf{P}^2(\mathbb{C})$ and $\mathcal{Q} = -\mathbf{P}^2(\mathbb{C})$.

The form $\mathbf{H}$ is the intersection form of $S^2 \times S^2$, where S^2 has the usual orientation and we take the product orientation. The module $\mathcal{H}_2(S^2 \times S^2;\mathbb{Z})$ has as basis $[S^2 \times \{p\}]$ and $[\{p\} \times S^2]$, for any $p \in S^2$.

REMARK 20.8. Both $\mathbf{P}^2(\mathbb{C})$ and $S^2 \times S^2$ are simply connected. By the Van Kampen theorem, the connected sum of two simply connected manifolds is simply connected. So it makes sense (and we will do it at some point) to restrict the discussion to simply connected manifolds.

The basic neutral classes are $\mathbf{H}$ and $\mathbf{U}_+ \perp \mathbf{U}_-$.

PROPOSITION 20.9. *Up to isomorphism of fibre bundles, there are two distinct fibre bundles over S^2 with fibre S^2 and orientable total space; $S^2 \times S^2$ and $\mathcal{P}\#\mathcal{Q} := S^2 \tilde{\times} S^2$ are the respective total spaces.*

Proof : By at theorem of Smale ([**S1**]), and also recall Section 7.5.2), $\mathrm{Diff}^+(S^2)$ retracts by deformation to $SO(3) \sim \mathbf{P}^3(\mathbb{R})$. Then there are exactly two such fibre bundles because $\pi_1(SO(3)) \sim \mathbb{Z}/2\mathbb{Z}$ (recall Section 6.5). The manifold $\mathcal{P}\#\mathcal{Q}$ can be obtained by the complex blowing up of $\mathbf{P}^2(\mathbb{C})$ at a point. It follows from the proof of Proposition 7.33 that it is the total space of a fibre bundle, as in the statement of the proposition. More precisely, let $\mathcal{D}$ be the unitary disk in an affine chart of $\mathcal{P}$ at a point $x_0 \sim 0$.

Then $\mathbf{B}_{\mathbb{C}}(\mathcal{D}, 0)$ is the oriented total space of a fibre bundle over the Riemann sphere $S^2 \sim \mathbf{P}^1(\mathbb{C})$ with fibre D^2; the fibres are given by the strict transform of the intersection with $\mathcal{D}$ of the complex lines through 0. Set $\mathcal{P}_0 := \mathcal{P} \setminus \mathrm{Int}(\mathcal{D})$. This $\mathcal{P}_0$ is the total space of a fibre bundle of the same type. Considering $\mathbf{P}^1(\mathbb{C}) \subset \mathcal{P}_0$, the fibres are given by the intersection with $\mathcal{P}_0$ of the complex lines passing through 0 and $x \in \mathbf{P}^1(\mathbb{C})$. The restriction of these fibres to $\partial\mathcal{D}$ induce the Hopf fibration $\mathfrak{h} : S^3 \to S^2$. Then $\mathbf{B}_{\mathbb{C}}(\mathcal{P}, x_0)$ is diffeomorphic to the double $D(\mathcal{P}_0) = \mathcal{P}_0 \amalg -\mathcal{P}_0/\mathrm{id}_{S^3}$ and hence to $\mathcal{P}\#\mathcal{Q}$. The fibration of $\mathcal{P}\#\mathcal{Q}$ with fibre S^2 is obtained by gluing "along the Hopf fibration" the two fibrations with fibre D^2 described so far. Finally, $S^2 \times S^2$ and $\mathcal{P}\#\mathcal{Q}$ are distinguished by the intersection forms.

■

Now we discuss a topological counterpart of the relation

$$\mathbf{H} \perp \mathbf{U}_{\pm} = \mathbf{U}_{\mp} \perp 2\mathbf{U}_{\pm} \ .$$

Proposition 20.10. *We have*

$$(S^2 \times S^2)\#\mathcal{Q} \sim \mathcal{P}\#2\mathcal{Q}, \ (S^2 \times S^2)\#\mathcal{P} \sim \mathcal{Q}\#2\mathcal{P} \ .$$

Proof : This proposition is analogous to Lemma 15.12 in the case of surfaces. As $S^2 \times S^2$ admits an orientation reversing diffeomorphism, the two relations are equivalent to each other. The second geometric proof of Lemma 15.12 applies *verbatim* to prove the first relation, provided that one replaces $\mathbb{R}$ with $\mathbb{C}$ everywhere.

■

A realization of indefinite even normal representatives, or of $\mathbf{E}_8$ itself, possibly by a simply connected smooth 4-manifold M, is a much more subtle and hard question. We will discuss later the following fundamental Rohlin's discovery (recall that algebra tells us that the signature of an even form is $\equiv 0 \mod (8)$); *if M is simply connected and its intersection form is even, then $\sigma(M) \equiv 0 \mod (16)$.*

Then $\mathbf{E}_8$ cannot be realized. If M is simply connected with indefinite and even intersection form, then this is necessarily isometric to a normal representative of the type

$$2a\mathbf{E}_8 \perp b\mathbf{H}$$

for some $a \in \mathbb{Z}$, $b \in \mathbb{N} \setminus \{0\}$. It is not evident (and ultimately false) that every such pair (a, b) can be realized. On the other hand, classical simply connected examples show the actual occurrence of $\mathbf{E}_8$.

Example 20.11. If we relax the requirement of dealing with normal representatives, it is not hard to make $\mathbf{E}_8$ visible. For example, by the indefinite and odd classification, the form of $M = 10\mathcal{P}\#\mathcal{Q}$ is isometric to $\mathbf{E}_8 \perp \mathbf{U}_+ \perp \mathbf{H}$. Nevertheless, this algebraic decomposition does not correspond to any connected sum decomposition of M.

A more substantial example, realizing a normal representative, is the so-called *Kummer variety*. Let the 4-torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ be realized as the product of two copies of $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ so that T^4 has a complex 2-manifold structure with "uniformizing" complex coordinates (w_1, w_2). The involution $\tau(w_1, w_2) = (-w_1, -w_2)$ descends to T^4 and has 16 fixed points. Let us perform the complex blow-up at such fixed points. We get a complex surface $\tilde{K}$, smoothly diffeomorphic to $T^4 \# 16\mathcal{Q}$. The exceptional complex surface over each fixed point is a Riemann sphere S with self-intersection number in $\tilde{K}$ equal to -1. The involution τ lifts to an involution $\tilde{\tau}$ of $\tilde{K}$ which is the identity on each exceptional sphere. We consider the quotient

$$K := \tilde{K}/\tilde{\tau} \ .$$

One verifies that K is a smooth complex surface. Using the natural projection, every exceptional sphere S maps to a 2-sphere S' embedded in K; the restriction of the projection on a suitable neighbourhood of each S in $\tilde{K}$ is a double covering of a neighbourhood in K of the corresponding sphere S'. Then the self-intersection number of every S' in K is equal to -2. One can verify that $\mathcal{H}_2(T^4; \mathbb{Z}) \sim \mathbb{Z}^6$ and is generated by six embedded 2-tori, while $\mathcal{H}_2(K; \mathbb{Z}) \sim \mathbb{Z}^{22}$ generated by the image of these tori together with the 16 spheres S'. Eventually, the intersection form of K is indefinite and even with normal representative $-2\mathbf{E}_8 \perp 3\mathbf{H}$.

20.2.1. On the intersection form of 4-manifolds with boundary. If $\partial M \neq \emptyset$, the intersection form $\sqcup : \mathcal{H}^2(M; \mathbb{Z}) \times \mathcal{H}^2(M; \mathbb{Z}) \to \mathbb{Z}$ and the $\mathbb{Z}$-linear map

$$\hat{\phi}^2 : \mathcal{H}^2(M; \mathbb{Z}) \to \mathrm{Hom}(\mathcal{H}_2(M; \mathbb{Z}), \mathbb{Z})$$

is defined as well. In general, the form is not unimodular. If $\beta := i_*(\alpha) \neq 0$ in $\mathcal{H}_2(M; \mathbb{Z})$ for some $\alpha \in \mathcal{H}_2(\partial M; \mathbb{Z})$, then $\beta \sqcup \gamma = 0$ for every γ. On the other hand, it follows from the results of Chapter 13 that

$$\hat{\phi}^2 : \mathcal{H}^2(M, \partial M; \mathbb{Z}) \to \mathrm{Hom}(\mathcal{H}_2(M; \mathbb{Z}), \mathbb{Z})$$

is an isomorphism. Hence the intersection form of M is unimodular if and only if $j_* : \mathcal{H}_2(M; \mathbb{Z}) \to \mathcal{H}_2(M, \partial M; \mathbb{Z})$ is an isomorphism. For simplicity, assume that M is part of a triad of the form $(M, \emptyset, V = \partial M)$ admitting an ordered handle decomposition with one 0-handle, some 2-handles, say k, and no 3- or 4-handles. In other words, by removing the 0-handle, we realize a surgery equivalence $S^3 \sim_\sigma V$. Hence V is connected and M is simply connected. We claim that every symmetric $\mathbb{Z}$-bilinear form (not necessarily unimodular) can be realized by such a 4-manifold. Let us sketch the argument. By using Section 9.3.1 we see that M retracts to a wedge of k 2-spheres. By using the bordism homotopy invariance and what we know about the bordism of S^2, we see that $\mathcal{H}_2(M; \mathbb{Z})$ has rank k; a geometric basis $\alpha_1, \dots, \alpha_k$ can be obtained by completing the core of every 2-handle with a Seifert surface of the corresponding attaching knot in S^3 (provided the handles have been ordered). The k-components framed link in S^3 which encodes the attaching of 2-handles carries a symmetric *linking matrix* made by the linking numbers of pairs of constituent knots and, along the diagonal, by the integers encoding the framing of every such a knot. With a bit of work one eventually realizes that this matrix equals the matrix of the intersection form of M for the above geometric basis. In Figure 1, we show a framed link in S^3 which realizes $\mathbf{E}_8$; ∂M is the *Poincaré sphere.*

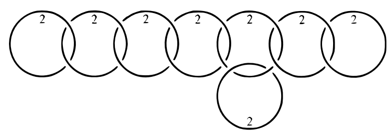

Figure 1. An $\mathbf{E}_8$-link.

Figure 1 is used with the permission of A. Scorpan ([**Sc**]).

PROPOSITION 20.12. *The intersection form of M is unimodular if and only if $\mathcal{H}_1(V;\mathbb{Z}) = \mathcal{H}_2(V;\mathbb{Z}) = 0$.*

Proof : As M is simply connected, $\mathcal{H}_3(M, \partial M;\mathbb{Z}) \sim \mathrm{Hom}(\mathcal{H}_1(M;\mathbb{Z}),\mathbb{Z}) = 0$. Hence by using the bordism long exact sequence of $(M, \partial M)$, we see that $i_* : \mathcal{H}_2(V;\mathbb{Z}) \to \mathcal{H}_2(M;\mathbb{Z})$ is injective; hence if the intersection form of M is unimodular, then $\mathcal{H}_2(V;\mathbb{Z}) = 0$. On the other hand, if $\mathcal{H}_1(V;\mathbb{Z}) = 0$, consider the dual handle decomposition; the cores of the 2-handles provide a basis of $\mathcal{H}_2(M, \partial M;\mathbb{Z})$. By capping each of them with a Seifert surface in V of the corresponding attaching knot, we get a further geometric basis of $\mathcal{H}_2(M;\mathbb{Z})$ dual to the previous one.

■

If the intersection form of M is unimodular, possibly by performing an elementary blow-up move (which replaces M with $M \# \pm \mathbf{P}^2(\mathbb{C})$, without modifying the boundary V), then we can assume that the unimodular intersection form of M is diagonalizable. If a 2-handle (corresponding to a constituent knot K_i) is slid over another, say corresponding to K_j, then the geometric basis, as above, changes by sending α_i to $\alpha_i \pm \alpha_j$, and the linking matrix changes by adding the j^{th} row to the i^{th} row, and the j^{th} column to the i^{th} column. The same discussion can be repeated (with some simplification) by replacing everywhere $\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$. It follows that if the form is non-degenerate and not totally isotropic, we can realize an orthogonal basis through handle sliding.

20.3. Ω_4

We already know that Ω_4 is nontrivial because $\chi_{(2)}(\mathbf{P}^2(\mathbb{C})) = 1$. More precisely, we have a surjective homomorphism defined by

$$\chi_{(2)} : \Omega_4 \to \mathbb{Z}/2\mathbb{Z},\ \chi_{(2)}([M]) := \chi_{(2)}(M)\ .$$

Pontryagin remarked that there is a subtler homomorphism induced by the signature. As usual

$$[M \# M'] = [M \amalg M'] = [M] + [M'] \in \Omega_4\ ,$$

so that every $\alpha \in \Omega_4$ can be represented by connected 4-manifolds and we can replace $\amalg$ with # to define the $\mathbb{Z}$-module operation on Ω_4.

PROPOSITION 20.13. *The map*

$$\sigma : \Omega_4 \to \mathbb{Z},\ \sigma(\alpha) := \sigma(M)\ ,$$

where M is any connected representative of the class α, is a well defined and surjective homomorphism.

Proof : As the signature is additive with respect to the connected sum, $\sigma(M) = -\sigma(-M)$ and $\sigma(\mathbf{P}^2(\mathbb{C})) = 1$, it is enough to show that if $[M] = 0 \in \Omega_4$, then $\sigma(M) = 0$. To compute the signature (that is the indices i_+, i_-), it is enough to extend the coefficients $\mathbb{Z} \subset \mathbb{Q}$. For every $\alpha \in \mathcal{H}_2(M;\mathbb{Q})$ there exists $m \in \mathbb{Z}$ such that $m\alpha = \alpha' \in \mathcal{H}_2(M;\mathbb{Z})$, and $\alpha \bullet \alpha = \alpha' \bullet \alpha'/(m^2)$. If for every $\alpha \in \mathcal{H}_2(M;\mathbb{Q})$, $\alpha \bullet \alpha = 0$, then $\sigma = 0$. Let $M = \partial W$, $i : M \to W$ be the inclusion. If $i_*(\alpha) = 0$, then $\alpha' \bullet \alpha' = 0$, hence $\alpha \bullet \alpha = 0$. So if for every α, $i_*(\alpha) = 0$, then $\sigma = 0$. Assume that $i_*(\alpha) \neq 0$. Then there is $b \in \mathcal{H}_3(W, M;\mathbb{Q})$ such that $\beta := \partial b \in \mathcal{H}_2(M;\mathbb{Q})$ and $\alpha \bullet \beta = 1$, $i_*(\beta) = 0$. Let V be the subspace of $\mathcal{H}_2(M;\mathbb{Q})$ generated by α and β. The matrix of the restriction of the intersection form on V has det $= -1$, hence its signature is equal to zero. As the restriction of the form to V is non-degenerate, also its restriction on the orthogonal space $V^\perp$ is non-degenerate. The we can iterate the construction until we no longer find classes such that $i_*(\alpha) \neq 0$. By the additivity of the signature with respect to the orthogonal direct sum, we conclude that $\sigma = 0$.

■

We are ready to state and prove the following theorem due to Rohlin. We will propose his original argument, as it is developed in the commentary to the four Rohlin papers in [**GM**]. See also [**Kirby**] for a somewhat different conclusion of the proof also based on Step 1 below. The statement is formally analogous to surface Theorems 15.14 and 15.15.

THEOREM 20.14. *The homomorphism induced by the signature $\sigma : \Omega_4 \to \mathbb{Z}$ is an isomorphism. Hence Ω_4 is generated by $[\mathbf{P}^2(\mathbb{C})]$ and is naturally isomorphic to the Witt group $W(\mathbb{Z})$.*

Proof : The restriction of σ to the submodule of Ω_4 generated by $[\mathbf{P}^2(\mathbb{C})]$ is an isomorphism to $\mathbb{Z}$. Hence it is enough to show that Ω_4 is generated by $[\mathbf{P}^2(\mathbb{C})]$. We will achieve this fact in several steps. Let M be as usual a compact, oriented, connected and boundaryless 4-manifold.

Step 1. This is similar to the first step in Rohlin's proof that $\Omega_3 = 0$. By Theorem 7.27, up to bordism, it is not restrictive to assume that $M \subset \mathbb{R}^7 \subset S^7$.

Step 2. We would like to construct along M a field v of unitary vectors tangent to S^7 and normal to M. This is not possible in general, however we are going to see that there is $\tilde{M} := M\#a\mathcal{P}\#b\mathcal{Q} \subset S^7$ for some $a, b \in \mathbb{N}$, which carries such a nowhere vanishing transverse field. A first obstruction is given by the Euler class $e \in \mathcal{H}^3(M;\mathbb{Z})$ of a bundle normal to M in S^7. On the other hand, $[M] = 0 \in \mathcal{H}^3(S^7;\mathbb{Z})$ and $e = i^*([M]) = 0$. This implies that such a field v can be defined on $M_0 = M \setminus \mathrm{Int}(B^4)$, where B^4 is a smooth 4-disk in M; in fact, M_0 has a 3-dimensional spine, v can be always constructed up to the 2-skeleton and the obstruction to extend it to the third skeleton belongs to $\pi_2(S^2)$ and vanishes because $e = 0$. The restriction of v to ∂M_0 defines an element of $\pi_3(S^2)$ which is in general nontrivial. This is the final effective obstruction to extend v on the whole of M. We know that $\pi_3(S^2) = \mathbb{Z}$ is generated by the Hopf map $\mathfrak{h} : S^3 \to S^2$. By transversality, we can perturb the field v and assume that it is defined on M' obtained by removing from M the interior of a finite number of disjoint 4-disks B_j embedded in $\mathrm{Int}(B)$ such that the restriction of v to every boundary ∂B_j is equal to $\pm\mathfrak{h}$. By using the field v we get an embedding of M' in the boundary $\partial N(M)$ of a tubular neighbourhood of M in S^7. By abstractly gluing to every boundary component of M' the mapping cylinder of the corresponding map $\pm\mathfrak{h}$, we get the 4-manifold $\tilde{M} := M\#a\mathcal{P}\#b\mathcal{Q}$ for some $a, b \in \mathbb{N}$. We claim that we can assume that $\tilde{M} \subset \partial N(M)$ by extending the given embedding of M'. For if $B_j \times D^3$ is a trivialized chart of $N(M)$ over the 4-ball B_j, the embedding of ∂B_j is, for instance, of the form $x \to (x, \mathfrak{h}(x))$ and $\mathcal{P}_0$ is the copy of $\mathcal{P} \setminus \mathrm{Int}(D^4)$ in $\tilde{M}$ corresponding to B_j, then an embedding of $\mathcal{P}_0$ is given (by using suitable homogeneous coordinates (x_0, x_1, x_2)) by:

$$(x_0, x_1, x_2) \to ((\frac{2x_0x_1}{\sum_{i=0}^2 |x_i|^2}, \frac{2x_0, x_2}{\sum_{i=0}^2 |x_i|^2}), \frac{x_1}{x_2}) \in B_j \times \mathbf{P}^1(\mathbb{C}) \ .$$

Clearly, the restriction $\tilde{v}$ to $\tilde{M}$ of a unitary normal field to the hypersurface $\partial N(M)$ in S^7 is nowhere vanishing along $\tilde{M}$.

Step 3. The field $\tilde{v}$ determines an embedding of a copy $\hat{M}$ of $\tilde{M}$ in $\partial N(\tilde{M})$ the boundary of a tubular neighbourhood $\pi : N(\tilde{M}) \to \tilde{M}$ of $\tilde{M}$ in S^7. Set $X := S^7 \setminus \mathrm{Int}(N(\tilde{M}))$. If $[\hat{M}] = 0$ in $\mathcal{H}_4(X;\mathbb{Z})$, then it is a boundary thanks to Proposition 13.9, and finally M would be bordant with $k\mathbf{P}^2(\mathbb{C})$ for some $k \in \mathbb{Z}$. Unfortunately, we cannot assume that $[\hat{M}] = 0$. However, we have

Claim. *There is an oriented surface F in $\tilde{M}$ such that the disjoint union of inclusions $j : \hat{M} \amalg \partial\pi^{-1}(F) \to \partial N(\tilde{M})$ represents zero in $\mathcal{H}_4(X;\mathbb{Z})$ (the 4-manifold $S := \partial\pi^{-1}(F)$ is oriented by the direct sum of the orientation of F and the orientation of the normal bundle of $\hat{M}$ in $\partial N(\tilde{M})$).*

Let us prove the claim. $\mathcal{H}_4(S^7;\mathbb{Z}) = 0$, more precisely $\Omega_4(S^7) \sim \Omega_4$. Hence there is an oriented triad $(W, \hat{M}, V)$ and a map $h : W \to S^7$ where the restriction to $\hat{M}$ is the inclusion and the restriction to V is a constant map. By transversality we can assume that the restriction of h to an open collar of $\hat{M}$ in W is an embedding in X transverse to $\partial N(\tilde{M})$, the image of V is in the interior of X, the restriction of h to the interior of W is transverse to $(N(\tilde{M})$, $\partial N(\tilde{N})$ and $\tilde{M}$. Then $F = h(\text{Int}(W)) \cap \tilde{M}$ is a surface in $\tilde{M}$ and $h(\text{Int}(W)) \cap \partial N(\tilde{M}) = \partial\pi^{-1}(F) := S$. Finally $(h^{-1}(X), h)$ realizes a bordism between $(\hat{M} \amalg \partial S, j)$ and $(V, h_{|V})$. The claim is proved.

With a slight abuse of notation, we write $[\hat{M} \amalg S]$ instead of $[\hat{M} \amalg S, j]$.

Step 4. *There is $\hat{M}' \sim \hat{M}\#p\mathcal{P}\#q\mathcal{Q}$, for some $p, q \in \mathbb{N}$, embedded in $\partial N(\tilde{M})$, such that $[\hat{M}'] = [\hat{M} \amalg S] \in \mathcal{H}_4(X; \mathbb{Z})$.*

This is enough to conclude. For by Step 3, $[\hat{M}'] = 0 \in \Omega_4$ and, by Step 2, $\hat{M}' \sim M\#k\mathcal{P}\#h\mathcal{Q}$, for some $k, h \in \mathbb{N}$. Let $\nu : E(\nu) \to \hat{M}$ be the rank-2 oriented normal bundle of $\hat{M}$ in $\partial N(\tilde{M})$. We can consider it as a complex line bundle. By pulling-back ν on $\tilde{M}$ via the field $\tilde{v}$, we see that $N(\tilde{M})$ can be identified with (a tubular neighbourhood of the zero section of) the bundle $\nu \oplus \epsilon$ over $\tilde{M}$, where ϵ is a trivial real line bundle. Using this fact, we can consider $\partial N(\tilde{M})$ as the total space of the projectivized bundle of the sum of complex line bundles over $\tilde{M}$, $\nu \oplus \epsilon_{\mathbb{C}}$, with $\epsilon_{\mathbb{C}}$ being the trivial complex line bundle. We can endow $\partial N(\tilde{M})$ with explicit homogeneous coordinates $[X_p, T_p]$ as follows: a point $P \in \partial N(\tilde{M})$ is represented by $[X_p, T_p]$, where $X_p \in E(\nu)$, $T_p \in E(\epsilon_C)$, $p = \pi(P) = \nu(X_p) = \epsilon_{\mathbb{C}}(T_p)$, $(X_p, T_p) \neq (0, 0)$, $[X_p, T_p] = [X'_p, T'_p]$ if and only if $(X'_p, X'_p) = \lambda(X_p, T_p)$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Let us denote by $\hat{F} \subset \hat{M}$ the copy of F in $\partial N(\tilde{M})$ determined by the above normal unitary field $\tilde{v}$ along $\tilde{M}$. Let μ be the normal bundle of $\hat{F}$ in $\hat{M}$, also considered as a complex line bundle. The total space $E(\mu)$ can be considered as being embedded in $\hat{M}$ as a tube around $\hat{F}$. On $E(\mu)$ we have the restriction of ν and $\mu^*(\mu) = \{(x, y) \in E(\mu) \times E(\mu);\ \mu(x) = \mu(y)\}$. The canonical section of $\mu^*(\mu)$ is $t(x) = (x, x)$. Let us keep the notation $\mu^*(\mu)$ for a bundle on $\hat{M}$ which extends $\mu^*(\mu)$, endowed with a section t which extends t and does not vanish on $\hat{M} \setminus \hat{F}$. Using the field $\tilde{v}$ as above, we can transport this configuration for $(\hat{M}, \hat{F})$ on $(\tilde{M}, F)$. Using the above homogeneous coordinates on $\partial N(\tilde{M})$, we have the following equation for the set $\hat{M} \cup \hat{F}$: $[X_p, T_p] \in \hat{M} \cup \hat{F}$ if and only if $X_p \otimes t(p) = 0 \in \nu \otimes \mu^*(\mu)$. We are going to define $\hat{M}'$ by a perturbed equation. Let σ be a section of $\nu \otimes \mu^*(\mu)$ over $\tilde{M}$, transverse to the zero section, such that its zero set is tranverse to $F \subset \tilde{M}$. Then the manifold $\hat{M}' := M_\sigma$ is defined such that $[X_p, T_p] \in M_\sigma$ if and only if $X_p \otimes t(p) = \sigma(p) \otimes T_p \in \nu \otimes \mu^*(\mu)$. This $\hat{M}'$ has the required properties. If Z is the finite set of zeros of σ on F, then the restriction $\pi : M_\sigma \to \tilde{M}$ realizes a system of (possibly anti) complex blow-ups of the points of Z. Hence, $M_\sigma \sim \hat{M}\#p\mathcal{P}\#q\mathcal{Q}$. This can be checked as follows. If $\pi(P) = p$ does not belong to F, then $t(p) \neq 0$ and the equation defines a unique $[X_p, T_p] \in \pi^{-1}(p)$. If $\pi(P) = p_0 \in F$, in suitable local complex coordinates u on F, and v on the fibre of μ, the equation has the form $Xv = \sigma(u, v)T$, where σ is a function such that if $\sigma(u, 0) = 0$ then $\frac{\partial}{\partial u}\sigma(u, 0)$ is invertible by transversality, hence M_σ is smooth. Moreover, we can take the charts in such a way that if $v = 0$ and $\sigma(u, 0) = 0$, then $\sigma(u, 0) = u$;

that is, we recover the formula of the complex blowing-up at a point. The sign depends on the orientation of the chart. Finally, using the 1-parameter family $M_{\tau\sigma}$, $\tau \in [0,1]$, we realize that $[\hat{M} \amalg S] = [M_\sigma] \in \mathcal{H}_4(\partial N(\tilde{M}); \mathbb{Z})$.

Theorem 20.14 is eventually achieved.

■

20.4. A classification up to odd stabilization

In this section we restrict to *simply connected* 4-manifolds.

THEOREM 20.15. *For every compact oriented simply connected boundaryless* 4*-manifold* M*, there exist* $(k,h),(m,n) \in \mathbb{N}\times\mathbb{N}$ *such that*

$$M\#k\mathcal{P}\#h\mathcal{Q} = m\mathcal{P}\#n\mathcal{Q}\ .$$

By using Proposition 20.10 one can slightly refine the statement in the following form:

$\cdots$ *there exists* $(k,m) \in \mathbb{N}\times\mathbb{N}$ *such that*

$$M\#(k+1)\mathcal{P}\#k\mathcal{Q} = (m+1)\mathcal{P}\#m\mathcal{Q}\ .$$

Theorem 20.15 is analogous to the results of Section 15.5 in the case of surfaces; however, we do not have here any *a priori* information about the integers k,h,m,n. By Theorem 20.14, for every M as above there is $l \in \mathbb{Z}$ such that $M\#l\mathbf{P}^2(\mathbb{C})$ is an oriented boundary and it is still simply connected; then Theorem 20.15 will readily follow by combining the next proposition with Proposition 20.10.

PROPOSITION 20.16. *Let* M *be simply connected and a boundary. Then there are* $(k_0,k_1),(h_0,h_1) \in \mathbb{N}\times\mathbb{N}$ *such that*

$$M\#k_0(S^2\times S^2)\#k_1(S^2\tilde{\times}S^2) \sim h_0(S^2\times S^2)\#h_1(S^2\tilde{\times}S^2)\ .$$

Proof : As M is a boundary, there is an oriented triad (W,M,S^4). Let us take an ordered handle decomposition of (W,M,S^4) without 0- and 5-handles. Hence it is of the form

$$(M\times[0,1])\cup\{\mathcal{H}^1\}\cup\{\mathcal{H}^2\}\cup\cdots\cup\{\mathcal{H}_4\}\cup([-1,0]\times S^4)$$

where every $\mathcal{H}_j$, $j=1,\dots,4$, denotes a pattern of a_j j-handles attached simultaneously at disjoint attaching tubes. We claim that we can modify the 5-manifold W without changing the boundary $M\amalg S^4$, in such a way that it is not restrictive to assume that $a_1 = a_4 = 0$. To do it, we apply the "trading" argument already used in the proof of Proposition 19.7. We can assume that the attaching tube of every 1-handle is contained in a smooth 4-disk of M. Then the new boundary component obtained by modifying M can be realized as well by a 3-handle trivially attached to M; thus we can trade every 1-handle with a 3-handle. By using the dual handle decomposition, we can trade every 4-handle with a 2-handle; so, up to reordering, we can assume that the ordered handle decomposition of (W,M,S^4) contains only

2- and 3- handles. Hence W can be obtained by gluing $(M \times [0,1]) \cup \{\mathcal{H}_2\}$ and $\{\mathcal{H}^3\} \cup ([-1,0] \times S^4)$ along diffeomorphic boundary components. Note that in terms of the dual decomposition, also $\{\mathcal{H}^3\} \cup ([-1,0] \times S^4)$ is obtained by attaching 2-handles. Then the following lemma allows us to conclude.

LEMMA 20.17. *Consider the cylinder* $(M \times [0,1], M_0, M_1)$, $M_j = M \times \{j\}$. *Let* $(Y, M_0, \hat{M}_1)$ *be obtained by attaching a* 2*-handle to* $M \times [0,1]$ *at* M_1. *Assume that* M *is simply connected. Then either* $\hat{M}_1 \sim M\#(S^2 \times S^2)$ *or* $\hat{M}_1 \sim M\#(S^2 \tilde{\times} S^2)$.

Proof : As $\dim M = 4$ and M is simply connected, the attaching 1-sphere of the handle is isotopic to a standard S^1 in a chart of M. Then it is easy to check that $M_1 \sim M\#\mathcal{F}$, where $\mathcal{F}$ is the total space of an oriented fibre bundle over S^2 with fibre S^2. Then we apply Proposition 20.9. The lemma and Proposition 20.16 are proved.

■

20.5. On the classification up to even stabilization

As in the previous section, we deal with simply connected 4-manifolds. Being very sketchy, we are going to discuss the following deeper result [**Wall3**], [**Wall4**].

THEOREM 20.18. *Let* M_0 *and* M_1 *be compact oriented simply connected boundaryless* 4*-manifolds with isometric intersection forms. Then there is* $k \in \mathbb{N}$ *such that* $M_0\#k(S^2 \times S^2) \sim M_1\#k(S^2 \times S^2)$.

A few comments are in order:

• In a sense, this is the strongest 4-dimensional analogue of surface classification (in terms of the intersection form), which had been obtained by means of the classical differential/topological methods used till the late 1970's.

• Theorem 20.18 implies Theorem 20.15. For up to a suitable odd stabilization $M\# \pm \mathbf{P}^2(\mathbb{C})$, this last has the same intersection form of some $k\mathcal{P}\#h\mathcal{Q}$. By applying to this couple of manifolds Theorem 20.18 and Proposition 20.10, we get Theorem 20.15. The proof of Theorem 20.18 is much more demanding; it incorporates the one of Theorem 20.15, together with more advanced tools in homotopy and homology theory, beyond the limits of the present text. So we will merely give some indications. A detailed proof can be found, for example, in [**Sc**].

• For our main application in Section 20.6, the simpler classification up to odd stabilization will suffice.

First, one proves the theorem under a stronger hypothesis. The idea is that the h-cobordism theorem also holds in dimension 5, up to even stabilization.

PROPOSITION 20.19. *Let M_0 and M_1 be compact oriented simply connected boundaryless 4-manifolds. Assume that they are h-cobordant. Then there is $k \in \mathbb{N}$ such that $M_0\#k(S^2 \times S^2) \sim M_1\#k(S^2 \times S^2)$.*

Sketch of proof: We know that the main difficulty to perform the stable proof of the h-cobordism theorem in dimension 5 is that we cannot apply the Whitney trick to eliminate couples of intersection points between the b-sphere S_b and the a-sphere S_a of two *algebraically* complementary handles. In particular, trying to construct a Whitney disk, we cannot avoid that such a generically immersed 2-disk D has self-intersection points. Let p such a point. Let us make the connected sum with a copy of $S^2 \times S^2$. This contains two 2-spheres S_1 and S_2 which transversely intersect at one point. By means of a thin embedded 1-handle we connect D with S_1, obtaining a new immersed 2-disk D' ($D' \sim D\#S_1$) which intersects transversely S_2 at one point q. Let c be a simple arc on D' which connects p and q and does not pass through other self-intersection points. By using another thin embedded 1-handle along c, we connect D' with a parallel copy of S_2 and get D" from which both the self-intersection points p and q have been eliminated. Hence up to a certain number of even stabilizations, we can assume that D is embedded and eventually provides a genuine Whitney disk.

■

The classification up to even stabilization is now a consequence of the "if" implication in the following deep Wall's theorem.

THEOREM 20.20. *Let M_0 and M_1 be compact oriented simply connected boundaryless 4-manifolds. Then they are h-cobordant if and only if they have isometric intersection forms.*

Here, we shall be even even more sketchy. "If" is the hard implication; it strengthens a classical Whitehead theorem (based on CW complex techniques) according to which M_0 and M_1 have the same homotopy type. If the intersection forms are isometric, then they have, in particular, the same signature. Hence M_0 is bordant with M_1 by Theorem 20.14 . Arguing as in the proof of Proposition 20.16, we know that there are triads (W, M_0, M_1) where W is obtained by gluing some V with boundary $\partial V = M_0 \amalg (M_0\#k(S^2 \times S^2)\#h(S^2 \tilde{\times} S^2))$ and some V' with boundary $\partial V' = (M_1\#k'(S^2 \times S^2)\#h'(S^2 \tilde{\times} S^2)) \amalg M_1$, via a diffeomorphism

$$\phi : M_0\#k(S^2 \times S^2)\#h(S^2 \tilde{\times} S^2) \to M_1\#k'(S^2 \times S^2)\#h'(S^2 \tilde{\times} S^2) \ .$$

As M_0 and M_1 are simply connected, then also W is so. The key point is to show that, by fully exploiting the hypothesis, among the triads of this kind there is one such that W is homologically trivial; by suitable algebraic/topological arguments, this is enough to conclude that the triad (W, M_0, M_1) is an h-cobordism.

20.6. Congruences modulo 16

To introduce the theme, let us begin with a bit of history. We have recalled in Section 17.4.3 that by the hardest application of Pontryagin method, in a series of four papers of 1951-52 (see [**GM**] for the translation in french and deep commentaries), Rohlin eventually computed the stable homotopy group

$$\pi_3^\infty = \pi_{n+3}(S^n) \sim \Omega_3^{\mathcal{F}}(S^n) \sim \mathbb{Z}/24\mathbb{Z},\ n \geq 5\ .$$

As a corollary, he obtained his celebrated congruence mod(16); a slightly weaker formulation of it is as follows.

Theorem 20.21. *Let M be a compact oriented boundaryless simply connected 4-manifold. Assume that its intersection form is even. Then $\sigma(M) \equiv 0$ mod (16).*

As the intersection form is even, the arithmetic of unimodular forms tells us that $\sigma(M) \equiv 0$ mod(8), so we can reformulate the result as

$$\frac{\sigma(M)}{8} \equiv 0 \text{ mod}(2)\ .$$

This improvement of the congruence mod (8) implies, in particular, that $\mathbf{E}_8$ cannot be realized by any simply connected 4-manifold. The derivation of Theorem 20.21 from stably $\pi_{n+3}(S^n) \sim \mathbb{Z}/24\mathbb{Z}$ is rather complicated and uses several facts less elementary than the ones covered by the present text. Just to give an idea, without any pretension to be understandable, let us sketch the argument by following [**MK**]. It is shown that $p_1(M) = 3\sigma(M)$, where $p_1(M)$ denotes the first Pontryagin number of $T(M)$. This follows because both p_1 and σ are bordism invariant, additive on connected sum and the formula holds for the generator of $\Omega_4 = \mathbb{Z}$. So it is enough to prove that $p_1(M) \equiv 0$ mod (48). One can assume that $M \subset \mathbb{R}^{4+n}$, $n \geq 5$. In the hypotheses of Rohlin's theorem, one can prove that M is almost parallelizable; that is, the tangent bundle of $M \setminus \{x_0\}$ admits a global trivialization. Let f be a nonvanishing section of the restriction to $M \setminus \{x_0\}$ of the $SO(n)$ normal bundle ν of M in $\mathbb{R}^{4+n}$. Let $\mathfrak{e}$ be the obstruction to extending f; it is identified with an element of $\pi_3(SO(n))$ (which is an infinite cyclic group), and the Pontryagin number $p_1(\nu)$ is identified with $\pm 2\mathfrak{e}$. Consider the J-homomorphism (Section 17.4.1) $J : \pi_3(SO(n)) \to \pi_{3+n}(S^n)$. One proves that $J(\mathfrak{e}) = 0$, hence $\mathfrak{e}$ is divisible by 24. Finally, one proves that $p_1(M) = -p_1(\nu)$ because $T(M) \oplus \nu = \epsilon^{4+n}$.

An interesting feature of this history is that in the second paper of the series, Rohlin outlined a proof of the *erroneous* result that stably $\pi_{n+3}(S^n) \sim \mathbb{Z}/12\mathbb{Z}$. Arguing as above, this would imply the not surprising congruence $\sigma(M) \equiv 0$ mod(8). In the fourth paper, after having established the isomorphism $\sigma : \Omega_4 \to \mathbb{Z}$ determined by the signature (i.e. Theorem 20.14), he first realized that this, combined with some claims in his earlier presumed proof, produced a contradiction; then he localized the mistake and corrected it, getting the right group $\mathbb{Z}/24\mathbb{Z}$. He pointed out that there was only one substantial mistake: a certain simply connected 4-manifold M has been constructed with a characteristic element $\omega \in \mathcal{H}^2(M;\mathbb{Z})$ of its intersection form which can be represented by a generic immersion $f : S^2 \to M$; then by an *abusive* application of the Whitney trick in dimension 4, he argued erroneously that ω was represented by an *embedded* $S^2 \subset M$. This was a quite fruitful mistake: his correction leads to the celebrated congruence mod(16) and provides a *concrete counterexample* to the applicability of Whitney's trick in dimension 4. By elaborating on this counterexample, the authors pointed out in [**KM**] (1961) an interesting extension. Recall that for every 4-manifold M and for every characteristic element $\omega \in \mathcal{H}^2(M;\mathbb{Z})$ of its intersection form, $\sigma(M) - \omega \sqcup \omega \equiv 0 \bmod(8)$. Then, assuming Theorem 20.21, the following theorem is proved in [**KM**].

THEOREM 20.22. *Let M be a compact oriented boundaryless simply connected 4-manifold. Let $\omega \in \mathcal{H}^2(M;\mathbb{Z})$ be a characteristic element of its intersection form that can be represented by an embedded 2-sphere. Then*

$$\frac{\sigma(M) - \omega \sqcup \omega}{8} \equiv 0 \bmod(2) \ .$$

If the intersection form is even, then we can take $\omega = 0$ and recover Rohlin's theorem. In general, a characteristic element ω as above can be represented by an oriented surface F embedded in M, but not necessarily by a 2-sphere. For example, take $M = \mathcal{P}\#8\mathcal{Q}$. If a_0 is the standard generator of $\mathcal{H}^2(\mathcal{P};\mathbb{Z})$ represented by a projective complex line, and similarly a_j for the jth-copy of $\mathcal{Q}$, then $\omega := 3a_0 + a_1 + \cdots + a_8$ is characteristic and $\omega \sqcup \omega - \sigma(M) = 8$, hence ω cannot be represented by a 2-sphere by Theorem 20.22. This motivates the following somewhat informal guess:

(1) Let M be a compact oriented boundaryless simply connected 4-manifold. Let $\omega \in \mathcal{H}^2(M;\mathbb{Z})$ be a characteristic element of its intersection form represented by an embedded oriented surface $F \subset M$. Then one expects a formula of the type

$$[\frac{\sigma(M) - \omega \sqcup \omega}{8}]_{(2)} = \alpha(F) \ ,$$

where $\alpha(F) \in \mathbb{Z}/2\mathbb{Z}$ represents an obstruction to surgery F "within M" to get an embedded S^2. Moreover, having in mind Pontryagin's computation of π_2^∞ depicted in Section 17.4.3 (recall also the study of immersions of surfaces in 3-manifolds in Section 19.7), it is predictable that $\alpha(F)$ is the Arf invariant of some quadratic enhancement of $\mathcal{H}_1(F;\mathbb{Z}/2\mathbb{Z})$ (see Section 15.6) associated to the embedding of F in M.

(2) Assuming the isomorphism $\sigma : \Omega_4 \to \mathbb{Z}$, in contrast with the above derivation of Theorem 20.21 from the homotopic information that $\pi_3^\infty = \mathbb{Z}/24\mathbb{Z}$, the definition of $\alpha(F)$ as well as the proof of the congruence should be geometric and possibly elementary.

Accordingly with Freedman-Kirby ([**FK**] (1978)), the realization therein of the above guess is derived, considerably different in details from one outlined by Casson in 1974 (unpublished). According to the historical appendix by Kharlamov and Viro in [**GM**], Rohlin announced such a formula at the Moskow IMC 1966; in a paper of 1972 [**Roh**] he used it to solve a conjecture of Gudkov concerning Hilbert's 16th problem, about the arrangements of ovals of planar even degree real algebraic curves (see also [**Wil**]). The study of this problem using a 4-manifold Y which is a two-sheeted covering of the complex projective plane, ramifying along the complexification of a given nonsingular real algebraic curve in $\mathbf{P}^2(\mathbb{R}) \subset \mathbf{P}^2(\mathbb{C})$, was introduced by Arnol'd [**A3**] (1971). The basic congruences mod(8) of Lemma 20.3 already imply nontrivial prohibitions for the oval configuration; the finer formula, as in the above guess, implies stronger prohibitions (see Section 20.6.4 below for more information).

All this holds under weaker hypotheses, relaxing the fact that M is simply connected; for example, $\Omega_1(M) = 0$ suffices to define the quadratic enhancement by using "membranes" (see below), and we can even avoid the use of membranes through the use of spin structures (see [**Kirby**]). However, we will keep that M is simply connected and follow the treatment of Matsumoto [**Mat**] given in a paper available in [**GM**]; it is accessible by the tools developed in the present text.

20.6.1. Quadratic enhancement for characteristic surfaces. In this section, M will be a compact oriented connected smooth 4-manifold such that $\Omega_1(M) = 0$ (this holds in particular if M is simply connected), and $F \subset M$ will be an orientable surface. Let c be a connected simple smooth circle on F. As $\Omega_1(M) = 0$ and using transversality, there exists a smooth map $f : P \to M$ such that:

- P is an oriented compact surface with one boundary component;
- $f(\partial P) = c$;
- The restriction of f to a collar C of ∂P in P is an embedding;
- $f(C \setminus \partial P) \subset M \setminus F$ and $f(C)$ is normal to F along c;
- f is a generic immersion of P in M;
- $f|(P \setminus \partial P)$ is transverse to F.

Such a map f is said to be a *membrane* along c. We simply write P instead of (P, f). If M is simply connected we can also assume that P is a 2-disk, but this is not so important at this point. For simplicity let us identify c with ∂P. The pull-back of $T(M)$ on P splits as

$$f^*T(M) = T(P) \oplus \nu(f) \ ,$$

where $\nu(f)$ is said the *normal bundle of the membrane* and is an oriented bundle of rank 2. As P retracts to a wedge of a finite number of S^1 (to one point if P is a disk), then $\nu(f)$ is isomorphic to a product bundle. Let us fix a global trivialization τ. This induces a trivialization of the restriction $\nu(f)|c$. Two trivializations of $\nu(f)$ differ by a map $g : P \to SO(2)$. The restriction $g|c$ represents 0 in $\Omega_1(SO(2))$, hence it is homotopically trivial (Section 13.3). Then the restricted trivialization τ_c does not depend on the choice of τ. The normal bundle ν_c of c in F defines a rank-1 orientable subbundle of $\nu(f)|c$. Then denote by $n(P)$ the *number of full twists made by ν_c with respect to τ_c, moving along c in the direction given by its orientation as ∂P*. It is not hard to check that $[n(p)]_{(2)} \in \mathbb{Z}/2\mathbb{Z}$ does not depend on the choice of the orientation of P.

Now, let $a \in \mathcal{H}_1(F;\mathbb{Z}/2\mathbb{Z})$. We know (Lemma 15.3) that $a = [c]$ for some simple smooth circle c on F. Given a membrane P along c, set

$$q_F(c,P) = [n(P)]_{(2)} + [P \bullet F]_{(2)} \in \mathbb{Z}/2\mathbb{Z} ,$$

where $P \bullet F$ is in fact the intersection number between $\text{Int}(P)$ and F.

Proposition 20.23. *Let $F \subset M$ be an oriented* characteristic surface *of M; that is, $\omega = [F] \in \mathcal{H}^2(M;\mathbb{Z})$ is a characteristic element of the intersection form of M. Then:*

(1) For every simple smooth circle c on F, $q_F(c) := q_F(c,P)$ does not depend on the choice of the membrane P along c.

(2) For every $a \in \mathcal{H}_1(F;\mathbb{Z}/2\mathbb{Z})$, for every simple smooth circle c representing a ($a = [c]$), then $q_F(a) := q_F(c)$ does not depend on the choice of the representative c.

(3) The function $q_F : \mathcal{H}_1(F;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ defined so far is a quadratic enhancement of the intersection form on $\mathcal{H}_1(F;\mathbb{Z}/2\mathbb{Z})$.

Proof : (1) Let P and P' be two membranes along c. Up to "spinning" P' along c, we can assume that P and P' glue along the common boundary c in such a way that: (i) $\Sigma = P \cup P'$ is a boundaryless surface generically immersed into M, and (ii) a tubular neighbourhood of c in Σ is an embedded annulus normal to F, made by two collars C and C' in P and P' respectively, opposite to each other. The membranes P and P' determine respective trivializations τ_c and τ'_c which induce opposite orientations on the fibres of the bundle. The difference between $-\tau'_c$ and τ_c along c is encoded by an element $d \in \pi_1(SO(2)) = \mathbb{Z}$. One verifies that

$$\Sigma \bullet \Sigma = d - 2P \bullet P' = d \text{ mod}(2) ,$$

$$\Sigma \bullet F = P \bullet F + P' \bullet F \text{ mod}(2)$$

(recall that the self-intersection of c in F, $c \bullet c = 0$ because F is orientable). As F is characteristic, then

$$\Sigma \bullet \Sigma = \Sigma \bullet F \text{ mod}(2) ;$$

hence

$$d = P \bullet F + P' \bullet F \text{ mod}(2) .$$

On the other hand,

$$n(P') = n(P) + d \text{ mod}(2) .$$

By combining these relations we eventually get

$$n(P) + P \bullet F = n(P') + P' \bullet F \text{ mod}(2)$$

as desired. Item (1) is proved.

To achieve (2) and (3), we can implement the method illustrated at the end of Section 15.6. We have defined a function that associates $q(c) \in \mathbb{Z}/2\mathbb{Z}$ to every simple smooth circle on F. It is clear that $q(c) = 0$ if c is the boundary of a 2-disk embedded in F. We additively extend this function to every not necessarily connected simple curve $c = c_1 \amalg \cdots \amalg c_k$ on F. If γ is now a curve generically immersed in F with a number $r(\gamma) \geq 0$ of normal crossings, every crossing can be simplified in two ways. Let us call a *state* s of γ a system of simplifications at every crossing. Performing these simplifications we get a simple curve c_s. Set

$$q_F(\gamma, s) = q_F(c_s) + [2r(\gamma)]_{(2)} \ .$$

Then it is enough to prove that $q_F(\gamma) := q_F(\gamma, s)$ does not depend on the choice of the state s. Arguing by induction of $r(\gamma)$, we localize the question at one crossing. If s and s' differ just at one crossing, then we can use membranes P and P' along the components of c_s and $c_{s'}$ which only differ locally at the crossing. By a direct computation, we can compute $q_F(\gamma, s)$ and $q_F(\gamma, s')$ by using P and P', getting the desired result.

■

For the definition of the Arf invariant of q_F, we refer to Section 15.6. In the next proposition, we show that the Arf invariant of q_F only depends on the characteristic element $\omega = [F] \in \mathcal{H}^2(M; \mathbb{Z})$.

PROPOSITION 20.24. *Let $F, F' \subset M$ be oriented characteristic surfaces of M representing the same characteristic element ω of the intersection form of M. Then Arf(q_F) = Arf($q_{F'}$), so that $\alpha(\omega) := \mathrm{Arf}(q_F) \in \mathbb{Z}/2\mathbb{Z}$ is well defined.*

Proof : We repeat an embedded bordism argument already employed in Sections 17.4.3, 19.7.1. We know that there is an orientable 3-dimensional triad (W, F, F') properly embedded in the triad $(M \times [0,1], M \times \{0\}, M \times \{1\})$, and we can assume that the restriction to (W, F, F') of the projection to $[0,1]$ is a Morse function. Consider the corresponding handle decomposition of (W, F, F') and the successive surgeries which produce F' from F. Attaching a 0-handle or attaching a 1-handle to different boundary connected components does not change the value of Arf. Attaching a 1-handle to a same connected component, the boundary is modified by an embedded connected sum with a copy of $T = S^1 \times S^1$; we realize that there is a basis l, m of $\mathcal{H}_1(T; \mathbb{Z}/2\mathbb{Z})$ such that the intersection form is represented by the standard matrix $\mathbf{H}$ and m is the co-core of the handle so that $q_T(m) = 0$. It follows that $\mathrm{Arf}(q_T) = 0$, so that the total Arf also does not change in this case. Finally, we consider the dual handle decomposition to also rule out 2- and 3-handles.

■

20.6.2. A digression in classical knot theory. Let us recall a few facts of classical knot theory (see for instance [**Kau**], [**Rolf**]) that we will use below in the proof of the main result. Let K be a knot in $S^3 = \partial D^4$ considered up to ambient isotopy. Every oriented proper surface $(S, \partial S) \subset (D^4, S^3)$ such that $\partial S = K$ is "characteristic" for $\mathcal{H}^2(D^4, S^3; \mathbb{Z}) = 0$. By a similar construction as above, we can define a quadratic form $q_S : \mathcal{H}_1(S; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ whose Arf invariant $\alpha(q_S) \in \mathbb{Z}/2\mathbb{Z}$ eventually depends only on the knot K, so that the *Arf invariant of the knot* $\mathrm{Arf}(K) := \alpha(q_S)$ is well defined. It can be computed using any oriented planar diagram $\mathcal{D}$ of K as follows. We can use as S the surface obtained by pushing in D^4 the Seifert surface of K in S^3 constructed through the Seifert algorithm (via the oriented simplification of the normal crossings of $\mathcal{D}$). If $\mathcal{D}'$ is a knot diagram which differs from $\mathcal{D}$ just by the over/under branches at one crossing, denote by K' the corresponding knot. Performing the simplification at the given crossing of $\mathcal{D}$ (or of $\mathcal{D}'$, the result is the same) we get a diagram $\mathcal{D}$" of a link with two oriented components K_1 and K_2. Then one realizes that the following relation holds, involving the linking number of K_1 and K_2:

$$\mathrm{Arf}(K) = \mathrm{Arf}(K') + [L(K_1, K_2)]_{(2)} \in \mathbb{Z}/2\mathbb{Z} \ .$$

The linking number mod (2) can be easily computed in terms of the diagram $\mathcal{D}$": the number c of crossings of $\mathcal{D}$" whose local branches do not belong to a same constituent knot is even and $[L(K_1, K_2)]_{(2)} = [c/2]_{(2)}$. Moreover, it is well known that one gets a diagram $\mathcal{D}_0$ for the unknot K_0 by switching some crossings of $\mathcal{D}$, and clearly $\mathrm{Arf}(K_0) = 0$; then the above relation allows to compute $\mathrm{Arf}(K)$ inductively, starting from $\mathcal{D}$.

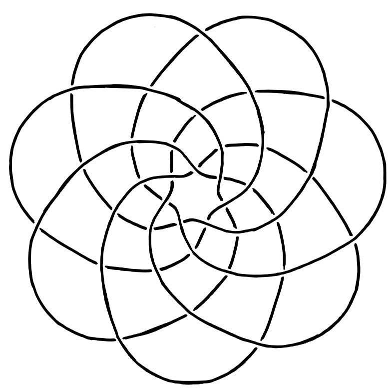

Figure 2. A standard diagram of $K(7, 6)$.

Let $T \subset \mathbb{R}^3$ be the standard torus obtained by rotation of the planar circle $\{x = 0,\ (y-2)^2 + z^2 = 1\}$ around the z-axis . For every couple (p, q) of coprime integers, the *torus knot* $K(p, q)$ is traced on T turning p times in the direction of the standard longitude of T, q times in the direction of the meridian. By projection to the (x, y)-coordinate plane, we get a standard diagram $\mathcal{D}(p,q)$ of $K(p,q)$. We will be interested in the case $K(s,s-1)$, where $s>1$ is odd (so that $(1-s^2)\equiv 0 \bmod (8)$). It is known in knot theory (for example, by applying the above method to the diagram $\mathcal{D}(s,s-1)$) that

$$\mathrm{Arf}(K(s,s-1)) = [\frac{1-s^2}{8}]_{(2)} \ .$$

20.6.3. The main results. We can state now the main result of this section.

THEOREM 20.25. *Let M be a compact oriented boundaryless simply connected 4-manifold. Let $\omega \in \mathcal{H}^2(M;\mathbb{Z})$ be a characteristic element of the intersection form of M. Then*

$$[\frac{\sigma(M)-\omega \sqcup \omega}{8}]_{(2)} = \alpha(\omega) \ .$$

Proof : The proof is based on the classification up to odd stabilization. First note that if $M = M_1 \# M_2$ is the connected sum of two simply connected manifolds, then a characteristic element ω of M is the sum $\omega = \omega_1 + \omega_2$ of characteristic elements of M_1 and M_2, respectively. If the theorem holds for two members of the triple (M,ω), (M_1,ω_1), (M_2,ω_2), then it holds also for the third. By Theorem 20.15 we have that

$$M\#(k\mathcal{P}\#h\mathcal{Q}) = m\mathcal{P}\#n\mathcal{Q}$$

for some $k,h,m,n\in\mathbb{N}$. Applying inductively the above remark, it is enough to prove the theorem for $\mathcal{P}$ and $\mathcal{Q}$. If $\mathbf{P}^1(\mathbb{C})\subset\mathcal{P}$ is a complex line, then every characteristic element of $\mathcal{P}$ is of the form $\omega = s[\mathbf{P}^1(\mathbb{C})]$, where s is an odd integer; to our aims it is not restrictive to assume that $s\geq 1$. The theorem clearly holds for $s=1$, so let us assume $s>1$. Then $\omega=[F]$, where F is any non singular complex projective curve in $\mathcal{P}$ defined as the zero set of a homegeneous polynomial of degree s in the homogeneous complex coordinates (z_1,z_2,z_3) on $\mathcal{P}$. One can indeed prove (by using the fibration theorem 6.11) that all these curves are isotopic to each other, but this is not so important for the present discussion. Let us consider the family of projective complex curves

$$F_\epsilon = \{z_1^s + z_2^{s-1}z_3 - \epsilon z_3^s = 0\}$$

where $\epsilon\in\mathbb{R}$, $\epsilon\geq 0$. For $\epsilon=0$, F_0 has one isolated singularity at the point $x_0=(0,0,1)$ and in the affine coordinates such that $z_3\neq 0$, it is defined by the equation $x^s+y^{s-1}=0$. The best reference for the study of such isolated singularities of complex planar curves is Milnor's celebrated book [**M6**]. Our case is particularly simple. There is a small round 4-disk D around $x_0=(0,0)$ in such affine chart, such that:

(1) $S^3=\partial D$ is transverse to F_0 and $K := F_0\cap S^3$ is a torus knot $K(s,s-1)$.
(2) The pair $(D, F_0\cap D)$ is homeomorphic to the pair (D, cK), where cK denotes the cone with base K and centre at x_0.

(3) $F_0 \cap (\mathcal{P} \setminus \mathrm{Int}(D))$ is a smooth properly embedded 2-disk. Hence F_0 is homeomorphic to S^2.

If $\epsilon > 0$ is small enough, then

(i) F_ϵ is nonsingular.

(ii) $F_\epsilon \pitchfork S^3$ is an isotopic copy of $K(s, s-1)$ and $F_\epsilon \cap D$ is properly embedded.

(iii) $F_\epsilon \cap (\mathcal{P} \setminus \mathrm{Int}(D))$ is a smooth properly embedded 2-disk.

Then it is clear that

$$\alpha(\omega) = \mathrm{Arf}(q_{F_\epsilon}) = \mathrm{Arf}(K(s, s-1)) = [\frac{1-s^2}{8}]_{(2)} = [\frac{\sigma(\mathcal{P}) - \omega \sqcup \omega}{8}]_{(2)}$$

and this achieves the case $M = \mathcal{P}$. By taking into account the change of orientation, the same argument holds as well for $M = \mathcal{Q}$ and the proof is complete.

■

20.6.4. Congruences mod(16) in Hilbert's 16th problem. We mentioned that Rohlin [**Roh**] applied the congruence mod(16) as in Theorem 20.25 to prove a conjecture of Gudkov about Hilbert's 16th problem, elaborating on the use of 4-manifold topology to this aim, introduced by Arnol'd [**A3**]. Let us outline some aspects of this application.

Let $X \subset \mathbf{P}^2(\mathbb{C})$ be a nonsingular complex curve defined by a real equation $f = 0$, where f is a real homogeneous polynomial of degree $2k$; assume that the real part $X_\mathbb{R} \subset \mathbf{P}^2(\mathbb{R})$ is also a regular nonempty curve. As the degree of the equation is even, the sign of f is well defined on every component of $\mathbf{P}^2(\mathbb{R}) \setminus X_\mathbb{R}$. It follows that every component of $X_\mathbb{R}$ is a two-sided *oval* dividing $\mathbf{P}^2(\mathbb{R})$ so that it has an interior part (diffeomorphic to a 2-disk) and an exterior (diffeomorphic to a Möbius band). We normalize the equation in such a way that $f < 0$ on the nonorientable component of $\mathbf{P}^2(\mathbb{R}) \setminus X_\mathbb{R}$. For the main application, one also assumes that $X_\mathbb{R}$ is an *M-curve*; that is, the number of ovals is maximal and equal to $g+1$, where $g := \frac{(2k-1)(2k-2)}{2}$ is the genus of X (considered as an oriented real surface), in accordance with Harnack inequality. In such a case, $X_\mathbb{R}$ divides X. For every $k \geq 1$, the problem consists in determining all possible arrangements of these $g+1$ ovals in $\mathbf{P}^2(\mathbb{R})$, considered up to smooth isotopy, when $(X, X_\mathbb{R})$ of degree $2k$, as above, varies. The problem has to be attacked from two sides: (i) to prove theorems giving restrictions on the possible relative positions of the ovals, and (ii) to construct examples of curves, hoping to eventually get all possibilities allowed by the theorems (for example, it is already nontrivial to construct M-curves for every k).

Gudkov's conjecture (and, hence, Rohlin's Theorem), concerns a restriction result. Its statement is elementary. An oval of $X_\mathbb{R}$ is *even* (*odd*) if it lies inside an even (odd) number of other ovals. Let us denote by p (n) the number of even (odd) ovals of $X_\mathbb{R}$. Then we have the following.

Theorem 20.26. *For every M-curve of even degree $2k$,*

$$p - n \equiv k^2 \bmod(8) \ .$$

This is achieved using 4-manifold topology and the arithmetic of unimodular forms. Let us describe first how Arnol'd proved his earlier weaker result mod(4). The most obvious covering of $\mathbf{P}^2(\mathbb{C})$ branched along the complex curve X is the nonsingular surface (defined by a real equation)

$$Z = \{(x_0 : x_1 : x_2 : x_3) \in \mathbf{P}^3(\mathbb{C});\ f(x_0, x_1, x_2) = x_3^{2k}\} \ ;$$

the map $Z \to \mathbf{P}^2(\mathbb{C})$ given by the projection onto (x_0, x_1, x_2) is a $2k$-fold covering of $\mathbf{P}^2(\mathbb{C})$ branched along X. We consider an intermediate covering $Z \to Y \to \mathbf{P}^2(\mathbb{C})$. To define Y, let G_r denote the (cyclic) group of rth roots of unity. We make $\omega \in G_{2k}$ act on Z by

$$(x_0 : x_1 : x_2 : x_3) \to (x_0 : x_1 : x_2 : \omega x_3) \ .$$

We can identify the covering $Z \to \mathbf{P}^2(\mathbb{C})$ with the quotient map $Z \to Z/G_{2k}$, and we set $Y = Z/G_k$.

Then Arnol'd proves:

(1) Y is *simply connected* and has a natural compact complex surface structure such that the maps

$$Z \to Y \to \mathbf{P}^2(\mathbb{C})$$

are holomorphic, the last being the required double covering branched along X.

(2) The covering involution θ on Y is holomorphic; its fixed point set $\mathrm{Fix}(\theta)$ is the inverse image of X via the covering map.

(3) The complex conjugation on Z induces an anti-holomorphic involution T on Y; its fixed point set $\mathrm{Fix}(T)$ is the inverse image of

$$B^+ = \{f \geq 0\} \subset \mathbf{P}^2(\mathbb{R}) \ ,$$

being in fact the double of B^+.

(4) The 2-manifolds $\mathrm{Fix}(\theta)$ and $\mathrm{Fix}(T)$ are naturally oriented, the respective classes α and β in $\mathcal{H}_2(Y, \mathbb{Z})$ have the same reduction mod(2).

(5) $\alpha \bullet \alpha = 2k^2$, $\beta \bullet \beta = 2(n - p)$, $\alpha \bullet \beta = 0$.

(6) If k is odd, then $\alpha + \beta$ is a characteristic element and the intersection form of Y is even. If k is even, then both α and β are characteristic elements.

By using Lemma 20.3, depending on the parity of k,

$$2k^2 + 2(n - p) \equiv 0,\ 2k^2 \equiv 2(n - p) \bmod(8) \ ;$$

hence, in any case,

$$n - p \equiv k^2 \bmod(4) \ .$$

The proofs are geometric in nature but also involve some facts about Stiefel-Whitney classes and spin structures. To illustrate how Theorem 20.25 allows to improve the result, let us assume, for simplicity, that k is odd.

Consider the involution $\theta \circ T : Y \to Y$. The surface $\text{Fix}(\theta \circ T)$ is orientable and Rohlin proves

(1) The class γ of $\text{Fix}(\theta \circ T)$ in $\mathcal{H}_2(Y, \mathbb{Z})$ is a characteristic element.

(2) $\gamma \bullet \gamma = 2(p - n - 1)$, while the signature $\sigma(Y) = 2 - 2k^2$.

(3) $\text{Arf}(\gamma) = 0$.

Assuming Arnol'd computations, these further verifications are geometric in nature.

Applying Theorem 20.25, we have

$$2(p - n - 1) - (2 - 2k^2) = 8\text{Arf}(\gamma) \equiv 0 \text{ mod}(16) ;$$

as the intersection of Y is even, we also have that

$$\sigma(Y) \equiv 0 \text{ mod}(16) .$$

Combining this facts, we get

$$p - n \equiv k^2 \text{ mod}(8)$$

as desired.

20.6.5. An extension to nonorientable characteristic surfaces. Elaborating on further applications of 4-manifold topology to Hilbert's 16th problem (studying so-called $(M - k)$-curves, $k \geq 1$), it is quite current to deal with *nonorientable characteristic surfaces*; that is, representing the reduction mod(2) of a characteristic element of the intersection form of some 4-manifold M. This strongly motivates the search for a further generalization of Theorem 20.25. We merely describe it.

Let $F \subset M$ be a not necessarily orientable characteristic surface. Assume that $\Omega_1(M) = 0$. Similarly to Section 19.7, using membranes we can define a quadratic enhancement

$$\hat{q}_F : \mathcal{H}_1(F; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/4\mathbb{Z}$$

of the intersection form by setting

$$\hat{q}_F([c]) = \hat{q}_F(c, P) = [\hat{n}(P)]_{(4)} + 2 \cdot ([P \bullet F]_{(2)} + c \bullet c) \in \mathbb{Z}/4\mathbb{Z}$$

where $\hat{n}(P)$ is the number of *half-twists* made by ν_c with respect to τ_c, moving along c. The fact that it is well defined is more complicated, but not much more so.

Similarly to the discussion made to define the integer Euler-Poincaré characteristic also for nonorientable manifolds, we can define geometrically the self-intersection number $F \bullet F \in \mathbb{Z}$, by identifying F with the zero section of its normal bundle in the oriented manifold M and fixing arbitrary compatible local orientations of F and F' at every point of $F \pitchfork F'$, F' being a section transverse to F. By usual arguments, this number does not depend on the arbitrary choices made to compute it. Recall the Arf-Brown invariant of $\hat{q}_F$ defined in Section 15.6. Here we denote it by $\hat{\alpha}(F) \in \mathbb{Z}/8\mathbb{Z}$. Recall that the multiplication by 2 determines injective homomorphisms $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/16\mathbb{Z}$.

THEOREM 20.27. *Let M be a compact oriented boundaryless simply connected 4-manifold. Let $F \subset M$ be a possibly nonorientable surface which represents the reduction mod(2) of any characteristic element ω of the intersection form of M. Then*

$$[\sigma(M) - F \bullet F]_{(16)} = 2 \cdot \hat{\alpha}(F) \ .$$

If F is oriented we recover Theorem 20.25, because $F \bullet F = \omega \sqcup \omega$, $\hat{q}_F = 2 \cdot q_F$, $\hat{\alpha}(F) = 4 \cdot \alpha(\omega)$.

Theorem 20.27 is due to Guillou-Marin [**GM**]. There are several difficulties to overcome. When F is nonorientable, $F \bullet F \in \mathbb{Z}$ cannot be identified with the intersection number of any bordism classes of M. So it is not clear how to reformulate Proposition 20.24. We should rather prove that $[F \bullet F + 2 \cdot \hat{\alpha}(F)]_{(16)}$ does not depend on the choice of the (possibly nonorientable) surface F representing $\omega_{(2)}$. Note also that, dealing with nonorientable surfaces, the embedded bordism argument used in the proof of Proposition 20.24 is not immediately available (recall Remark 13.11). In the already cited paper [**Mat**], Matsumoto gives another proof which, by an inductive argument, reduces the general statement to Theorem 20.25. In both proofs there are two further basic cases besides $\mathcal{P}$ and $\mathcal{Q}$, that is S^4 with suitably embedded real projective spaces as characteristic surface.

20.7. On the topological classification of smooth 4-manifolds

From Rohlin's theorem (1952) to Donaldson's work in 1982 [**Do**], no further prohibitions to the realization of unimodular forms by boundaryless simply connected smooth 4-manifolds appeared. Wall's Theorem 20.18 was the strongest one about the extent which the intersection form determines the differential topology of a boundaryless 4-manifold. At the beginning of the 80's, two parallel new waves revolutionized the subject. Since Donaldson's work, the introduction of new methods derived from gauge theory, of differential-geometric/analytic nature and strongly influenced by ideas of theoretical physics, have produced amazing new prohibitions and powerful smooth invariants distinguishing homeomorphic but not diffeomorphic smooth 4-manifolds. Let us recall a few new prohibitions.

(Donaldson 1982 [Do]) *If the intersection form of a simply connected, boundaryless smooth 4-manifold is definite, then it is diagonalizable; that is, of the form $k\mathbf{U}_\epsilon$.*

Donaldson's result means that the arithmetic complication of definite forms does not concern the intersection forms of smooth 4-manifolds; hence the problem of the 4-dimensional smooth realization is reduced to the indefinite and even case. With respect this, we recall:

(Furuta 2001 [Fu]) *If the intersection form of a simply connected, boundaryless smooth 4-manifold is indefinite and even, that is, of the type $2h\mathbf{E}_8 \perp a\mathbf{H}$, then $a \geq 2|h| + 1$.*

The following is still an open conjecture.

The so-called "11/8" Conjecture: *If the intersection form of a simply connected, boundaryless smooth* 4*-manifold is indefinite and even, that is, of the type* $2h\mathbf{E}_8 \perp a\mathbf{H}$*, then* $a \geq 3|h|$*.*

If the conjecture holds, then the rank must be at least 11/8 times $|\sigma|$. Furuta's theorem means that the rank is at least 10/8 times $|\sigma|$. If the form is indefinite and even we may assume that it is of non-positive signature by changing orientations if necessary, in which case $h \leq 0$. If $a \geq 3|h|$, then the form can be realized by means of $|h|K\#(a-3|h|)(S^2 \times S^2)$, where K is the Kummer complex surface of Example 20.11. Hence a confirmation of the conjecture would achieve the realization problem.

The other wave had a somewhat more conservative motivation. It was clear, at least since Rohlin's 'mistake', that there were actual obstructions to applying the Whitney trick in dimension 4; nevertheless, one wondered if such a 'technical' difficulty could be circumvented in some way to prove the 5-dimensional h-cobordism theorem. For example, in Wall's theorem 20.19 this is done by paying the price of performing even stabilizations. In this vein, in 1973-74, A. Casson introduced so-called "flexible handles", currently called "Casson handles" (see Lecture I in the second part of [**GM**]). Let M be a boundaryless simply connected 4-manifold and let $\alpha, \beta \in \mathcal{H}_2(M;\mathbb{Z})$ such that $\alpha \bullet \alpha = \beta \bullet \beta = 0$, $\alpha \bullet \beta = 1$. Then, by means of a certain 'infinite construction', he produced an open set V of M such that:

- V has the proper homotopy type of $S^2 \times S^2 \setminus \{*\}$, where "$*$" is a single point;
- $\mathcal{H}_2(V;\mathbb{Z})$ carries the submodule of $\mathcal{H}_2(M;\mathbb{Z})$ generated by α and β.

Moreover, he argued the following (Lecture III of the second part of [**GM**]).

If flexible handles V are diffeomorphic to the true $S^2 \times S^2 \setminus \{pt\}$, then we could carry out the Whitney process and cancel handles to trivialize 5*-dimensional simply connected h-cobordisms.*

Also, weaker information about the flexible handle (at least about its 'end') would have been of main importance for the realization problem:

• If such a flexible handle V is diffeomorphic to the true $S^2 \times S^2 \setminus \{*\}$, then we could split $M = M'\#(S^2 \times S^2)$, where M' is simply connected, and passing from M to M', we have surgered out a factor $\mathbf{H}$ of the intersection form of M.

• If V is diffeomorphic to $N \setminus \{*\}$, where N is a compact boundaryless 4-manifold, then $M = M'\#N$, where N has the homotopy type of $S^2 \times S^2$ and, again, carries α and β; so M' has the same properties as above.

• If the end of V coincides with the end of an open contractible manifold V^*, then by replacing V with V^*, we again get W' with α and β killed.

Before Donaldson's result, there were no known obstructions to realizing the arithmetic splitting of an indefinite and even form $2h\mathbf{E}_8 \perp a\mathbf{H}$ of a certain simply connected 4-manifold M by a splitting $M'\#a(S^2 \times S^2)$. After Donaldson, we know that this underlying hope was too optimistic; nevertheless, the main achievement of [**Fr**] (1982) was that *a flexible handle is a 'true'* $S^2 \times S^2 \setminus \{*\}$*, provided we work in the more flexible setting of almost smooth* 4*-manifolds.*

A topological manifold N is *almost smooth* if $N \setminus \{*\}$ has a smooth structure which, in general, cannot be extended over the whole N. Remarkably, more or less at the same time, it was proved in [**Q**] that *every compact boundaryless simply connected topological* 4*-manifold is almost smooth.*

This opens the way (via the solution of other hard technical issues) for a complete classification of topological simply connected 4-manifolds, which includes the fact that *every* unimodular symmetric form can be realized as the intersection form of a boundaryless simply connected almost smooth 4-manifold. Here we limit to state a few corollaries in our favourite smooth setting.

(1) **Topological 5-dimensional h-cobordism:** *Every smooth simply connected* 5*-dimensional h-cobordism* (W, M_0, M_1) *is* homeomorphic *to the product* $M_0 \times [0,1]$. *In particular* M_0 *and* M_1 *are* homeomorphic *to each other.*

(2) **A classification of smooth 4-manifolds up to homemorphism:** *Two compact smooth simply connected boundaryless* 4*-manifolds are* homeomorphic *if and only if they have isometric intersection forms.*

The new gauge theoretical prohibitions and smooth invariants, together with the above topological classifications, lead to a dramatic failure of the *smooth* 5-dimensional h-cobordism theorem and to the existence of plenty of not diffeomorphic smooth structures on certain topological 4-manifolds. In particular, the Kummer complex surface of Example 20.11 admits countably many not diffeomorphic smooth structures [**FS**]. Finally, we recall that the classification of topological 4-manifolds includes the solution of the 4-dimensional **topological** Poincaré conjecture:

Every compact boundaryless topological 4*-manifold which is homotopically equivalent to* S^4 *is homeomorphic to* S^4.

It is not known if every *smooth* boundaryless 4-manifold which is homeomorphic to S^4 is *diffeomorphic* to S^4. This smooth 4-dimensional Poincaré conjecture presumably is the main basic open question about smooth 4-manifolds.

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Appendix: baby categories

Along with the text, we make some (very moderate indeed) use of the language of categories. We collect in this appendix the few necessary notions.

A *category* **C** consists of three things:

(1) A class of *objects* X;
(2) For every ordered pair of objects (X, Y), a set $\mathrm{Hom}(X, Y)$ of *morphisms* (also called *arrows*) $f : X \mapsto Y$;
(3) For every ordered triple (X, Y, Z) of objects, a *composition function of arrows*

$$\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z),\ (f, g) \to g \circ f\ .$$

We require that the following properties are satisfied:

(1) *(Associativity)* Whenever the involved compositions make sense, we have $h \circ (g \circ f) = (h \circ g) \circ f$;
(2) *(Existence of the identity)* For every object X, there is a (necessarily unique) arrow $1_X \in \mathrm{Hom}(X, X)$ such that $1_X \circ f = f$, $g \circ 1_X = g$, whenever the compositions make sense.

A morphism $f \in \mathrm{Hom}(X, Y)$ is *an equivalence* in the category **C** if there exists a (necessarily unique) morphism $g \in \mathrm{Hom}(Y, X)$ such that $f \circ g = 1_X$ and $g \circ f = 1_Y$.

A fundamental example is the category of sets, denoted by **SET**, which has as objects the class of all sets, while $\mathrm{Hom}(X, Y)$ consists of the set of all maps from X to Y. The arrow 1_X is the identity map, while the equivalences are the bijective maps. We know a lot of sub-categories of **SET** obtained by specializing both objects and arrows: the categories of groups and group homomorphisms, of vector spaces (on a given scalar field) and linear maps, of topological spaces and continuous maps, of smooth manifolds and smooth maps, and so on. The equivalences are the isomorphisms, the homeomorphisms, the diffeomorphisms, and so on.

A single group G can be considered as a category with just G as a unique object, while $\mathrm{Hom}(G, G) \sim G$ by associating to every $h \in G$ the morphism by left multiplication by h, $L_h : G \to G,\ g \to hg$. In this category all morphisms are equivalences.

Not every category is a subcategory of **SET**. For example, starting from the category of topological spaces and continuous maps we can construct a new category with the same class of objects, and as arrows the *homotopy* *classes* of continuous maps from X to Y. The fact that associativity holds is left as an exercise.

If X is a path connected topological space, we can consider the category whose objects are the points of X and $\mathrm{Hom}(x, y)$ consists of the homotopy classes $[\alpha]$ of paths in X connecting x and y. One can verify that every morphism in this category is an equivalence (we say that it is a *groupoid*).

Given two categories $\mathbf{C}$ and $\mathbf{D}$, a *covariant functor* $\mathcal{F} : \mathbf{C} \Rightarrow \mathbf{D}$ fron $\mathbf{C}$ to $\mathbf{D}$ assigns to every object X of $\mathbf{C}$, an object $\mathcal{F}(X)$ of $\mathbf{D}$, to every arrow $f \in \mathrm{Hom}(X, Y)$ of $\mathbf{C}$, an arrow $\mathcal{F}(f) : \mathcal{F}(X) \mapsto \mathcal{F}(Y)$ of $\mathbf{D}$ in such a way that the following properties are satisfied:

(1) For every object X of $\mathbf{C}$, $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$;
(2) $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$, whenever the composition is defined.

A *contravariant functor* assigns to every $f \in \mathrm{Hom}(X, Y)$, an arrow $\mathcal{F}(f) \in \mathrm{Hom}(\mathcal{F}(Y), \mathcal{F}(X))$ in such a way that $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$. A basic example of contravariant functor if the functor from the category of vector spaces (on a given scalar field) to itself such that for every V, $\mathcal{F}(V) = V^*$ the dual space, and for every linear map $f : V \to W$, $\mathcal{F}(f) = f^t$ the transposed map of f, $f^t : W^* \to V^*$, $f^t(\phi) = \phi \circ f$.

Let $\mathcal{F}$ and $\mathcal{G}$ be two say covariant functors from $\mathbf{C}$ to $\mathbf{D}$. A *natural transformation* T from $\mathcal{F}$ to $\mathcal{G}$ is a rule assigning to every object X of $\mathbf{C}$, a morphism $T_X : \mathcal{F}(X) \mapsto \mathcal{G}(X)$ such that for every $f \in \mathrm{Hom}(X, Y)$ of $\mathbf{C}$, $\mathcal{G}(f) \circ T_X = T_Y \circ \mathcal{F}(f)$. If for every X, T_X is an equivalence, then T is called a *natural equivalence of functors.*

For example a Δ-complex mentioned in the text can be abstractly defined as being a contravariant functor from the category Δ to the category **SET**, where Δ has as objects the ordered sets $\Delta^n = \{0, 1, \ldots, n-1\}$, $n \in \mathbb{N}$, and as arrow the strictly increasing maps $\Delta^k \to \Delta^n$, $k \leq n$. Maps between Δ-complexes would be defined as natural transformations of the corresponding functors.

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