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**Calculus of Variations**

# Natural second-order regularity for parabolic systems with operators having $(p, \delta)$-structure and depending only on the symmetric gradient

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**Abstract**
In this paper we consider parabolic problems with stress tensor depending only on the symmetric gradient. By developing a new approximation method (which allows to use energy-type methods typical for linear problems) we provide an approach to obtain global regularity results valid for general potential operators with $(p, \delta)$-structure, for all $p > 1$ and for all $\delta > 0$. In this way we prove "natural" second order spatial regularity—up to the boundary—in the case of homogeneous Dirichlet boundary conditions. The regularity results, are presented with full details for the parabolic setting in the case $p > 2$. However, the same method also yields regularity in the elliptic case and for $1 < p \leq 2$, thus proving in a different way results already known.

**Mathematics Subject Classification** 35B65 · 35Q35 · 35K55

## 1 Introduction

In this paper we consider an initial boundary value problem for general nonlinear parabolic systems

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{u}) &= \mathbf{f} \quad && \text{in } I \times \Omega,\\
\mathbf{u} &= \mathbf{0} \quad && \text{on } I \times \partial\Omega,\\
\mathbf{u}(0) &= \mathbf{u}_0 \quad && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

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where the operator $\mathbf{S}$ depends only on the symmetric gradient $\mathbf{Du} = \frac{1}{2}((\nabla\mathbf{u})^\top + \nabla\mathbf{u})$ and has $(p, \delta)$-structure (cf. Definition 2.15). Here $I := (0, T)$ for some $T > 0$ is a finite time interval, and $\Omega \subset \mathbb{R}^3$ is a sufficiently smooth, bounded domain. The paradigmatic example for the operator in (1.1) is given via

$$\mathbf{S}(\mathbf{Du}) := (\delta + |\mathbf{Du}|)^{p-2}\mathbf{Du} \quad \delta \geq 0,\ 1 < p < \infty. \tag{1.2}$$

In this paper we only treat the case $p > 2$. However, the method of proof, based on an $(A, q)$-approximation (cf. Sect. 2.4) works for every $p \in (1, \infty)$. We focus to the case $p > 2$, since our main result in the case $p \in (1, 2]$ has been already proved in a different way (cf. [12]) and the method of the present paper simplifies a lot for these exponents. Note that the elliptic problem corresponding to (1.1) can be treated in the same way with much shorter proofs. Moreover, all our result possess corresponding analogues in $d$-dimensional domains $\Omega \subset \mathbb{R}^d$, $d \geq 2$. For simplicity we only treat the case $d = 3$.

Our main goal is to prove a result of "natural" second-order spatial regularity for weak solutions. This corresponds to proving, under appropriate (minimal) assumptions on the data, that weak solutions satisfy

$$\int_I \int_\Omega (\delta + |\mathbf{Du}|)^{p-2} |\nabla \mathbf{Du}|^2 \, d\mathbf{x}\, ds \leq C,$$

which can be also equivalently re-written as $\mathbf{F}(\mathbf{Du}) \in L^2(I; W^{1,2}(\Omega))$ with

$$\mathbf{F}(\mathbf{Du}) := (\delta + |\mathbf{Du}|)^{\frac{p-2}{2}} |\mathbf{Du}|. \tag{1.3}$$

We say "natural" as opposed to some recent results proving $\mathbf{S} \in L^2(I; W^{1,2}(\Omega))$, which is equivalent to proving that

$$\int_I \int_\Omega \left|\nabla\big((\delta + |\mathbf{Du}|)^{p-2}\mathbf{Du}\big)\right|^2 d\mathbf{x}\, ds \leq C,$$

which is called "optimal" second-order spatial regularity. The two notions of regularity are rather different in the spirit: the optimal regularity is linked with nonlinear versions of the singular integral theory, while the natural regularity is based on energy methods. This yields estimates in quasi-norms, which are of crucial relevance especially for the numerical analysis of the problem, and in particular to study optimal convergence rates of spatial discretizations (cf. Barrett and Liu [2]).

The problem has a long history and many result concern mainly the problem: (a) in the scalar or elliptic case; (b) with operators $\mathbf{S}$ depending on the full gradient; (c) the interior regularity. We refer to the classical results by DiBenedetto [20], Gilbarg and Trudinger [26], Ladyžhenskaja et al. [28, 29], Liebermann [30], Uhlenbeck [40], Ural'ceva [41], just to cite a few; or the ones linked more to applications Bensoussan and Frehse [9], Nečas [34], and Fuchs and Seregin [24]. Even if the studies started in the sixties, we observe that the field is still extremely active and very recent results are those in [3, 4, 17, 18].

Our treatment of the case of systems with dependence only on the symmetric gradient and up-to-the boundary is new, to the best of the author's knowledge. We extend the so called $A$-approximation technique from [32] such that it allows a treatment of all exponents $p \in (1, \infty)$. Here, we focus on the regularity of the quantity in (1.3). Thus, this work can be seen as a natural extension of previous results we have done in the case $p \in (1, 2]$ for the steady problem in [12] and for the unsteady continuous/discrete in [13]. Note that our approach allows to treat the full range of exponents $p \in (1, \infty)$, as in the scalar case, even if we give full details only in the case $p > 2$, as the case $p \in (1, 2)$ is already treated in a different way. Notice that the results in [18] hold only for $p > \frac{3}{2}$, which has been improved in [1], reaching $p > 4 - 2\sqrt{2}$. The limitation on $p > 3/2$ was also present in prior results of "natural" regularity in the symmetric gradient case [5], but it has then later removed completely in [11] to the case $p > 1$.

The techniques employed for $p > 2$ are rather different from those previously used in the case for $p < 2$, where calculations can be more easily justified by approximation of the system by means of adding the term $-\varepsilon \Delta \mathbf{u}$ (and then showing that estimates for a system with leading linear part could be made independent of $\varepsilon > 0$). Anyway, the technique we use can be also employed in the case $p \in (1, 2]$ to prove in an alternative way the regularity results already known. This requires some technical adjustments which are left for a further investigation, since the technicalities are complex enough already in the case $p > 2$. The introduction of a different regularization of the problem is due to the fact that for $p > 2$ the perturbation with the heat equation is not enough to justify the computations; hence, we developed a new (multiple) approximation technique, by a sequence of operators, such that the last is an affine one, which allows to use standard energy techniques leading to $W^{2,2}$-results.

### 1.1 Sketch of the proof of the main result

To prove the main regularity result (cf. Theorem 3.4) we proceed as follows: (a) we introduce a proper multiple approximation of the operator $\mathbf{S}$; (b) we prove interior and tangential estimates for second order derivatives by difference quotient methods; (c) we use the equations point-wise to recover the remaining derivative; (d) make again use of the point-wise equations and integration by parts in the full domain to obtain estimates independent of the approximation parameters; (e) and finally we pass to the limit with the multiple approximation parameters.

For the reader's convenience, we explain here the main ideas in the case that the operator $\mathbf{S}$ is given by (1.2) and that instead of (1.1) its steady counterpart is treated. Most of the calculations are elementary, but involved, and use various well-established techniques from the regularity theory of partial differential equations. Since they are linked in a quite intricate and delicate way and one has to be careful in tracking the dependence on various parameters, we sketch the proof now and then develop a full theory in the next sections.

A fundamental step in the approximation of general operators by ones with linear growth dates back to [32], where generalized Newtonian fluids are treated. The results proved there are obtained by using for $A \geq 1$ the following approximation[1] $\mathbf{S}^A$ defined via

$$\mathbf{S}^A(\mathbf{P}) = \begin{cases} (\delta + |\mathbf{P}^{\mathrm{sym}}|)^{p-2}\mathbf{P}^{\mathrm{sym}} & \text{if } |\mathbf{P}| \leq A, \\ c_2 \mathbf{P}^{\mathrm{sym}} + c_1 & \text{if } |\mathbf{P}| > A, \end{cases}$$

with appropriately chosen constants $c_i = c_i(A, \delta, p)$ to ensure an appropriate regularity of the stress tensor $\mathbf{S}^A$. Hence, the tensor $\mathbf{S}^A$ grows linearly for large $\mathbf{P}$. This can be also restated by writing that

$$\mathbf{S}^A(\mathbf{P}) := \frac{(\omega^A)'(|\mathbf{P}^{\mathrm{sym}}|)}{|\mathbf{P}^{\mathrm{sym}}|}\mathbf{P}^{\mathrm{sym}},$$

[1] The precise form of the approximation in [32] is slightly different, since there the potential of the stress tensor was depending on $|\mathbf{D}\mathbf{u}|^2$, instead of $|\mathbf{D}\mathbf{u}|$ here.

where $\omega^A : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is a regular N-function such that $(\omega^A)'(0) = 0$, $(\omega^A)'(t) = (\delta + t)^{p-2}t$ for $t \leq A$ and $(\omega^A)'(t) = c_2 t + c_1$ for $t > A$.

***Remark 1.1*** In Sect. 2 we will show that -roughly speaking- once the results is established for this explicit example, then it can be extended to a rather wide class of nonlinear operators.

To obtain results for the original problem we first consider the approximate problem

$$\begin{aligned} -\operatorname{div} \mathbf{S}^A(\mathbf{D}\mathbf{u}^A) &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u}^A &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

For regular enough $\mathbf{f}$ one can directly prove the existence of weak solutions satisfying

$$\int_\Omega |\mathbf{F}^A(\mathbf{D}\mathbf{u}^A)|^2 \, d\mathbf{x} \leq C,$$

where

$$\mathbf{F}^A(\mathbf{P}) := \sqrt{\frac{(\omega^A)'(|\mathbf{P}^{\mathrm{sym}}|)}{|\mathbf{P}^{\mathrm{sym}}|}} \mathbf{P}^{\mathrm{sym}}.$$

Note that $|\mathbf{F}^A(\mathbf{P})|^2 \sim \delta^2 + |\mathbf{P}^{\mathrm{sym}}|^2$, with constants depending on $A$. The special role of the quantity

$$a^A(t) := \frac{(\omega^A)'(t)}{t},$$

is evident from the definitions of $\mathbf{S}^A$ and $\mathbf{F}^A$.

The estimates for the second order spatial derivatives are obtained by using the difference quotient technique in the interior and along tangential directions (after appropriate localization of the equations). Once this step is done, one gets that the equations are satisfied almost everywhere. Thus, the equations can be used point-wise to determine (by ellipticity) estimations in the direction normal to the boundary. The outcome of this procedure, which is typical for second order elliptic equations, leads to the estimates (cf. Propositions 3.15, 3.16)

$$\delta^{p-2} \int_{\Omega_0} |\nabla \mathbf{D}\mathbf{u}^A|^2 \, d\mathbf{x} \leq \int_{\Omega_0} |\nabla \mathbf{F}^A(\mathbf{D}\mathbf{u}^A)|^2 \, d\mathbf{x} \leq C_1 \quad \forall\, \Omega_0 \subset\subset \Omega,$$
$$\delta^{p-2} \int_{\Omega} |\nabla \mathbf{D}\mathbf{u}^A|^2 \, d\mathbf{x} \leq \int_{\Omega} |\nabla \mathbf{F}^A(\mathbf{D}\mathbf{u}^A)|^2 \, d\mathbf{x} \leq C_2(A),$$

where the constant $C_1$ is independent of $A$. In addition, one gets that also tangential derivatives are regular up to the boundary with a bound independent of $A$. Note that the linear growth of the operator $\mathbf{S}^A$ results in an $L^2$-setting, which allows us to use the classical Korn inequality and to handle the dependence of the operator on the symmetric gradient (instead of on the full gradient) in the equations. An important feature of this step is that the proved regularity is sufficient to justify the following step and to remove the dependence on $A$ in the estimates in the direction normal to the boundary.

This is achieved by testing the equations locally near the boundary by second order derivatives in the normal direction, and adapting a method introduced by Seregin and Shilkin [37] for $1 < p < 2$ (cf. [12, 13]). This results in the estimate (cf. Propositions 3.20, 3.21)

$$\delta^{p-2} \int_{\Omega} |\nabla \mathbf{D}\mathbf{u}^A|^2 \, d\mathbf{x} \leq \int_{\Omega} |\nabla \mathbf{F}^A(\mathbf{D}\mathbf{u}^A)|^2 \, d\mathbf{x} \leq C_3,$$

for some $C_3$ which is independent of $A$.

The final step is the passage to the limit $A \to \infty$. By uniform boundedness it directly follows that $\mathbf{F}^A(\mathbf{Du}^A)$ has a weak limit $\widehat{\mathbf{F}} \in W^{1,2}(\Omega)$ and by using also the uniform bound on second order derivatives, it follows that $\mathbf{Du}^A \to \mathbf{Du}$ almost everywhere. Combining these two information, the definition of $\mathbf{F}^A$, and the lower semi-continuity of the norm it follows that

$$\widehat{\mathbf{F}} = \lim_{A\to\infty} \mathbf{F}^A(\mathbf{Du}^A) = \mathbf{F}(\mathbf{Du}) \ \text{ weakly in } W^{1,2}(\Omega) \ \text{ and a.e. in } \Omega,$$
$$\int_\Omega |\nabla \mathbf{F}(\mathbf{Du})|^2 \, d\mathbf{x} \le C_3.$$

It remains to prove that $\mathbf{u}$ is the unique solution of the steady version the original problem (1.1). From the construction of $\mathbf{S}^A$ follows $\mathbf{S}^A(\mathbf{P}) \to \mathbf{S}(\mathbf{P})$ for every $\mathbf{P} \in \mathbb{R}^{3\times 3}$. This fact, coupled with the almost everywhere convergence of $\mathbf{Du}^A$, implies that

$$\lim_{A\to\infty} \mathbf{S}^A(\mathbf{Du}^A(\mathbf{x})) \to \mathbf{S}(\mathbf{Du}(\mathbf{x})) \quad a.e. \ \mathbf{x} \in \Omega,$$

which is nevertheless *not* enough to infer directly that

$$\lim_{A\to\infty} \int_\Omega \mathbf{S}^A(\mathbf{Du}^A) \cdot \mathbf{Dw} \, d\mathbf{x} = \int_\Omega \mathbf{S}(\mathbf{Du}) \cdot \mathbf{Dw} \, d\mathbf{x} \quad \forall \, \mathbf{w} \in C_0^\infty(\Omega),$$

and to pass to the limit in the weak formulation. To this end we need -for instance- additionally an uniform bound on $\mathbf{S}^A(\mathbf{Du}^A)$ in $L^q(\Omega)$ for some $q > 1$. This implies that $\mathbf{S}^A(\mathbf{Du}^A) \rightharpoonup \widehat{\mathbf{S}}$ in $L^q(\Omega)$, and that the limit can be identified as $\widehat{\mathbf{S}} = \mathbf{S}(\mathbf{Du})$, by a classical result.

Observe that from the definition of $\mathbf{S}^A$ it follows (cf. Proposition 2.29, Lemmas 2.32, 2.35) that

$$|\mathbf{S}^A(\mathbf{Du}^A)| \le \begin{cases} c\,(\delta^{p-1} + |\mathbf{Du}^A|^{p-1}) & p > 2, \\ c\,\delta^{p-2}|\mathbf{Du}^A| & 1 < p \le 2, \end{cases}$$

while the proved estimate $\mathbf{F}(\mathbf{Du}^A) \in W^{1,2}(\Omega)$, which is uniformly with respect to $A$, implies by Sobolev embedding (in three-dimensions) that $\|\mathbf{F}(\mathbf{Du}^A)\|_6 \le C$. Using the properties of $\mathbf{F}^A$, it follows that (cf. Proposition 2.29, Lemmas 2.32, 2.35)

$$\begin{aligned} \|\mathbf{Du}^A\|_6 &\le C \qquad p > 2, \\ \|\mathbf{Du}^A\|_{3p} &\le C \quad 1 < p \le 2. \end{aligned}$$

Hence we get that $\mathbf{S}^A(\mathbf{Du}^A)$ is bounded uniformly in $L^{6/(p-1)}(\Omega)$ for $p > 2$ and in $L^{3p}(\Omega)$ for $1 < p \le 2$, which implies that the above argument to pass to the limit in the weak formulation works only for $1 < p < 7$.

To remove the restriction $p < 7$ in the regularity result[2] we introduce and perform a *multiple approximation* of the operator, which is roughly speaking the following: for given decreasing sequences $p > q_1 > q_2 > \cdots > q_N =: 2$ and $A_N > A_{N-1} > \cdots > A_1 \ge 1$ we

[2] The restriction depends on the space dimension and it is more stringent in the time-evolution case, due to different parabolic embedding results.

set

$$
\mathbf{S}^N(\mathbf{P}) := \begin{cases} (\delta + |\mathbf{P}^{\mathrm{sym}}|)^{p-2}\mathbf{P}^{\mathrm{sym}} & \text{if } |\mathbf{P}| \le A_1, \\ c_{2,q_1}|\mathbf{P}^{\mathrm{sym}}|^{q_1-2}\mathbf{P}^{\mathrm{sym}} + c_{1,q_1} & \text{if } A_1 < |\mathbf{P}| \le A_2, \\ \vdots & \vdots \\ c_{2,q_{N-1}}|\mathbf{P}^{\mathrm{sym}}|^{q_{N-1}-2}\mathbf{P}^{\mathrm{sym}} + c_{1,q_{N-1}} & \text{if } A_{N-1} < |\mathbf{P}| \le A_N, \\ c_{2,q_N}\mathbf{P}^{\mathrm{sym}} + c_{1,q_N} & \text{if } A_N < |\mathbf{P}|, \end{cases}
$$

where the various constants $c_{i,m}$ are chosen such that the operator $\mathbf{S}^N$ belongs to the class $C^1$. If the exponents $q_n$ are chosen such that

$$
\frac{3q_n}{q_{n-1}} > 1 \quad n = 1, \dots, N,
$$

it is possible to perform the limiting process step by step, sending to infinity $A_N$ (with $A_n$ for $n \le N-1$ fixed), then taking the limit $A_{N-1} \to \infty$ with the previous ones fixed, and so on. This procedure requires to prove the precise dependence of the lower and the upper bounds of the multiple approximation with respect to the parameters[3] $A_n$.

*Plan of the paper* The analysis of the approximate operators is the content of Sect. 2 of the paper, where the procedure is carried out with full details for general operators, derived from a potential and having $(p, \delta)$-structure. Moreover, for the derivation of the estimates for second derivatives, one also has to handle precisely the behavior of the related operators $\mathbf{F}^n$. In particular, we will see that a peculiar role is played by handling tensors derived from a potential $U$ satisfying $U'(t)/t \sim U''(t)$, which we call *balanced*. This allows us to reduce many of the estimations to computable explicit cases, cf. Remark 1.1.

Next, in Sect. 3 the existence and regularity for solution of the approximate problems is treated in detail. Particular care is given to the full justification of the calculations: the results are rather natural from a formal point of view, while the rigorous treatment of all integrals needs certain approximations and the application of difference quotients, in order to be sure that we do not work with infinite quantities. First, some $A_n$ dependent estimates are proved, in order to justify manipulating the system (1.4) point-wise and then to derive uniform estimates by (improved) generalized energy methods. The limiting process is carried out in the more technical parabolic case, using space-time compactness results and convergences (at the price of a more restrictive choice of the parameters $q_n$).

## 2 Nonlinear operators and N-functions

The goal of this section is to define an approximation, which possesses nice properties, for operators appearing in (1.1). The approximation is inspired by [32], while the proof of its properties is close to [36]. However, our notions are defined slightly different, which simplifies and shortens the argumentation.

### 2.1 Notation

We use $c, C$ to denote generic constants, which may change from line to line, but are not depending on the crucial quantities. Moreover, we write $f \sim g$ if and only if there exists constants $c, C > 0$ such that $c\, f \le g \le C\, f$.

[3] Moreover, some care has to be taken in the choice of the $A_n$ to ensure monotonicity of the resulting potentials.

For a bounded, sufficiently smooth domain $\Omega \subset \mathbb{R}^3$ we use the customary Lebesgue spaces $(L^p(\Omega), \|\,.\,\|_p)$, $p \in [1,\infty]$, and Sobolev spaces $(W^{k,p}(\Omega), \|\,.\,\|_{k,p})$, $p \in [1,\infty]$, $k \in \mathbb{N}$. We use the notation $(f, g) = \int_\Omega f g \, d\mathbf{x}$, whenever the right-hand side is well defined. We do not distinguish between scalar, vector-valued or tensor-valued function spaces in the notation if there is no danger of confusion. However, we denote scalar functions by roman letters, vector-valued functions by small boldfaced letters and tensor-valued functions by capital boldfaced letters. If the norms are considered on a set $M$ different from $\Omega$, this is indicated in the respective norms as $\|\,.\,\|_{p,M}$, $\|\,.\,\|_{k,p,M}$. We equip $W_0^{1,p}(\Omega)$ (based on the Poincaré lemma) with the gradient norm $\|\nabla\,.\,\|_p$. We denote by $|M|$ the 3-dimensional Lebesgue measure of a measurable set $M$. As usual the gradient of a vector field $\mathbf{v} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is denoted as $\nabla \mathbf{v} = (\partial_i v^j)_{i,j=1,2,3} = (\partial_i \mathbf{v})_{i=1,2,3}$, while its symmetric part is denoted as $\mathbf{Dv} := \frac{1}{2}\big(\nabla \mathbf{v} + \nabla \mathbf{v}^\top\big)$. The derivative of functions defined on tensors, i.e., $U : \mathbb{R}^{3\times 3} \to \mathbb{R}$, is denoted as $\partial U = (\partial_{ij} U)_{i,j=1,2,3}$ where $\partial_{ij}$ are the partial derivatives with respect to the canonical basis of $\mathbb{R}^{3\times 3}$.

### 2.2 N-functions

We start with a discussion of some non-trivial properties of N-functions that we need in the sequel. For a detailed discussion of Orlicz spaces and N-functions we refer to [27, 33, 35, 36].

**Definition 2.1** (*N-function and regular N-function*) A function $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is called N-function if $\varphi$ is continuous, convex, strictly positive for $t > 0$, and satisfies[4]

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

If $\varphi$ additionally belongs to $C^1(\mathbb{R}^{\geq 0}) \cap C^2(\mathbb{R}^{>0})$ and satisfies $\varphi''(t) > 0$ for all $t > 0$, we call $\varphi$ a regular N-function.

The use of regular N-functions is sufficient for our purposes. Thus, in the rest of the paper we restrict ourselves to this case. For a treatment in the general situation we refer to the above mentioned literature. Note that for a regular N-function we have $\varphi(0) = \varphi'(0) = 0$. Moreover, $\varphi'$ is increasing and $\lim_{t\to\infty} \varphi'(t) = \infty$.

The following notion plays an important role in the sequel.

**Definition 2.2** ($\Delta_2$-*condition*) A non-decreasing function $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is said to satisfy the $\Delta_2$-condition if for some constant $K \geq 2$ it holds

$$\varphi(2t) \leq K\varphi(t) \quad \forall\, t \geq 0. \tag{2.1}$$

We write $\varphi \in \Delta_2$ if $\varphi$ satisfies the $\Delta_2$-condition. The $\Delta_2$-constant of $\varphi$, denoted by $\Delta_2(\varphi)$, is the smallest constant $K \geq 2$ satisfying (2.1).

We have the following results.

**Lemma 2.3** *For a regular N-function $\varphi$ the following properties are satisfied:*

(i) *For all $t \geq 0$ there holds*

$$\varphi(t) \leq \varphi'(t)t \leq \varphi(2t).$$

[4] In the following we use the convention that $\frac{\varphi'(0)}{0} := 0$.

(ii) *If* $\varphi \in \Delta_2$, *then we have for all* $t \geq 0$

$$\varphi(t) \leq \varphi'(t)t \leq \Delta_2(\varphi)\,\varphi(t).$$

(iii) *It holds that* $\varphi \in \Delta_2$ *if and only if* $\varphi' \in \Delta_2$. *In this situation we have* $\Delta_2(\varphi) \leq 2\Delta_2(\varphi') \leq (\Delta_2(\varphi))^2$.

***Proof*** Assertion (i) is contained in [36,Lemma 5.1]. Assertion (ii) follows from (i). Assertion (iii) is proved in [36,Lemma 5.2]. □

For a regular N-function $\varphi$ we define the *complementary function* $\varphi^*$ by

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\,ds.$$

It is easily seen from this definition, using elementary properties of inverse functions (cf. proof of [36,Lemma 6.4]), that $\varphi^*$ is again a regular N-function. We have the following versions of Young inequality.

**Lemma 2.4** *(*Young type inequalities*) Let the regular N-function* $\varphi$ *be such that* $\varphi, \varphi^* \in \Delta_2$. *Then, for all* $t, u \geq 0$ *there holds*

$$\begin{aligned} tu &\leq \varepsilon\,\varphi(t) + (\Delta_2(\varphi^*))^M\,\varphi^*(u), \\ tu &\leq \varepsilon\,\varphi^*(t) + (\Delta_2(\varphi))^M\,\varphi(u), \\ t\varphi'(u) &\leq \varepsilon\,\varphi(t) + \Delta_2(\varphi)\,(\Delta_2(\varphi^*))^M\,\varphi^*(u), \\ \varphi'(t)u &\leq \varepsilon\,\varphi^*(t) + (\Delta_2(\varphi))^N\,\varphi(u) \end{aligned}$$

*for all* $\varepsilon \in (0, 1)$, $M \in \mathbb{N}$ *such that* $\varepsilon^{-1} \leq 2^M$, *and* $N \in \mathbb{N}$ *such that* $\Delta_2(\varphi)\,\varepsilon^{-1} \leq 2^N$.

***Proof*** The first two inequalities follow immediately from the classical Young inequality

$$tu \leq \varphi(t) + \varphi^*(u),$$

$\varphi, \varphi^* \in \Delta_2$, and $\psi(\varepsilon\,t) \leq \varepsilon\,\psi(t)$, valid for all convex functions $\psi$, $t \geq 0$ and $\varepsilon \in (0, 1)$. The last two inequalities follow from the first ones and the equivalence

$$(\Delta_2(\varphi^*))^{-1}\varphi(t) \leq \varphi^*(\varphi'(t)) \leq \Delta_2(\varphi)\,\varphi(t), \tag{2.2}$$

valid for all $t \geq 0$ (cf. [36,(5.17)]). □

In the study of nonlinear problems like (1.1) and of N-functions the property (2.3) below plays a fundamental role. To keep the presentation shorter we call functions satisfying it "balanced function".

**Definition 2.5** (*Balanced function*) We call a regular N-function $\varphi$ *balanced*, if there exist constants $\gamma_1 \in (0, 1]$ and $\gamma_2 \geq 1$ such that for all $t > 0$ there holds

$$\gamma_1\,\varphi'(t) \leq t\,\varphi''(t) \leq \gamma_2\,\varphi'(t). \tag{2.3}$$

The constants $\gamma_1$ and $\gamma_2$ are called *characteristics* of the balanced N-function $\varphi$, and will be denoted as $(\gamma_1, \gamma_2)$.

This property transmits itself to complementary functions.

**Lemma 2.6** *Let $\varphi$ be a balanced N-function with characteristics $(\gamma_1, \gamma_2)$. Then, the complementary N-function $\varphi^*$ is a balanced N-function with characteristics $(\gamma_2^{-1}, \gamma_1^{-1})$.*

***Proof*** The assertion is proved in [36,Lemma 6.4]. The proof uses only the condition (2.3), and the formula for the derivative of the inverse function applied to $(\varphi^*)'(t) = (\varphi')^{-1}(t)$. □

Balanced N-functions always satisfy the $\Delta_2$-condition (cf. [8]).

**Lemma 2.7** *For a balanced N-function $\varphi$ we have that $\varphi, \varphi^* \in \Delta_2$. In particular, for all $t \geq 0$ there holds*

$$\varphi(2t) \leq 2^{\gamma_2+1}\, \varphi(t),$$
$$\varphi^*(2t) \leq 2^{\frac{1}{\gamma_1}+1}\, \varphi^*(t),$$

*i.e., the $\Delta_2$-constants of $\varphi$ and $\varphi^*$ possess an upper bound depending only on $\gamma_1$ and $\gamma_2$.*

***Proof*** From condition (2.3) it follows for all $t > 0$ that

$$\frac{d}{dt}\log(\varphi'(t)) = \frac{\varphi''(t)}{\varphi'(t)} \leq \gamma_2 \frac{1}{t},$$

which implies by integration with respect to $t$ over $(s, 2s)$, $s > 0$, and using the exponential function that

$$\frac{\varphi'(2s)}{\varphi'(s)} \leq 2^{\gamma_2}.$$

A further integration with respect to $s$ over $(0, t)$, $t > 0$, proves, for all $t > 0$, that

$$\varphi(2t) \leq 2^{\gamma_2+1}\varphi(t),$$

showing the assertion for $\varphi$. The assertion for $\varphi^*$ follows analogously by using Lemma 2.6. □

**Corollary 2.8** *For a balanced N-function $\varphi$ we have*

$$\varphi(t) \sim \varphi'(t)\, t \sim \varphi''(t)\, t^2 \quad \text{for all } t > 0,$$

*with constants of equivalence depending only on the characteristics of $\varphi$.*

***Proof*** This follows immediately from Lemmas 2.3 and 2.7 since $\varphi$ is balanced. □

**Lemma 2.9** *Let $\varphi$ a balanced N-function with characteristics $(\gamma_1, \gamma_2)$. Let $U \in C^1(\mathbb{R}^{\geq 0}) \cap C^2(\mathbb{R}^{>0})$ with $U(0) = U'(0) = 0$ satisfy for some $c_0, c_1 > 0$ and for all $t > 0$*

$$c_0\, \varphi''(t) \leq U''(t) \leq c_1\, \varphi''(t). \tag{2.4}$$

*Then, also $U$ is a balanced N-function with characteristics $(\gamma_2 \frac{c_0}{c_1}, \gamma_1 \frac{c_1}{c_0})$, which satisfies for all $t \geq 0$*

$$\begin{aligned} c_0\, \varphi'(t) &\leq U'(t) \leq c_1\, \varphi'(t), \\ c_0\, \varphi(t) &\leq U(t) \leq c_1\, \varphi(t). \end{aligned} \tag{2.5}$$

***Proof*** The inequalities in (2.5) follow from (2.4) by integration using that $U'(0) = \varphi'(0) = 0$. From (2.5) and (2.3) it follows that $U$ is a balanced N-function with characteristics as indicated in the assertion. □

It turns out that the function $a_\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, defined for regular N-functions $\varphi$ via

$$a_\varphi(t) := \frac{\varphi'(t)}{t}, \tag{2.6}$$

plays an important role in the investigation of problem (1.1).

**Lemma 2.10** *Let $\varphi$ be a regular N-function such that $\varphi, \varphi^* \in \Delta_2$. Then, for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}$ there holds*

$$a_\varphi(|\mathbf{P}| + |\mathbf{P} - \mathbf{Q}|) \sim \int_0^1 a_\varphi(|\theta\, \mathbf{P} + (1-\theta)\mathbf{Q}|)\, d\theta,$$

*with constants of equivalence depending only on $\Delta_2(\varphi)$ and $\Delta_2(\varphi^*)$.*

***Proof*** This follows immediately from [36,Lemma 6.6] by using Lemma 2.3, the convexity of $\varphi$, $\varphi \in \Delta_2$, and $2^{-1}(|\mathbf{P}| + |\mathbf{Q}|) \leq |\mathbf{P}| + |\mathbf{P} - \mathbf{Q}| \leq 2(|\mathbf{P}| + |\mathbf{Q}|)$. □

It is convenient to introduce for all $p \in (1, \infty)$ and all $\delta \in [0, \infty)$ the function $\omega_{p,\delta} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ via

$$\omega(t) = \omega_{p,\delta}(t) := \int_0^t (\delta + s)^{p-2} s\, ds \quad \forall\, t \geq 0,$$

which is precisely the N-function associated with the definition of the tensor $\mathbf{S}$ from (1.2). If $p$ and $\delta$ are fixed we often simply write $\omega(t) := \omega_{p,\delta}(t)$. Nevertheless, we will track the possible dependence of constants in terms of these two parameters. Clearly, $\omega_{p,\delta}$ is a regular N-function for all $p \in (1, \infty)$ and all $\delta \in [0, \infty)$. The advantage is that we have exact control of all relevant constants for these functions. We have the following basic properties.

**Lemma 2.11** *For any $\delta \in [0, \infty)$ and for any $p \in (1, \infty)$ there holds*

$$\begin{aligned} \omega_{p,\delta}(t) \leq (\omega_{p,\delta})'(t)\, t \ &\leq 2^{p+1} \omega_{p,\delta}(t) \quad \forall\, t \geq 0, \\ \min\{1, p-1\}\, (\omega_{p,\delta})'(t) \leq (\omega_{p,\delta})''(t)\, t &\leq \max\{1, p-1\}\, (\omega_{p,\delta})'(t) \quad \forall\, t > 0. \end{aligned} \tag{2.7}$$

*In particular, $\omega_{p,\delta}$, $p \in (1, \infty)$, $\delta \geq 0$, are balanced N-functions with characteristics* $(\min\{1, p-1\}, \max\{1, p-1\})$ *and $\Delta_2$-constants depending only on $p$. Moreover, by the previous results also $(\omega_{p,\delta})^*$ are balanced N-functions with characteristics* $(\min\{1, (p-1)^{-1}\}, \max\{1, (p-1)^{-1}\})$ *and $\Delta_2$-constants depending only on $p$.*

***Proof*** The first assertion in (2.7) follows from Lemma 2.3 (ii) and [36,Lemma 5.3], since $\Delta_2((\omega_{p,0})') = 2^{p-1}$. The second assertion (2.7) follows from direct computations. □

## 2.3 Nonlinear operators with $(p, \delta)$-structure

In this section we collect the main results on nonlinear operators derived from a potential and having $(p, \delta)$-struct- ure.

**Definition 2.12** (*Operator derived from a potential*) We say that an operator $\mathbf{S} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\text{sym}}$ is *derived from a potential* $U : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, if $\mathbf{S}(\mathbf{0}) = \mathbf{0}$ and for all $\mathbf{P} \in \mathbb{R}^{3\times 3} \setminus \{\mathbf{0}\}$ there holds[5]

$$\mathbf{S}(\mathbf{P}) = \partial U(|\mathbf{P}^{\mathrm{sym}}|) = \frac{U'(|\mathbf{P}^{\mathrm{sym}}|)}{|\mathbf{P}^{\mathrm{sym}}|}\,\mathbf{P}^{\mathrm{sym}} = a_U(|\mathbf{P}^{\mathrm{sym}}|)\,\mathbf{P}^{\mathrm{sym}}$$

for some $U \in C^1(\mathbb{R}^{\geq 0}) \cap C^2(\mathbb{R}^{>0})$ satisfying $U(0) = U'(0) = 0$.

***Remark 2.13*** For ease of notation, in many cases we will also write $\mathbf{S} = \partial U$ for an operator derived from the potential $U$ and note that from its definition it follows that $\mathbf{S}(\mathbf{P}) = \mathbf{S}(\mathbf{P}^{\mathrm{sym}})$, for all $\mathbf{P} \in \mathbb{R}^{3\times 3}$.

Note also that we consider the operator $\mathbf{S}$ with domain $\mathbb{R}^{3\times 3}$, since we study the problem (1.1) in the setting of three space-dimensions. Clearly, the same definition and results below can be applied to a general operator defined on $\mathbb{R}^{d\times d}$, with $d \geq 2$.

***Remark 2.14*** Note that in investigations of the regularity of solution of (1.1), or its steady analogues, for operators derived from a potential $U$, the lower and upper bounds of the quantity

$$\frac{(a_U)'(t)\,t}{a_U(t)} = \frac{U''(t)\,t}{U'(t)} - 1,$$

play an important role (cf. the discussion in [3, 17–19].

If $U$ is a balanced N-function these bounds are closely related to the characteristics $(\gamma_1, \gamma_2)$ of $U$. In fact, we have

$$\gamma_1 - 1 \leq \frac{(a_U)'(t)\,t}{a_U(t)} \leq \gamma_2 - 1.$$

**Definition 2.15** (*Operator with $\varphi$-structure*) Let the operator $\mathbf{S}\colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$, belonging to $C^0(\mathbb{R}^{3\times 3}; \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \cap C^1(\mathbb{R}^{3\times 3} \setminus \{\mathbf{0}\}; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, satisfy $\mathbf{S}(\mathbf{P}) = \mathbf{S}\big(\mathbf{P}^{\mathrm{sym}}\big)$ and $\mathbf{S}(\mathbf{0}) = \mathbf{0}$.

We say that $\mathbf{S}$ has *$\varphi$-structure* if there exist a regular N-function $\varphi$ and constants $\gamma_3 \in (0, 1]$, $\gamma_4 > 1$ such that the inequalities

$$\sum_{i,j,k,l=1}^{3} \partial_{kl} S_{ij}(\mathbf{P})\,Q_{ij}\,Q_{kl} \geq \gamma_3\, a_\varphi(|\mathbf{P}^{\mathrm{sym}}|)\,|\mathbf{P}^{\mathrm{sym}}|^2, \tag{2.8a}$$

$$\big|\partial_{kl} S_{ij}(\mathbf{P})\big| \leq \gamma_4\, a_\varphi(|\mathbf{P}^{\mathrm{sym}}|), \tag{2.8b}$$

are satisfied for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}$ with $\mathbf{P}^{\mathrm{sym}} \neq \mathbf{0}$ and all $i, j, k, l = 1, 2, 3$. The constants $\gamma_3$, $\gamma_4$, and $\Delta_2(\varphi)$ are called the *characteristics* of $\mathbf{S}$ and will be denoted by $(\gamma_3, \gamma_4, \Delta_2(\varphi))$.

In the special case $\varphi = \omega_{p,\delta}$ with $p \in (1, \infty)$ and $\delta \in [0, \infty)$ we say that $\mathbf{S}$ has $(p, \delta)$-*structure* and call $(\gamma_3, \gamma_4, p)$ its characteristics.

Closely related to an operator with $\varphi$-structure is the function $\mathbf{F}_\varphi\colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ defined via

$$\mathbf{F}_\varphi(\mathbf{P}) := \sqrt{a_\varphi(|\mathbf{P}^{\mathrm{sym}}|)}\,\mathbf{P}^{\mathrm{sym}} = \frac{\sqrt{\varphi'(|\mathbf{P}^{\mathrm{sym}}|)|\mathbf{P}^{\mathrm{sym}}|}}{|\mathbf{P}^{\mathrm{sym}}|}\,\mathbf{P}^{\mathrm{sym}}, \tag{2.9}$$

[5] Here we use the notation (2.6) also for a more general function $U$ (not necessarily a balanced or even a regular N-function).

where the second representation holds only for $\mathbf{P}^{\text{sym}} \neq \mathbf{0}$. However, this form is convenient since it shows that $\mathbf{F}_\varphi$ is derived from the potential

$$\psi(t) := \int_0^t \sqrt{\varphi'(s)s}\, ds. \tag{2.10}$$

It is easily seen that $\psi \in C^1(\mathbb{R}^{\geq 0}) \cap C^2(\mathbb{R}^{>0})$. In the special case of an operator $\mathbf{S}$ with $(p, \delta)$-structure we have with $\omega = \omega_{p,\delta}$

$$\mathbf{F}(\mathbf{P}) := \mathbf{F}_\omega(\mathbf{P}) = \sqrt{a_\omega(|\mathbf{P}^{\text{sym}}|)}\, \mathbf{P}^{\text{sym}} = \big(\delta + |\mathbf{P}^{\text{sym}}|\big)^{\frac{p-2}{2}} \mathbf{P}^{\text{sym}}, \tag{2.11}$$

which is consistent with the notation used in the previous literature, as explained in the introduction, cf. (1.3).

To derive a very important result for operators with $\varphi$-structure we need the following result, which explains also the link (and the choice of a similar name) between the characteristics of a balanced N-function $\varphi$, and the characteristics of an operator derived from a potential $\varphi$.

**Proposition 2.16** *Let $\varphi$ be a balanced N-function with characteristics $(\gamma_1, \gamma_2)$. Let $\mathbf{T} = \partial\varphi$ be derived from the potential $\varphi$. Then, $\mathbf{T}$ has $\varphi$-structure with characteristics depending only on $\gamma_1$ and $\gamma_2$.*

***Proof*** It follows from Lemma 2.7 that the $\Delta_2$-constant of $\varphi$ depends only on $\gamma_2$. We have for all $\mathbf{P} \in \mathbb{R}^{3\times 3}$ with $\mathbf{P}^{\text{sym}} \neq \mathbf{0}$

$$\partial_{kl} T_{ij}(\mathbf{P}) = \frac{\varphi'(|\mathbf{P}^{\text{sym}}|)}{|\mathbf{P}^{\text{sym}}|} \left( \delta^{\text{sym}}_{ij,kl} - \frac{P^{\text{sym}}_{ij} P^{\text{sym}}_{kl}}{|\mathbf{P}^{\text{sym}}|^2} \right) + \varphi''(|\mathbf{P}^{\text{sym}}|)\, \frac{P^{\text{sym}}_{ij} P^{\text{sym}}_{kl}}{|\mathbf{P}^{\text{sym}}|^2}, \tag{2.12}$$

where $\delta^{\text{sym}}_{ij,kl} := \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Using (2.3) we obtain from this, for all $j, k, l, m$,

$$\big|\partial_{kl} T_{ij}(\mathbf{P})\big| \leq 2\, \frac{\varphi'(|\mathbf{P}^{\text{sym}}|)}{|\mathbf{P}^{\text{sym}}|} + \varphi''(|\mathbf{P}^{\text{sym}}|) \leq (2 + \gamma_2)\, a_\varphi(|\mathbf{P}^{\text{sym}}|),$$

which proves (2.8b). From (2.12), (2.3), and $\gamma_1 \leq 1$ we obtain for $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}$ with $\mathbf{P}^{\text{sym}} \neq \mathbf{0}$

$$\begin{aligned}
&\sum_{i,j,k,l=1}^{3} \partial_{kl} T_{ij}(\mathbf{P}) Q_{ij} Q_{kl} \\
&= \frac{\varphi'(|\mathbf{P}^{\text{sym}}|)}{|\mathbf{P}^{\text{sym}}|} \left( |\mathbf{Q}^{\text{sym}}|^2 - \frac{|\mathbf{P}^{\text{sym}} \cdot \mathbf{Q}^{\text{sym}}|^2}{|\mathbf{P}^{\text{sym}}|^2} \right) + \varphi''(|\mathbf{P}^{\text{sym}}|)\, \frac{|\mathbf{P}^{\text{sym}} \cdot \mathbf{Q}^{\text{sym}}|^2}{|\mathbf{P}^{\text{sym}}|^2} \\
&\geq \gamma_1\, \frac{\varphi'(|\mathbf{P}^{\text{sym}}|)}{|\mathbf{P}^{\text{sym}}|} \left( |\mathbf{Q}^{\text{sym}}|^2 - \frac{|\mathbf{P}^{\text{sym}} \cdot \mathbf{Q}^{\text{sym}}|^2}{|\mathbf{P}^{\text{sym}}|^2} \right) + \gamma_1\, \frac{\varphi'(|\mathbf{P}^{\text{sym}}|)}{|\mathbf{P}^{\text{sym}}|}\, \frac{|\mathbf{P}^{\text{sym}} \cdot \mathbf{Q}^{\text{sym}}|^2}{|\mathbf{P}^{\text{sym}}|^2} \\
&= \gamma_1\, a_\varphi(|\mathbf{P}^{\text{sym}}|)\, |\mathbf{Q}^{\text{sym}}|^2,
\end{aligned}$$

which proves (2.8a). □

We can now formulate the following crucial result for our investigations (cf. [36,Section 6]).

**Proposition 2.17** *Let $\varphi$ be a balanced N-function with characteristics $(\gamma_1, \gamma_2)$. Let $\mathbf{S}$ have $\varphi$-structure with characteristics $(\gamma_3, \gamma_4, \Delta_2(\varphi))$ and let $\mathbf{F}_\varphi$ be defined in (2.9). Then, we have for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}$ that*

$$\big(\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})\big) \cdot (\mathbf{P} - \mathbf{Q}) \sim a_\varphi(|\mathbf{P}^{\mathrm{sym}}| + |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|)\, |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|^2 \tag{2.13}$$

$$\sim |\mathbf{F}_\varphi(\mathbf{P}) - \mathbf{F}_\varphi(\mathbf{Q})|^2, \tag{2.14}$$

$$|\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})| \sim a_\varphi(|\mathbf{P}^{\mathrm{sym}}| + |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|)\, |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|, \tag{2.15}$$

*where the constants of equivalence depend only on $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_4$.*

***Proof*** First of all note that, due to Lemmas 2.7 and 2.6, $\varphi$ and $\varphi^*$ satisfy the $\Delta_2$-condition with $\Delta_2$-constants depending only on $\gamma_1$ and $\gamma_2$.

Using (2.8) and Lemma 2.10 we get that for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}$ with $\mathbf{P}^{\mathrm{sym}} \neq \mathbf{0}$

$$\begin{aligned}
&\big(\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})\big) \cdot (\mathbf{P} - \mathbf{Q}) \\
&= \int_0^1 \sum_{i,j,k,l=1}^{3} \partial_{kl} S_{ij}\big(\theta\mathbf{P} + (1-\theta)\mathbf{Q}\big)(P-Q)_{ij}(P-Q)_{kl}\, d\theta \\
&\sim \int_0^1 a_\varphi(|\theta\mathbf{P}^{\mathrm{sym}} + (1-\theta)\mathbf{Q}^{\mathrm{sym}}|)\, d\theta\ |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|^2 \\
&\sim a_\varphi(|\mathbf{P}^{\mathrm{sym}}| + |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|)\, |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|^2,
\end{aligned}$$

which proves (2.13) with constants of equivalence depending only on $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_4$. From (2.13) we immediately obtain, also using that $\mathbf{S}$ is symmetric,

$$\begin{aligned}
a_\varphi(|\mathbf{P}^{\mathrm{sym}}| + |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|)\, |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|^2 &\le c\,\big(\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})\big) \cdot (\mathbf{P} - \mathbf{Q}) \\
&\le c\,|\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})||\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|,
\end{aligned}$$

with constants depending only on $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_4$. This proves one inequality in (2.15). The other follows from

$$\begin{aligned}
|\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})| &= \bigg(\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}\int_0^1 \partial_{kl} S_{ij}\big(\theta\mathbf{P} + (1-\theta)\mathbf{Q}\big)(P-Q)_{kl}\, d\theta\Big)^2\bigg)^{\frac{1}{2}} \\
&\le c \int_0^1 a_\varphi(|\theta\mathbf{P}^{\mathrm{sym}} + (1-\theta)\mathbf{Q}^{\mathrm{sym}}|)\, d\theta\ |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}| \\
&\le c\, a_\varphi(|\mathbf{P}^{\mathrm{sym}}| + |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|)\, |\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|,
\end{aligned}$$

with constants depending only on $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_4$. Here, we used again (2.8b), the symmetry of $\partial_{kl} S_{ij}$ with respect to $k, l$, and Lemma 2.10.

To show (2.14) we use that $\mathbf{F}_\varphi$ defined in (2.9) possesses $\psi$-structure, where $\psi$ is defined in (2.10). We have using (2.3), for all $t > 0$, that

$$\psi''(t)\, t = \frac{\big(\varphi''(t)\, t + \varphi'(t)\big)\, t}{2\sqrt{\varphi'(t)\, t}} \sim \frac{\varphi'(t)\, t}{\sqrt{\varphi'(t)\, t}} = \psi'(t).$$

This shows that $\psi$ is a balanced N-function with characteristics $(\frac{1+\gamma_1}{2}, \frac{1+\gamma_2}{2})$. Thus, Proposition 2.16 yields that $\mathbf{F}_\varphi$ has $\psi$-structure with characteristics depending only on $\gamma_1$ and $\gamma_2$.

The already proven equivalence (2.15) reads in this case as

$$|\mathbf{F}_\varphi(\mathbf{P}) - \mathbf{F}_\varphi(\mathbf{Q})|^2 \sim \big(a_\psi(|\mathbf{P}^{\rm sym}| + |\mathbf{P}^{\rm sym} - \mathbf{Q}^{\rm sym}|)\big)^2 \, |\mathbf{P}^{\rm sym} - \mathbf{Q}^{\rm sym}|^2, \tag{2.16}$$

with constants of equivalence depending only on $\gamma_1$ and $\gamma_2$. From the definition of $\psi$ we get, for all $t > 0$,

$$\big(a_\psi(t)\big)^2 = \left(\frac{\sqrt{\varphi'(t)\, t}}{t}\right)^2 = a_\varphi(t),$$

which together with (2.16) yields (2.14). This finishes the proof. □

Let us finish this section by proving a useful result for the operator occurring in (1.1).

**Proposition 2.18** *Let the operator* $\mathbf{T} = \partial U$*, derived from a potential* $U$*, have* $\varphi$*-structure, with characteristics* $(\gamma_3, \gamma_4, \Delta_2(\varphi))$*. If* $\varphi$ *is a balanced N-function with characteristics* $(\gamma_1, \gamma_2)$*, then* $U$ *is a balanced N-function satisfying for all* $t > 0$

$$\frac{\gamma_3}{\gamma_2}\varphi''(t) \le U''(t) \le \frac{\gamma_4}{\gamma_1}\varphi''(t). \tag{2.17}$$

*The characteristics of* $U$ *is equal to* $\big(\frac{\gamma_3}{\gamma_4}\frac{\gamma_1^2}{\gamma_2}, \frac{\gamma_4}{\gamma_3}\frac{\gamma_2^2}{\gamma_1}\big)$.

***Proof*** For $\mathbf{P} = \frac{t}{\sqrt{3}}\mathbf{Id}$, $t > 0$, $\mathbf{Q} = \frac{1}{\sqrt{3}}\mathbf{Id}$ we get $|\mathbf{P}| = t$, $|\mathbf{Q}| = 1$. Thus, (2.8a), (2.8b), and the definition of $a_\varphi$ yield

$$\gamma_3 \frac{\varphi'(t)}{t} \le \sum_{i,j,k,l=1}^{3} \partial_{kl} T_{ij}(\mathbf{P}) Q_{ij} Q_{kl} = U''(t) \le \gamma_4 \frac{\varphi'(t)}{t}.$$

This implies (2.17), since $\varphi$ is balanced. The remaining assertions follow from Lemma 2.9. □

***Remark 2.19*** Proposition 2.18 states that $U$ is a balanced N-function with characteristics depending only on the characteristics of $\mathbf{S}$ and on the characteristics of $\varphi$. Consequently, Lemmas 2.7 and 2.6 yield that $U$ and $U^*$ satisfy the $\Delta_2$-condition, with $\Delta_2$-constants depending only on the characteristics of $\mathbf{S}$ and the characteristics of $\varphi$.

## 2.4 Approximations of a nonlinear operator

We now define the $(A, q)$-approximation and prove the relevant properties, needed in the sequel. Note that the $(A, q)$-approximation in the special case $p \ge 2$ and $q = 2$ was introduced in a slightly different form in [32] (in that reference the potential depends on $|\mathbf{P}^{\rm sym}|^2$). The idea behind is that the operator induced by the $(A, q)$-approximation for $q = 2$ has linear growth at infinity (cf. [32,Lemma 2.22]) and consequently, one can work on the level of this $(A, 2)$-approximation within the standard Hilbertian theory.

**Definition 2.20** $(A, q)$-(*approximation of a scalar real function*) Given a function $U \in C^1(\mathbb{R}^{\ge 0}) \cap C^2(\mathbb{R}^{>0})$ satisfying $U(0) = U'(0) = 0$ we define for $A \ge 1$ and $q \ge 2$ the $(A, q)$*-approximation* $U^{A,q} \in C^1(\mathbb{R}^{\ge 0}) \cap C^2(\mathbb{R}^{>0})$ via

$$U^{A,q}(t) := \begin{cases} U(t) & t \le A, \\ \alpha_{2,q}\, t^q + \alpha_{1,q}\, t + \alpha_{0,q} & t > A. \end{cases}$$

Consequently, the constants $\alpha_{i,q} = \alpha_{i,q}(U)$, $i = 0, 1, 2$, are given by

$$\alpha_{2,q} = \frac{1}{q(q-1)}\,\frac{U''(A)}{A^{q-2}},$$
$$\alpha_{1,q} = U'(A) - \frac{1}{q-1}\,U''(A)\,A,$$
$$\alpha_{0,q} = U(A) - U'(A)\,A + \frac{1}{q}U''(A)\,A^2.$$

***Remark 2.21*** If $\varphi$ is a regular N-function and $q = 2$, the definition of the $(A, 2)$-approximation $\varphi^{A,2}$ and the properties of $\varphi$ immediately imply that there exists a constant $c(A, \varphi)$ such that for all $t \geq 0$ there holds

$$a_{\varphi^{A,2}}(t) = \frac{(\varphi^{A,2})'(t)}{t} \leq c(A, \varphi).$$

Next, we define the $(A, q)$-approximation of an operator derived from a potential.

**Definition 2.22** ($(A, q)$-*approximation of an operator derived from a potential*) Let the operator $\mathbf{S} = \partial U$ be derived from the potential $U$. Then, we define for given $A \geq 1$ and $q \geq 2$ the $(A, q)$-*approximation* $\mathbf{S}^{A,q} := \partial U^{A,q}$ of $\mathbf{S}$ as the operator derived from the potential $U^{A,q}$, i.e., $\mathbf{S}^{A,q}$ satisfies $\mathbf{S}^{A,q}(\mathbf{0}) = \mathbf{0}$ and for all $\mathbf{P} \in \mathbb{R}^{3\times 3} \setminus \{\mathbf{0}\}$ there holds

$$\mathbf{S}^{A,q}(\mathbf{P}) := \partial U^{A,q}(|\mathbf{P}^{\mathrm{sym}}|) = \frac{(U^{A,q})'(|\mathbf{P}^{\mathrm{sym}}|)}{|\mathbf{P}^{\mathrm{sym}}|}\,\mathbf{P}^{\mathrm{sym}} = a_{U^{A,q}}(|\mathbf{P}^{\mathrm{sym}}|)\,\mathbf{P}^{\mathrm{sym}}.$$

As explained in the introduction, for an operator with $(p, \delta)$-structure, for large $p$, we need also multiple approximations, which we define now.

**Definition 2.23** (*Multiple approximation of an operator*) Let the operator $\mathbf{S}$ have $(p, \delta)$-structure for some $p \in (2, \infty)$ and $\delta \in [0, \infty)$ and let $\mathbf{S}$ be derived from the potential $U$. For given $N \in \mathbb{N}$ and $q_n \in [2, p]$, $n = 0, \ldots, N$ with $q_0 = p$, $q_N = 2$ and $q_n > q_{n+1}$, $n = 0, \ldots, N-1$, and $A_n \geq 1$, $n = 1, \ldots, N$ with $A_{n+1} \geq A_n + 1$, $n = 1, \ldots, N-1$, we set

$$U^0 := U, \quad \mathbf{S}^0 := \mathbf{S}, \quad \omega^0 := \omega_{p,\delta}, \quad \mathbf{F}^0 := \mathbf{F}_{\omega^0}, \quad a^0 := a_{\omega^0},$$

and then recursively

$$U^n := (U^{n-1})^{A_n,q_n}, \quad \mathbf{S}^n := \partial U^n, \quad \omega^n := (\omega^{n-1})^{A_n,q_n}, \quad \mathbf{F}^n := \mathbf{F}_{\omega^n}, \quad a^n := a_{\omega^n},$$

for $n = 1, \ldots, N$. We call $U^n$, $\mathbf{S}^n$, $\omega^n$, $\mathbf{F}^n$, and $a^n$, $n = 1, \ldots, N$, *multiple approximation* of $U$, $\mathbf{S}$, $\omega_{p,\delta}$, $\mathbf{F}$, and $a$, respectively.

***Remark 2.24*** As we will see later on (for the parabolic problem in three-space dimensions) strictly speaking the multiple approximation is not needed for $p \in (1, \frac{13}{3})$. Since in the definition of a multiple approximation the case $N = 1$ is included, also a single $(A, q)$-approximation is a special case of a multiple approximation. To unify the presentation we also call the $(A, 2)$-approximation for $p \in (1, \frac{13}{3})$ multiple approximation. In this case we have $U^1 = U^{A,2}$, $\mathbf{S}^1 = \mathbf{S}^{A,2} = \partial U^1$, $\omega^1 = (\omega^0)^{A,2} = (\omega_{p,\delta})^{A,2}$, $\mathbf{F}^1 = \mathbf{F}_{\omega^1}$, and $a^1 = a_{\omega^1}$.

In the following we derive various properties of multiple approximations for an operator $\mathbf{S}$ which is derived from a potential $U$ and has $(p, \delta)$-structure. In particular, we need to carefully track any possible dependence of the various constants and on the parameters $A_n$, $n = 1, \ldots, N$. We start with a single approximation, showing in particular independence of the characteristics of $\varphi^{A,q}$ on $A \geq 1$.

**Lemma 2.25** *Let $\varphi$ be a balanced N-function with characteristics $(\gamma_1, \gamma_2)$. Then, for all $A \geq 1$ and $q \geq 2$ the $(A, q)$-approximation $\varphi^{A,q}$ is a balanced N-function with characteristics $\big(\gamma_1, \max\{\gamma_2, q-1\}\big)$.*

***Proof*** By construction we have

$$\varphi^{A,q} \in C^1(\mathbb{R}^{\geq 0}) \cap C^2(\mathbb{R}^{>0}), \quad \varphi^{A,q}(0) = (\varphi^{A,q})'(0) = 0,$$
$$(\varphi^{A,q})''(t) > 0 \quad \text{for } t > 0.$$

For $t \leq A$ we have $\varphi^{A,q}(t) = \varphi(t)$, while $\frac{\varphi^{A,q}(t)}{t} = \alpha_{2,q}\, t^{q-1} + \alpha_{1,q} + \frac{\alpha_{0,q}}{t}$, for $t > A$, which implies that $\varphi^{A,q}$ is a regular N-function, since

$$\lim_{t \to 0^+} \frac{\varphi^{A,q}(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{\varphi^{A,q}(t)}{t} = \infty,$$

where we used in the first limit that $\varphi$ is an N-function. From $\varphi^{A,q}(t) = \varphi(t)$, for $t \leq A$, we get, for all $t \in (0, A]$, that

$$\gamma_1 (\varphi^{A,q})'(t) \leq (\varphi^{A,q})''(t)\, t \leq \gamma_2\, (\varphi^{A,q})'(t), \tag{2.18}$$

since $\varphi$ is balanced. On the other hand, for $t \geq A$ we have

$$\frac{(\varphi^{A,q})'(t)}{(\varphi^{A,q})''(t)\, t} = \frac{q\alpha_{2,q}\, t^{q-1} + \alpha_{1,q}}{q(q-1)\alpha_{2,q}\, t^{q-1}} =: g^A(t).$$

Observe that for fixed $A \geq 1$ there holds $\lim_{t\to\infty} g^A(t) = \frac{1}{q-1}$, while

$$g^A(A) = \frac{\varphi'(A)}{\varphi''(A)\, A} \in \left[\gamma_2^{-1}, \gamma_1^{-1}\right].$$

From $(g^A)'(t) = -\frac{1}{t^{q-2}} \frac{\alpha_{1,q}}{q\alpha_{2,q}}$ it follows that the sign of the derivative depends only on $\alpha_{1,q} = \varphi'(A) - \varphi''(A)\, A$. Thus, $g_A(t)$ is monotone. Distinguishing between $\alpha_{1,q} \geq 0$ and $\alpha_{1,q} \leq 0$, using $\gamma_1 \leq 1$ and $\gamma_2 \geq 1$, as well as (2.18), one easily sees that for all $t \geq 0$ there holds

$$\min\left\{\frac{1}{\gamma_2}, \frac{1}{q-1}\right\} \leq g^A(t) \leq \frac{1}{\gamma_1},$$

implying the assertion. □

**Corollary 2.26** *For all $A \geq 1$ and $q \geq 2$ the $(A, q)$-approximation $(\omega_{p,\delta})^{A,q}$ of $\omega_{p,\delta}$ with $p \in (1, \infty)$ and $\delta \in [0, \infty)$ is a balanced N-function with characteristics*

$$\big(\min\{1, p-1\}, \max\{1, p-1, q-1\}\big).$$

***Proof*** This follows immediately from Lemmas 2.11 and 2.25. □

We have the following analogue of Proposition 2.18 for $(A, q)$-approximations of $U$ and $\varphi$.

**Lemma 2.27** *Let $\varphi$ be a balanced N-function with characteristics $(\gamma_1, \gamma_2)$. Let the operator $\mathbf{S} = \partial U$ have $\varphi$-structure with characteristics $(\gamma_3, \gamma_4, \Delta_2(\varphi))$. For $A \geq 1$ and $q \geq 2$ let $U^{A,q}$ and $\varphi^{A,q}$ be the $(A, q)$-approximation of $U$ and $\varphi$, respectively. Then, there holds for all $t > 0$*

$$\frac{\gamma_3}{\gamma_2} (\varphi^{A,q})''(t) \leq (U^{A,q})''(t) \leq \frac{\gamma_4}{\gamma_1} (\varphi^{A,q})''(t).$$

***Proof*** By definition we have $U^{A,q}(t) = U(t)$ and $\varphi^{A,q}(t) = \varphi(t)$ for $t \le A$. Thus, the assertions for $t \le A$ follow from (2.17). For $t \ge A$ we have $(U^{A,q})''(t) = U''(A)\frac{t^{q-2}}{A^{q-2}}$ and $(\varphi^{A,q})''(t) = \varphi''(A)\frac{t^{q-2}}{A^{q-2}}$. Thus, for $t \ge A$ the assertion follows again from (2.17). □

The properties of the function $U^{A,q}$ proved in the previous lemmas allow us to show that the operator $\mathbf{S}^{A,q}$ has $\varphi^{A,q}$-structure.

**Proposition 2.28** *Let $\varphi$ be a balanced N-function with characteristics $(\gamma_1, \gamma_2)$. Let the operator $\mathbf{S} = \partial U$ have $\varphi$-structure with characteristics $(\gamma_3, \gamma_4, \Delta_2(\varphi))$. For $A \ge 1$ and $q \ge 2$ let $U^{A,q}$ and $\varphi^{A,q}$ be the $(A, q)$-approximation of $U$ and $\varphi$, respectively. Then, the operator $\mathbf{S}^{A,q} := \partial U^{A,q}$ has both $U^{A,q}$-structure and $\varphi^{A,q}$-structure, with characteristics depending only on $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and $q$.*

***Proof*** The operator $\mathbf{S}^{A,q}$ is derived from the potential $U^{A,q}$ which, according to Proposition 2.18 and Lemma 2.25, is a balanced N-function with characteristics $\big(\frac{\gamma_3}{\gamma_4}\frac{\gamma_1^2}{\gamma_2}, \max\{q-1, \frac{\gamma_4}{\gamma_3}\frac{\gamma_2^2}{\gamma_1}\}\big)$. Thus, the proof of Proposition 2.16 shows that $\mathbf{S}^{A,q}$ has $U^{A,q}$-structure with characteristics

$$\Big(\frac{\gamma_3}{\gamma_4}\frac{\gamma_1^2}{\gamma_2}, 2 + \max\Big\{q-1, \frac{\gamma_4}{\gamma_3}\frac{\gamma_2^2}{\gamma_1}\Big\}, \Delta_2(U^{A,q})\Big),$$

where $\Delta_2(U^{A,q})$ depends only on $\max\{q-1, \frac{\gamma_4}{\gamma_3}\frac{\gamma_2^2}{\gamma_1}\}$, according to Lemma 2.7. Now Lemma 2.27 yields that the operator $\mathbf{S}^{A,q}$ has $\varphi^{A,q}$-structure with characteristics

$$\Big(\frac{\gamma_3^2}{\gamma_4}\frac{\gamma_1^2}{\gamma_2^2}, \frac{\gamma_4}{\gamma_1}\Big(2 + \max\Big\{q-1, \frac{\gamma_4}{\gamma_3}\frac{\gamma_2^2}{\gamma_1}\Big\}\Big), \Delta_2(\varphi^{A,q})\Big).$$

Lemmas 2.25 and 2.7 yield that $\Delta_2(\varphi^{A,q})$ depends only on $\max\{\gamma_2, q-1\}$. This finishes the proof. □

We have the following crucial result (cf. Proposition 2.17).

**Proposition 2.29** *Let $\varphi$ be a balanced N-function with characteristics $(\gamma_1, \gamma_2)$. Let the operator $\mathbf{S} = \partial U$ have $\varphi$-structure with characteristics $(\gamma_3, \gamma_4, \Delta_2(\varphi))$. For $A \ge 1$ and $q \ge 2$ let $\varphi^{A,q}$ and $\mathbf{S}^{A,q}$ be the $(A, q)$-approximation of $\varphi$ and $\mathbf{S}$, respectively, and let $\mathbf{F}_{\varphi^{A,q}}$ be defined in (2.9). Then, we have for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}$ that*

$$\begin{aligned}(\mathbf{S}^{A,q}(\mathbf{P}) - \mathbf{S}^{A,q}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}) &\sim a_{\varphi^{A,q}}(|\mathbf{P}^{\text{sym}}| + |\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|)\,|\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|^2,\\ &\sim |\mathbf{F}_{\varphi^{A,q}}(\mathbf{P}) - \mathbf{F}_{\varphi^{A,q}}(\mathbf{Q})|^2,\\ |\mathbf{S}^{A,q}(\mathbf{P}) - \mathbf{S}^{A,q}(\mathbf{Q})| &\sim a_{\varphi^{A,q}}(|\mathbf{P}^{\text{sym}}| + |\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|)\,|\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|,\end{aligned}$$

*where the constants of equivalence depend only on $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and $q$.*

***Proof*** This is a direct consequence of Propositions 2.28 and 2.17. □

***Remark 2.30*** For the limiting processes, it is of fundamental relevance that in Proposition 2.29 the constants do not depend on $A \ge 1$.

Based on Proposition 2.29 we can prove the validity of equivalent expressions for $\nabla \mathbf{F}_{\varphi}(\mathbf{Du})$ which play a crucial role in the proof of regularity for the problem (1.1) (cf.

[12, 13, 37]). To this end, we define for a sufficiently smooth operator $\mathbf{S}: \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ the functions $\mathbb{P}_i: \mathbb{R}^{3\times 3} \to \mathbb{R}$, $i = 1, 2, 3$ via

$$\mathbb{P}_i(\mathbf{P}) := \partial_i \mathbf{S}(\mathbf{P}) \cdot \partial_i \mathbf{P} = \sum_{j,k,l,m=1}^{3} \partial_{jk} S_{lm}(\mathbf{P})\, \partial_i P_{jk}\, \partial_i P_{lm},$$

and emphasize that there is no summation over the index $i$.

If $\mathbf{S}^n$, $n \in \{1, \ldots, N\}$, is a multiple approximation of $\mathbf{S}$ we define analogously $\mathbb{P}_i^n: \mathbb{R}^{3\times 3} \to \mathbb{R}$, $i = 1, 2, 3$, for $n \in \{1, \ldots, N\}$, via

$$\mathbb{P}_i^n(\mathbf{P}) := \partial_i \mathbf{S}^n(\mathbf{P}) \cdot \partial_i \mathbf{P} = \sum_{j,k,l,m=1}^{3} \partial_{jk} S^n_{lm}(\mathbf{P})\, \partial_i P_{jk}\, \partial_i P_{lm}.$$

**Proposition 2.31** *Let the operator $\mathbf{S} = \partial U$ have $(p, \delta)$-structure for some $p \in (1, \infty)$ and $\delta \in [0, \infty)$, with characteristics $(\gamma_3, \gamma_4, p)$. For given $N \in \mathbb{N}$ let $\mathbf{S}^n$, $n \in \{1, \ldots, N\}$ be a multiple approximation of $\mathbf{S}$. If for a vector field $\mathbf{v}: \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ there holds $\mathbf{F}^n(\mathbf{Dv}) \in W^{1,2}(\Omega)$, then we have for $i = 1, 2, 3$ and a.e. in $\Omega$ the following equivalences*

$$\begin{aligned} |\partial_i \mathbf{F}^n(\mathbf{Dv})|^2 &\sim a^n(|\mathbf{Dv}|)\, |\partial_i \mathbf{Dv}|^2 \\ &\sim \mathbb{P}_i^n(\mathbf{Dv}), \\ |\partial_i \mathbf{S}^n(\mathbf{Dv})|^2 &\sim a^n(|\mathbf{Dv}|)\, \mathbb{P}_i^n(\mathbf{Dv}). \end{aligned} \tag{2.19}$$

*where the constants of equivalence depend only on $\gamma_3, \gamma_4, p$, and $q_n$, for $n = 1, \ldots, N$.*

***Proof*** For $h > 0$ and $i = 1, 2, 3$ let $\Delta_i^+ \mathbf{v}(\mathbf{x}) := \mathbf{v}(\mathbf{x} + h\,\mathbf{e}_i) - \mathbf{v}(\mathbf{x})$ and $d_i^+ \mathbf{v}(\mathbf{x}) := h^{-1} \Delta_i^+ \mathbf{v}(\mathbf{x})$ be the classical increments and difference quotients in direction $\mathbf{e}_i$ of the canonical basis. The standard theory of difference quotients (cf. [16]) and $\mathbf{F}^n(\mathbf{Dv}) \in W^{1,2}(\Omega)$ yield that $d_i^+ \mathbf{F}^n(\mathbf{Dv}) \to \partial_i \mathbf{F}^n(\mathbf{Dv})$ a.e. and in $L^2_{\mathrm{loc}}(\Omega)$ as $h \to 0$ and

$$\int_{\Omega_h} |d_i^+ \mathbf{F}^n(\mathbf{Dv})|^2\, dx \le c,$$

where the constant $c$ is independent of $h$, and where we used the notation

$$\Omega_h := \left\{ \mathbf{x} \in \Omega \,\middle|\, d(\mathbf{x}, \partial\Omega) > 2h \right\}.$$

Thus, Proposition 2.29, Lemma 2.35 for $p \le 2$, and Lemma 2.32 for $p > 2$ yield that

$$\int_{\Omega_h} |d_i^+ \mathbf{Dv}|^2\, dx \le c,$$

with $c$ independent of $h$ (even if it may depend on $\delta$ and $A_n$). Consequently, we obtain that $\mathbf{Dv} \in W^{1,2}(\Omega)$, and $d_i^+ \mathbf{Dv} \to \partial_i \mathbf{Dv}$, $\Delta_i^+ \mathbf{Dv} \to \mathbf{0}$ a.e. and in $L^2_{\mathrm{loc}}(\Omega)$, as $h \to 0$. Proposition 2.29 yields

$$|d_i^+ \mathbf{F}^n(\mathbf{Dv})(\mathbf{x})|^2 \sim a^n(|\mathbf{Dv}(\mathbf{x})| + |\Delta_i^+ \mathbf{Dv}(\mathbf{x})|)\, |d_i^+ \mathbf{Dv}(\mathbf{x})|^2,$$

which implies, using the above proved convergences, $(2.19)_1$ as $h \to 0$.

Proposition 2.28 shows that $\mathbf{S}^n$ has $\omega^n$-structure, which implies

$$\mathbb{P}_i^n(\mathbf{D}\mathbf{v}) = \sum_{j,k,l,m=1}^{3} \partial_{jk} S_{lm}^n(\mathbf{D}\mathbf{v})\, \partial_i D_{jk}\mathbf{v}\, \partial_i D_{lm}\mathbf{v} \sim a^n(|\mathbf{D}\mathbf{v}|)|\partial_i \mathbf{D}\mathbf{v}|^2,$$

showing $(2.19)_2$. To prove $(2.19)_3$ we use the definition of $\mathbb{P}_i^n$ and $(2.19)_{1,2}$ to get

$$a^n(|\mathbf{D}\mathbf{v}|)\left(\mathbb{P}_i^n(\mathbf{D}\mathbf{v})\right)^2 \le a^n(|\mathbf{D}\mathbf{v}|)\,|\partial_i \mathbf{S}^n(\mathbf{D}\mathbf{v})|^2 |\partial_i \mathbf{D}\mathbf{v}|^2 \sim \mathbb{P}_i^n(\mathbf{D}\mathbf{v})\,|\partial_i \mathbf{S}^n(\mathbf{D}\mathbf{v})|^2,$$

which implies

$$a^n(|\mathbf{D}\mathbf{v}|)\,\mathbb{P}_i^n(\mathbf{D}\mathbf{v}) \le c\,|\partial_i \mathbf{S}^n(\mathbf{D}\mathbf{v})|^2.$$

On the other hand, the fact that $\mathbf{S}^n$ has $\omega^n$-structure and $(2.19)_{1,2}$ imply that $\mathbb{P}_i(\mathbf{P}) = 0$ if and only if $\mathbf{P} = \mathbf{0}$. Consequently, we obtain

$$\begin{aligned}|\partial_i \mathbf{S}^n(\mathbf{D}\mathbf{v})|^2 &\le \sum_{k,l=1}^{3} |\partial_{kl} \mathbf{S}^n(\mathbf{D}\mathbf{v})\, \partial_i D_{kl}\mathbf{v}|^2 \le c\,\left(a^n(|\mathbf{D}\mathbf{v}|)\right)^2 |\partial_i \mathbf{D}\mathbf{v}|^2 \\ &\le c\, a^n(|\mathbf{D}\mathbf{v}|)\,|\partial_i \mathbf{F}^n(\mathbf{D}\mathbf{v})|^2 \le c\, a^n(|\mathbf{D}\mathbf{v}|)\,\mathbb{P}_i^n(\mathbf{D}\mathbf{v}).\end{aligned}$$

Note that all constants just depend on the quantities indicated in the formulation of the assertion. This yields the reverse estimate, proving $(2.19)_3$. □

To derive a priori estimates and to perform the limiting process we need estimates, which do not involve the parameters $A_n$, for $n = 1, \ldots, N$. We restrict ourselves to the case that $\varphi = \omega_{p,\delta} = \omega$ with $p \in (1, \infty)$ $\delta \in [0, \infty)$ and distinguish between the cases $p \le 2$ and $p > 2$ for the sake of a simpler presentation.

## 2.5 Some estimates specific to the case $p > 2$

In this section we prove some results, which are specific of the case $p > 2$. In particular, in the case $p \ge \frac{13}{3}$ we need multiple approximations, which makes the results more technical.

**Lemma 2.32** *For given $p > 2$, $\delta > 0$, and $N \in \mathbb{N}$ let $a^n$, $n \in \{1, \ldots, N\}$, be a multiple approximation of $a^0$. Then, there exist $\widehat{A}_n = \widehat{A}_n(\delta, p, q_1, \ldots, q_n) \ge 1$ such that for all $A_n \ge \max\{\delta, \widehat{A}_n\}$ the function $a^n$ is non-decreasing and satisfies for all $t \ge 0$*

$$\begin{aligned}\frac{1}{(p-1)2^{q_1-2}}\,\delta^{p-2} \le \frac{1}{p-1}\,\frac{\delta^{p-q_n}}{2^{q_1-2}}\, a_{\omega_{q_n,\delta}}(t) &\le a^n(t),\\ a^n(t) &\le \frac{p-1}{q_n-1}\, 2^{p-2}\,(A_{n-1})^{p-q_{n-1}}\, a_{\omega_{q_{n-1},\delta}}(t), \qquad (2.20)\\ a^n(t) &\le \frac{p-1}{q_n-1}\, 2^{p-2}\, a^0(t).\end{aligned}$$

***Proof*** For ease of presentation we show the assertion just in the first two cases, i.e., $n = 1, 2$. The remaining cases follow in the same way and are left to the interested reader.

The case $(\mathbf{n} = \mathbf{1})$ : For simplicity we set $A := A_1$ and $q := q_1$. For $t \le A$ we have $a^1(t) = a^0(t) = (\delta + t)^{p-2}$. Thus, $q_0 = p > q \ge 2$ implies

$$\delta^{p-2} \le \delta^{p-q}\,(\delta + t)^{q-2} \le (\delta + t)^{p-2},$$

and $\frac{1}{(p-1)2^{q-2}} \leq 1 \leq \frac{p-1}{q-1}(1+\delta)^{p-2}$, which proves (2.20) for $n=1$ and $t \leq A$. Moreover, $(\delta+t)^{p-2}$ is an increasing function in $t$.

For $t \geq A$ we have

$$a^1(t) = q\,\alpha_{2,q}\,t^{q-2}\big(1 + \frac{\alpha_{1,q}}{q\,\alpha_{2,q}\,t^{q-1}}\big) =: q\,\alpha_{2,q}\,t^{q-2} g^A(t), \tag{2.21}$$

where $\alpha_{i,q} = \alpha_{i,q}(\omega^0)$, $i=1,2$. The expressions for the coefficients $\alpha_{i,q}$, $i=1,2$, imply $g^A(A) = (q-1)\frac{(\omega^0)'(A)}{(\omega^0)''(A)A}$, $\lim_{t\to\infty} g^A(t) = 1$, and

$$(g^A)'(t) = \frac{\alpha_{1,q}(1-q)}{q\,\alpha_{2,q}\,t^q} = (q-1)^2\,\frac{A^{q-1}}{t^q}\Big(\frac{1}{q-1} - \frac{(\omega^0)'(A)}{(\omega^0)''(A)A}\Big). \tag{2.22}$$

From the properties of $\omega^0$ it follows that $\lim_{t\to\infty}\frac{(\omega^0)'(t)}{(\omega^0)''(t)t} = \frac{1}{q-1}$ and that $\frac{(\omega^0)'(t)}{(\omega^0)''(t)t}$ is strictly monotone increasing. Thus, for $A \geq \widehat{A}(p,q,\delta)$ the expression in the parenthesis in (2.22) is non-negative and thus $g^A$ is a non-decreasing function. Consequently, we get that $a^1$ is a non-decreasing function, since $q\,\alpha_{2,q}\,t^{q-2}$ is increasing for $q>2$ (non-decreasing for $q=2$). Using these properties and that $\omega^0$ is balanced we obtain that for $t \geq A \geq \widehat{A}$ there holds

$$\frac{q-1}{p-1} \leq g^A(t) \leq 1. \tag{2.23}$$

It remains to estimate the factor in front of $g^A$ in (2.21). From the expression for $\alpha_{2,q}$ we get that $q\,\alpha_{2,q}\,t^{q-2} = \frac{(\omega^0)''(A)}{q-1}\big(\frac{t}{A}\big)^{q-2}$. Thus, using (2.7), $t \geq A \geq \delta \geq 0$, and $2 \leq q < p$ we obtain

$$\begin{aligned} q\,\alpha_{2,q}\,t^{q-2} &\leq \frac{p-1}{q-1}\frac{(\delta+A)^{p-2}}{A^{q-2}}\,t^{q-2} = \frac{p-1}{q-1}\Big(\frac{\delta}{A}+1\Big)^{p-2} A^{p-q}\,t^{q-2} \\ &\leq \frac{p-1}{q-1}\,2^{p-2}\,A^{p-q}\,(\delta+t)^{q-2} = \frac{p-1}{q-1}\,2^{p-2}\Big(\frac{\delta+t}{A}\Big)^{q-p}(\delta+t)^{p-2} \\ &\leq \frac{p-1}{q-1}\,2^{p-2}\,(\delta+t)^{p-2}. \end{aligned} \tag{2.24}$$

For $A \geq \delta$ we get that $t \geq 2^{-1}(\delta+t)$ for all $t \geq A$. Using this, the definition of $\omega^0$, (2.7), $2 \leq q < p$ and $t \geq A \geq \delta \geq 0$ we obtain

$$\begin{aligned} q\,\alpha_{2,q}\,t^{q-2} &\geq \frac{1}{q-1}\frac{(\delta+A)^{p-2}}{A^{q-2}}\,t^{q-2} \geq \frac{1}{q-1}(\delta+A)^{p-q}\frac{1}{2^{q-2}}(\delta+t)^{q-2} \\ &\geq \frac{\delta^{p-q}}{q-1}\frac{1}{2^{q-2}}\,a_{q,\delta}(t) \geq \frac{\delta^{p-2}}{p-1}\frac{1}{2^{q-2}}. \end{aligned} \tag{2.25}$$

The inequalities (2.23)–(2.25) imply (2.20) for $n=1$ and $t \geq A \geq \max\{\delta, \widehat{A}\}$. This completes the proof for $n=1$.

($\mathbf{n=2}$) : For simplicity we set $B := A_2 \geq A_1 =: A$ and $r := q_2$, $q := q_1$. For $t \leq B$ we have $a^2(t) = a^1(t)$. Thus, $p > q > r \geq 2$ implies

$$\delta^{p-2} \leq \delta^{p-r}\,(\delta+t)^{r-2} \leq \delta^{p-q}\,(\delta+t)^{q-2},$$

which together with (2.20) for $n=1$ shows $(2.20)_1$ for $n=2$ and $t \leq B$. The estimate $(2.20)_{2,3}$ for $n=2$ and $A \leq t \leq B$ follows from $a^2(t) = a^1(t)$, (2.21), (2.23), the estimates in (2.24) and $r < q$; while for $t \le A$ it follows from $a^2(t) = a^0(t)$, $\delta \le A$, $2 \le r < q < p$, and

$$(\delta+t)^{p-2} = (\delta+t)^{p-q}\,(\delta+t)^{q-2} \le 2^{p-q}\,(\delta+t)^{q-2}.$$

For $t \ge B$ we have

$$a^2(t) = r\,\alpha_{2,r}\,t^{r-2}\Big(1 + \frac{\alpha_{1,r}}{q\,\alpha_{2,r}\,t^{r-1}}\Big) =: r\,\alpha_{2,r}\,t^{r-2}h^B(t), \tag{2.26}$$

where $\alpha_{i,r} = \alpha_{i,r}(\omega_1)$, $i = 1, 2$. The expressions of the coefficients $\alpha_{i,r}$, $1 = 1, 2$, imply $h^B(B) = (r-1)\frac{(\omega^1)'(B)}{(\omega^1)''(B)B}$, $\lim_{t\to\infty} h^B(t) = 1$, and

$$(h^B)'(t) = \frac{\alpha_{1,r}(1-r)}{r\,\alpha_{2,r}\,t^r} = (r-1)^2\,\frac{B^{r-1}}{t^r}\Big(\frac{1}{r-1} - \frac{(\omega^1)'(B)}{(\omega^1)''(B)B}\Big). \tag{2.27}$$

In the proof of Lemma 2.25 we showed that $\lim_{t\to\infty} \frac{(\omega^1)'(t)}{(\omega^1)''(t)\,t} = \frac{1}{q-1}$ and that $\frac{(\omega^1)'(t)}{(\omega^1)''(t)\,t}$ is strictly monotone. Thus, for $B \ge \widehat{B}(p, q, r, \delta)$ the expression in the parenthesis in (2.27) is non-negative and thus $h^B$ is a non-decreasing function. Consequently, we get that $a^2$ is a non-decreasing function, since $r\,\alpha_{2,r}\,t^{r-2}$ is increasing for $r > 2$. Using these properties and Lemma 2.25 for $\omega^1$ we obtain that for $t \ge B \ge \widehat{B}$ there holds

$$\frac{r-1}{p-1} \le h^B(t) \le 1. \tag{2.28}$$

It remains to estimate the factor in front of $h^B$ in (2.26). From the expressions for $\alpha_{2,r}$ and $(\omega^1)''(B)$ we get that $r\,\alpha_{2,r}\,t^{r-2} = \frac{(\omega^1)''(B)}{r-1}\big(\frac{t}{B}\big)^{r-2} = \frac{(\omega^0)''(A)}{r-1}\big(\frac{B}{A}\big)^{q-2}\big(\frac{t}{B}\big)^{r-2}$. Thus, using (2.7), the definition of $\omega^1$, $2 \le r < q < p$ and $0 \le \delta \le A \le B \le t$ we obtain

$$\begin{aligned} r\,\alpha_{2,r}\,t^{r-2} &\le \frac{p-1}{r-1}\Big(\frac{\delta}{A}+1\Big)^{p-2}\Big(\frac{t}{B}\Big)^{r-q} A^{p-q}\,t^{q-2} \\ &\le \frac{p-1}{r-1}\,2^{p-2}\,A^{p-q}\,(\delta+t)^{q-2} = \frac{p-1}{r-1}\,2^{p-2}\Big(\frac{\delta+t}{A}\Big)^{q-p}(\delta+t)^{p-2} \\ &\le \frac{p-1}{r-1}\,2^{p-2}\,(\delta+t)^{p-2}. \end{aligned}$$

For $B \ge A \ge \delta$ we get that $t \ge 2^{-1}(\delta+t)$ for all $t \ge B$. Using this, the definition of $\omega^0$, (2.7), $2 \le r < q < p$ and $t \ge B \ge A \ge \delta \ge 0$ we obtain

$$\begin{aligned} r\,\alpha_{2,r}\,t^{r-2} &\ge \frac{1}{r-1}\,\frac{(\delta+A)^{p-2}}{A^{q-2}}\,B^{q-r}\,t^{r-2} \ge \frac{1}{r-1}(\delta+A)^{p-q}\,B^{q-r}\,\frac{1}{2^{r-2}}\,(\delta+t)^{r-2} \\ &\ge \frac{\delta^{p-r}}{r-1}\,\frac{1}{2^{r-2}}\,a_{r,\delta}(t) \ge \frac{\delta^{p-2}}{r-1}\,\frac{1}{2^{r-2}}. \end{aligned}$$

The last two inequalities and (2.28) imply (2.20) for $n = 2$ and $t \ge B \ge \max\{A, \widehat{B}\}$. This completes the proof for $n = 2$. □

In the proof of regularity we will need mainly the following corollary.

**Corollary 2.33** *Let the operator* $\mathbf{S} = \partial U$*, derived from the potential* $U$*, have* $(p, \delta)$*-structure with* $p > 2$ *and* $\delta > 0$*, and characteristics* $(\gamma_3, \gamma_4, p)$*. For* $N \in \mathbb{N}$ *let* $\omega^n$, $\mathbf{F}^n$, $\mathbf{S}^n$,

*$n \in \{1, \dots, N\}$, be a multiple approximation of $\omega^0$, $\mathbf{F}^0$, $\mathbf{S}^0$. Then, for all $A_n \geq \max\{\delta, \widehat{A}_n\}$ with $\widehat{A}_n$ from Lemma 2.32 and all $t \geq 0$ there holds*

$$\begin{aligned} \frac{\delta^{p-2}}{(p-1)2^{q_1-1}}\, t^2 \leq \frac{\delta^{p-q_n}}{(p-1)2^{q_1-2}}\, \omega_{q_n,\delta}(t) \leq \omega^n(t) \leq \frac{p-1}{q_n-1}\, 2^{p-2}\, \omega^0(t), \\ \omega^n(t) \leq \frac{p-1}{q_n-1}\, 2^{p-2}\, (A_{n-1})^{p-q_{n-1}}\, \omega_{q_{n-1},\delta}(t), \\ (\omega^n)^*(t) \leq \frac{(p-1)2^{q_1-3}}{\delta^{p-2}}\, t^2. \end{aligned} \tag{2.29}$$

*Moreover, for all $\mathbf{P} \in \mathbb{R}^{3\times 3}$ there holds*

$$\begin{aligned} |\mathbf{F}^n(\mathbf{P})|^2 \sim \omega^n(|\mathbf{P}^{\mathrm{sym}}|), \\ c\,\delta^{p-q_n}\, |\mathbf{F}_{\omega_{q_n,\delta}}(\mathbf{P})|^2 \leq |\mathbf{F}^n(\mathbf{P})|^2, \\ |\mathbf{S}^n(\mathbf{P})| \leq c\,(A_{n-1})^{p-q_{n-1}}\, (\omega_{q_{n-1},\delta})'(|\mathbf{P}^{\mathrm{sym}}|) \end{aligned} \tag{2.30}$$

*with constants $c$ depending only on $\gamma_3$, $\gamma_4$, $q_n$, $q_1$ and $p$.*

***Proof*** The proof of the estimates $(2.29)_{1,2}$ is a direct application of the previous lemma, the definition in (2.6), $\omega^n(0) = \omega_{q_n,\delta}(0) = \omega_{q_{n-1},\delta}(0) = 0$ and integration.

Estimate $(2.29)_3$ follows from $(2.29)_{1,2}$ and the equivalent definition of the complementary function since

$$\begin{aligned} (\omega^n)^*(t) &:= \sup_{s\geq 0} s\,t - \omega^n(s) \\ &\leq \sup_{s\geq 0} s\,t - \frac{\delta^{p-2}}{(p-1)2^{q_1-1}}\, s^2 \\ &= \frac{(p-1)2^{q_1-3}}{\delta^{p-2}}\, t^2. \end{aligned}$$

The estimates (2.30) follow immediately from the definition of multiple approximations of $\mathbf{S}$, $\mathbf{F}$, $\omega$, $a$; Proposition 2.29 and (2.29). □

Let us finish this section with a more technical estimate needed in the proof of regularity in the case $p > 2$.

**Lemma 2.34** *For given $p > 2$, $\delta \geq 0$ and $N \in \mathbb{N}$ let $a^n$, $\omega^n$ and $n \in \{1, \dots, N\}$, be a multiple approximation of $a^0$ and $\omega^0$ with $A_n \geq \max\{\delta, 1\}$, respectively. Then, there exists a constant $c = c(p, q_1, \dots, q_n)$ such that for all $s, t \geq 0$ there holds*

$$a^n(t)\, s^2 \leq c\,\big(\delta^p + \omega^n(s) + \omega^n(t)\big). \tag{2.31}$$

***Proof*** The assertion follows essentially from Young inequality and the expressions for the coefficients of the approximations. However, we have to distinguish several cases. For ease of presentation we show the assertion just in the first two cases, i.e., $n = 1, 2$. The remaining cases follow in the same way and are left to the interested reader.

The case ($\mathbf{n = 1}$) For simplicity we set $A := A_1$ and $q := q_1$. For $t \leq A$ we have $a^1(t) = a^0(t) = (\delta + t)^{p-2}$. For $s, t \leq A$ Young inequality with $\frac{p}{2}$ and $\frac{p}{p-2}$, $\delta \geq 0$ and $p > 2$ yield

$$(\delta + t)^{p-2}\, s^2 \leq c\,\big((\delta + t)^p + s^p\big) \leq c\,\big(\delta^p + (\delta + t)^{p-2} t^2 + (\delta + s)^{p-2} s^2\big),$$

which implies (2.31) for $s, t \leq A$, since Corollary 2.8 implies

$$(\delta + t)^{p-2} t^2 \sim \omega^0(t) = \omega^1(t), \tag{2.32}$$

valid for $t \leq A$.

Next, assume that $s, t \geq A$. Since $\omega^1$ is balanced with characteristics $(1, p-1)$, the definition of $\omega^1$ implies for $t \geq A$ that there holds

$$a^1(t) = \frac{(\omega^1)'(t)}{t} \sim (\omega^1)''(t) = \frac{(\omega^0)''(A)}{A^{q-2}}\, t^{q-2}, \tag{2.33}$$

with constants of equivalence depending only on $p$. This and Young inequality with $\frac{q}{2}$ and $\frac{q}{q-2}$ imply

$$a^1(t)\, s^2 \leq c(p,q)\, \frac{(\omega^0)''(A)}{A^{q-2}} \left(t^{q-2}\, t^2 + s^{q-2}\, s^2\right) = c\left((\omega^1)''(t)\, t^2 + (\omega^1)''(s)\, s^2\right),$$

which yields (2.31) for $s, t \geq A$, since Corollary 2.8 shows $(\omega^1)''(t)\, t^2 \sim \omega^1(t)$.

Next, assume $s \leq A \leq t$. Using (2.33), Young inequality with $\frac{p}{p-2}$ and $\frac{p}{2}$, $\delta \geq 0$, (2.32), (2.33), and again $(\omega^1)''(t)\, t^2 \sim \omega^1(t)$ we obtain

$$\begin{aligned} a^1(t)\, s^2 &\leq c(p) \left( \left( \frac{(\omega^0)''(A)}{A^{q-2}} \right)^{\frac{p}{p-2}} t^{\frac{q-2}{p-2}\, p} + s^{p-2}\, s^2 \right) \\ &\leq c \left( \frac{(\omega^0)''(A)}{A^{q-2}}\, t^{q-2}\, t^2 \left( \frac{(\omega^0)''(A)}{A^{q-2}} \right)^{\frac{p}{p-2}-1} t^{\frac{q-2}{p-2}\, p-q} + \omega^1(s) \right) \\ &\leq c \left( \omega^1(t) \left( \frac{(\omega^0)''(A)}{A^{q-2}} \right)^{\frac{2}{p-2}} t^{2\frac{q-p}{p-2}} + \omega^1(s) \right). \end{aligned} \tag{2.34}$$

Using $(\omega^0)''(A) \sim (\delta + A)^{p-2}$, $\max\{1, \delta\} \leq A$ and $q < p$ we get

$$\left( \frac{(\omega^0)''(A)}{A^{q-2}} \right)^{\frac{2}{p-2}} t^{2\frac{q-p}{p-2}} \leq c\, \frac{(\delta + A)^2}{A^2}\, \frac{t^{2\frac{q-p}{p-2}}}{A^{2\frac{q-2}{p-2}-2}} \leq c \left( \frac{t}{A} \right)^{2\frac{q-p}{p-2}} \leq c,$$

which together with the last estimate implies (2.31) for $s \leq A \leq t$.

Finally, for $t \leq A \leq s$ we get

$$a^1(t)\, s^2 = (\delta + t)^{p-2}\, s^2 = (\delta + t)^{p-2} \left( \frac{(\omega^0)''(A)}{A^{q-2}} \right)^{-\frac{2}{q}} \left( \frac{(\omega^0)''(A)}{A^{q-2}} \right)^{\frac{2}{q}} s^2.$$

We use Young inequality with $\frac{q}{q-2}$ and $\frac{q}{2}$, (2.33) and again $(\omega^1)''(t)\, t^2 \sim \omega^1(t)$ to arrive at

$$a^1(t)\, s^2 \leq c(p,q) \left( (\delta + t)^{q\frac{p-2}{q-2}} \left( \frac{(\omega^0)''(A)}{A^{q-2}} \right)^{\frac{-2}{q-2}} + \frac{(\omega^0)''(A)}{A^{q-2}}\, s^{q-2}\, s^2 \right).$$

From $(\omega^0)''(A) \sim (\delta + A)^{p-2}$, $p > q \geq 2$, $t \leq A$ and $\delta \geq 0$ we obtain

$$\begin{aligned} (\delta + t)^{q\frac{p-2}{q-2}} \left( \frac{(\omega^0)''(A)}{A^{q-2}} \right)^{\frac{-2}{q-2}} &\leq c\, (\delta + t)^p\, (\delta + A)^{q\frac{p-2}{q-2} - p + 2 - 2\frac{p-2}{q-2}} \\ &\leq c\, (\delta + t)^p \leq c\, (\delta^p + (\delta + t)^{p-2}\, t^2), \end{aligned}$$

which together with (2.32), (2.33) and the last estimate yields (2.31) for $t \leq A \leq s$. This finishes the proof for $n = 1$.

The case ($\mathbf{n = 2}$) For simplicity we additionally set $B := A_2$ and $r := q_2$. For $s, t \leq B$ we have $a^2(t) = a^1(t)$. Thus, the assertion (2.31) for $n = 2$ is already proved in the case

$n = 1$ above. In the case $s, t \geq B$ we proceed exactly as in the case $s, t \geq A$ for $n = 1$ just replacing $\omega^1$, $\omega^0$, $q$ and $A$ by $\omega^2$, $\omega^1$, $r$ and $B$, respectively.

In the case $s \leq B \leq t$ we have to distinguish between $s \leq A$ and $A \leq s \leq B$. In the former case we use the analogue of (2.33) for $a^2$ and proceed as in (2.34) to arrive at

$$a^2(t)\, s^2 \leq c(p) \Big( \omega^2(t) \Big( \frac{(\omega^1)''(B)}{B^{r-2}} \Big)^{\frac{2}{p-2}} t^{2\frac{r-p}{p-2}} + \omega^2(s) \Big).$$

Using $(\omega^1)''(B) = \frac{(\omega^0)''(A)}{A^{q-2}} B^{q-2} \leq (p-1) \frac{(\delta+A)^{p-2}}{A^{q-2}} B^{q-2}$, $1 \leq A \leq B$ and $r < q < p$ we get the estimate

$$\Big( \frac{(\omega^1)''(B)}{B^{r-2}} \Big)^{\frac{2}{p-2}} t^{2\frac{r-p}{p-2}} \leq c\, \frac{(\delta + A)^2}{A^2} \Big( \frac{A}{B} \Big)^{2\frac{p-q}{p-2}} \Big( \frac{t}{B} \Big)^{2\frac{r-p}{p-2}} \leq c,$$

which together with the last estimate implies (2.31) for $s \leq A \leq B \leq t$. For $A \leq s \leq B \leq t$ we have

$$a^2(t)\, s^2 \leq \frac{(\omega^1)''(B)}{B^{r-2}}\, t^{r-2} \Big( \frac{(\omega^0)''(A)}{A^{q-2}} \Big)^{-\frac{2}{q}} \Big( \frac{(\omega^0)''(A)}{A^{q-2}} \Big)^{\frac{2}{q}} s^2.$$

Using Young inequality with $\frac{q}{q-2}$ and $\frac{q}{2}$, the analogue of (2.33) for $a^2$ and $a^1$, and $(\omega^2)''(t)\, t^2 \sim \omega^2(t)$ we get

$$\begin{aligned} a^2(t)\, s^2 &\leq c(p, q) \Big( t^{q\frac{r-2}{q-2}} \Big( \frac{(\omega^0)''(A)}{A^{q-2}} \Big)^{\frac{-2}{q-2}} \Big( \frac{(\omega^1)''(B)}{B^{r-2}} \Big)^{\frac{q}{q-2}} + \frac{(\omega^0)''(A)}{A^{q-2}} s^{q-2} s^2 \Big) \\ &\leq c \Big( \omega^2(t)\, t^{2\frac{r-q}{q-2}} \frac{A^2}{((\omega^0)''(A))^{\frac{2}{q-2}}} \Big( \frac{(\omega^1)''(B)}{B^{r-2}} \Big)^{\frac{2}{q-2}} + \omega_2(s) \Big). \end{aligned}$$

From $(\omega^1)''(B) = \frac{(\omega^0)''(A)}{A^{q-2}} B^{q-2}$, $q > r \geq 2$ and $t \geq B$ we obtain

$$t^{2\frac{r-q}{q-2}} \frac{A^2}{((\omega^0)''(A))^{\frac{2}{q-2}}} \Big( \frac{(\omega^1)''(B)}{B^{r-2}} \Big)^{\frac{2}{q-2}} = \Big( \frac{t}{B} \Big)^{2\frac{r-q}{q-2}} \leq 1,$$

which together with the last estimate yields (2.31) for $A \leq s \leq B \leq t$.

In the case $t \leq B \leq s$ we have to distinguish between $t \leq A$ and $A \leq t \leq B$. In the former case we have

$$a^2(t)\, s^2 = (\delta + t)^{p-2} \Big( \frac{B^{r-2}}{(\omega^1)''(B)} \Big)^{\frac{2}{r}} \Big( \frac{(\omega^1)''(B)}{B^{r-2}} \Big)^{\frac{2}{r}} s^2,$$

which by Young inequality with $\frac{r}{r-2}$ and $\frac{r}{2}$, and the analogue of (2.33) for $a_2$ yields

$$\begin{aligned} a^2(t)\, s^2 &\leq c(p, r) \Big( (\delta + t)^{r\frac{p-2}{r-2}} \Big( \frac{B^{r-2}}{(\omega^1)''(B)} \Big)^{\frac{2}{r-2}} + \omega^2(s) \Big) \\ &\leq c \Big( \big( \delta^p + a^0(t) \big) (\delta + t)^{2\frac{p-r}{r-2}} \frac{B^2}{((\omega^1)''(B))^{\frac{2}{r-2}}} + \omega^2(s) \Big). \end{aligned}$$

From $(\omega^1)''(B) = \frac{(\omega^0)''(A)}{A^{q-2}} B^{q-2}$, $q > r \geq 2$, $\delta \geq 0$ and $B \geq A \geq t$ we obtain

$$(\delta + t)^{2\frac{p-r}{r-2}} \frac{B^2}{((\omega^1)''(B))^{\frac{2}{r-2}}} \leq \Big( \frac{\delta + t}{\delta + A} \Big)^{2\frac{p-r}{r-2}} \Big( \frac{A}{B} \Big)^{2\frac{q-r}{r-2}} \leq 1,$$

which together with $a^0(t) = a^2(t)$, $t \le A$, and the last estimate yields (2.31) for $t \le A \le B \le s$. For $A \le t \le B \le s$ we have

$$a^2(t)\, s^2 \le \frac{(\omega^0)''(A)}{A^{q-2}}\, t^{q-2} \Big(\frac{B^{r-2}}{(\omega^1)''(B)}\Big)^{\frac{2}{r}} \Big(\frac{(\omega^1)''(B)}{B^{r-2}}\Big)^{\frac{2}{r}} s^2,$$

which, by Young inequality with $\frac{r}{r-2}$ and $\frac{r}{2}$ and (2.33) for $a^2$ and $a^1$, yields

$$\begin{aligned} a^2(t)\, s^2 &\le c(p,r) \left( t^{r\frac{q-2}{r-2}} \frac{((\omega^0)''(A))^{\frac{r}{r-2}}}{A^{r\frac{q-2}{r-2}}} \frac{B^2}{((\omega^1)''(B))^{\frac{2}{r-2}}} + \omega^2(s) \right) \\ &\le c \left( \omega^2(t)\, t^{2\frac{q-r}{r-2}} \frac{((\omega^0)''(A))^{\frac{2}{r-2}}}{A^{2\frac{q-2}{r-2}}} \frac{B^2}{((\omega^1)''(B))^{\frac{2}{r-2}}} + \omega^2(s) \right). \end{aligned}$$

Using $(\omega^0)''(A) \sim (\delta + A)^{p-2}$ and $(\omega^1)''(B) = \frac{(\omega^0)''(A)}{A^{q-2}} B^{q-2}$, $q > r \ge 2$ and $B \ge t$ we obtain

$$t^{2\frac{q-r}{r-2}} \frac{((\omega^0)''(A))^{\frac{2}{r-2}}}{A^{2\frac{q-2}{r-2}}} \frac{B^2}{((\omega^1)''(B))^{\frac{2}{r-2}}} \le c \Big(\frac{t}{B}\Big)^{2\frac{q-r}{r-2}} \le 1,$$

which together with the last estimate yields (2.31) for $A \le t \le B \le s$. This finishes the proof of the case $n = 2$. □

### 2.6 Some estimates specific of the case $1 < p \le 2$

For completeness we deduce estimates for the case $p \in (1, 2]$, which are the counterpart of those proved in the previous section and which can be used to prove the regularity results also in the case $p \in (1, 2]$. Note that in this case it is enough to use a single approximation with $q = 2$.

**Lemma 2.35** *For $p \in (1, 2]$ and $\delta > 0$ let $\omega = \omega_{p,\delta}$ and $a = a_\omega$. For $A \ge 1$ and $q = 2$ we set $\omega^A := \omega^{A,2}$ and $a^A := a_{\omega^{A,2}}$. Then, the function $a^A$ is non-increasing and satisfies for all $t \ge 0$*

$$\begin{aligned} (p-1)\, a(t) &\le a^A(t) \le \delta^{p-2}, \\ (p-1)\, (\delta + A)^{p-2} &\le a^A(t). \end{aligned} \tag{2.35}$$

***Proof*** The statement is clear for $t \le A$ using $a^A(t) = a(t) = (\delta + t)^{p-2}$, $0 \le \delta, t \le A$, and $p \le 2$. Moreover, $(\delta + t)^{p-2}$ is a non-increasing function in $t$.

For $t \ge A$ we have $a^A(t) = \omega''(A) + \frac{\omega'(A) - \omega''(A)\, A}{t}$. Thus, we get that $a^A(A) = (\delta + A)^{p-2}$, $\lim_{t\to\infty} a^A(t) = (\delta + A)^{p-3} \big(\delta + (p-1)A\big)$ and $(a^A)'(t) = -\frac{\omega'(A) - \omega''(A)\, A}{t^2} \le 0$ in view of (2.7), and $p \le 2$, hence proving that $a^A$ is non-increasing also for $t > A$ (contrary to the case $p > 2$ we do not have any restriction on the choice of $A$). This yields

$$(\delta + A)^{p-2} \ge a^A(t) \ge (\delta + A)^{p-3}((p-1)A + \delta) \ge (p-1)\, (\delta + A)^{p-2},$$

which implies the assertions using $\delta^{p-2} \ge (\delta + A)^{p-2}$ and $(\delta + A)^{p-2} \ge (\delta + t)^{p-2}$ in view of $t \ge A$, and $p \le 2$. □

As in the case $p > 2$ we would need mainly the following corollary in the proof of regularity.

**Corollary 2.36** *Let the operator* $\mathbf{S}$*, derived from the potential* $U$*, have* $(p, \delta)$*-structure for some* $p \in (1, 2]$ *and* $\delta > 0$*, with characteristics* $(\gamma_3, \gamma_4, p)$*. Denote* $\omega = \omega_{p,\delta}$*,* $\mathbf{F} = \mathbf{F}_\omega$ *and for* $A \geq 1$ *set* $\omega^A := \omega^{A,2}$*,* $\mathbf{F}^A := \mathbf{F}_{\omega^A}$ *and* $\mathbf{S}^A := \mathbf{S}^{A,2}$*. Then, there holds for all* $t \geq 0$ *that*

$$
\begin{aligned}
&(p-1)\,\omega(t) \leq \omega^A(t) \leq \frac{\delta^{p-2}}{2}\,t^2,\\
&\frac{p-1}{2}\,(\delta + A)^{p-2}\,t^2 \leq \omega^A(t),\\
&(\omega^A)^*(t) \leq (p-1)\,(\Delta_2(\omega^*))^M\,\omega^*(t),
\end{aligned}
\tag{2.36}
$$

*where* $M \in \mathbb{N}_0$ *is chosen such that* $(p-1)^{-1} \leq 2^M$*. Moreover, for all* $\mathbf{P} \in \mathbb{R}^{3\times 3}$ *there holds*

$$
\begin{aligned}
&|\mathbf{F}^A(\mathbf{P})|^2 \sim \omega^A(|\mathbf{P}^{\mathrm{sym}}|),\\
&c\,|\mathbf{F}(\mathbf{P})|^2 \leq |\mathbf{F}^A(\mathbf{P})|^2,\\
&|\mathbf{S}^A(\mathbf{P})| \leq c\,\delta^{p-2}|\mathbf{P}^{\mathrm{sym}}|,
\end{aligned}
\tag{2.37}
$$

*with constants* $c$ *depending only on* $\gamma_3, \gamma_4$*, and* $p$*.*

***Proof*** Assertions $(2.36)_{1,2}$ follow from $(2.35)_{1,2}$, the definition of $a, a^A$, $\omega(0) = \omega^A(0) = 0$ and integration. Using the first inequality in $(2.36)_1$ we get for all $t \geq 0$ that

$$
\begin{aligned}
(\omega^A)^*(t) &= \sup_{s\geq 0}\; s\,t - \omega^A(s)\\
&\leq (p-1)\,\sup_{s\geq 0}\; s\,\frac{t}{p-1} - \omega(s)\\
&= (p-1)\,\omega^*\Big(\frac{t}{p-1}\Big) \leq (p-1)\,(\Delta_2(\omega^*))^M\,\omega^*(t),
\end{aligned}
$$

with $M \in \mathbb{N}_0$ as chosen above. This proves $(2.36)_2$. The inequalities in (2.37) follow from Proposition 2.29 with $\mathbf{Q} = \mathbf{0}$, the definition of $a^A$, the fact that $\omega, \omega^A$ are balanced, $(2.36)_1$, the equivalences for $\mathbf{F}$ and $\mathbf{S}$ in Proposition 2.17, and Lemma 2.35. □

## 3 On the existence and uniqueness of regular solutions

In this section we prove our main result, namely Theorem 3.4, i.e., the existence and uniqueness of regular solutions of (1.1), solely based on appropriate assumptions on the regularity (but not on the size) of the data. From now on we will restrict to the case $p > 2$, but with a few (but non completely trivial) changes the same arguments can be applied also to the case $p \in (1, 2]$, where a single approximation would be enough. Even if the theory of approximation gives a unified approach valid for all $p$, we decided to focus on the case $p > 2$ since many estimates should be changed, starting already from the a priori estimates and we think that explaining the steps that need to be changed in the case $p \leq 2$ would fragment the presentation in such a way that the readability of the paper would be much more difficult. Since the result in the case $p \leq 2$ is already contained in [12, 13, 37], using a different approximation, we preferred to skip them. Nevertheless, they will be presented in a forthcoming paper [14].

**Definition 3.1** (*Regular solution*) Let the operator $\mathbf{S}$ in (1.1), derived from a potential $U$, have $(p, \delta)$-structure for some $p \in (1, \infty)$, and $\delta \in [0, \infty)$ fixed but arbitrary. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2,1}$ boundary, and let $I = (0, T)$, $T \in (0, \infty)$, be a finite time interval. Then, we say that $\mathbf{u}$ is a regular solution of (1.1) if $\mathbf{u} \in L^p(I; W_0^{1,p}(\Omega))$ satisfies for all $\psi \in C_0^\infty(0,T)$ and all $\mathbf{w} \in W_0^{1,p}(\Omega)$

$$\int_0^T \Big(\frac{\partial \mathbf{u}(t)}{\partial t}, \mathbf{w}\Big)\,\psi(t) + (\mathbf{S}(\mathbf{Du}(t)), \mathbf{Dw})\,\psi(t)\,dt = \int_0^T (\mathbf{f}(t), \mathbf{w})\,\psi(t)\,dt,$$

and fulfils

$$\begin{aligned} \mathbf{u} &\in L^\infty(I; W_0^{1,2}(\Omega)) \cap W^{1,2}(I; L^2(\Omega)), \\ \mathbf{F}(\mathbf{Du}) &\in L^\infty(I; L^2(\Omega)) \cap L^2(I; W^{1,2}(\Omega)), \end{aligned}$$

***Remark 3.2*** Note that we are focusing on the "natural" second order spatial regularity, especially we are proving that $\mathbf{F}(\mathbf{Du}) \in L^2(I; W^{1,2}(\Omega))$. In the parabolic case it is possible, at the price of some more restrictive hypotheses on the data, also to prove that $\mathbf{F}(\mathbf{Du}) \in W^{1,2}(I; L^2(\Omega))$. This result can be obtained independently on what we prove later on and nevertheless implies also some simplifications of the argument concerning the treatment of the time derivative. The regularity of $\frac{\partial \mathbf{F}(\mathbf{Du})}{\partial t}$ would be needed in case of time-discretization to prove optimal convergence results, as done in [13].

***Remark 3.3*** To formulate clearly the dependence on the data in the various estimates we introduce the quantity

$$|||\mathbf{u}_0, \mathbf{f}|||^2 := \int_\Omega |\mathbf{u}_0|^2 + \omega(|\mathbf{Du}_0|)\,d\mathbf{x} + \int_0^T \int_\Omega |\mathbf{f}|^2\,d\mathbf{x}\,dt.$$

Using the equivalences $\omega_{p,\delta}(t) + \delta^p \sim t^p + \delta^p$ and $\omega^*(t) \sim (\delta^{p-1} + t)^{p'-2} t^2$, valid for all $p \in (1,\infty)$, $t, \delta \geq 0$ with constants of equivalence just depending on $p$, together with Korn and Poincarè inequalities, one easily checks that $|||\mathbf{u}_0, \mathbf{f}|||$ is finite if $\mathbf{u}_0 \in W_0^{1,p}(\Omega)$ and $\mathbf{f} \in L^2(I \times \Omega)$, provided that $p \geq \frac{6}{5}$.

We can now state the main result of this paper.

**Theorem 3.4** *Let the operator* $\mathbf{S}$ *in* (1.1)*, derived from a potential* $U$*, have* $(p,\delta)$*-structure for some* $p \in (2,\infty)$*, and*[6] $\delta \in (0,\infty)$ *fixed but arbitrary. Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with* $C^{2,1}$ *boundary, and let* $I = (0,T)$*,* $T \in (0,\infty)$*, be a finite time interval. Assume that* $\mathbf{u}_0 \in W_0^{1,p}(\Omega)$ *and* $\mathbf{f} \in L^2(I \times \Omega)$*.*

*Then, the system* (1.1) *has a unique regular solution with norms depending only on the characteristics of* $\mathbf{S}$*,* $\delta^{-1}$*,* $T$*,* $\Omega$*, and* $|||\mathbf{u}_0, \mathbf{f}|||$*.*

To prove Theorem 3.4 we use an approximate problem, obtained by replacing the operator $\mathbf{S} = \partial U$ with $(p,\delta)$-structure by the last item $\mathbf{S}^N = \partial U^N$ of a special multiple approximation $\mathbf{S}^n$, $n = 1, \ldots, N$, of $\mathbf{S}$, i.e.,

$$\mathbf{S}^N(\mathbf{P}) = \frac{(\omega^N)'(|\mathbf{P}^{\mathrm{sym}}|)}{|\mathbf{P}^{\mathrm{sym}}|}\mathbf{P}^{\mathrm{sym}} = a^N(|\mathbf{P}^{\mathrm{sym}}|)\mathbf{P}^{\mathrm{sym}},$$

which we define now.

[6] For a discussion of the case $\delta = 0$ we refer to Remark 3.17.

**Definition 3.5** (*Special multiple approximation*) Let the operator $\mathbf{S} = \partial U$, derived from a potential $U$, have $(p, \delta)$-structure for some $p > 2$ and $\delta > 0$ with characteristics $(\gamma_3, \gamma_4, p)$. We call $\mathbf{S}^n$, $U^n$, $\omega^n$, and $a^n$, $n = 1, \ldots, N$, a special multiple approximation of $\mathbf{S}$, $U$, $\omega_{p,\delta}$, and $a_{\omega_{p,\delta}}$ if it is a multiple approximation generated by $N := \lceil \frac{p-2}{2} \rceil$, exponents

$$q_n := p - 2n \quad \text{for } n = 1, \ldots, N-1 \ \text{ and } \ q_N := 2,$$

and parameters $A_n$, $n = 1, \ldots, N$ satisfying the conditions in Definition 2.23 and in Corollary 2.33.

***Remark 3.6*** (i) Let $\mathbf{S}^n$, $n = 1, \ldots, N$, be a special multiple approximation as in Definition 3.5. Lemma 2.11 and a successive application of Proposition 2.28, and Lemma 2.7 yields that for each $n = 1, \ldots, N$ the operator $\mathbf{S}^n$ has $\omega^n$-structure with characteristics depending only on the characteristics of $\mathbf{S}$, i.e., on $\gamma_3, \gamma_4, p$, due to the special choice of $q_j$, $j = 1, \ldots, N$. The special choice of $q_j$, $j = 1, \ldots, N$, Lemmas 2.11 and 2.25 imply that the characteristics of $\omega^n$, $n = 1, \ldots, N$, depends only on $p$. Thus, the constants in Proposition 2.31 as well as in Proposition 2.29 and Corollary 2.33 applied to $\mathbf{S}^n$, $n = 1, \ldots, N$, depend only on the characteristics of $\mathbf{S}$.

(ii) In view of (i), Lemma 2.32, and Remark 2.21 the operator $\mathbf{S}^N$ has $(2, \delta)$-structure with characteristics depending on $p, \gamma_3, \gamma_4, \omega^{N-1}$, and $A_N$.

In view of the previous remark we can work in the $W^{1,2}$-setting, which is sufficient to justify all forthcoming computations, which is the main reason for the introduction of these approximations.

### 3.1 The approximate problem and some global regularity in time

We have the following result on existence and uniqueness of solutions with certain additional regularity of the approximate problem.

**Proposition 3.7** *Let the operator $\mathbf{S} = \partial U$, derived from the potential $U$, have $(p, \delta)$-structure for some $p \in (2, \infty)$ and $\delta \in (0, \infty)$. Assume that $\mathbf{u}_0 \in W_0^{1,p}(\Omega)$ and $\mathbf{f} \in L^2(I \times \Omega)$. Let $\mathbf{S}^N$ be the last item of a special*[7] *multiple approximation $\mathbf{S}^n$, $n = 1, \ldots, N$, of $\mathbf{S}$ as in Definition 3.5. Then, the approximate problem*

$$\begin{aligned} \frac{\partial \mathbf{u}^N}{\partial t} - \operatorname{div} \mathbf{S}^N(\mathbf{D}\mathbf{u}^N) &= \mathbf{f} && \text{in } I \times \Omega, \\ \mathbf{u}^N &= \mathbf{0} && \text{on } I \times \partial\Omega, \\ \mathbf{u}^N(0) &= \mathbf{u}_0 && \text{in } \Omega, \end{aligned} \tag{3.1}$$

*possesses a unique strong solution $\mathbf{u}^N$, i.e., $\mathbf{u}^N \in W^{1,2}(I; L^2(\Omega))$ with $\mathbf{F}^N(\mathbf{D}\mathbf{u}^N) \in L^\infty(I; L^2(\Omega))$, which satisfies for all $\psi \in C_0^\infty(0, T)$ and all $\mathbf{w} \in W_0^{1,2}(\Omega)$*

$$\int_0^T \Big(\frac{\partial \mathbf{u}^N(t)}{\partial t}, \mathbf{w}\Big)\, \psi(t) + (\mathbf{S}^N(\mathbf{D}\mathbf{u}^N(t)), \mathbf{D}\mathbf{w})\, \psi(t)\, dt = \int_0^T (\mathbf{f}(t), \mathbf{w})\, \psi(t)\, dt. \tag{3.2}$$

[7] Observe that the results of this section, prior to the passage to the limit, are in fact valid for any sequence of parameter $q_n$ as described in the definition of the multiple approximation.

*In addition, the solution* $\mathbf{u}^N$ *satisfies the estimate*

$$\operatorname{esssup}_{t\in I} \|\mathbf{u}^N(t)\|_2^2 + \|\mathbf{F}^N(\mathbf{D}\mathbf{u}^N(t))\|_2^2 + \delta^{p-2}\|\nabla\mathbf{u}^N(t)\|_2^2 \\ + \delta^{p-2}\int_0^T \|\mathbf{D}\mathbf{u}^N(t)\|_2^2\,dt + \int_0^T \left\|\frac{\partial\mathbf{u}^N(t)}{\partial t}\right\|_2^2 dt \le C\left(1 + |||\mathbf{u}_0,\mathbf{f}|||^2\right), \tag{3.3}$$

*with* $C$ *depending only on the characteristics of* $\mathbf{S}$, $\delta^{p-2}$, *and* $\Omega$.

***Proof*** The proof is based on a standard Faedo-Galerkin approximation of (3.1). The existence of solutions of the Galerkin approximations follows from the standard Carathéodory theory for systems of ordinary differential equations. As pointed out in Remark 3.6, the operator $\mathbf{S}^N$ has $(2,\delta)$ structure, hence the system can be treated essentially as the heat equation. In particular, once the existence of the Galerkin solution $\mathbf{u}_k^N$, $k \in \mathbb{N}$, is obtained, passing to the limit as $k \to \infty$ can be done within the standard theory of evolutionary problems with monotone operators.

Since this is a standard procedure, we just derive the a priori estimates necessary for it.

The first a priori estimate, derived by using $\mathbf{u}_k^N$ as test function in the Galerkin approximation for $\mathbf{u}_k^N$, is the following one:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_k^N\|_2^2 + c\,\|\mathbf{F}^N(\mathbf{D}\mathbf{u}_k^N)\|_2^2 &\le c_\varepsilon \int_\Omega (\omega^N)^*(|\mathbf{f}|)\,d\mathbf{x} + \varepsilon \int_\Omega \omega^N(|\mathbf{u}_k^N|)\,d\mathbf{x} \\ &\le c_\varepsilon \int_\Omega (\omega^N)^*(|\mathbf{f}|)\,d\mathbf{x} + \varepsilon\, C \int_\Omega \omega^N(|\mathbf{D}\mathbf{u}_k^N|)\,d\mathbf{x},\end{aligned}$$

where we used in the first line Proposition 2.29 with $\mathbf{Q} = \mathbf{0}$ together with Young inequality, and in the second line

$$\int_\Omega \omega^N(|\mathbf{u}_k^N|)\,d\mathbf{x} \le C_P \int_\Omega \omega^N(|\nabla\mathbf{u}_k^N|)\,d\mathbf{x} \le C_P C_K \int_\Omega \omega^N(|\mathbf{D}\mathbf{u}_k^N|)\,d\mathbf{x},$$

which follows from modular versions of Poincarè and Korn inequalities in Orlicz spaces (cf. [7, 22, 38]). Moreover, we absorb the last term on the right-hand side of the previous estimate using $\int_\Omega \omega^N(|\mathbf{D}\mathbf{u}_k^N|)\,d\mathbf{x} \sim \|\mathbf{F}^N(\mathbf{D}\mathbf{u}_k^N)\|_2^2$ in view of Corollary 2.33. Note that all constants are independent of $A_n$, $n = 1, \ldots, N$, and depend only on the characteristics of $\mathbf{S}$ and on $\Omega$, due to Remark 3.6 and [7, 22, 38]. Moreover, from Corollary 2.33 and the definition of $|||\mathbf{u}_0,\mathbf{f}|||$, it follows that

$$\int_0^T\int_\Omega (\omega^N)^*(|\mathbf{f}|)\,d\mathbf{x}\,ds \le c(p)\,\delta^{p-2}\,|||\mathbf{u}_0,\mathbf{f}|||^2 < \infty.$$

Hence, after the limiting procedure $k \to \infty$ we arrive at

$$\operatorname{esssup}_{t\in I}\|\mathbf{u}^N(t)\|_2^2 + \int_0^T \|\mathbf{F}^N(\mathbf{D}\mathbf{u}^N(s))\|_2^2\,ds + \delta^{p-2}\int_0^T \|\mathbf{D}\mathbf{u}^N(s)\|_2^2\,ds \le c\,|||\mathbf{u}_0,\mathbf{f}|||^2,$$

where we also used Corollary 2.33. Next, we take $\frac{\partial\mathbf{u}_k^N}{\partial t}$ as test function in the Galerkin approximation, use the fact that $\mathbf{S}^N = \partial U^N$ is derived from the potential $U^N \sim \omega^n$ in view of Lemmas 2.27, 2.9 and Proposition 2.18, to arrive at

$$\int_0^t \left\| \frac{\partial \mathbf{u}_k^N(s)}{\partial t} \right\|_2^2 ds + \int_\Omega \omega^N(|\mathbf{D}\mathbf{u}_k^N(t)|)\, d\mathbf{x} \le c \int_\Omega \omega^N(|\mathbf{D}\mathbf{u}_0^k|)\, d\mathbf{x} + c \int_0^t \|\mathbf{f}(s)\|_2^2\, ds.$$

Since $p > 2$, we use Corollary 2.33 to arrive at

$$\int_\Omega \omega^N(|\mathbf{D}\mathbf{u}_0^k|)\, d\mathbf{x} \le c(p) \int_\Omega \omega(|\mathbf{D}\mathbf{u}_0^k|)\, d\mathbf{x}.$$

These properties together with Corollary 2.33 imply, after the limiting procedure $k \to \infty$, that for a.e. $t \in [0, T]$

$$\|\mathbf{F}^N(\mathbf{D}\mathbf{u}^N(t))\|_2^2 + \delta^{p-2}\|\nabla \mathbf{u}^N(t)\|_2^2 + \int_0^T \left\| \frac{\partial \mathbf{u}^N(s)}{\partial t} \right\|_2^2 ds \le c\left(1 + |||\mathbf{u}_0, \mathbf{f}|||^2\right),$$

where we also used Corollary 2.33 and Korn inequality.

The uniqueness of the solution $\mathbf{u}^N$ follows in a standard manner. □

***Remark 3.8*** Note that by the fundamental theorem of calculus of variations the weak formulation (3.2) is equivalent to

$$\left(\frac{\partial \mathbf{u}^N(t)}{\partial t}, \mathbf{w}\right) + (\mathbf{S}^N(\mathbf{D}\mathbf{u}^N(t)), \mathbf{D}\mathbf{w}) = (\mathbf{f}(t), \mathbf{w}), \tag{3.4}$$

being satisfied for a.e. $t \in I$ and all $\mathbf{w} \in W_0^{1,2}(\Omega)$.

In order to prove existence and uniqueness of regular solutions to (1.1), by taking the various limits $A_n \to \infty$, we need to prove further regularity for the solution $\mathbf{u}^N$, namely on the second order spatial derivatives, in such a way that $\mathbf{D}\mathbf{u}^N$ converges almost everywhere. The regularity in the spatial variables requires an ad hoc treatment (localization) for the Dirichlet boundary value problem. To do this we adapt the argument introduced in [32] (treating the case $p > 2$) and that in [12, 13] (treating the case $p < 2$). We sketch the relevant steps, pointing out the main new aspects which are present in the time-dependent case.

### 3.2 Description and properties of the boundary

We assume that the boundary $\partial\Omega$ is of class $C^{2,1}$, that is for each point $P \in \partial\Omega$ there are local coordinates such that in these coordinates we have $P = 0$ and $\partial\Omega$ is locally described by a $C^{2,1}$-function, i.e., there exist $R_P,\ R'_P \in (0, \infty)$, $r_P \in (0, 1)$ and a $C^{2,1}$-function $g_P : B^2_{R_P}(0) \to B^1_{R'_P}(0)$ such that

(b1) $\mathbf{x} \in \partial\Omega \cap (B^2_{R_P}(0) \times B^1_{R'_P}(0)) \iff x_3 = g_P(x_1, x_2)$,

(b2) $\Omega_P := \{(x', x_3) \,|\, x' = (x_1, x_2) \in B^2_{R_P}(0),\ g_P(x') < x_3 < g_P(x') + R'_P\} \subset \Omega$,

(b3) $\nabla g_P(0) = \mathbf{0}$, and $\forall\, x' = (x_1, x_2)^\top \in B^2_{R_P}(0) \quad |\nabla g_P(x')| < r_P$,

where $B^k_r(0)$ denotes the $k$-dimensional open ball with center 0 and radius $r > 0$. Note that $r_P$ can be made arbitrarily small if we make $R_P$ small enough. In the sequel we will also use, for $0 < \lambda < 1$, the scaled open sets $\lambda\,\Omega_P \subset \Omega_P$, defined as follows

$$\lambda\,\Omega_P := \{(x', x_3) \,|\, x' = (x_1, x_2)^\top \in B^2_{\lambda R_P}(0),\ g_P(x') < x_3 < g_P(x') + \lambda R'_P\}.$$

To localize near $\partial\Omega \cap \partial\Omega_P$, for $P \in \partial\Omega$, we fix smooth functions $\xi_P : \mathbb{R}^3 \to \mathbb{R}$ such that
($\ell$1) $\chi_{\frac{1}{2}\Omega_P}(\mathbf{x}) \le \xi_P(\mathbf{x}) \le \chi_{\frac{3}{4}\Omega_P}(\mathbf{x})$,

where $\chi_A(\mathbf{x})$ is the indicator function of the measurable set $A$. For the remaining interior estimate we localize by a smooth function $0 \le \xi_0 \le 1$ with $\operatorname{spt}\xi_0 \subset \Omega_0$, where $\Omega_0 \subset \Omega$ is an open set such that $\operatorname{dist}(\partial\Omega_0, \partial\Omega) > 0$. Since the boundary $\partial\Omega$ is compact, we can use an appropriate finite sub-covering which, together with the interior estimate, yields the global estimate.

Let us introduce the tangential derivatives near the boundary. To simplify the notation we fix $P \in \partial\Omega$, $h \in (0, \frac{R_P}{16})$, and simply write $\xi := \xi_P$, $g := g_P$. We use the standard notation $\mathbf{x} = (x', x_3)^\top$ and denote by $\mathbf{e}^i$, $i = 1, 2, 3$ the canonical orthonormal basis in $\mathbb{R}^3$. In the following lower-case Greek letters take values 1, 2. For a function $f$ with $\operatorname{spt} f \subset \operatorname{spt}\xi$ we define for $\alpha = 1, 2$ tangential translations:

$$f_\tau(x', x_3) = f_{\tau_\alpha}(x', x_3) := f\big(x' + h\,\mathbf{e}^\alpha, x_3 + g(x' + h\,\mathbf{e}^\alpha) - g(x')\big),$$

tangential differences $\Delta^+ f := f_\tau - f$, and tangential difference quotients $d^+ f := h^{-1}\Delta^+ f$. For simplicity we denote $\nabla g := (\partial_1 g, \partial_2 g, 0)^\top$ and use the operations $(\cdot)_\tau$, $(\cdot)_{-\tau}$, $\Delta^+(\cdot)$, $\Delta^+(\cdot)$, $d^+(\cdot)$ and $d^-(\cdot)$ also for vector-valued and tensor-valued functions, intended as acting component-wise.

We will use the following properties of the difference quotients, all proved in [6].

**Lemma 3.9** *Let* $\mathbf{v} \in W^{1,1}(\Omega)$ *be such that* $\operatorname{spt}\mathbf{v} \subset \operatorname{spt}\xi$. *Then*

$$\begin{aligned}
\nabla d^\pm \mathbf{v} &= d^\pm \nabla\mathbf{v} + (\partial_3 \mathbf{v})_\tau \otimes d^\pm \nabla g,\\
\mathbf{D} d^\pm \mathbf{v} &= d^\pm \mathbf{D}\mathbf{v} + (\partial_3 \mathbf{v})_\tau \overset{s}{\otimes} d^\pm \nabla g,\\
\operatorname{div} d^\pm \mathbf{v} &= d^\pm \operatorname{div}\mathbf{v} + (\partial_3 \mathbf{v})_{\pm\tau} d^\pm \nabla g,\\
\nabla \mathbf{v}_{\pm\tau} &= (\nabla\mathbf{v})_{\pm\tau} + (\partial_3 \mathbf{v})_{\pm\tau} d^\pm \nabla g,
\end{aligned}$$

*where* $(\mathbf{v} \otimes \mathbf{w})_{ij} := v_i w_j$, $i, j = 1, 2, 3$, *and* $\mathbf{v} \overset{s}{\otimes} \mathbf{w} := \frac{1}{2}\big(\mathbf{v} \otimes \mathbf{w} + (\mathbf{v} \otimes \mathbf{w})^\top\big)$.

As for the classical difference quotients, $L^q$-uniform (with respect to $h > 0$) bounds for $d^+ f$ imply that $\partial_\tau f$ belongs to $L^q(\operatorname{spt}\xi)$.

**Lemma 3.10** *It holds*[8] *that, if* $f \in W^{1,1}(\Omega)$, *then we have for* $\alpha = 1, 2$

$$d^+ f \to \partial_\tau f = \partial_{\tau_\alpha} f := \partial_\alpha f + \partial_\alpha g\, \partial_3 f \quad \text{as } h \to 0, \tag{3.5}$$

*almost everywhere in* $\operatorname{spt}\xi$, *(cf.* [32]*).*

*If we define, for* $0 < h < R_P$

$$\Omega_{P,h} = \left\{\mathbf{x} \in \Omega_P \,\middle|\, x' \in B^2_{R_P - h}(0)\right\},$$

*and if* $f \in W^{1,q}_{\mathrm{loc}}(\Omega)$, $1 \le q < \infty$, *then*

$$\int\limits_{\Omega_{P,h}} |d^+ f|^q \, d\mathbf{x} \le c \int\limits_{\Omega_P} |\partial_\tau f|^q \, d\mathbf{x}.$$

[8] Note that $\partial_\tau f$ denotes a tangential derivative, and to avoid confusion with time derivatives, the latter will be always denoted as $\frac{\partial f}{\partial t}$.

*Moreover, if* $d^+ f \in L^q(\Omega_{P,h_0})$, $1 < q < \infty$, *and if*

$$\exists\, c_1 > 0 : \int\limits_{\Omega_{P,h_0}} |d^+ f|^q \, d\mathbf{x} \le c_1 \quad \forall\, h_0 \in (0, R_P) \text{ and } \forall\, h \in (0, h_0),$$

*then* $\partial_\tau f \in L^q(\Omega_P)$ *and*

$$\int\limits_{\Omega_P} |\partial_\tau f|^q \, d\mathbf{x} \le c_1.$$

***Remark 3.11*** All assertions of the previous lemma also hold in Orlicz spaces generated by N-functions $\phi \in \Delta_2$, as can be easily seen by adapting the proof carried out in [23] to this situation.

The following variant of formula of integration by parts will be often used.

**Lemma 3.12** *Let* $\operatorname{spt} g \cup \operatorname{spt} f \subset \operatorname{spt} \xi = \operatorname{spt} \xi_P$ *and* $0 < h < \frac{R_P}{16}$. *Then*

$$\int\limits_\Omega f g_{-\tau} \, d\mathbf{x} = \int\limits_\Omega f_\tau g \, d\mathbf{x}.$$

*Consequently,* $\int\limits_\Omega f d^+ g \, d\mathbf{x} = \int\limits_\Omega (d^- f) g \, d\mathbf{x}$. *Moreover, if in addition* $f$ *and* $g$ *are smooth enough and at least one vanishes on* $\partial\Omega$, *then*

$$\int\limits_\Omega f \partial_\tau g \, d\mathbf{x} = -\int\limits_\Omega (\partial_\tau f) g \, d\mathbf{x}.$$

Also the following properties of the difference quotient will be used in the sequel.

**Lemma 3.13** *Let* $\operatorname{spt} g \subset \operatorname{spt} \xi$. *Then*

$$(d^- g)_\tau = -d^+ g, \quad (d^+ g)_{-\tau} = -d^- g, \quad d^- g_\tau = -d^+ g.$$

**Lemma 3.14** *Let* $\operatorname{spt} g \cup \operatorname{spt} f \subset \operatorname{spt} \xi$. *Then*

$$d^\pm (f g) = f_{\pm\tau} \, d^\pm g + (d^\pm f) \, g.$$

### 3.3 A first regularity result in space

We start proving spatial regularity for the approximate problem. The estimates proved in this intermediate step are uniform with respect to $A_n$, $n = 1, \ldots, N$, only (a) in the interior of $\Omega$ and (b) in the case of tangential derivatives. On the contrary estimates depend on $A_n$ in the normal direction. Nevertheless, this allows later on to use the equations point-wise to prove in a different way estimates independent of $A_n$, $n = 1, \ldots, N$, in the normal direction. Thus, we can pass to the limit with $A_n \to \infty$, to treat the original problem in the non-degenerate case.

We observe that by using a translation method, the result below is proved rigorously for the solutions we constructed.

**Proposition 3.15** *Let the operator* $\mathbf{S} = \partial U$, *derived from the potential* $U$, *have* $(p, \delta)$-*structure for some* $p \in (2, \infty)$, *and* $\delta \in (0, \infty)$ *with characteristics* $(\gamma_3, \gamma_4, p)$. *Let* $\Omega \subset \mathbb{R}^3$

*be a bounded domain with $C^{2,1}$ boundary and assume that $\mathbf{u}_0 \in W_0^{1,p}(\Omega)$ and $\mathbf{f} \in L^2(I \times \Omega)$. Let $\mathbf{S}^N$ be the last item of the special multiple approximation $\mathbf{S}^n$, $n = 1, \ldots, N$, of $\mathbf{S}$ from Definition 3.5.*

*Then, the unique strong solution $\mathbf{u}^N$ of the approximate problem (3.1) satisfies for a.e. $t \in I$*

$$\begin{aligned}\int_0^t \int_\Omega \xi_0^2 |\nabla \mathbf{F}^N(\mathbf{Du}^N(s))|^2 + \delta^{p-2} \xi_0^2 |\nabla^2 \mathbf{u}^N(s)|^2 \, d\mathbf{x}\, ds &\leq c_0, \\ \int_0^t \int_\Omega \xi_P^2 |\partial_\tau \mathbf{F}^N(\mathbf{Du}^N(s))|^2 + \delta^{p-2} \xi_P^2 |\partial_\tau \nabla \mathbf{u}^N(s)|^2 \, d\mathbf{x}\, ds &\leq c_P,\end{aligned} \tag{3.6}$$

*where $c_0 = c_0(\delta^{2-p}\|\mathbf{f}\|_2^2, |||\mathbf{u}_0, \mathbf{f}|||, \|\xi_0\|_{1,\infty}, \gamma_3, \gamma_4, p)$, while the constant related to the neighborhood of $P$ is such that $c_P = c_P(\delta^{2-p}\|\mathbf{f}\|_2^2, |||\mathbf{u}_0, \mathbf{f}|||, \|\xi_P\|_{1,\infty}, \|g_P\|_{C^{2,1}}, \gamma_3, \gamma_4, p)$.*

*Here, $\xi_0(\mathbf{x})$ is a cut-off function with support in the interior of $\Omega$ and, for arbitrary $P \in \partial\Omega$, the tangential derivative*

*is defined locally in $\Omega_P$ via (3.5).*

Propositions 3.15 and 3.7 imply $\mathbf{u}^N(t) \in W^{2,2}(\Omega)$ and $\frac{\partial \mathbf{u}^N}{\partial t}(t) \in L^2(\Omega)$ for a.e. $t \in I$. Hence, equations (3.1) hold point-wise a.e. in $I \times \Omega$.

We employ this to deduce the following result, by using the equations in a point-wise sense, yielding however a critical dependence on the approximation of the operator.

**Proposition 3.16** *Under the assumptions of Proposition 3.15 there exists a constant $C_1 > 0$ such that, provided in the local description of the boundary there holds $r_P < C_1$ in (b3), where $\xi_P$ is a cut-off function with support in $\Omega_P$, there holds for a.e. $t \in I$*

$$\int_0^t \int_\Omega \xi_P^2 |\partial_3 \mathbf{F}^N(\mathbf{Du}^N(s))|^2 + \delta^{p-2} \xi_P^2 |\partial_3 \mathbf{Du}^N(s)|^2 \, d\mathbf{x}\, ds \leq C_N, \tag{3.7}$$

*where $C_N = C_N(\delta^{2-p}, \delta^{2-p}\|\mathbf{f}\|_2^2, |||\mathbf{u}_0, \mathbf{f}|||, \|\xi_P\|_{1,\infty}, \|g_P\|_{C^{2,1}}, \gamma_3, \gamma_4, p, A_N, \omega_{N-1})$.*

***Remark 3.17*** We consider only the case $\delta > 0$ and in the estimates of the two above propositions all dependencies on $\delta$ are traced in a precise and explicit way, showing how they deteriorate in the degenerate case. The degenerate problem could be treated by assuming more stringent assumptions on the regularity of the data (namely the regularity of the right-hand side $\mathbf{f}$). The same phenomenon is well-known to happen even for the $p$-Laplace problem. In that case sharpness of additional assumptions and links with the fractional regularity of the solution are proved and discussed in detail by Brasco and Santambrogio [15] and the references therein.

***Proof of Proposition 3.15*** Fix $P \in \partial\Omega$ and define in $\Omega_P$

$$\mathbf{w} := d^-(\xi^2 d^+(\mathbf{u}^N|_{\frac{1}{2}\Omega_P})),$$

where $\xi := \xi_P$, $g := g_P$, and $h \in (0, \frac{R_P}{16})$ and use the function $\mathbf{w}$ extended by zero outside of $\Omega_P$ as a test function in (3.4). This yields, using the properties of the difference quotient in Lemmas 3.9, 3.12, 3.14, for a.e. $t \in I$

$$\int_0^t \int_\Omega \xi^2 d^+ \frac{\partial \mathbf{u}^N(s)}{\partial t} \cdot d^+ \mathbf{u}^N(s) \, d\mathbf{x}\, ds + \int_0^t \int_\Omega \xi^2 d^+ \mathbf{S}^N(\mathbf{Du}^N(s)) \cdot d^+ \mathbf{Du}^N(s) \, d\mathbf{x}\, ds$$

$$
\begin{aligned}
&= -\int_0^t \int_\Omega \xi^2 d^+ \mathbf{S}^N(\mathbf{D}\mathbf{u}^N(s)) \cdot \big(\partial_3 \mathbf{u}^N(s)\big)_\tau \overset{s}{\otimes} d^+ \nabla g \, d\mathbf{x}\, ds \\
&\quad - 2 \int_0^t \int_\Omega d^+ \mathbf{S}^N(\mathbf{D}\mathbf{u}^N(s)) \cdot \xi \nabla \xi \overset{s}{\otimes} d^+ \mathbf{u}^N \, d\mathbf{x}\, ds \\
&\quad + \int_0^t \int_\Omega \mathbf{S}^N((\mathbf{D}\mathbf{u}^N)_\tau) \cdot \big(2\xi\, \partial_3 \xi\, d^+ \mathbf{u}^N\big) \overset{s}{\otimes} d^+ \nabla g \, d\mathbf{x}\, ds \\
&\quad + \int_0^t \int_\Omega \mathbf{S}^N((\mathbf{D}\mathbf{u}^N)_\tau) \cdot \big(\xi^2 d^+ \partial_3 \mathbf{u}^N\big) \overset{s}{\otimes} d^+ \nabla g \, d\mathbf{x}\, ds \\
&\quad + \int_0^t \int_\Omega \mathbf{f}(s) \cdot d^- (\xi^2 d^+ \mathbf{u}^N(s))\, d\mathbf{x}\, ds =: \sum_{j=1}^5 \int_0^t \mathcal{I}_j(s)\, ds.
\end{aligned}
$$

Proposition 2.29 yields for a.e. $s \in I$ the following equivalence

$$
\int_\Omega \xi^2 \big|d^+ \mathbf{F}^N(\mathbf{D}\mathbf{u}^N(s))\big|^2 \, d\mathbf{x} \sim \int_\Omega \xi^2 d^+ \mathbf{S}^N(\mathbf{D}\mathbf{u}^N(s)) \cdot d^+ \mathbf{D}\mathbf{u}^N(s)\, d\mathbf{x},
$$

with constants depending only on the characteristics of $\mathbf{S}$, due to Remark 3.6. This equivalence provides the "natural" quantity on the left-hand side. We estimate the integrals $\mathcal{I}_j$, $j = 1, \dots, 5$, similarly as in [6]. Note that all constants in the following can depend on the characteristics of $\mathbf{S}$ and that other dependencies will be indicated.

We start estimating the first one as

$$
\begin{aligned}
\mathcal{I}_1 &\le c \int_\Omega \xi^2 |d^+ \mathbf{D}\mathbf{u}^N|\, a^N(|\mathbf{D}\mathbf{u}^N| + |\Delta^+ \mathbf{D}\mathbf{u}^N|) |(\nabla \mathbf{u}^N)_\tau|\, |d^+ \nabla g|\, d\mathbf{x} \\
&\le c\, \|g\|_{C^{1,1}} \Bigg( \int_\Omega \xi^2 \, a^N(|\mathbf{D}\mathbf{u}^N| + |\Delta^+ \mathbf{D}\mathbf{u}^N|) |d^+ \mathbf{D}\mathbf{u}^N|^2 \, d\mathbf{x} \Bigg)^{1/2} \times \\
&\quad \times \Bigg( \int_\Omega \xi^2 a^N(|\mathbf{D}\mathbf{u}^N| + |\Delta^+ \mathbf{D}\mathbf{u}^N|) |(\nabla \mathbf{u}^N)_\tau|^2 \, d\mathbf{x} \Bigg)^{1/2} \\
&\le \varepsilon\, \|\xi\, d^+ \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)\|_2^2 + C\Big(\delta^p + \int_\Omega \omega^N(|\mathbf{D}\mathbf{u}^N|)\, d\mathbf{x}\Big),
\end{aligned}
$$

where we used Proposition 2.29, Hölder and Young inequalities, Lemma 2.34, the convexity and $\Delta_2$-condition of the balanced N-function $\omega^N$, the substitution theorem and Korn inequality. The constant $C$ depends on $\|g\|_{C^{1,1}}$ and $\varepsilon^{-1}$. Note that in view of $\int_\Omega \omega^N(|\mathbf{D}\mathbf{u}^N|)\, d\mathbf{x} \sim \|\mathbf{F}^N(\mathbf{D}\mathbf{u}^N)\|_2^2$, estimate (3.3), and the substitution theorem the right-hand side of the last estimate is finite. This comment also applies to the estimates of the other terms $\mathcal{I}_j$, $j = 1, \dots, 5$.

The second term is estimated more or less in the same way

$$
\mathcal{I}_2 \le c \int_\Omega \xi |d^+ \mathbf{D}\mathbf{u}^N|\, a^N(|\mathbf{D}\mathbf{u}^N| + |\Delta^+ \mathbf{D}\mathbf{u}^N|) |\nabla \xi|\, |d^+ \mathbf{u}^N|\, d\mathbf{x}
$$

$$\leq c\,\|\nabla\xi\|_\infty^2\bigg(\int\limits_\Omega \xi^2\, a^N(|\mathbf{Du}^N|+|\Delta^+\mathbf{Du}^N|)|d^+\mathbf{Du}^N|^2\,d\mathbf{x}\bigg)^{1/2}\times$$

$$\times\bigg(\int\limits_{\Omega\cap\operatorname{spt}\xi} a^N(|\mathbf{Du}^N|+|\Delta^+\mathbf{Du}^N|)|d^+\mathbf{u}^N|^2\,d\mathbf{x}\bigg)^{1/2}$$

$$\leq \varepsilon\,\|\xi\, d^+\mathbf{F}^N(\mathbf{Du}^N)\|^2 + C(\varepsilon^{-1},\|\xi\|_{1,\infty})\,\Big(\delta^p+\int\limits_\Omega \omega^N(|\mathbf{Du}^N|)\,d\mathbf{x}\Big),$$

where we additionally used Remark 3.11.

To estimate the integral $\mathcal{I}_3$ we use that due to Proposition 2.29 there holds $|\mathbf{S}^N(\mathbf{P})|\leq c\,(\omega^N)'(|\mathbf{P}^{\mathrm{sym}}|)$. Using this, Young inequality, (2.2), the substitution theorem, Remark 3.11 and Korn inequality we get

$$|\mathcal{I}_3|\leq c(\|\xi\|_{1,\infty},\|g\|_{C^{2,1}})\int\limits_\Omega (\omega^N)^*(|\mathbf{S}^N((\mathbf{Du}^N)_\tau)|)+\omega^N(|d^+\mathbf{u}^N|)\,d\mathbf{x}$$

$$\leq C(\|\xi\|_{1,\infty},\|g\|_{C^{2,1}})\int\limits_\Omega \omega^N(|\mathbf{Du}^N|)\,d\mathbf{x}.$$

The integral $\mathcal{I}_4$ is estimated by using Lemmas 3.12–3.14 to obtain

$$|\mathcal{I}_4| = \bigg|\int\limits_\Omega \big(-\xi^2 d^+\mathbf{S}^N(\mathbf{Du}^N)d^+\nabla g+\mathbf{S}^N(\mathbf{Du}^N)d^+(\nabla g)\,d^-(\xi^2)\big)\overset{s}{\otimes}\partial_3\mathbf{u}^N$$

$$+\mathbf{S}^N(\mathbf{Du}^N)(\xi^2)_{-\tau}\,d^-d^+\nabla g\overset{s}{\otimes}\partial_3\mathbf{u}^N\,d\mathbf{x}\bigg|$$

$$\leq \varepsilon\,\|\xi\, d^+\mathbf{F}^N(\mathbf{Du}^N)\|_2^2 + c(\varepsilon^{-1},\|\xi\|_{1,\infty},\|g\|_{C^{2,1}})\,\Big(\delta^p+\int\limits_\Omega \omega^N(|\mathbf{Du}^N|)\,d\mathbf{x}\Big),$$

where the first term was treated as $\mathcal{I}_1$, while the other two were treated as $\mathcal{I}_3$ .

On the other hand, the integral related to the right-hand side can be estimated as follows

$$\mathcal{I}_5\leq c(\varepsilon^{-1})\,\delta^{2-p}\|\mathbf{f}\|_2^2+\varepsilon\,\delta^{p-2}\int\limits_\Omega |d^-(\xi^2 d^+\mathbf{u}^N)|^2\,d\mathbf{x}$$

$$\leq c(\varepsilon^{-1})\,\delta^{2-p}\|\mathbf{f}\|_2^2+c(\|\xi\|_{1,\infty},\|g\|_{C^{1,1}})\,\delta^{p-2}\int\limits_\Omega |\mathbf{Du}^N|^2\,d\mathbf{x}$$

$$+\varepsilon\, c\,\delta^{p-2}\int\limits_\Omega \xi^2\,|d^+\mathbf{Du}^N|^2\,d\mathbf{x}$$

$$\leq c(\varepsilon^{-1})\,\delta^{2-p}\|\mathbf{f}\|_2^2+c(\|\xi\|_{1,\infty},\|g\|_{C^{1,1}})\,\delta^{p-2}\int\limits_\Omega |\mathbf{Du}^N|^2\,d\mathbf{x}$$

$$+\varepsilon\, c\int\limits_\Omega |d^+\mathbf{F}^N(\mathbf{Du}^N)|^2\,d\mathbf{x},$$

where we used standard properties of the difference quotient in $L^2$, Korn inequality, the substitution theorem, as well as Proposition 2.29, and Lemma 2.32, which yield $\delta^{p-2}|d^+\mathbf{Du}^N|^2\leq c\,|d^+\mathbf{F}^N(\mathbf{Du}^N)|^2$.

Observing that $d^+\frac{\partial \mathbf{u}^N}{\partial t} = \frac{\partial d^+\mathbf{u}^N}{\partial t}$, choosing $\varepsilon > 0$ sufficiently small, and using (3.3) we proved that for a.e. $t \in I$

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \xi^2|d^+\mathbf{u}^N(t)|^2\,d\mathbf{x} + c\int_0^t\int_\Omega \xi^2\big|d^+\mathbf{F}^N(\mathbf{D}\mathbf{u}^N(s))\big|^2\,d\mathbf{x}\,ds \\
&\quad\le \frac{1}{2}\int_\Omega \xi^2|d^+\mathbf{u}_0|^2\,d\mathbf{x} + c(\|\xi\|_{1,\infty}, \|g\|_{C^{2,1}}, \delta^{2-p}\|\mathbf{f}\|_2^2, |||\mathbf{u}_0, \mathbf{f}|||, \gamma_3, \gamma_4, p) \\
&\quad\le C_0(\|\xi\|_{1,\infty}, \|g\|_{C^{2,1}}, \delta^{2-p}\|\mathbf{f}\|_2^2, |||\mathbf{u}_0, \mathbf{f}|||, \gamma_3, \gamma_4, p),
\end{aligned} \tag{3.8}$$

where we also used the assumption on the data. Since $C_0$ does not depend on $h > 0$, it follows by Lemma 3.10 that for a.e. $t \in I$

$$\int_0^t\int_\Omega \xi^2\big|\partial_\tau\mathbf{F}^N(\mathbf{D}\mathbf{u}^N(s))\big|^2\,d\mathbf{x}\,ds \le \int_0^t\int_\Omega \xi^2\big|d^+\mathbf{F}^N(\mathbf{D}\mathbf{u}^N(s))\big|^2\,d\mathbf{x}\,ds \le C_0,$$

proving the estimate for the first term in $(3.6)_2$. Next, observe that Proposition 2.29 and Lemma 2.32 imply

$$\delta^{p-2}\int_0^t\int_\Omega \xi^2|d^+\mathbf{D}\mathbf{u}^N(s)|^2\,d\mathbf{x}\,ds \le \int_0^t\int_\Omega \xi^2\big|d^+\mathbf{F}^N(\mathbf{D}\mathbf{u}^N(s))\big|^2\,d\mathbf{x}\,ds \le C_0.$$

Now we proceed exactly as in the proof of [11,(3.12)–(3.14)] for the special choice $\phi(t) = t^2$ to get

$$\begin{aligned}
&\delta^{p-2}\int_0^t\int_\Omega \xi^2|d^+\nabla\mathbf{u}^N(s)|^2\,d\mathbf{x}\,ds \\
&\le \delta^{p-2}\int_0^t\int_\Omega \xi^2|d^+\mathbf{D}\mathbf{u}^N(s)|^2\,d\mathbf{x}\,ds + c(\|\xi\|_{1,\infty}, \|g\|_{C^{1,1}})\,\delta^{p-2}\int_0^t\int_\Omega |\mathbf{D}\mathbf{u}^N(s)|^2\,d\mathbf{x}\,ds \\
&\le C_0 + c(\|\xi\|_{1,\infty}, \|g\|_{C^{1,1}})\,\delta^{p-2}\int_0^t\int_\Omega |\mathbf{D}\mathbf{u}^N(s)|^2\,d\mathbf{x}\,ds.
\end{aligned} \tag{3.9}$$

This, the a priori estimate (3.3), and Lemma 3.10 finally shows for a.e. $t \in I$

$$\delta^{p-2}\int_0^t\int_\Omega \xi^2|\partial_\tau\nabla\mathbf{u}^N|^2\,d\mathbf{x} \le C(\|\xi\|_{1,\infty}, \|g\|_{C^{2,1}}, \delta^{2-p}\|\mathbf{f}\|^2, |||\mathbf{u}_0, \mathbf{f}|||, \gamma_3, \gamma_4, p),$$

proving the estimate for the second term in $(3.6)_2$.

The same argument used with a test function $\xi_0$ with compact support in $\Omega$, and standard difference quotients can be used to prove $(3.6)_1$. □

**Corollary 3.18** *Under the assumptions of Proposition 3.15 there holds a.e. in $I \times \Omega$*

$$\big|\partial_\tau\mathbf{F}^N(\mathbf{D}\mathbf{u}^N)\big|^2 \sim a^N(|\mathbf{D}\mathbf{u}^N|)|\partial_\tau\mathbf{D}\mathbf{u}^N|^2$$

*with constants depending only on the characteristics of* $\mathbf{S}$.

***Proof*** Proposition 2.29 implies

$$\big|d^+\mathbf{F}^N(\mathbf{Du}^N)\big|^2 \sim a^N(|\mathbf{Du}^N| + |\Delta^+\mathbf{Du}^N|)|d^+\mathbf{Du}^N|^2.$$

The estimates (3.8), (3.9) and Lemma 3.10 yield that a.e. in $I\times\Omega$ there holds $d^+\mathbf{F}^N(\mathbf{Du}^N) \to \partial_\tau \mathbf{F}^N(\mathbf{Du}^N)$ and $d^+\mathbf{Du}^N \to \partial_\tau \mathbf{Du}^N$ as $h \to 0$. These observations immediately imply the assertion. □

Now we prove the result on the regularity in the "normal" direction from (3.7), which is valid up to the boundary, but is dependent on the chosen multiple approximation.

***Proof of Proposition 3.16*** Thanks to the previous results we can re-write the equations in (3.1) a.e. in $I \times \Omega$ as follows

$$-\frac{\partial u_i^N}{\partial t} + \sum_{k=1}^{3} \partial_{k3} S_{i3}^N(\mathbf{Du}^N)\partial_3 D_{k3}\mathbf{u}^N + \sum_{\alpha=1}^{2} \partial_{3\alpha} S_{i3}^N(\mathbf{Du}^N)\partial_3 D_{3\alpha}\mathbf{u}^N = \mathfrak{f}_i,$$

where

$$\mathfrak{f}_i := -f_i - \sum_{\gamma,\sigma=1}^{2} \partial_{\gamma\sigma} S_{i3}^N(\mathbf{Du}^N)\partial_3 D_{\gamma\sigma}\mathbf{u}^N - \sum_{k,l=1}^{3} \partial_{kl} S_{i\beta}^N(\mathbf{Du}^N)\partial_\beta D_{kl}\mathbf{u}^N,$$

for $i = 1, 2, 3$. We now proceed as in [12,Eq. (3.3)] and multiply these equations by $\partial_3 \widehat{D}_{i3}\mathbf{u}^N$, where $\widehat{D}_{\alpha\beta}\mathbf{u}^N = 0$, for $\alpha, \beta = 1, 2$, $\widehat{D}_{\alpha 3}\mathbf{u}^N = \widehat{D}_{3\alpha}\mathbf{u}^N = 2D_{\alpha 3}\mathbf{u}^N$, for $\alpha = 1, 2$, $\widehat{D}_{33}\mathbf{u}^N = D_{33}\mathbf{u}^N$ and sum over $i = 1, 2, 3$. Since $\mathbf{S}^N$ has $\omega^N$-structure we get

$$-\sum_{i=1}^{3} \frac{\partial u_i^N}{\partial t}\partial_3 \widehat{D}_{i3}\mathbf{u}^N + \gamma\, a^N(|\mathbf{Du}^N|)|\mathfrak{b}|^2 \le |\mathfrak{f}||\mathfrak{b}| \quad \text{a.e. in } I \times \Omega,$$

where $\mathfrak{b}_i := \partial_3 D_{i3}\mathbf{u}^N$ and where the constant $\gamma$ just depends on the characteristics of $\mathbf{S}$.

By straightforward manipulations (cf. [11,Sections 3.2 and 4.2]) we obtain that a.e. in $I \times \Omega_P$ it holds

$$\begin{aligned} |\mathfrak{f}| &\le c\left(|\mathbf{f}|a^N(|\mathbf{Du}^N|\right)\left(|\partial_\tau \nabla \mathbf{u}^N| + \|\nabla g\|_\infty |\nabla^2 \mathbf{u}^N|\right), \\ |\mathfrak{b}| &\ge 2|\widetilde{\mathfrak{b}}| - |\partial_\tau \nabla \mathbf{u}^N| - \|\nabla g\|_\infty |\nabla^2 \mathbf{u}^N|, \end{aligned}$$

for $\widetilde{\mathfrak{b}}_i := \partial_{33}^2 u_i^N$, $i = 1, 2, 3$. Consequently we get a.e. in $I \times \Omega_P$

$$\begin{aligned} &-\sum_{i=1}^{3} \frac{\partial u_i^N}{\partial t}\partial_3 \widehat{D}_{i3}\mathbf{u}^N + 2\gamma\, a^N(|\mathbf{Du}^N|)|\widetilde{\mathfrak{b}}|^2 \\ &\le c\left[|\mathbf{f}| + a^N(|\mathbf{Du}^N|)\left(|\partial_\tau \nabla \mathbf{u}^N| + \|\nabla g\|_\infty |\nabla^2 \mathbf{u}^N|\right)\right]|\mathfrak{b}|. \end{aligned}$$

We then add on both sides, for $\alpha = 1, 2$ and $i, k = 1, 2, 3$, the term (which is finite a.e.)

$$2\gamma\, a^N(|\mathbf{Du}^N|)\, |\partial_\alpha \partial_i u_k^N|^2,$$

use the estimate $|\mathfrak{b}| \le |\nabla^2 \mathbf{u}^N|$ and Young inequality, yielding

$$\begin{aligned} &-\sum_{i=1}^{3} \frac{\partial u_i^N}{\partial t}\partial_3 \widehat{D}_{i3}\mathbf{u}^N + 2\gamma\, a^N(|\mathbf{Du}^N|)|\nabla^2 \mathbf{u}^N|^2 \\ &\le \gamma a^N(|\mathbf{Du}^N|)|\nabla^2 \mathbf{u}^N|^2 + \frac{c\,|\mathbf{f}|^2}{a^N(|\mathbf{Du}^N|)} + c\, a^N(|\mathbf{Du}^N|)\big(|\partial_\tau \nabla \mathbf{u}^N|^2 + \|\nabla g\|_\infty^2 |\nabla^2 \mathbf{u}^N|^2\big), \end{aligned}$$

where in the right-hand side we used also the definition of the tangential derivative (cf. (3.5)). Next, we choose the sets $\Omega_P$ such that $\|\nabla g\|_\infty = \|\nabla g_P(x_1, x_2)\|_{\infty,\Omega_P}$ is small enough, so that we can absorb the last term from the right-hand side. We finally arrive at the following pointwise inequality

$$
\begin{aligned}
&-\sum_{i=1}^{3} \frac{\partial u_i^N}{\partial t} \partial_3 \widehat{D}_{i3} \mathbf{u}^N + \gamma\, a^N(|\mathbf{D}\mathbf{u}^N|) |\nabla^2 \mathbf{u}^N|^2 \\
&\leq c \left( \frac{|\mathbf{f}|^2}{a^N(|\mathbf{D}\mathbf{u}^N|)} + a^N(|\mathbf{D}\mathbf{u}^N|)\, |\partial_\tau \nabla \mathbf{u}^N|^2 \right) \quad \text{a.e. in } I \times \Omega_P.
\end{aligned}
\tag{3.10}
$$

We multiply (3.10) by $\xi^2$, and integrate for a.e. $t \in I$ over the sub-domain

$$
(0, t) \times \Omega_{P,\varepsilon} := (0, t) \times \left\{ \mathbf{x} \in \Omega_P \,\middle|\, g_P + \varepsilon < x_3 < g_P + R'_P \right\},
$$

for $0 < \varepsilon < R'_P$. This shows, using also Young inequality, that

$$
\begin{aligned}
&\gamma \int_0^t \int_{\Omega_{P,\varepsilon}} \xi^2 a^N(|\mathbf{D}\mathbf{u}^N|) |\nabla^2 \mathbf{u}^N|^2 d\mathbf{x}\, ds \\
&\leq \int_0^t \int_{\Omega_{P,\varepsilon}} c\, \xi^2 \left( \frac{|\mathbf{f}|^2}{a^N(|\mathbf{D}\mathbf{u}^N|)} + a^N(|\mathbf{D}\mathbf{u}^N|)\, |\partial_\tau \nabla \mathbf{u}^N|^2 \right) + \xi^2 \left| \frac{\partial \mathbf{u}^N}{\partial t} \right| |\nabla^2 \mathbf{u}^N|\, d\mathbf{x}\, ds \\
&\leq \int_0^t \int_{\Omega} c\, \xi^2 \left( \frac{|\mathbf{f}|^2 + \left| \frac{\partial \mathbf{u}^N}{\partial t} \right|^2}{a^N(|\mathbf{D}\mathbf{u}^N|)} + a^N(|\mathbf{D}\mathbf{u}^N|)\, |\partial_\tau \nabla \mathbf{u}^N|^2 \right) d\mathbf{x}\, ds \\
&\quad + \frac{\gamma}{2} \int_0^t \int_{\Omega_{P,\varepsilon}} \xi^2 a^N(|\mathbf{D}\mathbf{u}^N|) |\nabla^2 \mathbf{u}^N|^2\, d\mathbf{x}\, ds.
\end{aligned}
$$

Now we absorb the last term from the right-hand side in the left-hand side. Moreover, we use that $a^N$ is bounded from below by $c\,\delta^{p-2}$ (cf. Lemma 2.32), the assumption on $\mathbf{f}$ and (3.3) to estimate the first term on the right-hand side. To handle the second term we first use that $a^N$ is bounded from above by a constant $c$ depending on $p, \gamma_3, \gamma_4, \omega^{N-1}$, and $A_N$ (cf. Remark 2.21, Remark 3.6) and then we use the estimate $(3.6)_2$. These estimates result in

$$
\int_0^T \int_{\Omega_{P,\varepsilon}} \xi^2 a^N(|\mathbf{D}\mathbf{u}^N|) |\nabla^2 \mathbf{u}^N|^2\, d\mathbf{x}\, dt \leq C(A_N, \omega_{N-1}, \delta^{2-p}, |||\mathbf{u}_0, \mathbf{f}|||).
$$

By monotone convergence for $\varepsilon \to 0$, this shows that

$$
\int_0^T \int_{\Omega} \xi^2 a^N(|\mathbf{D}\mathbf{u}^N|) |\nabla^2 \mathbf{u}^N|^2\, d\mathbf{x}\, dt \leq C(A_N, \omega_{N-1}, \delta^{2-p}, c_P).
$$

Using Lemma 2.32 we finally get also

$$
\delta^{p-2} \int_0^T \int_{\Omega} \xi^2 |\nabla^2 \mathbf{u}^N|^2\, d\mathbf{x}\, dt \leq C(A_N, \omega_{N-1}, \delta^{2-p}, c_P).
$$

The last two estimates together with Proposition 2.31 and the definition of the tangential derivatives (cf. (3.5)) finish the proof of Proposition 3.16. □

### 3.4 Uniform estimates for the second order spatial derivatives

We now improve the estimate in the normal direction in the sense that we will show that they are bounded uniformly with respect to the parameters $A_n$, for all $n = 1, \dots, N$. The used method is an adaption to the time evolution problem and the case $p > 2$ of the treatment in [12, 37] in the case $p < 2$ (cf. [13]). In particular, it involves a technical steps to justify the treatment of the time derivative, which is an adaptation of the method used in [31, 39].

**Lemma 3.19** *Let $\partial\Omega \in C^{2,1}$ and let $\mathbf{v} \in L^2(I; W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \cap W^{1,2}(I; L^2(\Omega))$. Then, for all $t \in [0, T]$ it holds*

$$-\int_0^t \int_\Omega \frac{\partial \mathbf{v}}{\partial t} \partial_{33}^2 \mathbf{v} \, d\mathbf{x} \, dt = \frac{1}{2} \|\partial_3 \mathbf{v}(t)\|_2^2 - \frac{1}{2} \|\partial_3 \mathbf{v}(0)\|_2^2.$$

***Proof*** Note that the assumptions on $\mathbf{v}$ already imply, by parabolic interpolation, that $\mathbf{v} \in C(\overline{I}; W_0^{1,2}(\Omega))$. We give an elementary proof, by heat regularization, since the direct integration by parts is not justified under the given assumptions. In fact, we have that $\frac{\partial \mathbf{v}}{\partial t} = \mathbf{0}$ on the boundary, but is it not clear if this holds also in the sense of traces. Let us define $\boldsymbol{\phi} := \frac{\partial \mathbf{v}}{\partial t} - \Delta \mathbf{v} \in L^2(I \times \Omega)$ and $\boldsymbol{\psi} := \mathbf{v}(0) \in W_0^{1,2}(\Omega)$ and approximate these functions by sequences of smooth and compactly supported functions $\boldsymbol{\phi}_n$ and $\boldsymbol{\psi}_n$, respectively. Let $\mathbf{v}_n$ be the solution of boundary initial value problem

$$\begin{aligned} \frac{\partial \mathbf{v}_n}{\partial t} - \Delta \mathbf{v}_n &= \boldsymbol{\phi}_n && \text{in } I \times \Omega, \\ \mathbf{v}_n &= \mathbf{0} && \text{on } I \times \partial\Omega, \\ \mathbf{v}_n(0) &= \boldsymbol{\psi}_n && \text{in } \Omega. \end{aligned} \tag{3.11}$$

By energy methods, one obtains directly that there exists a unique solution $\mathbf{v}_n$ belonging to $L^2(I; W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \cap W^{1,2}(I; L^2(\Omega))$. Moreover, we have

$$\begin{aligned} &\|\mathbf{v}_n - \mathbf{v}_k\|_{L^2(I; W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \cap W^{1,2}(I; L^2(\Omega))} \\ &\quad \le c\, \|\boldsymbol{\phi}_n - \boldsymbol{\phi}_k\|_{L^2(I \times \Omega)} + c\, \|\boldsymbol{\psi}_n - \boldsymbol{\psi}_k\|_{W^{1,2}(\Omega)} \quad \text{for } k, n \in \mathbb{N}, \end{aligned}$$

which implies that $(\mathbf{v}_n)$ is a Cauchy sequence in the spaces on the left-hand side. Let $\mathbf{u}$ be the limit in $L^2(I; W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \cap W^{1,2}(I; L^2(\Omega))$ of the sequence $\mathbf{v}_n$. By passing to the limit in (3.11) we see that $\mathbf{u} - \mathbf{v}$ is a solution of (3.11) with vanishing data. Thus, by uniqueness we proved that

$$\mathbf{v}_n \to \mathbf{v} \quad \text{in } L^2(I; W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \cap W^{1,2}(I; L^2(\Omega)) \cap C(\overline{I}; W_0^{1,2}(\Omega)). \tag{3.12}$$

Next, testing by the "second order time derivative" of $\mathbf{v}_n$, which can be justified with the help of an Galerkin approximation (cf. [10, 21]), one gets that

$$\left\| \frac{\partial \mathbf{v}_n(t)}{\partial t} \right\|_2^2 + \int_0^t \left\| \frac{\partial \nabla \mathbf{v}_n(s)}{\partial t} \right\|_2^2 ds \le c \int_0^t \left\| \frac{\partial \boldsymbol{\phi}_n(s)}{\partial t} \right\|_2^2 ds + c\, \|\boldsymbol{\psi}_n\|_{W^{2,2}(\Omega)}^2 \le c(n).$$

This proves that $\frac{\partial \mathbf{v}_n}{\partial t} \in L^2(I; W^{1,2}(\partial\Omega)) \hookrightarrow L^2(I; W^{1/2}(\partial\Omega))$. Thus, $\frac{\partial \mathbf{v}_n}{\partial t} = \mathbf{0}$ holds in the sense of traces in $L^2(I; W^{1/2}(\partial\Omega))$ and we obtain

$$-\int_0^t \int_\Omega \frac{\partial \mathbf{v}_n}{\partial t} \partial_{33}^2 \mathbf{v}_n \, d\mathbf{x}\, dt = \int_0^t \int_\Omega \frac{\partial^2 \mathbf{v}_n}{\partial t \partial_3} \partial_3 \mathbf{v}_n \, d\mathbf{x}\, dt = \frac{1}{2}\|\partial_3 \mathbf{v}_n(t)\|_2^2 - \frac{1}{2}\|\partial_3 \mathbf{v}_n(0)\|_2^2.$$

Passing with $n \to \infty$, which is justified by (3.12), we proved the assertion. □

**Proposition 3.20** *Let the same hypotheses as in Theorem 3.4 be satisfied with $\delta > 0$ and let the local description $g_P$ of the boundary and the localization function $\xi_P$ satisfy (b1)–(b3) and ($\ell$1) (cf. Sect. 3.2). Then, there exists a constant $C_2 > 0$ such that the solution $\mathbf{u}^N \in L^\infty(I; W_0^{1,2}(\Omega)) \cap L^2(I; W^{2,2}(\Omega))$ of the approximate problem (3.1), ensured in Proposition 3.7, satisfies for every $P \in \partial\Omega$ and for a.e. $t \in I$*

$$\int_0^t \int_\Omega \xi_P^2 |\partial_3 \mathbf{F}^N(\mathbf{D}\mathbf{u}^N(s))|^2 \, d\mathbf{x}\, ds \le C,$$

*provided $r_P < C_2$ in (b3), with $C$ depending on the characteristics of $\mathbf{S}$, $\delta^{2-p}\|\mathbf{f}\|_2^2$, $|||\mathbf{u}_0, \mathbf{f}|||$, $\|\xi_P\|_{1,\infty}$, $\|g_P\|_{C^{2,1}}$, and $C_2$.*

***Proof*** We adapt the strategy in [12,Proposition 3.2] to the time-dependent problem. Fix an arbitrary point $P \in \partial\Omega$ and a local description $g = g_P$ of the boundary and the localization function $\xi = \xi_P$ satisfying (b1)–(b3) and ($\ell$1). In the following constants $c, C$ can always depend on the characteristics of $\mathbf{S}^N$, hence on those of $\mathbf{S}$, i.e., on $\gamma_3, \gamma_4$, and $p$. First we observe that Proposition 2.31 and Remark 3.6 yield that there exists a constant $C_0$, depending only on the characteristics of $\mathbf{S}$ such that

$$\frac{1}{C_0}|\partial_3 \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2 \le \mathbb{P}_3^N(\mathbf{D}\mathbf{u}^N) \quad \text{a.e. in } I \times \Omega.$$

Thus, we get, using also the symmetry of both $\mathbf{D}\mathbf{u}^N$ and $\mathbf{S}^N$,

$$\begin{aligned}
&\frac{1}{C_0}\int_0^t \int_\Omega \xi^2 |\partial_3 \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2 \, d\mathbf{x}\, ds \\
&\le \int_0^t \int_\Omega \xi^2 \partial_3 S_{\alpha\beta}^N(\mathbf{D}\mathbf{u}^N)\, \partial_3 D_{\alpha\beta}\mathbf{u}^N \, d\mathbf{x}\, ds + \int_0^t \int_\Omega \xi^2 \partial_3 S_{3\alpha}^N(\mathbf{D}\mathbf{u}^N)\, \partial_\alpha D_{33}\mathbf{u}^N \, d\mathbf{x}\, ds \\
&\quad + \int_0^t \int_\Omega \sum_{j=1}^3 \xi^2 \partial_3 S_{j3}^N(\mathbf{D}\mathbf{u}^N)\, \partial_3^2 u_j^N \, d\mathbf{x}\, ds \\
&=: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned} \tag{3.13}$$

The terms $\mathcal{J}_1$ and $\mathcal{J}_2$ can be estimated exactly as in [12], if one replaces $\phi''(|\mathbf{D}\mathbf{u}^N|)$ used there with the equivalent quantity $a^N(|\mathbf{D}\mathbf{u}^N|)$. Let us sketch the main steps. All missing details can be found in [12]. To treat $\mathcal{J}_2$ we multiply and divide by $\sqrt{a^N(|\mathbf{D}\mathbf{u}^N|)}$, use Proposition 2.31 and Young inequality, to show that, for any given $\lambda > 0$, it holds

$$|\mathcal{J}_2| \le \lambda \int_0^t \int_\Omega \xi^2 |\partial_3 \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2 \, d\mathbf{x}\, ds + c_{\lambda^{-1}} \sum_{\beta=1}^2 \int_0^t \int_\Omega \xi^2 |\partial_\beta \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2 \, d\mathbf{x}\, ds,$$

for some constant $c_{\lambda^{-1}}$ depending on $\lambda^{-1}$. To treat the term $\mathcal{J}_1$ we first use the algebraic identity $\partial_3 D_{\alpha\beta}\mathbf{u}^N = \partial_\alpha D_{3\beta}\mathbf{u}^N + \partial_\beta D_{3\alpha}\mathbf{u}^N - \partial_\beta\partial_\alpha u_3^N$. The first two terms in the resulting equation are treated as $\mathcal{J}_2$, while in the term with $\partial_\beta\partial_\alpha u_3^N$ we use the definition of tangential derivatives (3.5). This results in three terms, where one is again treated as $\mathcal{J}_2$. This procedure leads to[9]

$$
\begin{aligned}
|\mathcal{J}_1|7 \leq\ & \lambda \int_0^t \int_\Omega \xi^2 |\partial_3 \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2 \, d\mathbf{x}\, ds \\
& + c_{\lambda^{-1}} \left(1 + \|\nabla g\|_\infty^2\right) \sum_{\beta=1}^{2} \int_0^t \int_\Omega \xi^2 |\partial_\beta \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2 \, d\mathbf{x}\, ds \\
& + \int_0^t \int_\Omega \xi^2 |\partial_3 \mathbf{S}^N(\mathbf{D}\mathbf{u}^N)|\, |\nabla^2 g|\, |\mathbf{D}\mathbf{u}^N| \, d\mathbf{x}\, ds \\
& + \left| \int_0^t \int_\Omega \xi^2 \partial_3 S_{\alpha\beta}^N(\mathbf{D}\mathbf{u}^N)\, \partial_\alpha \partial_{\tau_\beta} u_3^N \, d\mathbf{x}\, ds \right| .
\end{aligned}
\tag{3.14}
$$

In the last but one term in (3.14) we multiply and divide by $\sqrt{a^N(|\mathbf{D}\mathbf{u}^N|)}$, use Proposition 2.31, Young inequality and $a^N(|\mathbf{D}\mathbf{u}^N|)|\mathbf{D}\mathbf{u}^N|^2 \sim |\mathbf{F}^N(|\mathbf{D}\mathbf{u}^N|)|^2$ (cf. Proposition 2.29), yielding that it is estimated by

$$
\lambda \int_0^t \int_\Omega \xi^2 |\partial_3 \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2 \, d\mathbf{x}\, ds + c_{\lambda^{-1}} \|\nabla^2 g\|_\infty^2 \int_0^t \int_\Omega |\mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2 \, d\mathbf{x}\, ds.
$$

To handle the last term in (3.14) we want to perform the crucial partial integration, which reads (neglecting the localization $\xi$)

$$
\begin{aligned}
\int_0^t \int_\Omega \partial_3 S_{\alpha\beta}^N(\mathbf{D}\mathbf{u}^N)\, \partial_\alpha \partial_{\tau_\beta} u_3^N \, d\mathbf{x}\, ds &= \int_0^t \int_\Omega \partial_\alpha S_{\alpha\beta}^N(\mathbf{D}\mathbf{u}^N)\, \partial_3 \partial_{\tau_\beta} u_3^N \, d\mathbf{x}\, ds \\
&= \int_0^t \int_\Omega \partial_\alpha S_{\alpha\beta}^N(\mathbf{D}\mathbf{u}^N)\, \partial_{\tau_\beta} D_{33}\mathbf{u}^N \, d\mathbf{x}\, ds.
\end{aligned}
$$

This partial integration replaces the term with $\partial_3 \mathbf{S}^N(\mathbf{D}\mathbf{u}^N)$, which cannot be estimated in terms of tangential derivatives, by a term with $\partial_\alpha \mathbf{S}^N(\mathbf{D}\mathbf{u}^N)$, which can be estimated in terms of tangential derivatives. Again, we multiply and divide by $\sqrt{a^N(|\mathbf{D}\mathbf{u}^N|)}$, use Proposition 2.31, Young inequality, Corollary 3.18, and the definition of the tangential derivatives, yielding

[9] The estimated terms correspond to the terms $A$ and $B_3$ in [12].

that the last term is estimated by

$$c\sum_{\alpha=1}^{2}\int_0^t\int_\Omega |\partial_\alpha \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2\,d\mathbf{x}\,ds + c\sum_{\beta=1}^{2}\int_0^t\int_\Omega |\partial_{\tau_\beta} \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2\,d\mathbf{x}\,ds$$
$$\leq c\sum_{\alpha=1}^{2}\int_0^t\int_\Omega |\partial_{\tau_\alpha} \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2\,d\mathbf{x}\,ds + c\,\|\nabla g\|_\infty^2\int_0^t\int_\Omega |\partial_3 \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2\,d\mathbf{x}\,ds.$$

The presence of the localization $\xi$ leads to several additional terms, which all can be handled as in [12]. All together we arrive at the following estimate

$$\begin{aligned}|\mathcal{J}_1|+|\mathcal{J}_2| \leq\; &\big(\lambda + c_{\lambda^{-1}}\,\|\nabla g\|_\infty^2\big)\int_0^t\int_\Omega \xi^2|\partial_3 \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2\,d\mathbf{x}\,ds\\ &+ c_{\lambda^{-1}}\sum_{\beta=1}^{2}\int_0^t\int_\Omega \xi^2|\partial_{\tau_\beta} \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2\,d\mathbf{x}\,ds\\ &+ c_{\lambda^{-1}}\big(1+\|\nabla\xi\|_\infty^2\big)\int_0^t\int_\Omega |\mathbf{F}^N(|\mathbf{D}\mathbf{u}^N|)|^2\,d\mathbf{x}\,ds.\end{aligned} \tag{3.15}$$

In this estimate we used for the terms with $\partial_\beta \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)$ the definition of the tangential derivative in (3.5) to get

$$\begin{aligned}\int_\Omega \xi^2|\partial_\beta \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2\,d\mathbf{x} \leq &\int_\Omega \xi^2|\partial_{\tau_\beta} \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2\,d\mathbf{x}\\ &+ \|\nabla g\|_\infty^2\int_\Omega \xi^2|\partial_3 \mathbf{F}^N(\mathbf{D}\mathbf{u}^N)|^2\,d\mathbf{x}.\end{aligned} \tag{3.16}$$

Also the term $\mathcal{J}_3$ is treated essentially as in [12]. Since in this step the equation (3.1) is used, in addition we have to handle the term with the time derivative. More precisely, we re-write the equations (3.1) as follows

$$\partial_3 S^N_{j3}(\mathbf{D}\mathbf{u}^N) = \frac{\partial u^N_j}{\partial t} - f^j - \partial_\beta S^N_{j\beta}(\mathbf{D}\mathbf{u}^N) \quad \text{a.e. in } I\times\Omega,$$

multiply it by $\partial_{33}\mathbf{u}^N$, use the algebraic identity

$$\partial_j\partial_k u^N_i = \partial_j D_{ik}\mathbf{u}^N + \partial_k D_{ij}\mathbf{u}^N - \partial_i D_{jk}\mathbf{u}^N, \tag{3.17}$$

treat all terms without the time derivative as $I_3$ in [12,p. 186] and integrate by parts the term involving $\frac{\partial \mathbf{u}^N}{\partial t}$, use Lemma 3.19, to get the following

$$\begin{aligned}\mathcal{J}_3 &= \sum_{j=1}^{3}\int_0^t\int_\Omega \xi^2\frac{\partial u^N_j}{\partial t}\,\partial^2_{33}u^N_j - \xi^2\big(f^j + \partial_\beta S^N_{j\beta}(\mathbf{D}\mathbf{u}^N)\big)\big(2\partial_3 D_{j3}\mathbf{u}^N - \partial_j D_{33}\mathbf{u}^N\big)\,d\mathbf{x}\,ds\\ &= -\frac{1}{2}\int_\Omega \xi^2|\partial_3\mathbf{u}^N(t)|^2\,d\mathbf{x} + \frac{1}{2}\int_\Omega \xi^2|\partial_3\mathbf{u}^N(0)|^2\,d\mathbf{x} - 2\sum_{j=1}^{3}\int_0^t\int_\Omega \xi\,\partial_3\xi\,\frac{\partial u^N_j}{\partial t}\partial_3 u^N_j\,d\mathbf{x}\,ds\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{3}\int_0^t\int_\Omega \xi^2\big(f_j+\partial_\beta S^N_{j\beta}(\mathbf{Du}^N)\big)\big(2\partial_3 D_{j3}\mathbf{u}^N-\partial_j D_{33}\mathbf{u}^N\big)\,d\mathbf{x}\,ds\\
&\le -\frac{1}{2}\int_\Omega \xi^2|\partial_3\mathbf{u}^N(t)|^2\,d\mathbf{x}+\frac{1}{2}\int_\Omega \xi^2|\partial_3\mathbf{u}^N(0)|^2\,d\mathbf{x}\\
&+\big(\lambda+c_{\lambda^{-1}}\,\|\nabla g\|_\infty^2\big)\int_0^t\int_\Omega \xi^2|\partial_3\mathbf{F}^N(\mathbf{Du}^N)|^2\,d\mathbf{x}\,ds\\
&+c_{\lambda^{-1}}\sum_{\beta=1}^{2}\int_0^t\int_\Omega \xi^2|\partial_{\tau_\beta}\mathbf{F}^N(\mathbf{Du}^N)|^2\,d\mathbf{x}\,ds+c\int_0^t\int_\Omega \xi^3|\partial_3\mathbf{u}^N|^2\,d\mathbf{x}\,ds\\
&+c\,\|\nabla\xi\|_\infty^2\int_0^t\int_\Omega\Big|\frac{\partial\mathbf{u}^N}{\partial t}\Big|^2\,d\mathbf{x}\,ds+c_{\lambda^{-1}}\int_0^t\int_\Omega\frac{|\mathbf{f}|^2}{a^N(|\mathbf{Du}^N|)}\,d\mathbf{x}\,ds,
\end{aligned}
\tag{3.18}
$$

where we used again (3.16). Now we choose in the estimates (3.15), (3.18) first $\lambda > 0$ small enough and then the covering of the boundary $\partial\Omega$ such that $\|\nabla g\|_\infty$ is small enough in order to absorb in the left-hand side of (3.13) the term involving $\partial_3\mathbf{F}^N(\mathbf{Du}^N)$. This way we obtain the following estimate

$$
\begin{aligned}
&\int_\Omega \xi^3|\partial_3\mathbf{u}^N(t)|^2\,d\mathbf{x}+\frac{1}{C_0}\int_0^t\int_\Omega \xi^2|\partial_3\mathbf{F}^N(\mathbf{Du}^N(s))|^2\,d\mathbf{x}\,ds\\
&\le\int_\Omega \xi^3|\partial_3\mathbf{u}_0|^2\,d\mathbf{x}+c\sum_{\beta=1}^{2}\int_0^T\int_\Omega \xi^2|\partial_{\tau_\beta}\mathbf{F}^N(\mathbf{Du}^N(s))|^2\,d\mathbf{x}\,ds\\
&\quad+c\int_0^T\int_\Omega\frac{|\mathbf{f}(s)|^2}{a^N(|\mathbf{Du}^N(s)|)}+|\mathbf{F}^N(\mathbf{Du}^N(s))|^2+\Big|\frac{\partial\mathbf{u}^N(s)}{\partial t}\Big|^2\,d\mathbf{x}\,ds\\
&\quad+c\int_0^T\int_\Omega \xi^3|\partial_3\mathbf{u}^N(s)|^2\,d\mathbf{x}\,ds
\end{aligned}
$$

with constants depending only on the characteristics of $\mathbf{S}$, $\|g\|_{C^{2,1}}$, and $\|\xi\|_{1,\infty}$.

Using the uniform estimates (3.3), (3.6) and the lower bound in Lemma 2.32, which yields

$$
\int_0^T\int_\Omega\frac{|\mathbf{f}|^2}{a^N(|\mathbf{Du}^N|)}\,d\mathbf{x}\,ds\le C\delta^{2-p}\int_0^T\int_\Omega|\mathbf{f}|^2\,d\mathbf{x}\,ds,
$$

we get from the last estimate the assertion of Proposition 3.20. □

Choosing now an appropriate finite covering of the boundary (for the details see also [11]), Propositions 3.15 and 3.20 yield the following result:

**Proposition 3.21** *Let the same hypotheses as in Theorem 3.4 with $\delta > 0$ be satisfied. Then, it holds for a.e. $t \in I$*

$$\int_0^t \|\nabla \mathbf{F}^N(\mathbf{D}\mathbf{u}^N(s))\|_2^2\, ds \le C$$

*with $C$ depending only on the characteristics of $\mathbf{S}$, $|||\mathbf{u}_0, \mathbf{f}|||$, $\delta^{2-p}\|\mathbf{f}\|_2^2$, $\delta^{p-2}$ and $\partial\Omega$. In particular is $C$ independent of $A_n$, $n = 1, \ldots, N$.*

### 3.5 Multiple passage to the limit

From Propositions 3.7 and 3.21 we obtain the following estimate, uniform with respect to $A_n \ge 1, n = 1, \ldots, N$, and valid for a.e. $t \in I$.

$$\|\mathbf{u}^N(t)\|_{1,2}^2 + \|\mathbf{F}^N(\mathbf{D}\mathbf{u}^N(t))\|_2^2 + \int_0^t \left\|\frac{\partial \mathbf{u}^N(s)}{\partial t}\right\|_2^2 + \|\nabla \mathbf{F}^N(\mathbf{D}\mathbf{u}^N(s))\|_2^2\, ds \le C \quad (3.19)$$

with $C$ depending only on the data of the problem (1.1).

Note that the functions $\mathbf{u}^N$ and $\mathbf{F}^N$ depend (implicitly) on the parameters $A_n$. Since these parameters are relevant for the various limiting processes, we now start to write these dependencies in an explicit way. The uniform estimates for $\mathbf{u}^N(t, \mathbf{x}, A_1, \ldots, A_N)$ and $\mathbf{F}^N(\mathbf{D}\mathbf{u}^N(t, \mathbf{x}, A_1, \ldots, A_N))$ are inherited by taking appropriate limits of the various $A_n$. In particular, we will define (when the limit exists in appropriate spaces)

$$\mathbf{u}^{N-1}(t, \mathbf{x}, A_1, \ldots, A_{N-1}) := \lim_{A_N \to \infty} \mathbf{u}^N(t, \mathbf{x}, A_1, \ldots, A_{N-1}, A_N),$$

and then inductively

$$\mathbf{u}^{n-1}(t, \mathbf{x}, A_1, \ldots, A_{n-1}) = \lim_{A_n \to \infty} \mathbf{u}^n(t, \mathbf{x}, A_1, \ldots, A_{n-1}, A_n) \quad n = 1, \ldots, N,$$

in such a way that the function $\mathbf{u} := \mathbf{u}^0$ will be shown to be the unique regular solution to the initial boundary value problem (1.1).

***Proof of Theorem 3.4*** From estimate (3.19) we obtain that $\mathbf{u}^N$ is uniformly bounded in $W^{1,2}(I; L^2(\Omega)) \cap L^\infty(I; W^{1,2}(\Omega))$ and that $\mathbf{F}^N(\mathbf{D}\mathbf{u}^N)$ is uniformly bounded in $L^\infty(I; L^2(\Omega)) \cap L^2(I; W^{1,2}(\Omega))$.

These bounds directly imply that there exists a sequence $A_{N_k} \to \infty$ (which we call again $A_N$), a vector field $\mathbf{u}^{N-1}(t, \mathbf{x}, A_1, \ldots, A_{N-1})$, and a tensor field $\widehat{\mathbf{F}^{N-1}}$

$$\begin{aligned} \lim_{A_N \to \infty} \mathbf{u}^N &= \mathbf{u}^{N-1} && \text{weakly in } W^{1,2}(I; L^2(\Omega)), \\ \lim_{A_N \to \infty} \mathbf{u}^N &= \mathbf{u}^{N-1} && \text{weakly* in } L^\infty(I; W^{1,2}(\Omega)), \\ \lim_{A_N \to \infty} \mathbf{F}^N(\mathbf{D}\mathbf{u}^N) &= \widehat{\mathbf{F}^{N-1}} && \text{weakly in } L^2(I; W^{1,2}(\Omega)), \\ \lim_{A_N \to \infty} \mathbf{F}^N(\mathbf{D}\mathbf{u}^N) &= \widehat{\mathbf{F}^{N-1}} && \text{weakly* in } L^\infty(I; L^2(\Omega)). \end{aligned} \quad (3.20)$$

From $\|\mathbf{F}^N(\mathbf{D}\mathbf{u}^N)\|_{L^2(I; W^{1,2}(\Omega)} \le C$ it follows, using Proposition 2.31, the lower bound on $a^N$ proved in Lemma 2.32, and the identity (3.17), that

$$\delta^{p-2}\|\nabla^2 \mathbf{u}^N\|_{L^2(I; W^{2,2}(\Omega))} \le C \quad (3.21)$$

with $C$ depending only on the data of the problem (1.1), but independent of $A_N$. The estimates (3.19), (3.21) and the Aubin-Lions compactness lemma imply that (up to a further subsequence)

$$\lim_{A_N\to\infty} \mathbf{Du}^N = \mathbf{Du}^{N-1} \quad \text{a.e. in } I\times\Omega \text{ and strongly in } L^2(I\times\Omega),$$

for all fixed $A_n$, with $n = 1, \dots, N-1$. Next, we observe that since

$$\lim_{A_N\to\infty} a^N(t) = a^{N-1}(t),$$

uniformly for $t$ belonging to compact sets in $\mathbb{R}^{\geq 0}$ (but in reality even more since $a^N(t) = a^{N-1}(t)$ for all $0 \leq t \leq A_N$), it follows by the definition of $\mathbf{F}^N$ and $\mathbf{S}^N$ that a.e. in $I\times\Omega$ and for all fixed $A_n$, $n = 1, \dots, N-1$, there holds

$$\begin{aligned}
\lim_{A_N\to\infty} \mathbf{F}^N(\mathbf{Du}^N(A_1,\dots,A_{N-1},A_N)) &= \mathbf{F}^{N-1}(\mathbf{Du}^{N-1}(A_1,\dots,A_{N-1})),\\
\lim_{A_N\to\infty} \mathbf{S}^N(\mathbf{Du}^N(A_1,\dots,A_{N-1},A_N)) &= \mathbf{S}^{N-1}(\mathbf{Du}^{N-1}(A_1,\dots,A_{N-1})).
\end{aligned} \tag{3.22}$$

In fact, by the definition of multiple approximation it follows that for all given $\mathbf{P}\in\mathbb{R}^{3\times 3}$ and for all fixed $A_1,\dots,A_{N-1}$ it holds

$$\begin{aligned}
\lim_{A_N\to\infty} \mathbf{F}^N(\mathbf{P},A_1,\dots,A_{N-1},A_N) &= \mathbf{F}^{N-1}(\mathbf{P},A_1,\dots,A_{N-1}),\\
\lim_{A_N\to\infty} \mathbf{S}^N(\mathbf{P},A_1,\dots,A_{N-1},A_N) &= \mathbf{S}^{N-1}(\mathbf{P},A_1,\dots,A_{N-1}),
\end{aligned}$$

hence

$$\begin{aligned}
&\mathbf{F}^N(\mathbf{Du}^N(t,\mathbf{x},A_1,\dots,A_{N-1},A_N)) - \mathbf{F}^{N-1}(\mathbf{Du}^{N-1}(t,\mathbf{x},A_1,\dots,A_{N-1})),\\
&\quad = \mathbf{F}^N(\mathbf{Du}^N(t,\mathbf{x},A_1,\dots,A_{N-1},A_N)) - \mathbf{F}^{N-1}(\mathbf{Du}^N(t,\mathbf{x},A_1,\dots,A_{N-1},A_N)),\\
&\qquad + \mathbf{F}^{N-1}(\mathbf{Du}^N(t,\mathbf{x},A_1,\dots,A_{N-1},A_N)) - \mathbf{F}^{N-1}(\mathbf{Du}^{N-1}(t,\mathbf{x},A_1,\dots,A_{N-1})),
\end{aligned}$$

and the first line on the right-hand side vanishes for large enough $A_N$, by the properties of the multiple approximation; while the second one converges to zero due to the continuity of $\mathbf{F}^{N-1}$ and the point-wise convergence of $\mathbf{Du}^N$. The same argument applies also to $\mathbf{S}^N$.

The classical result stating that the weak limit in Lebesgue spaces and the a.e. limit coincide (cf. [25]) and (3.20) imply that

$$\widehat{\mathbf{F}^{N-1}} = \mathbf{F}^{N-1}(\mathbf{Du}^{N-1}(A_1,\dots,A_{N-1})) \quad \text{in } L^2(0,T;W^{1,2}(\Omega)).$$

This identification, the convergences in (3.20), and the lower semicontinuity of norms proves that, for a.e. $t\in I$, it holds

$$\begin{aligned}
&\|\mathbf{u}^{N-1}(t)\|_{W^{1,2}}^2 + \|\mathbf{F}^{N-1}(\mathbf{Du}^{N-1}(t))\|_2^2\\
&\quad + \int_0^t \left\| \frac{\partial \mathbf{u}^{N-1}(s)}{\partial t} \right\|_2^2 + \|\nabla \mathbf{F}^{N-1}(\mathbf{Du}^{N-1}(s))\|_2^2 \, ds \leq C
\end{aligned} \tag{3.23}$$

with a constant $C$ depending on the data of the problem (1.1), but independent of $A_n$, for $n = 1, \dots, N-1$.

We have now to pass to the limit in the weak formulation (3.2) of the approximate problem (3.1). Since, in view of (3.20), we easily deal with the time derivative and the right-hand side $\mathbf{f}$, the crucial point is the justification of the limit

$$\int_0^T (\mathbf{S}^N(\mathbf{D}\mathbf{u}^N(t)), \mathbf{D}\mathbf{w})\,\psi(t)\,dt \to \int_0^T (\mathbf{S}^{N-1}(\mathbf{D}\mathbf{u}^{N-1}(t)), \mathbf{D}\mathbf{w})\,\psi(t)\,dt, \tag{3.24}$$

for all $\psi \in C_0^\infty(I)$ and all $\mathbf{w} \in C_0^\infty(\Omega)$. At the moment we already know that $\lim_{A_N\to\infty} \mathbf{S}^N(\mathbf{D}\mathbf{u}^N) = \mathbf{S}^{N-1}(\mathbf{D}\mathbf{u}^{N-1})$ holds a.e. in $I \times \Omega$. Thus, to conclude it is sufficient to show that $\mathbf{S}^N(\mathbf{D}\mathbf{u}^N)$ is bounded uniformly with respect to $N$ in $L^q(I \times \Omega)$, for some $q > 1$. To this end we observe that Corollary 2.33, Proposition 2.17, the definition of $\mathbf{F}_{\omega_{q_N,\delta}}$ in (2.11), and $q_N \geq 2$ imply that for all $\mathbf{P} \in \mathbb{R}^{3\times 3}$ there holds

$$\begin{aligned} |\mathbf{F}^N(\mathbf{P})|^2 &\geq c\,\delta^{p-q_N}|\mathbf{F}_{\omega_{q_N,\delta}}(\mathbf{P})|^2 = c\,\delta^{p-q_N}(\delta + |\mathbf{P}^{\mathrm{sym}}|)^{q_N-2}|\mathbf{P}^{\mathrm{sym}}|^2 \\ &\geq c\,\delta^{p-q_N}|\mathbf{P}^{\mathrm{sym}}|^{q_N}. \end{aligned} \tag{3.25}$$

The a priori bound (3.19) and parabolic embedding imply that $\mathbf{F}^N(\mathbf{D}\mathbf{u}^N)$ is bounded in $L^{\frac{10}{3}}(I \times \Omega)$ by a constant depending only on the data of problem (1.1). This together with (3.25) and $\frac{5}{3}q \geq q + \frac{4}{3}$, valid for all $q \geq 2$, implies

$$\|\mathbf{D}\mathbf{u}^N\|_{L^{q_N+\frac{4}{3}}(I\times\Omega)} \leq C$$

with a constant independent of $A_N$. Corollary 2.33 also implies that

$$\begin{aligned} |\mathbf{S}^N(\mathbf{D}\mathbf{u}^N(t,\mathbf{x}))| &\leq c\,A_{N-1}^{p-q_{N-1}}\,(\omega_{q_{N-1},\delta})'(|\mathbf{D}\mathbf{u}^N(t,\mathbf{x})|) \\ &\leq C\,A_{N-1}^{p-q_{N-1}}\,\big(\delta^{q_{N-1}-1} + |\mathbf{D}\mathbf{u}^N(t,\mathbf{x})|^{q_{N-1}-1}\big). \end{aligned}$$

Hence, the latter estimates prove that

$$\|\mathbf{S}^N(\mathbf{D}\mathbf{u}^N)\|_{L^{(4/3+q_N)/(q_{N-1}-1)}(I\times\Omega)} \leq C(A_{N-1}),$$

where the constant $C$ depends on the data of the problem (1.1), on $A_{N-1}$, but is independent of $A_N$. Thus, we can infer that there exists $\widehat{\mathbf{S}^{N-1}}$ such that (up possibly to a further relabelled sub-sequence)

$$\lim_{A_N\to\infty} \mathbf{S}^N(\mathbf{D}\mathbf{u}^N) = \widehat{\mathbf{S}^{N-1}} \quad \text{weakly in } L^{(4/3+q_N)/(q_{N-1}-1)}(I \times \Omega), \tag{3.26}$$

provided $(4/3 + q_N)/(q_{N-1} - 1) > 1$, which is equivalent to

$$q_{N-1} - q_N < \frac{7}{3},$$

which motivated the choice of $q_n$ in Definition 3.5. Using again the classical result stating that the weak limit in Lebesgue spaces and the a.e. limit coincide (cf. [25]) we infer from (3.22) and (3.26) that

$$\widehat{\mathbf{S}^{N-1}} = \mathbf{S}^{N-1}(\mathbf{D}\mathbf{u}^{N-1}(A_1, \ldots, A_{N-1})) \quad \text{in } L^{(\frac{4}{3}+q_N)/(q_{N-1}-1)}(I \times \Omega),$$

which in turn implies (3.24). Thus we proved that $\mathbf{u}^{N-1}$ satisfies (3.23) and

$$\int_0^T \Big(\frac{\partial \mathbf{u}^{N-1}(t)}{\partial t}, \mathbf{w}\Big)\,\psi(t)\,dt + \int_0^T (\mathbf{S}^{N-1}(\mathbf{D}\mathbf{u}^{N-1}(t)), \mathbf{D}\mathbf{w})\,\psi(t)\,dt = \int_0^T (\mathbf{f}(t), \mathbf{w})\,\psi(t)\,dt,$$

for all $\psi \in C_0^\infty(I)$ and all $\mathbf{w} \in C_0^\infty(\Omega)$.

At this point we can repeat exactly the same argument by replacing $N$ with $N-1$. Thus, one obtains inductively that for all $n = 1, \dots, N-1$ there holds

$$\int_0^T \Big(\frac{\partial \mathbf{u}^{n-1}(t)}{\partial t}, \mathbf{w}\Big)\, \psi(t)\, dt + \int_0^T (\mathbf{S}^{n-1}(\mathbf{D}\mathbf{u}^{n-1}(t)), \mathbf{D}\mathbf{w})\, \psi(t)\, dt = \int_0^T (\mathbf{f}(t), \mathbf{w})\, \psi(t)\, dt,$$

for all $\psi \in C_0^\infty(I)$ and all $\mathbf{w} \in C_0^\infty(\Omega)$. After $N$ iterations we find, using also the density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ in the last step, that the vector field $\mathbf{u}^0 =: \mathbf{u}$ is a regular solution of the original problem problem (1.1). This finishes the proof of Theorem 3.4. □

Let us finish with stating the corresponding result to Theorem 3.4 in the steady case. This result can be proved, with many simplifications due to the absence of the time derivative and the better embedding results in the steady case (cf. Sect. 1.1), exactly in the same way as the unsteady result Theorem 3.4. Thus, we have the following result:

**Theorem 3.22** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2,1}$ boundary, and assume that $\mathbf{f} \in L^2(\Omega)$. Let the operator $\mathbf{S}$, derived from a potential $U$, have $(p, \delta)$-structure for some $p \in (2, \infty)$, and $\delta \in (0, \infty)$ fixed but arbitrary.*

*Then, there exists a unique regular solution of the steady version of the system (1.1), i.e., $\mathbf{u} \in W_0^{1,p}(\Omega)$ fulfils for all $\mathbf{w} \in C_0^\infty(\Omega)$*

$$\int_\Omega \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}\mathbf{w}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{w}\, d\mathbf{x},$$

*and satisfies $\mathbf{F}(\mathbf{D}\mathbf{u}) \in W^{1,2}(\Omega)$ with norm depending only on the characteristics of $\mathbf{S}$, $\delta^{-1}$, $\Omega$, and $\|\mathbf{f}\|_2$.*

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