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Calculus of Variations

Local uniqueness of ground states for the generalized Choquard equation

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Abstract

We consider the generalized Choquard equation of the type

$$-\Delta Q + Q = I(|Q|^p)|Q|^{p-2}Q,$$

for $3 \leq n \leq 5$, with $Q \in H^1_{rad}(\mathbb{R}^n)$, where the operator I is the classical Riesz potential defined by $I(f)(x) = (-\Delta)^{-1} f(x)$ and the exponent $p \in (2, 1 + 4/(n-2))$ is energy subcritical. We consider Weinstein-type functional restricted to rays passing through the ground state. The corresponding real valued function of the path parameter has an appropriate analytic extension. We use the properties of this analytic extension in order to show local uniqueness of ground state solutions. The uniqueness of the ground state solutions for the case $p = 2$, i.e. for the case of Hartree–Choquard, is well known. The main difficulty for the case $p > 2$ is connected with a possible lack of control on the L^p norm of the ground states as well on the lack of Sturm's comparison argument.

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1 Main results

In this work we study the uniqueness of the ground states for generalized Choquard equation

$$-\Delta u + u = I(|u|^p)|u|^{p-2}u. \tag{1.1}$$

Here and below $I(f)$ is the Riesz potential defined by

$$I(f)(x) = (-\Delta)^{-1} f(x) = G_0 * f(x), \; G_0(|y|) = \frac{1}{(n-2)|\mathbb{S}^{n-1}|}\frac{1}{|y|^{n-2}}, \tag{1.2}$$

where $3 \le n \le 5$ and $|\mathbb{S}^{n-1}| = n\pi^{n/2}/\Gamma(1+n/2)$ being the surface measure of the unit sphere in $\mathbb{R}^n$. The active study of the existence and qualitative behavior of the ground states Q is closely connected with stability/instability properties of the corresponding standing waves $U(t,x) = e^{i\omega t}u(x)$ that are solutions of the Cauchy problem for NLS

$$\begin{aligned} &i\partial_t U + \Delta U + I(|U|^p)|U|^{p-2}U = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n,\\ &U(0,x) = u(x). \end{aligned} \tag{1.3}$$

The study of the H^1-evolution dynamics of this Cauchy problem is motivated by the important question of orbital stability/instability properties of the standing waves. The existence of ground states is studied in [4, 17, 18], while [20, 21] treat the decay and scattering properties of the ground states. A detailed classification result for linearized stability properties of the standing waves is obtained in [5]. Considering linearization of (1.3) around standing waves, one can apply the classification results from [5] and deduce that linearized stability holds for $p \in (1+2/n, 1+4/n)$, while linearized instability is fulfilled for $p \in [1+4/n, 1+4/(n-2))$. The ground states in this case can be obtained (see Theorem 2 in [5]) via the minimization problem

$$\mathcal{E}_\sigma = \inf_{u\in H^1,\ \|u\|^2_{L^2}=\sigma} E_p(u). \tag{1.4}$$

Here and below

$$E_p(u) = \frac{1}{2}\|\nabla u\|^2_{L^2} - \frac{1}{2p}D(|u|^p, |u|^p), \tag{1.5}$$

where

$$D(|u|^p, |u|^p) = \langle I(|u|^p), |u|^p\rangle_{L^2} = \left\|(-\Delta)^{-1/2}|u|^p\right\|^2_{L^2}. \tag{1.6}$$

Since the local uniqueness of ground states for $n = 3$ and $p < 7/3$ is already discussed in [7] and since our goal is to study the general case $3 \le n \le 5$ and $2 < p < 1+4/(n-2)$, we shall turn back to the approach in [17] where the ground states are associated with Weinstein-type functional

$$W_p(u) = \frac{\|\nabla u\|^2_{L^2} + \|u\|^2_{L^2}}{\sqrt[p]{D(|u|^p, |u|^p)}}. \tag{1.7}$$

Namely, we define

$$\mathcal{W} = \inf_{u\in H^1_{rad}\setminus\{0\}} W_p(u), \tag{1.8}$$

where

$$H^1_{rad}(\mathbb{R}^n) = \{u \in H^1(\mathbb{R}^n);\, u(x) = u(|x|)\}.$$

One can use the Gagliardo–Nirenberg type inequality

$$D(|u|^p, |u|^p) \leq C\|u\|_{L^2}^{n+2-p(n-2)}\|\nabla u\|_{L^2}^{np-(n+2)} \tag{1.9}$$

and verify that $\mathcal{W}$ is a positive constant. Nontrivial minimizer $u \in H^1_{rad}$ of (1.8) exists (see [5, 17]) and it can be normalized (multiplying it by appropriate constant) so that it satisfies the Euler–Lagrange equation (1.1) and the condition

$$\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 = D(|u|^p, |u|^p). \tag{1.10}$$

As a consequence, it satisfies the Pohozaev identity

$$\frac{\|\nabla u\|^2}{np-n-2} = \frac{D(|u|^p, |u|^p)}{2p}.$$

Summarizing, we have the following relations

$$\frac{\|u\|^2}{\beta} = \frac{\|\nabla u\|^2}{\gamma} = \frac{D(|u|^p, |u|^p)}{p} = k_{\mathcal{W}}, \tag{1.11}$$

where

$$\beta = \frac{n+2-p(n-2)}{2}, \ \gamma = \frac{np-n-2}{2} = p - \beta \tag{1.12}$$

and

$$k_{\mathcal{W}} = \frac{1}{p}\mathcal{W}^{p/(p-1)}.$$

Now we can state our first main result, which treats the local uniqueness of minimizers Q of (1.8), satisfying the normalization condition (1.10).

Theorem 1.1 *Assume $n \geq 3$ and $2 < p < 1+4/(n-2)$. Then one can find $\varepsilon \in (0,1)$, so that for any two radial positive minimizers $Q_1, Q_2 \in H^1_{rad}$ of (1.8), satisfying the normalization condition (1.10) and such that*

$$\|Q_1 - Q_2\|_{L^2_{rad}} \leq \varepsilon,$$

we have $Q_1 = Q_2$.

Remark 1.1 Note that the Pohozaev normalization conditions (1.11) are obtained as a consequence of the fact that Q is a minimizer of (1.8) and satisfies (1.10), so there is universal constant $R > 0$, so that

$$\|Q\|_{H^1} \leq R \tag{1.13}$$

for any minimizer Q satisfying (1.10).

Another important question is the nondegeneracy of the ground state. The degeneracy of the ground state means that the kernel of the operator

$$L_+ = -\Delta + 1 - pI(Q^{p-1}\cdot)Q^{p-1} - (p-1)I(Q^p)Q^{p-2} \tag{1.14}$$

is non trivial on H^1_{rad}. Here and below $Q(|x|) \in H^1_{rad}(\mathbb{R}^n)$ is a radial positive solution of (1.1), so that setting $A(|x|) = (-\Delta)^{-1}Q^p(|x|)$ and $r = |x|$ we have the following ordinary differential system

$$
\begin{aligned}
&-\partial_r^2 Q(r) - \frac{n-1}{r}\partial_r Q(r) + Q(r) = A(r)Q(r)^{p-1} \\
&-\partial_r^2 A(r) - \frac{n-1}{r}\partial_r A(r) = Q^p(r).
\end{aligned}
\tag{1.15}
$$

The operator L_+ becomes

$$
L_+ = -\partial_r^2 - \frac{n-1}{r}\partial_r + 1 - pI(Q^{p-1}\cdot)Q^{p-1} - (p-1)AQ^{p-2}. \tag{1.16}
$$

Our next result treats the dimension of the kernel of L_+ in $H^1_{rad}(\mathbb{R}^n)$. To be more precise, if $h \in H^1_{rad}(\mathbb{R}^n) \cap \mathrm{Ker} L_+$, then we can have stronger regularity properties (see Proposition 2.2 below)

$$
h \in H^s_q,\ s \in [0, p+1),\ 1 < q < \infty,
$$

where the Sobolev space H^s_q, defined for $s \in \mathbb{R}$ and q as above, is the closure of the Schwartz functions under the norm $\|f\|_{H^s_q(\mathbb{R}^n)} = \|(1-\Delta)^{s/2} f\|_{L^q(\mathbb{R}^n)}$. If $h \in H^1_{rad}(\mathbb{R}^n)$ is a radial solution of the equation $L_+ h = 0$, then the couple of h and $B = (-\Delta)^{-1} Q^{p-1} h$ satisfies the system of nonlinear second-order differential equations

$$
\left.
\begin{aligned}
&h''(r) + \frac{n-1}{r}h'(r) = h(r) - pBQ^{p-1} - (p-1)AQ^{p-2}h, \\
&B''(r) + \frac{n-1}{r}B'(r) = -Q^{p-1}h.
\end{aligned}
\right\}
\tag{1.17}
$$

Our key point in the proof of Theorem 1.1 is the following.

Theorem 1.2 *There is no classical solution* (h, B) *of the Cauchy problem* (1.17) *with initial data*

$$
h(0) > 0,\ h'(0) = 0,\ B(0) < 0,\ B'(0) = 0.
$$

Now we can give some more precise information about the kernel of L_+.

Corollary 1.1 *If* $n \geq 3$, $2 < p < 1 + 4/(n-2)$, *then*

$$
\dim\left(\mathrm{Ker} L_+ \cap H^1_{rad}\right) \leq 1.
$$

Remark 1.2 It is well-known that nondegeneracy of the ground states plays crucial role in the applications (for example blow-up in mass-critical defocusing case, spectral stability/instability of ground states). The existence of nodal solutions is discussed in [8] and in [9]. Their results imply existence of non-trivial non radial solutions to (1.1) that minimize the energy functional over Nehari manifold. In the case $p > 2$ one can expect that these non-radial solutions are minimizers of the Weinstein functional over H^1 without radiality assumption. However, the existence of non-trivial radial solution to (1.17) remains an open problem. It is interesting to recall that uniqueness and nondegeneracy hold for $n = 3$ and $p > 2$ close to 2 [23]. Even in the case of degeneracy one can use appropriate modification of nondegeneracy assumption in order to control the spectral stability/instability as in [5].

Remark 1.3 Note that we treat the case $p > 2$. The analysis of the local uniqueness and a result similar to Corollary 1.1 in the interval $1 + \frac{2}{n} < p < 2$ is also important. However, we prefer to concentrate on the case $p > 2$ since our proofs use essentially the exponential decay of the ground state. In the case $1 + \frac{2}{n} < p < 2$, only polynomial decay occurs.

1.1 Overview on existing results and ideas to prove the main results

There are different methods to prove the uniqueness of positive radial minimizers of nonlinear elliptic equations with local-type nonlinearities. The method of McLeod and Serin [15, 16] and the subsequent refinements due to Kwong [11] are also based on Sturm's oscillation argument and therefore they work effectively for local type nonlinearities. In our case the nonlinearities involve the non-local Riesz potential and consequently we have met essential difficulties in following this strategy. The classical case $p = 2, n = 3$ has been studied in [13] (see [12] too), the approach is based on shooting method and the fact that the Riesz potential behaves like

$$I(|u|^2)(x) = \frac{\|u\|^2_{L^2}}{4\pi|x|} + o\left(|x|^{-1}\right), \quad x \to \infty \tag{1.18}$$

so that the conditions (1.11) become

$$\frac{\|u\|^2}{3} = \|\nabla u\|^2 = \frac{D(|u|^2, |u|^2)}{4}$$

in this case. Indeed, taking any two solutions u_1, u_2, we use the previous normalization conditions and from (1.18) we deduce

$$I(|u_1|^2)(x) - I(|u_2|^2)(x) = o\left(|x|^{-1}\right), \quad x \to \infty.$$

This gives the possibility to apply Sturm's argument and to follow shooting method to deduce uniqueness. If $p > 2$ and $n \geq 3$, then (1.18) becomes

$$I(|u|^p)(x) = c_n \frac{\|u\|^p_{L^p}}{|x|^{n-2}} + o\left(|x|^{-n+2}\right), \quad x \to \infty, \quad c_n = \frac{1}{(n-2)|\mathbb{S}^{n-1}|}$$

and obviously we lose uniqueness of the asymptotics of Riesz potential at infinity, since in this case the L^p norm is not presented in Pohozaev normalization conditions (1.11). Another key point in [13] is the application of Newton's formula (see Theorem 9.7 in [14]) valid for $n \geq 3$ and any radial functions $f(|x|)$ that is sufficiently regular and decaying at infinity

$$\begin{aligned} I(f)(x) &= \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} \frac{f(|y|)dy}{|x-y|^{n-2}} \\ &= \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} \frac{f(|y|)dy}{\max(|x|^{n-2}, |y|^{n-2})}. \end{aligned} \tag{1.19}$$

One can assume that Q is a radial positive minimizer of Lemma 5.1 satisfying (1.1). In the case $p = 2$ the Newton's identity enables one to take radial $\xi \in \operatorname{Ker} L_+$, where L_+ is the operator

$$L_+ = -\Delta + 1 - 2I(Q\cdot)Q - I(Q^2),$$

and rewrite $L_+\xi = 0$ as $\mathscr{L}_+\xi = cQ$, with c being a real constant and

$$\mathscr{L}_+\xi = -\Delta\xi + \xi - \left(|x|^{-1} * |Q|^2\right)\xi + 2Q(r)\int_0^r s\left(1 - \frac{s}{r}\right)Q(s)\xi(s)ds.$$

One can check that similar application of the Newton's relation with $p > 2$ will lead to relation $\mathscr{L}_+\xi = c(x)Q$, where $c(x)$ is not a constant and

$$\mathscr{L}_+\xi = -\Delta\xi + \xi - c_n(p-1)\left(|x|^{-(n-2)} * |Q|^p\right)Q^{p-2}\xi$$

$$+\frac{p}{n-2}Q(r)^{p-1}\int_0^r s\left(1-\frac{s^{n-2}}{r^{n-2}}\right)Q^{p-1}(s)\xi(s)ds.$$

The fact that $c(x)$ is not a constant is the reason why we can not follow directly the approach developed in [12, 13] and therefore we are trying to obtain only local uniqueness of ground state following a different idea. The case $n=3$ and $2<p<2+\delta$ with $\delta>0$ small is studied in [23]. Since the nondegeneracy property of L_+ is fulfilled for $p=2, n=3$, the author in [23] shows that Q_p, the ground state for $p \searrow 2$, is close to Q_2 and obtains nondegeneracy (and uniqueness) for p sufficiently close to 2. The result in Theorem 1.1 guarantees local uniqueness of minimizers in the general case $p\in(2,5)$ for $n=3$ and also for

$$p \in \begin{cases} (2,3), & \text{if } n=4; \\ (2,7/3), & \text{if } n=5. \end{cases}$$

The case $n=3$ and $5/3<p<1+4/3$ was announced [7], where local uniqueness is established. Here we give detailed proof in the case $n\geq 3, 2<p<1+4/(n-2)$ that clarifies some missing points in the proofs for the particular case $n=3$, $p<1+4/n$ studied in [7]. The approach in [7] is based on the construction of analytic function

$$K(z)=\frac{D((Q+zh)^p,(Q+zh)^p)}{\left(\|\nabla Q\|^2+\sigma+z^2(\|\nabla h\|^2+1)\right)^p}, \tag{1.20}$$

where $h\in \mathrm{Ker}L_+$ is orthogonal to Q and $\sigma=\|Q\|^2_{L^2}$. The construction of analytic extension of K depends essentially on the asymptotic behaviors of Q and h at infinity. In this work we continue to use the analytic extension of K, but a more precise asymptotic analysis is applied following the approach in [6]. Recall that [6] proves the uniqueness of the ground states associated with energy functional (1.5) with constraint $\|u\|_{L^p}=const$.

Another delicate point is the fact that the kernel of L_+ might be nontrivial. The key novelty in our work is the fact that $dim\mathrm{Ker}L_+\leq 1$ obtained in Corollary 1.1. On the other hand, the lack of Sturm's comparison argument for nonlocal ODE causes essential difficulties in treating the nondegeneracy of L_+ or to show nonexistence of nontrivial solutions of (1.17). Our approach to obtain the local uniqueness of the minimizer might allow degeneracy of L_+, however Theorem 1.2 guarantees that the Kernel of L_+ has at most one nontrivial solution. To show this fact we switch to new unknown functions

$$\begin{aligned} \xi_B(r) &= -\int_r^\infty \tau^{n-1}B(\tau)Q^p(\tau)d\tau, \\ \xi_h(r) &= -\int_r^\infty \tau^{n-1}A(\tau)Q^{p-1}(\tau)h(\tau)d\tau \end{aligned} \tag{1.21}$$

and consider the ODE system (3.8) for these quantities in the place of the ODE system (1.17). Here the key advantage of using the new quantities ξ_B, ξ_h is the fact that the initial conditions for ξ_B, ξ_h can be connected with the orthogonality conditions (see Lemma 2.3 and (2.22))

$$\begin{aligned} &h\perp Q,\quad h\perp L_+(Q), \\ &\xi_B(0) = -\int_0^\infty \tau^{n-1}B(\tau)Q^p(\tau)d\tau = 0. \end{aligned} \tag{1.22}$$

Next, we explain the main idea to prove that there is no solution of (1.17) having initial data

$$h(0)>0>B(0),\quad h'(0)=B'(0)=0.$$

We start with the following observation. If the first zero R_0 of $\xi_h(r)$ is finite, then

$$\begin{aligned} &\xi_h(r) > 0, r \in (0, R_0), \xi_h(R_0) = 0, \\ &\xi_h'(R_0) \le 0. \end{aligned} \tag{1.23}$$

Once h, B are given so that they satisfy (1.17) and (1.1), we define ξ_h, ξ_B and R_0. Then we are able to control the sign of ξ_B' on $(0, R_0)$.

Crucial point now is to introduce the combination

$$\xi_h(r) + \nu \xi_B(r)$$

assuming $\nu > 0$ chosen appropriately large.

Consider the set

$$N = \{\nu > 0; \xi_h'(r) + \nu \xi_B'(r) < 0, \ \forall r \in (0, R_0]\}.$$

On one hand, we have $\xi_B'(r) < 0$, for $r \in (0, R_0)$ as established in Lemma 3.5. If $R_0 < \infty$, then this Lemma gives

$$\xi_B'(R_0) < 0. \tag{1.24}$$

Further, Lemma 3.8 implies that the set N is connected and $N = (\nu_0, \infty)$ for some $\nu_0 > 0$. Hence we are able to find ν_0 so that

$$\begin{aligned} &\xi_h'(r) + \nu_0 \xi_B'(r) < 0, r \in (0, R_0), \\ &\xi_h'(R_0) + \nu_0 \xi_B'(R_0) = 0. \end{aligned} \tag{1.25}$$

On the other hand, the property (1.23) guarantees that

$$\xi_h'(R_0) \le 0 \tag{1.26}$$

and together with (1.24) this gives

$$\xi_h'(R_0) + \nu_0 \xi_B'(R_0) < 0$$

which contradicts (1.25). The contradiction shows that $R_0 = \infty$. But in this case ξ_B' and B have permanent sign on $(0, \infty)$ that is impossible due to orthogonality condition (1.22).

Outline of the paper

We organize the work as follows. Section 2 is devoted to the proof of Theorem 1.1, once the crucial result given in Theorem 1.2 is assumed to be acquired. Namely, we start by displaying our analysis on the first and the second linearization of a suitable version of the Weinstein functional (1.7), summarizing some properties of the ground states arising from (1.1). At this point we set up the main steps of the proof, introducing the definition of local uniqueness and how to show it by using an extension of the Weinstein functional in the complex plane (see (1.20)). As aforementioned, we utilize Corollary 1.1, which ensures that the space generated by the H^1 radial functions lying in the kernel of L_+, defined as in (1.14), has dimension one at most. Such a result is a straightforward consequence of Theorem 1.2, which will be established in Sect. 3. Finally, from Sect. 4 to "Appendix 7" a wide set of ancillary tools, mandatory for the proof of the main results, are developed.

2 Proof of Theorem 1.1

2.1 Preliminary facts and scheme of the proof

Lemma 5.1 guarantees that we have to show the local uniqueness of the minimizer Q, associated with the minimization problem

$$\mathcal{W}_\sigma = \inf_{u \in H^1_{rad}, \|u\|^2_{L^2} = \sigma} W_p(u), \tag{2.1}$$

where

$$\sigma = \beta k_{\mathcal{W}}, \; k_{\mathcal{W}} = \frac{1}{p} \mathcal{W}^{p/(p-1)}, \; \beta = \frac{n+2-p(n-2)}{2}.$$

Any minimizer Q has to satisfy the Euler–Lagrange equation

$$-\Delta Q + Q = A Q^{p-1}, \quad A = I(Q^p) = (-\Delta)^{-1}(Q^p) \tag{2.2}$$

as well the normalization conditions (1.11), i.e.

$$\frac{\|Q\|^2}{\beta} = \frac{\|\nabla Q\|^2}{\gamma} = \frac{D(Q^p, Q^p)}{p} = k_{\mathcal{W}}, \tag{2.3}$$

with

$$\gamma = \frac{np-n-2}{2} = p - \beta.$$

Let us start with the local regularity of ground states.

This question is discussed in Theorem 2 in [18] and therefore, if $Q \in H^1(\mathbb{R}^n)$ is a solution to

$$(1-\Delta)Q = |Q|^{p-2} Q (-\Delta)^{-1}(|Q|^p), \tag{2.4}$$

then $Q \in W^{2,q}_{loc}(\mathbb{R}^n)$. By using a bootstrap argument carefully, we can verify the following global elliptic bounds.

Proposition 2.1 *If*

$$2 < p < 1 + \frac{4}{n-2}, \quad n = 3, 4, 5 \tag{2.5}$$

and $Q \in H^1(\mathbb{R}^n)$ *is solution to* (2.4)*, then for any* $s \in [0, 1+p)$ *and for any* $q \in (1, \infty)$ *we have* [1]

$$\|Q\|_{H^s_q(\mathbb{R}^n)} \lesssim 1. \tag{2.6}$$

Corollary 2.1 *If the assumption* (2.5) *is fulfilled, then* $A = I(Q^p) = (-\Delta)^{-1}(Q^p)$ *satisfies*

$$\|D^s A\|_{L^q(\mathbb{R}^n)} \lesssim 1, \quad \forall s \in [0, p+2), \; q \in \left(\max\left(\frac{n}{n-2+s}, 1 \right), \infty \right). \tag{2.7}$$

[1] Here $H^s_q(\mathbb{R}^n)$ is the Sobolev space obtained as completion of smooth compactly supported functions with respect to the norm $\|f\|_{H^s_q(\mathbb{R}^n)} = \|(1-\Delta)^{s/2} f\|_{L^q(\mathbb{R}^n)}$.

We can use the regularity result in Proposition 2.1 and see that

$$Q \in H^s_q(\mathbb{R}^n), \ \forall s \in [0, 1+p), \ \forall q \in (1, \infty). \tag{2.8}$$

Further Corollary 2.1 gives (2.7). Using the Sobolev embedding we can see that for $n = 3, 4, 5, \ p > 2$ we have

$$H^s_q(\mathbb{R}^n) \subset C^3(\mathbb{R}^n), \ s \in (3, 1+p) \tag{2.9}$$

and $q \in (2, \infty)$. Therefore, we can consider positive radial minimizers Q that are strictly decreasing in $r = |x|$ and such that

$$\left.\begin{array}{l} Q \in H^2_{rad}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n) \\ A = (-\Delta)^{-1}(Q^p) \in L^q_{rad}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n), \ \nabla A \in H^1_{rad}(\mathbb{R}^n), \end{array}\right\} \tag{2.10}$$

where $q \in (n/(n-2), \infty)$. The regularity of the radial functions Q, A and the positiveness of Q imply

$$Q(0) > 0, \, Q'(0) = 0, \, A(0) > 0, \, A'(0) = 0.$$

The asymptotic behavior of Q is established in Corollary 4.1 as follows

$$Q(|x|) = c^\diamond(Q) G(|x|) \left(1 + O\left(e^{-|x|}\right)\right) \quad |x| \to \infty, \tag{2.11}$$

with $c^\diamond(Q) > 0$ and G being the fundamental solution of $1 - \Delta$. The minimizers of W_p are maximizers of $\frac{1}{W_p^p}$ so we can consider the functional

$$K(\varepsilon, h) = \frac{1}{W_p(Q + \varepsilon h)^p}, \tag{2.12}$$

which is well defined for $(\varepsilon, h) \in [-\varepsilon_0, \varepsilon_0] \times \{h \in H^1_{rad}, \|h\|_{L^2(\mathbb{R}^n)} = 1\}$ and $\varepsilon_0 > 0$ small. Then

$$K(\varepsilon, h) = \frac{D(|Q + \varepsilon h|^p, |Q + \varepsilon h|^p)}{\left(\|\nabla(Q + \varepsilon h)\|^2_{L^2} + \|Q + \varepsilon h\|^2_{L^2}\right)^p}$$

has Taylor expansion

$$K(\varepsilon, h) = K(0, h) + \partial_\varepsilon K(0, h)\varepsilon + \frac{1}{2}\partial^2_\varepsilon K(0, h)\varepsilon^2 + o(\varepsilon^2), \tag{2.13}$$

where

$$\left.\begin{array}{l} K(0, h) = D(Q^p, Q^p)^{1-p}, \\ \partial_\varepsilon K(0, h) = -2pD(Q^p, Q^p)^{-p}\langle Q, L_- h\rangle_{L^2}, \\ \dfrac{1}{2}\partial^2_\varepsilon K(0, h) = -2p(p-1)D(Q^p, Q^p)^{-p-1}D(Q^p, Q^{p-1}h)^2 - \\ - pD(Q^p, Q^p)^{-p}\langle L_+ h, h\rangle_{L^2} \end{array}\right\} \tag{2.14}$$

and

$$\left.\begin{array}{l} L_- h = -\Delta h + h - (-\Delta)^{-1}(Q^p)Q^{p-2}h, \\ L_+ h = -\Delta h + h - pQ^{p-1}(-\Delta)^{-1}(Q^{p-1}h) - \\ - (p-1)Q^{p-2}h(-\Delta)^{-1}(Q^p). \end{array}\right\} \tag{2.15}$$

It will be convenient to introduce the following.

Definition 2.1 If

$$\mathcal{A} \subseteq H^1_{rad}(\mathbb{R}^n),$$

then we shall say that local uniqueness of the minimizer Q holds on $\mathcal{A}$, if there exists $\varepsilon_0 = \varepsilon_0(\mathcal{A}) > 0$, so that

$$W_p\,(Q + \varepsilon h) > W_p(Q)$$

is fulfilled for any $h \in \mathcal{A}$ and for any $\varepsilon \in (0, \varepsilon_0]$.

Now we are ready to explain the scheme of the proof.

Step I Proof of (2.13) and coercive estimate needed for Step II.

Step II Dichotomy property (see Lemma 2.4): reduction of the proof to check uniqueness only on the one-dimensional space (due to Corollary 1.1)

$$\mathrm{Ker} L_+ \cap Q^\perp \cap (\Delta Q)^\perp.$$

Step III Check of the fact that $K(z) = K(z, h)$ is analytic near the origin and it is Hölder continuous in $h \in H^1_{rad}$.

Step IV Construction of analytic extension of $K(z)$ in appropriate domain (see Fig. 1) in the complex plane and verification that $K(z)$ can not be a constant.

2.2 Step I

As we promised above first we verify (2.13). We take $h \in H^1(\mathbb{R}^n)$. We have the expansions

$$\begin{aligned} D(|Q+\varepsilon h|^p, |Q+\varepsilon h|^p) &= D(Q^p, Q^p) + 2pD(Q^p, Q^{p-1}h)\varepsilon \\ &+ \left[p(p-1)D(Q^p, Q^{p-2}h^2) + p^2 D(Q^{p-1}h, Q^{p-1}h)\right]\varepsilon^2 + o(\varepsilon^2) \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\left(\|\nabla Q\|^2_{L^2} + \sigma + 2(\langle \nabla Q, \nabla h\rangle_{L^2} + \langle Q, h\rangle_{L^2})\,\varepsilon + \left(\|\nabla h\|^2_{L^2} + \|h\|^2_{L^2}\right)\varepsilon^2\right)^p} \\ &= \frac{1}{(\|\nabla Q\|^2_{L^2} + \sigma)^p} - \frac{2p(\langle \nabla Q, \nabla h\rangle_{L^2} + \langle Q, h\rangle_{L^2})\varepsilon}{(\|\nabla Q\|^2_{L^2} + \sigma)^{p+1}} \\ &\quad - \left[\frac{p}{(\|\nabla Q\|^2_{L^2} + \sigma)^{p+1}}\left(\|\nabla h\|^2_{L^2} + \|\,h\|^2_{L^2}\right)\right]\varepsilon^2 \\ &\quad + \left[\frac{4p(p+1)(\langle \nabla Q, \nabla h\rangle_{L^2} + \langle Q, h\rangle_{L^2})^2}{2(\|\nabla Q\|^2_{L^2} + \sigma)^{p+2}}\right]\varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

Hence

$$\begin{aligned} K(0, h) &= D(Q^p, Q^p)\frac{1}{(\|\nabla Q\|^2_{L^2} + \sigma)^p}, \\ \partial_\varepsilon K(0, h) &= -D(Q^p, Q^p)\frac{2p(\langle \nabla Q, \nabla h\rangle_{L^2} + \langle Q, h\rangle_{L^2})}{(\|\nabla Q\|^2_{L^2} + \sigma)^{p+1}} + \\ &+ 2pD(Q^p, Q^{p-1}h)\frac{1}{(\|\nabla Q\|^2_{L^2} + \sigma)^p}. \end{aligned}$$

From Lemma 5.1 we know that Q has to satisfy

$$D(Q^p, Q^p) = \|\nabla Q\|_{L^2}^2 + \|Q\|_{L^2}^2. \tag{2.16}$$

Thus

$$\begin{aligned}&\partial_\varepsilon K(0,h)\\ &\quad = -2pD(Q^p, Q^p)^{-p}\left(\langle \nabla Q, \nabla h\rangle_{L^2} + \langle Q, h\rangle_{L^2} - D(Q^p, Q^{p-1}h)\right)\end{aligned}$$

and we obtain the second identity in (2.13) as well as the Euler–Largange equation (2.2) together with the Pohozaev normalization conditions (2.3). Using (2.16), we obtain further

$$\begin{aligned}\frac{1}{2}\partial_\varepsilon^2 K(0,h) &= -2pD(Q^p, Q^{p-1}h)\frac{2p(\langle \nabla Q, \nabla h\rangle_{L^2} + \langle Q, h\rangle_{L^2})}{(\|\nabla Q\|_{L^2}^2 + \sigma)^{p+1}}\\ &\quad - D(Q^p, Q^p)\left[\frac{p}{(\|\nabla Q\|_{L^2}^2 + \sigma)^{p+1}}\left(\|\nabla h\|_{L^2}^2 + \| h\|_{L^2}^2\right)\right]\\ &\quad + 4D(Q^p, Q^p)\left[\frac{p(p+1)(\langle \nabla Q, \nabla h\rangle_{L^2} + \langle Q, h\rangle_{L^2})^2}{2(\|\nabla Q\|_{L^2}^2 + \sigma)^{p+2}}\right]\\ &\quad + \left[p(p-1)D(Q^p, Q^{p-2}h^2) + p^2 D(Q^{p-1}h, Q^{p-1}h)\right]\frac{1}{(\|\nabla Q\|_{L^2}^2 + \sigma)^p}\\ &= -\left(4p^2 - 2p(p+1)\right)D(Q^p, Q^p)^{-p-1}D(Q^p, Q^{p-1}h)^2 -\\ &\quad - pD(Q^p, Q^p)^{-p}(\|\nabla h\|_{L^2}^2 + \| h\|_{L^2}^2)\\ &\quad + pD(Q^p, Q^p)^{-p}\left[(p-1)D(Q^p, Q^{p-2}h^2) + pD(Q^{p-1}h, Q^{p-1}h)\right]\\ &= -2p(p-1)D(Q^p, Q^p)^{-p-1}D(Q^p, Q^{p-1}h)^2 - pD(Q^p, Q^p)^{-p}\langle L_+h, h\rangle_{L^2}.\end{aligned}$$

These relations imply

$$\begin{cases}\dfrac{1}{2}\partial_\varepsilon^2 K(0,h) = -2p(p-1)D(Q^p, Q^p)^{-p-1}D(Q^p, Q^{p-1}h)^2\\ \qquad - pD(Q^p, Q^p)^{-p}\langle L_+h, h\rangle_{L^2}.\end{cases} \tag{2.17}$$

The fact that Q is a minimizer of W_p satisfying the corresponding Euler–Lagrange equation (2.2) implies $L_- Q = 0$, so we have

$$\partial_\varepsilon K(0,h) = \langle Q, L_- h\rangle_{L^2} = 0. \tag{2.18}$$

Let us recall some of the known properties of the operators $L_\pm$.

Lemma 2.1 (see Lemma 1 in [5]) *The operator L_- is self-adjoint and non-negative on H^1_{rad}.*

The control of the sign of $\langle L_+h, h\rangle_{L^2}$ in (2.17) is realised on the space orthogonal to $L_+(Q)$ as stated in the next.

Lemma 2.2 *The operator L_+ satisfies the following properties*

(a) *L_+ is self-adjoint on H^1_{rad};*
(b) *L_+ has exactly one negative eigenvalue;*
(c) *L_+ is non-negative on a space of codimension 1. More precisely,*

$$\langle L_+h, h\rangle_{L^2} \geq 0 \tag{2.19}$$

for $h \perp L_+(Q)$.

Proof Most of the assertions are already established in [5] (see Lemma 1 and Lemma 6 in this work). For completeness we shall sketch the idea of the proof. The operator L_+ has the representation

$$\begin{aligned} &L_+ = -\Delta + 1 - \mathfrak{K}, \\ &\mathfrak{K}(h) = pQ^{p-1}(-\Delta)^{-1}(Q^{p-1}h) + (p-1)Q^{p-2}h(-\Delta)^{-1}(Q^p). \end{aligned} \tag{2.20}$$

One can easily show that it is a symmetric Δ-bounded operator on L^2_{rad} so L_+ is self-adjoint. Moreover $\mathfrak{R}$ maps any bounded domain in H^1_{rad} into a precompact set in L^2.

We turn to the proof of the inequality (2.19). The relation (2.15) implies

$$\left.\begin{aligned} L_+Q &= -2(p-1)Q^{p-1}(-\Delta)^{-1}(Q^p), \\ \langle L_+(Q), h\rangle_{L^2} &= -2(p-1)D(Q^p, Q^{p-1}h). \end{aligned}\right\} \tag{2.21}$$

Then we quote the identity (2.17) and note that

$$\begin{aligned} \frac{1}{2}\partial_\varepsilon^2 K(0,h) &= -pD(Q^p, Q^p)^{-p-1}D(Q^p, Q^{p-1}h)\langle L_+(Q), h\rangle_{L^2} \\ &-pD(Q^p, Q^p)^{-p}\langle L_+h, h\rangle_{L^2} = -pD(Q^p, Q^p)^{-p}\langle L_+h, h\rangle_{L^2} \end{aligned}$$

for $h \perp L_+(Q)$. Finally, we recall that $K(\varepsilon, h)$ has local maximum at $\varepsilon = 0$ so we have

$$\partial_\varepsilon^2 K(0,h) \le 0.$$

Therefore, we have (2.19) and hence L_+ can not have 2 negative eigenvalues. Thus, the existence of at least one negative eigenvalue follows from (2.21) so

$$\langle L_+(Q), Q\rangle_{L^2} = -2(p-1)D(Q^p, Q^p) < 0.$$

□

Therefore, we consider the set

$$\begin{aligned} &\{h \in H^1_{rad}; h \perp L_+(Q), \partial_\varepsilon^2 K(0,h) = 0\} \\ &\quad = \{h \in H^1_{rad}; h \perp L_+(Q), \langle L_+h, h\rangle_{L^2} = 0\}. \end{aligned}$$

The relation (2.21) guarantees that $h \perp L_+(Q)$ if and only if

$$D(Q^p, Q^{p-1}h) = 0$$

and the last identity can be rewritten in two equivalent forms

$$\left.\begin{aligned} D(Q^p, Q^{p-1}h) &= \int_0^\infty A(r)Q^{p-1}(r)h(r)r^{n-1}dr = 0, \\ D(Q^p, Q^{p-1}h) &= \int_0^\infty Q^p(r)B(r)r^{n-1}dr = 0, \end{aligned}\right\} \tag{2.22}$$

where

$$A(r) = (-\Delta)^{-1}(Q^p)(r), B(r) = (-\Delta)^{-1}(Q^{p-1}h)(r). \tag{2.23}$$

Then the fact that $K(\varepsilon, h)$ has a local maximum at $\varepsilon = 0$ implies

$$\langle L_+h, h\rangle_{L^2} \ge 0, \quad \forall h \perp L_+(Q) \tag{2.24}$$

for any $h \in H^1_{rad}$ with $\|h\|_{L^2} = 1$. One can verify the following

$$\{h \in H^1_{rad}; h \perp L_+(Q),\ \langle L_+ h, h\rangle_{L^2} = 0\} = \{h \in H^1_{rad}; h \perp L_+(Q),\ L_+ h = 0\}, \quad (2.25)$$

that follows from the stronger coerciveness property.

Lemma 2.3 *Assume*

$$2 < p < 1 + \frac{4}{n-2}.$$

Then we have

$$\langle L_+ h, h\rangle_{L^2} \geq C\|h\|^2_{H^1_{rad}},\ \forall\, h \in \mathcal{H}_Q, \quad (2.26)$$

where

$$\mathcal{H}_Q = \left\{h \in H^1_{rad},\ h \perp L_+(Q),\ h \perp \mathrm{Ker} L_+\right\}. \quad (2.27)$$

Proof To check this coercive estimate we follow [1, 22]. More precisely, we assume

$$\inf_{h \in \mathcal{H}_Q,\ \langle(-\Delta+1)h, h\rangle_{L^2}=1} \langle L_+ h, h\rangle_{L^2} = 0. \quad (2.28)$$

where $\mathcal{H}_Q$ is defined by (2.27). Using (2.20) we see that (2.28) is equivalent to

$$\inf_{h \in \mathcal{H}_Q, \langle \mathfrak{K} h, h\rangle_{L^2}=1} \langle(-\Delta+1)h, h\rangle_{L^2} = 1 \quad (2.29)$$

and equip the space

$$\mathrm{Ker} L_+ \cap H^1_{rad} \cap \{L_+(Q)\}^{\perp} = \{h \in H^1_{rad}; h \perp L_+(Q), L_+(h) = 0\} \quad (2.30)$$

with orthonormal basis. Since this space has maximal dimension $k \leq 1$ (see Corollary 1.1), we consider only the case $k = 1$, since the case $k = 0$ is similar. For this purpose, let the vector $\mathfrak{e} \neq 0$ generate the space (2.30). The minimization problem (2.28) has a minimization sequence $\{h_k\}_{k\in\mathbb{N}}$, satisfying all constraints. On the other hand, we have the representation (2.20) with operator $\mathfrak{K}$ being a compact operator in H^1_{rad}. In conclusion, taking a subsequence of $\{h_k\}$ we prove its convergence in H^1_{rad} to some $h^* \in H^1_{rad}$, satisfying $\langle \mathfrak{K} h^*, h^*\rangle_{L^2} = 1$, $h^* \perp L_+(Q)$, $h^* \perp \mathrm{Ker} L_+$ and

$$(-\Delta + 1)h^* = \lambda L_+(Q) + \lambda_1 \mathfrak{e} + \lambda_2 \mathfrak{K} h^*.$$

Multiplying by h^* and using (2.29) we find $\lambda_2 = 1$ so

$$L_+ h^* = \lambda L_+(Q) + \lambda_1 \mathfrak{e}.$$

Multiplying by $\mathfrak{e}$, we get $\lambda_1 = 0$. Hence, we have

$$L_+ h^* = \lambda L_+(Q). \quad (2.31)$$

Multiplying now by Q, we see that $\lambda = 0$ so $h^* \in \mathrm{Ker} L_+$ and this contradicts the properties $h^* \perp \mathrm{Ker} L_+$ and $\|h^*\|_{H^1_{rad}} = 1$. Therefore, we have the estimate (2.26). □

The coercive estimate (2.26) implies the following.

Corollary 2.2 *We have the relation*

$$\{h \in H^1_{rad}; h \perp L_+(Q), \partial^2_\varepsilon K(0, h) = 0\} = \{h \in H^1_{rad}; h \perp L_+(Q), L_+ h = 0, h \perp Q\}. \quad (2.32)$$

Proof We already know from identity (2.17) that

$$\{h \in H^1_{rad}; h \perp L_+(Q), \partial^2_\varepsilon K(0,h) = 0\} = \{h \in H^1_{rad}; h \perp L_+(Q), \langle L_+h, h\rangle_{L^2} = 0\}.$$

Combining the decomposition $h = h_1 + h_1^\perp$ with $h_1 \in \mathrm{Ker} L_+$ and $h_1^\perp \perp \mathrm{Ker} L_+$, the relation

$$\langle L_+(h_1 + h_1^\perp), (h_1 + h_1^\perp)\rangle_{L^2} = \langle L_+(h_1^\perp), (h_1^\perp)\rangle_{L^2}$$

and the coercive estimate (2.26) of Lemma 2.3, we conclude that

$$\langle L_+h, h\rangle_{L^2} = 0, h \perp L_+(Q)$$

implies $h_1^\perp = 0$ and $h \in \mathrm{Ker} L_+$. So

$$\{h \in H^1_{rad}; h \perp L_+(Q), \partial^2_\varepsilon K(0,h) = 0\} = \{h \in H^1_{rad}; h \perp L_+(Q), L_+h = 0\}.$$

It remains to show that

$$\{h \in H^1_{rad}; h \perp L_+(Q), L_+h = 0\} \subset \{h \perp Q\}.$$

For the purpose we use (7.1) of Lemma 7.1 and can write

$$L_+\left(SQ + \frac{2}{p-1}Q\right) = -2Q.$$

Since $Q \in \mathrm{Im} L_+$, we deduce

$$\langle h, Q\rangle_{L^2} = -\frac{1}{2}\langle L_+(h), \left(SQ + \frac{2}{p-1}Q\right)\rangle_{L^2} = 0.$$

The proof is complete. □

To see that the negativeness of the second derivative $\partial^2_\varepsilon K(0,h)$ implies the local uniqueness (as stated in Theorem 1.1) we turn to the next.

Corollary 2.3 *If*

$$\{h \in H^1_{rad}; h \perp L_+(Q), \partial^2_\varepsilon K(0,h) = 0\} = \{0\}, \tag{2.33}$$

then the local uniqueness holds.

Proof Indeed in this case, the set $\mathcal{H}_Q$ in (2.27) coincides with

$$\left\{h \in H^1_{rad},\ h \perp L_+(Q)\right\}$$

and the estimate (2.26) implies

$$\partial^2_\varepsilon K(0,h) = -2pD(Q^p, Q^p)^{-p}\langle L_+h, h\rangle_{L^2} \leq -C\|h\|^2_{H^1_{rad}},\ \forall\, h \in \mathcal{H}_Q. \tag{2.34}$$

Further, we look for $\varepsilon_0 > 0$ so that taking arbitrary $v \in H^1_{rad}$ with $\|v\|_{L^2} = 1$ and $v \perp Q$ we have

$$K(\varepsilon, v) < K(0, v) \tag{2.35}$$

provided $0 < \varepsilon \leq \varepsilon_0$. To verify this, we take $v \in H^1_{rad}$ with $\|v\|_{L^2} = 1$ and we represent v as $\alpha L_+(Q) + h_1$, where $h_1 \perp L_+(Q)$. If $\alpha = 0$, then (2.34) yields (2.35). If $\alpha \neq 0$, then $|\alpha| \leq 1/\|L_+(Q)\|_{L^2}$ and we use the representation $Q = \mu L_+(Q) + w_Q$, where $\mu < 0$ and $w_Q \perp L_+(Q)$. This relation shows that

$$v = \nu Q + h,\ \ h \perp L_+(Q),\ \nu \in \mathbb{R},\ \nu = \frac{\alpha}{\mu} \neq 0.$$

Then the assumption $v \perp Q$ implies $h \neq 0$ and we have the relations

$$K(\varepsilon, v) = \frac{1}{W_p(Q+\varepsilon v)^p} = \frac{1}{W_p(Q+\varepsilon \nu Q+\varepsilon h)^p} = K\left(\frac{\varepsilon}{1+\varepsilon \nu}, h\right).$$

Then using the coercive estimate (2.26) we arrive again at (2.35).

□

2.3 Step II

Our goal is to establish the local uniqueness of the minimizer Q, using the L^2 norm as a measure for the distance between two minimizers. Proposition 4.1 shows that

$$\|Q_1 - Q_2\|_{L^2_{rad}} \leq \varepsilon$$

implies a similar bound in H^1_{rad}. Then our goal is to show that for any $R > 2$ there exists $\varepsilon_0 > 0$, so that for any h in the set

$$\{g \in H^1_{rad}(\mathbb{R}^n);\ \|g\|_{L^2} = 1,\ \|g\|_{H^1_{rad}} \leq R\},$$

we have

$$W_p\,(Q+\varepsilon h) > W_p(Q),$$

for $0 < \varepsilon \leq \varepsilon_0$. Using the Taylor expansion (2.13) and the Corollary 2.2, we can conclude that the local uniqueness of Q is fulfilled on

$$\mathcal{B}_{rad}(R) \cap (\mathrm{Ker} L_+)^{\perp}\,, \tag{2.36}$$

where

$$\mathcal{B}_{rad}(R)\ = \{g \in H^1_{rad}(\mathbb{R}^n);\ \|g\|_{L^2} = 1,\ \|g\|_{H^1_{rad}} \leq R,\ g \perp Q, g \perp L_+(Q)\}.$$

Since $\mathrm{Ker} L_+$ is at most one dimensional we can take $\mathfrak{e}$ as the unit vector generating this kernel. Then using the coerciveness, we deduce the local uniqueness of Q on the set

$$\mathcal{B}_{rad}(R) \cap \{g;\ dist_{H^1_{rad}}(g, \mathrm{Ker} L_+) \geq \delta\} \tag{2.37}$$

for small $\delta > 0$. Indeed, if h is in the set (2.37), then $h = k + k^{\perp}$, where $k \in \mathrm{Ker} L_+$ and $k^{\perp} \perp \mathrm{Ker} L_+$ with

$$dist_{H^1_{rad}}(h, \mathrm{Ker} L_+) = \|k^{\perp}\|_{H^1_{rad}} \geq \delta$$

and

$$\langle L_+ h, h\rangle_{L^2} = \langle L_+ k^{\perp}, k^{\perp}\rangle_{L^2} \geq C\|k^{\perp}\|^2_{H^1} \geq C\delta$$

due to (2.26). So it remains to verify the local uniqueness of the minimizer Q on the domain

$$\mathcal{B}_{rad}(R) \cap \{g;\ dist_{H^1_{rad}}(g, \mathrm{Ker} L_+) \leq \delta\} \tag{2.38}$$

choosing sufficiently small δ.

Lemma 2.4 *Let $\mathfrak{e}$ generate $\mathrm{Ker} L_+$ and $\|\mathfrak{e}\|_{L^2} = 1$. We have the following two possibilities:*

(a) *there exists $\delta > 0$ so that local uniqueness of the minimizer Q is valid in (2.38);*

(b) *for any sequence* $\{\varepsilon_k\}_{k\in\mathbb{N}},\ \varepsilon_k \to 0$ *there exists subsequence* $\{\varepsilon_{m_k}\}_{k\in\mathbb{N}}$, *so that*

$$W_p\left(Q+\varepsilon_{m_k}\mathfrak{e}\right) = W_p(Q); \tag{2.39}$$

or

$$W_p\left(Q-\varepsilon_{m_k}\mathfrak{e}\right) = W_p(Q). \tag{2.40}$$

Proof Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$, be a sequence with $\varepsilon_k \to 0$. If the property (a) is not true, then for any $\delta = 1/m$, $m \in \mathbb{N}$ and for any $k \in \mathbb{N}$, we can find $h_{k,m} \in \mathcal{B}_{rad}(R)$ so that

$$W_p\left(Q+\varepsilon_k(h_{k,m})\right) = W_p(Q), \tag{2.41}$$

$dist_{H^1_{rad}}(h_{k,m}, \mathrm{Ker}L_+) < 1/m$ and

$$\|h_{k,m}\|_{H^1_{rad}} \le R. \tag{2.42}$$

Fix k and make the projection

$$h_{k,m} = g_{k,m} + g^{\perp}_{k,m},\quad g_{k,m} \in \mathrm{Ker}L_+,\ g^{\perp}_{k,m} \perp \mathrm{Ker}L_+$$

with

$$dist_{H^1_{rad}}(h_{k,m}, \mathrm{Ker}L_+) = \|g^{\perp}_{k,m}\|_{H^1_{rad}} < \frac{1}{m}. \tag{2.43}$$

Since the dimension of $\mathrm{Ker}L_+$ is at most 1 (due to Corollary 1.1) and we have (2.42), we can find a subsequence of $\{1/m\}_{m\in\mathbb{N}}$ that shall be denoted again as $\{1/m\}_{m\in\mathbb{N}}$ so that

$$\lim_{m\to\infty} g_{k,m} \to \pm\lambda\mathfrak{e} \in \mathrm{Ker}L_+, \tag{2.44}$$

where the convergence is in H^1_{rad} and $\lambda \ge 0$. The relation $\|h_{k,m}\|_{L^2} = 1$ and (2.43) yields $\lambda = 1$. Thus, we can justify the limit $m \to \infty$ in (2.41) so we obtain (2.39) or (2.40). Therefore the property (b) is established and the proof of the Lemma is complete. □

2.4 Step III

Let us assume that option (b) of Lemma 2.4 holds. Applying a bootstrap argument as in Proposition 2.1, we arrive at the following.

Proposition 2.2 *If* (2.5) *is fulfilled and* $h \in \mathrm{Ker}L_+ \cap H^1_{rad}(\mathbb{R}^n)$, *then for any* $s \in [0, 1+p)$ *and for any* $q \in (1,\infty)$ *we have*

$$\|h\|_{H^s_q(\mathbb{R}^n)} \lesssim 1. \tag{2.45}$$

Moreover, $B = (-\Delta)^{-1}(Q^{p-1}h)$ has regularity described in (2.7). For simplicity, we shall use a weaker regularity (similar to the one proposed in (2.10))

$$\left.\begin{aligned} &h \in H^2_{rad}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)\\ &B = (-\Delta)^{-1}(Q^{p-1}h) \in L^q_{rad}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n),\ \nabla B \in H^1_{rad}(\mathbb{R}^n).\end{aligned}\right\} \tag{2.46}$$

Then our goal is to take $\mathfrak{e}$ that generates $\mathrm{Ker}L_+$, normalized by $\|\mathfrak{e}\|_{L^2} = 1$ and show that one can find $\varepsilon_0 > 0$ so that

$$W_p(Q+\varepsilon h) > W_p(Q) \tag{2.47}$$

for $0 < \varepsilon \leq \varepsilon_0$. This conclusion is in contradiction with the assumption that (b) of Corollary 2.4 holds and therefore the statement of Theorem 1.1 will be established. Thanks to regularity property (2.46), we see that $h(r) = \mathfrak{e}(r)$ and $B(r) = I(Q^{p-1}\mathfrak{e})(r) \in C^2((0,\infty)) \cap C^1([0,\infty))$ satisfy the system of nonlinear second-order differential equations

$$\left.\begin{aligned} h''(r) + \frac{n-1}{r}h'(r) &= h(r) - pBQ^{p-1} - (p-1)I(Q^p)Q^{p-2}h, \\ B''(r) + \frac{n-1}{r}B'(r) &= -Q^{p-1}h. \end{aligned}\right\} \tag{2.48}$$

We can apply the asymptotic expansion (4.25), (4.27) and we find

$$\|h\|_{H^2_{rad}} + \|G^{-1}h\|_{L^\infty(|x|>1)} + \|G_0^{-1}B\|_{L^\infty(|x|>1)} \leq C. \tag{2.49}$$

Here and below $G(|x|) = G_n(|x|)$ is the fundamental solution of $(1-\Delta)$ having asymptotics

$$G_n(|x|) = c_\infty \frac{e^{-|x|}}{|x|^{(n-1)/2}}, \ |x| \to \infty, c_\infty > 0.$$

Following the scheme of the proof of the Theorem, we shall show that the function $K(\varepsilon) = K_h(\varepsilon)$ can be extended to an analytic function $K(z) = K_h(z)$ for complex z near the origin.

Lemma 2.5 *Let $h = \mathfrak{e}$ generate* $\mathrm{Ker}L_+$ *and $\|\mathfrak{e}\|_{L^2} = 1$. Then there exists $\varepsilon_0 > 0$ so that the function*

$$K_h(\varepsilon) = K(\varepsilon, h) = \frac{1}{W_p\,(Q+\varepsilon h)^p}, \quad \varepsilon \in [0, \varepsilon_0]$$

can be extended as analytic function

$$K_h : z \in \{z \in \mathbb{C}; |z| \leq \varepsilon_0\} \to \frac{D((Q+zh)^p, (Q+zh)^p)}{\left(\|\nabla Q\|^2 + \sigma + z^2(\|\nabla h\|^2+1)\right)^p}.$$

Proof We have the relation

$$\frac{1}{W_p\,(Q+\varepsilon h)^p} = \frac{D(|Q+\varepsilon h|^p, |Q+\varepsilon h|^p)}{\left((\|\nabla Q\|^2 + \varepsilon^2\|\nabla h\|^2) + (\sigma+\varepsilon^2)\right)^p}.$$

We obviously have the analyticity of

$$z \to \frac{1}{\left((\|\nabla Q\|^2+\sigma) + z^2(1+\|\nabla h\|^2)\right)^p} \tag{2.50}$$

near $z = 0$. In fact, (2.49) implies $\|h\|_{H^1} \leq C$ and hence there exists $\varepsilon_0 > 0$ so that (2.50) is analytic in $\{|z| \leq \varepsilon_0\}$. More delicate is the analyticity of the map

$$z \to D((Q+zh)^p, (Q+zh)^p).$$

In this case, we use (2.49) again and find the estimate

$$|h(r)|/Q(r) \leq C.$$

Then $\mathrm{Re}\,(1 + zh(r)/Q(r)) > 1/2$ for $|z|$ small and the function

$$z \to \left(1 + z\frac{h(r)}{Q(r)}\right)^p$$

is analytic near the origin, say $\{|z| < \varepsilon_0\}$ with $\varepsilon_0 < 1/(2C)$. Then,

$$z \to \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(1 + z\frac{h(|x|)}{Q(|x|)}\right)^p \left(1 + z\frac{h(|y|)}{Q(|y|)}\right)^p \frac{Q(x)^p Q(y)^p dx dy}{|x-y|} \tag{2.51}$$

is analytic in the same disk. This completes the proof. □

Remark 2.1 As mentioned in Sect. 1.1, the proof of the above lemma relies on the formula (1.20) introduced in the paper [7], which is a consequence of the orthogonality property $Q \perp h$ in H^1_{rad}, if $h \in \mathrm{Ker} L_+$. Specifically, we know that the operator L_- is self-adjoint on H^1_{rad} (see Lemma 2.1) and also that $\langle Q, L_- h\rangle_{L^2} = 0$, by (2.18). Moreover, since the operator L_+ is also self-adjoint on H^1_{rad} (see Lemma 2.2) and $h \in \mathrm{Ker} L_+$, we get

$$\langle Q, L_+ h\rangle_{L^2} = 0 = \langle L_+ Q, h\rangle_{L^2}.$$

An application of (2.15), (2.21) in combination with the fact that $Q \in \mathrm{Im} L_+$ due to (7.1), gives then the desired

$$\langle Q, h\rangle_{L^2} = 0 = \langle \nabla Q, \nabla h\rangle_{L^2}.$$

2.5 Step IV

Let us summarize the properties of the function

$$K(z) = \frac{1}{W_p\,(Q + z\mathfrak{e})^p}. \tag{2.52}$$

(i) $K(z)$ is analytic in z in a small neighborhood of the origin in $\mathbb{C}$;
(ii) the coefficients of the series expansion of $K(z)$ are real numbers;
(iii) for σ close to 0 in $\mathbb{R}$ we have local minimum at the origin: $K(\sigma) \geq K(0)$ and in the case (b) of Lemma 2.4 all partial derivatives $\partial_\sigma^m K(0)$ are identically zero, so $K(z)$ is a constant.

Then the final step in the proof of Theorem 1.1 is the following.

Lemma 2.6 *Let $h = \mathfrak{e}$ generate* $\mathrm{Ker} L_+$ *and $\|\mathfrak{e}\|_{L^2} = 1$. The function $K(z) = K_h(z)$ can not be a constant.*

Proof If $K(z)$ is a constant, then

$$K(z) = K(0) \tag{2.53}$$

near $z = 0$. Further, setting

$$w = w(z) = 1 + z\frac{h(r)}{Q(r)},$$

we have on the line $\{\mathrm{Re} z = \mathrm{Im} z\}$ the property

$$\mathrm{Re}\, w(z) = 1 + \mathrm{Re} z\frac{h(r)}{Q(r)} = 1 + \mathrm{Im} z\frac{h(r)}{Q(r)} = 1 + \mathrm{Im}\, w(z).$$

Since the principal value of Log w can be defined on the line $\mathrm{Re} w(z) = 1 + \mathrm{Im} w(z)$ as well as on its small neighborhood

$$\Lambda_\delta = \{|\mathrm{Re} z - \mathrm{Im} z| < \delta, \mathrm{Re} z > 0, \mathrm{Im} z > 0\}.$$

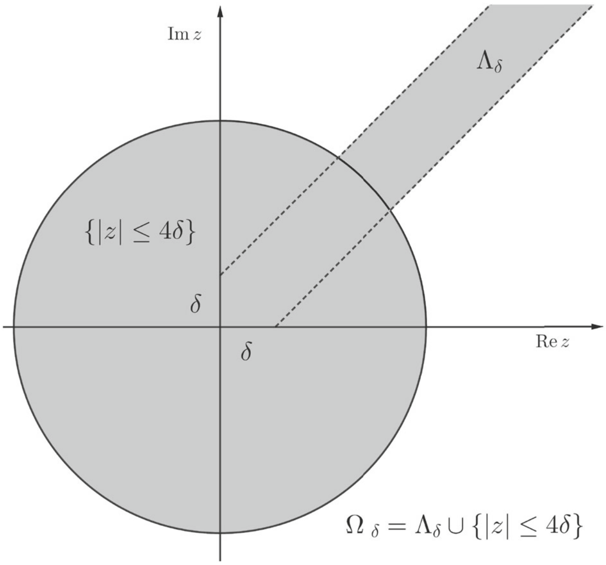

Fig. 1 The domain of analyticity of $K(z)$, here depicted by the shaded region Ω_δ

Indeed, we have

$$\mathrm{Re} w - \mathrm{Im} w = 1 + (\mathrm{Re} z - \mathrm{Im} z)\frac{h_\theta(r)}{Q(r)} \geq 1 - \delta\frac{|h(r)|}{Q(r)} \geq 1 - \delta C.$$

In conclusion we have analytic extension of

$$z \to D((Q + zh)^p, (Q + zh)^p)$$

in the domain

$$\Omega_\delta = \{|z| \leq 4\delta\} \cup \Lambda_\delta.$$

Our next step is to show that $K(z)$ can be extended as analytic function in Ω_δ. Indeed, we can show the analyticity of $\mathrm{Arg}(\sigma + z^2)$ on Ω_δ. For $|z| < 4\delta$ and $\delta < \sqrt{\sigma}/8$ one has $\mathrm{Re}\ (\sigma + z^2) > 3\sigma/4$. For $|z| > 4\delta$ and $z \in \Lambda_\delta$ it is easy to see that $\mathrm{Re}\ z > 2\delta$, then we have

$$\begin{aligned} \mathrm{Im}\ (\sigma + z^2) &= 2(\mathrm{Re}\ z)(\mathrm{Im}\ z) = 2(\mathrm{Re}\ z)^2 + 2\,\mathrm{Re}\ z(\mathrm{Im}\ z - \mathrm{Re}\ z) \\ &> 2(\mathrm{Re}\ z)^2 - 2\,\mathrm{Re}\ z\delta = 2\,\mathrm{Re}\ z(\mathrm{Re}\ z - \delta) > 4\delta(2\delta - \delta) = 4\delta^2. \end{aligned}$$

This shows that we can extend $K(z)$ as analytic function in the domain Ω_δ, so we can extend the relation (2.53) in the whole Ω_δ.

Choosing $z(R) = R + iR$ with $R \to \infty$, we can use the relation

$$\frac{1 + z(R)h(|x|)/Q(|x|)}{\sqrt{\sigma + z(R)^2}} \to \frac{h(|x|)}{Q(|x|)},$$

combined with Lebesgue dominated convergence theorem to conclude that

$$\lim_{R\to\infty} K(z(R)) = \frac{1}{W_p(\sqrt{\sigma}\ h)^p}.$$

The relation

$$W_p(\sqrt{\sigma}\, h)^p = \frac{1}{K(0)} = W_p(Q)^p$$

shows that $u(|x|) = \sqrt{\sigma}\, h$ is a minimizer of W_p, satisfying the constraint condition $\|u\|_{L^2}^2 = \sigma$. Hence the same is true for $|u(|x|)|$ and both of them satisfy the equation

$$-\Delta u + u = I(|u|^p)|u|^{p-2}u. \tag{2.54}$$

Since h is orthogonal to Q, there exists $r_0 > 0$, such that $h(r_0) = u(r_0) = 0$. Now we can use the following.

Lemma 2.7 *If u and $|u|$ solve (2.54), $u \in C^1(0,\infty)$ and there exists $r_0 > 0$, such that $u(r_0) = 0$, then $u(r) \equiv 0$.*

Proof If $u'(r_0) = 0$, then the Cauchy problem for the ODE (2.54) implies the assertion. If $u'(r_0) < 0$, then $|u(r)|$ is not differentiable in r_0. The proof is now completed.

Therefore, we are in position to apply Lemma 2.7 and to conclude that $u(r) = 0$ for any $r > 0$. This is an obvious contradiction since $\|h\|_{L^2} = 1$ and completes the proof. □

3 On radial solutions in the kernel of L_+

3.1 ODE system and its initial data

We recall that (Q, A) are solutions of

$$\left.\begin{aligned} Q''(r) + \frac{n-1}{r}Q'(r) &= Q(r) - A(r)Q^{p-1}, \\ A''(r) + \frac{n-1}{r}A'(r) &= -Q^p. \end{aligned}\right\} \tag{3.1}$$

If $h \in H^1_{rad}(\mathbb{R}^n)$ is a radial solution of the equation $L_+ h = 0$, then h and $B = (-\Delta)^{-1}(Q^{p-1}h)$ are sufficiently regular as in (2.46) so the pair (h, B) is a classical solution to the nonlinear ordinary differential equations system

$$\left.\begin{aligned} h''(r) + \frac{n-1}{r}h'(r) &= h(r) - pBQ^{p-1} - (p-1)AQ^{p-2}h, \\ B''(r) + \frac{n-1}{r}B'(r) &= -Q^{p-1}h. \end{aligned}\right\} \tag{3.2}$$

We can transform the system (3.2) into two equivalent forms replacing (h, B) by the normalized quantities

$$\widetilde{h}(r) = \frac{h(r)}{Q(r)},\ \widetilde{B}(r) = \frac{B(r)}{A(r)} \tag{3.3}$$

and the new unknown functions

$$\begin{aligned} \xi_B(r) &= -\int_r^\infty \tau^{n-1} B(\tau) Q^p(\tau)d\tau, \\ \xi_h(r) &= -\int_r^\infty \tau^{n-1} A(\tau) Q^{p-1}(\tau) h(\tau) d\tau. \end{aligned} \tag{3.4}$$

We note that we have the asymptotic expansions

$$\widetilde{h}(r) = \widetilde{h}_0(1 + O(r^2)), \tag{3.5}$$
$$\widetilde{B}(r) = \widetilde{B}_0(1 + O(r^2)), \tag{3.6}$$

near the origin.

Lemma 3.1 *We have the following properties*

(a) *the normalized quantities in* (3.3) *satisfy the system*

$$\left.\begin{aligned} \left[r^{n-1} Q^2(r)\widetilde{h}'\right]' &= -r^{n-1} A Q^p (p\widetilde{B} + (p-2)\widetilde{h}), \\ \left[r^{n-1} A^2(r)\widetilde{B}'\right]' &= -r^{n-1} A Q^p (\widetilde{h} - \widetilde{B}); \end{aligned}\right\} \tag{3.7}$$

(b) *the quantities* $\xi_h(r), \xi_B(r)$ *defined in* (3.4) *satisfy the following system*

$$\left.\begin{aligned} -\mathcal{L}_0(\xi_h)(r) &= \frac{1}{r^{n-1}Q^2(r)}\left[p\xi_B(r) + (p-2)\xi_h(r)\right], \\ -\mathcal{L}_0(\xi_B)(r) &= \frac{1}{r^{n-1}Q^2(r)}(\alpha(r)\xi_h(r) - \alpha(r)\xi_B(r)), \end{aligned}\right\} \tag{3.8}$$

where

$$\mathcal{L}_0(f)(r) = \left(\frac{f'(r)}{r^{n-1}AQ^p}\right)' \tag{3.9}$$

and

$$\alpha(r) = \frac{Q^2(r)}{A^2(r)} \in (0, 1].$$

Proof It is easy to obtain a system satisfied by $\widetilde{h}$ and $\widetilde{B}$. Indeed, we use the relations

$$Q^2\widetilde{h}' = h'Q - hQ', \ \ A^2\widetilde{B}' = B'A - BA'$$

and arrive at the system (3.7). Integrating over (r, ∞), we find

$$\begin{aligned} r^{n-1}Q^2(r)\widetilde{h}'(r) &= -p\xi_B(r) - (p-2)\xi_h(r), \\ r^{n-1}A^2(r)\widetilde{B}'(r) &= -\xi_h(r) + \xi_B(r). \end{aligned} \tag{3.10}$$

From

$$\begin{aligned} \xi_B'(r) &= r^{n-1}A(r)Q^p(r)\widetilde{B}(r), \\ \xi_h'(r) &= r^{n-1}A(r)Q^p(r)\widetilde{h}(r), \end{aligned}$$

we arrive at

$$\left.\begin{aligned} \left(\frac{\xi_h'(r)}{r^{n-1}AQ^p}\right)' &= -\frac{1}{r^{n-1}Q^2(r)}(p\xi_B(r) + (p-2)\xi_h(r)), \\ \left(\frac{\xi_B'(r)}{r^{n-1}AQ^p}\right)' &= -\frac{1}{r^{n-1}A^2(r)}(\xi_h(r) - \xi_B(r)) \end{aligned}\right\}$$

and thus we obtain (3.8). This completes the proof. □

Corollary 3.1 *For any* $\nu > 0$ *we have the equation*

$$- \mathcal{L}\left(\nu\xi_B(r) + \xi_h(r)\right)(r) = a(\nu, r)\xi_h(r), \tag{3.11}$$

where

$$\begin{cases} -\mathcal{L}(f)(r) := -\mathcal{L}_0(f)(r) + V(\nu, r)f = \\ -\left(\dfrac{f'(r)}{r^{n-1}AQ^p}\right)' + V(\nu, r)f(r), \\ V(\nu, r) = \dfrac{1}{r^{n-1}Q^2(r)}\left(\alpha(r) - \dfrac{p}{\nu}\right) \\ a(\nu, r) = \dfrac{1}{r^{n-1}Q^2(r)}\left[\nu\alpha(r) + (p-2) + \left(\alpha(r) - \dfrac{p}{\nu}\right)\right]. \end{cases} \tag{3.12}$$

Proof We have the identities

$$\begin{aligned} &-\left(\frac{(\nu\xi_B(r) + \xi_h(r))'}{r^{n-1}AQ^p}\right)' \\ &= \frac{1}{r^{n-1}Q^2(r)}\left[\nu\alpha(r)\xi_h(r) - \nu\alpha(r)\xi_B(r) + p\xi_B(r) + (p-2)\xi_h(r)\right] \\ &= \frac{1}{r^{n-1}Q^2(r)}\left[(\nu\alpha(r) + (p-2))\xi_h(r) - (\nu\alpha(r) - p)\xi_B(r)\right] \\ &= \frac{1}{r^{n-1}Q^2(r)}\left[(\nu\alpha(r) + (p-2) + \left(\alpha(r) - \frac{p}{\nu}\right))\xi_h(r)\right] \\ &\quad - \frac{1}{r^{n-1}Q^2(r)}\left[\left(\alpha(r) - \frac{p}{\nu}\right)(\nu\xi_B(r) + \xi_h(r))\right] \end{aligned}$$

and the proof is completed now. □

An important point in the proof of Lemma 3.1 is the following inequality

Lemma 3.2

$$A(r) \geq Q(r), \quad \forall r > 0. \tag{3.13}$$

Proof Indeed the inequality is true for r large due to asymptotic expansions of Sect. 4.3. For this we can define

$$r_* = \inf\{\tau;\, A(r) \geq Q(r), \quad \forall r > \tau\}.$$

If $r_* > 0$, then $A(r_*) = Q(r_*)$ and we have two possibilities:

Case A there exists $r_1 \in (0, r_*)$ so that $A(r) < Q(r), \quad \forall r \in (r_1, r_*)$ and $A(r_1) = Q(r_1)$. Then we use (3.1) and find

$$-\Delta(A - Q) + (A - Q)Q^{p-1} = Q \tag{3.14}$$

and applying the maximum principle for the interval (r_1, r_*) we arrive at a contradiction with the fact that Q is positive.

Case B $A(r) < Q(r), \quad \forall r \in [0, r_*)$. Thanks to regularity results of Proposition 2.1 we can assert that $A(r), Q(r) \in C^2[0, \infty)$. Since we have the Fuchs–Painleve system (3.1), we can apply Theorem 6.1 and deduce that $A(r), Q(r)$ can be extended as even functions. Using even extensions of A, Q on the real line, we deduce

$$A(r) < Q(r), \quad \forall r \in (-r_*, r_*), A(r_*) = Q(r_*), A(-r_*) = Q(-r_*).$$

Again an application of the maximum principle for (3.14) leads to a contradiction. □

Our next step is to study the asymptotic behavior of ξ_h, ξ_B near the origin and near infinity. By using the orthogonality conditions (2.22) and the definitions (3.4) of ξ_h, ξ_B, we find

$$\begin{aligned} \xi_h(0) &= -\int_0^\infty \tau^{n-1} A(\tau) Q^p(\tau) \widetilde{h}(\tau) d\tau = 0, \\ \xi_B(0) &= -\int_0^\infty \tau^{n-1} A(\tau) Q^p(\tau) \widetilde{B}(\tau) d\tau = 0. \end{aligned} \tag{3.15}$$

Therefore, we have the relations

$$\begin{aligned} \xi_h(r) &= \int_0^r \tilde{h}(\tau) \tau^{n-1} A(\tau) Q^p(\tau) d\tau, \\ \xi_B(r) &= \int_0^r \tilde{B}(\tau) \tau^{n-1} A Q^p(\tau) d\tau \end{aligned} \tag{3.16}$$

and hence we have the asymptotic expansion, given by next Lemma.

Lemma 3.3 *We have the following asymptotics near $r = 0$*

$$\begin{aligned} \xi_B(r) &= \frac{r^n}{n} A_0 Q_0^p \widetilde{B}_0 \left(1 + O(r^2)\right), \\ \xi_h(r) &= \frac{r^n}{n} A_0 Q_0^p \widetilde{h}_0 \left(1 + O(r^2)\right); \end{aligned} \tag{3.17}$$

where $A_0 = A(0)$, $Q_0 = Q(0)$.

Moreover, (3.4) imply that $\xi_h(r)$ and $\xi_B(r)$ have exponential decay at infinity. From Lemma 3.1 we know that $(\tilde{h}, \tilde{B})$ satisfies the equations in the system (3.7). We take initial data

$$\tilde{h}(0) = \tilde{h}_0, \tilde{B}(0) = \tilde{B}_0$$

so that

$$\tilde{h}_0 > 0 > \tilde{B}_0. \tag{3.18}$$

From (3.16) we arrive at the following ordering rules.

Lemma 3.4 *We have the following properties*

(a) *If*

$$\tilde{h}(t) > 0, \quad \forall t \in (0, T), \tag{3.19}$$

for some $T > 0$, then

$$\xi_h(t) > 0, \quad \forall t \in (0, T). \tag{3.20}$$

(b) *If*

$$\tilde{B}(t) < 0, \quad \forall t \in (0, T), \tag{3.21}$$

for some $T > 0$, then

$$\xi_B(t) < 0, \quad \forall t \in (0, T). \tag{3.22}$$

(c) *If*

$$\widetilde{h}(t) > 0 > \widetilde{B}(t), \quad \forall t \in (0, T) \tag{3.23}$$

for some $T > 0$, *then*

$$\xi_h(t) > 0 > \xi_B(t), \quad \forall t \in (0, T). \tag{3.24}$$

3.2 Scheme of the proof of Theorem 1.2

Let us make a plan of the proof of Theorem 1.2.

(a) We take initial data satisfying

$$h(0) > 0 > B(0), \quad h'(0) = B'(0) = 0 \tag{3.25}$$

and assume classical solution (ξ_h, ξ_B) exists.

(b) We define R_0 as the first zero of $\xi_h(r)$. Note that this is well-defined, since (3.25) implies positiveness of $\xi_h(r)$ for small $r > 0$. We have two cases:

(1) If $R_0 < \infty$, then

$$\begin{aligned} &\xi_h(r) > 0, r \in (0, R_0), \xi_h(R_0) = 0, \\ &\xi_h'(R_0) \leq 0. \end{aligned} \tag{3.26}$$

(2) If $R_0 = \infty$, then we require

$$\begin{aligned} &\xi_h(r) > 0, r \in (0, \infty), \lim_{r \to \infty} \xi_h(r) = 0, \\ &\lim_{r \to \infty} \xi_h'(r) = 0. \end{aligned} \tag{3.27}$$

(c) One can show that the second option in the previous point is impossible, while in the case $R_0 < \infty$ we can control the sign of ξ_B' on $(0, R_0)$ (see Lemma 3.5 below). To be more precise, the case $R_0 = \infty$ can be excluded, since permanent sign of ξ_B' means permanent sign of $\tilde{B}$ and this contradicts the orthogonality condition (3.15) (see Corollary 3.2 below).

(d) The previous point guarantees that there is a fixed $R_0 < \infty$, so that (3.26) holds. Once R_0 is fixed, we can find sufficiently large $\nu_0 > 0$, so that

$$\begin{aligned} &\xi_h'(r) + \nu_0 \xi_B'(r) < 0, r \in (0, R_0), \\ &\xi_h'(R_0) + \nu_0 \xi_B'(R_0) = 0. \end{aligned} \tag{3.28}$$

The precise statement and proof are given in Lemma 3.8.

(e) The relations (3.26) and (3.28) lead to a contradiction. In fact, formally from (3.26) and (3.28) we can have

$$\xi_h'(R_0) = \nu_0 \xi_B'(R_0) = 0.$$

However, in this case we can use the fact that

$$\xi_B'(r) < 0, 0 < r < R_0$$

and deduce that

$$\begin{aligned} &\xi_h'(r) + \nu \xi_B'(r) < 0, r \in (0, R_0), \\ &\xi_h'(R_0) + \nu \xi_B'(R_0) = 0 \end{aligned} \tag{3.29}$$

for any $\nu \geq \nu_0$. However, for ν large we prove in Lemma 3.6 below that

$$\xi_h'(R_0) + \nu \xi_B'(R_0) < 0 \tag{3.30}$$

holds. This is a clear contradiction with (3.29).

3.3 Proof of Theorem 1.2

Proof of Theorem 1.2 We define the maps

$$(h, B) \to (\widetilde{h}, \widetilde{B}) \to (\xi_h, \xi_B)$$

as in (3.3), (3.4). We start with point c) in the scheme of Sect. 3.2. □

Lemma 3.5 *Assume the initial data $\tilde{h}_0$, $\tilde{B}_0$ satisfy* (3.18) *and ξ_h, ξ_B are $C^2(0,\infty) \cap C([0,\infty))$ functions defined in* (3.16) *so that (ξ_h, ξ_B) is a classical solution of*

$$\left.\begin{aligned} -\left(\frac{\xi_h'(r)}{r^{n-1}AQ^p}\right)' &= \frac{1}{r^{n-1}Q^2(r)}(p\xi_B(r) + (p-2)\xi_h(r)),\\ -\left(\frac{\xi_B'(r)}{r^{n-1}AQ^p}\right)' &= \frac{1}{r^{n-1}A^2(r)}(\xi_h(r) - \xi_B(r)). \end{aligned}\right\} \tag{3.31}$$

Let $0 < R_0 \leq \infty$ be the first zero of $\xi_h(r)$, satisfying (3.26) *or* (3.27)*. Then we have*

$$\xi_B(r) < 0, \quad r \in (0, R_0) \tag{3.32}$$

and

$$\xi_B'(r) < 0, \quad r \in (0, R_0). \tag{3.33}$$

If $R_0 < \infty$, then

$$\xi_B'(r) < 0, \quad r \in (0, R_0]. \tag{3.34}$$

Proof We prove (3.33), since then (3.32) follows. To prove (3.33) we argue by a contradiction. Let $R_0 < \infty$. If $r_1 \in (0, R_0]$ is the first zero of ξ_B', such that $\xi_B'(r_1) = 0$ and ξ_B' is negative on $(0, r_1)$, then we can multiply the second equation in (3.31) by $\xi_B(r)$ and integrate by parts in $(0, r_1)$. Note that at this point it is crucial to use the asymptotics (3.17) near the origin. In this way we find

$$\int_0^{r_1} \frac{(\xi_B'(r))^2}{r^{n-1}AQ^p} dr = \int_0^{r_1} \frac{1}{r^{n-1}A^2(r)}(\xi_h(r) - \xi_B(r))\xi_B(r) dr. \tag{3.35}$$

The different signs in (3.35) lead to contradiction. The case $R_0 = \infty$ can be treated in a similar way. To assure the integration by parts on $(0, \infty)$ we use the exponential decay of ξ_h, ξ_B that follows from the definition (3.4) and the asymptotics of h, B.

This completes the proof. □

As it is mentioned in the point (c) in the scheme of Sect. 1.2, the above Lemma implies

Corollary 3.2 *Assume the initial data $\tilde{h}_0$, $\tilde{B}_0$ satisfy* (3.18) *and ξ_h, ξ_B are $C^2(0,\infty) \cap C([0,\infty))$ functions defined in* (3.16) *so that (ξ_h, ξ_B) is a classical solution of* (3.8)*. Then $R_0 < \infty$ and* (3.26) *is satisfied.*

Lemma 3.6 *Assume the initial data $\tilde{h}_0$, $\tilde{B}_0$ satisfy (3.18) and ξ_h, ξ_B are $C^2(0,\infty)\cap C([0,\infty))$ functions defined in (3.16) so that (ξ_h, ξ_B) is a classical solution of (3.8). Then there exists $\nu_0 > 0$, so that for any $\nu > \nu_0$ we have*

$$R(\nu) > R_0, \tag{3.36}$$

where $R(\nu) > 0$ is the first positive critical point of $\xi_h + \nu\xi_B$ and

$$\xi_h'(r) + \nu\xi_B'(r) < 0, \quad r \in (0, R(\nu)). \tag{3.37}$$

Proof We use Corollary 3.1, especially we recall that the Eq. (3.11) is fulfilled. We take

$$\nu_0 > \frac{p}{\inf_{[0,R_0]} \alpha(r)}$$

and then (3.11) yields

$$-\left(\frac{(\xi_h(r) + \nu\xi_B(r))'}{r^{n-1}AQ^p}\right)' + V(\nu, r)(\xi_h(r) + \nu\xi_B(r)) = a(\nu, r)\xi_h(r), \tag{3.38}$$

where

$$\begin{cases} V(\nu, r) = \dfrac{1}{r^{n-1}Q^2(r)}\left(\alpha(r) - \dfrac{p}{\nu}\right) > 0, \\ a(\nu, r) = \dfrac{1}{r^{n-1}Q^2(r)}\left[\nu\alpha(r) + (p-2) + \left(\alpha(r) - \dfrac{p}{\nu}\right)\right] > 0 \end{cases}$$

for $r \in (0, R_0)$, $\nu > \nu_0$. We can follow the proof of Lemma 3.5 so multiplying the equation by $\xi_h(r) + \nu\xi_B(r)$ and integrating over $(0, R_0)$, we arrive at contradiction if $R(\nu) \le R_0$. This completes the proof.

□

Consider the set

$$N = \{\nu > 0; \xi_h'(r) + \nu\xi_B'(r) < 0, \ \forall r \in (0, R_0]\}.$$

Given any $\nu \in N$, we denote by $R(\nu) > 0$ the first positive zero of $\xi_h'(r) + \nu\xi_B'(r)$, satisfying (3.37).

Lemma 3.7 *The set N is connected and open. More precisely, if $\nu_* \in N$, then for any $\nu > \nu_*$ we have $\nu \in N$.*

Proof Set

$$u_\nu(r) = \xi_h(r) + \nu\xi_B(r).$$

If $\nu_* \in N$, then we have

$$u_{\nu_*}'(r) < 0 \, \forall r \in (0, R(\nu_*)), R(\nu_*) > R_0.$$

Take any $\nu > \nu_*$. For any $r \in (0, R_0]$ we have

$$u_\nu'(r) = \xi_h'(r) + \nu\xi_B'(r) = u_{\nu_*}'(r) + (\nu - \nu_*)\xi_B'(r).$$

From $\nu_* \in N$ we have $u_{\nu_*}'(r) < 0$ for any $r \in (0, R_0]$. The inequality $\xi_B'(r) \le 0$ for $r \in (0, R_0]$ is established in Lemma 3.5. So the definition of N leads to $\nu \in N$. The fact that N is open follows from the strict inequality in the definition of N.

□

Finally, we show that $N \not\supseteq (0, \infty)$. More precisely, we can complete the point (d) in the scheme.

Lemma 3.8 *There exists $\nu_0 > 0$, so that $N = (\nu_0, \infty)$. Moreover, we have*

$$R(\nu_0) = R_0$$

and

$$\xi_h'(R_0) + \nu_0 \xi_B'(R_0) = 0.$$

Proof of Lemma 3.8 We know from (3.4) that

$$\xi_h'(r) + \nu \xi_B'(r) = r^{n-1} Q^{p-1}(r)(B(0)Q(0) + \nu A(0)h(0) + o(1))$$

as $r \searrow 0$. This fact and the choice $h(0) > 0 > B(0)$ imply that

$$\xi_h'(r) + \nu \xi_B'(r) > 0$$

for and $r > 0$ close to 0 and $\nu > 0$ small. Hence N can not be $(0, \infty)$. Let $N = (\nu_0, \infty)$ with $\nu_0 > 0$. If

$$\xi_h'(r) + \nu_0 \xi_B'(r) < 0, \ \forall r \in [0, R_0],$$

then we can find $\delta > 0$ so that $\nu_0 - \delta \in N$. This contradicts the fact that

$$N = (\nu_0, \infty).$$

□

As it is mentioned in the point (e) of the scheme we arrive at a contradiction. This proves Theorem 1.2. □

4 Asymptotics at infinity

In this section we aim to find more precise asymptotic expansions for two elliptic equations

$$-\Delta B(|x|) = H(|x|), \tag{4.1}$$
$$(-\Delta + 1)h(|x|) = F(|x|), \tag{4.2}$$

assuming sufficiently fast decay of the radial source terms F, H. Taking into account the regularity properties of Q obtained in Proposition 2.1 and the regularity of A obtained in Corollary 2.1, we shall assume

$$\left.\begin{aligned} &h(r) \in C^2(0,\infty) \cap C^1([0,\infty)),\ F(r) \in C(0,\infty),\\ &h(r) = O(1),\ h'(r) = O(1),\ F(r) = O(e^{-br}),\ \ b > 1, \quad \text{as } r \to \infty, \end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned} &B(r) \in C^2(0,\infty) \cap C^1([0,\infty)),\ H(r) \in C(0,\infty),\\ &B(r) = o(1),\ H(r) = O(r^{-a}),\ \ a > n \ \text{ as } r \to \infty. \end{aligned}\right\} \tag{4.4}$$

The asymptotic expansions concern solutions to these equations represented as follows

$$B = G_0 * H, \quad h = G * F,$$

where G_0 and G are the corresponding fundamental solutions.

4.1 Estimates and asymptotics of *B*

Lemma 4.1 *If B, H satisfy*

$$\left.\begin{aligned} &B(r) \in C^2(0,\infty) \cap C^1([0,\infty)),\ \ H(r) \in C(0,\infty),\\ &B(r) = o(1),\ \ H(r) = O(r^{-a}),\ \ a > n \ \ as\, r \to \infty \end{aligned}\right\} \tag{4.5}$$

and

$$-\left(r^{n-1}B'(r)\right)' = r^{n-1}H(r), \tag{4.6}$$

then we have

$$\left.\begin{aligned} \left|B(r) - G_0(r)\|H\|_{L^1(\mathbb{R}^n)}\right| &\lesssim r^{-n+2}\int_r^\infty |H(\tau)|\tau^{n-1}d\tau,\\ \left|B'(r) - G_0'(r)\|H\|_{L^1(\mathbb{R}^n)}\right| &\lesssim r^{-n+1}\int_r^\infty |H(\tau)|\tau^{n-1}d\tau, \end{aligned}\right\} \tag{4.7}$$

where

$$\|H\|_{L^1(\mathbb{R}^n)} = |\mathbb{S}^{n-1}|\int_0^\infty \tau^{n-1}H(\tau)d\tau.$$

Proof Integrating (4.6), we get

$$r^{n-1}B'(r) = -\int_0^r \tau^{n-1}H(\tau)d\tau. \tag{4.8}$$

Therefore, we have

$$B'(r) = -r^{-n+1}\int_0^\infty \tau^{n-1}H(\tau)d\tau + r^{-n+1}\int_r^\infty \tau^{n-1}H(\tau)d\tau.$$

Since

$$G_0(r) = \frac{1}{(n-2)|\mathbb{S}^{n-1}|}\frac{1}{r^{n-2}},\ \ G_0'(r) = -\frac{1}{|\mathbb{S}^{n-1}|}\frac{1}{r^{n-1}}, \tag{4.9}$$

we obtain the second estimate in (4.7). Since $B(r) = o(1)$, we get

$$B(r) = \frac{1}{n-2}\int_0^\infty \frac{H(\tau)}{[\max(r,\tau)]^{n-2}}\tau^{n-1}d\tau \tag{4.10}$$

so we can write

$$\begin{aligned} B(r) &= \frac{1}{n-2}\int_0^r \frac{H(\tau)}{r^{n-2}}\tau^{n-1}d\tau + \frac{1}{n-2}\int_r^\infty \frac{H(\tau)}{\tau^{n-2}}\tau^{n-1}d\tau\\ &= \frac{1}{n-2}\int_0^\infty \frac{H(\tau)}{r^{n-2}}\tau^{n-1}d\tau - \frac{1}{n-2}\int_r^\infty \frac{H(\tau)}{r^{n-2}}\tau^{n-1}d\tau + \frac{1}{n-2}\int_r^\infty \frac{H(\tau)}{\tau^{n-2}}\tau^{n-1}d\tau \end{aligned}$$

and we find

$$\left|B(r) - G_0(r)\int_{\mathbb{R}^n} H(|x|)dx\right| \lesssim G_0(r)\int_r^\infty |H(\tau)|\tau^{n-1}d\tau.$$

Therefore, we have the first asymptotic estimate in (4.7). ◻

The kernel $G(|x|)$ of $(1-\Delta)^{-1}$ is a radial positive decreasing function given by

$$G(|x|) = G_n(|x|) = (2\pi)^{-n/2}\frac{K_{(n-2)/2}(|x|)}{|x|^{(n-2)/2}}, \tag{4.11}$$

where $K_\nu(r)$ is the modified Bessel function of order $\nu > -1/2$. We have the following asymptotic expansion valid if $r > 0$ tends to ∞

$$\begin{aligned} G(|x|) = G_n(|x||) &\sim \frac{e^{-|x|}}{|x|^{(n-1)/2}}\left(1 + O(|x|^{-1}\right), \\ G'(|x|) &\sim -\frac{e^{-|x|}}{|x|^{(n-1)/2}}\left(1 + O(|x|^{-1}\right) \end{aligned} \tag{4.12}$$

near infinity. Near $r > 0$ close to 0 we have the asymptotic expansion (see (27), p.83, Chapter 7.2, [3])

$$K_\nu(r) = c_0 r^{-\nu}(1 + O(r)), c_0 > 0. \tag{4.13}$$

Hence

$$\begin{aligned} G(|x|) &= \frac{c}{|x|^{n-2}}(1 + O(|x|)) \sim G_0(|x|), \\ G'(|x|) &= -\frac{c(n-2)}{|x|^{n-1}}(1 + O(|x|)) \end{aligned} \tag{4.14}$$

near $x = 0$ with $c > 0$. Concerning the fact that $G(r)$ is positive and decreasing, we can deduce this property by applying a combination between the maximum principle and the fact that $G(r)$ is a solution of the elliptic equation

$$-G''(r) - \frac{n-1}{r}G'(r) + G(r) = 0, \quad r \in (0, \infty).$$

More precisely, the function $\widetilde{G}(r) = r^k G(r)$ satisfies the elliptic equation

$$-\widetilde{G}''(r) - \frac{n-1+2k}{r}\widetilde{G}'(r) + \widetilde{G}(r) + \frac{(n-2-k)k}{4r^2}\widetilde{G}(r) = 0$$

so the maximum principle implies

$$r^k G(r) \searrow, \quad \forall k \in [0, n-2]. \tag{4.15}$$

Remark 4.1 The asymptotic behavior of ground states that are solutions of nonlocal elliptic equations is studied in [2, 19] and these asymptotics are important to treat stability and scattering.

4.2 Estimates and asymptotics of *h*

Lemma 4.2 *If* h, F *satisfy*

$$\left.\begin{aligned} &h(r) \in C^2(0,\infty) \cap C^1([0,\infty)), \;\; F(r) \in C(0,\infty), \\ &h(r) = O(1), h'(r) = O(1), \;\; F(r) = O(e^{-br}), \;\; b > 1, \quad as\; r \to \infty \end{aligned}\right\} \tag{4.16}$$

and $(1-\Delta)h = F$. *Then for* $r \to \infty$ *we have*

$$\left.\begin{aligned} h(r) &= c^\diamond\, G(r) + O\left(r^{-(n-1)/2}e^{-r}\int_r^\infty \tau^{(n-1)/2}e^{\tau}|F(\tau)|d\tau\right), \\ h'(r) &= c^\diamond\, G'(r) + O\left(r^{-(n-1)/2}e^{-r}\int_r^\infty \tau^{(n-1)/2}e^{\tau}|F(\tau)|d\tau\right). \end{aligned}\right\} \tag{4.17}$$

Here

$$c^\diamond = \int_0^\infty G^\diamond(\tau)\tau^{n-1}G(\tau)F(\tau)d\tau,\ G^\diamond(r) = \int_0^r \rho^{-n+1}\frac{1}{G^2(\rho)}d\rho. \tag{4.18}$$

Proof We can make the substitution $h(r) = G(r)v(r)$, so that the Eq. (4.2) is transformed into

$$-r^{n-1}G(r)\left(v''(r) + \frac{n-1}{r}v'(r) + \frac{2G'(r)}{G(r)}\,v'(r)\right) = r^{n-1}F(r).$$

Introducing the function $\xi(r)$ so that

$$\frac{\xi'(r)}{\xi(r)} = \frac{n-1}{r} + \frac{2G'(r)}{G(r)},$$

we find $\xi(r) = G^2(r)r^{n-1}$ we can rewrite the equation for v in the form

$$-\left(\xi(r)v'(r)\right)' = G^{-1}(r)\xi(r)F(r) \equiv G(r)r^{n-1}F(r).$$

Integrating over (r, R), we deduce

$$r^{n-1}G^2(r)v'(r) - R^{n-1}G^2(R)v'(R) = \int_r^R \tau^{n-1}G(\tau)F(\tau)d\tau.$$

Using (4.16), we arrive at

$$R^{n-1}G^2(R)v'(R) \sim e^{-2R}\left(e^R R^{(n-1)/2}h'(R) + e^R R^{(n-1)/2}h(R)\right) \to 0$$

as $R \to \infty$. Hence

$$r^{n-1}G^2(r)v'(r) = \int_r^\infty \tau^{n-1}G(\tau)F(\tau)d\tau. \tag{4.19}$$

Integrating over $(0, r)$, we find

$$\begin{aligned} v(r) &= \int_0^r \rho^{-n+1}\frac{1}{G^2(\rho)}\int_\rho^\infty \tau^{n-1}G(\tau)F(\tau)d\tau \\ &= \int_0^\infty \left(\int_0^{\min(r,\tau)} \rho^{-n+1}\frac{1}{G^2(\rho)}d\rho\right)\tau^{n-1}G(\tau)F(\tau)d\tau \\ &= \int_0^\infty G^\diamond(\min(r,\tau))\tau^{n-1}G(\tau)F(\tau)d\tau. \end{aligned}$$

The function $G^\diamond$ is is positive increasing and has asymptotics

$$G^\diamond(r) = e^{2r}(c_0 + o(1))$$

as $r \to \infty$. From the above relations we find

$$v(r) = \int_0^\infty G^\diamond(\min(r,\tau))\tau^{n-1}G(\tau)F(\tau)d\tau$$

$$= \int_0^\infty G^\diamond(\tau)\tau^{n-1}G(\tau)F(\tau)d\tau - \int_r^\infty G^\diamond(\tau)\tau^{n-1}G(\tau)F(\tau)d\tau +$$
$$+ \int_r^\infty G^\diamond(r)\tau^{n-1}G(\tau)F(\tau)d\tau$$

and this gives

$$v(r) = \int_0^\infty G^\diamond(\tau)\tau^{n-1}G(\tau)F(\tau)d\tau \left(1 + O\left(\int_r^\infty \tau^{(n-1)/2}e^\tau |F(\tau)|d\tau\right)\right).$$

Hence we have the first asymptotic expansion in (4.17). To check the second one we use (4.19) and get

$$\left(\frac{h(r)}{G(r)}\right)' = v'(r) = \frac{1}{r^{n-1}G^2(r)}\int_r^\infty \tau^{n-1}G(\tau)F(\tau)d\tau$$

so we have

$$\left|h' - \frac{G'(r)h(r)}{G(r)}\right| = \left|\frac{1}{r^{n-1}G(r)}\int_r^\infty \tau^{n-1}G(\tau)F(\tau)d\tau\right|$$
$$\le \frac{1}{r}\int_r^\infty \tau|F(\tau)|d\tau \le \frac{1}{r}\ r^{-(n-3)/2}e^{-r}\int_r^\infty \tau\tau^{(n-3)/2}e^\tau|F(\tau)|d\tau$$

due to (4.15). Using the first inequality in (4.17) we arrive at the second one. □

4.3 Asymptotics of the ground state *Q* and vector *h* in the kernel of L_+

The ground state for the Choquard problem is described by the following elliptic system

$$(1-\Delta)Q = AQ^{p-1}, \tag{4.20}$$
$$-\Delta A = Q^p. \tag{4.21}$$

The kernel of L_+ can be described by the following linear elliptic system

$$(1-\Delta)h = pBQ^{p-1} + (p-1)AQ^{p-2}h, \tag{4.22}$$
$$-\Delta B = Q^{p-1}h. \tag{4.23}$$

Thanks to regularity properties obtained in Proposition 2.1 and Corollary 2.1 we can assume that regularity properties of h, Q are given by (4.3), while (4.4) represents the regularity of B, A. By the asymptotic expansions of Lemmas 4.1 and 4.2, we achieve

Corollary 4.1 *If $Q \in H^1_{rad}(\mathbb{R}^n)$ is a positive solution of* (4.20) *and $h \in H^1_{rad}(\mathbb{R}^n)$ is a solution of* (4.22)*, then we have*

$$Q(|x|) = c^\diamond(Q)G(|x|)\left(1 + O\left(e^{-(p-2)|x|}\right)\right),\ \ |x| \to \infty, \tag{4.24}$$

with

$$c^\diamond(Q) = \int_0^\infty G^\diamond(\tau)\tau^{n-1}G(\tau)A(\tau)Q(\tau)^{p-1}d\tau > 0$$

and

$$h(|x|) = c^\diamond(h,Q)G(|x|)\left(1 + O\left(e^{-|x|}\right)\right),\ \ |x| \to \infty \tag{4.25}$$

with

$$c^{\diamond}(h,Q)=\int_0^{\infty}G^{\diamond}(\tau)\tau^{n-1}G(\tau)\left[pB(\tau)Q(\tau)^{p-1}+(p-1)AQ^{p-2}h\right]d\tau.$$

Moreover, if $A\in H^1_{rad}(\mathbb{R}^n)$ is a positive solution of (4.21) and $B\in H^1_{rad}(\mathbb{R}^n)$ is a solution of (4.23), then

$$A(|x|)=d(Q)G_0(|x|)\left(1+O\left(e^{-p|x|}\right)\right),\quad |x|\to\infty, \tag{4.26}$$

where

$$d(Q)=\int_{\mathbb{R}^n}Q(|x|)^{p-1}dx=|\mathbb{S}^{n-1}|\int_0^{\infty}Q(\tau)^{p-1}\tau^{n-1}d\tau$$

and

$$B(|x|)=d(h,Q)G_0(|x|)\left(1+O\left(e^{-|x|}\right)\right),\quad |x|\to\infty, \tag{4.27}$$

where

$$d(h,Q)=\int_{\mathbb{R}^n}Q(|x|)^{p-2}h(|x|)dx=|\mathbb{S}^{n-1}|\int_0^{\infty}Q(\tau)^{p-2}h(\tau)\tau^{n-1}d\tau.$$

Now we are ready to check that the annihilation of both coefficients $c^{\diamond}(h,Q)$ and $d(h,Q)$ in the asymptotics of h, B implies $h\equiv 0$, $B\equiv 0$.

Corollary 4.2 *If h, B are solutions to (4.22), then the conditions*

$$\lim_{r\to\infty}G^{-1}(r)h(r)=\lim_{r\to\infty}G_0^{-1}(r)B(r)=0$$

imply $h\equiv 0$, $B\equiv 0$.

Proof Following the approach in [6], we define the decreasing function

$$\Phi(r)=\sup_{\rho>r}\left|G^{-1}(\rho)h(\rho)\right|+\left|G_0^{-1}(\rho)B(\rho)\right|.$$

The estimate (4.7) applied with $H=hQ^{p-1}$ and the assumption

$$\lim_{r\to\infty}G_0^{-1}(r)B(r)=0$$

give

$$\begin{aligned}\frac{|B(r)|}{G_0(r)}&\lesssim\int_r^{\infty}\frac{|h(\tau)|Q^{p-1}}{G_0(\tau)}\tau d\tau\lesssim\int_r^{\infty}\tau^{n-1}|h(\tau)|Q^{p-1}(\tau)d\tau\\ &\lesssim\int_r^{\infty}\tau^{n-1}\Phi(\tau)G(\tau)^p d\tau\lesssim\int_r^{\infty}\tau^{(n-1)-(n-1)p/2}\Phi(\tau)e^{-p\tau}d\tau.\end{aligned}$$

In this way we arrive at

$$\frac{|B(r)|}{G_0(r)}\lesssim\int_r^{\infty}\tau^{(2-p)(n-1)/2}e^{-p\tau}\Phi(\tau)d\tau. \tag{4.28}$$

In a similar manner, we use (4.17) with $F=pBQ^{p-1}+(p-1)AQ^{p-2}h$ and the assumption

$$\lim_{r\to\infty}G^{-1}(r)h(r)=0$$

and obtain

$$\frac{|h(r)|}{G(r)} \lesssim \int_r^\infty \tau^{(n-1)/2} e^\tau |B(\tau)| Q^{p-1}(\tau) d\tau + \int_r^\infty \tau^{(n-1)/2} e^\tau A(\tau) Q^{p-2}(\tau) |h(\tau)| d\tau$$
$$\lesssim \int_r^\infty \tau^{(n-1)/2} e^\tau \tau^{-n+2} \Phi(\tau) \tau^{(1-n)(p-1)/2} e^{-(p-1)\tau} d\tau.$$

From these estimates and (4.28) we find

$$\Phi(r) \lesssim \int_r^\infty \tau^{(2-p)(n-1)/2} e^{-(p-2)\tau} \Phi(\tau) d\tau$$

and therefore we can use Gronwall Lemma A.1 from [6] and deduce $\Phi \equiv 0$. □

Proposition 4.1 *If Q_1, Q_2 are radial minimizers of* (1.8)*, satisfying $\|Q_j\|_{H^1} \lesssim 1$, $j = 1, 2$, the normalization condition* (1.10) *and such that*

$$\|Q_1 - Q_2\|_{L^2_{rad}} \leq \varepsilon,$$

then we have

$$\|Q_1 - Q_2\|_{H^1} \lesssim \varepsilon. \tag{4.29}$$

Proof We know that if Q_j, $j = 1, 2$ are minimizers, then we have (4.20) and

$$(1 - \Delta) Q_j = A(Q_j) |Q_j|^{p-2} Q_j,$$
$$A(Q_j) = (-\Delta)^{-1}(|Q_j|^p).$$

The regularity estimate of Proposition 2.1

$$\|Q_j\|_{H^s(\mathbb{R}^n)} \lesssim 1, \tag{4.30}$$

for $s \in [0, p+1)$. Then we have

$$(1 - \Delta)(Q_1 - Q_2) = A(Q_1)|Q_1|^{p-2} Q_1 - A(Q_2)|Q_2|^{p-2} Q_2. \tag{4.31}$$

Using Sobolev inequalities, we find

$$\left\|(-\Delta)^{-1/2} g\right\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \tag{4.32}$$

and

$$\|f|g|^{p-1}\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \lesssim \|f\|_{L^{2np/(n+2)}(\mathbb{R}^n)} \|g\|^{p-1}_{L^{2np/(n+2)}(\mathbb{R}^n)}.$$

Multiplying (in L^2) the equation (4.31) by $Q_1 - Q_2$ and applying the above estimate, we get

$$\|\nabla(Q_1 - Q_2)\|^2_{L^2(\mathbb{R}^n)} + \|Q_1 - Q_2\|^2_{L^2(\mathbb{R}^n)}$$
$$\lesssim \|Q_1 - Q_2\|^2_{L^{2np/(n+2)}} \left(\|Q_1\|_{L^{2np/(n+2)}} + \|Q_2\|_{L^{2np/(n+2)}}\right)^{2(p-1)}.$$

To be more precise, in the above bounds we are using (4.32) again and via the Gagliardo–Nirenberg inequality

$$\|g\|_{L^{2np/(n+2)}} \lesssim \|\nabla g\|_{L^2}^{(np-(n+2))/(2p)} \|g\|_{L^2}^{(n+2-p(n-2))/(2p)}$$

we find

$$\|\nabla(Q_1 - Q_2)\|^2_{L^2(\mathbb{R}^n)} \lesssim \|\nabla(Q_1 - Q_2)\|_{L^2}^{(np-(n+2))/p} \varepsilon^{((n+2)-p(n-2))/p} \tag{4.33}$$

and this estimate implies (4.29). □

Following the proof of the Proposition 4.1, we get

Corollary 4.3 *If* $h_1, h_2 \in \mathrm{Ker} L_+ \cap H^1_{rad}(\mathbb{R}^n)$, *then we have*

$$\|h_1 - h_2\|_{H^1(\mathbb{R}^n)} \lesssim \|h_1 - h_2\|_{L^2(\mathbb{R}^n)}. \tag{4.34}$$

5 Appendix I: Properties of the ground states

We start with the following variational statement.

Lemma 5.1 *Assume* $n \geq 3,\ p \in (2, 1 + 4/(n-2))$. *Then we have the following properties:*

(a) *we have the identities*

$$\mathcal{W} = \mathcal{W}^* = \mathcal{W}_\sigma, \tag{5.1}$$

where

$$\begin{aligned} \mathcal{W}^* &= \inf\{W_p(u); u \in H^1_{rad} \setminus \{0\}, \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} = D(|u|^p, |u|^p)\}, \\ \mathcal{W}_\sigma &= \inf\{W_p(u); u \in H^1_{rad} \setminus \{0\}, \|u\|^2_{L^2} = \sigma, \sigma = \beta k_{\mathcal{W}}\}; \end{aligned} \tag{5.2}$$

(b) *we have also the identity*

$$\begin{aligned} &\{u \in H^1_{rad} \setminus \{0\}; W_p(u) = \mathcal{W}, \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} = D(|u|^p, |u|^p)\} \\ &= \{u \in H^1_{rad} \setminus \{0\}; W_p(u) = \mathcal{W}, \|u\|^2_{L^2} = \sigma = \beta k_{\mathcal{W}}\}. \end{aligned} \tag{5.3}$$

Proof of Lemma 5.1. To show the first identity in (5.1) it is sufficient to use the obvious inequality $\mathcal{W} \leq \mathcal{W}^*$ and deduce the opposite inequality $\mathcal{W}^* \leq \mathcal{W}$ from the following observation: for any $\varepsilon > 0$ the property $\mathcal{W} + \varepsilon \geq W_p(u) \geq \mathcal{W}$ implies that for any real nonzero constant μ the function μu is also satisfies $\mathcal{W} + \varepsilon \geq W_p(\mu u) \geq \mathcal{W}$. If we choose μ so that μu satisfies the constraint condition and take $\varepsilon \to 0$, we get $\mathcal{W}^* \leq \mathcal{W}$. In this way we deduce $\mathcal{W}^* = \mathcal{W}$. Similar argument shows that $\mathcal{W} = \mathcal{W}_\sigma$. Let

$$W_p(u) = \mathcal{W},\ \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} = D(|u|^p, |u|^p).$$

Then for any $h \in S(\mathbb{R}^n)$ we have (see Step I, proof of Theorem 1.1 in Sect. 2 for more detailed calculation)

$$\frac{d}{d\varepsilon} W_p(u + \varepsilon h)\bigg|_{\varepsilon=0} = D(|u|^p, |u|^p)^{-1/p} \left(\langle -\Delta u + u - I(|u|^p)|u|^{p-2}u, h\rangle_{L^2}\right)$$

and we deduce the Eq. (1.1). From this equation we deduce the normalization conditions (1.11) so we have

$$\|u\|^2_{L^2} = \sigma = \beta k_{\mathcal{W}}.$$

In this way, we obtain the inclusion

$$\begin{aligned} &\{u \in H^1_{rad} \setminus \{0\}; W_p(u) = \mathcal{W}, \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} = D(|u|^p, |u|^p)\} \\ &\subseteq \{u \in H^1_{rad} \setminus \{0\}; W_p(u) = \mathcal{W}, \|u\|^2_{L^2} = \sigma = \beta k_{\mathcal{W}}\}. \end{aligned} \tag{5.4}$$

Vice versa we have to show

$$\begin{aligned} &\{u \in H^1_{rad} \setminus \{0\}; W_p(u) = \mathcal{W}, \|u\|^2_{L^2} = \sigma = \beta k_{\mathcal{W}}\} \\ &\subseteq \{u \in H^1_{rad} \setminus \{0\}; W_p(u) = \mathcal{W}, \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} = D(|u|^p, |u|^p)\}. \end{aligned} \tag{5.5}$$

If

$$W_p(u) = \mathcal{W},$$

then we have

$$\frac{d}{d\varepsilon} W_p(u + \varepsilon h)\bigg|_{\varepsilon=0} = D(|u|^p, |u|^p)^{-1/p} \left(\langle -\Delta u + u - \Lambda I(|u|^p)|u|^{p-2}u, h\rangle_{L^2} \right),$$

where

$$\Lambda = \frac{\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2}{D(|u|^p, |u|^p)}.$$

Then (5.5) easily follows from the implications

$$\Lambda \lesseqgtr 1 \implies \|u\|_{L^2}^2 \gtreqless \sigma = \beta k_{\mathcal{W}}. \tag{5.6}$$

To be more precise, if $\Lambda > 1$, then we can find $\mu > 1$ so that μu satisfies

$$\|\nabla(\mu u)\|_{L^2}^2 + \|\mu u\|_{L^2}^2 = D(|\mu u|^p, |\mu u|^p).$$

Since μu is also minimizer for W_p, we see that (5.4) implies

$$\mu^2 \|u\|_{L^2}^2 = \sigma,$$

so $\|u\|_{L^2}^2 < \sigma$. Similarly, $\Lambda < 1 \implies \|u\|_{L^2}^2 > \sigma$ and we have (5.6) that implies (5.5). This completes the proof. □

6 Appendix II: Fuchs–Painleve series expansions of ground states

The equation

$$-\Delta u + u = I(u^p)u^{p-1} \tag{6.1}$$

can be rewritten as a system of nonlinear second-order differential equations

$$\begin{aligned} Q''(r) + \frac{n-1}{r} Q'(r) &= Q - A(r) Q^{p-1}, \\ A''(r) + \frac{n-1}{r} A'(r) &= -Q^p. \end{aligned} \tag{6.2}$$

Our goal will be to verify that imposing special initial data

$$\begin{aligned} &Q(0) = Q_0 > 0, \ \ Q'(0) = 0, \\ &A(0) = A_0, \ \ \ A'(0) = 0, \end{aligned} \tag{6.3}$$

we can find a unique real analytic (near $r = 0$) solution to this Cauchy problem. Then we can consider the following more general problem

$$\begin{aligned} &Y''(r) + \frac{c}{r} Y'(r) = F(r, Y), \\ &Y(0) = Y'(0) = 0, \end{aligned} \tag{6.4}$$

where we have shifted the initial data to zero, but we assume that $F(r,0) \neq 0$ may be a nontrivial source term. To be more precise, here $Y(t) \in C^2([0,1);\mathbb{R}^n)$ is a vector-valued function, while F satisfies the assumptions

$$F(r,Y) \text{is real analytic near } r=0, Y=0 \tag{6.5}$$

and

$$F(0,0) \neq 0. \tag{6.6}$$

As in Theorem 11.1.1 in [10] we can state the following Fuchs–Painleve type result.

Theorem 6.1 *If the conditions* (6.5) *and* (6.6) *are fulfilled, then the Cauchy problem* (6.4) *has a unique real analytic solution*

$$Y(r) = \sum_{k=2}^{\infty} Y_k r^k$$

near $r=0$.

This result applied to the Cauchy problem (6.2), (6.3) gives the following series expansions near $r=0$

$$Q(r) = Q_0 + \sum_{k=1}^{\infty} Q_{2k} r^{2k}, \quad A(r) = A_0 + \sum_{k=1}^{\infty} A_{2k} r^{2k}. \tag{6.7}$$

To be more precise, we can take more general initial data

$$\begin{aligned} &Q(0) = Q_0 > 0, \;\; Q'(0) = Q_1, \\ &A(0) = A_0, \quad A'(0) = A_1, \end{aligned} \tag{6.8}$$

and we can take

$$\begin{aligned} &\begin{pmatrix} Q(r) \\ A(r) \end{pmatrix} = \begin{pmatrix} Q_0 + Q_1 r \\ A_0 + A_1 r \end{pmatrix} + Y(r), Y(r) = \begin{pmatrix} Y_1(r) \\ Y_2(r) \end{pmatrix} \\ &F(r,Y) = \begin{pmatrix} Q_0 + Q_1 r + Y_1(r) - (A_0 + A_1 r + Y_2(r))(Q_0 + Q_1 r + Y_1(r))^{p-1} \\ -(Q_0 + Q_1 r + Y_1(r))^p \end{pmatrix}. \end{aligned}$$

Then assuming $Q_0 > 0$, we see that $F(r,Y)$ is real analytic near $r=0$, $Y=0$ and we have the equation

$$Y''(r) + \frac{n-1}{r} Y'(r) = F(r,Y)$$

with zero initial data. Applying the Fuchs–Painleve Theorem 6.1 we see that $Y(r)$ is real analytic near $r=0$ and rewriting the equation in Y in the form

$$rY''(r) + (n-1)Y'(r) = rF(r,Y),$$

we see that $Y'(0) = 0$, so we have (6.3).

7 Appendix III: Q is in the image of L_+

Following [22] we have to prove the following.

Lemma 7.1 *If $S = x \cdot \nabla_x$ is the scaling operator in $\mathbb{R}^n$, then*

$$L_+\left(SQ + \frac{2}{p-1}Q\right) = -2Q. \tag{7.1}$$

Proof The scaling operator $S = x \cdot \nabla_x$ satisfies the commutator relations

$$\begin{aligned}
&[(-\Delta), S] = 2(-\Delta),\\
&[(-\Delta)^{-1}, S] = -2(-\Delta)^{-1}.
\end{aligned} \tag{7.2}$$

Indeed the first relation in (7.2) is trivial, while the second one follows from

$$[AB, C] = A[B, C] + [A, C]B,$$

applied with $A = (-\Delta)$, $B = (-\Delta)^{-1}$ and $C = S$. Since

$$\begin{aligned}
L_+h &= (-\Delta + 1)h\\
&\quad -pQ^{p-1}(-\Delta)^{-1}(Q^{p-1}h) - (p-1)Q^{p-2}h(-\Delta)^{-1}(Q^p)
\end{aligned}$$

and

$$L_+(Q) = -2(p-1)Q^{p-1}(-\Delta)^{-1}(Q^p),$$

we shall use the relations

$$\begin{aligned}
L_+(SQ) &= (-\Delta + 1)SQ\\
&\quad -pQ^{p-1}(-\Delta)^{-1}(Q^{p-1}SQ) - (p-1)(Q^{p-2}SQ)(-\Delta)^{-1}(Q^p)\\
&= S(-\Delta+1)Q - 2\Delta Q - Q^{p-1}(-\Delta)^{-1}(SQ^p) - (SQ^{p-1})(-\Delta)^{-1}(Q^p)\\
&= S(-\Delta+1)Q - 2\Delta Q - \left[S\left(Q^{p-1}(-\Delta)^{-1}(Q^p)\right) + 2Q^{p-1}(-\Delta)^{-1}(Q^p)\right]\\
&= S\left[(-\Delta+1)Q - Q^{p-1}(-\Delta)^{-1}(Q^p)\right] - 2\Delta Q - 2Q^{p-1}(-\Delta)^{-1}(Q^p)\\
&= 2\left[(-\Delta+1)Q - Q^{p-1}(-\Delta)^{-1}(Q^p)\right] - 2Q + 4Q^{p-1}(-\Delta)^{-1}(Q^p)\\
&= -2Q - L_+\left(\frac{2}{p-1}Q\right),
\end{aligned}$$

because (7.2) implies

$$\begin{aligned}
S\left(Q^{p-1}(-\Delta)^{-1}(Q^p)\right) &= (SQ^{p-1})(-\Delta)^{-1}(Q^p) + Q^{p-1}S\left((-\Delta)^{-1}(Q^p)\right)\\
&= (SQ^{p-1})(-\Delta)^{-1}(Q^p) + Q^{p-1}(-\Delta)^{-1}(SQ^p) - 2Q^{p-1}(-\Delta)^{-1}(Q^p).
\end{aligned}$$

Then the proof is completed. □

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