COMPACTNESS AND BLOW UP RESULTS FOR DOUBLY PERTURBED YAMABE PROBLEMS ON MANIFOLDS WITH UMBILIC BOUNDARY

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Abstract. Given a compact Riemannian manifold with umbilic boundary, the Yamabe boundary problem studies if there exist conformal scalar-flat metrics such that ∂M has constant mean curvature. In this paper we address to the stability of this problem with respect of perturbation of mean curvature of the boundary and scalar curvature of the manifold. In particular we prove that the Yamabe boundary problem is stable under perturbation of the mean curvature and the scalar curvature from below, while it is not stable if one of the two curvatures is perturbed from above.

1. Introduction

Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ with boundary. In [17] Escobar asked if there exists a conformal metric $\tilde{g} = u^{\frac{4}{n-2}}g$ for which M has zero scalar curvature and constant boundary mean curvature.

This problem can be understood as a generalization of the Riemann mapping theorem and it is equivalent to finding a positive solution to the following nonlinear boundary value problem

$$(1.1) \qquad \begin{cases} L_g u = 0 & \text{in } M \\ B_g u + (n-2)u^{\frac{n}{n-2}} = 0 & \text{on } \partial M \end{cases}.$$

Here $L_g = \Delta_g - \frac{n-2}{4(n-1)}R_g$ and $B_g = -\frac{\partial}{\partial\nu} - \frac{n-2}{2}h_g$ are respectively the conformal Laplacian and the conformal boundary operator, R_g is the scalar curvature of the manifold, h_g is the mean curvature of the ∂M and ν is the outer normal with respect to ∂M. If M is of *positive type*, that is when

$$Q(M, \partial M) := \inf_{u \in H^1 \smallsetminus 0} \frac{\int\limits_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)}R_g u^2\right) dv_g + \int\limits_{\partial M} \frac{n-2}{2}h_g u^2 d\sigma_g}{\left(\int\limits_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g\right)^{\frac{n-2}{n-1}}}$$

is strictly positive, equation (1.1) could have multiple solutions, and the question of compactness of solution arises naturally. In fact, if the boundary of M is umbilic, and the Weyl tensor W_g never vanishes on the boundary, the full set of solutions of (1.1) is compact. This is proved in [9], for dimensions $n > 8$, and in [12], for dimensions $n = 6, 7, 8$. We recall that the boundary of M is called umbilic if the trace-free second fundamental form of ∂M is zero everywhere.

Also, the authors show in [10] that the problem is stable for perturbation from below of the mean curvature, while in [11], with Pistoia, they prove that there is a blow up phenomenon when perturbing the mean curvature from above. This recalls a similar result from the Yamabe problems on boundaryless manifolds, in

which perturbations form below of the scalar curvature preserve the compactness of the set of solutions (see [4, 5]).

At this point it is interesting to study what happens when one perturbs both the scalar and the mean curvature, and to investigate compactness versus blow up of solutions in this framework. Thus, we study the linearly perturbed problem

$$(1.2) \qquad \begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} R_g u + \varepsilon_1 \alpha u = 0 & \text{in } M \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u + \varepsilon_2 \beta u = (n-2) u^{\frac{n}{n-2}} & \text{on } \partial M \end{cases}$$

or, in a more compact form,

$$\begin{cases} L_g u - \varepsilon_1 \alpha u = 0 & \text{in } M \\ B_g u - \varepsilon_2 \beta u + (n-2) u^{\frac{n}{n-2}} = 0 & \text{on } \partial M \end{cases},$$

where $\varepsilon_1, \varepsilon_2$ are small positive parameters and $\alpha, \beta : M \to \mathbb{R}$ are smooth functions. Here we choose ε_1 sufficiently small such that $-L_g + \varepsilon_1 \alpha$ is still a positive definite operator.

Our aim is to prove that, if we linearly perturb the mean curvature term h_g with a negative smooth function, and jointly we perturb the scalar curvature term R_g with another negative smooth function, the set of solution is still compact. On the contrary, if one between scalar and mean curvature is perturbed from above, the compactness of solutions is lost. Our main results read as

Theorem 1. *Let (M, g) be a smooth, n-dimensional Riemannian manifold of positive type not conformally equivalent to the standard ball with regular umbilic boundary ∂M.*

Let $\alpha, \beta : M \to \mathbb{R}$ be smooth functions such that $\alpha, \beta < 0$ on ∂M. Suppose that $n \geq 8$ and that the Weyl tensor W_g is not vanishing on ∂M. Then, there exists a positive constant C such that for any $\varepsilon_1, \varepsilon_2 \geq 0$ small enough and for any $u > 0$ solution of (1.2) it holds

$$C^{-1} \leq u \leq C \text{ and } \|u\|_{C^{2,\eta}(M)} \leq C$$

for some $0 < \eta < 1$. The constant C does not depend on $u, \varepsilon_1, \varepsilon_2$.

Theorem 2. *Let (M, g) be a smooth, n-dimensional Riemannian manifold of positive type not conformally equivalent to the standard ball with regular umbilic boundary ∂M.*

Let $\alpha, \beta : M \to \mathbb{R}$ be smooth functions. Suppose that $n \geq 8$ and that the Weyl tensor W_g is not vanishing on ∂M. If $\alpha > 0$ on ∂M or $\beta > 0$ on ∂M, then there exists a sequence of solutions $u_{\varepsilon_1, \varepsilon_2}$ of (1.2) which blows up at a point of the boundary when $(\varepsilon_1, \varepsilon_2) \to (0, 0)$.

Let us shortly comment these two results.

- In a series of papers [4, 5, 6] Druet, Hebey and Robert studied the stability of classical Yamabe problem under perturbation of scalar curvature terms. They proved that the set of solutions of $-\Delta_g u + \frac{n-2}{4(n-1)} a(x) u = c u^{\frac{n+2}{n-2}}$ in M is compact if $a(x) \leq R_g(x)$ on M, thus the problem is stable perturbing R_g from below, while they found counterexamples to compactness when $a(x)$ is greater than $R_g(x)$. In [10, 11] the same problem is studied in the case of boundary Yamabe equations by perturbing the mean curvature term and a matching compactness versus blow up phenomenon appears. So there is a strong analogy between the role of R_g in classical case and h_g in boundary case. We continue here the same analysis, by perturbing both the curvature terms at the same time, to complete the study. It appears that the problem is stable only when perturbing both terms with non positive functions, while it is enough to perturb from above one between h_g and R_g to lose compactness of the solutions.

- We worked here in the framework of umbilic boundary manifolds. In a recent paper [14], we studied the case of manifold with non umbilic boundary, that is when the trace-free second fundamental form is non zero in any point of ∂M. In this case it is possible to have compactness also for positive small perturbation of the scalar curvature. We want to remind that in the case of non umbilic boundary the compactness of solution for the unperturbed case was proved by Almaraz [1] and Kim, Musso and Wei [16].
- In the unperturbed case the compactness of solutions for umbilic manifolds has been proved for dimensions $n \geq 6$ (see [9, 12]). It should be possible to apply the same technique also in the perturbed case to extend Theorem 1 to $n = 6, 7$. It is less clear to us if Theorem 2 could be extended to lower dimensions. Case $n = 5$ remains open also for the unperturbed problem.
- Our theorems consider only perturbations that are everywhere positive or everywhere negative on M. However, in Remark 27 it is shown that it is possible to construct a sign changing α such that for any β the set of solutions in no more compact, or a sign changing β such that for any α the set of solutions in no more compact. We do not know if it would be possible to craft a sign changing perturbation for which compactness is preserved.

The paper is organized as follows. Hereafter we recall some basic definitions and all the preliminary notions useful to achieve the result. Section 2 is devoted to the proof of the compactness theorem, while in Section 3 we prove the non compactness result.

1.1. **Notations and preliminary definitions.**

Remark 3 (Notations). We will use the indices $1 \leq i, j, k, m, p, r, s \leq n - 1$ and $1 \leq a, b, c, d \leq n$. Moreover we use the Einstein convention on repeated indices. We denote by g the Riemannian metric, by R_{abcd} the full Riemannian curvature tensor, by R_{ab} the Ricci tensor and by R_g and h_g respectively the scalar curvature of (M, g) and the mean curvature of ∂M; moreover the Weyl tensor of (M, g) will be denoted by W_g. The bar over an object (e.g. $\bar{W}_g$) will means the restriction to this object to the metric of ∂M.

Finally, on the half space $\mathbb{R}_+^n = \{y = (y_1, \ldots, y_{n-1}, y_n) \in \mathbb{R}^n, y_n \geq 0\}$ we set $B_r(y_0) = \{y \in \mathbb{R}^n, |y - y_0| \leq r\}$ and $B_r^+(y_0) = B_r(y_0) \cap \{y_n > 0\}$. When $y_0 = 0$ we will use simply $B_r = B_r(y_0)$ and $B_r^+ = B_r^+(y_0)$. On the half ball B_r^+ we set $\partial' B_r^+ = B_r^+ \cap \partial \mathbb{R}_+^n = B_r^+ \cap \{y_n = 0\}$ and $\partial^+ B_r^+ = \partial B_r^+ \cap \{y_n > 0\}$. On $\mathbb{R}_+^n$ we will use the following decomposition of coordinates: $(y_1, \ldots, y_{n-1}, y_n) = (\bar{y}, y_n) = (z, t)$ where $\bar{y}, z \in \mathbb{R}^{n-1}$ and $y_n, t \geq 0$.

Fixed a point $q \in \partial M$, we denote by $\psi_q : B_r^+ \to M$ the Fermi coordinates centered at q. We denote by $B_g^+(q, r)$ the image of B_r^+. When no ambiguity is possible, we will denote $B_g^+(q, r)$ simply by B_r^+, omitting the chart ψ_q.

We introduce the following notation for integral quantities which recur often in the paper

$$I_m^\alpha := \int_0^\infty \frac{s^\alpha ds}{(1 + s^2)^m}.$$

By direct computation (see [1, Lemma 9.4]) it holds

$$(1.3) \qquad I_m^\alpha = \frac{2m}{\alpha+1} I_{m+1}^{\alpha+2} \text{ for } \alpha + 1 < 2m$$

$$I_m^\alpha = \frac{2m}{2m-\alpha-1} I_{m+1}^\alpha \text{ for } \alpha + 1 < 2m$$

$$I_m^\alpha = \frac{2m-\alpha-3}{\alpha+1} I_m^{\alpha+2} \text{ for } \alpha + 3 < 2m$$

Also, we have the following integral identities:

$$(1.4) \qquad \int_0^\infty \frac{t^k dt}{(1+t)^m} = \frac{k!}{(m-1)(m-2)\cdots(m-1-k)}$$

$$\int_0^\infty \frac{dt}{(1+t)^m} = \frac{1}{m-1}$$

and, by change of variables

$$(1.5) \qquad \int_{\mathbb{R}_+^n} \frac{|\bar{y}|^\alpha y_n^\beta}{[(1+y_n)^2 + |\bar{y}|^2]^\gamma} d\bar{y} dy_n = \omega_{n-2} I_\gamma^{\alpha+n-2} \int_0^\infty \frac{y_n^\beta}{(1+y_n)^{2\gamma-\alpha-n+1}} dy_n$$

where ω_{n-2} is the volume of $\mathbb{S}^{n-1}$.

We shortly recall here the well known function $U(y) := \dfrac{1}{[(1+y_n)^2 + |\bar{y}|^2]^{\frac{n-2}{2}}}$
which is also called the standard bubble and which is the unique solution, up to translations and rescaling, of the nonlinear critical problem.

$$(1.6) \qquad \begin{cases} -\Delta U = 0 & \text{on } \mathbb{R}_+^n; \\ \frac{\partial U}{\partial y_n} = -(n-2)U^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}_+^n. \end{cases}$$

We set

$$(1.7) \qquad j_l := \partial_l U = -(n-2)\frac{y_l}{[(1+y_n)^2 + |\bar{y}|^2]^{\frac{n}{2}}}$$

$$\partial_k \partial_l U = (n-2)\left\{ \frac{n y_l y_k}{[(1+y_n)^2 + |\bar{y}|^2]^{\frac{n+2}{2}}} - \frac{\delta^{kl}}{[(1+y_n)^2 + |\bar{y}|^2]^{\frac{n}{2}}} \right\}$$

$$(1.8) \qquad j_n := y^b \partial_b U + \frac{n-2}{2}U = -\frac{n-2}{2}\frac{|y|^2-1}{[(1+y_n)^2 + |\bar{y}|^2]^{\frac{n}{2}}}.$$

and we recall that $j_1, \ldots, j_n$ are a basis of the space of the H^1 solutions of the linearized problem

$$(1.9) \qquad \begin{cases} -\Delta\phi = 0 & \text{on } \mathbb{R}_+^n, \\ \frac{\partial\phi}{\partial t} + nU^{\frac{2}{n-2}}\phi = 0 & \text{on } \partial\mathbb{R}_+^n, \\ \phi \in H^1(\mathbb{R}_+^n). \end{cases}$$

Given a point $q \in \partial M$, we introduce now the function γ_q which arises from the second order term of the expansion of the metric g on M (see 1.18). The choice of this function plays a twofold role in this paper. On the one hand, using the function γ_q we are able to perform the estimates of Lemmas 12, 13 and Proposition 14. On the other hand, it gives the correct correction to the standard bubble in order to perform finite dimensional reduction.

For the proof of the following Lemma we refer to [9, Lemma 3] and [1, Proposition 5.1].

Lemma 4. *Assume $n \geq 5$. Given a point $q \in \partial M$, there exists a unique $\gamma_q : \mathbb{R}^n_+ \to \mathbb{R}$ classical solution of the linear problem*

$$(1.10) \qquad \begin{cases} -\Delta\gamma = \left[\frac{1}{3}\bar{R}_{ijkl}(q)y_k y_l + R_{ninj}(q)y_n^2\right]\partial^2_{ij}U & \text{on } \mathbb{R}^n_+ \\ \frac{\partial\gamma}{\partial y_n} = -nU^{\frac{2}{n-2}}\gamma & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

which is $L^2(\mathbb{R}^n_+)$-orthogonal to the functions $j_1, \ldots, j_n$ (that is $\int_{\mathbb{R}^n_+} \gamma_q j_l dx = 0$), where $j_1, \ldots, j_n$ are defined in (1.7) and (1.8).

Moreover it holds

$$(1.11) \qquad |\nabla^\tau \gamma_q(y)| \leq C(1+|y|)^{4-\tau-n} \text{ for } \tau = 0, 1, 2.$$

$$(1.12) \qquad \int_{\mathbb{R}^n_+} \gamma_q \Delta\gamma_q dy \leq 0,$$

$$(1.13) \qquad \int_{\partial\mathbb{R}^n_+} U^{\frac{n}{n-2}}(t,z)\gamma_q(t,z)dz = 0$$

$$(1.14) \qquad \gamma_q(0) = \frac{\partial\gamma_q}{\partial y_1}(0) = \cdots = \frac{\partial\gamma_q}{\partial y_{n-1}}(0) = 0.$$

Finally the map $q \mapsto \gamma_q$ is $C^2(\partial M)$ in classical sense.

1.2. **Expansion of the metric.** Since the boundary ∂M is umbilic, given $q \in \partial M$ there exists a conformally related metric $\tilde{g}_q = \Lambda_q^{\frac{4}{n-2}}g$ such that some geometric quantities at q have a simpler form which will be summarized later in this paragraph. We have

$$\Lambda_q(q) = 1, \ \frac{\partial\Lambda_q}{\partial y_k}(q) = 0 \text{ for all } k = 1, \ldots, n-1.$$

Also, we have that $\tilde{u}_q := \Lambda_q u$, is a solution of (1.2) if and only if u solves the following problem

$$(1.15) \qquad \begin{cases} L_{\tilde{g}_q}u - \varepsilon_1\tilde{\alpha}u = 0 & \text{in } M \\ B_{\tilde{g}_q}u + (n-2)u^{\frac{n}{n-2}} - \varepsilon_2\tilde{\beta}u = 0 & \text{on } \partial M \end{cases}.$$

where $\tilde{\alpha} := \Lambda_q^{-\frac{4}{n-2}}\alpha$ and $\tilde{\beta} := \Lambda_q^{-\frac{2}{n-2}}\beta$.

In the following expansion and in section 2, in order to simplify notations, we will omit the *tilde* symbols, since we will always work in the conformal metric $\tilde{g}$, while in Section 3 we will switch between metrics, so we will keep g and $\tilde{g}$ explicitly indicated.

With this metric we have the following expansions.

Remark 5. In Fermi conformal coordinates around $q \in \partial M$, it holds (see [20])

$$(1.16) \qquad |\det g_q(y)| = 1 + O(|y|^N) \text{ with } N \text{ arbitrarily large}$$

$$(1.17) \qquad |h_{ij}(y)| = O(|y^4|) \qquad |h_g(y)| = O(|y^4|)$$

$$(1.18) \quad g_q^{ij}(y) = \delta^{ij} + \frac{1}{3}\bar{R}_{ikjl}y_k y_l + R_{ninj}y_n^2$$

$$+ \frac{1}{6}\bar{R}_{ikjl,m}y_k y_l y_m + R_{ninj,k}y_n^2 y_k + \frac{1}{3}R_{ninj,n}y_n^3$$

$$+ \left(\frac{1}{20}\bar{R}_{ikjl,mp} + \frac{1}{15}\bar{R}_{iksl}\bar{R}_{jmsp}\right) y_k y_l y_m y_p$$

$$+ \left(\frac{1}{2}R_{ninj,kl} + \frac{1}{3}\mathrm{Sym}_{ij}(\bar{R}_{iksl}R_{nsnj})\right) y_n^2 y_k y_l$$

$$+ \frac{1}{3}R_{ninj,nk}y_n^3 y_k + \frac{1}{12}\left(R_{ninj,nn} + 8R_{nins}R_{nsnj}\right)y_n^4 + O(|y|^5)$$

$$(1.19) \qquad R_{g_q}(y) = O(|y|^2) \text{ and } \partial_{ii}^2 R_{g_q} = -\frac{1}{6}|\bar{W}|^2$$

$$(1.20) \qquad \partial_{tt}^2 \bar{R}_{g_q} = -2R_{ninj}^2 - 2R_{ninj,ij}$$

$$(1.21) \qquad \bar{R}_{kl} = R_{nn} = R_{nk} = R_{nn,kk} = 0$$

$$(1.22) \qquad R_{nn,nn} = -2R_{nins}^2.$$

All the quantities above are calculate in $q \in \partial M$, unless otherwise specified.

If we choose ε_1 sufficiently small in order to have that $-L_g + \varepsilon_1\alpha$ is a positive definite operator, we can define an equivalent scalar product on H^1 as
(1.23)
$$\langle\langle u,v\rangle\rangle_g = \int_M \left(\nabla_g u \nabla_g v + \frac{n-2}{4(n-1)}R_g uv + \varepsilon_1 \alpha uv\right)d\mu_g + \frac{n-2}{2}\int_{\partial M} h_g uv\, d\nu_g$$

which leads to the norm $\|\cdot\|_g$ equivalent to the usual one.

With this norm we have that Λ_q is an isometry. In fact, by (1.23), for any $u, v \in H^1(M)$,

$$\langle\langle \Lambda_q u, \Lambda_q v\rangle\rangle_g = \langle\langle u,v\rangle\rangle_{\tilde{g}_q} \text{ and, consequently, } \|\Lambda_q u\|_g = \|u\|_{\tilde{g}_q}.$$

1.3. Variational framework.

Given $1 \le t \le \frac{2(n-1)}{n-2}$ we have the well known embedding

$$i : H^1(M) \to L^t(\partial M).$$

We define, by the scalar product $\langle\langle\cdot,\cdot\rangle\rangle_g$,

$$i_\alpha^* : L^{\frac{2(n-1)}{n-2}}(\partial M) \to H^1(M)$$

in the following sense: given $f \in L^{\frac{2(n-1)}{n-2}}(\partial M)$ there exists a unique $v \in H^1(M)$ such that

$$(1.24) \qquad v = i_\alpha^*(f) \iff \langle\langle v,\varphi\rangle\rangle_g = \int_{\partial M} f\varphi\, d\sigma \text{ for all } \varphi$$

$$\iff \begin{cases} -\Delta_g v + \frac{n-2}{4(n-1)}R_g v + \varepsilon_1\alpha = 0 & \text{on } M; \\ \frac{\partial v}{\partial \nu} + \frac{n-2}{2}h_g v = f & \text{on } \partial M. \end{cases}$$

So Problem (1.2) is equivalent to find $v \in H^1(M)$ such that

$$v = i_\alpha^*(f(v) - \varepsilon_2\beta v)$$

where

$$f(v) = (n-2)\left(v^+\right)^{\frac{n}{n-2}}.$$

Notice that, if $v \in H_g^1$, then $f(v) \in L^{\frac{2(n-1)}{n}}(\partial M)$.

Problem (1.2) has also a variational structure and a positive solution for (1.2) is a critical point for the following functional defined on $H^1(M)$

$$J_{\varepsilon_1,\varepsilon_2,g}(v) = J_g(v) := \frac{1}{2}\int_M |\nabla_g v|^2 + \frac{n-2}{4(n-1)}R_g v^2 + \varepsilon_1\alpha v^2 d\mu_g + \frac{n-2}{4}\int_{\partial M} h_g v^2 d\sigma_g$$
$$+ \frac{1}{2}\int_{\partial M}\varepsilon_2\beta v^2 d\sigma_g - \frac{(n-2)^2}{2(n-1)}\int_{\partial M}\left(v^+\right)^{\frac{2(n-1)}{n-2}} d\sigma_g.$$

We remark that, defined

$$\tilde{J}_{\varepsilon_1,\varepsilon_2,\tilde{g}}(v) = \tilde{J}_{\tilde{g}}(v) := \frac{1}{2}\int_M |\nabla_{\tilde{g}} v|^2 + \frac{n-2}{4(n-1)}R_{\tilde{g}} v^2 + \varepsilon_1\tilde{\alpha} v^2 d\mu_{\tilde{g}} + \frac{n-2}{4}\int_{\partial M} h_{\tilde{g}} v^2 d\sigma_{\tilde{g}}$$
$$+ \frac{1}{2}\int_{\partial M}\varepsilon_2\tilde{\beta} v^2 d\sigma_{\tilde{g}} - \frac{(n-2)^2}{2(n-1)}\int_{\partial M}\left(v^+\right)^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}},$$

we have

(1.25) $$J_g(\Lambda_q u) = \tilde{J}_{\tilde{g}}(u).$$

Given $q \in \partial M$ and $\psi_q^\partial : \mathbb{R}_+^n \to M$ the Fermi coordinates in a neighborhood of q; we define

$$W_{\delta,q}(\xi) = U_\delta\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right)\chi\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right) =$$
$$= \frac{1}{\delta^{\frac{n-2}{2}}}U\left(\frac{y}{\delta}\right)\chi(y) = \frac{1}{\delta^{\frac{n-2}{2}}}U(x)\chi(\delta x)$$

where $y = (z,t)$, with $z \in \mathbb{R}^{n-1}$ and $t \geq 0$, $\delta x = y = \left(\psi_q^\partial\right)^{-1}(\xi)$ and χ is a radial cut off function, with support in ball of radius R. In an analogous way, given γ_q as in Lemma 4 we define

$$V_{\delta,q}(\xi) = \frac{1}{\delta^{\frac{n-2}{2}}}\gamma_q\left(\frac{1}{\delta}\left(\psi_q^\partial\right)^{-1}(\xi)\right)\chi\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right)$$

and, given j_a defined in (1.7) and (1.8) we define

$$Z_{\delta,q}^b(\xi) = \frac{1}{\delta^{\frac{n-2}{2}}}j_b\left(\frac{1}{\delta}\left(\psi_q^\partial\right)^{-1}(\xi)\right)\chi\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right).$$

By means of $\langle\langle\cdot,\cdot\rangle\rangle_g$ it is possible to decompose H^1 in the direct sum of the following two subspaces

$$\tilde{K}_{\delta,q} = \text{Span}\left\langle\Lambda_q Z_{\delta,q}^1,\ldots,\Lambda_q Z_{\delta,q}^n\right\rangle$$
$$\tilde{K}_{\delta,q}^\perp = \left\{\varphi \in H^1(M) \; : \; \langle\langle\varphi,\Lambda_q Z_{\delta,q}^b\rangle\rangle_g = 0, \; b = 1,\ldots,n\right\}$$

and to define the projections

$$\tilde{\Pi} = H^1(M) \to \tilde{K}_{\delta,q} \text{ and } \tilde{\Pi}^\perp = H^1(M) \to \tilde{K}_{\delta,q}^\perp.$$

Notice that, since Λ_q is an isometry, we have that $\varphi \in \tilde{K}_{\delta,q}$ if and only if $\Lambda_q^{-1}\varphi \in K_{\delta,q}$ and the same holds for $\tilde{K}_{\delta,q}^\perp$.

In Section 3, we will look for a solution $\tilde{u}_q = \Lambda_q u$ of (1.2) which has the form

$$\tilde{u}_q = \Lambda_q\left(W_{\delta,q} + \delta^2 V_{\delta,q} + \phi\right) = \tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi}$$

where $\tilde{\phi} \in \tilde{K}_{\delta,q}^\perp$. By means of i_α^* this is equivalent to the following pair of equations

(1.26)
$$\tilde{\Pi}\left\{\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi} - i_\alpha^*\left[f(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi}) - \varepsilon_2\beta(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi})\right]\right\} = 0$$

(1.27)
$$\tilde{\Pi}^\perp\left\{\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi} - i_\alpha^*\left[f(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi}) - \varepsilon_2\beta(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi})\right]\right\} = 0.$$

2. The compactness result

In this section, firstly we recall a Pohozaev type identity which will give us a fundamental sign condition to rule out the possibility of blowing up sequence (see subsection 2.4). A recall of preliminary results on blow up points is collected in subsection 2.2, while a careful analysis of blow up sequences is performed in subsection 2.3. The proof of Theorem 1 is given in subsection 3.1. Throughout this section we work in $\tilde{g}$ metric. For the sake of readability we will omit the *tilde* symbol throughout this section.

2.1. A Pohozaev type identity.

We will use this version of a local Pohozaev type identity (see [1, 9]).

Theorem 6 (Pohozaev Identity). *Let u be a C^2-solution of the following problem*

$$\begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} R_g u + \varepsilon_1 \alpha = 0 = 0 & in \ B_r^+ \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u + \varepsilon_2 \beta u = (n-2) u^{\frac{n}{n-2}} & on \ \partial' B_r^+ \end{cases}$$

for $B_r^+ = \psi_q^{-1}(B_g^+(q,r))$ for $q \in \partial M$. Let us define

$$P(u,r) := \int\limits_{\partial^+ B_r^+} \left(\frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma_r + \frac{r(n-2)^2}{2(n-1)} \int\limits_{\partial(\partial' B_r^+)} u^{\frac{2(n-1)}{n-2}} d\bar{\sigma}_g,$$

and

$$\hat{P}(u,r) := - \int\limits_{B_r^+} \left(y^a \partial_a u + \frac{n-2}{2} u \right) [(L_g - \Delta) u] dy + \frac{n-2}{2} \int\limits_{\partial' B_r^+} \left(\bar{y}^k \partial_k u + \frac{n-2}{2} u \right) h_g u d\bar{y}$$

$$+ \varepsilon_1 \int\limits_{B_r^+} \left(y^a \partial_a u + \frac{n-2}{2} u \right) \alpha u dy$$

$$+ \frac{n-2}{2} \varepsilon_2 \int\limits_{\partial' B_r^+} \left(\bar{y}^k \partial_k u + \frac{n-2}{2} u \right) \beta u d\bar{y}.$$

Then $P(u,r) = \hat{P}(u,r)$.
   Here $a = 1, \ldots, n$, $k = 1, \ldots, n-1$ and $y = (\bar{y}, y_n)$, where $\bar{y} \in \mathbb{R}^{n-1}$ and $y_n \geq 0$.

2.2. Isolated and isolated simple blow up points.

Here we recall the definitions of some type of blow up points, and we give the basic properties about the behavior of these blow up points (see [1, 7, 15, 21]). We will omit the proofs of the well known results.

Let $\{u_i\}_i$ be a sequence of positive solution to

$$(2.1) \qquad \begin{cases} L_{g_i} u - \varepsilon_{1,i} \alpha_i u = 0 & in \ M \\ B_{g_i} u + (n-2) u^{\frac{n}{n-2}} - \varepsilon_{2,i} \beta_i u = 0 & on \ \partial M \end{cases} .$$

where $\alpha_i = \Lambda_{x_i}^{-\frac{4}{n-2}} \alpha \to \Lambda_{x_0}^{-\frac{4}{n-2}} \alpha$, $\beta_i = \Lambda_{x_i}^{-\frac{2}{n-2}} \beta \to \Lambda_{x_0}^{-\frac{2}{n-2}} \beta$, $x_i \to x_0$, $g_i \to g_0$ in the C_{loc}^3 topology and $0 \leq \varepsilon_{1,i}, \varepsilon_{2,i} < \bar{\varepsilon}$.

Definition 7. 1) We say that $x_0 \in \partial M$ is a blow up point for the sequence u_i of solutions of (2.1) if there is a sequence $x_i \in \partial M$ of local maxima of $u_i|_{\partial M}$ such that $x_i \to x_0$ and $u_i(x_i) \to +\infty$.

Shortly we say that $x_i \to x_0$ is a blow up point for $\{u_i\}_i$.

2) We say that $x_i \to x_0$ is an *isolated* blow up point for $\{u_i\}_i$ if $x_i \to x_0$ is a blow up point for $\{u_i\}_i$ and there exist two constants $\rho, C > 0$ such that

$$u_i(x) \leq C d_{\bar{g}}(x, x_i)^{\frac{2-n}{2}} \text{ for all } x \in \partial M \smallsetminus \{x_i\}, \ d_{\bar{g}}(x, x_i) < \rho.$$

Given $x_i \to x_0$ an isolated blow up point for $\{u_i\}_i$, and given $\psi_i : B_\rho^+(0) \to M$ the Fermi coordinates centered at x_i, we define the spherical average of u_i as

$$\bar{u}_i(r) = \frac{2}{\omega_{n-1} r^{n-1}} \int_{\partial^+ B_r^+} u_i \circ \psi_i d\sigma_r$$

and

$$w_i(r) := r^{\frac{2-n}{2}} \bar{u}_i(r)$$

for $0 < r < \rho$.

3) We say that $x_i \to x_0$ is an *isolated simple* blow up point for $\{u_i\}_i$ solutions of (2.1) if $x_i \to x_0$ is an isolated blow up point for $\{u_i\}_i$ and there exists ρ such that w_i has exactly one critical point in the interval $(0, \rho)$.

Given $x_i \to x_0$ a blow up point for $\{u_i\}_i$, we set

$$M_i := u_i(x_i) \quad \text{and} \quad \delta_i := M_i^{\frac{2}{2-n}}.$$

Obviously $M_i \to +\infty$ and $\delta_i \to 0$.

We recall two propositions whose proofs can be found in [2] and in [7].

Proposition 8. *Let $x_i \to x_0$ be an* isolated *blow up point for $\{u_i\}_i$ and ρ as in Definition 7. We set*

$$v_i(y) = M_i^{-1}(u_i \circ \psi_i)(M_i^{\frac{2}{2-n}} y), \text{ for } y \in B^+_{\rho M_i^{\frac{n-2}{2}}}(0).$$

Then, given $R_i \to \infty$ and $c_i \to 0$, up to subsequences, we have

(1) $|v_i - U|_{C^2\left(B_{R_i}^+(0)\right)} < c_i$;

(2) $\lim\limits_{i \to \infty} \dfrac{R_i}{\log M_i} = 0.$

Proposition 9. *Let $x_i \to x_0$ be an isolated simple blow-up point for $\{u_i\}_i$. Let η be small. If $0 < \bar{\varepsilon} \le 1$ is small enough and $0 \le \varepsilon_{1,i}, \varepsilon_{2,i} \le \bar{\varepsilon}$, there exist $C, \rho > 0$ such that*

$$M_i^{\lambda_i}|\nabla^k u_i(\psi_i(y))| \le C|y|^{2-k-n+\eta}$$

for $y \in B_\rho^+(0) \setminus \{0\}$ and $k = 0, 1, 2$. Here $\lambda_i = \left(\frac{2}{n-2}\right)(n-2-\eta) - 1$.

Proposition 10. *Let $x_i \to x_0$ be an isolated simple blow-up point for $\{u_i\}_i$ and $\alpha, \beta < 0$. Then $\varepsilon_{1,i} \delta_i \to 0$ and $\varepsilon_{2,i} \to 0$.*

Proof. We compute the Pohozaev identity in a ball of radius r and we set $\frac{r}{\delta_i} =: R_i \to \infty$. We estimate any term of $P(u_i, r_i)$ and $\hat{P}(u_i, r_i)$.

Set

$$I_1(u, r) := \int_{\partial^+ B_r^+} \left(\frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2}|\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma_r$$

$$I_2(u, r) := \frac{r(n-2)^2}{2(n-1)} \int_{\partial(\partial' B_r^+)} u^{\frac{2(n-1)}{n-2}} d\bar{\sigma}_g,$$

we have $P(u_i, r) = I_1(u_i, r) + I_2(u_i, r)$

By Proposition 9 we obtain

$$I_1(u_i, r) = M_i^{-2\lambda_i} I_1(M_i^{\lambda_i} u_i, r) \le c M_i^{-2\lambda_i} \int_{\partial^+ B_r^+} |y|^{2(2-n+\eta)} d\sigma_r \le c\delta_i^{\lambda_i(n-2)};$$

$$I_2(u_i, r) \le c M^{-\lambda_i \frac{2(n-1)}{n-2}} \le c\delta_i^{\lambda_i(n-2)}.$$

Then

$$(2.2) \qquad\qquad P(u_i, r) \leq \delta_i^{\lambda_i(n-2)}.$$

In a similar way we decompose

$$\hat{P}(u_i, r) := -\int_{B_r^+} \left(y^a \partial_a u + \frac{n-2}{2} u \right) [(L_g - \Delta) u] dy + \frac{n-2}{2} \int_{\partial' B_r^+} \left(\bar{y}^k \partial_k u + \frac{n-2}{2} u \right) h_g u d\bar{y}$$

$$+ \varepsilon_1 \int_{B_r^+} \left(y^a \partial_a u + \frac{n-2}{2} u \right) \alpha u dy$$

$$+ \frac{n-2}{2} \varepsilon_2 \int_{\partial' B_r^+} \left(\bar{y}^k \partial_k u + \frac{n-2}{2} u \right) \beta u d\bar{y} =: I_3(u, r) + I_4(u, r) + I_5(u, r) + I_6(u, r).$$

The terms I_3, I_4 and I_6 are been estimated in [10]. For the sake of completeness we report here the main steps of the estimates. By Proposition 9 and by definition of v_i we have

$$|\nabla^k v_i(s)| \leq M_i^{\eta \frac{2}{n-2}} |1 + |s||^{2-k-n} = \delta_i^{-\eta} |1 + |s||^{2-k-n}.$$

So, recalling that $|h_{g_i}(\delta_i s)| \leq O(\delta_i^4 |s|^4)$, we get

$$(2.3) \qquad |I_4(u_i, r)| = \frac{n-2}{2} \delta_i \int_{\partial' B_{R_i}^+} \left(\bar{s}^k \partial_k v_i + \frac{n-2}{2} v_i \right) h_{g_i}(\delta_i s) v_i d\bar{s} \leq c \delta_i^{5-2\eta}.$$

Using the expansion of the metric, it easy to check that

$$(2.4) \qquad\qquad |I_3(u_i, r)| \leq c \delta_i^{2-2\eta}.$$

Finally, by Claim 1 of Proposition 8, by (1.5) and by (1.3) we get

$$\lim_{i \to \infty} \int_{\partial' B_{R_i}^+} \left(\bar{s}^k \partial_k v_i + \frac{n-2}{2} v_i \right) \beta_i(\delta_i s) v_i d\bar{s}$$

$$= \beta(x_0) \int_{\mathbb{R}^{n-1}} \left(\bar{s}^k \partial_k U + \frac{n-2}{2} U \right) U d\bar{s}$$

$$= \frac{n-2}{2} \beta(x_0) \int_{\mathbb{R}^{n-1}} \frac{1 - |\bar{s}|^2}{[1 + |\bar{s}|^2]^{n-1}} d\bar{s}$$

$$= \frac{n-2}{2} \omega_{n-2} \beta(x_0) \left[I_{n-1}^{n-2} - I_{n-1}^n \right] = -\frac{n-2}{n-1} \beta(x_0) \omega_{n-2} I_{n-1}^n =: B > 0,$$

so

$$(2.5) \qquad\qquad I_6(u_i, r) = \varepsilon_{2,i} \delta_i (B + o(1)).$$

In a similar way we proceed for I_5. In fact, by (1.5), (1.3) and (1.4), we have

$$(2.6) \quad \lim_{i\to\infty} \int_{B_{R_i}^+} \left(\bar{s}^a \partial_a v_i + \frac{n-2}{2} v_i \right) \alpha_i(\delta_i s) v_i d\bar{s} =$$

$$= \alpha(x_0) \int_{\mathbb{R}_+^n} \left(\bar{s}^a \partial_a U + \frac{n-2}{2} U \right) U d\bar{s}$$

$$= \frac{n-2}{2} \alpha(x_0) \int_{\mathbb{R}^+} \int_{\mathbb{R}^{n-1}} \frac{1 - |s_n|^2 - |\bar{s}|^2}{[|1+s_n|^2 + |\bar{s}|^2]^{n-1}} d\bar{s} ds_n$$

$$= \frac{n-2}{2} \alpha(x_0) \omega_{n-2} \left[I_{n-1}^{n-2} \int_0^\infty \frac{1-t^2}{(1+t)^{n-1}} dt - I_{n-1}^n \int_0^\infty \frac{1}{(1+t)^{n-3}} dt \right]$$

$$= \frac{n-2}{2} \alpha(x_0) \omega_{n-2} \left[I_{n-1}^{n-2} \frac{n-5}{(n-3)(n-4)} - I_{n-1}^n \frac{1}{n-4} \right]$$

$$= -\frac{2(n-2)}{(n-1)(n-4)} \alpha(x_0) I_{n-1}^n \omega_{n-2} =: A > 0$$

and thus

$$(2.7) \qquad I_5(u_i, r) = \varepsilon_{1,i} \delta_i^2 (A + o(1)).$$

Concluding, by (2.2), (2.4), (2.3), (2.5), (2.5). we get

$$-c\delta_i^{2-2\eta} + (A + o(1))\varepsilon_{1,i}\delta_i^2 + (B + o(1))\varepsilon_{2,i}\delta_i \leq \delta_i^{\lambda_i(n-2)}$$

which is possible only if $\varepsilon_{1,i}\delta_i, \varepsilon_{2,i} \to 0$. $\qquad\square$

Since $\varepsilon_{1,i}\delta_i, \varepsilon_{2,i} \to 0$ by Prop. 10, the proof of the next proposition is analogous to Prop. 4.3 in [1]

Proposition 11. *Let $x_i \to x_0$ be an isolated simple blow-up point for $\{u_i\}_i$ and $\alpha, \beta < 0$. Then there exist $C, \rho > 0$ such that*

(1) $M_i u_i(\psi_i(y)) \leq C|y|^{2-n}$ *for all $y \in B_\rho^+(0) \smallsetminus \{0\}$;*

(2) $M_i u_i(\psi_i(y)) \geq C^{-1} G_i(y)$ *for all $y \in B_\rho^+(0) \smallsetminus B_{r_i}^+(0)$ where $r_i := R_i M_i^{\frac{2}{2-n}}$ and G_i is the Green's function which solves*

$$\begin{cases} L_{g_i} G_i = 0 & in \ B_\rho^+(0) \smallsetminus \{0\} \\ G_i = 0 & on \ \partial^+ B_\rho^+(0) \\ B_{g_i} G_i = 0 & on \ \partial' B_\rho^+(0) \smallsetminus \{0\} \end{cases}$$

and $|y|^{n-2} G_i(y) \to 1$ as $|z| \to 0$.

By Proposition 8 and Proposition 11 we have that, if $x_i \to x_0$ is an isolated simple blow-up point for $\{u_i\}_i$, then it holds

$$v_i \leq CU \text{ in } B_{\rho M_i^{\frac{2}{2-n}}}^+(0).$$

2.3. **Blowup estimates.** Our aim is to provide a fine estimate for the approximation of the rescaled solution near an isolated simple blow up point. In this section $x_i \to x_0$ is an isolated simple blowup point for a sequence $\{u_i\}_i$ of solutions of (2.1). We will work in the conformal Fermi coordinates in a neighborhood of x_i.

Set $\tilde{u}_i = \Lambda_{x_i}^{-1} u_i$ and

$$(2.8) \quad \delta_i := \tilde{u}_i^{\frac{2}{2-n}}(x_i) = u_i^{\frac{2}{2-n}}(x_i) = M_i^{\frac{2}{2-n}} \quad v_i(y) := \delta_i^{\frac{n-2}{2}} u_i(\delta_i y) \text{ for } y \in B_{\frac{R}{\delta_i}}^+(0).$$

Then v_i satisfies

$$(2.9) \qquad \begin{cases} L_{\hat{g}_i} v_i - \varepsilon_{1,i} \alpha_i(\delta_i y) v_i = 0 & \text{in } B^+_{\frac{R}{\delta_i}}(0) \\ B_{\hat{g}_i} v_i + (n-2) v_i^{\frac{n}{n-2}} - \varepsilon_{2,i} \beta_i(\delta_i y) v_i = 0 & \text{on } \partial' B^+_{\frac{R}{\delta_i}}(0) \end{cases}$$

where $\hat{g}_i := \tilde{g}_i(\delta_i y) = \Lambda_{x_i}^{\frac{4}{n-2}}(\delta_i y) g(\delta_i y)$, and $\alpha_i(y) = \Lambda_{x_i}^{-\frac{2}{n-2}}(y)\alpha(y)$, $\beta_i(y) = \Lambda_{x_i}^{-\frac{2}{n-2}}(y)\beta(y)$.

The estimates that follow are similar to the ones of [1, Lemma 6.1], [9, Section 4] and [10, Section 5], where the main differences concern the terms which contain the linear perturbations.

Lemma 12. *Assume $n \geq 8$. Let γ_{x_i} be defined in (1.10). There exist $R, C > 0$ such that*

$$|v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)| \leq C \left(\delta_i^3 + \varepsilon_{1,i} \delta_i^2 + \varepsilon_{2,i} \delta_i \right)$$

for $|y| \leq R/\delta_i$.

Proof. Let y_i such that

$$\mu_i := \max_{|y| \leq R/\delta_i} |v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)| = |v_i(y_i) - U(y_i) - \delta_i^2 \gamma_{x_i}(y_i)|.$$

We can assume, without loss of generality, that $|y_i| \leq \frac{R}{2\delta_i}$.

In fact, suppose that there exists $c > 0$ such that $|y_i| > \frac{c}{\delta_i}$ for all i. Then, since $v_i(y) \leq CU(y)$, and by (1.11), we get the inequality

$$|v_i(y_i) - U(y_i) - \delta_i^2 \gamma_{x_i}(y_i)| \leq C \left(|y_i|^{2-n} + \delta_i^2 |y_i|^{4-n} \right) \leq C \delta_i^{n-2}$$

which proves the Lemma. So, in the next we will suppose $|y_i| \leq \frac{R}{2\delta_i}$. This will be useful later.

By contradiction, suppose that

$$(2.10) \qquad \max \left\{ \mu_i^{-1} \delta_i^3, \mu_i^{-1} \varepsilon_{1,i} \delta_i^2, \mu_i^{-1} \varepsilon_{2,i} \delta_i \right\} \to 0 \text{ when } i \to \infty.$$

Defined

$$w_i(y) := \mu_i^{-1} \left(v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y) \right) \text{ for } |y| \leq R/\delta_i,$$

we have, by direct computation, that w_i satisfies

$$(2.11) \qquad \begin{cases} L_{\hat{g}_i} w_i = A_i & \text{in } B^+_{\frac{R}{\delta_i}}(0) \\ B_{\hat{g}_i} w_i + b_i w_i = F_i & \text{on } \partial' B^+_{\frac{R}{\delta_i}}(0) \end{cases}$$

where

$$b_i = (n-2) \frac{v_i^{\frac{n}{n-2}} - (U + \delta_i^2 \gamma_{x_i})^{\frac{n}{n-2}}}{v_i - U - \delta_i^2 \gamma_{x_i}},$$

$$Q_i = -\frac{1}{\mu_i} \left\{ (L_{\hat{g}_i} - \Delta)(U + \delta_i^2 \gamma_{x_i}) + \delta_i^2 \Delta \gamma_{x_i} \right\},$$

$$A_i = Q_i + \frac{\varepsilon_{1,i} \delta_i^2}{\mu_i} \alpha_i(\delta_i y) v_i(y),$$

$$\bar{Q}_i = -\frac{1}{\mu_i} \left\{ (n-2)(U + \delta_i^2 \gamma_{x_i})^{\frac{n}{n-2}} - (n-2) U^{\frac{n}{n-2}} - n\delta_i^2 U^{\frac{2}{n-2}} \gamma_{x_i} - \frac{n-2}{2} h_{\hat{g}_i}(U + \delta_i^2 \gamma_{x_i}) \right\},$$

$$F_i = \bar{Q}_i + \frac{\varepsilon_{2,i} \delta_i}{\mu_i} \beta_i(\delta_i y) v_i(y).$$

We will estimate terms b_i, A_i, F_i obtaining that the sequence w_i converges in $C^2_{\text{loc}}(\mathbb{R}^n_+)$ to some w solution of

$$(2.12) \qquad \begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^n_+ \\ \frac{\partial}{\partial \nu} w + nU^{\frac{n}{n-2}} w = 0 & \text{on } \partial \mathbb{R}^n_+ \end{cases},$$

then we will derive a contradiction using (2.10).

Since $v_i \to U$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n_+)$ we have, at once,

$$(2.13) \qquad b_i \to nU^{\frac{2}{n-2}} \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^n_+);$$

$$(2.14) \qquad |b_i(y)| \leq (1+|y|)^{-2} \text{ for } |y| \leq R/\delta_i.$$

We proceed now by estimating Q_i and $\bar{Q}_i$. As in [10, Lemma 11], using the expansion of the metric and the decays properties of U and γ_{x_i} we obtain

$$(2.15) \qquad Q_i = O(\mu_i^{-1} \delta_i^3 (1+|y|)^{3-n})$$

and

$$(2.16) \qquad \bar{Q}_i = O(\mu_i^{-1} \delta_i^4 (1+|y|)^{5-n}).$$

Since $|v_i(y)| \leq CU(y)$ from (2.15) and (2.16) we get

$$(2.17) \qquad \begin{aligned} A_i &= O(\mu_i^{-1} \delta_i^3 (1+|y|)^{3-n}) + O(\mu_i^{-1} \varepsilon_{1,i} \delta_i^2 (1+|y|)^{2-n}), \\ F_i &= O(\mu_i^{-1} \delta_i^4 (1+|y|)^{5-n}) + O(\mu_i^{-1} \varepsilon_{2,i} \delta_i (1+|y|)^{2-n}). \end{aligned}$$

In light of (2.10) we also have $A_i \in L^p(B^+_{R/\delta_i})$ and $F_i \in L^p(\partial' B^+_{R/\delta_i})$ for all $p \geq 2$.

Finally we remark that $|w_i(y)| \leq 1$, so by (2.10) (2.13), (2.14), (2.17) and by standard elliptic estimates we conclude that, up to subsequence, $\{w_i\}_i$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^n_+)$ to some w solution of (2.12) as claimed at the beginning of the proof.

The next step is to prove that $|w(y)| \leq C(1+|y|^{-1})$ for $y \in \mathbb{R}^n_+$. Consider G_i the Green function for the conformal Laplacian $L_{\hat{g}_i}$ defined on B^+_{r/δ_i} with boundary conditions $B_{\hat{g}_i} G_i = 0$ on $\partial' B^+_{r/\delta_i}$ and $G_i = 0$ on $\partial^+ B^+_{r/\delta_i}$. It is well known that $G_i = O(|\xi - y|^{2-n})$. By the Green formula and by (2.17) we have

$$w_i(y) = -\int_{B^+_{\frac{R}{\delta_i}}} G_i(\xi, y) A_i(\xi) d\mu_{\hat{g}_i}(\xi) - \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} \frac{\partial G_i}{\partial \nu}(\xi, y) w_i(\xi) d\sigma_{\hat{g}_i}(\xi)$$
$$+ \int_{\partial' B^+_{\frac{R}{\delta_i}}} G_i(\xi, y) (b_i(\xi) w_i(\xi) - F_i(\xi)) d\sigma_{\hat{g}_i}(\xi),$$

so

$$|w_i(y)| \leq \frac{\delta_i^3}{\mu_i} \int_{B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n} (1+|\xi|)^{3-n} d\xi + \frac{\varepsilon_{1,i} \delta_i^2}{\mu_i} \int_{B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n} (1+|\xi|)^{2-n} d\xi$$
$$+ \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} |\xi - y|^{1-n} w_i(\xi) d\sigma(\xi)$$
$$+ \int_{\partial' B^+_{\frac{R}{\delta_i}}} |\bar{\xi} - y|^{2-n} \left((1+|\bar{\xi}|)^{-2} + \frac{\delta_i^4}{\mu_i} (1+|\bar{\xi}|)^{5-n} + \frac{\varepsilon_{2,i} \delta_i}{\mu_i} (1+|\bar{\xi}|)^{2-n} \right) d\bar{\xi}.$$

Notice that in the third integral we used that $|y| \leq \frac{R}{2\delta_i}$ to estimate $|\xi - y| \geq |\xi| - |y| \geq \frac{R}{2\delta_i}$ on $\partial^+ B^+_{R/\delta_i}$. Moreover, since $v_i(\xi) \leq CU(\xi)$, we get

$$(2.18) \qquad |w_i(\xi)| \leq \frac{C}{\mu_i} \left((1+|\xi|)^{2-n} + \delta_i^2 (1+|\xi|)^{4-n} \right) \leq C \frac{\delta_i^{n-2}}{\mu_i} \text{ on } \partial^+ B^+_{R/\delta_i};$$

hence

$$(2.19) \qquad \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} |\xi - y|^{1-n} w_i(\xi) d\sigma(\xi) \leq C \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} \frac{\delta_i^{2n-3}}{\mu_i} d\sigma_{\hat{g}_i}(\xi) \leq C \frac{\delta_i^{n-2}}{\mu_i}.$$

For the other terms we use the formula

$$(2.20) \qquad \int_{\mathbb{R}^m} |\xi - y|^{l-m}(1 + |\xi|)^{-\eta} d\xi \leq C(1 + |y|)^{l-\eta}$$

where $y \in \mathbb{R}^{m+k} \supseteq \mathbb{R}^m$, $\eta, l \in \mathbb{N}$, $0 < l < \eta < m$ (see [1, Lemma 9.2] and [3, 8]) . We get

$$(2.21) \qquad \frac{\delta_i^3}{\mu_i} \int_{B_{\frac{R}{\delta_i}}^+} |\xi - y|^{2-n}(1 + |\xi|)^{3-n} d\xi \leq C \frac{\delta_i^3}{\mu_i}(1 + |y|)^{5-n},$$

$$(2.22) \qquad \frac{\varepsilon_{1,i}\delta_i^2}{\mu_i} \int_{B_{\frac{R}{\delta_i}}^+} |\xi - y|^{2-n}(1 + |\xi|)^{2-n} d\xi \leq C \frac{\varepsilon_{1,i}\delta_i^2}{\mu_i}(1 + |y|)^{4-n},$$

$$(2.23) \qquad \int_{\partial' B_{\frac{R}{\delta_i}}^+} |\bar{\xi} - y|^{2-n}(1 + |\bar{\xi}|)^{-2} d\bar{\xi} \leq (1 + |y|)^{-1},$$

$$(2.24) \qquad \frac{\delta_i^4}{\mu_i} \int_{\partial' B_{\frac{R}{\delta_i}}^+} |\bar{\xi} - y|^{2-n}(1 + |\bar{\xi}|)^{5-n} d\bar{\xi} \leq C \frac{\delta_i^4}{\mu_i}(1 + |y|)^{6-n},$$

$$(2.25) \qquad \frac{\varepsilon_{2,i}\delta_i}{\mu_i} \int_{\partial' B_{\frac{R}{\delta_i}}^+} |\bar{\xi} - y|^{2-n}(1 + |\bar{\xi}|)^{2-n} d\bar{\xi} \leq C \frac{\varepsilon_{2,i}\delta_i}{\mu_i}(1 + |y|)^{3-n}.$$

By the previous estimates we infer that, for $|y| \leq \frac{R}{2\delta_i}$,

$$|w_i(y)| \leq C\left((1 + |y|)^{-1} + \frac{\delta_i^3}{\mu_i}(1 + |y|)^{5-n} + \frac{\varepsilon_{1,i}\delta_i^2}{\mu_i}(1 + |y|)^{4-n} + \frac{\varepsilon_{2,i}\delta_i}{\mu_i}(1 + |y|)^{3-n}\right)$$

so by assumption (2.10) we prove

$$(2.26) \qquad |w(y)| \leq C(1 + |y|)^{-1} \text{ for } y \in \mathbb{R}_+^n$$

as claimed.

Finally we notice that, since $v_i \to U$ near 0, and, by (1.14), we have $w_i(0) \to 0$ as well as $\frac{\partial w_i}{\partial y_j}(0) \to 0$ for $j = 1, \ldots, n - 1$. This implies that

$$(2.27) \qquad w(0) = \frac{\partial w}{\partial y_1}(0) = \cdots = \frac{\partial w}{\partial y_{n-1}}(0) = 0.$$

We are ready now to prove the contradiction. In fact, it is known (see [1, Lemma 2]) that any solution of (2.12) that decays as (2.26) is a linear combination of $\frac{\partial U}{\partial y_1}, \ldots, \frac{\partial U}{\partial y_{n-1}}, \frac{n-2}{2}U + y^b \frac{\partial U}{\partial y_b}$. This fact, combined with (2.27), implies that $w \equiv 0$.

Now, on the one hand $|y_i| \leq \frac{R}{2\delta_i}$, so estimate (2.26) holds; on the other hand, since $w_i(y_i) = 1$ and $w \equiv 0$, we get $|y_i| \to \infty$, obtaining

$$1 = w_i(y_i) \leq C(1 + |y_i|)^{-1} \to 0$$

which gives us the contradiction. $\qquad \square$

Lemma 13. *Assume $n \geq 8$ and $\alpha, \beta < 0$. There exists $R, C > 0$ such that*

$$\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i \leq C\delta_i^3$$

for $|y| \leq R/\delta_i$.

Proof. We proceed by contradiction, supposing that

$$(2.28) \qquad \left(\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i\right)^{-1}\delta_i^3 \to 0 \text{ when } i \to \infty.$$

Thus, by Lemma 12, we have

$$|v_i(y) - U(y) - \delta_i^2\gamma_{x_i}(y)| \le C(\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i) \text{ for } |y| \le R/\delta_i.$$

We define, similarly to Lemma 12,

$$w_i(y) := \frac{1}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}\left(v_i(y) - U(y) - \delta_i^2\gamma_{x_i}(y)\right) \text{ for } |y| \le R/\delta_i,$$

and we have that w_i satisfies (2.11) where b_i is as in Lemma 12 and

$$Q_i = -\frac{1}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}\left\{(L_{\hat{g}_i} - \Delta)(U + \delta_i^2\gamma_{x_i}) + \delta_i^2\Delta\gamma_{x_i}\right\},$$

$$A_i = Q_i + \frac{\varepsilon_{1,i}\delta_i^2}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}\alpha_i(\delta_i y)v_i(y),$$

$$\bar{Q}_i = -\frac{1}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}\left\{(n-2)(U + \delta_i^2\gamma_{x_i})^{\frac{n}{n-2}} - (n-2)U^{\frac{n}{n-2}} - n\delta_i^2 U^{\frac{2}{n-2}}\gamma_{x_i} - \frac{n-2}{2}h_{\hat{g}_i}(U + \delta_i^2\gamma_{x_i})\right\},$$

$$F_i = \bar{Q}_i + \frac{\varepsilon_{2,i}\delta_i}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}\beta_i(\delta_i y)v_i(y).$$

As before, b_i satisfies inequality (2.14) while

$$(2.29) \quad A_i = O\left(\frac{\delta_i^3}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}(1 + |y|)^{3-n}\right) + O\left(\frac{\varepsilon_{1,i}\delta_i^2}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}(1 + |y|)^{2-n}\right)$$

$$F_i = O\left(\frac{\delta_i^4}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}(1 + |y|)^{5-n}\right) + O\left(\frac{\varepsilon_{2,i}\delta_i}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}(1 + |y|)^{2-n}\right),$$

so by classic elliptic estimates we can prove that the sequence w_i converges in $C^2_{\text{loc}}(\mathbb{R}^n_+)$ to some w.

We proceed as in Lemma 12 to deduce that, by (2.28) and since $\frac{\varepsilon_{1,i}\delta_i^2}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i} \le 1$, $\frac{\varepsilon_{2,i}\delta_i}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i} \le 1$,

$$|w_i(y)| \le C\left((1 + |y|)^{-1} + \frac{\delta_i^3(1 + |y|)^{5-n}}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i} + \frac{\varepsilon_{1,i}\delta_i^2(1 + |y|)^{4-n}}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i} + \frac{\varepsilon_{2,i}\delta_i(1 + |y|)^{3-n}}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}\right)$$

$$(2.30)$$

$$\le C\left((1 + |y|)^{-1}\right) \text{ for } |y| \le \frac{R}{2\delta_i}.$$

Now let j_n be defined as in (1.8). Indeed, since w_i satisfies (2.11), integrating by parts we obtain

$$(2.31) \quad \int_{\partial' B^+_{\frac{R}{\delta_i}}} j_n F_i d\sigma_{\hat{g}_i} = \int_{\partial' B^+_{\frac{R}{\delta_i}}} j_n\left[B_{\hat{g}_i}w_i + b_i w_i\right] d\sigma_{\hat{g}_i}$$

$$= \int_{\partial' B^+_{\frac{R}{\delta_i}}} w_i\left[B_{\hat{g}_i}j_n + b_i j_n\right] d\sigma_{\hat{g}_i} + \int_{\partial^+ B^+_{\frac{R}{\delta_i}}}\left[\frac{\partial j_n}{\partial \eta_i}w_i - \frac{\partial w_i}{\partial \eta_i}j_n\right] d\sigma_{\hat{g}_i}$$

$$+ \int_{B^+_{\frac{R}{\delta_i}}}\left[w_i L_{\hat{g}_i}j_n - j_n L_{\hat{g}_i}w_i\right] d\mu_{\hat{g}_i}$$

where η_i is the inward unit normal vector to $\partial^+ B^+_{\frac{R}{\delta_i}}$. One can check easily that

$$\lim_{i \to +\infty} \int_{\partial' B^+_{\frac{R}{\delta_i}}} j_n \bar{Q}_i d\sigma_{\hat{g}_i} = 0.$$

Also, since $\beta_i(\delta_i y) = \Lambda_{x_i}^{-\frac{2}{n-2}}(\delta_i y)\beta(\delta_i y)$, and $\beta < 0$, by Proposition 8, we have

$$\beta_i(\delta_i y)v_i(y) \to \beta(x_0)U(y) \text{ for } i \to +\infty.$$

and thus, as in Proposition 10

(2.32)
$$\lim_{i \to +\infty} \int_{\partial' B^+_{\frac{R}{\delta_i}}} \beta_i(\delta_i y)v_i(y)j_n(y) = \frac{n-2}{2}\beta(x_0) \int_{\mathbb{R}^{n-1}} \frac{1 - |\bar{y}|^2}{(1 + |\bar{y}|^2)^{n-1}} =: B > 0$$

so

(2.33)
$$\int_{\partial' B^+_{\frac{R}{\delta_i}}} j_n F_i d\sigma_{\hat{g}_i} = \frac{\varepsilon_{2,i}\delta_i}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}(B + o(1)).$$

By (2.31) and (2.33) we derive a contradiction. Indeed, by the decay of j_n and by the decay of w_i, given by (2.30) and by (2.28), we have

(2.34)
$$\lim_{i \to +\infty} \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} \left[\frac{\partial j_n}{\partial \eta_i} w_i - \frac{\partial w_i}{\partial \eta_i} j_n \right] d\sigma_{\hat{g}_i} = 0.$$

Since $\Delta j_n = 0$, one can check that

(2.35)
$$\lim_{i \to +\infty} \int_{B^+_{\frac{R}{\delta_i}}} w_i L_{\hat{g}_i} j_n d\mu_{\hat{g}_i} = 0.$$

Also, we can prove that

(2.36)
$$\lim_{i \to +\infty} \int_{B^+_{\frac{R}{\delta_i}}} j_n Q_i d\mu_{\hat{g}_i} = 0.$$

Finally

(2.37)
$$\lim_{i \to +\infty} \int_{\partial' B^+_{\frac{R}{\delta_i}}} w_i \left[B_{\hat{g}_i} j_n + b_i j_n \right] d\sigma_{\hat{g}_i} = \int_{\partial \mathbb{R}^n_+} w \left[\frac{\partial j_n}{\partial y_n} + nU^{\frac{2}{n-2}} j_n \right] d\sigma_{\hat{g}_i} = 0$$

since $\frac{\partial j_n}{\partial y_n} + nU^{\frac{2}{n-2}} j_n = 0$ when $y_n = 0$.

In light of (2.34) (2.36) and (2.35) we infer, by (2.31), that

(2.38)
$$\int_{\partial' B^+_{\frac{R}{\delta_i}}} j_n F_i d\sigma_{\hat{g}_i} = - \int_{B^+_{\frac{R}{\delta_i}}} \left[j_n A_i w_i \right] d\mu_{\hat{g}_i} + o(1).$$

$$= -\frac{\varepsilon_{1,i}\delta_i^2}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i} \int_{B^+_{\frac{R}{\delta_i}}} j_n(y)\alpha_i(\delta_i y)v_i(y)d\mu_{\hat{g}_i} + o(1).$$

Again we have $\alpha_i(\delta_i y)v_i(y) \to \alpha(x_0)U(y)$ for $i \to +\infty$ and $\alpha < 0$, so, proceeding as in Proposition 10, we have

(2.39)
$$\lim_{i \to \infty} \int_{B^+_{\frac{R}{\delta_i}}} j_n(y)\alpha_i(\delta_i y)v_i(y)d\mu_{\hat{g}_i} = \alpha(x_0) \lim_{i \to \infty} \int_{\mathbb{R}^n_+} \left(s^a \partial_a v_i + \frac{n-2}{2}v_i \right) v_i ds =: A > 0,$$

so

(2.40)
$$\int_{B^+_{\frac{R}{\delta_i}}} \left[j_n A_i w_i \right] = -\frac{\varepsilon_{1,i}\delta_i^2}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}(A + o(1))$$

and, by (2.38), (2.33) and (2.40), we have

$$(2.41) \qquad \frac{\varepsilon_{2,i}\delta_i}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}(B + o(1)) = -\frac{\varepsilon_{1,i}\delta_i^2}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}(A + o(1)).$$

Since $\frac{\varepsilon_{2,i}\delta_i}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}$ and $\frac{\varepsilon_{1,i}\delta_i^2}{\varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i}$ cannot vanish simultaneously while $i \to \infty$, equation (2.41) leads us to a contradiction. □

The above lemmas are the core of the following proposition, in which we iterate the procedure of Lemma 12, to obtain better estimates of the rescaled solution v_i of (2.9) around the isolated simple blow up point $x_i \to x_0$.

Proposition 14. *Assume $n \geq 8$ and $\alpha, \beta < 0$. Let γ_{x_i} be defined in (1.10). There exist $R, C > 0$ such that*

$$\left| \nabla_{\bar{y}}^{\tau} \left(v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y) \right) \right| \leq C\delta_i^3 (1 + |y|)^{5-\tau-n}$$

$$\left| y_n \frac{\partial}{\partial_n} \left(v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y) \right) \right| \leq C\delta_i^3 (1 + |y|)^{5-n}$$

for $|y| \leq \frac{R}{2\delta_i}$. Here $\tau = 0, 1, 2$ and $\nabla_{\bar{y}}^{\tau}$ is the differential operator of order τ with respect the first $n-1$ variables.

Proof. In analogy with Lemma 12, we set

$$w_i(y) := v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y) \text{ for } |y| \leq R/\delta_i,$$

and we have that w_i satisfies (2.11) where b_i is defined as before,

$Q_i = -\left\{ (L_{\hat{g}_i} - \Delta)(U + \delta_i^2 \gamma_{x_i}) + \delta_i^2 \Delta \gamma_{x_i} \right\},$

$A_i = Q_i + \varepsilon_{1,i}\delta_i^2 \alpha_i(\delta_i y) v_i(y),$

$\bar{Q}_i = -\left\{ (n-2)(U + \delta_i^2 \gamma_{x_i})^{\frac{n}{n-2}} - (n-2)U^{\frac{n}{n-2}} - n\delta_i^2 U^{\frac{2}{n-2}}\gamma_{x_i} - \frac{n-2}{2}h_{\hat{g}_i}(U + \delta_i^2 \gamma_{x_i}) \right\},$

$F_i = \bar{Q}_i + \varepsilon_{2,i}\delta_i \beta_i(\delta_i y) v_i(y).$

As before, b_i satisfies inequality (2.14) and

$$(2.42) \qquad A_i = O(\delta_i^3 (1 + |y|)^{3-n}) + O(\varepsilon_{1,i}\delta_i^2 (1 + |y|)^{2-n}),$$

$$(2.43) \qquad F_i = O(\delta_i^4 (1 + |y|)^{5-n}) + O(\varepsilon_{2,i}\delta_i (1 + |y|)^{2-n}).$$

We define the Green function G_i as in the previous lemma and again, by Green's formula, by (2.42), (2.43), and Lemmas 12 and 13, we have

$$(2.44) \qquad |w_i(y)| \leq C\delta_i^3 \text{ on } B_{R/\delta_i}^+ \quad \text{and} \quad |w_i(\xi)| \leq C\delta_i^{n-2} \text{ on } \partial^+ B_{R/\delta_i}^+.$$

By this we show that $\int_{\partial' B_{\frac{R}{\delta_i}}^+} |\bar{\xi} - y|^{2-n} b_i(\xi) w_i(\xi) d\bar{\xi} \leq \delta_i^3 (1 + |y|)^{-1}$, while one can manage the other terms in Green's formula as in the previous lemmas. So we obtain

$$(2.45) \qquad |w_i(y)| \leq C\delta_i^3 (1 + |y|)^{-1} \text{ for } |y| \leq \frac{R}{2\delta_i}.$$

Now we can iterate the procedure until we reach

$$(2.46) \qquad |w_i(y)| \leq C\delta_i^3 (1 + |y|)^{5-n} \text{ for } |y| \leq \frac{R}{2\delta_i},$$

which proves the first claim for $\tau = 0$. The other claims follow similarly. □

2.4. **Sign estimates of Pohozaev identity terms.** In this section, we want to estimate $P(u_i, r)$, where $\{u_i\}_i$ is a family of solutions of (2.1) which has an isolated simple blow up point $x_i \to x_0$. This estimate, given in the following Proposition 16, is a crucial point for the proof of the vanishing of the Weyl tensor at an isolated simple blow up point.

Since the leading term of $P(u_i, r)$ will be $-\int_{B^+_{r/\delta_i}} \left(y^b \partial_b u + \frac{n-2}{2} u\right) [(L_{\hat{g}_i} - \Delta)v] \, dy$, we set

$$(2.47) \qquad R(u,v) = -\int_{B^+_{r/\delta_i}} \left(y^b \partial_b u + \frac{n-2}{2} u\right) [(L_{\hat{g}_i} - \Delta)v] \, dy,$$

and we recall the following result

Lemma 15. *For $n \geq 8$ we have*

$$R(U + \delta^2 \gamma_q, U + \delta^2 \gamma_q) = \begin{cases} \delta^4 \frac{(n-2)\omega_{n-2} I_n^n}{(n-1)(n-3)(n-5)(n-6)} \left[\frac{(n-2)}{6}|\bar{W}(q)|^2 + \frac{4(n-8)}{(n-4)} R^2_{ninj}(q)\right] \\ \qquad -2\delta^4 \int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q \, dy + o(\delta^4) \text{ for } n > 8 \\ \\ \delta_i^4 \omega_6 I_8^8 \left[\frac{1}{35}|\bar{W}(q)|^2 + \frac{1089}{34020} R^2_{8i8j}(q)\right] + o(\delta^4) \text{ for } n = 8 \end{cases}.$$

Proof. For the proof we refer to [9] for the case $n > 8$ and to [12] for the case $n = 8$. $\qquad \square$

Proposition 16. *Let $x_i \to x_0$ be an isolated simple blow-up point for u_i solutions of (2.1). Let $\alpha, \beta < 0$ and $n \geq 8$. Then, fixed r, we have, for i large*

$$\hat{P}(u_i, r) \geq \delta_i^4 \left[C_1 |\bar{W}(x_i)|^2 + C_2 R^2_{nlnj}(x_i)\right] + o(\delta_i^4)$$

where $C_1, C_2 > 0$.

Proof. We recall that the definition of $\hat{P}$ is given in Theorem 6 and we take $v_i(y)$ as in (2.8). By Proposition 14 and by (1.11) of Lemma 4, for $|y| < R/\delta_i$ we have

$$|v_i(y) - U(y)| = O(\delta_i^3(1 + |y|^{5-n})) + O(\delta_i^2(1 + |y|^{4-n})) = O(\delta_i^2(1 + |y|^{4-n})),$$

$$|y_a \partial_a v_i(y) - y_a \partial_a U(y)| = O(\delta_i^3(1 + |y|^{5-n})) + O(\delta_i^2(1 + |y|^{4-n})) = O(\delta_i^2(1 + |y|^{4-n})),$$

so

$$\int_{B^+_r} \left(y^a \partial_a u_i + \frac{n-2}{2} u_i\right) \varepsilon_{1,i} \alpha_i u_i \, dy = \varepsilon_{1,i} \delta_i^2 \int_{B^+_{r/\delta_i}} \left(y^a \partial_a v_i + \frac{n-2}{2} v_i\right) \alpha_i(\delta_i y) v_i \, dy + \varepsilon_{1,i} \delta_i^2 o(\delta_i^2)$$

and, recalling that $\alpha_i(\delta_i y) \to \alpha(x_0) < 0$ and proceeding as in Proposition 10 we get

$$\lim_{i \to \infty} \int_{B^+_r} \left(y^a \partial_a v_i + \frac{n-2}{2} v_i\right) \alpha_i v_i \, dy = \frac{n-2}{2} \alpha(x_0) \int_{\mathbb{R}^n_+} \frac{1 - |y|^2}{[(1 + y_n)^2 + |\bar{y}|^2]^{n-1}} \, dy > 0.$$

Analogously

$$\int_{\partial' B^+_r} \left(\bar{y}^k \partial_k u_i + \frac{n-2}{2} u_i\right) \varepsilon_{2,i} \beta_i u_i \, d\bar{y} = \varepsilon_{2,i} \delta_i \int_{\partial' B^+_{r/\delta_i}} \left(\bar{y}^k \partial_k v_i + \frac{n-2}{2} v_i\right) \beta_i(\delta_i y) U \, d\bar{y} + \varepsilon_{2,i} \delta_i O(\delta_i^2)$$

and again we get

$$\lim_{i \to \infty} \int_{\partial' B^+_{r/\delta_i}} \left(\bar{y}^k \partial_k v_i + \frac{n-2}{2} v_i\right) \beta_i(\delta_i y) v_i \, d\bar{y} = \frac{n-2}{2} \beta(x_0) \int_{\mathbb{R}^{n-1}} \frac{1 - |\bar{y}|^2}{[1 + |\bar{y}|^2]^{n-1}} \, d\bar{y} > 0.$$

Finally, since $h_{g_i}(\delta_i y) = O(\delta_i^4 |y|^4)$ we have $\int_{\partial' B_{r/\delta_i}^+} \left(y^b \partial_b v_i + \frac{n-2}{2} v_i \right) h_{g_i}(\delta_i y) v_i d\bar{y} = O(\delta_i^5)$. So, for i sufficiently large we obtain

$$\hat{P}(u_i, r) \geq - \int_{B_{r/\delta_i}^+} \left(y^b \partial_b v_i + \frac{n-2}{2} v_i \right) [(L_{\hat{g}_i} - \Delta) v_i] \, dy + O(\delta_i^5).$$

Now define, in analogy with Proposition 14,

$$w_i(y) := v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y).$$

Recalling (2.47), we have

$$\hat{P}(u_i, r) \geq R(U, U) + R(U, \delta_i^2 \gamma_{x_i}) + R(\delta_i^2 \gamma_{x_i}, U) + R(w_i, U) + R(U, w_i)$$
$$+ R(w_i, w_i) + R(\delta_i^2 \gamma_{x_i}, \delta_i^2 \gamma_{x_i}) + R(w_i, \delta_i^2 \gamma_{x_i}) + R(\delta_i^2 \gamma_{x_i}, w_i) + O(\delta_i^5).$$

By [9] we have that $\hat{P}(u_i, r) \geq R(U, U) + R(U, \delta_i^2 \gamma_{x_i}) + R(\delta_i^2 \gamma_{x_i}, U) + R(\delta_i^2 \gamma_{x_i}, \delta_i^2 \gamma_{x_i}) + O(\delta_i^5)$ and by Lemma 15 we conclude the proof. $\square$

Proposition 17. *Assume $n \geq 8$ and $\alpha, \beta < 0$. Let $x_i \to x_0$ be an isolated simple blow-up point for u_i solutions of (2.1). Then $|W(x_0)| = 0$.*

Proof. By Proposition 11 and Proposition 9, and since $M_i = \delta_i^{\frac{2-n}{2}}$ we have,

$$P(u_i, r) := \frac{1}{M_i^{2\lambda_i}} \int_{\partial^+ B_r^+} \left(\frac{n-2}{2} M_i^{\lambda_i} u_i \frac{\partial M_i^{\lambda_i} u_i}{\partial r} - \frac{r}{2} |\nabla M_i^{\lambda_i} u_i|^2 + r \left| \frac{\partial M_i^{\lambda_i} u_i}{\partial r} \right|^2 \right) d\sigma_r$$

$$+ \frac{r(n-2)^2}{(n-1) M_i^{\lambda_i \frac{2(n-1)}{n-2}}} \int_{\partial(\partial' B_r^+)} \left(M_i^{\lambda_i} u_i \right)^{\frac{2(n-1)}{n-2}} d\bar{\sigma}_g.$$

$$\leq \frac{C}{M_i^{\lambda_i \frac{2(n-1)}{n-2}}} \leq C \delta_i^{(n-1)\lambda_i} \leq C \delta_i^{n-2}.$$

On the other hand recalling Proposition 16 and Theorem 6 we have

$$P(u_i, r) = \hat{P}(u_i, r) \geq \delta_i^4 \left[C_1 |\bar{W}(x_i)|^2 + C_2 R_{nlnj}^2(x_i) \right] + o(\delta_i^4),$$

so we get $\left[C_1 |\bar{W}(x_i)|^2 + C_2 R_{nlnj}^2(x_i) \right] \leq \delta_i^2$. Recalling that when the boundary is umbilic $W(q) = 0$ if and only if $\bar{W}(q) = 0$ and $R_{nlnj}(q) = 0$ (see [20, page 1618]) we conclude the proof. $\square$

Remark 18. Let $x_i \to x_0$ be an isolated blow up point for u_i solutions of (2.1). We set

$$(2.48) \qquad P'(u, r) := \int_{\partial^+ B_r^+} \left(\frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma_r,$$

so

$$P(u_i, r) = P'(u_i, r) + \frac{r(n-2)^2}{(n-1)} \int_{\partial(\partial' B_r^+)} u_i^{\frac{2(n-1)}{n-2}} d\bar{\sigma}_g$$

and, keeping in mind that for i large $M_i u_i \leq C|y|^{2-n}$ by Proposition 11, we have
(2.49)

$$\left| r \int_{\partial(\partial' B_r^+)} u_i^{\frac{2(n-1)}{n-2}} d\bar{\sigma}_g \right| \leq \frac{Cr}{M_i^{\frac{2(n-1)}{n-2}}} \int_{\substack{y_n = 0 \\ |\bar{y}| = r}} \frac{1}{|y|^{2(n-1)}} d\bar{\sigma}_g \leq \frac{C(r)}{M_i^{\frac{2(n-1)}{n-2}}} = C(r) \delta_i^{n-2}$$

for i sufficiently large.

Using Proposition 16, (2.49), and since $n \geq 8$ we get

$$(2.50) \qquad P'(u_i, r) = P(u_i, r) - \frac{r(n-2)^2}{(n-1)} \int\limits_{\partial(\partial' B_r^+)} u_i^{\frac{2(n-1)}{n-2}} d\bar{\sigma}_g \geq C\delta_i^4 + o(\delta^4)$$

where $C > 0$.

Proposition 19. *Let $x_i \to x_0$ be an isolated blow up point for u_i solutions of (2.1). Assume $n \geq 8$ and $|W(x_0)| \neq 0$. Then x_0 is isolated simple.*

For the proof of this Lemma we refer to [1, 9]

2.5. **Proof of Theorem 1.** Before the proof of Theorem, we summarize a result which proves that only isolated blow up points may occur to a blowing up sequence of solution. For the proof of this result we refer to [19, Proposition 5.1], [23, Lemma 3.1], [15, Proposition 1.1], [1, Proposition 4.2] for the first claims, [9] for the last claim when $n > 8$ and to [12] in the case $n = 8$.

Proposition 20. *Given $K > 0$ and $R > 0$ there exist two constants $C_0, C_1 > 0$ (depending on K, R and (M, g)) such that if u is a solution of*

$$(2.51) \qquad \begin{cases} L_g u - \varepsilon_1 \alpha = 0 & in \ M \\ B_g u - \varepsilon_2 \beta u + (n-2) u^{\frac{n}{n-2}} = 0 & on \ \partial M \end{cases}$$

and $\max_{\partial M} u > C_0$, then there exist $q_1, \ldots, q_N \in \partial M$, with $N = N(u) \geq 1$ with the following properties: for $j = 1, \ldots, N$

(1) *set $r_j := R u(q_j)^{1-p}$ then $\{ B_{r_j} \cap \partial M \}_j$ are a disjoint collection;*
(2) *we have $\left| u(q_j)^{-1} u(\psi_j(y)) - U(u(q_j)^{p-1} y) \right|_{C^2(B_{2r_j}^+)} < K$ (here ψ_j are the Fermi coordinates at point q_j;*
(3) *we have*

$$u(x) d_{\bar{g}} \left(x, \{q_1, \ldots, q_n\} \right)^{\frac{1}{p-1}} \leq C_1 \ for \ all \ x \in \partial M$$

$$u(q_j) d_{\bar{g}} \left(q_j, q_k \right)^{\frac{1}{p-1}} \geq C_0 \ for \ any \ j \neq k.$$

In addition, if $n \geq 8$ and $W(x) \neq 0$ for any $x \in \partial M$, there exists $d = d(K, R)$ such that

$$\min_{\substack{i \neq j \\ 1 \leq i, j \leq N(u)}} d_{\bar{g}}(q_i(u), q_j(u)) \geq d.$$

Here $\bar{g}$ is the geodesic distance on ∂M.

We prove now the main result

Proof of Theorem 1. . By contradiction, suppose that $x_i \to x_0$ is a blowup point for u_i solutions of (1.2). Let $q_1^i, \ldots q_{N(u_i)}^i$ be the sequence of points given by Proposition 20. By Claim 3 of Proposition 20 there exists a sequence of indices $k_i \in 1, \ldots N$ such that $d_{\bar{g}} \left(x_i, q_{k_i}^i \right) \to 0$. Up to relabeling, we say $k_i = 1$ for all i. Then also $q_1^i \to x_0$ is a blow up point for u_i. By Proposition 20 and Proposition 19 we have that $q_1^i \to x_0$ is an isolated simple blow up point for u_i. Then by Proposition 17 we deduce that $W(x_0) = 0$, contradicting the assumption of the theorem. This concludes the proof. $\square$

3. The non compactness result

In this section we perform the Ljapunov-Schmidt finite dimensional reduction, which relies on three steps. First, we start finding a solution of the infinite dimensional problem (1.27) with the ansatz $\Lambda_q u = \tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q} + \tilde{\phi}$ where $\tilde{\phi} \in \tilde{K}_{\delta,q}^{\perp}$. This is done in subsection 3.1. Then, we study the finite dimensional reduced problem in subsection 3.2, and in the last subsection we give the proof of Theorem 2.

3.1. The finite dimensional reduction.

Let us define the linear operator $L : \tilde{K}_{\delta,q}^{\perp} \to \tilde{K}_{\delta,q}^{\perp}$ as

$$(3.1) \qquad L(\tilde{\phi}) = \tilde{\Pi}^{\perp} \left\{ \tilde{\phi} - i_{\alpha}^{*} \left(f'(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q})[\tilde{\phi}] \right) \right\},$$

and let us define a nonlinear term $N(\tilde{\phi})$ and a remainder term R as

$$(3.2)$$
$$N(\tilde{\phi}) = \tilde{\Pi}^{\perp} \left\{ i_{\alpha}^{*} \left(f(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q} + \tilde{\phi}) - f(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) - f'(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q})[\tilde{\phi}] \right) \right\}$$

$$(3.3)$$
$$R = \tilde{\Pi}^{\perp} \left\{ i_{\alpha}^{*} \left(f(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) \right) - \tilde{W}_{\delta,q} - \delta^2 \tilde{V}_{\delta,q} \right\},$$

With these operators the infinite dimensional equation (1.27) becomes

$$L(\tilde{\phi}) = N(\tilde{\phi}) + R - \tilde{\Pi}^{\perp} \left\{ i_{\alpha}^{*} \left(\varepsilon_2 \beta (\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q} + \tilde{\phi}) \right) \right\}.$$

In this subsection we will find, for any δ, q given, a function $\tilde{\phi}$ which solves equation (1.27).

Lemma 21. *It holds*

$$\|R\|_g = \left\{ \begin{array}{ll} O\left(\delta^3 \log \delta\right) + O(\varepsilon_1 \delta^2) & \text{if } n = 8 \\ O\left(\delta^3\right) + O(\varepsilon_1 \delta^2) & \text{if } n > 8 \end{array} \right. .$$

Proof. Several estimates for this proof has been calculated in [13], which we refer to. We report here only the main steps.

Take the unique Γ such that

$$\Gamma = i_{\alpha}^{*} \left(f(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) \right),$$

that is the unique Γ which solves

$$\left\{ \begin{array}{ll} -\Delta_g \Gamma + \frac{n-2}{4(n-1)} R_g \Gamma + \varepsilon_1 \alpha \Gamma = 0 & \text{on } M; \\ \frac{\partial \Gamma}{\partial \nu} + \frac{n-2}{2} h_g \Gamma = (n-2) \left((\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q})^{+} \right)^{\frac{n}{n-2}} & \text{on } \partial M. \end{array} \right.$$

Let us call $a := \frac{n-2}{4(n-1)} R_g$. We have, by (1.23) that

$$\|R\|_g^2 \leq \left\| i_g^{*} \left(f(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) - \tilde{W}_{\delta,q} - \delta^2 \tilde{V}_{\delta,q} \right) \right\|_g^2 = \|\Gamma - \tilde{W}_{\delta,q} - \delta^2 \tilde{V}_{\delta,q}\|_g^2$$

$$= \int_M \left[\Delta_g (\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) - a(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) \right] (\Gamma - \tilde{W}_{\delta,q} - \delta^2 \tilde{V}_{\delta,q}) d\mu_g$$

$$- \int_M \varepsilon_1 \alpha (\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q})(\Gamma - \tilde{W}_{\delta,q} - \delta^2 \tilde{V}_{\delta,q}) d\mu_g$$

$$- \int_{\partial M} h_g (\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q})(\Gamma - \tilde{W}_{\delta,q} - \delta^2 \tilde{V}_{\delta,q}) d\sigma_g$$

$$+ \int_{\partial M} \left[f(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) - \frac{\partial}{\partial \nu}(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) \right] (\Gamma - \tilde{W}_{\delta,q} - \delta^2 \tilde{V}_{\delta,q}) d\sigma_g$$

$$=: I_1 + I_2 + I_3 + I_4.$$

We have that

$$I_3 = \int_{\partial M} h_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q})(\Lambda_q^{-1} R) d\sigma_{\tilde{g}}$$
$$\leq C |h_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q})|_{L_{\tilde{g}}^{\frac{2(n-1)}{n}}(\partial M)} \|\Lambda_q^{-1} R\|_{\tilde{g}},$$

and, by change of variables and by (1.17), we get

$$|h_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q})|_{L_{\tilde{g}}^{\frac{2(n-1)}{n}}(\partial M)} = \begin{cases} O(\delta^3 \log \delta) & \text{if } n = 8 \\ O(\delta^3) & \text{if } n > 8 \end{cases}.$$

Similarly for I_1 we have

$$I_1 \leq \left| \Delta_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q}) - \tilde{a}(W_{\delta,q} + \delta^2 V_{\delta,q}) \right|_{L_{\tilde{g}}^{\frac{2n}{n+2}}(M)} \|\Lambda_q^{-1} R\|_{\tilde{g}}$$

Since $R_{\tilde{g}}(0) = 0$ (see [20, page 1609]), we get

$$\left| \tilde{a}(W_{\delta,q} + \delta^2 V_{\delta,q}) \right|_{L_{\tilde{g}}^{\frac{2n}{n+2}}(M)} = \begin{cases} O(\delta^3 \log \delta) & \text{if } n = 8 \\ O(\delta^3) & \text{if } n > 8 \end{cases}$$

and, using the expansion of the metric $\tilde{g}$ and (1.10), one can show that

$$\left| \Delta_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q}) \right|_{L_{\tilde{g}}^{\frac{2n}{n+2}}(M)} = \begin{cases} O(\delta^3 \log \delta) & \text{if } n = 8 \\ O(\delta^3) & \text{if } n > 8 \end{cases};$$

thus we get

$$I_1 + I_3 = \begin{cases} O(\delta^3 \log \delta)\|R\|_g & \text{if } n = 8 \\ O(\delta^3)\|R\|_g & \text{if } n > 8 \end{cases}.$$

For the integral I_4 we have

$$I_4 \leq C(n-2) \left| \left((W_{\delta,q} + \delta^2 V_{\delta,q})^+ \right)^{\frac{n}{n-2}} - (W_{\delta,q})^{\frac{n}{n-2}} - \delta^2 \frac{\partial}{\partial \nu} V_{\delta,q} \right|_{L_{\tilde{g}}^{\frac{2(n-1)}{n}}(\partial M)} \|R\|_g$$

$$+ C \left| (n-2)(W_{\delta,q})^{\frac{n}{n-2}} - \frac{\partial}{\partial \nu} W_{\delta,q} \right|_{L_{\tilde{g}}^{\frac{2(n-1)}{n}}(\partial M)} \|R\|_g.$$

and, since U solves (1.6), we get immediately

$$(3.4) \qquad \left| (n-2)(W_{\delta,q})^{\frac{n}{n-2}} - \frac{\partial}{\partial \nu} W_{\delta,q} \right|_{L_{\tilde{g}}^{\frac{2(n-1)}{n}}(\partial M)} = O(\delta^3).$$

Estimating the other terms requires more care, but, expanding $\left((U + \delta^2 \gamma_q)^+ \right)^{\frac{n}{n-2}}$ near U, using (1.10) and the decay estimates (1.11), one can show that (see [13] for all the details)

$$(3.5) \quad \left| \left((W_{\delta,q} + \delta^2 V_{\delta,q})^+ \right)^{\frac{n}{n-2}} - (W_{\delta,q})^{\frac{n}{n-2}} - \delta^2 \frac{\partial}{\partial \nu} V_{\delta,q} \right|_{L_{\tilde{g}}^{\frac{2(n-1)}{n}}(\partial M)} = O(\delta^3).$$

Thus by (3.4) and (3.5) we have

$$I_4 = O(\delta^3).$$

For I_2 we have

$$I_2 \leq \varepsilon_1 \left| \tilde{a}(W_{\delta,q} + \delta^2 V_{\delta,q}) \right|_{L_{\tilde{g}}^{\frac{2n}{n+2}}(M)} \|\Lambda_q^{-1} R\|_{\tilde{g}}$$

Now by change of variables we have

$$\left| W_{\delta,q} + \delta^2 V_{\delta,q} \right|_{L_{\tilde{g}}^{\frac{2n}{n+2}}(M)} = O(\delta^2) \left| (1 + |x|)^{2-n} \right|_{L^{\frac{2n}{n+2}}(B_{1/\delta})} = O(\delta^2)$$

so

$$I_2 = O(\varepsilon_1 \delta^2) \|R\|_g$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

The following lemma is a standard tool in finite dimensional reduction, so we refer to [18, 22] for its proof.

Lemma 22. *Given* $(\varepsilon_1, \varepsilon_2)$, *for any pair* (δ, q) *there exists a positive constant* $C = C(\delta, q)$ *such that for any* $\varphi \in \tilde{K}_{\delta,q}^{\perp}$ *it holds*

$$\|L(\varphi)\|_g \geq C\|\varphi\|_g.$$

It is also standard to prove that N is a contraction, that is there exists $\eta < 1$ such that, for any $\varphi_1, \varphi_2 \in \tilde{K}_{\delta,q}^{\perp}$ it holds

$$(3.6) \qquad \|N(\varphi)\|_g \leq \eta\|\varphi\|_g \text{ and } \|N(\varphi_1) - N(\varphi_2)\|_g \leq \eta\|\varphi_1 - \varphi_2\|_g$$

By Lemma 21, Lemma 22, and (3.6) we prove the last result of this subsection.

Proposition 23. *Given* $(\varepsilon_1, \varepsilon_2)$, *for any pair* (δ, q) *there exists a unique* $\tilde{\phi} = \tilde{\phi}_{\delta,q} \in \tilde{K}_{\delta,q}^{\perp}$ *which solves* (1.27) *such that*

$$\|\tilde{\phi}\|_g = \begin{cases} O\left(\delta^3 \log \delta + \varepsilon_1 \delta^2 + \varepsilon_2 \delta\right) & \text{if } n = 8 \\ O\left(\delta^3 + \varepsilon_1 \delta^2 + \varepsilon_2 \delta\right) & \text{if } n > 8 \end{cases}.$$

The map $q \mapsto \phi$ *is* C^1.

Proof. By Lemma 22, by (3.6) and by the properties of i_α, there exists $C > 0$ such that

$$\left\|L^{-1}\left(N(\tilde{\phi}) + R - \Pi^{\perp}\left\{i_\alpha^*\left(\varepsilon_2\beta(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi})\right)\right\}\right)\right\|_g$$
$$\leq C\left((\eta\|\tilde{\phi}\|_g + \|R\|_g + \left\|i_\alpha^*\left(\varepsilon_2\beta(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi})\right)\right\|_g\right).$$

Now it is easy to estimate that

$$\left\|i_\alpha^*\left(\varepsilon_2\beta(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi})\right)\right\|_g \leq \varepsilon_2\left(\left\|\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q}\right\|_{L_g^{\frac{2(n-1)}{n}}(\partial M)} + \left\|\tilde{\phi}\right\|_g\right)$$
$$(3.7) \qquad\qquad\qquad\qquad \leq C\left(\varepsilon_2\delta + \varepsilon_2\left\|\tilde{\phi}\right\|_g\right).$$

If $n > 8$, by Lemma 21 and by the previous estimates, for the map

$$T(\tilde{\phi}) := L^{-1}\left(N(\tilde{\phi}) + R - \Pi^{\perp}\left\{i_\alpha^*\left(\varepsilon_2\beta(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi})\right)\right\}\right)$$

it holds

$$\|T(\tilde{\phi})\|_g \leq C\left((\eta + \varepsilon_2)\|\tilde{\phi}\|_g + \varepsilon_2\delta + \varepsilon_1\delta^2 + \delta^3\right).$$

It is possible to choose $\rho > 0$ such that T is a contraction from the ball $\|\tilde{\phi}\|_g \leq \rho(\varepsilon_2\delta + \varepsilon_1\delta^2 + \delta^3)$ in itself, so, by the fixed point Theorem, there exists a unique $\tilde{\phi}$ with $\|\tilde{\phi}\|_g = O(\varepsilon_2\delta + \varepsilon_1\delta^2 + \delta^3)$ which solves (1.27). In addition by the implicit function Theorem it is possible to prove the regularity of the map $q \mapsto \tilde{\phi}$. The case $n = 8$ follows verbatim. $\square$

3.2. **The reduced functional.** Once we solved (1.27), we show that we can find a critical point of $J_g\left(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi}\right)$ by solving a finite dimensional problem depending only on (δ, q).

Lemma 24. *Assume* $n \geq 8$. *It holds*

$$\left|J_g\left(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q} + \tilde{\phi}\right) - J_g\left(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q}\right)\right|$$
$$\leq C\left(\left\|\tilde{\phi}\right\|_g^2 + \delta^2\left\|\tilde{\phi}\right\|_g + \varepsilon_2\delta\left\|\tilde{\phi}\right\|_g\right)$$

where the constant C *does not depend on* $q \in \partial M$.

Proof. We have, for some $\theta \in (0, 1)$

$$\tilde{J}_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q} + \phi) - \tilde{J}_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q}) = \tilde{J}'_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q})[\phi]$$
$$+ \frac{1}{2}\tilde{J}''_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q} + \theta\phi)[\phi, \phi]$$
$$= \int_M \left(\nabla_{\tilde{g}} W_{\delta,q} + \delta^2 \nabla_{\tilde{g}} V_{\delta,q}\right) \nabla_{\tilde{g}}\phi + \left(\frac{n-2}{4(n-1)}R_{\tilde{g}} + \varepsilon_1\tilde{\alpha}\right)\left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)\phi d\mu_{\tilde{g}}$$
$$- (n-2)\int_{\partial M}\left(\left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}}\phi d\sigma_{\tilde{g}} + \frac{n-2}{2}\int_{\partial M} h_{\tilde{g}_q}\left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)\phi d\sigma_{\tilde{g}}$$
$$+ \int_{\partial M}\varepsilon_2\tilde{\beta}\left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)\phi d\sigma_{\tilde{g}} + \frac{1}{2}\|\phi\|^2_{\tilde{g}}$$
$$- \frac{n}{2}\int_{\partial M}\left(\left(W_{\delta,q} + \delta^2 V_{\delta,q} + \theta\phi_{\delta,q}\right)^+\right)^{\frac{2}{n-2}}\phi^2_{\delta,q} d\sigma_{\tilde{g}_q} + \frac{1}{2}\int_{\partial M}\varepsilon_2\tilde{\beta}|\phi|^2 d\sigma_{\tilde{g}}.$$

Now, by Holder inequality we have

$$\left|\int_M W_{\delta,q}\phi d\mu_{\tilde{g}}\right| \leq C|W_{\delta,q}|_{L^{\frac{2n}{n+2}}_{\tilde{g}}}|\phi|_{L^{\frac{2n}{n-2}}_{\tilde{g}}} \leq C\delta^2\|\phi\|_{\tilde{g}}$$

and

$$\delta^2\left|\int_M V_{\delta,q}\phi d\mu_{\tilde{g}}\right| \leq C\delta^2|V_{\delta,q}|_{L^2_{\tilde{g}}}|\phi|_{L^2_{\tilde{g}}} \leq C\delta^2\|\phi\|_{\tilde{g}}.$$

Immediately we have $\left|\int_{\partial M}\varepsilon_2\tilde{\beta}|\phi|^2 d\sigma_{\tilde{g}}\right| \leq C\varepsilon_2\|\phi\|^2_{\tilde{g}}$, and following the proof of [13, Lemma 8] we obtain that

$$\left|\int_M\left(\nabla_{\tilde{g}} W_{\delta,q} + \delta^2\nabla_{\tilde{g}} V_{\delta,q}\right)\nabla_{\tilde{g}}\phi - (n-2)\int_{\partial M}\left(\left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}}\phi d\sigma_{\tilde{g}}\right|$$
$$\leq C\delta^2\|\phi\|_{\tilde{g}}$$

and

$$\left|\int_{\partial M}\left(\left(W_{\delta,q} + \delta^2 V_{\delta,q} + \theta\phi\right)^+\right)^{\frac{2}{n-2}}\phi^2_{\delta,q} d\sigma\right| \leq C\|\phi\|^2_{\tilde{g}}.$$

Finally by (1.17) we have

$$\left|\int_{\partial M} h_{\tilde{g}_q}\left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)\phi d\sigma_{\tilde{g}}\right| \leq C\delta^3\|\phi\|_{\tilde{g}}$$

and, proceeding similarly to (3.7),

$$\left|\int_{\partial M}\varepsilon_2\tilde{\beta}\left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)\phi d\sigma_{\tilde{g}}\right| \leq C\varepsilon_2\delta\|\phi\|_{\tilde{g}}.$$

This proves the first estimate. Using the result of Proposition 23 we complete the proof. $\square$

Lemma 25. *Let $n \geq 8$. It holds*

$$J_g(\tilde{W}_{\delta,q} + \delta^2\tilde{V}_{\delta,q}) = A + \varepsilon_1\delta^2\alpha(q)B + \varepsilon_2\delta\beta(q)C + \delta^4\varphi(q) + O(\varepsilon_1\delta^4) + O(\varepsilon_2\delta^3) + O(\delta^5)$$

where

$$A = \frac{1}{2} \int_{\mathbb{R}^n_+} |\nabla U(y)|^2 dy - \frac{(n-2)^2}{2(n-1)} \int_{\mathbb{R}^{n-1}} U(\bar{y}, 0)^{\frac{2(n-1)}{(n-2)}} d\bar{y}$$

$$B = \frac{1}{2} \alpha(q) \int_{\mathbb{R}^n} U(y)^2 dy$$

$$C = \frac{1}{2} \beta(q) \int_{\mathbb{R}^{n-1}} U(\bar{y}, 0)^2 d\bar{y}$$

$$\varphi(q) = \frac{1}{2} \int_{\mathbb{R}^n_+} v_q \Delta v_q dy - \frac{n-2}{96(n-1)} |\bar{W}(q)|^2 \int_{\mathbb{R}^n_+} |\bar{y}|^2 U^2(\bar{y}, y_n) dy$$

$$- \frac{(n-2)(n-8)}{2(n^2-1)} R^2_{ninj}(q) \int_{\mathbb{R}^n_+} \frac{y_n^2 |\bar{y}|^4}{((1+y_n)^2 + |\bar{y}|^2)^n} dy.$$

Here $\bar{W}(q)$ is the Weyl tensor restricted to boundary.

Proof. The main estimates of this proof are proved in [11, Lemma 8], which we refer to for a detailed proof, here we limit ourselves to estimate the perturbation terms. We have

$$J_g(\tilde{W}_{\delta,q} + \delta^2 \tilde{V}_{\delta,q}) = \frac{1}{2} \int_M |\nabla_{\tilde{g}_q}(W_{\delta,q} + \delta^2 V_{\delta,q})|^2 d\mu_{\tilde{g}} + \frac{n-2}{8(n-1)} \int_M R_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q})^2 d\mu_{\tilde{g}}$$

$$+ \frac{1}{2} \varepsilon_1 \int_M \Lambda_q^{-\frac{4}{n-2}} \alpha(W_{\delta,q} + \delta^2 V_{\delta,q})^2 d\mu_{\tilde{g}}$$

$$+ \frac{1}{2} \varepsilon_2 \int_{\partial M} \Lambda_q^{-\frac{2}{n-2}} \beta(W_{\delta,q} + \delta^2 V_{\delta,q})^2 d\sigma_{\tilde{g}}$$

$$- \frac{(n-2)^2}{2(n-1)} \int_{\partial M} \left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_q}$$

$$+ \frac{n-2}{4} \int_{\partial M} h_{\tilde{g}_q}(W_{\delta,q} + \delta^2 V_{\delta,q})^2 d\sigma_{\tilde{g}_q}$$

We easily compute the terms involving ε_1 and ε_2, taking in account that $\Lambda_q(q) = 1$ and the expansion of the volume form given by (1.16), getting

$$\frac{1}{2} \varepsilon_1 \int_M \Lambda_q^{-\frac{4}{n-2}} \alpha(W_{\delta,q} + \delta^2 V_{\delta,q})^2 d\mu_{\tilde{g}} = \frac{1}{2} \varepsilon_1 \delta^2 \alpha(q) \int_{\mathbb{R}^n} U(y)^2 dy + O(\varepsilon_1 \delta^4),$$

and

$$\frac{1}{2} \varepsilon_2 \int_{\partial M} \Lambda_q^{-\frac{2}{n-2}} \beta(W_{\delta,q} + \delta^2 V_{\delta,q})^2 d\mu_{\tilde{g}} = \frac{1}{2} \varepsilon_2 \delta \beta(q) \int_{\mathbb{R}^{n-1}} U(\bar{y}, 0)^2 d\bar{y} + O(\varepsilon_2 \delta^3).$$

The remaining terms are estimated in [11, Lemma 8], and it holds

$$\frac{1}{2} \int_M |\nabla_{\tilde{g}_q}(W_{\delta,q} + \delta^2 V_{\delta,q})|^2 d\mu_{\tilde{g}} + \frac{n-2}{8(n-1)} \int_M R_{\tilde{g}}(W_{\delta,q} + \delta^2 V_{\delta,q})^2 d\mu_{\tilde{g}}$$

$$- \frac{(n-2)^2}{2(n-1)} \int_{\partial M} \left(W_{\delta,q} + \delta^2 V_{\delta,q}\right)^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_q} + \frac{n-2}{4} \int_{\partial M} h_{\tilde{g}_q}(W_{\delta,q} + \delta^2 V_{\delta,q})^2 d\sigma_{\tilde{g}_q}$$

$$= A + \delta^4 \varphi(q) + O(\delta^5)$$

which completes the proof. $\qquad \square$

3.3. Proof of Theorem 2. At first we provide a sign estimate for function $\varphi(q)$ defined in the previous paragraph.

Lemma 26. *Assume $n \geq 8$ and that the Weyl tensor W_g is not vanishing on ∂M. Then the function $\varphi(q)$ defined in Lemma 25 is strictly negative on ∂M.*

Proof. We can write the function $\varphi(q)$ defined in Lemma 25 as

$$\varphi(q) = \frac{1}{2} \int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q dt dz - C_1 |\bar{W}(q)|^2 - (n-8)C_2 R^2_{ninj}(q),$$

where C_1, C_2 are positive constants. If $n > 8$, since in umbilic boundary manifolds $W(q) = 0$ if and only if $\bar{W}(q)$ and $R_{ninj}(q)$ are both zero (see [20, page 1618]), by our assumption at least one among $|\bar{W}(q)|$ and $R^2_{nlnj}(q)$ is strictly positive. Since by (1.12) the term involving γ_q is non positive, the lemma is proved.

When $n = 8$ the term involving R^2_{8i8j} vanishes. However in [12] a refined analysis of the term $\int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q dt dz$ was performed, leading to the following improvement of estimate (1.12):

$$\int_{\mathbb{R}^8_+} \gamma_q \Delta \gamma_q dy \leq -C_3 R^2_{8i8j}(q),$$

where $C_3 > 0$. This was possible by a more precise description of function γ_q as sum of an harmonic function with explicit rational functions, proved in Lemma 19 of the cited paper.

Thus for $n = 8$ we have

$$\varphi(q) \leq -C_1 |\bar{W}(q)|^2 - C_3 R^2_{8i8j}(q) < 0,$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 2. We give a detailed proof in the case $\alpha > 0$. The case $\beta > 0$ is analogous and we will emphasize the difference at the end of the proof.

If $\alpha > 0$ we choose

$$\delta = \sqrt{\lambda \varepsilon_1}$$
$$\varepsilon_2 = o(\varepsilon_1^2)$$

where $\lambda \in \mathbb{R}^+$. With this choice, by Lemma 24, we have that

$$\left| J_g \left(\tilde{W}_{\sqrt{\lambda \varepsilon_1}, q} + \lambda \varepsilon_1 \tilde{V}_{\sqrt{\lambda \varepsilon_1}, q} + \tilde{\phi} \right) - J_g \left(\tilde{W}_{\sqrt{\lambda \varepsilon_1}, q} + \lambda \varepsilon_1 \tilde{V}_{\sqrt{\lambda \varepsilon_1}, q} \right) \right| = o(\varepsilon_1^2)$$

and that, by Lemma 25,

$$J_g \left(\tilde{W}_{\sqrt{\lambda \varepsilon_1}, q} + \lambda \varepsilon_1 \tilde{V}_{\sqrt{\lambda \varepsilon_1}, q} \right) = A + \varepsilon_1^2 \left(\lambda \alpha(q) B + \lambda^2 \varphi(q) \right) + o(\varepsilon_1^2).$$

We recall a result which is a key tool in Ljapunov-Schmidt procedure, and which is proved, for instance, in [11, Lemma 9] and which relies on the estimates of Lemma 24.

Remark. Given $(\varepsilon_1, \varepsilon_2)$, if $(\bar{\lambda}, \bar{q}) \in (0, +\infty) \times \partial M$ is a critical point for the reduced functional $I_{\varepsilon_1, \varepsilon_2}(\lambda, q) := J_g \left(\tilde{W}_{\sqrt{\lambda \varepsilon_1}, q} + \lambda \varepsilon_1 \tilde{V}_{\sqrt{\lambda \varepsilon_1}, q} + \tilde{\phi} \right)$, then the function $\tilde{W}_{\sqrt{\bar{\lambda} \varepsilon_1}, \bar{q}} + \bar{\lambda} \varepsilon_1 \tilde{V}_{\sqrt{\lambda \varepsilon_1}, \bar{q}} + \tilde{\phi}$ is a solution of (1.2).

To conclude the proof it lasts to find a pair $(\bar{\lambda}, \bar{q})$ which is a critical point for $I_{\varepsilon_1, \varepsilon_2}(\lambda, q)$.

Let us call $G(\lambda, q) := \lambda \alpha(q) B + \lambda^2 \varphi(q)$. We have that $\alpha(q)B$ is strictly positive on ∂M, by our assumptions, while by Lemma 26, φ is strictly negative on ∂M. At this point there exists a compact set $[a, b] \subset \mathbb{R}^+$ such that the function G admits an absolute maximum in $(a, b) \times \partial M$, which also is the absolute maximum value of G on $\mathbb{R}^+ \times \partial M$. This maximum is also C^0-stable, in the sense that, if (λ_0, q_0) is the maximum point for G, for any function $f \in C^1([a, b] \times \partial M)$ with $\|f\|_{C^0}$ sufficiently small, then the function $G + f$ on $[a, b] \times \partial M$ admits a maximum point $(\bar{\lambda}, \bar{q})$ close to (λ_0, q_0). By the C_0 stability of this maximum (λ_0, q_0), and by Lemma 25, given ε_1 sufficiently small (and $\varepsilon_2 = o(\varepsilon_1^2)$), there exists a pair $(\lambda_{\varepsilon_1}, q_{\varepsilon_1})$ which

is a maximum point for $J_g\left(\tilde{W}_{\sqrt{\lambda\varepsilon_1},q} + \lambda\varepsilon_1\tilde{V}_{\sqrt{\lambda\varepsilon_1},q}\right)$, and, in turn, that there exists a pair $\left(\bar{\lambda}_{\varepsilon_1}, \bar{q}_{\varepsilon_1}\right)$ which is a maximum point for $I_{\varepsilon_1,\varepsilon_2}(\lambda, q)$. This implies, in light of the above Remark, that $\tilde{W}_{\sqrt{\bar{\lambda}_{\varepsilon_1}\varepsilon_1},\bar{q}_{\varepsilon_1}} + \bar{\lambda}_{\varepsilon_1}\varepsilon_1\tilde{V}_{\sqrt{\bar{\lambda}_{\varepsilon_1}\varepsilon_1},\bar{q}_{\varepsilon_1}} + \tilde{\phi}$ is a solution of (1.2), and the proof for the case $\alpha > 0$ is complete.

For the case $\beta > 0$ we choose

$$\delta = \lambda\varepsilon_2^{\frac{1}{3}} \text{ and } \varepsilon_1 = o(\varepsilon_2^{\frac{2}{3}})$$

in order to have

$$J_g\left(\tilde{W}_{\lambda\varepsilon_2^{\frac{1}{3}},q} + \lambda^2\varepsilon_2^{\frac{2}{3}}\tilde{V}_{\delta,q}\right) = A + \varepsilon_2^{\frac{4}{3}}\left(\lambda\beta(q)C + \lambda^4\varphi(q)\right) + o(\varepsilon_2^{\frac{4}{3}}),$$

and the proof follows identically. $\qquad\square$

Remark 27. We give an example of sign changing perturbation $\alpha(q)$ such that problem (1.2) admits blowing up sequences of solutions. Since ∂M is compact, there exists a $q_0 \in \partial M$ maximum point for φ. We take a $\alpha \in C^2(\partial M)$ which has a positive local maximum in q_0, and that is negative somewhere. We choose

$$\delta = \sqrt{\lambda\varepsilon_1}$$
$$\varepsilon_2 = o(\varepsilon_1^2)$$

as in the previous proof. By construction, the pair $(\lambda_0, q_0) = \left(-\frac{B\alpha(q_0)}{2\varphi(q_0)}, q_0\right)$ is a C^0-stable critical point for $G(\lambda, q)$, in fact $\nabla_{\lambda,q}G(\lambda_0, q_0) = 0$ and the Hessian matrix is negative definite. Then we can repeat the arguments of Theorem 2. The construction of a sign changing β is completely analogous.

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