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EQUILIBRIUM STATES OF ENDOMORPHISMS OF $\mathbb{P}^k$: SPECTRAL STABILITY AND LIMIT THEOREMS

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Abstract. We establish the existence of a spectral gap for the transfer operator induced on $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$ by a generic holomorphic endomorphism and a suitable continuous weight and its perturbations on various functional spaces, which is new even in dimension one. Thanks to the spectral gap, we establish an exponential speed of convergence for the equidistribution of the backward orbits of points towards the conformal measure and the exponential mixing. Moreover, as an immediate consequence, we obtain a full list of statistical properties for the equilibrium states: CLT, Berry-Esseen Theorem, local CLT, ASIP, LIL, LDP, almost sure CLT. Many of these properties are new even in dimension one, some even in the case of zero weight function (i.e., for the measure of maximal entropy).

Contents

Notation. Throughout the paper, $\mathbb{P}^k$ denotes the complex projective space of dimension k endowed with the standard Fubini-Study form ω_{FS}. This is a Kähler $(1,1)$-form normalized so that ω_{FS}^k is a probability measure. We will use the metric and distance $\mathrm{dist}(\cdot,\cdot)$ on $\mathbb{P}^k$ induced by ω_{FS} and the standard ones on $\mathbb{C}^k$ when we work on open subsets of $\mathbb{C}^k$. We denote by $\mathbb{B}_{\mathbb{P}^k}(a,r)$ (resp. $\mathbb{B}^k_r$, $\mathbb{D}(a,r)$, $\mathbb{D}_r$) the open ball of center a and radius r in $\mathbb{P}^k$ (resp. the open ball of center 0 and radius r in $\mathbb{C}^k$, the open disc of center a and radius r in $\mathbb{C}$, and the open disc of center 0 and radius r in $\mathbb{C}$). Leb denotes the standard Lebesgue measure on a Euclidean space or on a sphere.

The pairing $\langle \cdot,\cdot \rangle$ is used for the integral of a function with respect to a measure or more generally the value of a current at a test form. If S and R are two $(1,1)$-currents,

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we will write $|R| \le S$ when $\Re(\xi R) \le S$ for every function $\xi\colon \mathbb{P}^k \to \mathbb{C}$ with $|\xi| \le 1$, i.e., all currents $S - \Re(\xi R)$ with ξ as before are positive. Notice that this forces S to be real and positive. We also write other inequalities such as $|R| \le |R_1| + |R_2|$ if $|R| \le S_1 + S_2$ whenever $|R_1| \le S_1$ and $|R_2| \le S_2$. Recall that $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$ and $dd^c = \frac{i}{\pi}\partial\overline{\partial}$. The notations $\lesssim$ and $\gtrsim$ stand for inequalities up to a multiplicative constant. The function identically equal to 1 is denoted by $\mathbb{1}$. We often use the Cauchy-Schwarz's inequality $|i\alpha \wedge \beta| \lesssim i\alpha \wedge \bar{\alpha} + i\beta \wedge \bar{\beta}$, which is valid for every $(1,0)$-form α and every $(0,1)$-form β. We also use the function $\log^\star(\cdot) := 1 + |\log(\cdot)|$.

Consider a holomorphic endomorphism $f\colon \mathbb{P}^k \to \mathbb{P}^k$ of algebraic degree $d \ge 2$ satisfying the Assumption (**A**) in the Introduction. Denote respectively by T, $\mu = T^k$, $\mathrm{supp}(\mu)$ the Green $(1,1)$-current, the measure of maximal entropy (also called the Green measure or the equilibrium measure), and the small Julia set of f. If S is a positive closed $(1,1)$-current on $\mathbb{P}^k$, its dynamical potential is denoted by u_S and is defined in Sect. 2.3.

We also consider a weight ϕ which is a continuous function on $\mathbb{P}^k$. We often assume that ϕ is real. The transfer operator (Perron-Frobenius operator) $\mathcal{L} = \mathcal{L}_\phi$ is introduced in the Introduction together with the scaling ratio $\lambda = \lambda_\phi$, the conformal measure m_ϕ, the density function $\rho = \rho_\phi$, the equilibrium state $\mu_\phi = \rho m_\phi$, the pressure $P(\phi)$. The measures m_ϕ and μ_ϕ are probability measures. The operator L is a suitable modification of $\mathcal{L}$ and is introduced and used in Sect. 5.

The oscillation $\Omega(\cdot)$, the modulus of continuity $m(\cdot,\cdot)$, the semi-norms $\|\cdot\|_{\log^p}$ and $\|\cdot\|_*$ of a function are defined in Sect. 2.1. Other norms and semi-norms $\|\cdot\|_p$, $\|\cdot\|_{p,\alpha}$, $\|\cdot\|_{\langle p,\alpha\rangle}$, $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ for $(1,1)$-currents and functions are introduced in Sect. 3 and the norms $\|\cdot\|_{\diamond_1}$, $\|\cdot\|_{\diamond_2}$ in Sect. 4.4. The semi-norms we consider are almost norms: they vanish only on constant functions. It is easy to make them norms by adding a suitable functional such as $g \mapsto |\langle m_\phi, g\rangle|$. However, for simplicity, it is more convenient to work directly with these semi-norms. The versions of these semi-norms for currents are actually norms. The positive numbers q_0, q_1, q_2 are defined in Lemmas 3.7, 3.10, 3.14 and the families of weights $\mathcal{P}(q, M, \Omega)$ and $\mathcal{Q}_0$ are introduced in Sects. 4.1 and 4.3.

1 Introduction and results

Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of the complex projective space $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$, with $k \ge 1$. It is induced by a map from $\mathbb{C}^{k+1}$ to $\mathbb{C}^{k+1}$ whose components are homogeneous polynomials of the same degree d without non-trivial common zeros. We call d the *algebraic degree* of f and assume that $d \ge 2$, see, e.g., [DS101] for details. Denote by μ the unique measure of maximal entropy for the dynamical system $(\mathbb{P}^k, f)$ [Mik83, BD09, DS101, BM01]. The support $\mathrm{supp}(\mu)$ of μ is called *the small Julia* set of f. Given a *weight*, i.e., a real-valued continuous function ϕ, as

$$P(\phi) := \sup\Big\{\mathrm{Ent}_f(\nu) + \langle \nu, \phi\rangle\Big\},$$

where the supremum is taken over all Borel f-invariant probability measures ν and $\mathrm{Ent}_f(\nu)$ denotes the metric entropy of ν. An *equilibrium state* for ϕ is then an invariant probability measure μ_ϕ realizing a maximum in the above formula, that is,

$$P(\phi) = \mathrm{Ent}_f(\mu_\phi) + \langle \mu_\phi, \phi \rangle.$$

Define also the *Perron-Frobenius* (or *transfer*) operator $\mathcal{L}$ with weight ϕ as (we often drop the index ϕ for simplicity)

$$\mathcal{L}g(y) := \mathcal{L}_\phi g(y) := \sum_{x \in f^{-1}(y)} e^{\phi(x)} g(x), \tag{1.1}$$

where $g\colon \mathbb{P}^k \to \mathbb{R}$ is a continuous test function and the points x in the sum are counted with multiplicity. A *conformal measure* is an eigenvector for the dual operator $\mathcal{L}^*$ acting on positive measures.

The following result was obtained in [BD23]. We refer to that paper for references to earlier related results.

Theorem 1.1. *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and satisfying the Assumption* (**A**) *below. Let ϕ be a real-valued $\log^q$-continuous function on $\mathbb{P}^k$, for some $q > 2$, such that $\Omega(\phi) := \max \phi - \min \phi < \log d$. Then ϕ admits a unique equilibrium state μ_ϕ, whose support is equal to the small Julia set of f. This measure μ_ϕ is K-mixing and mixing of all orders, and repelling periodic points of period n (suitably weighted) are equidistributed with respect to μ_ϕ as n goes to infinity. Moreover, there is a unique conformal measure m_ϕ associated to ϕ. We have $\mu_\phi = \rho m_\phi$ for some strictly positive continuous function ρ on $\mathbb{P}^k$ and the preimages of points by f^n (suitably weighted) are equidistributed with respect to m_ϕ as n goes to infinity.*

Recall that a function is $\log^q$-continuous if its oscillation on a ball of radius r is bounded by a constant times $(\log^\star r)^{-q}$, see Sect. 2.1 for details. We made use of the following technical assumption for f:

(**A**) the local degree of the iterate $f^n := f \circ \cdots \circ f$ (n times) satisfies

$$\lim_{n \to \infty} \frac{1}{n} \log \max_{a \in \mathbb{P}^k} \deg(f^n, a) = 0.$$

Here, $\deg(f^n, a)$ is the multiplicity of a as a solution of the equation $f^n(z) = f^n(a)$. When k and d are fixed, the endomorphism f is parametrized by a finite number of complex coefficients. By [DS102], condition (**A**) is satisfied by generic maps, i.e., for parameters outside a countable family of proper algebraic subsets of the parameter space. We assume (**A**) also throughout all the current paper. As in [BD23], we will see that the quantity $\log d$ in Theorem 1.1 naturally appears as the gap between the topological degree of f and the other dynamical degrees. It can be seen as the first constraint for perturbing the system without changing its expanding behaviour on the small Julia set.

A reformulation of Theorem 1.1 is the following: given ϕ as in the statement, there exist a *scaling ratio* $\lambda > 0$ and a continuous function $\rho = \rho_\phi\colon \mathbb{P}^k \to \mathbb{R}$ such that, for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, the following uniform convergence holds:

$$\lambda^{-n}\mathcal{L}^n g \to c_g \rho \tag{1.2}$$

for some constant c_g depending on g. By duality, this is equivalent to the convergence, uniform on probability measures ν,

$$\lambda^{-n}(\mathcal{L}^*)^n \nu \to m_\phi, \tag{1.3}$$

where m_ϕ is a conformal measure associated to the weight ϕ. The equilibrium state μ_ϕ is then given by $\mu_\phi = \rho m_\phi$, and we have $c_g = \langle m_\phi, g\rangle$.

We aim here at establishing an exponential speed of convergence in (1.2), when g satisfies necessary regularity properties. This requires to build a suitable (semi-)norm for (or equivalently, a suitable functional space on) which the operator $\lambda^{-1}\mathcal{L}$ turns out to be a contraction. Observe that condition (**A**) is necessary for the uniform convergence. As an example, if f admits a point a outside its small Julia set such that $f^{-1}(\{a\}) = \{a\}$, then (1.3) fails for the Dirac mass at a, thus (1.2) fails as well.

Establishing the following statement (Theorem 1.2) is then our main goal in the current paper. As far as we know, this is the first time that the existence of a spectral gap for the Perron-Frobenius operator with weight is proved in this context even in dimension 1, except for hyperbolic endomorphisms or for weights with ad-hoc conditions (see for instance [Rue92, MS00]). Observe that, while in Theorem 1.1 ϕ is required to just be $\log^q$-continuous, here it may a priori have to be (slightly) more regular. An important and specific feature of our norms, which will be highlighted below, is their dependence on the map f.

Theorem 1.2. *Let f, q, ϕ, ρ, m_ϕ be as in Theorem 1.1 and $\mathcal{L}$, λ the above Perron-Frobenius operator and scaling factor associated to ϕ. Let $A > 0$ and $0 < \Omega < \log d$ be two constants. Then, for every constant $0 < \gamma \le 1$, there exist two explicit equivalent norms for functions on $\mathbb{P}^k$: $\|\cdot\|_{\diamond_1}$, depending on f, γ, q and independent of ϕ, and $\|\cdot\|_{\diamond_2}$, depending on f, ϕ, γ, q, such that*

$$\|\cdot\|_\infty + \|\cdot\|_{\log^q} \lesssim \|\cdot\|_{\diamond_1} \simeq \|\cdot\|_{\diamond_2} \lesssim \|\cdot\|_{\mathcal{C}^\gamma}.$$

Moreover, there exists a positive constant $\beta = \beta(f, \gamma, q, A, \Omega) < 1$, independent of ϕ and n, such that when $\|\phi\|_{\diamond_1} \le A$ and $\Omega(\phi) \le \Omega$ we have

$$\left\|\lambda^{-1}\mathcal{L} g\right\|_{\diamond_2} \le \beta \, \|g\|_{\diamond_2}$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$ with $\langle m_\phi, g\rangle = 0$. Furthermore, given any constant $1 < \delta < d^{\gamma/(2\gamma+2)}$, when A is small enough the norm $\|\cdot\|_{\diamond_2}$ can be chosen so that we can take $\beta = 1/\delta$.

According to this theorem, on the space of functions with bounded $\|\cdot\|_{\diamond_2}$ norm, the operator $\lambda^{-1}\mathcal{L}$ admits a spectral gap. It acts as the identity of the line spanned by ρ while its norm on the invariant hyperplane $\{g: \langle m_\phi, g\rangle = 0\}$ is bounded by $\beta < 1$.

The construction of the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ is quite involved. We use here ideas from the theory of interpolation between Banach spaces [Tri95] combined with techniques from pluripotential theory and complex dynamics. Roughly speaking, an idea from interpolation theory allows us to reduce the problem to the case where $\gamma = 1$. The definition of the above norms in this case requires a control of the derivatives of g (in the distributional sense), and this is where we use techniques from pluripotential theory. This also explains why these norms are bounded by the $\mathcal{C}^1$ norm. Note that we should be able to bound the derivatives of $\mathcal{L}g$ in a similar way. A quick expansion of the derivatives of $\mathcal{L}g$ using (1.1) gives an idea of the difficulties that one faces.

Let us highlight two among these difficulties. First, the objects from complex analysis and geometry are too rigid for perturbations with a non-constant weight: the operators f_*, d, and dd^c do not commute with the operator $\mathcal{L}$. In particular, the dd^c-method developed by the second author and Sibony (see for instance [DS101]) cannot be applied in this context, even for small perturbations of the weight $\phi = 0$. Moreover, there may be critical points on the support of the measure, which cause a loss in the regularity of functions under the operators f_* and $\mathcal{L}$ (see Sect. 3). Notice that we do not assume that our potential degenerates at the critical points.

Our solution to these problems is to define a new invariant functional space and norm in this mixed real-complex setting, that we call the *dynamical Sobolev space* and *semi-norm*, taking into account both the regularity of the function (to cope with the rigidity of the complex objects) and the action of f (to take into account the critical dynamics), see Definitions 3.9 and 3.12. The construction of this norm requires the definition of several intermediate semi-norms and the precise study of the action of the operator f_* with respect to them, and is carried out in Sect. 3. Some of the intermediate estimates already give new or more precise convergence properties for the operator f_* and the equilibrium measure μ, see for instance Theorem 3.2.

A spectral gap for the Perron-Frobenius operator and its perturbations is one of the most desirable properties in dynamics. It allows us to obtain several statistical properties of the equilibrium state. In the present setting, we have the following result. The Berry-Esseen Theorem, ASIP, local CLT, and LDP will be defined in Sect. 5, see also the end of Sect. 5.6 for references for the LIL and the almost sure CLT.

Theorem 1.3. *Let f, ϕ, μ_ϕ, m_ϕ, $\|\cdot\|_{\diamond_1}$ be as in Theorems 1.1 and 1.2, λ the scaling ratio associated to ϕ, and assume that $\|\phi\|_{\diamond_1} < \infty$. Then the equilibrium state μ_ϕ is exponentially mixing for observables with bounded $\|\cdot\|_{\diamond_1}$ norm and the preimages of points by f^n (suitably weighted) equidistribute exponentially fast towards m_ϕ as n goes to infinity. The measure μ_ϕ satisfies the LDP for all observables with finite $\|\cdot\|_{\diamond_1}$ norm, the ASIP, CLT, Berry-Esseen Theorem, almost sure CLT, LIL for all observables with finite $\|\cdot\|_{\diamond_1}$ norm which are not coboundaries, and the local CLT for all observables with finite $\|\cdot\|_{\diamond_1}$ norm which are not $(\|\cdot\|_{\diamond_1}, \phi)$-multiplicative cocycles.*

Moreover, the pressure $P(\phi) = \log \lambda$ *is analytic in the following sense: for* $\|\psi\|_{\diamond_1} < \infty$ *and* t *sufficiently small, the function* $t \mapsto P(\phi + t\psi)$ *is analytic.*

In particular, all the properties in Theorem 1.3 hold when the weight ϕ and the observable are Hölder continuous and satisfy the necessary coboundary/cocycles requirements. Under such assumptions, some of the above properties were previously obtained in [PUH89, DU911, DU912, DPU96, Hay99, DNS07, SUZ15] when $k = 1$, see also [UZ13, SUZ14] for mixing, CLT, LIL when $k \geq 1$, and [Dup10] for the ASIP when $k \geq 1$ and $\phi = 0$. Note that the LDP, the Berry-Esseen Theorem, and the local CLT are new even for $\phi = 0$ (the first for all $k \geq 1$, the second and the third for all $k > 1$). The proof of Theorem 1.3 exploits the spectral gap established in Theorem 1.2 and is based on the theory of perturbed operators [Nag57, PP90, Rou83, Bro96, Gou15]. Observe in particular that the fine control given for instance by the local CLT is simply impossible to prove using weaker arguments, such as martingales, see, e.g., [Gou15, p. 163].

Outline of the organization of the paper. In Sect. 2, we introduce some notations and establish comparison principles for currents and potentials that will be the technical key in the construction of our norms. In Sect. 3, we introduce the main (semi-)norms that we will need, and study the action of the operator f_* with respect to these (semi-)norms. Section 4 is dedicated to the proof of Theorem 1.2. Finally, in Sect. 5, we develop the statistical study of the equilibrium states. This section contains the proof of Theorem 1.3 and more precise statements.

2 Preliminaries and comparison principles

2.1 Some definitions. We collect here some notions that we will use throughout the paper.

DEFINITION 2.1. Given a subset U of $\mathbb{P}^k$ or $\mathbb{C}^k$ and a real-valued function $g\colon U \to \mathbb{R}$, define the *oscillation* $\Omega_U(g)$ of g as

$$\Omega_U(g) := \sup_U g - \inf_U g$$

and its continuity modulus $m_U(g, r)$ at distance r as

$$m_U(g, r) := \sup_{x,y \in U\colon \operatorname{dist}(x,y) \leq r} |g(x) - g(y)|.$$

We may drop the index U when there is no possible confusion.

The following semi-norms will be the main building blocks for all the semi-norms that we will construct and study in the sequel.

DEFINITION 2.2. The semi-norm $\|\cdot\|_{\log^p}$ is defined for every $p > 0$ and $g\colon \mathbb{P}^k \to \mathbb{R}$ as

$$\|g\|_{\log^p} := \sup_{a \neq b \in \mathbb{P}^k} |g(a) - g(b)| \cdot (\log^\star \operatorname{dist}(a,b))^p = \sup_{r>0, a \in \mathbb{P}^k} \Omega_{\mathbb{B}_{\mathbb{P}^k}(a,r)}(g) \cdot (1 + |\log r|)^p,$$

where $\mathbb{B}_{\mathbb{P}^k}(a,r)$ denotes the ball of center a and radius r in $\mathbb{P}^k$. The definition can be extended to functions on any metric space.

DEFINITION 2.3. The norm $\|\cdot\|_*$ of a $(1,1)$-current R is given by

$$\|R\|_* := \inf \|S\|$$

where the infimum is taken over all positive closed $(1,1)$-currents S such that $|R| \leq S$, see the Notation at the beginning of the paper. When such a current S does not exist, we put $\|R\|_* := +\infty$. The semi-norm $\|\cdot\|_*$ of an integrable function $g\colon \mathbb{P}^k \to \mathbb{R}$ is given by

$$\|g\|_* := \|dd^c g\|_* .$$

Note that when R is a real closed $(1,1)$-current the above norm is equivalent to the usual one defined as

$$\|R\|_* := \inf(\|S^+\| + \|S^-\|)$$

where the infimum is taken over all positive closed $(1,1)$-currents $S^\pm$ on $\mathbb{P}^k$ such that $R = S^+ - S^-$. In particular, for $R = dd^c g$, the currents S^+ and S^- are cohomologous and thus have the same mass, i.e., $\|S^+\| = \|S^-\|$, see [DS101, App. A.4] for details.

In this paper, we only consider continuous functions g. So the above semi-norms (and the others that we will introduce later) are almost norms: they only vanish when g is constant. In particular, they are norms on the space of functions g satisfying $\langle \nu, g \rangle = 0$ for some fixed probability measure ν. For convenience, we will use later $\nu = m_\phi$ or $\nu = \mu_\phi$ to obtain a spectral gap for the Perron-Frobenius operator and to study the statistical properties of μ_ϕ.

2.2 Approximations for Hölder continuous functions. We will need the following property for Hölder continuous functions, see Definition 3.12 and Remark 3.13.

LEMMA 2.4. *Let $0 < \gamma \leq 1$ be a constant. Then, for every $\mathcal{C}^\gamma$ function $g\colon \mathbb{P}^k \to \mathbb{R}$, $s \geq 1$, and $0 < \epsilon \leq 1$, there exist a $\mathcal{C}^s$ function $g^{(1)}_\epsilon$ and a continuous function $g^{(2)}_\epsilon$ such that*

$$g = g^{(1)}_\epsilon + g^{(2)}_\epsilon, \qquad \|g^{(1)}_\epsilon\|_{\mathcal{C}^s} \leq c \|g\|_\infty (1/\epsilon)^{s/\gamma} \qquad \textit{and} \qquad \|g^{(2)}_\epsilon\|_\infty \leq c \|g\|_{\mathcal{C}^\gamma} \epsilon,$$

where $c = c(\gamma, s)$ is a positive constant independent of g and ϵ.

Proof. Using a partition of unity, we can reduce the problem to the case where g is supported by the unit ball of an affine chart $\mathbb{C}^k \subset \mathbb{P}^k$. Consider a smooth non-negative function χ with support in the unit ball of $\mathbb{C}^k$ whose integral with respect to the Lebesgue measure is 1. For $\ell > 0$, consider the function $\chi_\ell(z) := \ell^{-2k}\chi(z/\ell)$ which has integral 1 and tends to the Dirac mass at 0 when ℓ tends to 0. We consider $\ell := \epsilon^{1/\gamma}$ and define an approximation of g using the standard convolution operator $g_\ell := g * \chi_\ell$, and define $g_\epsilon^{(1)} := g_\ell$ and $g_\epsilon^{(2)} := g - g_\ell$. It remains to bound $\|g_\epsilon^{(1)}\|_{\mathcal{C}^s}$ and $\|g_\epsilon^{(2)}\|_\infty$.

By standard properties of the convolution we have, for some constant $\kappa > 0$,

$$\|g_\epsilon^{(2)}\|_\infty \lesssim m(g, \kappa\ell) \lesssim \|g\|_{\mathcal{C}^\gamma}\,(\kappa\ell)^\gamma \lesssim \|g\|_{\mathcal{C}^\gamma}\,\epsilon,$$

and, by definition of g_ℓ,

$$\|g_\epsilon^{(1)}\|_{\mathcal{C}^s} \lesssim \|g\|_\infty\,\|\chi_\ell\|_{\mathcal{C}^s}\,\mathrm{Leb}(\mathbb{B}_\ell^k) \lesssim \|g\|_\infty\,\ell^{-s} \lesssim \|g\|_\infty\,\epsilon^{-s/\gamma}.$$

The lemma follows. □

2.3 Dynamical potentials. Let T denote the Green $(1,1)$-current of f. It is positive closed and of unit mass, and can be defined as

$$T := \lim_{n\to\infty} \frac{1}{d^n}(f^n)^*(\omega_{\mathrm{FS}}),$$

see for instance [DS101, Th. 1.16 and Def. 1.17]. Let S be any positive closed $(1,1)$-current of mass m on $\mathbb{P}^k$. As the cohomology $H^{1,1}(\mathbb{P}^k, R)$ has dimension 1, there is a unique function $u_S\colon \mathbb{P}^k \to \mathbb{R}\cup\{-\infty\}$ which is p.s.h. modulo mT (i.e., locally written as $v - mv_T$ with v, v_T p.s.h. and $dd^c v_T = T$) and such that

$$S = mT + dd^c u_S \qquad \text{and} \qquad \langle \mu, u_S\rangle = 0.$$

Locally, u_S is the difference between a potential of S and a potential of mT. We call it *the dynamical potential* of S. Observe that the dynamical potential of T is zero, i.e., $u_T = 0$.

Recall that T has Hölder continuous potentials. So, u_S is locally the difference between a p.s.h. function and a Hölder continuous one. We refer the reader to [DS101, BD23] for details. In this paper, we only need currents S such that u_S is continuous.

2.4 Complex Sobolev functions. In our study, we will be naturally lead to consider currents of the form $i\partial u\wedge\bar\partial u$, where u is a function. These currents are always positive when u is real valued. In this section, we study the regularity of u under the assumption that $i\partial u\wedge\bar\partial u \le dd^c v$ for some v of given regularity. Recall that, given a smoothly bounded open set $\Omega\subset\mathbb{C}^k$, the Sobolev space $W^{1,2}(\Omega)$ is defined as the space of functions $u\colon \Omega\to\mathbb{R}$ such that $\|u\|_{W^{1,2}(\Omega)} := \|u\|_{L^2(\Omega)} + \|\partial u\|_{L^2(\Omega)} < \infty$, where the reference measure is the standard Lebesgue measure on Ω. The Poincaré-Wirtinger's inequality (see for instance [ABM14, Cor. 5.4.1] implies that $\|u\|_{W^{1,2}(\Omega)} \lesssim \|\partial u\|_{L^2(\Omega)} + \left|\int_\Omega u\, d\,\mathrm{Leb}\right|$. We will need the following lemmas for functions on $\mathbb{C}$.

LEMMA 2.5. *There is a universal positive constant c such that*

$$\int_K |u| d\,\mathrm{Leb} \leq c\,\mathrm{Leb}(K)(\log^\star \mathrm{Leb}(K))^{1/2},$$

for every compact set $K \subset \mathbb{D}_1$ with $\mathrm{Leb}(K) > 0$ *and function $u\colon \mathbb{D}_1 \to \mathbb{R}$ such that* $\|u\|_{W^{1,2}(\mathbb{D}_1)} \leq 1$.

Proof. By Trudinger-Moser's inequality [Mos71], there are positive constants c_0 and α such that

$$\int_{\mathbb{D}_1} e^{2\alpha|u|^2} d\,\mathrm{Leb} \leq c_0.$$

Let m denote the restriction of the measure Leb to K multiplied by $1/\mathrm{Leb}(K)$. This is a probability measure. It follows from Cauchy-Schwarz's inequality that

$$\int e^{\alpha|u|^2} dm \leq \Big(\int e^{2\alpha|u|^2} dm\Big)^{1/2} \Big(\int \mathbb{1} dm\Big)^{1/2} \lesssim \mathrm{Leb}(K)^{-1/2}.$$

Observe that the function $t \mapsto e^{\alpha t^2}$ is convex on $\mathbb{R}^+$ and its inverse is the function $t \mapsto \alpha^{-1/2}(\log t)^{1/2}$. By Jensen's inequality, we obtain

$$\int |u| dm \leq \alpha^{-1/2}\Big[\log \int e^{\alpha|u|^2} dm\Big]^{1/2} \lesssim (\log^\star \mathrm{Leb}(K))^{1/2}.$$

The lemma follows. □

LEMMA 2.6. *Let $u\colon \mathbb{D}_2 \to \mathbb{R}$ be a continuous function and $\chi\colon \mathbb{D}_2 \to \mathbb{R}$ a smooth function with compact support in $\mathbb{D}_2$ and equal to 1 on $\overline{\mathbb{D}}_1$. Set $\chi_z := \partial\chi/\partial z$. Then we have, for all $0 < r < s < 1$,*

$$u(0) - u(r) = \frac{1}{2\pi}\Big\langle i\partial u, \chi(s^{-1}z)\frac{r}{\overline{z}(\overline{z} - r)} d\overline{z}\Big\rangle + \frac{1}{2\pi}\Big\langle u, \chi_z(s^{-1}z)\frac{r}{s\overline{z}(\overline{z} - r)} idz \wedge d\overline{z}\Big\rangle. \tag{2.1}$$

Proof. Denote by δ_ξ the Dirac mass at $\xi \in \mathbb{C}$. Observe that

$$\frac{i}{2\pi}\partial \frac{d\overline{z}}{\overline{z} - \overline{\xi}} = dd^c \log|z - \xi| = \delta_\xi,$$

where the equalities are in the sense of currents on $\mathbb{C}$. Hence, for $|\xi| < s$,

$$\frac{i}{2\pi}\partial\Big[\frac{\chi(s^{-1}z)d\overline{z}}{\overline{z} - \overline{\xi}}\Big] = \frac{\chi_z(s^{-1}z)idz \wedge d\overline{z}}{2\pi s(\overline{z} - \overline{\xi})} + \chi(s^{-1}z)\delta_\xi = \frac{\chi_z(s^{-1}z)idz \wedge d\overline{z}}{2\pi s(\overline{z} - \overline{\xi})} + \delta_\xi.$$

Applying this identity for $\xi = 0$ and $\xi = r$, and since $u(0) - u(r) = \langle u, \delta_0 - \delta_r \rangle$, we obtain

$$\begin{aligned} u(0) - u(r) &= \frac{1}{2\pi} \Big\langle u, i\partial \Big[\chi(s^{-1}z) \Big(\frac{1}{\overline{z}} - \frac{1}{\overline{z} - r} \Big) d\overline{z} \Big] \Big\rangle \\ &\quad - \frac{1}{2\pi s} \Big\langle u, \chi_z(s^{-1}z) \Big(\frac{1}{\overline{z}} - \frac{1}{\overline{z} - r} \Big) i dz \wedge d\overline{z} \Big\rangle \\ &= \frac{1}{2\pi} \Big\langle i\partial u, \chi(s^{-1}z) \frac{r}{\overline{z}(\overline{z} - r)} d\overline{z} \Big\rangle + \frac{1}{2\pi} \Big\langle u, \chi_z(s^{-1}z) \frac{r}{s\overline{z}(\overline{z} - r)} i dz \wedge d\overline{z} \Big\rangle. \end{aligned}$$

The assertion is proved. □

The following is a main result in this section. It will be a crucial technical tool in the construction of the norms with respect to which the transfer operator has a spectral gap.

PROPOSITION 2.7. *Let $u \colon \mathbb{B}^k_5 \to \mathbb{R}$ be continuous and such that $\|\partial u\|_{L^2(\mathbb{B}^k_5)} < \infty$. Assume that $i\partial u \wedge \overline{\partial} u \leq dd^c v$ where $v \colon \mathbb{B}^k_5 \to \mathbb{R}$ is continuous, p.s.h., and such that*

$$\int_0^1 m_{\mathbb{B}^k_4}(v, t) (\log \log^\star t)^4 t^{-1} dt < +\infty. \tag{2.2}$$

Then there is a positive constant c, independent of u and v, such that, for all $0 < r \leq 1/2$, we have

$$\begin{aligned} m_{\mathbb{B}^k_1}(u, r) &\leq c \Big(\int_0^{r^{1/2}} m_{\mathbb{B}^k_4}(v, t) (\log \log^\star t)^2 t^{-1} dt \Big)^{1/2} \\ &\quad + c m_{\mathbb{B}^k_4}(v, r)^{1/3} \Omega_{\mathbb{B}^k_4}(v)^{1/6} (\log^\star r)^{1/2} + c \Omega_{\mathbb{B}^k_4}(v)^{1/2} r^{1/2} (\log^\star r)^{1/2}. \end{aligned} \tag{2.3}$$

Proof. After some preliminary simplifications, the main idea of the proof will be to reduce the desired estimate to the evaluation of integrals that can treated thanks to the previous two lemmas. In particular, we will use Cauchy-Schwarz's inequality (see the Notation at the beginning of the paper) to bound integrals containing ∂u with integrals containing the term $i\partial u \wedge \overline{\partial} u$. By the assumption, these can be bounded by integrals containing the term $dd^c v$. As a last step, we will relate these integrals involving $dd^c v$ to the RHS of (2.3).

As a first simplification, observe that if $u_1, u_2, v_1, v_2 \colon \mathbb{B}^k_5 \to \mathbb{R}$ satisfy $i\partial u_1 \wedge \overline{\partial} u_1 \leq dd^c v_1$ and $i\partial u_2 \wedge \overline{\partial} u_2 \leq dd^c v_2$, Cauchy-Schwarz's inequality implies that

$$i\partial(u_1 + u_2) \wedge \overline{\partial}(u_1 + u_2) \lesssim i\partial u_1 \wedge \overline{\partial} u_1 + i\partial u_2 \wedge \overline{\partial} u_2 \leq dd^c(v_1 + v_2).$$

With the same idea one can prove that, by means of a standard regularization, we can assume that u and v are smooth. Let $x \in \mathbb{B}^k_1$ and $y \in \mathbb{B}^k_2$ be such that $\mathrm{dist}(x, y) \leq r \leq 1/2$. We need to bound $|u(x) - u(y)|$ by the RHS of (2.3). We can assume without loss of generality that $\mathrm{dist}(x, y) = r$. We use the change of coordinates $z \mapsto z - x$ in order to assume that $x = 0$. So, to obtain (2.3), it is enough to show for $\|y\| \leq r$ that

(we change the size of ball taking into account $\|x\|$)

$$|u(0)-u(y)| \leq c\Big(\int_0^{r^{1/2}} m_{\mathbb{B}^k_3}(v,t)(\log\log^\star t)^2 t^{-1} dt\Big)^{1/2} \\ + c m_{\mathbb{B}^k_3}(v,r)^{1/3}\Omega_{\mathbb{B}^k_3}(v)^{1/6}(\log^\star r)^{1/2} + c\Omega_{\mathbb{B}^k_3}(v)^{1/2} r^{1/2}(\log^\star r)^{1/2}.$$

By restricting to the complex line through 0 and y we can reduce the problem to the case of dimension 1. Up to a further rotation in $\mathbb{C}$, we can assume that $y=r$ in $\mathbb{C}$ and that u, v are defined on $\mathbb{D}_4$. By subtracting constants, we can assume that $v(0)=0$ and $\int_{\mathbb{D}_3} u(z) i dz\wedge d\bar z = 0$. Finally, by multiplying u and v by suitable constants γ and γ^2 and since the inequalities $i\partial u\wedge\bar\partial u \leq dd^c v$ and (2.3) are preserved by such scaling, we can assume that $m_{\mathbb{D}_3}(v,1)=1/8$, which implies that $|v|\leq 1$ on $\mathbb{D}_3$. Thus, in order to establish (2.3), it is enough to show that $|u(0)-u(r)|$ is bounded by a constant times

$$\Big(\int_0^{r^{1/2}} m_{\mathbb{D}_3}(v,t)(\log\log^\star t)^2 t^{-1} dt\Big)^{1/2} + m_{\mathbb{D}_3}(v,r)^{1/3}(\log^\star r)^{1/2} + r^{1/2}(\log^\star r)^{1/2}. \quad (2.4)$$

Since v is bounded, by Chern-Levine-Nirenberg's inequality [CLN69] the mass of $dd^c v$ on $\mathbb{D}_2$ is bounded by a constant. Thus, by the hypotheses on u and v, the L^2-norm of ∂u on $\mathbb{D}_2$ is bounded by a constant and therefore, by Poincaré-Wirtinger's inequality, $\|u\|_{W^{1,2}(\mathbb{D}_2)}$ is also bounded by a constant.

Fix a smooth function $0\leq \chi(z)\leq 1$ with compact support in $\mathbb{D}_2$ and such that $\chi=1$ on $\overline{\mathbb{D}}_1$. Define $\chi_z := \partial\chi/\partial z$ and set

$$s := r\min\Big\{r^{-1/2}, m_{\mathbb{D}_3}(v,r)^{-1/3}\Big\}.$$

We have

$$\sqrt{2} r \leq s \leq r^{1/2} < 1 \quad (2.5)$$

because $m_{\mathbb{D}_3}(v,r)\leq m_{\mathbb{D}_3}(v,1)=1/8$ and $0<r\leq 1/2$. The functions u and χ satisfy the assumptions of Lemma 2.6. Thus, (2.1) holds for the above s and r. The second term in the RHS of (2.1) is an integral over $\mathbb{D}_{2s}\setminus\mathbb{D}_s$ because χ_z has support in $\mathbb{D}_2\setminus\mathbb{D}_1$. Moreover, for $z\in\mathbb{D}_{2s}\setminus\mathbb{D}_s$, we have $\frac{r}{s\bar z(z-r)} = O(rs^{-3})$ because of (2.5). Using that $\|u\|_{W^{1,2}(\mathbb{D}_2)}\lesssim 1$ and that $\mathrm{Leb}(\mathbb{D}_{2s})\lesssim s^2$, Lemma 2.5 implies that the considered term has modulus bounded by a constant times

$$rs^{-1}(\log^\star s)^{1/2} \lesssim \max\Big\{r^{1/2}, m_{\mathbb{D}_3}(v,r)^{1/3}\Big\}(\log^\star r)^{1/2}.$$

The last expression is bounded by the sum in (2.4).

In order to conclude, it remains to bound the first term in the RHS of (2.1). As we will see, we will need to study the integral of some functions near two singularities, where the behaviour of the function is comparable. To simplify the notations, choose a smooth decreasing function $h(t)$ defined for $t>0$ and such that

$h(t) := (-\log t)(\log(-\log t))^2$ for t small enough and $h(t) = 1$ for t large enough. Define also the function $\eta(z) := h(|z|) + h(|z-1|)$. The function h describes the singularities and will be used to simplify some notations below, see for instance (2.8). The function η will account for the two singularities. We will also use the function $\tilde{v}(z) := v(z) - r^{-1}v(r)\Re(z)$. This function satisfies $dd^c\tilde{v} = dd^c v$ and $\tilde{v}(0) = \tilde{v}(r) = 0$. By Cauchy-Schwarz's inequality we have for the first term in the RHS of (2.1)

$$\Big|\Big\langle i\partial u, \chi(s^{-1}z)\frac{r}{\overline{z}(\overline{z}-r)}d\overline{z}\Big\rangle\Big|^2 \leq \Big\langle i\partial u\wedge\overline{\partial}u, \chi(s^{-1}z)\eta(r^{-1}z)\Big\rangle \int \frac{\chi(s^{-1}z)}{\eta(r^{-1}z)}\frac{r^2}{|z^2(z-r)^2|}idz\wedge d\overline{z}.$$

Using the change of variable $z \mapsto rz$, the fact that $0 \leq \chi \leq 1$, and the definition of η we see that the last integral is bounded by

$$\int_{\mathbb{C}} \frac{idz\wedge d\overline{z}}{\Big[h(|z|) + h(|z-1|)\Big]|z^2(z-1)^2|}.$$

Using polar coordinates for z and for $z-1$ and the definition of h it is not difficult to see that the last integral is finite. Therefore, since $i\partial u\wedge\overline{\partial}u \leq dd^c v = dd^c\tilde{v}$, we get

$$\Big|\Big\langle i\partial u, \chi(s^{-1}z)\frac{r}{\overline{z}(\overline{z}-r)}d\overline{z}\Big\rangle\Big|^2 \lesssim \Big\langle dd^c\tilde{v}, \chi(s^{-1}z)\eta(r^{-1}z)\Big\rangle.$$

Define $\hat{v}(z) := \tilde{v}(sz)$. The RHS in the last expression is then equal to

$$\Big\langle dd^c\hat{v}, \chi(z)\eta(r^{-1}sz)\Big\rangle = \Big\langle dd^c\hat{v}, \chi(z)h(r^{-1}s|z|)\Big\rangle + \Big\langle dd^c\hat{v}, \chi(z)h(|r^{-1}sz-1|)\Big\rangle.$$

In order to conclude the proof of the proposition, it is enough to show that each term in the last sum is bounded by a constant times

$$\int_0^s m_{\mathbb{D}_3}(v,t)(\log^\star|\log t|)^2 t^{-1}dt + m_{\mathbb{D}_3}(v,r)^{2/3}\log^\star r. \tag{2.6}$$

We will only consider the first term. The second term can be treated in a similar way using the coordinate $z' := z - rs^{-1}$. Since h is decreasing, the first term we consider is bounded by

$$\Big\langle dd^c\hat{v}, \chi(z)h(|z|)\Big\rangle.$$

Claim *We have*

$$\Big\langle dd^c\hat{v}, \chi(z)h(|z|)\Big\rangle = \int_{\mathbb{D}_2\setminus\{0\}} \hat{v}(z)dd^c[\chi(z)h(|z|)]. \tag{2.7}$$

We assume the claim for now and conclude the proof of the proposition. Notice that the assumption (2.2) will be used in the proof of this claim.

By the definition of h, we have

$$|h'(t)| \lesssim t^{-1}(\log(-\log t))^2 \quad \text{and} \quad |h''(t)| \lesssim t^{-2}(\log(-\log t))^2 \quad \text{for } t \to 0.$$

It follows that, near $z=0$, we have

$$|dd^c(\chi(z)h(|z|))| \lesssim |z^{-2}| \cdot |\log(-\log|z|)|^2 idz \wedge d\overline{z}. \tag{2.8}$$

Hence, we can bound the RHS of (2.7) by a constant times

$$\begin{aligned}\int_{\mathbb{D}_2\setminus\{0\}} |\hat{v}(z)z^{-2}|(\log\log^\star|z|)^2 idz\wedge d\overline{z} &= \int_{\mathbb{D}_{2s}\setminus\{0\}} |\tilde{v}(z)z^{-2}|(\log\log^\star|z/s|)^2 idz\wedge d\overline{z}\\ &\lesssim \int_{\mathbb{D}_{2s}\setminus\{0\}} |\tilde{v}(z)z^{-2}|(\log\log^\star|z|)^2 idz\wedge d\overline{z},\end{aligned}$$

where we used the change of variable $z\mapsto sz$ and the fact that $\log^\star|z/s| \lesssim \log^\star|z|$ for $0<|z|<2s<2$. Moreover, by the definition of $\tilde{v}$ and using that $v(0)=\tilde{v}(0)=0$, we have for $|z|<2s$

$$|\tilde{v}(z)| \le m_{\mathbb{D}_3}(v,|z|) + r^{-1}|v(r)\Re(z)| \le m_{\mathbb{D}_3}(v,|z|) + m_{\mathbb{D}_3}(v,r)r^{-1}|z|.$$

Therefore, using polar coordinates, we see that the last integral is bounded by a constant times

$$\int_0^{2s} m_{\mathbb{D}_3}(v,t)(\log\log^\star t)^2 t^{-1}dt + m_{\mathbb{D}_3}(v,r)r^{-1}s(\log\log^\star(2s))^2.$$

The first term in this sum is bounded by a constant times the integral in (2.6) because $m_{\mathbb{D}_3}(v,t') \le 4m_{\mathbb{D}_3}(v,t)$ for $s/2\le t\le s\le t'\le 2s$. The second one is bounded by a constant times the second term in (2.6) by the definition of s and (2.5). The proposition follows. □

Proof of the claim. Observe that $h(|z|)$ tends to infinity when z tends to 0. Let $\vartheta:\mathbb{R}\to\mathbb{R}$ be a smooth increasing concave function such that $\vartheta(t)=t$ for $t\le 0$ and $\vartheta(t)=1$ for $t\ge 2$. Define $\vartheta_n(t):=\vartheta(t-n)+n$. This is a sequence of smooth functions increasing to the identity. Define $l(z):=\chi(z)h(|z|)$. Using an integration by parts, we see that the LHS of (2.7) is equal to

$$\begin{aligned}\lim_{n\to\infty}\left\langle dd^c\hat{v}, \vartheta_n(l(z))\right\rangle &= \lim_{n\to\infty}\int_{\mathbb{D}_3}\hat{v}(z)dd^c\vartheta_n(l(z))\\ &= \lim_{n\to\infty}\int_{\mathbb{D}_3}\hat{v}(z)\vartheta_n'(l(z))dd^c l(z)\\ &\quad + \lim_{n\to\infty}\int_{\mathbb{D}_3}\hat{v}(z)\vartheta_n''(l(z))dl(z)\wedge d^c l(z).\end{aligned}$$

The first term in the last sum converges to the RHS of the identity in the claim using Lebesgue's dominated convergence theorem and (2.2). Indeed, we have

$|dd^c l(z)| \lesssim |z^{-2}|(\log\log^\star|z|))^2 idz\wedge d\overline{z}$ (see (2.8)) and the sequence $\vartheta_n'(l(z))$ satisfies $0\le\vartheta_n'(l(z))\le 1$ and $\vartheta_n'(l(z))\to 1$ as $n\to\infty$.

We need to show that the second term tends to 0. Since $\chi(z)=1$ for z near 0, for n large enough, the considered term has an absolute value bounded by a constant times

$$\begin{aligned}&\lim_{n\to\infty}\int_{\{h(|z|)>n\}}|\hat{v}(z)|i\partial h(|z|)\wedge\overline{\partial}h(|z|)\\ &\qquad\lesssim\lim_{n\to\infty}\int_{\{h(|z|)>n\}}|\hat{v}(z)z^{-2}|(\log\log^\star|z|))^4 idz\wedge d\overline{z}.\end{aligned}$$

Using the arguments as at the end of the proof of Proposition 2.7 and the assumption (2.2) on v, we see that the last integrand is an integrable function on $\mathbb{D}_1$. Since the set $\{h(|z|)>n\}$ decreases to $\{0\}$ when n tends to infinity, the last limit is zero according to Lebesgue's dominated convergence theorem. This ends the proof of the claim. □

Corollary 2.8. *Let S_0 be a positive closed $(1,1)$-current on $\mathbb{P}^k$ of unit mass, whose dynamical potential u_S satisfies $\|u_S\|_{\log^p}\le 1$ for some $p>3/2$. Let $\mathcal{F}(S_0)$ denote the set of all continuous functions $g\colon\mathbb{P}^k\to\mathbb{R}$ such that $i\partial g\wedge\bar{\partial}g\le S_0$. Then for any positive number $q<\frac{p}{3}-\frac{1}{2}$ we have $\|g\|_{\log^q}\le c$ for some positive constant $c=c(p,q)$ independent of S_0. In particular, the family $\mathcal{F}(S_0)$ is equicontinuous.*

Proof. Notice that (2.2) is satisfied for all v such that $\|v\|_{\log^p}<\infty$ for some $p>1$. It follows that if u and v are as in Proposition 2.7 and v is $\log^p$-continuous for some $p>3/2$ then u is $\log^q$-continuous on $\mathbb{B}_1^k$ for all q as in the statement, with $\|u\|_{\mathbb{B}_1^k,\log^q}\le c\|v\|^{1/2}_{\mathbb{B}_5^k,\log^p}$ for some positive constant c independent of u, v. The result is thus deduced from Proposition 2.7 by means of a finite cover of $\mathbb{P}^k$. □

3 Some semi-norms and equidistribution properties

In this section, we consider the action of the operator $(f^n)_*$ on functions and currents. We also introduce the semi-norms which are crucial in our study. Some results and ideas here are of independent interest. Recall that we always assume that f satisfies the Assumption (**A**) in the Introduction.

3.1 Bounds with respect to the semi-norm $\|\cdot\|_{\log^p}$. In this section, we study the action of the operator f_* on functions with bounded semi-norm $\|\cdot\|_{\log^p}$. We first prove that, with respect to this semi-norm, the operator f_* is Lipschitz.

Lemma 3.1. *For every constant $A>1$, there exists a positive constant $c=c(A)$ such that for every $n\ge 0$, $p>0$, and continuous function $g\colon\mathbb{P}^k\to\mathbb{R}$, we have*

$$\left\|d^{-kn}(f^n)_*g\right\|_{\log^p}\le c^pA^{pn}\|g\|_{\log^p}.$$

Proof. We have

$$\left\| d^{-kn}(f^n)_* g \right\|_{\log^p} = \sup_{x,y\in\mathbb{P}^k} d^{-kn} \left|(f^n)_* g(x) - (f^n)_* g(y)\right| \cdot (\log^\star \operatorname{dist}(x,y))^p.$$

We need to bound the RHS by $c^p A^{pn} \|g\|_{\log^p}$.

Applying [DS102, Cor. 4.4] inductively to some iterate of f, we see that the Assumption (**A**) implies:

(**A′**) for every constant $\kappa > 1$, there are an integer $n_\kappa \geq 0$ and a constant $c_\kappa > 0$ independent of n such that for all $x, y \in \mathbb{P}^k$ and $n \geq n_\kappa$ we can write $f^{-n}(x) = \{x_1, \dots, x_{d^{kn}}\}$ and $f^{-n}(y) = \{y_1, \dots, y_{d^{kn}}\}$ (counting multiplicity) with the property that

$$\operatorname{dist}(x_j, y_j) \leq c_\kappa \operatorname{dist}(x,y)^{1/\kappa^n} \quad \text{for} \quad j = 1, \dots, d^{kn}.$$

Fix $\kappa < A$. We have, for $n \geq n_\kappa$,

$$\begin{aligned}
&d^{-kn} \left|(f^n)_* g(x) - (f^n)_* g(y)\right| (\log^\star \operatorname{dist}(x,y))^p \\
&\quad\leq \max_j |g(x_j) - g(y_j)| \, (\log^\star \operatorname{dist}(x,y))^p \\
&\quad\leq \max_j \frac{\|g\|_{\log^p}}{(\log^\star \operatorname{dist}(x_j,y_j))^p} (\log^\star \operatorname{dist}(x,y))^p \\
&\quad= \max_j \Big(\frac{\log^\star \operatorname{dist}(x,y)}{\kappa^n \log^\star \operatorname{dist}(x_j,y_j)}\Big)^p \|g\|_{\log^p} \kappa^{pn}.
\end{aligned}$$

We need to check that the expression in the last parentheses is bounded by a constant. Fix a large constant $M > 0$. Since $\log^\star \operatorname{dist}(x_j, y_j)$ is bounded from below by 1, when $\log^\star \operatorname{dist}(x,y)$ is bounded by $2M\kappa^n$ the considered expression is bounded by some constant c as desired. Assume now that $\log^\star \operatorname{dist}(x,y) \geq 2M\kappa^n$. Since M is large, we deduce that $\log \operatorname{dist}(x,y) \leq -2M\kappa^n + 1 \leq -M\kappa^n$. Hence, by (**A′**), since M is large, we have

$$\log \operatorname{dist}(x_j, y_j) \leq \log c_\kappa + \kappa^{-n} \log \operatorname{dist}(x,y) \leq \frac{1}{2} \kappa^{-n} \log \operatorname{dist}(x,y).$$

It is now clear that $\kappa^n \log^\star \operatorname{dist}(x_j, y_j) \geq \frac{1}{2} \log^\star \operatorname{dist}(x,y)$ which implies that the considered expression is bounded, as desired. This implies the lemma for $n \geq n_\kappa$.

As the multiplicity of f^n at a point is at most d^{kn}, we also have (see again [DS102, Cor. 4.4]):

(**A″**) there is a constant $c_0 > 0$ such that for every $n \geq 0$, for all $x, y \in \mathbb{P}^k$, we can write $f^{-n}(x) = \{x_1, \dots, x_{d^{kn}}\}$ and $f^{-n}(y) = \{y_1, \dots, y_{d^{kn}}\}$ (counting multiplicity) with the property that

$$\operatorname{dist}(x_j, y_j) \leq c_0 \operatorname{dist}(x,y)^{1/d^{kn}} \quad \text{for} \quad j = 1, \dots, d^{kn}.$$

Hence, when $n \le n_\kappa$, it is enough to use ($\mathbf{A''}$) instead of ($\mathbf{A'}$). Since n_κ is fixed it is clear that $\frac{\log^\star \mathrm{dist}(x,y)}{\log^\star \mathrm{dist}(x_j,y_j)} \lesssim d^{kn_\kappa}$, which is bounded. The proof is complete. $\square$

We will need the following result which is an improvement of [DS102, Th. 1.1] in the case where f satisfies the Assumption ($\mathbf{A}$). By duality, this result implies an exponential equidistribution of $d^{-kn}(f^n)^*\nu$ towards μ for every probability measure ν. The assumption ($\mathbf{A}$) is necessary here to get the estimate in the norm $\|\cdot\|_\infty$.

Theorem 3.2. *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \ge 2$ satisfying Assumption* ($\mathbf{A}$). *Consider a real number $p > 0$. Let $g\colon \mathbb{P}^k \to \mathbb{R}$ be such that $\|dd^c g\|_* \le 1$, $\langle \mu, g\rangle = 0$ and $\|g\|_{\log^p} \le 1$. Then, for every constant $d^{-p/(p+1)} < \eta < 1$, there is a positive constant c independent of g such that for every $n \ge 0$*

$$\left\| d^{-kn}(f^n)_* g \right\|_\infty \le c\eta^n.$$

Proof. Set $g_n := d^{-kn}(f^n)_* g$. Recall (see, e.g., [Sko72, DNS10]) that there exists a positive constant c_0 independent of g and n such that

$$\int_{\mathbb{P}^k} e^{d^n|g_n|} \le c_0, \tag{3.1}$$

where the integral above is taken with respect to the Lebesgue measure associated to the volume form ω_{FS}^k on $\mathbb{P}^k$.

Fix a constant $A > 1$ such that $\eta > (A/d)^{p/(p+1)}$. Suppose by contradiction that for infinitely many n there exists a point $a_n \in \mathbb{P}^k$ such that $|g_n(a_n)| \ge 3\eta^n$ for some g as above. Choose $r := e^{-c_A A^n \eta^{-n/p}}$ with c_A the constant given by Lemma 3.1 (we write c_A instead of c in order to avoid confusion). By that lemma, when $\mathrm{dist}(z, a_n) < r$, we have

$$|g_n(z)| \ge |g_n(a_n)| - |g_n(z) - g_n(a_n)| \ge 3\eta^n - c_A^p A^{pn}(1 + |\log r|)^{-p} \ge \eta^n.$$

This implies that

$$c_0 \ge \int_{\mathbb{P}^k} e^{d^n|g_n|} \ge \int_{\mathrm{dist}(z,a_n)<r} e^{d^n|g_n(z)|} \gtrsim r^{2k} e^{d^n\eta^n} \gtrsim e^{-2kc_A A^n \eta^{-n/p} + d^n\eta^n}.$$

By the choice of A, the last expression diverges when n tends to infinity. This is a contradiction. The theorem follows. $\square$

3.2 The semi-norm $\|\cdot\|_p$. In this section, we combine the semi-norms $\|\cdot\|_{\log^p}$ and $\|\cdot\|_*$ to build a new semi-norm $\|\cdot\|_p$ and study its first properties. For our convenience, we will use dynamical potentials of currents.

For every positive closed $(1,1)$-current S on $\mathbb{P}^k$ we first define

$$\|S\|'_p := \|S\| + \|u_S\|_{\log^p},$$

where u_S is the dynamical potential of S. When R is any $(1,1)$-current we define

$$\|R\|_p = \inf \|S\|'_p,$$

where the infimum is taken over all positive closed $(1,1)$-currents S such that $|R| \le S$, and we set $\|R\|_p := \infty$ when no such S exists. By the compactness of positive closed currents with bounded $\|\cdot\|'_p$-norm, this infimum is actually a minimum when it is finite. Finally, for all $g \colon \mathbb{P}^k \to \mathbb{R}$, define

$$\|g\|_p := \|dd^c g\|_p.$$

The following lemma shows in particular that the norm $\|\cdot\|_p$ is equivalent to the norm $\|\cdot\|'_p$ when both are defined. We will thus just consider the norm $\|\cdot\|_p$ in the sequel.

Lemma 3.3. *Let S be a positive closed $(1,1)$-current on $\mathbb{P}^k$ and let $g \colon \mathbb{P}^k \to \mathbb{R}$ be a continuous function. Then*

$$\|u_S\|_p \le 2\|S\|'_p, \qquad \|S\|_p \le \|S\|'_p \le c\,\|S\|_p\,, \qquad \textit{and} \qquad \|g\|_{\log^p} \le c\,\|g\|_p$$

for some positive constant $c = c(p)$ independent of S and g.

Proof. Define $m := \|S\|$. We have $\|S\|'_p \ge m$. Since $u_T = 0$, we have $\|T\|_p = \|T\|'_p = 1$ and

$$\|u_S\|_p = \|dd^c u_S\|_p = \|S - mT\|_p \le \|S\|_p + m\|T\|_p \le \|S\|'_p + m \le 2\|S\|'_p.$$

This proves the first assertion in the lemma.

We prove now the second assertion. The first inequality is true by definition. For the second one, it is enough to prove that if $\tilde{S}$ is also positive closed, and such that $S \le \tilde{S}$, then $\|S\|'_p \le c\left\|\tilde{S}\right\|'_p$ for some constant c independent of S, $\tilde{S}$. It is clear that $\|S\| \le \left\|\tilde{S}\right\|$. So we can assume that $\left\|\tilde{S}\right\| = 1$ and we only need to check that $\|u_S\|_{\log^p}$ is bounded by $c(1 + \|u_{\tilde{S}}\|_{\log^p})$ for some constant c.

We cover $\mathbb{P}^k$ with a finite family of open sets of the form $V_j := \Phi_j(\mathbb{B}^k_{1/2})$ where Φ_j is an injective holomorphic map from $\mathbb{B}^k_4$ to $\mathbb{P}^k$. Observe, in particular, that all the Φ_j's and their inverses are Lipschitz. In particular, there exists a constant $L > 1$ such that

$$L^{-1}\,\mathrm{dist}_{\mathbb{B}^k_4}(x,y) \le \mathrm{dist}_{\mathbb{P}^k}(\Phi_j(x), \Phi_j(y)) \le L\,\mathrm{dist}_{\mathbb{B}^k_4}(x,y) \tag{3.2}$$

for every j and every $x, y \in \mathbb{B}^k_{1/2}$. Let h_j denote a potential of T on $\Phi_j(\mathbb{B}^k_4)$. This is a Hölder continuous function. By definition of dynamical potential, $v_j := u_{\tilde{S}} + h_j$ is a potential of $\tilde{S}$ on $\Phi_j(\mathbb{B}^k_4)$. By [BD23, Corollary 2.5], we have that $\Omega(u_S)$ is bounded by a constant. By (3.2), a comparison principle [BD23, Corollary 2.7] implies that for all r smaller than some constant $r_0 > 0$

$$m_{\mathbb{P}^k}(u_S, r) \lesssim \max_j m_{V_j}(v_j, c'\sqrt{r}) + \sqrt{r} \lesssim (1 + \|u_{\tilde{S}}\|_{\log^p})|\log r|^{-p}$$

for some positive constant c' depending on L. This proves the second assertion.

Finally, let us consider the last inequality in the statement. By linearity, we can assume that there exists a positive closed $(1,1)$-current $\tilde{S}$ of mass $\left\|\tilde{S}\right\| = 1$ such that $|dd^c g| \leq \tilde{S}$ and prove that $\|g\|_{\log^p} \leq c(1 + \left\|u_{\tilde{S}}\right\|_{\log^p})$ for some positive constant c independent of g and $\tilde{S}$. Observe that $dd^c g + \tilde{S}$ is a positive closed current and $dd^c g + \tilde{S} \leq 2\tilde{S}$. So we can apply the arguments in the previous paragraph to $g + u_{\tilde{S}}$, $2u_{\tilde{S}}$ instead of u_S, $u_{\tilde{S}}$. We obtain $\|g + u_{\tilde{S}}\|_{\log^p} \lesssim 1 + \|u_{\tilde{S}}\|_{\log^p}$. This implies the last assertion, and completes the proof of the lemma. $\square$

Lemma 3.4. *There is a positive constant $c = c(p)$ such that for all continuous functions $g, h\colon \mathbb{P}^k \to \mathbb{R}$ with finite $\|\cdot\|_p$ semi-norms and any $\mathcal{C}^2$ convex function $\chi : \mathbb{R} \to \mathbb{R}$ we have*

$$\|\chi(g)\|_p \leq c\|\chi'(g)\|_\infty \|g\|_p \quad \textit{and} \quad \left\|\partial g \wedge \bar{\partial} h\right\|_p \leq c(\Omega(g)\|h\|_p + \|g\|_p \Omega(h)).$$

Proof. We can write $|dd^c g| \leq S$ for some positive closed $(1,1)$-current S with $\|S\| \lesssim \|g\|_p$ and $\|u_S\|_{\log^p} \lesssim \|g\|_p$. We first prove the first inequality. Set $A := \|\chi'(g)\|_\infty$. We have

$$dd^c \chi(g) = \chi'(g) dd^c g + \frac{1}{\pi}\chi''(g) i\partial g \wedge \overline{\partial} g = \Big[\chi'(g) dd^c g + AS + \frac{1}{\pi}\chi''(g) i\partial g \wedge \overline{\partial} g\Big] - [AS].$$

Write the last expression as $R^+ - R^-$ where R^+ (resp. R^-) is the expression in the first (resp. second) brackets. Using the definition of A, the inequality $|dd^c g| \leq S$, the convexity of χ, and the fact that $i\partial g \wedge \overline{\partial} g$ is always positive, we deduce that both R^+ and R^- are positive currents. Clearly, the current R^- is closed and its mass is $\lesssim A \|g\|_p$. The current R^+ is cohomologous to R^- because $dd^c \chi(g)$ is an exact current. It follows that R^+ is also a positive closed current of mass $\lesssim A \|g\|_p$.

We have $\|u_{R^-}\|_{\log^p} = A\|u_S\|_{\log^p} \lesssim A \|g\|_p$. This and the above estimate for the mass imply that $\|R^-\|_p \lesssim A \|g\|_p$. On the other hand, the above identities imply that $\chi(g) + u_{R^-}$ differs from u_{R^+} by a constant. We deduce that

$$\|u_{R^+}\|_{\log^p} \leq \|u_{R^-}\|_{\log^p} + \|\chi(g)\|_{\log^p} \lesssim \|u_{R^-}\|_{\log^p} + A\|g\|_{\log^p} \lesssim A \|g\|_p .$$

Therefore, we also have $\|R^+\|_p \lesssim A \|g\|_p$. It is now clear that $\|dd^c \chi(g)\|_p \lesssim A \|g\|_p$. Hence, we get the first assertion in the lemma.

For the second assertion, we first consider the case where $g = h$. We can replace g by $g - \min g$ in order to assume that $\min g = 0$ and hence $\|g\|_\infty = \Omega(g)$. The above computation applied with $\chi(t) = t^2$ gives

$$0 \leq i\partial g \wedge \bar{\partial} g = \pi\Big(dd^c g^2 - 2g dd^c g\Big) \lesssim dd^c g^2 + 2\|g\|_\infty S.$$

We thus have

$$\|i\partial g \wedge \bar{\partial} g\|_p \lesssim \|dd^c g^2\|_p + \|g\|_\infty \|S\|_p \lesssim \|dd^c g^2\|_p + \|g\|_\infty \|g\|_p .$$

We obtain the desired estimate by applying the first assertion in the lemma to $\|dd^c g^2\|_p$, using the function $\chi(t) := t^2$.

Finally, let us consider the second inequality for g and h. As above, we can assume that $\min h = 0$ and hence $\|h\|_\infty = \Omega(h)$. It follows from Cauchy-Schwarz's inequality that

$$\left|\partial g\wedge\bar\partial h \pm \partial h\wedge\bar\partial g\right| \lesssim \frac{\|h\|_\infty}{\|g\|_\infty} i\partial g\wedge\bar\partial g + \frac{\|g\|_\infty}{\|h\|_\infty} i\partial h\wedge\bar\partial h.$$

The assertion thus follows from the particular case considered above. This completes the proof of the lemma. □

REMARK 3.5. Lemma 3.4 can be applied to a non-convex $\mathcal{C}^2$ function χ as it can be written as the difference of two convex functions. We can also apply it to χ Lipschitz and convex because such a function can be approximated by smooth convex functions with bounded first derivatives.

3.3 The dynamical norm $\|\cdot\|_{p,\alpha}$. In this section, we define the main norms $\|\cdot\|_{p,\alpha}$ for $(1,1)$-currents that we will use to quantify the convergence (1.2). Based on the results in the previous sections, we will see later that these norms satisfy the inequalities

$$\|\cdot\|_q \lesssim \|\cdot\|_{p,\alpha} \lesssim \|\cdot\|_p$$

for some explicit q depending on p, α, and d. In particular, the new norms are at the same time weaker than the previous norm $\|\cdot\|_p$, but still inherit the main properties of a similar norm $\|\cdot\|_q$ which are obtained in the previous section.

DEFINITION 3.6. Given a positive closed $(1,1)$-current S on $\mathbb{P}^k$ and a real number α such that $d^{-1}\leq\alpha<1$, we define the current S_α by

$$S_\alpha := \sum_{n=0}^{\infty}\alpha^n \frac{(f^n)_*(S)}{d^{(k-1)n}}\cdot$$

For any $(1,1)$-current R on $\mathbb{P}^k$ and real number $p>0$, we define (see the Notation at the beginning of the paper)

$$\|R\|_{p,\alpha} := \inf\Big\{c\in\mathbb{R}\colon \exists S \text{ positive closed}\colon \|S\|_p\leq 1, |R|\leq cS_\alpha\Big\} \tag{3.3}$$

and we set $\|R\|_{p,\alpha} := \infty$ if such a number c does not exist.

Recall that the mass of $d^{-(k-1)n}(f^n)_*(S)$ is independent of n. Hence, we have $\|S_\alpha\| = \sum_{n\geq 0}\alpha^n$. Note also that when $\|R\|_{p,\alpha}$ is finite, by compactness, the infimum in (3.3) is actually a minimum and that, by definition we have $\|\xi R\|_{p,\alpha}\leq\|R\|_{p,\alpha}$ for every $(1,1)$-current R and every $\xi\colon\mathbb{P}^k\to\mathbb{C}$ with $|\xi|\leq 1$. We have the following lemma where the assumption $d^{-1}\leq\alpha<d^{-1/(p+1)}$ is equivalent to $0<q_0\leq p$.

LEMMA 3.7. *Let α and p be positive and such that $d^{-1}\leq\alpha<d^{-1/(p+1)}$. Then, for every $0<q<q_0:=\frac{|\log\alpha|}{\log d}(p+1)-1$, there are positive constants $c_1=c_1(p,\alpha)$ and*

$c_2 = c_2(p,\alpha,q)$ *such that, for every* $(1,1)$*-current* R,

$$\|R\|_{p,\alpha} \leq c_1 \|R\|_p \qquad \text{and} \qquad \|R\|_q \leq c_2 \|R\|_{p,\alpha}.$$

Proof. The first inequality holds by the definition of $\|\cdot\|_{p,\alpha}$ and Lemma 3.3. We prove the second inequality. Consider a current R such that $\|R\|_{p,\alpha} = 1$. We have to show that $\|R\|_q$ is bounded by a constant.

From the definition of $\|\cdot\|_{p,\alpha}$, we can find a positive closed current S such that $\|S\|_p = 1$ and $|R| \leq S_\alpha$. By the definition of the norm $\|\cdot\|_q$ and Lemma 3.3 applied to S_α, it is enough to show that $\|S_\alpha\|'_q$ is bounded. Denote by u_α the dynamical potential of S_α. Since the mass of S_α is bounded, we only need to show that $\|u_\alpha\|_{\log^q}$ is bounded. By definition of S_α and the invariance of μ, we have

$$u_\alpha = \sum_{n=0}^{\infty} \alpha^n \frac{(f^n)_* u_S}{d^{(k-1)n}}\cdot$$

It follows that, for every positive number N,

$$m(u_\alpha, r) \leq \sum_{n\leq N} \alpha^n d^{-(k-1)n} \|(f^n)_* u_S\|_{\log^p} (\log^\star r)^{-p} + 2\sum_{n>N} \alpha^n d^{-(k-1)n} \|(f^n)_* u_S\|_\infty .$$

Fix constants $A > 1$ close enough to 1, $\eta > d^{-p/(p+1)}$ close enough to $d^{-p/(p+1)}$ and $\alpha' > \alpha A^p$ close enough to α. In particular, by the assumption on α and the choice of η, we have that $\alpha d \eta$ is close to $\alpha d^{1/(p+1)}$ and smaller than 1. By Lemma 3.1 and Theorem 3.2 we know that $\|(f^n)_* u_S\|_{\log^p} \lesssim d^{kn} A^{pn}$ and $\|(f^n)_* u_S\|_\infty \lesssim d^{kn}\eta^n$. This, the above estimate on $m(u_\alpha, r)$, and the fact that $\alpha' d > \alpha d \geq 1$ imply that

$$m(u_\alpha, r) \lesssim \sum_{n\leq N} \alpha^n d^n A^{pn} (\log^\star r)^{-p} + \sum_{n>N} \alpha^n d^n \eta^n \lesssim (\alpha' d)^N (\log^\star r)^{-p} + (\alpha d \eta)^N.$$

Finally, choose $N = \frac{p+1}{\log d} \log\log^\star r$. Observe that if we replace α' by α and $\alpha d\eta$ by $\alpha d^{1/(p+1)}$, the last sum is equal to $2(\log^\star r)^{-q_0}$. So, this sum is bounded by a constant times $(\log^\star r)^{-q}$ for $q < q_0$ because α' is chosen close to α and $\alpha d\eta$ is close to $\alpha d^{1/(p+1)}$. This concludes the proof of the lemma. □

The following shifting property of the norm $\|\cdot\|_{p,\alpha}$ is very useful when we work with the action of f, and is the key property that we need of this norm.

LEMMA 3.8. *For every* $n \geq 0$ *and every* $(1,1)$*-current* R *on* $\mathbb{P}^k$, *we have*

$$\left\| d^{-kn} (f^n)_* R \right\|_{p,\alpha} \leq \frac{1}{d^n \alpha^n} \|R\|_{p,\alpha}.$$

Proof. We can assume that $\|R\|_{p,\alpha} = 1$, so that there is a positive closed current S with $\|S\|_p = 1$ and $|R| \leq S_\alpha$, see the Notation at the beginning of the paper. Consider any function $\xi\colon \mathbb{P}^k \to \mathbb{C}$ such that $|\xi| \le 1$ and define $\xi_n := \xi \circ f^n$. Since $|\xi_n| \le 1$, we have

$$\Re\Big(\xi d^{-kn} (f^n)_* R\Big) = \frac{1}{d^n}\Re\Big(\frac{(f^n)_* (\xi_n R)}{d^{(k-1)n}}\Big) \le \frac{1}{d^n \alpha^n} \sum_{j=0}^{\infty} \alpha^{n+j} \frac{(f^n)_*}{d^{(k-1)n}} \frac{(f^j)_*}{d^{(k-1)j}} S \le \frac{1}{d^n \alpha^n} S_\alpha.$$

The lemma follows. □

3.4 The dynamical Sobolev semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$. We can now define the first semi-norm for functions $g\colon \mathbb{P}^k \to \mathbb{R}$ with respect to which we will be able to prove the existence of a spectral gap for the transfer operator. We can also define this norm for 1-forms. Observe that this will not be the final norm, as it is only bounded by the $\|\cdot\|_{\mathcal{C}^1}$ norm, and not by the Hölder norms as in Theorem 1.2.

Definition 3.9. Let p and α be real numbers such that $p > 0$ and $d^{-1} \le \alpha < 1$. For any function $g\colon \mathbb{P}^k \to \mathbb{R}$ we set

$$\|g\|_{\langle p,\alpha\rangle} := \Big\| i\partial g \wedge \bar\partial g \Big\|_{p,\alpha}^{1/2}.$$

The following two lemmas give the main properties of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$ that we will need in Sect. 5, together with Lemma 3.8. Recall that q_0 is defined in Lemma 3.7. Note that the hypothesis $p > 3/2$ ensures that $d^{-1} < d^{-5/(2p+2)}$ and the hypothesis on α ensures that $q_0 > 3/2$, and hence that q_1 as in the statement below is positive.

Lemma 3.10. *Let α and p be positive numbers such that $p > 3/2$ and $d^{-1} \le \alpha < d^{-5/(2p+2)}$. Then, for every $0 < q < q_1 := \frac{q_0}{3} - \frac{1}{2}$, there are positive constants $c_1 = c_1(p,\alpha,q)$ and $c_2 = c_2(p,\alpha)$ such that for every $g\colon \mathbb{P}^k \to \mathbb{R}$ we have*

$$\|g\|_{\log^q} \le c_1 \|g\|_{\langle p,\alpha\rangle}, \quad \|g\|_{\langle p,\alpha\rangle} \le c_2 \|g\|_p, \quad \text{and} \quad \|g\|_{\langle p,\alpha\rangle} \le c_2 \|g\|_{\mathcal{C}^1}.$$

Proof. We can assume that $\|g\|_{\langle p,\alpha\rangle} \le 1$. By the definition of the norm $\|\cdot\|_{\langle p,\alpha\rangle}$ and Lemma 3.7, $\Big\| i\partial g \wedge \bar\partial g \Big\|_{q'}$ is bounded by a constant for any $q' < q_0$. Therefore, we have $i\partial g \wedge \bar\partial g \le R$ for some positive closed current R such that $\|R\|$ and $\|u_R\|_{\log^{q'}}$ are bounded by a constant. The first inequality follows from Corollary 2.8. The second assertion follows from Lemmas 3.7 and 3.4. The last assertion follows from Definition 3.9. □

Lemma 3.11. *Let α and p be positive numbers such that $d^{-1} \le \alpha < 1$. Then for all functions $g, h\colon \mathbb{P}^k \to \mathbb{R}$ we have*

$$\|gh\|_{\langle p,\alpha\rangle} \le \sqrt{2}\Big(\|g\|_{\langle p,\alpha\rangle} \|h\|_\infty + \|g\|_\infty \|h\|_{\langle p,\alpha\rangle} \Big).$$

Proof. Using an expansion of $i\partial(gh)\wedge\bar\partial(gh)$ and Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\left\|i\partial(gh)\wedge\bar\partial(gh)\right\|_{p,\alpha} &\le \|h\|_\infty^2\left\|i\partial g\wedge\bar\partial g\right\|_{p,\alpha}+\|g\|_\infty^2\left\|i\partial h\wedge\bar\partial h\right\|_{p,\alpha}\\ &\quad+\|g\|_\infty\|h\|_\infty\left\|i\partial g\wedge\bar\partial h+i\partial h\wedge\bar\partial g\right\|_{p,\alpha}\\ &\le \|h\|_\infty^2\left\|i\partial g\wedge\bar\partial g\right\|_{p,\alpha}+\|g\|_\infty^2\left\|i\partial h\wedge\bar\partial h\right\|_{p,\alpha}\\ &\quad+\|g\|_\infty\|h\|_\infty\Big(\frac{\|h\|_\infty}{\|g\|_\infty}\left\|i\partial g\wedge\bar\partial g\right\|_{p,\alpha}+\frac{\|g\|_\infty}{\|h\|_\infty}\left\|i\partial h\wedge\bar\partial h\right\|_{p,\alpha}\Big)\\ &\le 2\|h\|_\infty^2\left\|i\partial g\wedge\bar\partial g\right\|_{p,\alpha}+2\|g\|_\infty^2\left\|i\partial h\wedge\bar\partial h\right\|_{p,\alpha}.\end{aligned}$$

The assertion follows from Definition 3.9. □

3.5 The semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$.

The following semi-norm defines the final space of functions that we will use in our study of the transfer operator. We use here some ideas from the theory of interpolation between Banach spaces, see also [Tri95].

DEFINITION 3.12. For all real numbers $d^{-1}\le\alpha<1$, $\gamma>0$ and $p>0$, we define for a continuous function $g\colon\mathbb{P}^k\to\mathbb{R}$

$$\begin{aligned}\|g\|_{\langle p,\alpha\rangle,\gamma}:=\inf\Big\{c\ge 0\colon \forall\, 0<\epsilon\le 1\ \exists\, g_\epsilon^{(1)},g_\epsilon^{(2)}\colon\mathbb{P}^k\to\mathbb{R}\quad :\\ g=g_\epsilon^{(1)}+g_\epsilon^{(2)},\left\|g_\epsilon^{(1)}\right\|_{\langle p,\alpha\rangle}\le c(1/\epsilon)^{1/\gamma},\left\|g_\epsilon^{(2)}\right\|_\infty\le c\epsilon\Big\}.\end{aligned}\tag{3.4}$$

When such a number c does not exist, we set $\|g\|_{\langle p,\alpha\rangle,\gamma}:=\infty$.

REMARK 3.13. Lemma 2.4 applied for $s=1$ implies that $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}\lesssim\|\cdot\|_{\mathcal{C}^\gamma}$ because $\|\cdot\|_{\langle p,\alpha\rangle}\lesssim\|\cdot\|_{\mathcal{C}^1}$, see Lemma 3.10. We also see that the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ formally coincides with the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$ for $\gamma=\infty$. In this sense, we may think of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$ a limit of the semi-norms $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ for $\gamma\to\infty$.

The following two lemmas are the counterparts of Lemmas 3.10 and 3.11 for the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$. Recall that q_1 is defined in Lemma 3.10.

LEMMA 3.14. *For all positive numbers p, α, γ, q satisfying $p>3/2$, $d^{-1}\le\alpha<d^{-5/(2p+2)}$ and $q<q_2:=\frac{\gamma}{\gamma+1}q_1$, there is a positive constant $c=c(p,\alpha,\gamma,q)$ such that*

$$\|g\|_{\log^q}\le c\,\|g\|_{\langle p,\alpha\rangle,\gamma}\qquad\text{and}\qquad\|g\|_{\langle p,\alpha\rangle,\gamma}\le\|g\|_{\langle p,\alpha\rangle}$$

for every continuous function $g\colon\mathbb{P}^k\to\mathbb{R}$. Moreover, if $\chi: I\to\mathbb{R}$ is a Lipschitz function with Lipschitz constant κ on an interval $I\subset\mathbb{R}$ containing the image of g, then we have

$$\|\chi(g)\|_{\langle p,\alpha\rangle,\gamma}\le\kappa\|g\|_{\langle p,\alpha\rangle,\gamma}.$$

Proof. Let us prove the first inequality. We can assume that $\|g\|_{\langle p,\alpha\rangle,\gamma} \leq 1$. Lemma 3.10 implies that $g_\epsilon^{(1)}$ has $\|\cdot\|_{\log^{q'}}$ semi-norm bounded by a constant times $(1/\epsilon)^{1/\gamma}$ when $q' < q_1$. Therefore, we have for $r > 0$

$$m(g,r) \leq m(g_\epsilon^{(1)},r) + m(g_\epsilon^{(2)},r) \lesssim \frac{(1/\epsilon)^{1/\gamma}}{(\log^\star r)^{q'}} + \epsilon.$$

Choosing $\epsilon = (\log^\star r)^{-K}$ with $K = q'/(1+1/\gamma)$ gives

$$m(g,r) \lesssim (\log^\star r)^{-q'+K/\gamma} + (\log^\star r)^{-K} = 2(\log^\star r)^{-q'/(1+1/\gamma)}.$$

The first assertion of the lemma follows by choosing q' close enough to q_1.

The second inequality follows from the definition of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$, by taking $g_\epsilon^{(2)} = 0$ in the decomposition $g = g_\epsilon^{(1)} + g_\epsilon^{(2)}$ for every ϵ.

We prove now the last assertion. Since we can approximate χ uniformly by smooth functions χ_n with $|\chi_n'| \leq \kappa$, we can assume for simplicity that χ is smooth. Define $h := \chi(g)$ and recall that we are assuming that $\|g\|_{\langle p,\alpha\rangle,\gamma} \leq 1$. For every $0 < \epsilon \leq 1$, we have the decomposition

$$g = g_\epsilon^{(1)} + g_\epsilon^{(2)} \quad \text{with} \quad \left\|g_\epsilon^{(1)}\right\|_{\langle p,\alpha\rangle} \leq (1/\epsilon)^{1/\gamma} \quad \text{and} \quad \left\|g_\epsilon^{(2)}\right\|_\infty \leq \epsilon.$$

Write

$$h = h_\epsilon^{(1)} + h_\epsilon^{(2)} \quad \text{with} \quad h_\epsilon^{(1)} := \chi(g_\epsilon^{(1)}) \quad \text{and} \quad h_\epsilon^{(2)} := h - h_\epsilon^{(1)}.$$

We have

$$\|h_\epsilon^{(2)}\|_\infty = \|\chi(g) - \chi(g_\epsilon^{(1)})\|_\infty \lesssim \kappa\|g - g_\epsilon^{(1)}\|_\infty = \kappa\|g_\epsilon^{(2)}\|_\infty \leq \kappa\epsilon$$

and also

$$\|h_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} = \|i\partial\chi(g_\epsilon^{(1)}) \wedge \overline{\partial}\chi(g_\epsilon^{(1)})\|_{p,\alpha}^{1/2} \leq \kappa\|i\partial g_\epsilon^{(1)} \wedge \overline{\partial} g_\epsilon^{(1)}\|_{p,\alpha}^{1/2} = \kappa\|g_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} \leq \kappa(1/\epsilon)^{1/\gamma}.$$

It follows that $\|h\|_{\langle p,\alpha\rangle,\gamma} \leq \kappa$. This completes the proof of the lemma. □

LEMMA 3.15. *For all positive numbers p, α, γ such that $d^{-1} \leq \alpha < 1$ we have*

$$\|gh\|_{\langle p,\alpha\rangle,\gamma} \leq 3\Big(\|g\|_{\langle p,\alpha\rangle,\gamma} \|h\|_\infty + \|g\|_\infty \|h\|_{\langle p,\alpha\rangle,\gamma} \Big)$$

for every continuous functions $g, h\colon \mathbb{P}^k \to \mathbb{R}$.

Proof. We can assume that $\|g\|_{\langle p,\alpha\rangle,\gamma} = \|h\|_{\langle p,\alpha\rangle,\gamma} = 1$. For every $0 < \epsilon \leq 1$, we need to find a decomposition $gh = L_\epsilon^{(1)} + L_\epsilon^{(2)}$ with $L_\epsilon^{(1)}$, $L_\epsilon^{(2)}$ such that

$$\|L_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} \leq 3(\|g\|_\infty + \|h\|_\infty)(1/\epsilon)^{1/\gamma} \quad \text{and} \quad \|L_\epsilon^{(2)}\|_\infty \leq 3(\|g\|_\infty + \|h\|_\infty)\epsilon.$$

Case 1. Assume that $\|g\|_\infty \leq 3\epsilon$ and $\|h\|_\infty \leq 3\epsilon$. Choose $L_\epsilon^{(1)} = 0$ and $L_\epsilon^{(2)} = gh$. Clearly, these functions satisfy the desired estimates.

Case 2. Assume now that $\|g\|_\infty + \|h\|_\infty \geq 3\epsilon$. By the definition of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$, we have the decompositions $g = g_\epsilon^{(1)} + g_\epsilon^{(2)}$ and $h = h_\epsilon^{(1)} + h_\epsilon^{(2)}$ with

$$\|g_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} \leq (1/\epsilon)^{1/\gamma}, \quad \|g_\epsilon^{(2)}\|_\infty \leq \epsilon, \quad \|h_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} \leq (1/\epsilon)^{1/\gamma}, \quad \|h_\epsilon^{(2)}\|_\infty \leq \epsilon.$$

Observe that $\left\|g_\epsilon^{(1)}\right\|_\infty \leq \|g\|_\infty + \epsilon$ and $\left\|h_\epsilon^{(1)}\right\|_\infty \leq \|h\|_\infty + \epsilon$, which imply that

$$\left\|g_\epsilon^{(1)}\right\|_\infty + \left\|h_\epsilon^{(1)}\right\|_\infty \leq 2(\|g\|_\infty + \|h\|_\infty).$$

Set

$$L_\epsilon^{(1)} := g_\epsilon^{(1)} h_\epsilon^{(1)} \qquad \text{and} \qquad L_\epsilon^{(2)} := g_\epsilon^{(1)} h_\epsilon^{(2)} + g_\epsilon^{(2)} h_\epsilon^{(1)} + g_\epsilon^{(2)} h_\epsilon^{(2)}.$$

The desired estimate for $\|L_\epsilon^{(1)}\|_{\langle p,\alpha\rangle}$ follows from Lemma 3.11 and the one for $\|L_\epsilon^{(2)}\|_\infty$ is obtained by a direct computation. This ends the proof of the lemma. □

4 Spectral gap for the transfer operator

In this section we prove our main Theorem 1.2. Theorem 1.1 and (1.2) give the scaling ratio λ, the density function ρ as an eigenfunction for the operator $\mathcal{L} = \mathcal{L}_\phi$, and the probability measures m_ϕ and μ_ϕ, all under the hypothesis that $\|\phi\|_{\log^q} < \infty$ for some $q > 2$. The semi-norms $\|\cdot\|_{\langle p,\alpha\rangle}$ and $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ were introduced in Sects. 3.4 and 3.5, respectively.

4.1 Some preliminary results. For positive real numbers q, M, and Ω with $q > 2$ and $\Omega < \log d$, consider the following set of weights

$$\mathcal{P}(q, M, \Omega) := \left\{\phi\colon \mathbb{P}^k \to \mathbb{R} \ : \ \|\phi\|_{\log^q} \leq M, \ \Omega(\phi) \leq \Omega\right\}$$

and the uniform topology induced by the sup norm. Observe that this family is equicontinuous. The following two lemmas were obtained in [BD23], see Lemmas 4.6 and 4.7 therein. Since we will use them several times in this section, we restate them here. We use the index ϕ or parameter ϕ for objects which depend on ϕ, e.g., we write λ_ϕ, $\mathcal{L}_\phi$, ρ_ϕ instead of λ, $\mathcal{L}$, ρ.

LEMMA 4.1. *Let q, M, and Ω be positive real numbers such that $q > 2$ and $\Omega < \log d$. The maps $\phi \mapsto \lambda_\phi$, $\phi \mapsto m_\phi$, $\phi \mapsto \mu_\phi$, and $\phi \mapsto \rho_\phi$ are continuous on $\phi \in \mathcal{P}(q, M, \Omega)$ with respect to the standard topology on $\mathbb{R}$, the weak topology on measures, and the uniform topology on functions. In particular, ρ_ϕ is bounded from above and below by positive constants which are independent of $\phi \in \mathcal{P}(q, M, \Omega)$. Moreover, $\|\lambda_\phi^{-n}\mathcal{L}_\phi^n\|_\infty$ is bounded by a constant which is independent of n and of $\phi \in \mathcal{P}(q, M, \Omega)$.*

LEMMA 4.2. *Let q, M, and Ω be positive real numbers such that $q > 2$ and $\Omega < \log d$. Let $\mathcal{F}$ be a uniformly bounded and equicontinuous family of real-valued functions on*

$\mathbb{P}^k$. Then the family

$$\left\{\lambda_\phi^{-n}\mathcal{L}_\phi^n(g)\ :\ n\geq 0,\ \phi\in\mathcal{P}(q,M,\Omega),\ g\in\mathcal{F}\right\}$$

is equicontinuous. Moreover, $\left\|\lambda_\phi^{-n}\mathcal{L}_\phi^n(g)-\langle m_\phi,g\rangle\right\|_\infty$ tends to 0 uniformly on $\phi\in\mathcal{P}(q,M,\Omega)$ and $g\in\mathcal{F}$ when n goes to infinity.

4.2 Main result and first step of the proof. The following is the main result of this section. We will use it in order to prove Theorem 1.2 with a suitable norm and another value of β.

Theorem 4.3. *Let f, ϕ, m_ϕ, ρ be as in Theorem 1.1, $\mathcal{L}$ the Perron-Frobenius operator associated to ϕ as in (1.1), and λ the associated scaling ratio as in (1.2). Let p, α, γ, A, Ω be positive constants and q_2 as in Lemma 3.14 such that $p>3/2$, $d^{-1}\leq\alpha<d^{-5/(2p+2)}$, $\Omega<\log(d\alpha)$, and $q_2>2$. Assume that $\|\phi\|_{\langle p,\alpha\rangle,\gamma}\leq A$ and $\Omega(\phi)\leq\Omega$. Then we have*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle,\gamma}\leq c,\quad \|\rho\|_{\langle p,\alpha\rangle,\gamma}\leq c,\quad \textit{and}\quad \|1/\rho\|_{\langle p,\alpha\rangle,\gamma}\leq c$$

for some positive constant $c=c(p,\alpha,\gamma,A,\Omega)$ independent of ϕ and n. Moreover, for every constant $0<\beta<1$ there is a positive integer $N=N(p,\alpha,\gamma,A,\Omega,\beta)$ independent of ϕ such that

$$\left\|\lambda^{-N}\mathcal{L}^N g\right\|_{\langle p,\alpha\rangle,\gamma}\leq\beta\,\|g\|_{\langle p,\alpha\rangle,\gamma} \tag{4.1}$$

for every function $g\colon\mathbb{P}^k\to\mathbb{R}$ with $\langle m_\phi,g\rangle=0$. Furthermore, there exists $N'=N'(p,\alpha,\gamma,\Omega)$ such that for any given constant $1<\delta<(d\alpha)^{\gamma/(2\gamma+2)}$, when A is small enough (depending on the choice of N') (4.1) holds with $N=N'$ and $\beta=\delta^{-N'}$.

Notice that Lemma 3.14 and the assumption $q_2>2$ imply that $\|\phi\|_{\log^q}<\infty$ for some $q>2$. Hence, the scaling ratio λ, the density function ρ, and the measures m_ϕ and μ_ϕ are well defined by Theorem 1.1. Notice also that $q_2>2$ implies that the condition $\alpha<d^{-5/(2p+2)}$ is automatically satisfied.

The proof of Theorem 4.3 will be reduced to a comparison between suitable currents and their norms. Theorem 4.3 will then follow from some interpolation techniques (see Sect. 4.3). A crucial estimate that we will need here is the following.

Proposition 4.4. *Let f be as in Theorem 1.1. Take $0<\alpha<1$ and $p>0$. Given n functions $\phi^{(j)}\colon\mathbb{P}^k\to\mathbb{R}$ for $j=1,\dots,n$, set*

$$\Phi_m:=\alpha^{-m}d^{(k-1)m}e^{\sum_{j=1}^m\max(\phi^{(j)})}\quad\textit{and}\quad\mathcal{L}_{m,n}:=\mathcal{L}_{\phi^{(m)}}\circ\cdots\circ\mathcal{L}_{\phi^{(n)}}.$$

Then there exists a positive constant $c = c(p,\alpha)$, independent of $\phi^{(j)}$, such that

$$\begin{aligned}\|\mathcal{L}_{1,n} g\|_{\langle p,\alpha\rangle} &\leq c\,\|\mathcal{L}_{1,n}\mathbb{1}\|_\infty^{1/2}\,\Phi_n^{1/2}\,\|g\|_{\langle p,\alpha\rangle}\\ &\quad + c\sum_{m=1}^{n}\left\|\phi^{(m)}\right\|_{\langle p,\alpha\rangle} m^{3/2}\Phi_m^{1/2}\,\|\mathcal{L}_{1,m}\mathbb{1}\|_\infty^{1/2}\,\|\mathcal{L}_{m+1,n}g\|_\infty\end{aligned}$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$.

Proof. By Definition 3.9 of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$, we need to bound the $\|\cdot\|_{p,\alpha}$-norm of the current $i\partial\mathcal{L}_{1,n}g\wedge\bar\partial\mathcal{L}_{1,n}g$.

Consider the product space $(\mathbb{P}^k)^{n+1}$ and denote by $(x_0,\ldots,x_n)$ its elements. Define the manifold $\Gamma_n \subset (\mathbb{P}^k)^{n+1}$ by

$$\Gamma_n := \Big\{(x, f(x),\ldots,f^n(x)) : x\in\mathbb{P}^k\Big\},$$

which can also be seen as the graph of the map $(f, f^2,\ldots,f^n)$ in the product space $(\mathbb{P}^k)^{n+1}$. Denote by π_n the restriction to Γ_n of the projection of $(\mathbb{P}^k)^{n+1}$ to its last component.

We have, using a direct computation,

$$\partial\Big(e^{\phi^{(n)}(x_0)+\cdots+\phi^{(1)}(x_{n-1})}g(x_0)\Big) = \Theta_1+\Theta_2$$

with

$$\Theta_1 := \mathbb{h}_0\partial g(x_0),\quad \Theta_2 := \mathbb{h}_0 g(x_0)\sum_{m=0}^{n-1}\partial\phi^{(n-m)}(x_m),\quad\text{and}$$

$$\mathbb{h}_0 := e^{\phi^{(n)}(x_0)+\cdots+\phi^{(1)}(x_{n-1})}.$$

Using Cauchy-Schwarz's inequality (see the Notation at the beginning of the paper), we obtain

$$\begin{aligned}i\partial\mathcal{L}_{1,n}g\wedge\bar\partial\mathcal{L}_{1,n}g &= i(\pi_n)_*(\Theta_1+\Theta_2)\wedge(\pi_n)_*(\overline{\Theta}_1+\overline{\Theta}_2)\\ &\leq 2i(\pi_n)_*(\Theta_1)\wedge(\pi_n)_*(\overline{\Theta}_1)+2i(\pi_n)_*(\Theta_2)\wedge(\pi_n)_*(\overline{\Theta}_2).\end{aligned}$$

We need to bound the norm $\|\cdot\|_{p,\alpha}$ of the two terms in the last sum by the square of the RHS of the inequality in the proposition.

For the first term, using again Cauchy-Schwarz's inequality, the definition of Φ_m as in the statement, and Lemma 3.8, we get (notice that $(\pi_n)_*(\mathbb{h}_0)=\mathcal{L}_{1,n}\mathbb{1}$)

$$\begin{aligned}\left\|i(\pi_n)_*(\Theta_1)\wedge(\pi_n)_*(\overline{\Theta}_1)\right\|_{p,\alpha} &\leq \left\|(\pi_n)_*(\mathbb{h}_0)(\pi_n)_*\Big(\mathbb{h}_0 i\partial g(x_0)\wedge\bar\partial g(x_0)\Big)\right\|_{p,\alpha}\\ &\leq \|\mathcal{L}_{1,n}\mathbb{1}\|_\infty\, e^{\sum_{j=1}^n \max\phi^{(j)}}\left\|(f^n)_*(i\partial g\wedge\bar\partial g)\right\|_{p,\alpha}\\ &\leq \|\mathcal{L}_{1,n}\mathbb{1}\|_\infty\,\Phi_n\left\|i\partial g\wedge\bar\partial g\right\|_{p,\alpha}.\end{aligned}$$

This gives the desired estimate for the first term.

For the second term, observe that $i(\pi_n)_*(\Theta_2)\wedge(\pi_n)_*(\overline{\Theta}_2)$ is equal to

$$\sum_{0\le m,m'<n} i(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\partial\phi^{(n-m)}(x_m)\Big)\wedge(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\overline{\partial}\phi^{(n-m')}(x_{m'})\Big)$$
$$\le 2\sum_{0\le m'\le m<n}\Big|i(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\partial\phi^{(n-m)}(x_m)\Big)\wedge(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\overline{\partial}\phi^{(n-m')}(x_{m'})\Big)\Big|,$$

see the Notation at the beginning of the paper for the meaning of the absolute value signs. Using Cauchy-Schwarz's inequality, we can bound the current in the absolute value signs by

$$(m-m'+1)^{-2} i(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\partial\phi^{(n-m)}(x_m)\Big)\wedge(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\overline{\partial}\phi^{(n-m)}(x_m)\Big)$$
$$+(m-m'+1)^{2} i(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\partial\phi^{(n-m')}(x'_m)\Big)\wedge(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\overline{\partial}\phi^{(n-m')}(x'_m)\Big)$$

and deduce that $i(\pi_n)_*(\Theta_2)\wedge(\pi_n)_*(\overline{\Theta}_2)$ is bounded by a constant times

$$\sum_{0\le m<n}(n-m)^3 i(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\partial\phi^{(n-m)}(x_m)\Big)\wedge(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\overline{\partial}\phi^{(n-m)}(x_m)\Big).$$

Therefore, in order to get the proposition, setting $\eta:=(\pi_n)_*\Big(\mathbb{h}_0 g(x_0)\partial\phi^{(n-m)}(x_m)\Big)$ we only need to show that

$$\|i\eta\wedge\overline{\eta}\|_{p,\alpha}\le\Big\|\phi^{(n-m)}\Big\|^2_{\langle p,\alpha\rangle}\Phi_{n-m}\,\|\mathcal{L}_{1,n-m}\mathbb{1}\|_\infty\,\|\mathcal{L}_{n-m+1,n}g\|^2_\infty.$$

Consider the map $\pi':\Gamma_n\to(\mathbb{P}^k)^{n-m+1}$ defined by $\pi'(x):=x':=(x_m,\dots,x_n)$. Denote by Γ' the image of Γ_n by π'. Consider also the map $\pi'':\Gamma'\to\mathbb{P}^k$ defined by $\pi''(x'):=x_n$. Both $\pi':\Gamma_n\to\Gamma'$ and $\pi'':\Gamma'\to\mathbb{P}^k$ are ramified coverings, respectively of degrees d^{km} and $d^{k(n-m)}$, and we have $\pi_n=\pi''\circ\pi'$.

With the notation as above, we see that

$$\eta=\pi''_*\Big(\mathcal{L}_{n-m+1,n}g(x_m)\mathbb{h}_m\partial\phi^{(n-m)}(x_m)\Big)\quad\text{with}\quad\mathbb{h}_m:=e^{\phi^{(n-m)}(x_m)+\cdots+\phi^{(1)}(x_{n-1})}.$$

It follows from Cauchy-Schwarz's inequality that

$$i\eta\wedge\overline{\eta}\le\|\mathcal{L}_{n-m+1,n}g\|^2_\infty\pi''_*(\mathbb{h}_m)\pi''_*\Big(\mathbb{h}_m i\partial\phi^{(n-m)}(x_m)\wedge\overline{\partial}\phi^{(n-m)}(x_m)\Big)$$
$$\le\|\mathcal{L}_{n-m+1,n}g\|^2_\infty\|\pi''_*(\mathbb{h}_m)\|_\infty\|\mathbb{h}_m\|_\infty(f^{n-m})_*\Big(i\partial\phi^{(n-m)}\wedge\overline{\partial}\phi^{(n-m)}\Big).$$

Thus, by Lemma 3.8 and the definition of π'', we get

$$\|i\eta\wedge\overline{\eta}\|_{p,\alpha}\le\|\mathcal{L}_{n-m+1,n}g\|^2_\infty\|\mathcal{L}_{1,n-m}\mathbb{1}\|_\infty\Phi_{n-m}\|i\partial\phi^{(n-m)}\wedge\overline{\partial}\phi^{(n-m)}\|_{p,\alpha}.$$

This ends the proof of the proposition. □

4.3 Proof of Theorem 4.3. We will need the following elementary lemma.

LEMMA 4.5. *Let $\phi, \phi^{(j)}, \psi \colon \mathbb{P}^k \to \mathbb{R}$ be such that $\sum_{j=1}^\infty \left\| \phi - \phi^{(j)} \right\|_\infty \le a$ for some positive constant a. Define $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ by $\vartheta(t) := t^{-1}(e^t - 1)$ and $\vartheta(0) = 1$, which is a smooth increasing function. Then we have*

(i) $\|\mathcal{L}_\phi - \mathcal{L}_\psi\|_\infty \le \vartheta(\|\phi - \psi\|_\infty) \|\mathcal{L}_\phi\|_\infty \|\phi - \psi\|_\infty$;
(ii) *for every $n \ge 1$,* $\left\| \mathcal{L}^n - \mathcal{L}_{\phi^{(1)}} \circ \cdots \circ \mathcal{L}_{\phi^{(n)}} \right\|_\infty \le \vartheta(a) a \|\mathcal{L}^n\|_\infty$.

Proof. Observe that for $\phi, \psi \colon \mathbb{P}^k \to \mathbb{R}$ we have, using the definition of $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$,

$$\|\mathcal{L}_\phi(g) - \mathcal{L}_\psi(g)\|_\infty \le \left\| (1 - e^{\psi - \phi}) \|g\|_\infty \right\|_\infty \|\mathcal{L}_\phi(\mathbb{1})\|_\infty = \left\| 1 - e^{\psi - \phi} \right\|_\infty \|\mathcal{L}_\phi\|_\infty \|g\|_\infty .$$

The first item in the lemma follows. For the second item, notice that

$$\begin{aligned}
&\left\| \left(\phi + \phi \circ f + \cdots + \phi \circ f^{n-1} \right) - \left(\phi^{(n)} + \phi^{(n-1)} \circ f + \cdots + \phi^{(1)} \circ f^{n-1} \right) \right\|_\infty \\
&\quad \le \sum_{j=1}^{n} \left\| \phi - \phi^{(j)} \right\|_\infty .
\end{aligned}$$

Therefore, using this estimate and the expansions of $\mathcal{L}^n(g)$ and $\mathcal{L}_{\phi^{(1)}} \circ \cdots \circ \mathcal{L}_{\phi^{(n)}}(g)$, we obtain the result in the same way as the first item. □

We continue the proof of Theorem 4.3 We first prove the following result.

PROPOSITION 4.6. *Under the hypotheses of Theorem 4.3, there exists a positive integer $N_0 = N_0(p, \alpha, \gamma, A, \Omega, \beta)$ independent of ϕ and g and such that (4.1) holds for all $N \ge N_0$.*

By subtracting from ϕ a constant, we can assume that ϕ belongs to the family of weights

$$\mathcal{Q}_0 := \left\{ \phi \colon \mathbb{P}^k \to \mathbb{R} \ : \ \min \phi = 0, \ \|\phi\|_{\langle p, \alpha \rangle, \gamma} \le A, \ \Omega(\phi) \le \Omega \right\}.$$

Observe that we can apply Lemmas 4.1 and 4.2 because, by Lemma 3.14 and the assumptions on α and p, the family $\mathcal{Q}_0$ is contained in $\mathcal{P}_0(q, M, \Omega) := \{\phi \in \mathcal{P}(q, M, \Omega) \colon \min \phi = 0\}$ for suitable $q > 2$ and M. Observe also that $\|\phi\|_\infty = \Omega(\phi) \le \Omega$ and $\|\phi\|_\infty \lesssim \|\phi\|_{\langle p, \alpha \rangle, \gamma} \le A$.

Consider two constants $K \ge 1$ and $K' \ge 1$ whose values will be specialised later, depending on β. We will not fix A but we assume $A \le A_0$ for some fixed constant $A_0 > 0$. For a large part of this section we can take $A = A_0$, but at the end of the proof of Theorem 4.3 we will consider $A \to 0$. This is the reason why we will keep the constant A in the estimates below. Note that the constants hidden in the signs $\lesssim$ below are independent of the parameters A, β, K, K', n and also of the constant $0 < \epsilon \le 1$ and the integer j that we consider now.

Since $\|\phi\|_{\langle p,\alpha\rangle,\gamma} \leq A$, for every $j \geq 1$ there are functions $\phi^{(j)}$ and $\psi^{(j)}$ such that

$$\phi = \phi^{(j)} + \psi^{(j)}, \quad \left\|\phi^{(j)}\right\|_{\langle p,\alpha\rangle} \leq A(Kj^2)^{1/\gamma}(1/\epsilon)^{1/\gamma}, \quad \text{and} \quad \left\|\psi^{(j)}\right\|_\infty \leq AK^{-1}j^{-2}\epsilon. \tag{4.2}$$

Observe that $\|\phi^{(j)}\|_\infty$ is bounded by a constant since $\|\phi\|_\infty$ is bounded by a constant.

We can assume for simplicity that $\|g\|_{\langle p,\alpha\rangle,\gamma} \leq 1$, which implies that $\Omega(g)$ is bounded by a constant. Since $\langle m_\phi, g\rangle = 0$ by hypothesis, we deduce that $\|g\|_\infty$ is bounded by a constant. By the definition of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$, we can find two functions $g_\epsilon^{(1)}$ and $g_\epsilon^{(2)}$ satisfying

$$g = g_\epsilon^{(1)} + g_\epsilon^{(2)}, \quad \left\|g_\epsilon^{(1)}\right\|_{\langle p,\alpha\rangle} \leq K'^{1/\gamma}(1/\epsilon)^{1/\gamma}, \quad \left\|g_\epsilon^{(2)}\right\|_\infty \leq 2K'^{-1}\epsilon,$$
$$\langle m_\phi, g_\epsilon^{(1)}\rangle = \langle m_\phi, g_\epsilon^{(2)}\rangle = 0.$$

Notice that without the condition $\langle m_\phi, g_\epsilon^{(2)}\rangle = 0$ we would not need the coefficient 2 in the above estimate of $\|g_\epsilon^{(2)}\|_\infty$. We obtain this condition by adding to $g_\epsilon^{(2)}$ a suitable constant and subtracting the same constant from $g_\epsilon^{(1)}$. The condition $\langle m_\phi, g_\epsilon^{(1)}\rangle = 0$ is deduced from the hypothesis $\langle m_\phi, g\rangle = 0$ when we have $\langle m_\phi, g_\epsilon^{(2)}\rangle = 0$. Since $\|g\|_\infty$ is bounded by a constant, $\|g_\epsilon^{(1)}\|_\infty$ is also bounded by a constant.

Define as above $\mathcal{L}_{m,n} := \mathcal{L}_{\phi^{(m)}} \circ \cdots \circ \mathcal{L}_{\phi^{(n)}}$, where the $\phi^{(j)}$'s are as in (4.2), and write

$$\lambda^{-n}\mathcal{L}^n g = \lambda^{-n}\mathcal{L}_{1,n}g_\epsilon^{(1)} + \lambda^{-n}\Big(\mathcal{L}^n g_\epsilon^{(1)} - \mathcal{L}_{1,n}g_\epsilon^{(1)}\Big) + \lambda^{-n}\mathcal{L}^n g_\epsilon^{(2)} =: G_{n,\epsilon}^{(a)} + G_{n,\epsilon}^{(b)} + G_{n,\epsilon}^{(c)}. \tag{4.3}$$

Lemma 4.7. *When K and K' are large enough, we have for every $n \geq 1$*

$$\left\|G_{n,\epsilon}^{(b)}\right\|_\infty \leq \frac{1}{2}\beta\epsilon \quad \text{and} \quad \left\|G_{n,\epsilon}^{(c)}\right\|_\infty \leq \frac{1}{2}\beta\epsilon.$$

Proof. The above estimate on $\|\psi^{(j)}\|_\infty$ implies that $\sum \|\psi^{(j)}\|_\infty \lesssim AK^{-1}\epsilon$. Therefore, using Lemma 4.5 and the fact that the sequence $\lambda^{-n}\mathcal{L}^n\mathbb{1}$ is bounded uniformly on n and ϕ (see Lemma 4.1), we get

$$\left\|G_{n,\epsilon}^{(b)}\right\|_\infty \lesssim AK^{-1}\epsilon\|g_\epsilon^{(1)}\|_\infty \lesssim AK^{-1}\epsilon \leq A_0K^{-1}\epsilon \tag{4.4}$$

because $\|g_\epsilon^{(1)}\|_\infty$ is bounded by a constant. So we get the first estimate in the lemma when K is large enough (depending on A_0 and β).

For the second estimate, using again that the sequence $\lambda^{-n}\mathcal{L}^n\mathbb{1}$ is uniformly bounded, we obtain

$$\left\|G_{n,\epsilon}^{(c)}\right\|_\infty \lesssim \|g_\epsilon^{(2)}\|_\infty \leq 2K'^{-1}\epsilon. \tag{4.5}$$

The result follows provided that K' is large enough. $\square$

LEMMA 4.8. *When $K \geq 1$ and K' are fixed, there is a constant $0 < \epsilon_0 \leq 1$ independent of ϕ and g such that, for all $0 < \epsilon \leq \epsilon_0$ and all n large enough, also independent of ϕ and g,*

$$\left\| G_{n,\epsilon}^{(a)} \right\|_{\langle p,\alpha \rangle} \leq \beta (1/\epsilon)^{1/\gamma}.$$

Proof. Fix an $\epsilon_0 > 0$ small enough. We will apply Proposition 4.4.

First recall that, by the definition of the transfer operator $\mathcal{L}$ and the positivity of the continuous function ρ satisfying $\lambda\rho = \mathcal{L}\rho$, we have $\lambda \geq d^k e^{\min \phi}$. Indeed, denoting by y_0 a point of minimum of ρ and by $x_1, \ldots, x_{d^k}$ its preimages (counting multiplicities), we have

$$\lambda \rho(y_0) = \sum_\ell e^{\phi(x_\ell)} \rho(x_\ell) \geq d^k e^{\min \phi} \rho(y_0),$$

which gives the desired inequality.

Using the definition of the $\phi^{(j)}$'s and the $\psi^{(j)}$'s, and the estimate $\sum \|\psi^{(j)}\|_\infty \lesssim AK^{-1}\epsilon \leq A_0$, we obtain

$$\begin{aligned}\Phi_m &= \alpha^{-m} d^{(k-1)m} e^{\sum_{j=1}^m \max \phi^{(j)}} \leq \alpha^{-m} d^{(k-1)m} e^{m \max \phi + \sum_{j=1}^m \|\psi^{(j)}\|_\infty} \\ &\lesssim \alpha^{-m} d^{(k-1)m} e^{m \max \phi} \lesssim \alpha^{-m} d^{-m} \lambda^m e^{m\Omega(\phi)} \leq \alpha^{-m} d^{-m} \lambda^m e^{m\Omega}.\end{aligned}$$

By Lemmas 4.5 and 4.1, we have

$$\|\mathcal{L}_{1,m}\mathbb{1}\|_\infty \lesssim \|\mathcal{L}^m \mathbb{1}\|_\infty \lesssim \lambda^m \quad \text{and} \quad \|\mathcal{L}_{m+1,n}\mathbb{1}\|_\infty \lesssim \|\mathcal{L}^{n-m}\mathbb{1}\|_\infty \lesssim \lambda^{n-m}$$

and also, again by Lemma 4.5,

$$\begin{aligned}\|\mathcal{L}_{m+1,n} g_\epsilon^{(1)}\|_\infty &\leq \|\mathcal{L}^{n-m} g_\epsilon^{(1)}\|_\infty + \|\mathcal{L}_{m+1,n} g_\epsilon^{(1)} - \mathcal{L}^{n-m} g_\epsilon^{(1)}\|_\infty \\ &\lesssim \|\mathcal{L}^{n-m} g_\epsilon^{(1)}\|_\infty + \|\mathcal{L}^{n-m}\mathbb{1}\|_\infty \|g_\epsilon^{(1)}\|_\infty AK^{-1}\epsilon \\ &\lesssim \|\mathcal{L}^{n-m} g\|_\infty + \|\mathcal{L}^{n-m} g_\epsilon^{(2)}\|_\infty + \lambda^{n-m} AK^{-1}\epsilon \\ &\lesssim \|\mathcal{L}^{n-m} g\|_\infty + \lambda^{n-m} K'^{-1}\epsilon + \lambda^{n-m} AK^{-1}\epsilon.\end{aligned}$$

This, Proposition 4.4 (applied with $g_\epsilon^{(1)}$ instead of g), and the estimates in the definitions of $g_\epsilon^{(1)}$ and $\phi^{(j)}$ allow us to bound $\left\| G_{n,\epsilon}^{(a)} \right\|_{\langle p,\alpha \rangle}$ by a constant times

$$\begin{aligned}&\Big[K'^{1/\gamma} \Big(\frac{e^\Omega}{d\alpha} \Big)^{n/2} \\ &\quad + \sum_{m=1}^n AK^{1/\gamma} m^{2/\gamma + 3/2} \Big(\frac{e^\Omega}{d\alpha} \Big)^{m/2} \Big(\|\lambda^{-n+m} \mathcal{L}^{n-m} g\|_\infty + (K'^{-1} + AK^{-1})\epsilon_0 \Big) \Big] \epsilon^{1/\gamma}.\end{aligned} \tag{4.6}$$

Recall that g belongs to a uniformly bounded and equicontinuous family of functions, see Lemma 3.14. It follows from Lemma 4.2 that $\|\lambda^{-n+m}\mathcal{L}^{n-m} g\|_\infty$ tends to 0, uniformly on ϕ and g, when $n-m$ tends to infinity. This and the fact that $e^\Omega < d\alpha$ imply that, when n tends to infinity, the sum between brackets in (4.6) converges to

$$\sum_{m=1}^{\infty} AK^{1/\gamma} m^{2/\gamma+3/2}\Big(\frac{e^\Omega}{d\alpha}\Big)^{m/2}(K'^{-1}+AK^{-1})\epsilon_0.$$

This sum is smaller than β because ϵ_0 is chosen small enough. Therefore, we get the estimate in the lemma for n large enough, independently of ϕ and g, because the last convergence is uniform on ϕ and g. □

Proof of Proposition 4.6. Take N large enough, independent of ϕ. It suffices to show that, for every $0<\epsilon\leq 1$, we can write

$$\lambda^{-N}\mathcal{L}^N g = G^{(1)}_{N,\epsilon} + G^{(2)}_{N,\epsilon} \quad \text{with} \quad \Big\|G^{(1)}_{N,\epsilon}\Big\|_{\langle p,\alpha\rangle} \leq \beta(1/\epsilon)^{1/\gamma} \quad \text{and} \quad \Big\|G^{(2)}_{N,\epsilon}\Big\|_\infty \leq \beta\epsilon.$$

We apply Lemmas 4.7 and 4.8 to $n:=N$. When $\epsilon\leq\epsilon_0$, it is enough to choose $G^{(1)}_{N,\epsilon} := G^{(a)}_{N,\epsilon}$ and $G^{(2)}_{N,\epsilon} := G^{(b)}_{N,\epsilon} + G^{(c)}_{N,\epsilon}$. Assume now that $\epsilon_0\leq\epsilon\leq 1$ and choose $G^{(1)}_{N,\epsilon}:=0$ and $G^{(2)}_{N,\epsilon} := \lambda^{-N}\mathcal{L}^N g$. With N large enough, we have $\|G^{(2)}_{N,\epsilon}\|_\infty \leq \beta\epsilon_0\leq\beta\epsilon$ because $\|\lambda^{-n}\mathcal{L}^n g\|_\infty$ tends to 0 uniformly on ϕ and g when n goes to infinity, see Lemma 4.2. Thus, we have the desired decomposition of $\lambda^{-N}\mathcal{L}^N g$ and hence the property (4.1) for all N large enough. □

Proposition 4.9. *Under the hypotheses of Theorem 4.3, there is a positive constant $c=c(p,\alpha,\gamma,A,\Omega)$ independent of ϕ and n such that*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle,\gamma}\leq c, \quad \|\rho\|_{\langle p,\alpha\rangle,\gamma}\leq c, \quad \textit{and} \quad \|1/\rho\|_{\langle p,\alpha\rangle,\gamma}\leq c.$$

Proof. We prove the first inequality. It is enough to consider only n large enough. We will use the above computations for $K=K'=\epsilon_0=1$. Consider any function $g\colon \mathbb{P}^k\to\mathbb{R}$ such that $\|g\|_{\langle p,\alpha\rangle,\gamma}\leq 1$. We do not assume that $\langle m_\phi, g\rangle = 0$. As before, for any $0<\epsilon\leq 1$ we can write

$$g = g^{(1)}_\epsilon + g^{(2)}_\epsilon \quad \text{with} \quad \Big\|g^{(1)}_\epsilon\Big\|_{\langle p,\alpha\rangle} \leq (1/\epsilon)^{1/\gamma} \quad \text{and} \quad \Big\|g^{(2)}_\epsilon\Big\|_\infty \leq \epsilon.$$

We also consider as above the decomposition

$$\lambda^{-n}\mathcal{L}^n g = G^{(1)}_{n,\epsilon} + G^{(2)}_{n,\epsilon} \quad \text{with} \quad G^{(1)}_{n,\epsilon} := G^{(a)}_{n,\epsilon} \quad \text{and} \quad G^{(2)}_{n,\epsilon} := G^{(b)}_{n,\epsilon} + G^{(c)}_{n,\epsilon},$$

see (4.3). The computations in Lemmas 4.7 and 4.8 give $\|G^{(2)}_{n,\epsilon}\|_\infty \lesssim \epsilon$ and $\|G^{(1)}_{n,\epsilon}\|_{\langle p,\alpha\rangle} \lesssim \epsilon^{1/\gamma}$, for n large enough. We use here the fact that $\|\lambda^{-n+m}\mathcal{L}^{n-m}g\|_\infty$ is bounded by a constant, see Lemma 4.2. Therefore, $\|\lambda^{-n}\mathcal{L}^n g\|_{\langle p,\alpha\rangle,\gamma}$ is bounded by a constant. Thus, the first inequality in the proposition holds.

Consider now the second inequality. Observe that

$$\rho = \lim_{n\to\infty}\lambda^{-n}\mathcal{L}^n\mathbb{1} = \mathbb{1} + \sum_{n=0}^{\infty}\lambda^{-n}\mathcal{L}^n g \quad \text{with} \quad g:=\lambda^{-1}\mathcal{L}\mathbb{1}-\mathbb{1}.$$

The $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ implies that $\langle m_\phi, g\rangle = 0$. Fix an integer N as in Proposition 4.6, and write $n = mN + m'$, with $m, m' \in \mathbb{N}$ and $m' < N$. Proposition 4.6 and the first inequality in the present proposition imply that $\|\lambda^{-n}\mathcal{L}^n g\|_{\langle p,\alpha\rangle,\gamma} \lesssim \beta^m \lesssim \beta^{n/N}$. We deduce that $\|\rho\|_{\langle p,\alpha\rangle,\gamma}$ is bounded by a constant.

For the last inequality in the lemma, observe that ρ is bounded from above and below by positive constants which are independent of ϕ, see Lemma 4.1. The result is then a consequence of Lemma 3.14 applied to the function $\chi(t) := 1/t$. The proof is complete. □

End of the proof of Theorem 4.3. By Propositions 4.6 and 4.9, it only remains to prove the last assertion in this theorem. We continue to use the computations in Lemmas 4.7 and 4.8 and take $K = 1$ and $\epsilon_0 = 1$. We also fix δ' and d' such that $\delta < \delta' = d'^{\gamma/(2\gamma+2)}$ and $d' < d\alpha$. The constant K' will be chosen below. Recall that the implicit constants in (4.4), (4.5), and (4.6) are independent of N, A, β, K, K', ϵ. As above, for every N sufficiently large we consider the decomposition

$$\lambda^{-N}\mathcal{L}^N g = G^{(1)}_{N,\epsilon} + G^{(2)}_{N,\epsilon} \quad \text{with} \quad G^{(1)}_{N,\epsilon} := G^{(a)}_{N,\epsilon} \quad \text{and} \quad G^{(2)}_{N,\epsilon} := G^{(b)}_{N,\epsilon} + G^{(c)}_{N,\epsilon}.$$

Take $A \to 0$, which also implies that $\Omega(\phi) \to 0$. So we can fix an Ω as small as needed and assume that $d' < e^{-\Omega} d\alpha$. Then, the estimates (4.4) and (4.5) in Lemma 4.7 and (4.6) in Lemma 4.8 give

$$\|G^{(1)}_{N,\epsilon}\|_{\langle p,\alpha\rangle} \leq c\Big(K'^{1/\gamma} d'^{-N/2} + A\Big)\epsilon^{1/\gamma} \quad \text{and} \quad \|G^{(2)}_{N,\epsilon}\|_\infty \leq c(K'^{-1} + A)\epsilon$$

for every N sufficiently large, where c is a positive constant independent of N, A, K', ϵ. With N large enough and A small enough, setting $K' = \delta'^N$ and since $\delta < \delta'$, we get

$$\|G^{(1)}_{N,\epsilon}\|_{\langle p,\alpha\rangle} \leq \delta^{-N}\epsilon^{1/\gamma} \quad \text{and} \quad \|G^{(2)}_{N,\epsilon}\|_\infty \leq \delta^{-N}\epsilon.$$

In other words, fixing N is large enough we can take $\beta = \delta^{-N}$ when A is small enough. This completes the proof of the theorem. □

4.4 Proof of Theorem 1.2. The statement is a consequence of Theorem 4.3, namely, of the estimate (4.1). Note that the constant β in Theorem 1.2 is not the one in (4.1). Given ϕ as in the statement, we first choose α sufficiently close to 1 so that $\Omega(\phi) < \log(\alpha d)$. Then, we choose p large enough so that $q < q_2$, where q and γ are as in the statement and q_2 is defined in Lemma 3.14 (this also implies that $\alpha < d^{-5/(2p+2)}$ since $q > 2$). Recall that the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ is almost a norm. Define

$$\|\cdot\|_{\diamond_1} := \|\cdot\|_\infty + \|\cdot\|_{\langle p,\alpha\rangle,\gamma}\,.$$

This is now a norm, which is independent of ϕ. By Lemmas 3.14 and 2.4, we have

$$\|\cdot\|_\infty + \|\cdot\|_{\log^q} \lesssim \|\cdot\|_{\diamond_1} \lesssim \|\cdot\|_{\mathcal{C}^\gamma}\,.$$

By Lemma 4.1, the quantities $\|\lambda^{-n}\mathcal{L}^n\|_\infty$, $\|\rho\|_\infty$, and $\|1/\rho\|_\infty$ are bounded by a constant independent of ϕ when $\|\phi\|_{\diamond_1} \le A$. By Theorem 4.3, $\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle,\gamma}$, $\|\rho\|_{\langle p,\alpha\rangle,\gamma}$, and $\|1/\rho\|_{\langle p,\alpha\rangle,\gamma}$ are also bounded by a constant independent of ϕ. We deduce that $\|\lambda^{-n}\mathcal{L}^n\|_{\diamond_1}$, $\|\rho\|_{\diamond_1}$, and $\|1/\rho\|_{\diamond_1}$ satisfy the same property.

Let N and β_0 be as in Theorem 4.3 (we write β_0 instead of β to distinguish it from the constant that we use now for Theorem 1.2). Fix a constant β such that $\beta_0^{1/N} < \beta < 1$ and consider the following norms

$$\|g\|_\diamond := |c_g| + \|g'\|_{\langle p,\alpha\rangle,\gamma} \quad \text{and} \quad \|g\|_{\diamond_2} := |c_g| + \sum_{n=0}^{\infty} \beta^{-n} \left\|\lambda^{-n}\mathcal{L}^n g'\right\|_{\langle p,\alpha\rangle,\gamma}$$

for every function $g\colon \mathbb{P}^k \to \mathbb{R}$, where $c_g := \langle m_\phi, g\rangle$ and $g' := g - c_g\rho$.

Lemma 4.10. *We have $\|gh\|_{\diamond_1} \le 3\|g\|_{\diamond_1}\|h\|_{\diamond_1}$ for all functions $g,h\colon \mathbb{P}^k \to \mathbb{R}$. Moreover, both of the norms $\|\cdot\|_\diamond$ and $\|\cdot\|_{\diamond_2}$ are equivalent to $\|\cdot\|_{\diamond_1}$.*

Proof. The first assertion is a direct consequence of Lemma 3.15. We prove now the second assertion. Since m_ϕ is a probability measure and $\|\rho\|_{\langle p,\alpha\rangle,\gamma}$ is bounded, we have

$$\begin{aligned}\|g\|_\diamond &= |c_g| + \|g - c_g\rho\|_{\langle p,\alpha\rangle,\gamma} \le |c_g| + \|g\|_{\langle p,\alpha\rangle,\gamma} + |c_g|\|\rho\|_{\langle p,\alpha\rangle,\gamma}\\ &\lesssim \|g\|_\infty + \|g\|_{\langle p,\alpha\rangle,\gamma} = \|g\|_{\diamond_1}.\end{aligned}$$

Conversely, assume that $\|g\|_\diamond \le 1$, then $|c_g| \le 1$ and $\|g'\|_{\langle p,\alpha\rangle,\gamma} \le 1$. It follows that $\|g\|_{\langle p,\alpha\rangle,\gamma}$ is bounded by a constant because it is bounded by $\|g'\|_{\langle p,\alpha\rangle,\gamma} + |c_g|\,\|\rho\|_{\langle p,\alpha\rangle,\gamma}$. By Lemma 3.14, $\Omega(g)$ is also bounded by a constant. This and the inequality $|\langle m_\phi, g\rangle| = |c_g| \le 1$ imply that $\|g\|_\infty$ is bounded by a constant. We deduce that $\|\cdot\|_\diamond$ is equivalent to $\|\cdot\|_{\diamond_1}$.

Observe that $\|\cdot\|_\diamond \le \|\cdot\|_{\diamond_2}$. To complete the proof, it is enough to show that $\|g\|_{\diamond_2} \lesssim \|g\|_\diamond$ for every function g. Recall that ρ is invariant by $\lambda^{-1}\mathcal{L}$ and $\langle m_\phi, \rho\rangle = 1$. Therefore, we have $\langle m_\phi, g'\rangle = 0$. Theorem 4.3 and Proposition 4.9 imply that $\|\lambda^{-n}\mathcal{L}^n g'\|_{\langle p,\alpha\rangle,\gamma} \lesssim \beta_0^{n/N}\|g'\|_{\langle p,\alpha\rangle,\gamma}$ for every N such that (4.1) holds. Hence

$$\|g\|_{\diamond_2} \lesssim |c_g| + \|g'\|_{\langle p,\alpha\rangle,\gamma} \sum_{n=0}^{\infty} \beta^{-n}\beta_0^{n/N} \lesssim |c_g| + \|g'\|_{\langle p,\alpha\rangle,\gamma} = \|g\|_\diamond.$$

The last infinite sum is finite because $\beta > \beta_0^{1/N}$. This ends the proof of the lemma. □

Consider now a function g with $c_g = \langle m_\phi, g\rangle = 0$, which implies $g = g'$. We also have $\langle m_\phi, \lambda^{-1}\mathcal{L}g\rangle = 0$ because m_ϕ is invariant by $\lambda^{-1}\mathcal{L}^*$. From the definition of $\|\cdot\|_{\diamond_2}$, we get

$$\left\|\lambda^{-1}\mathcal{L}g\right\|_{\diamond_2} = \beta(\|g\|_{\diamond_2} - \|g\|_\diamond) \le \beta\,\|g\|_{\diamond_2}.$$

This is the desired contraction. Finally, the last assertion in Theorem 1.2 is a direct consequence of the last assertion in Theorem 4.3 by taking α close enough to 1. The proof of Theorem 1.2 is now complete.

4.5 Spectral gap in the limit case. The semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$ can be seen as the limit of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$ as γ goes to infinity, see Remark 3.13. In order to complete our study, we will prove here a spectral gap with respect to this limit norm. The following is an analogue of Theorem 4.3.

Theorem 4.11. *Let f, ϕ, λ, m_ϕ, ρ be as in Theorem 1.1 and $\mathcal{L}$ the Perron-Frobenius operator associated to ϕ. Let p, α, A, Ω be positive constants and q_1 as in Lemma 3.10 such that $p>3/2$, $d^{-1}\le\alpha<d^{-5/(2p+2)}$, $\Omega<\log(d\alpha)$, and $q_1>2$. Assume that $\|\phi\|_{\langle p,\alpha\rangle}\le A$ and $\Omega(\phi)\le\Omega$. Then we have*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle}\le c,\quad \|\rho\|_{\langle p,\alpha\rangle}\le c,\quad \text{and}\quad \|1/\rho\|_{\langle p,\alpha\rangle}\le c$$

for some positive constant $c=c(p,\alpha,A,\Omega)$ independent of ϕ and n. Moreover, for every constant $0<\beta<1$ there is a positive integer $N=N(p,\alpha,A,\Omega,\beta)$ independent of ϕ such that

$$\left\|\lambda^{-N}\mathcal{L}^N g\right\|_{\langle p,\alpha\rangle}\le\beta\,\|g\|_{\langle p,\alpha\rangle}$$

for every function $g\colon\mathbb{P}^k\to\mathbb{R}$ with $\langle m_\phi,g\rangle=0$. Furthermore, for every given constant $1<\delta<(d\alpha)^{1/2}$, when A is small enough we can take $\beta=\delta^{-N}$.

Notice that Lemma 3.10 and the assumption $q_1>2$ imply that $\|\phi\|_{\log^q}$ is finite for some $q>2$. Hence, the scaling ratio λ, the density function ρ, and the measures m_ϕ and μ_ϕ are well defined by Theorem 1.1. Notice also that $q_1>2$ implies that the condition $\alpha<d^{-5/(2p+2)}$ is automatically satisfied.

Proof. The proof follows the same lines as the one of Theorem 4.3. It is however simpler because the definition of the semi-norm $\|\cdot\|_{\langle p,\alpha\rangle}$ is simpler than the one of $\|\cdot\|_{\langle p,\alpha\rangle,\gamma}$. In particular, we do not need any decomposition of $\lambda^{-n}\mathcal{L}^n g$. Applying directly Proposition 4.4 with $\phi^{(j)}:=\phi$ for all $j\ge1$ and recalling that $\|\mathbb{1}_n\|_\infty\lesssim\lambda^n$ we obtain

$$\|\lambda^{-n}\mathcal{L}^n g\|_{\langle p,\alpha\rangle}\lesssim\|g\|_{\langle p,\alpha\rangle}\Big(\frac{e^\Omega}{d\alpha}\Big)^{n/2}+\|\phi\|_{\langle p,\alpha\rangle}\sum_{m=1}^{n}m^{3/2}\Big(\frac{e^\Omega}{d\alpha}\Big)^{m/2}\|\lambda^{-n+m}\mathcal{L}^{n-m}g\|_\infty.$$

With this estimate, the rest of the proof is the same as that of Theorem 4.3. □

As in the last section, we obtain the following counterpart of Theorem 1.2 as a consequence of the last result.

Theorem 4.12. *Let f, p, α, A, Ω, ϕ, ρ, λ, m_ϕ and $\mathcal{L}$ be as in Theorem 4.11. Then there is an explicit norm $\|\cdot\|_{\diamond_0}$, depending on f, p, α, ϕ and equivalent to $\|\cdot\|_\infty+\|\cdot\|_{\langle p,\alpha\rangle}$, such that when $\|\phi\|_{\langle p,\alpha\rangle}\le A$ and $\Omega(\phi)\le\Omega$ we have*

$$\|\lambda^{-n}\mathcal{L}^n\|_{\langle p,\alpha\rangle}\le c,\quad \|\rho\|_{\langle p,\alpha\rangle}\le c,\quad \|1/\rho\|_{\langle p,\alpha\rangle}\le c,\quad \text{and}\quad \left\|\lambda^{-1}\mathcal{L}g\right\|_{\diamond_0}\le\beta\,\|g\|_{\diamond_0}$$

for every $g\colon\mathbb{P}^k\to\mathbb{R}$ with $\langle m_\phi,g\rangle=0$, and for some positive constants $c=c(f,p,\alpha,A,\Omega)$ and $\beta=\beta(f,p,\alpha,A,\Omega)$ with $\beta<1$, both independent of ϕ, n, and g.

Furthermore, given any constant $1<\delta<(d\alpha)^{1/2}$, when A is small enough, the norm $\|\cdot\|_{\diamond_0}$ can be chosen so that we can take $\beta=1/\delta$.

Note that Lipschitz functions have finite $\|\cdot\|_{\langle p,\alpha\rangle}$ semi-norm (this follows from Lemma 3.10, since Lipschitz functions can be uniformly approximated by $\mathcal{C}^1$ ones whose norm is dominated by the Lipschitz constant, see also the proof of Lemma 3.14). So the last theorem can be applied to Lipschitz functions. For such functions we can take any p large enough and α close to 1. The rate of contraction is then almost equal to $d^{-1/2}$ when A is small enough (i.e., when ϕ is close to a constant function). This rate is likely optimal as it corresponds to known results obtained in the setting of zero weight, see [DS101].

5 Statistical properties of equilibrium states

Theorem 1.3 follows from Theorem 1.2 by means of more standard arguments. We work under the hypotheses of Theorems 1.2 and 1.3 and with the equivalent norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ as in Theorem 1.2, see Sect. 4.4. As the arguments are mostly classical, we will only refer to the existence literature for the details.

5.1 Exponential equidistribution of preimages of points. The following consequence of Theorem 1.2 gives a quantitative version of the equidistribution of preimages in Theorem 1.1. Because of Lemma 2.4 and of the definition of the norm $\|\cdot\|_{\diamond_1}$, it applies in particular to Hölder continuous test functions, see Remark 3.13.

Theorem 5.1. *Under the hypotheses of Theorem 1.2, for every $x\in\mathbb{P}^k$, as n tends to infinity the points in $f^{-n}(x)$, with suitable weights, are equidistributed exponentially fast with respect to the conformal measure m_ϕ. More precisely, we have*

$$\Big|\Big\langle \lambda^{-n}\sum_{f^n(a)=x} e^{\phi(a)+\cdots+\phi(f^{n-1}(a))}\delta_a-\rho(x)m_\phi, g\Big\rangle\Big|\le c\beta^n\|g\|_{\diamond_1},$$

for all $g\colon\mathbb{P}^k\to\mathbb{R}$ of finite $\|\cdot\|_{\diamond_1}$-norm, where $0<\beta<1$ is the constant in Theorem 1.2 and c is a positive constant independent of x and g.

5.2 Multiple decorrelation. The speed of mixing in Theorem 1.1 is not controlled. We establish here some uniform exponential bound for the speed of mixing of the system $(\mathbb{P}^k,f,\mu_\phi)$ for more regular observables. The following property is implied by the spectral gap of the transfer operator, see, e.g., [Gou07, Lem. B.2] and [Pen02].

Theorem 5.2 (Multiple decorrelation). *Under the hypotheses of Theorem 1.2, for all integers $m,m'\ge 0$, there is a positive constant $c=c(m,m')$ such that, for every $N\in\mathbb{N}$, every increasing sequences $(k_j)_{1\le j\le m}$ and $(l_j)_{1\le j\le m'}$, and all functions*

$g_1, \dots, g_m, h_1, \dots, h_{m'} \colon \mathbb{P}^k \to \mathbb{R}$ *of finite* $\|\cdot\|_{\diamond_1}$*-norm we have*

$$\Big|\langle \mu_\phi, \prod_{j=1}^m g_j \circ f^{k_j} \cdot \prod_{j=1}^{m'} h_j \circ f^{N+l_j}\rangle - \langle \mu_\phi, \prod_{j=1}^m g_j \circ f^{k_j}\rangle \cdot \langle \mu_\phi, \prod_{j=1}^{m'} h_j \circ f^{l_j}\rangle\Big| \\ \le c\beta^{N-k_m}\Big(\prod_{j=1}^m \|g_j\|_{\diamond_1}\Big) \cdot \Big(\prod_{j=1}^{m'} \|h_j\|_{\diamond_1}\Big).$$

Here, the constant $0<\beta<1$ *is the one from Theorem* 1.2.

As a consequence of Theorem 5.2 we have the following quantitative version of the mixing in Theorem 1.1. In the case of Hölder continuous weight ϕ and observable g, this was established in [Hay99] for $k=1$ (see also [DPU96] for a uniform sub-exponential speed) and in [SUZ14] for $k>1$, see also [DS101] for the case when ϕ is constant.

Corollary 5.3 (Exponential mixing of all orders). *Under the hypotheses of Theorem* 1.2, *for every integer* $r \ge 0$, *there is a positive constant* $c = c(r)$ *such that, for all functions* $g_0, \dots, g_r \colon \mathbb{P}^k \to \mathbb{R}$ *of finite* $\|\cdot\|_{\diamond_1}$*-norm and integers* $0 =: n_0 \le n_1 \le \dots \le n_r$, *we have*

$$\Big|\,\langle \mu_\phi, g_0\,(g_1 \circ f^{n_1}) \dots (g_r \circ f^{n_r})\rangle - \prod_{j=0}^r \langle \mu_\phi, g_j\rangle\,\Big| \le c\beta^n\Big(\prod_{j=0}^{r-1}\|g_j\|_{\diamond_1}\Big)\,\|g_r\|_{L^1(\mu_\phi)},$$

where $n := \min_{0\le j<r}(n_{j+1}-n_j)$ *and the constant* $0<\beta<1$ *is the one from Theorem* 1.2.

5.3 Properties of perturbed Perron-Frobenius operators. The next statistical properties will be proved by means of spectral methods, and more precisely by the introduction of suitable (complex) perturbations of the operator $\mathcal{L} = \mathcal{L}_\phi$. This method was originally developed by Nagaev [Nag57] in the context of Markov chains.

Definition 5.4. Given functions $\phi, g \colon \mathbb{P}^k \to \mathbb{R}$, $h \colon \mathbb{P}^k \to \mathbb{C}$, and a parameter $\theta \in \mathbb{C}$ we set

$$\mathcal{L}_{\phi+\theta g} h := \mathcal{L}_{\phi+\theta g}\Re h + i\mathcal{L}_{\phi+\theta g}\Im h,$$

where the operator in the RHS is the linear extension of (1.1) in the case with complex weight.

Since from now on we fix ϕ and g, we will just denote the above operator by $\mathcal{L}_{[\theta]}$ when no possible confusion arises. In particular, we have $\mathcal{L}_{[0]} = \mathcal{L}_\phi$. By means of Definition 5.4, we extend the operator $\mathcal{L}$ to complex weights and complex test functions. We naturally extend the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ to these function spaces by setting

$$\|h\|_{\diamond_1} := \|\Re h\|_{\diamond_1} + \|\Im h\|_{\diamond_1} \quad \text{and} \quad \|h\|_{\diamond_2} := \|\Re h\|_{\diamond_2} + \|\Im h\|_{\diamond_2}.$$

We will be in particular interested in the case where θ is small or pure imaginary. The next elementary lemma collects the main properties of the family of operators $\mathcal{L}_{[\theta]}$ that we need, see for instance [Bro96, Prop. 5.1] for a proof.

LEMMA 5.5. *Assume that $\|g\|_{\diamond_1}$ is finite. Then the following assertions hold for both of the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$.*

(i) *For every $\theta \in \mathbb{C}$, $\mathcal{L}_{[\theta]}$ is a bounded operator;*
(ii) *The map $\theta \mapsto \mathcal{L}_{[\theta]}$ is analytic in θ;*
(iii) *For every $n \in \mathbb{N}$, $\theta \in \mathbb{C}$, and $h\colon \mathbb{P}^k \to \mathbb{C}$, we have*

$$\mathcal{L}^n_{[\theta]} h = \mathcal{L}^n_{[0]}(e^{\theta S_n(g)} h), \quad \textit{where} \quad S_0(g) := 0 \textit{ and } S_n(g) := \sum_{j=0}^{n-1} g \circ f^j \textit{ for } n \geq 1.$$

Recall that the operator $\lambda^{-1}\mathcal{L}_{[0]}$ has ρ as its unique (up to a multiplicative constant) eigenfunction of eigenvalue 1. It is a contraction with respect to the norm $\|\cdot\|_{\diamond_2}$ (which is equivalent to $\|\cdot\|_{\diamond_1}$) on the space of functions whose integrals with respect to m_ϕ are zero, see Theorem 1.2. The following is then a consequence of the Rellich perturbation method described in [DS58, Ch. VII], see also [Bro96, Prop. 5.2] and [Kat13, Emi82]. Note that the last assertion of Theorem 1.3 is a direct consequence of the analyticity of α given by the fourth item.

PROPOSITION 5.6. *Assume that $\|g\|_{\diamond_1}$ is finite and let $0 < \beta < 1$ be the constant in Theorem 1.2. Then, for all $\beta < \beta' < 1$, the following holds for θ sufficiently small and all $n \in \mathbb{N}$: there exists a decomposition*

$$\lambda^{-n}\mathcal{L}^n_{[\theta]} = \alpha(\theta)^n \Phi_\theta + \Psi^n_\theta$$

as operators on $\{h : \|h\|_{\diamond_1} < \infty\}$ such that

(i) *$\alpha(\theta)$ is the (only) largest eigenvalue of $\mathcal{L}_{[\theta]}$, $\alpha(0) = 1$ and $|\alpha(\theta)| > \beta'$;*
(ii) *Φ_θ is the projection on the (one dimensional) eigenspace associated to $\alpha(\theta)$ and we have $\Phi_0(h) = \langle m_\phi, h\rangle \rho$;*
(iii) *Ψ_θ is a bounded operator on $\{h : \|h\|_{\diamond_1} < \infty\}$ whose spectral radius is $< \beta'$ and*

$$\Psi_\theta \circ \Phi_\theta = \Phi_\theta \circ \Psi_\theta = 0;$$

(iv) *the maps $\theta \mapsto \Psi_\theta$, $\theta \mapsto \Phi_\theta$, and $\theta \mapsto \alpha(\theta)$ are analytic.*

The last property that we will need is the second order expansion of $\alpha(\theta)$ for θ near 0. It is a consequence of the above results by means of standard arguments, see for instance [GH88] and [PP90, Chap. 4].

LEMMA 5.7. *Assume that $\|g\|_{\diamond_1}$ is finite and $\langle \mu_\phi, g\rangle = 0$. Let $\sigma^2 \geq 0$ be given by*

$$\sigma^2 := \left\langle \mu_\phi, g^2 \right\rangle + 2 \sum_{n \geq 1} \langle \mu_\phi, g \cdot (g \circ f^n)\rangle \tag{5.1}$$

and $\alpha(\theta)$ be given by Proposition 5.6. Then we have

$$\alpha(\theta) = e^{\frac{\sigma^2\theta^2}{2} + o(\theta^2)} = 1 + \frac{\theta^2\sigma^2}{2} + o(\theta^2).$$

5.4 Central limit theorem (CLT) and Berry-Esseen theorem. The following result is a version of the Berry-Esseen Theorem and is a consequence of the previous section and known arguments, see, e.g., [Gou15, Th. 3.7] and [Nag57, GH88]. Recall that the CLT is a weaker version of the Berry-Esseen Theorem, which only asks for the convergence to 0 of the LHS of (5.2). In the case of Hölder continuous weight ϕ and observable g, the CLT was established in [DPU96] for $k = 1$ and in [SUZ14] for $k > 1$, see also [DS101] for the case when ϕ is constant.

Theorem 5.8. *Under the hypotheses of Theorem 1.2, consider a function $g\colon \mathbb{P}^k \to \mathbb{R}$ such that $\|g\|_{\diamond_1} < \infty$ and $\langle \mu_\phi, g\rangle = 0$. Assume that g is not a coboundary. Then g satisfies the Berry-Esseen Theorem with variance $\sigma^2 > 0$ given by (5.1). Namely, there exists a constant $C > 0$ such that, for all $n \in \mathbb{N}$ and any interval $I \subset \mathbb{R}$, we have*

$$\Big| \mu_\phi \Big\{ \frac{1}{\sqrt{n}} S_n(g) \in I \Big\} - \frac{1}{\sqrt{2\pi}\sigma} \int_I e^{-\frac{t^2}{2\sigma^2}} dt \Big| \le \frac{C}{\sqrt{n}} \cdot \tag{5.2}$$

Recall that g is a coboundary if there exists $h \in L^2(\mu_\phi)$ such that $g = h \circ f - h$, and this is the case if and only if $\sigma = 0$. Note that if g is such a coboundary, then

$$S_n(g) = (h\circ f - h) + (h \circ f^2 - h \circ f) + \cdots + (h \circ f^n - h \circ f^{n-1}) = h \circ f^n - h$$

for some $h \in L^2(\mu_\phi)$, hence $n^{-1/2} S_n(g)$ converges almost surely to 0 and therefore g cannot satisfy the CLT. We also have the following characterization of coboundaries with bounded $\|\cdot\|_{\diamond_1}$ norm that will be used in Sect. 5.5. The proof is standard, see for instance [FMT03, Lem. 3.4 and Cor. 3.5].

PROPOSITION 5.9. *Let g be a coboundary and assume that $\|g\|_{\diamond_1} < \infty$. Then there exists a function $\tilde{h}$ such that $\|\tilde{h}\|_{\diamond_1} < \infty$ and $g = \tilde{h} \circ f - \tilde{h}$ on the small Julia set of f.*

5.5 Local central limit theorem (LCLT). We establish here an improvement of the CLT for observables satisfying a necessary cocycle condition. Our result is new for $k = 1$, ϕ non-constant, and for $k > 1$, even when $\phi = 0$; for $k = 1$ and $\phi = 0$, see [DNS07]. In this section, we will reserve the letter x for points of the real line. We need the following definition. Recall that the supports of μ_ϕ and m_ϕ are both equal to the small Julia set of f, see Theorem 1.1.

DEFINITION 5.10. Let $g\colon \mathbb{P}^k \to \mathbb{R}$ be a measurable function. We say that g is a *multiplicative cocycle* if there exist $t > 0$, $s \in \mathbb{R}$, and a measurable function $\xi\colon \mathbb{P}^k \to \mathbb{C}$, not equal to zero μ_ϕ-almost everywhere, such that $e^{itg(z)}\xi(z) = e^{is}\xi(f(z))$. We say that g is a $(\mathcal{C}^0, \phi)$*-multiplicative cocycle* (resp. $(\|\cdot\|_{\diamond_1}, \phi)$*-multiplicative cocycle*) if there exist $t > 0$, $s \in \mathbb{R}$, and $\xi\colon \mathbb{P}^k \to \mathbb{C}$, not identically zero on the small Julia set of

f, which is continuous (resp. with finite $\|\cdot\|_{\diamond_1}$ norm), such that $e^{itg(z)}\xi(z) = e^{is}\xi(f(z))$ on the small Julia set of f.

Observe that the last condition above is required to hold only on the small Julia set.

Theorem 5.11. *Under the hypotheses of Theorem 1.2, let $g\colon \mathbb{P}^k \to \mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite, $\langle \mu_\phi, g\rangle = 0$, and g is not a $(\|\cdot\|_{\diamond_1}, \phi)$-multiplicative cocycle. Then g satisfies the LCLT with variance $\sigma^2 > 0$ given by (5.1). Namely, for every bounded interval $I \subset \mathbb{R}$ the convergence*

$$\lim_{n\to\infty} \Big| \sigma\sqrt{n}\mu_\phi\{x + S_n(g) \in I\} - \frac{1}{\sqrt{2\pi}} e^{-x^2/(2\sigma^2 n)}|I| \Big| = 0$$

holds uniformly in $x \in \mathbb{R}$. Here $|I|$ denotes the length of I.

Remark 5.12. The LCLT is a refined version of the CLT. Notice that it requires a stronger assumption on the observable g, see Proposition 5.15.

We need some properties of multiplicative cocycles. Note that the following lemma still holds if we only assume that ϕ and g have bounded $\|\cdot\|_{\log^q}$ norms for some $q > 2$.

Lemma 5.13. *There is a positive constant c independent of g, t, and n such that $\|\lambda^{-n}\mathcal{L}^n_{[it]}\|_\infty \le c$ for all $n \ge 0$ and $t \in \mathbb{R}$. Let K be a compact subset of $\mathbb{R}$. Let $\mathcal{F}$ be a uniformly bounded and equicontinuous family of functions on $\mathbb{P}^k$. Then the family*

$$\mathcal{F}^K_{\mathbb{N}} := \Big\{ \lambda^{-n}\mathcal{L}^n_{[it]} h \ : \ t \in K,\ h \in \mathcal{F},\ n \in \mathbb{N} \Big\}$$

is also uniformly bounded and equicontinuous. Furthermore, if $K \subset (0,\infty)$ and g is not a $(\mathcal{C}^0, \phi)$-multiplicative cocycle, then $\|\lambda^{-n}\mathcal{L}^n_{[it]} h\|_\infty$ tends to 0 when n goes to infinity, uniformly in $t \in K$ and $h \in \mathcal{F}$.

Proof. Define $\phi_t := \phi + itg$. Observe that $|e^{\phi_t}| = e^\phi$. It follows that $\|\mathcal{L}^n_{[it]}\|_\infty \le \|\mathcal{L}^n\|_\infty \le c\lambda^n$ for some positive constant c, according to (1.2). Using this, we can follow the proof of [BD23, Lemma 3.9], with ϕ_t instead of ϕ, and obtain that $\mathcal{F}^K_{\mathbb{N}}$ is uniformly bounded and equicontinuous. Indeed, as t belongs to the compact set K, every ϕ_t has uniformly bounded $\|\cdot\|_{\diamond_1}$-norm, hence they form an equicontinuous family. It remains to prove the last assertion in the lemma.

Let $\mathcal{F}^K_\infty$ denote the family of the limit functions of all sequences $\lambda^{-n_j}\mathcal{L}^{n_j}_{[it_j]}(h_j)$ with $t_j \in K$, $h_j \in \mathcal{F}$, and n_j going to infinity. By Arzelà-Ascoli theorem, this is a uniformly bounded and equicontinuous family of functions which is compact for the uniform topology. Define

$$M := \max \Big\{ |l(a)/\rho(a)| \ : \ l \in \mathcal{F}^K_\infty,\ a \text{ in the small Julia set} \Big\}.$$

The following claim follows from similar arguments as in [BD23, Th. 3.1 and Prop. 4.1].

Claim *The following assertions hold.*

(i) *If* $M=0$, *then* $\mathcal{F}_\infty^K$ *only contains the zero function.*
(ii) *There are* $t\in K$ *and* $l\in\mathcal{F}_\infty^K$ *such that* $|\lambda^{-N}\mathcal{L}_{[it]}^N l|=M\rho$ *on the small Julia set for every* $N\geq 0$.

Assume now that g is not a $(\mathcal{C}^0,\phi)$-multiplicative cocycle. By (i), we only need to show that $M=0$. Assume by contradiction that $M\neq 0$. Consider t and l as in (ii). Define $\xi(a):=l(a)/\rho(a)$ for $a\in\mathbb{P}^k$ and $\vartheta(a):=e^{itg(a)}\xi(a)/\xi(f(a))$ for a in the small Julia set. These functions are continuous and we have $|\xi(a)|=M$ and $|\vartheta(a)|=1$ on the small Julia set. We have for a in the small Julia set

$$\begin{aligned} M\rho(a)=|\lambda^{-n}\mathcal{L}_{[it]}^n l(a)| &= \Big|\lambda^{-n}\sum_{b\in f^{-n}(a)} e^{\phi_t(b)+\cdots+\phi_t(f^{n-1}(b))}l(b)\Big| \\ &\leq \lambda^{-n}\sum_{b\in f^{-n}(a)} e^{\phi(b)+\cdots+\phi(f^{n-1}(b))}M\rho(b)=M\rho(a). \end{aligned}$$

So, the last inequality is an equality. Using the function ξ, this equality gives

$$\Big|\sum_{b\in f^{-n}(a)}\vartheta(b)\dots\vartheta(f^{n-1}(b))e^{\phi(b)+\cdots+\phi(f^{n-1}(b))}\rho(b)\Big|=\sum_{b\in f^{-n}(a)}e^{\phi(b)+\cdots+\phi(f^{n-1}(b))}\rho(b).$$

Here, we removed the factors $|\xi(f^n(b))|$ and M as they are both equal to $|\xi(a)|$ and independent of $b\in f^{-n}(a)$. As $|\vartheta|=1$, we deduce that if b and b' are two points in $f^{-n}(a)$ then

$$\vartheta(b)\dots\vartheta(f^{n-1}(b))=\vartheta(b')\dots\vartheta(f^{n-1}(b')).$$

This and a similar equality for $f(b)$, $f(b')$, $n-1$ instead of b, b', n imply that $\vartheta(b)=\vartheta(b')$. We conclude that ϑ is constant on $f^{-n}(a)$ for every n. As $f^{-n}(a)$ tends to the small Julia set when n going to infinity, it follows that ϑ is constant. From the definition of ϑ, and since ξ is continuous, we obtain that g is a $(\mathcal{C}^0,\phi)$-multiplicative cocycle for a suitable real number s such that $\vartheta=e^{is}$. This is a contradiction. So we have $M=0$, as desired. □

Recall that ϕ and g have bounded $\|\cdot\|_{\diamond_1}$ norms.

LEMMA 5.14. *Let* K *be a compact subset of* $\mathbb{R}$. *There is a positive constant* c *such that* $\|\lambda^{-n}\mathcal{L}_{[it]}^n\|_{\diamond_1}\leq c$ *for every* $n\geq 0$ *and* $t\in K$. *If* $K\subset\mathbb{R}\setminus\{0\}$ *and* g *is not a* $(\|\cdot\|_{\diamond_1},\phi)$*-multiplicative cocycle, then there are constants* $c>0$ *and* $0<r<1$ *such that* $\|\lambda^{-n}\mathcal{L}_{[it]}^n\|_{\diamond_1}\leq cr^n$ *for every* $t\in K$ *and* $n\geq 0$.

Proof. Consider the functional ball $\mathcal{F}:=\{h\ :\ \|h\|_{\diamond_1}\leq 1\}$. By Arzelà-Ascoli theorem this ball is compact for the uniform topology. Define $\mathcal{F}_\infty^K$ as in the proof of Lemma 5.13. Using that lemma, we can follow the proof of Proposition 4.9 and obtain that $\|\lambda^{-n}\mathcal{L}_{[it]}^n\|_{\langle p,\alpha\rangle,\gamma}$ is bounded uniformly on n and $t\in K$. It follows that a similar property holds for the norm $\|\cdot\|_{\diamond_1}$. This gives the first assertion in the lemma. We also obtain that the family $\mathcal{F}_\infty^K$ is bounded in the $\|\cdot\|_{\diamond_1}$ norm and, using again Arzelà-Ascoli theorem, we obtain that it is compact in the uniform topology.

Consider now the second assertion and assume that g is not a $(\|\cdot\|_{\diamond_1},\phi)$-multiplicative cocycle. Let $K\subset\mathbb{R}\setminus\{0\}$ be a compact set. We first show that $\mathcal{F}_\infty^K$ reduces to $\{0\}$. Assume by contradiction that this is not true. Consider t and l as in the assertion (ii) of the Claim in the proof of Lemma 5.13. Recall that both $\|l\|_{\diamond_1}$ and $\|1/\rho\|_{\diamond_1}$ are finite, see Theorem 1.2. By Lemma 3.15, the function $\xi:=l/\rho$ satisfies the same property and we conclude, as at the end of the proof of Lemma 5.13, that g is a $(\|\cdot\|_{\diamond_1},\phi)$-multiplicative cocycle. This contradicts the hypothesis.

So $\mathcal{F}_\infty^K$ is reduced to $\{0\}$. By definition of $\mathcal{F}_\infty^K$, we obtain that $\lambda^{-n}\mathcal{L}_{[it]}^n h$ converges to 0 uniformly on $h\in\mathcal{F}$ and $t\in K$. Using this property, we can follow the proof of Proposition 4.6 (take $\beta=1/4$) to obtain that $\|\lambda^{-N}\mathcal{L}_{[it]}^N h\|_{\langle p,\alpha\rangle,\gamma}\leq 1/4$ for N large enough and for all $h\in\mathcal{F}$ and $t\in K$. Indeed, observe that the proof of Proposition 4.6 is given in the assumption that $\langle m_\phi,g\rangle=0$. We do not have this assumption here, so we need to check a priori that the sequence converges to 0 uniformly. When N is large enough, we also have $\|\lambda^{-N}\mathcal{L}_{[it]}^N h\|_\infty\leq 1/4$. Therefore, we have $\|\lambda^{-N}\mathcal{L}_{[it]}^N\|_{\diamond_1}\leq 1/2$ which implies the desired property with $r=2^{-1/N}$. □

The following characterizations of multiplicative cocycles now follows from the above lemmas.

Proposition 5.15. *Let $g\colon\mathbb{P}^k\to\mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite. Then the following properties are equivalent*:

- (ia) *g is a multiplicative cocycle*;
- (ib) *g is a $(\mathcal{C}^0,\phi)$-multiplicative cocycle*;
- (ic) *g is a $(\|\cdot\|_{\diamond_1},\phi)$-multiplicative cocycle*;
- (iia) *there exists a number $t>0$ such that the spectral radius with respect to the norm $\|\cdot\|_{\diamond_1}$ of $\lambda^{-1}\mathcal{L}_{[it]}$ is ≥ 1.*
- (iib) *there exists a number $t>0$ such that the spectral radius with respect to the norm $\|\cdot\|_{\diamond_1}$ of $\lambda^{-1}\mathcal{L}_{[it]}$ is equal to* 1.

Moreover, every coboundary with finite $\|\cdot\|_{\diamond_1}$ norm is a $(\|\cdot\|_{\diamond_1},\phi)$-multiplicative cocycle.

Once the characterization of multiplicative cocycles in Proposition 5.15 is established, in order to prove Theorem 5.11 one can follow the proof of [DNS07, Th. C], which is based on [Bre92, Theorem 10.17], see also [GH88].

5.6 Almost sure invariant principle (ASIP) and consequences. We can now prove the ASIP for observables which are not coboundaries. The ASIP was proved by Dupont [Dup10] in the case where $\phi=0$ for observables which are Hölder continuous, or admit analytic singularities, by using [PS75], see also Przytycki-Urbański-Zdunik [PUH89] for $k=1$ and $\phi=0$.

Theorem 5.16. *Under the hypotheses of Theorem 1.2, let $g\colon\mathbb{P}^k\to\mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite and $\langle\mu_\phi,g\rangle=0$. Assume that g is not a coboundary. Then g satisfies* *the ASIP with variance $\sigma^2 > 0$ given by (5.1) and error rate*

$$O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}).$$

Namely, there exist, on some probability space $\mathcal{X}$, two processes $(X_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ such that

(i) *the processes $(g, g\circ f, g\circ f^2, \dots)$ and $(X_0, X_1, X_2 \dots)$ have the same distribution;*

(ii) *the random variables B_j are i.i.d., with distribution equal to the Gaussian distribution $\mathcal{N}(0,\sigma^2)$ centered at 0 and with variance σ^2;*

(iii) $\left|\sum_{j=0}^n X_j - \sum_{j=0}^n B_j\right| = O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4})$ *almost everywhere on $\mathcal{X}$.*

General criteria that allow one to establish the ASIP in various contexts and weaker rates are given in [PS75]. Theorem 5.16 is a consequence of [CM15, Th. 3.2]. It is likely that the rate is not optimal, see for instance [C+20]. We also observe that, by [Gou10, Th. 1.2] and [Gou15, Th. 5.2], it is also possible to obtain a version of Theorem 5.16 in the more general case of random variables with values in $\mathbb{R}^d$, with error rate $o(n^\ell)$ for every $\ell > 1/4$.

The Law of Iterated Logarithms (LIL) and the Almost Sure Central Limit Theorem (ASCLT) are general consequences of the ASIP, see for instance [PS75, LP89, CG07] for the definitions of these properties and their deduction from the ASIP. The LIL was established in [SUZ14] in the case where both the weight ϕ and the observable g are Hölder continuous.

5.7 Large deviation principle (LDP). We conclude the statistical study of $(\mathbb{P}^k, f, \mu_\phi)$ with the following property, which gives very precise estimates on the measure of the set where the partial sums are far from the mean value. This is new in this generality for all $k \geq 1$, even for $\phi = 0$ (see [CR11] for the case when $k = 1$ and some kind of weak hyperbolicity is assumed). The LDP in particular implies the Large Deviation Theorem (which only requires an upper bound for the measure in (5.3) below), which is proved in [DNS10] in the case $\phi = 0$, see also [PS96, DS101].

Theorem 5.17. *Under the hypotheses of Theorem 1.2, let $g\colon \mathbb{P}^k \to \mathbb{R}$ be such that $\|g\|_{\diamond_1}$ is finite and $\langle \mu_\phi, g\rangle = 0$. Assume that g is not a coboundary. Then g satisfies the LDP. Namely, there exists a non-negative, strictly convex function c which is defined on a neighbourhood of $0 \in \mathbb{R}$ and vanishes only at 0 such that, for all $\epsilon > 0$ sufficiently small,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\phi \Big\{ x \in \mathbb{P}^k \colon \frac{S_n(g)(x)}{n} > \epsilon \Big\} = -c(\epsilon). \tag{5.3}$$

Theorem 5.17 is a consequence of Theorem 1.2, the results in Sect. 5.3, and [HH01, Lem. XIII.2] (a local version of Gärtner-Ellis Theorem [Gar77, Ell84, DZ98] which is due to Bougerol-Lacroix [BL85]). Notice that the symmetric statement for the measure of the set where $n^{-1}S_n(g) < -\epsilon$ in (5.3) can be obtained by applying the same arguments as above to the sequence of random variables $-S_n(g)$.

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