Γ-convergence for power-law functionals with variable exponents *

Michela Eleuteri
Università degli Studi di Modena e Reggio Emilia
Dipartimento di Scienze Fisiche Informatiche e Matematiche
via Campi 213/b, 41125 Modena (Italy)
michela.eleuteri@unimore.it

and Francesca Prinari
Università degli Studi di Ferrara
Dipartimento di Matematica e Informatica
via Machiavelli 30, 44121 Ferrara (Italy)
francesca.prinari@unife.it

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Abstract

We study the Γ-convergence of the functionals $F_n(u) := ||f(\cdot, u(\cdot), Du(\cdot))||_{p_n(\cdot)}$ and $\mathcal{F}_n(u) := \int_\Omega \frac{1}{p_n(x)} f^{p_n(x)}(x, u(x), Du(x))dx$ defined on $X \in \{L^1(\Omega, \mathbb{R}^d), L^\infty(\Omega, \mathbb{R}^d), C(\Omega, \mathbb{R}^d)\}$ (endowed with their usual norms) with effective domain the Sobolev space $W^{1,p_n(\cdot)}(\Omega, \mathbb{R}^d)$. Here $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, $N, d \geq 1$ and the measurable functions $p_n : \Omega \to [1, +\infty)$ satisfy the conditions $\operatorname{ess\,sup}_\Omega p_n \leq \beta \operatorname{ess\,inf}_\Omega p_n < +\infty$ for a fixed constant $\beta > 1$ and $\operatorname{ess\,inf}_\Omega p_n \to +\infty$ as $n \to +\infty$. We show that when $f(x, u, \cdot)$ is level convex and lower semicontinuous and it satisfies a uniform growth condition from below, then, as $n \to \infty$, the sequences $(F_n)_n$ Γ-converges in X to the functional F represented as $F(u) = ||f(\cdot, u(\cdot), Du(\cdot))||_\infty$ on the effective domain $W^{1,\infty}(\Omega, \mathbb{R}^d)$. Moreover we show that the Γ-$\lim_n \mathcal{F}_n$ is given by the functional $\mathcal{F}(u) := \begin{cases} 0 & \text{if } ||f(\cdot, u(\cdot), Du(\cdot))||_\infty \leq 1, \\ +\infty & \text{otherwise in } X. \end{cases}$

Keywords: Γ-convergence, Lebesgue-Sobolev spaces with variable exponent, power-law functionals, supremal functionals, Young measures, level convex functions.

Contents

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1 Introduction

The classical functionals of the Calculus of Variations are represented in the integral form

$$\mathcal{H}_n(u,A) = \int_A f_n(x, Du(x))dx,$$

and are defined on some subset of a Sobolev space $W^{1,q}(\Omega)$, where $A \subseteq \Omega \subseteq \mathbb{R}^N$ with A and Ω open sets. When the sequence of Borel functions (f_n) satisfies a uniform growth condition of order $q > 1$, often named *standard growth condition*

$$\alpha(|\xi|^q - 1) \leq f_n(x,\xi) \leq \beta(|\xi|^q + 1) \tag{1.1}$$

with $0 < \alpha \leq \beta$, then it is possible to apply a general compactness procedure to get that there exists a subsequence $(\mathcal{H}_{k_n})_n$ Γ-converging with respect to the L^q-norm to a functional $\mathcal{H}_0$ that can be represented in the integral form. If the growth condition of order q is not uniformly satisfied, then the Γ-limit of the sequence $\mathcal{H}_n(u,A)$ can lose the additivity property with respect to the union of disjoint open sets and may assume a different representation form. This is the case, for instance, when (1.1) is replaced by the so called *non-standard growth condition* (considered for the first time in the pioneering papers by Marcellini [24, 25] and more recently in [22], [23], [27]; for some examples of functionals with general growth we refer to [26]). In the case of integral functionals exhibiting a gap between the coercivity and the growth exponent, in [33] Mingione and Mucci show that in the relaxation procedure energy concentrations may appear leading to a measure representation of the relaxed functional with a nonzero singular part.
A different situation appears for example in [28]: Garroni, Nesi and Ponsiglione consider the case when

$$f_n(x,\xi) := \frac{1}{n}(f(x,\xi))^n \tag{1.2}$$

where $f(x,\xi) := a(x)|\xi|$ with $a \in L^\infty(\Omega)$ satisfying the condition $\operatorname{ess\,inf}_{x\in\Omega} a(x) > 0$. This sequence (f_n) does not verify uniformly a q-growth condition and in [28, Proposition 2.1] it is shown that, when Ω is the unitary cube of $\mathbb{R}^N$, the Γ-limit (with respect to the L^1-convergence) of the sequence $(\mathcal{I}_n)$ defined by

$$\mathcal{I}_n(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{n}(a(x)|Du(x)|)^n dx, & \text{if } u \in W^{1,n}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases} \tag{1.3}$$

is given by

$$\mathcal{I}(u) := \begin{cases} 0 & \text{if } ||a(x)|Du(x)|||_\infty \leq 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

Moroever, in the same paper (see Proposition 2.6 therein), it is proved that the sequence of the L^n-norms

$$I_n(u) := \begin{cases} \displaystyle\left(\int_\Omega (a(x)|Du(x)|)^n dx\right)^{1/n} & \text{if } u \in W^{1,n}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{1.5}$$

Γ-converges (with respect to the L^1-convergence) to the functional I represented in the *supremal* form

$$I(u) := \begin{cases} \operatorname*{ess\,sup}_{\Omega} a(x)|Du(x)| & \text{if } u \in W^{1,\infty}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{1.6}$$

Recently, the class of functionals represented in the general supremal form has being studied with growing interest. They appear in a very natural way in variational problems where the relevant quantities do not express a mean property and the values of the energy densities on very small subsets of Ω cannot be neglected. Their study was introduced by Aronsson in the 1960s (see [2], [3], [4]). In the seminal papers [32] and [29], the supremal functional $F(u) = ||Du||_\infty$ appears in the variational problem of finding the best Lipschitz extension u in Ω of a function g defined on the boundary $\partial\Omega$. There after, several mathematical model have been formulated by means of a supremal functional: for example, models describing dielectric breakdown in a composite material (see [28]) or polycrystal plasticity (see [11]). Also the problem of image reconstruction and enhancement can been formulated as an L^∞ problem (see [13]). A recent application involving a supremal functional has been given in [30] where, in order to lay the rigorous mathematical foundations of the Fluorescent Optical Tomography (FOT), the author poses FOT as a minimisation problem in L^∞ with PDE constraints.

The above mentioned results contained in [28] have been generalized in different directions:

- when $X = C(\overline{\Omega}, \mathbb{R}^d)$ is endowed with the uniform topology, in [15], [37] and [38] the authors study the Γ-convergence of the family of integral functional $F_p : X \to [0, +\infty]$ given by

$$F_p(u) := \begin{cases} \left(\int_\Omega f^p(x, u(x), Du(x)) dx \right)^{1/p}, & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^d), \\ +\infty, & \text{otherwise in X} \end{cases}$$

  gradually weakening the assumptions on f:

  - in [15, Theorem 3.1] the function f is a normal integrand satisfying a superlinear growth condition and a generalized Jensen inequality for gradient Young measures; in particular the Γ-convergence result therein holds when the sub level sets $\{\xi \in \mathbb{R}^{d\times N} : f(x, u, \xi) \le \lambda\}$ are closed and convex for every $\lambda \in \mathbb{R}$;
  - in [37] the function $f = f(x, \xi)$ is assumed to be a Carathéodory integrand satisfying the linear growth condition (1.1) with $q = 1$, while in [38] the continuity assumption on f with respect to the gradient variable has dropped and the function $f = f(x, \xi)$ is assumed to be only $\mathcal{L}^N \times \mathcal{B}_{d\times N}$-measurable;

- in the papers [11] and [1] the space X coincides with the class of the functions U in $L^1(\Omega, \mathbb{R}^{d\times N})$ or in $L^\infty(\Omega, \mathbb{R}^{d\times N})$, constrained to satisfy a general rank-constant differential constraint $\mathcal{A}U = 0$ and the functionals $F_p : X \to [0, +\infty]$ are defined by

$$F_p(U) := \begin{cases} \left(\int_\Omega f^p(x, U(x)) dx \right)^{1/p}, & \text{if } u \in L^p(\Omega, \mathbb{R}^d) \cap X, \\ +\infty, & \text{otherwise in } X. \end{cases}$$

  In [11] the authors consider the case when $f(x, \xi) = a(x)|\xi|$ while in [1] the Γ-convergence is studied in the wider class of $\mathcal{A}_\infty$-quasiconvex function f (see Definition 3.2 therein). These results have been extended in [10] in the setting of variable exponent Lebesgue space when $f(x, \cdot)$ is quasiconvex in the sense of Morrey.

A different generalization of the results contained in [28] has been given by Bocea-Mihilescu in [9]: they show the Γ-convergence of the sequences (1.3) and (1.5) respectively to the functionals $\mathcal{I}$ and I given by (1.4) and (1.6) respectively when Ω is a Lipschitz connected open set satisfying $\mathcal{L}^N(\Omega) = 1$ and the constant sequence (n) is replaced by a sequence $(p_n) = (p_n(x))$ of Lipschitz continuous functions satisfying

$$p_n^- := \operatorname*{ess\,inf}_{\Omega} p_n \to +\infty$$

as $n \to +\infty$ and

$$p_n^+ = \operatorname*{ess\,sup}_{\Omega} p_n \leq \beta p_n^-$$

for a fixed constant $\beta > 1$.

Inspired by [9], in our paper we consider the general case when $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{N\times d} \to [0, +\infty)$ in formula (1.2) is a Borel function satisfying the coercivity assumption

$$f(x, u, \xi) \geq \alpha|\xi|^\gamma \qquad \text{for a.e } x \in \Omega, \text{ for every } (u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{Nd} \tag{1.7}$$

(with $\alpha, \gamma > 0$) and such that its sub level sets $\{\xi \in \mathbb{R}^N : f(x, u, \xi) \leq t\}$ are closed and convex for any $t \in \mathbb{R}$. The last assumption is sufficient to ensure the lower semicontinuity with respect to the weak* lower semicontinuity of the supremal functional

$$F(u) := \operatorname*{ess\,sup}_{\Omega} f(x, u(x), Du(x))$$

in the space $W^{1,\infty}(\Omega, \mathbb{R}^d)$ (see Theorem 3.4 in [7]).

Under the previous hypotheses, in Theorems 4.1 and 4.2 we show that, if we consider $X \in \{L^1(\Omega, \mathbb{R}^d), L^\infty(\Omega, \mathbb{R}^d), C(\Omega, \mathbb{R}^d)\}$, endowed with their usual norms, then the sequence of functionals $F_n : X \to [0, +\infty]$ defined by

$$F_n(u) := \begin{cases} ||f(\cdot, u(\cdot), Du(\cdot))||_{p_n(\cdot)} & \text{if } u \in W^{1,p_n(\cdot)}(\Omega, \mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases}$$

as $n \to +\infty$ Γ-converges to the functional F given by

$$F(u) := \begin{cases} \operatorname*{ess\,sup}_{\Omega} f(x, u(x), Du(x)) & \text{if } u \in W^{1,\infty}(\Omega, \mathbb{R}^d), \\ +\infty & \text{otherwise in } X. \end{cases}$$

In particular, thanks to the more general growth condition (1.7) on f, we get an improvement of Theorem 3.1 in [15] (see Corollary 4.3).

Moreover in Theorems 4.4 and 4.5 we show that the sequence of the integral functionals $\mathcal{F}_n : X \to [0, +\infty]$ defined by

$$\mathcal{F}_n(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{p_n(x)} f^{p_n(x)}(x, u(x), Du(x))dx & \text{if } u \in W^{1,p_n(\cdot)}(\Omega, \mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases}$$

as $n \to +\infty$, Γ-converges to the functional defined by

$$\mathcal{F}(u) := \begin{cases} 0 & \text{if } u \in W^{1,\infty}(\Omega, \mathbb{R}^d) \text{ and } ||f(x, u(x), Du(x))||_\infty \leq 1, \\ +\infty & \text{otherwise in } X. \end{cases}$$

Note that the proofs of the previous results are given in the vectorial setting and in general case $\mathcal{L}^N(\Omega) \in (0, +\infty)$. Moreover we do not need any connectedness hypothesis on Ω and only

when $X = L^1(\Omega,\mathbb{R}^d)$ we assume that $\partial\Omega$ is Lipschitz regular. We point out that in [9] the weak lower semicontinuity in $L^q(\Omega)$ of the integral functionals (1.3) and (1.5) was enough to prove the Γ-convergence results therein. Instead, the more general class of our variational functionals requires as key tool the use of *gradient Young measures*: indeed, this instrument turns to be crucial in order to show the Γ-liminf inequality, combined with a Jensen type inequality satisfied by level convex functions, see Theorem 2.5. Moreover, in our paper, we deal with more general topologies instead of treating only with the strong convergence in $L^1(\Omega)$ as in [9]. This allows us to remove the regularity assumptions on $\partial\Omega$ in the case of $X = \{L^\infty(\Omega,\mathbb{R}^d), C(\Omega,\mathbb{R}^d)\}$.
We devote a forthcoming paper to study the homogenization of supremal functionals of the form

$$F_\epsilon(u) := \operatorname*{ess\,sup}_{\Omega} g\left(\frac{x}{\epsilon}, Du(x)\right)$$

where $g(x,\xi) := (f(x,\xi))^{p(x)}$. With this aim, we will proceed our analysis by discussing the Γ-convergence of the sequence of functionals

$$H_n(u) := \begin{cases} \left(\int_\Omega \frac{1}{np(x)} f^{np(x)}(\cdot,u(\cdot),Du(\cdot))dx\right)^{1/n} & \text{if } u \in W^{1,p_n(\cdot)}(\Omega,\mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases}$$

already considered in the case $f(\xi) = |\xi|$ by Zhikov in [40] (see also [41]) and, more recently, by Bocea and Mihilescu in [9] in the case when $f(x,\xi) = a(x)|\xi|$.

2 Preliminary results

2.1 Variable exponents Lebesgue-Sobolev spaces

In this section we collect some basic results concerning variable exponent Lebesgue and Sobolev spaces. For more details we refer to the monograph [18], see also [31], [19], [20], [21].
For the purpose of our paper, we consider the case when $\Omega \subset \mathbb{R}^N$ is an open set (where $N \geq 1$) and denote by $\mathcal{L}^N(\Omega)$ the N-dimensional Lebesgue measure of the set Ω. In the sequel we will consider functions $u : \Omega \to \mathbb{R}^d$, with $d \geq 1$ and we denote by k any dimension different from Nd.

Definition 2.1. *For any (Lebesgue) measurable function $p : \Omega \to [1,+\infty]$ we define*

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \qquad\qquad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

Such function p is called variable exponent on Ω. If $p^+ < +\infty$ then we call p a bounded variable exponent.

In the sequel we need to introduce the Lebesgue spaces with variable exponents, $L^{p(\cdot)}(\Omega)$. They differ from the classical L^p spaces because now the exponent p is not constant but it is a variable exponent in the sense specified above. Originally the spaces $L^{p(\cdot)}$ have been introduced in the case $1 \leq p^- \leq p^+ < +\infty$ by Orlicz [35] in 1931 and, in the case $p^+ = \infty$, by Sharpudinov [39] and later (in the higher dimensional case), by Kováčik and Rákosník [31].
In the sequel we consider the case $p^+ < +\infty$. In this case the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ can be defined as

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\}.$$

Let us note that in the case $p^+ = +\infty$ the space above defined may even fail to be a vector space (see [16] Section 2) and a different definition of the variable Lebesgue spaces has been given in order to preserve the vectorial structure of the space (we refer to [18], Definition 3.2.1). On the other hand if $p^+ < +\infty$, it is possible to show that $L^{p(\cdot)}(\Omega)$ is a Banach space endowed with the *Luxemburg norm*

$$\|u\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}$$

(see Theorem 3.2.7 in [18]). Moreover if $p^+ < +\infty$ the space $L^{p(\cdot)}(\Omega)$ is separable and the space $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ while if $1 < p^- \le p^+ < +\infty$ the space $L^{p(\cdot)}(\Omega)$ is reflexive and uniformly convex (see Lemma 3.4.1, Theorem 3.4.7, Theorem 3.4.9 and Theorem 3.4.12 in [18]). Finally, by Corollary 3.3.4 in [18], if $0 < \mathcal{L}^N(\Omega) < +\infty$ and p and q are variable exponents such that $p \le q$ a.e. in Ω, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous. The embedding constant is less or equal to $2\max\left\{\mathcal{L}^N(\Omega)^{\left(\frac{1}{q}-\frac{1}{p}\right)^+}, \mathcal{L}^N(\Omega)^{\left(\frac{1}{q}-\frac{1}{p}\right)^-}\right\}$.
For any variable exponent p, we define p' by setting

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

with the convention that, if $p(x) = +\infty$ then $p'(x) = 1$. The function p is called *the dual variable exponent of* p.
We have the following result (for more details see Lemma 3.2.20 in [18]).

Theorem 2.2. *(Hölder's inequality) Let p, q, s be measurable exponents such that*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$$

a.e. in Ω. Then

$$\|fg\|_{s(\cdot)} \le \left(\left(\frac{s}{p}\right)^+ + \left(\frac{s}{q}\right)^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}$$

for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, where in the case $s = p = q = \infty$, we use the convention $\frac{s}{p} = \frac{s}{q} = 1$.
In particular, in the case $s = 1$, we have

$$\left|\int_\Omega f\, g\, dx\right| \le \int_\Omega |f|\, |g|\, dx \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

We moreover introduce the *modular* of the space $L^{p(\cdot)}(\Omega)$ which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx.$$

Thanks to Lemma 3.2.4 in [18], for every $u \in L^{p(\cdot)}(\Omega)$

$$\|u\|_{p(\cdot)} \le 1 \Longleftrightarrow \rho_{p(\cdot)}(u) \le 1 \tag{2.1}$$

$$\|u\|_{p(\cdot)} \le 1 \Longrightarrow \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)} \tag{2.2}$$

$$\|u\|_{p(\cdot)} > 1 \Longrightarrow \|u\|_{p(\cdot)} \le \rho_{p(\cdot)}(u). \tag{2.3}$$

The following further results hold in the special case $p^+ < +\infty$. By Lemma 3.2.5 in [18], for every $u \in L^{p(\cdot)}(\Omega)$ it holds

$$\min\left\{ \left(\rho_{p(\cdot)}(u)\right)^{\frac{1}{p^-}}, \left(\rho_{p(\cdot)}(u)\right)^{\frac{1}{p^+}} \right\} \leq \|u\|_{p(\cdot)} \leq \max\left\{ \left(\rho_{p(\cdot)}(u)\right)^{\frac{1}{p^-}}, \left(\rho_{p(\cdot)}(u)\right)^{\frac{1}{p^+}} \right\}. \tag{2.4}$$

In particular, we get that

$$\|1\|_{p(\cdot)} \leq \max\{\left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{p^-}}, \left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{p^+}}\}. \tag{2.5}$$

Moroever, from (2.4), taking into account (2.2), (2.3), it follows that for every $u \in L^{p(\cdot)}(\Omega)$

$$\|u\|_{p(\cdot)} > 1 \Longrightarrow \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+} \tag{2.6}$$

$$\|u\|_{p(\cdot)} < 1 \Longrightarrow \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}. \tag{2.7}$$

Finally, thanks to Lemma 3.4.2 in [18], for every $u \in L^{p(\cdot)}(\Omega)$

$$\|u\|_{p(\cdot)} < 1 \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 \tag{2.8}$$

$$\|u\|_{p(\cdot)} = 1 \Longleftrightarrow \rho_{p(\cdot)}(u) = 1 \tag{2.9}$$

$$\|u\|_{p(\cdot)} > 1 \Longleftrightarrow \rho_{p(\cdot)}(u) > 1. \tag{2.10}$$

We conclude this part by recalling the definition of variable exponent Sobolev spaces. For more details we refer to [16] (see also [18], Definition 8.1.2).

Definition 2.3. *Let $k, d \in \mathbb{N}$, $k \geq 0$, and let p be a measurable exponent. We define*

$$W^{k,p(\cdot)}(\Omega, \mathbb{R}^d) := \{u : \Omega \to \mathbb{R}^d : u, \partial_\alpha u \in L^{p(\cdot)}(\Omega, \mathbb{R}^d) \quad \forall \alpha \text{ multi-index such that } |\alpha| \leq k \},$$

where

$$L^{p(\cdot)}(\Omega, \mathbb{R}^d) := \{u : \Omega \to \mathbb{R}^d : \ |u| \in L^{p(\cdot)}(\Omega)\}.$$

We define the semimodular on $W^{k,p(\cdot)}(\Omega)$ by

$$\rho_{W^{k,p(\cdot)}(\Omega)}(u) := \sum_{0 \leq |\alpha| \leq k} \rho_{L^{p(\cdot)}(\Omega)}(|\partial_\alpha u|)$$

which induces a norm by

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \ \rho_{W^{k,p(\cdot)}(\Omega)}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

For $k \in \mathbb{N} \setminus \{0\}$, the space $W^{k,p(\cdot)}(\Omega)$ is called *Sobolev space* and its elements are called *Sobolev functions*. Clearly $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$.

2.2 Level convex functions

Definition 2.4. *We say that $f : \mathbb{R}^k \to \mathbb{R}$ is level convex if for every $t \in \mathbb{R}$ the level set $\left\{\xi \in \mathbb{R}^k : f(\xi) \leq t\right\}$ is convex.*

We recall Jensen's inequality introduced by Barron, Jensen, and Liu in [6] for lower semicontinuous and level convex functions (see also [7] Theorem 1.2).

Theorem 2.5. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a lower semicontinuous and level convex function, and let μ be a probability measure supported on the open set $\Omega \subseteq \mathbb{R}^N$. Then for every function $u \in L^1_\mu(\Omega; \mathbb{R}^k)$ we have*

$$f\left(\int_\Omega u(\xi) d\mu(\xi)\right) \leq \mu\text{-}\operatorname*{ess\,sup}_{\xi \in \Omega}(f \circ u)(\xi). \tag{2.11}$$

2.3 Young measures

In this section we briefly recall some results on the theory of Young measures (see e.g. [5], [8]). If $\Omega \subseteq \mathbb{R}^N$ is an open set (not necessarily bounded) and $d \geq 1$, we denote by $C_c(\Omega;\mathbb{R}^d)$ the set of continuous functions with compact support in Ω, endowed with the supremum norm. The dual of the closure of $C_c(\Omega;\mathbb{R}^d)$ may be identified with the set of $\mathbb{R}^d$-valued Radon measures with finite mass $\mathcal{M}(\Omega;\mathbb{R}^d)$, through the duality

$$\langle \mu, \varphi \rangle := \int_\Omega \varphi(\xi)\, d\mu(\xi)\,, \qquad \mu \in \mathcal{M}(\Omega;\mathbb{R}^d)\,, \qquad \varphi \in C_c(\Omega;\mathbb{R}^d)\,.$$

Definition 2.6. *A map $\mu : \Omega \mapsto \mathcal{M}(\Omega;\mathbb{R}^d)$ is said to be weak*-measurable if $x \mapsto \langle \mu_x, \varphi \rangle$ are measurable for all $\varphi \in C_c(\Omega;\mathbb{R}^d)$.*

Definition 2.7. *Let (V_n) be a bounded sequence in $L^1(\Omega,\mathbb{R}^d)$. We say that (V_n) is equi-integrable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that, for every measurable $E \subset \Omega$, if $\mathcal{L}^N(E) < \delta$, then*

$$\sup_n \int_E |V_n(x)|\, dx < \varepsilon\,.$$

For every $1 < p < +\infty$ we say that (V_n) is p-equi-integrable if $(|V_n|^p)$ is equi-integrable.

We are in position to state the main result concerning Young Measures (for a proof see [34, Theorem 3.1]).

Theorem 2.8 (*Fundamental Theorem on Young Measures*)**.** *Let $E \subseteq \mathbb{R}^N$ be a measurable set of finite measure and let (V_n) be a sequence of measurable functions, $V_n : E \mapsto \mathbb{R}^d$. Then there exists a subsequence (V_{n_k}) and a weak*-measurable map $\mu : E \mapsto \mathcal{M}(\Omega;\mathbb{R}^d)$ such that the following statements hold:*

(1) $\mu_x \geq 0$, $\|\mu_x\|_{\mathcal{M}(\Omega;\mathbb{R}^d)} = \int_{\mathbb{R}^d} d\mu_x \leq 1$ *for a.e.* $x \in E$*;*

(2) $\forall \varphi \in C_c(\Omega;\mathbb{R}^d)$

$$\varphi(V_{n_k}) \overset{*}{\rightharpoonup} \bar{\varphi}$$

where

$$\bar{\varphi}(x) := \langle \mu_x, \varphi \rangle; \qquad \text{for a.e. } x \in E\,;$$

(3) for every compact subset $K \subset \mathbb{R}^d$*, if* $\operatorname{dist}(V_{n_k} K) \to 0$ *in measure, then*

$$\operatorname{supp} \mu_x \subset K \qquad \text{for a.e. } x \in E\,;$$

(4) $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ *for a.e.* $x \in E$ *if and only if*

$$\lim_{M\to\infty} \sup_k \mathcal{L}^N(\{|V_{n_k}| \geq M\}) = 0\,; \tag{2.12}$$

(5) if $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ *for a.e.* $x \in E$ *then in (3) we may replace "if" with "if and only if";*

(6) if $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ *for a.e.* $x \in E$ *and* $A \subseteq E$ *is measurable and if* $\varphi \in C(\mathbb{R}^d)$ *is such that* $(\varphi(V_{n_k}))$ *is equi-integrable in* $L^1(A,\mathbb{R}^d)$ *then*

$$\varphi(V_{n_k}) \rightharpoonup \bar{\varphi}.$$

The map $\mu : E \mapsto \mathcal{M}(\mathbb{R}^d)$ is called *Young measure generated by the sequence* (V_{n_k}).

From now on, for the sake of simplicity, we denote by (V_n) the sequence (V_{n_k}) generating the corresponding Young measure.

Remark 2.9. Condition (2.12) holds if there exists any $q \geq 1$ such that

$$\sup_{n \in \mathbb{N}} \|V_n\|_q < +\infty.$$

Indeed, by Chebyshev's inequality,

$$\sup_{n \in \mathbb{N}} \mathcal{L}^N(\{|V_n| \geq M\}) \leq \frac{1}{M^q} \int_E |V_n|^q dx \leq \frac{C}{M^q}.$$

In particular, if (V_n) an equi-integrable sequence in $L^1(E, \mathbb{R}^d)$, as a consequence of Theorem 2.8(4)-(6) (with $\varphi = Id$), it generates the Young measure $\mu = (\mu_x)$ satisfying $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ such that $V_{n_k} \rightharpoonup \bar{V}$ weakly in $L^1(E, \mathbb{R}^d)$, where

$$\bar{V}(x) = \int_{\mathbb{R}^d} \xi \, d\mu_x(\xi) \qquad \text{for a.e. } x \in E\,.$$

The following Corollary 2.11 allows us to treat limits of integrals in the form $\int_E f(x, V_n(x), DV_n(x))dx$ without any convexity assumption of $f(x, u, \cdot)$ (see Corollary 3.3 in [34]). First we recall the following definitions.

Definition 2.10. *A function $f : \Omega \times \mathbb{R}^k \to \mathbb{R}$ is said to be a normal integrand if*

- *f is $\mathcal{L}^N \otimes \mathcal{B}_k$-measurable;*
- *$f(x, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$;*

A function $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ is said to be a normal integrand if

- *f is $\mathcal{L}^N \otimes \mathcal{B}_d \otimes \mathcal{B}_k$-measurable;*
- *$f(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$;*

Corollary 2.11. *Suppose that the sequence of measurable functions $V_n : E \mapsto \mathbb{R}^d$ generates the Young measure (μ_x).*

1. *If $f : E \times \mathbb{R}^d \mapsto \mathbb{R}$ is a normal integrand such that the negative part $f(x, V_n(x))^-$ is weakly relatively compact in $L^1(E, \mathbb{R}^d)$, then*

$$\liminf_{n \to \infty} \int_E f(x, V_n(x))\, dx \geq \int_E \bar{f}(x)\, dx\,,$$

where

$$\bar{f}(x) := \langle \mu_x, f(x, \cdot) \rangle = \int_{\mathbb{R}^d} f(x, y)\, d\mu_x(y)\,;$$

2. *if f is a Carathéodory integrand such that $(|f(\cdot, V_n(\cdot))|)$ is equi-integrable, then*

$$\lim_{n \to \infty} \int_E f(x, V_n(x))\, dx = \int_E \bar{f}(x)\, dx < +\infty.$$

Remark 2.12. If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^d)$, then the sequence $(Du_n)_n$ is equi-integrable and generates a Young measure $\mu = (\mu_x)$ such that $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ for a.e. $x \in \Omega$ and

$$Du(x) = \int_{\mathbb{R}^{Nd}} \xi \, d\mu_x(\xi) \, .$$

Such a Young measure μ is usually called a $W^{1,p}$*-gradient Young measure*, see [36].
Moreover, by Corollary 3.4 in [34], the couple (u_n, Du_n) generates the Young measure $x \to \delta_{u(x)} \otimes \mu(x)$, and, if $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ is a normal integrand bounded from below then, by Corollary 2.11 (1), it follows that

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n(x), Du_n(x)) \, dx \geq \int_\Omega \int_{\mathbb{R}^{Nd}} f(x, u(x), \xi) d\mu_x(\xi) \, dx. \tag{2.13}$$

2.4 Γ-convergence

We recall the sequential characterization of the Γ-limit when X is a metric space.

Proposition 2.13 ([17] Proposition 8.1)**.** *Let X be a metric space and let $\varphi_n : X \to \mathbb{R} \cup \{\pm\infty\}$ for every $n \in \mathbb{N}$. Then (φ_n) Γ-converges to φ with respect to the strong topology of X (and we write $\Gamma(X)\text{-}\lim_{n\to\infty} \varphi_n = \varphi$) if and only if*

(i) (Γ-liminf inequality) *for every $x \in X$ and for every sequence (x_n) converging to x, it is*

$$\varphi(x) \leq \liminf_{n\to\infty} \varphi_n(x_n);$$

(ii) (Γ-limsup inequality) *for every $x \in X$, there exists a sequence (x_n) converging to $x \in X$ such that*

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x_n).$$

We recall that the $\Gamma\text{-}\lim_{n\to\infty} \varphi_n$ is lower semicontinuous on X (see [17] Proposition 6.8).
Finally we recall also that the function $\varphi = \Gamma(w^*\text{-}X)\text{-}\lim_{n\to\infty} \varphi_n$ is weakly* lower semicontinuous on X (see [17] Proposition 6.8) and when $\varphi_n = \psi$ $\forall n \in \mathbb{N}$ then φ coincides with the weakly* lower semicontinuous (l.s.c.) envelope of ψ, i.e.

$$\varphi(x) = \sup \big\{ h(x) : \ \forall\, h : X \to \mathbb{R} \cup \{\pm\infty\} \ \ w^* \text{ l.s.c.}, \ h \leq \psi \text{ on } X \big\} \tag{2.14}$$

(see Remark 4.5 in [17]).
We will say that a family (φ_p) Γ-converges to φ, with respect to the topology considered on X as $p \to \infty$, if (φ_{p_n}) Γ-converges to φ for all sequences (p_n) of positive numbers diverging to ∞ as $n \to \infty$.

Finally we state the fundamental theorem of Γ-convergence.

Theorem 2.14. *Let (φ_n) be an equi-coercive sequence Γ-converging on X to the function φ with respect to the topology of X. Then we have the convergence of minima*

$$\min_X \varphi = \lim_{n\to\infty} \inf_X \varphi_n.$$

Moreover we have also the convergence of minimizers: if (x_n) is such that $\lim_{n\to\infty} \varphi_n(x_n) = \lim_{n\to\infty} \inf_X \varphi_n$ then, up to subsequences, $(x_n) \to x$ and x is a minimizer for φ.

For an introduction to Γ-convergence we refer to the books [17] and [12].

3 Some technical lemmas

We devote this section to show some auxiliary results necessary in order to prove the main theorems of this paper. First of all we recall the following lemma (see [18, Lemma 3.2.6]). We need to assume $s < p^-$ in order to ensure that $\frac{p(x)}{s} \geq 1$ for a.e. $x \in \Omega$.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p : \Omega \to [1, +\infty)$ be a bounded variable exponent. Then*

$$\||u|^s\|_{\frac{p(\cdot)}{s}}^{1/s} = \|u\|_{p(\cdot)} \qquad \text{for all } u \in L^{p(\cdot)}(\Omega) \text{ and } s \in (1, p^-).$$

Proof. By definition

$$\begin{aligned} \||u|^s\|_{\frac{p(\cdot)}{s}} &= \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|u(x)|^s}{\lambda}\right)^{\frac{p(x)}{s}} dx \leq 1\right\} \\ &= \inf\left\{\lambda^s > 0 : \int_\Omega \left(\frac{|u(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\} = \|u\|_{p(\cdot)}^s. \end{aligned}$$

□

Lemma 3.2. *Let $p : \Omega \to [1, +\infty)$ be a bounded variable exponent such that $\mathcal{L}^N(\Omega) < +\infty$. Assume that there exists $\beta > 1$ such that $p^+ \leq \beta p^-$. Then for every $1 \leq q \leq p^-$ and for every $u \in L^{p(\cdot)}(\Omega)$.*

$$||u||_q \leq \max\left\{\left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}-\frac{1}{p^-}}, \left(\mathcal{L}^N(\Omega)\right)^{\beta\left(\frac{1}{q}-\frac{1}{p^+}\right)}\right\}\left[1 + \frac{q}{p^+(\cdot)}(\beta - 1)\right]^{1/q} ||u||_{p(\cdot)}.$$

In particular, if $u \in L^{p(\cdot)}(\Omega, \mathbb{R}^d)$, then $u \in L^q(\Omega, \mathbb{R}^d)$ for every $1 \leq q \leq p^-$.

Proof. Let $q \geq 1$. By using Hölder's inequality we have that

$$\int_\Omega |u(x)|^q dx \leq \left[\frac{1}{(\frac{p(\cdot)}{p(\cdot)-q})^-} + \frac{1}{(\frac{p(\cdot)}{q})^-}\right] ||1||_{\frac{p(\cdot)}{p(\cdot)-q}} |||u|^q||_{\frac{p(\cdot)}{q}} = \left[\frac{p^- - q}{p^+} + \frac{q}{p^-}\right] ||1||_{\frac{p(\cdot)}{p(\cdot)-q}} |||u|^q||_{\frac{p(\cdot)}{q}}.$$

By (2.5)

$$\|1\|_{\frac{p(\cdot)}{p(\cdot)-q}} \leq \max\left\{\left(\mathcal{L}^N(\Omega)\right)^{\frac{p^- - q}{p^+}}, \left(\mathcal{L}^N(\Omega)\right)^{\frac{p^+ - q}{p^-}}\right\} \leq \max\left\{\left(\mathcal{L}^N(\Omega)\right)^{1-\frac{q}{p^-}}, \left(\mathcal{L}^N(\Omega)\right)^{\beta\left(1-\frac{q}{p^+}\right)}\right\} =: C$$

and we get that

$$\begin{aligned} \int_\Omega |u(x)|^q dx &\leq C\left(\frac{p^+ - q}{p^+} + \frac{q}{p^-}\right) |||u|^q||_{\frac{p(\cdot)}{q}} \\ &= C\left[1 + \frac{q}{p^+}\left(\frac{p^+}{p^-} - 1\right)\right] |||u|^q||_{\frac{p(\cdot)}{q}} \\ &\leq C\left[1 + \frac{q}{p^+}(\beta - 1)\right] |||u|^q||_{\frac{p(\cdot)}{q}}, \end{aligned}$$

where we used the fact that $p^+ \leq \beta p^-$; this in turn implies

$$||u||_q \leq C^{1/q}\left[1 + \frac{q}{p^+}(\beta - 1)\right]^{1/q} |||u|^q||_{\frac{p(\cdot)}{q}}^{1/q}.$$

By Lemma 3.1, we get

$$||u||_q \leq \max\left\{ \left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}-\frac{1}{p^-}}, \left(\mathcal{L}^N(\Omega)\right)^{\beta\left(\frac{1}{q}-\frac{1}{p^+}\right)} \right\} \left[1+\frac{q}{p^+}(\beta-1)\right]^{1/q} ||u||_{p(\cdot)}.$$

□

In [9, Lemma 2], by assuming that $\mathcal{L}^N(\Omega) = 1$ and that the sequence of Lipschitz continuous functions $p_n : \Omega \to (1, +\infty)$ satisfies the conditions:

$$p_n^- \to +\infty \qquad \text{as } n \to +\infty \tag{3.1}$$

$$\exists\, \beta > 1 : \; p_n^+ \leq \beta\, p_n^- < +\infty \quad \forall n \in \mathbb{N}, \tag{3.2}$$

the authors show that the L^∞-norm is the limit of the $L^{p_n(\cdot)}$-norms. We improve their result by showing that if the limit of the $L^{p_n(\cdot)}$-norms of a measurable function u is finite, then $u \in L^\infty(\Omega)$. Here we consider a sequence of bounded variable exponents $p_n : \Omega \to [1, +\infty)$, according to Definition 2.1, and we drop the Lipschitz continuity assumption required in [9]. Moreover, for sake of completeness, we give the detailed proof when $\mathcal{L}^N(\Omega) \in (0, +\infty)$.

Proposition 3.3. *Assume $\mathcal{L}^N(\Omega) < +\infty$ and let $p_n : \Omega \to [1, +\infty)$ be a sequence of bounded variable exponents satisfying* (3.1) *and* (3.2). *Let $u : \Omega \to \bar{\mathbb{R}}$ be a measurable function. Then the following properties are equivalent:*

(i) $u \in L^\infty(\Omega)$*;*

(ii) $\lim_{n\to\infty} ||u||_{p_n(\cdot)} \in \mathbb{R}$.

Moreover if (i) or (ii) holds, then

$$||u||_\infty = \lim_{n\to\infty} ||u||_{p_n(\cdot)}. \tag{3.3}$$

Proof. $\boxed{(i) \Longrightarrow (ii)}$ Note that, in order to show (3.3), it sufficient to prove that

$$\lim_{n\to\infty} ||u||_{p_n(\cdot)} = 1 \quad \forall u \in L^\infty(\Omega) \text{ s.t. } ||u||_\infty = 1.$$

Let $u \in L^\infty(\Omega)$ such that $||u||_\infty = 1$. Then for every $n \in \mathbb{N}$ we get

$$\int_\Omega |u(x)|^{p_n(x)} dx \leq \mathcal{L}^N(\Omega). \tag{3.4}$$

Since $|u(x)| \leq 1$ for a.e. $x \in \Omega$, we have that for every $n \in \mathbb{N}$

$$\int_\Omega |u(x)|^{p_n(x)} dx \geq \int_\Omega |u(x)|^{p_n^+} dx$$

and, thanks to (3.4), we get that

$$1 = \lim_{n\to\infty} (\mathcal{L}^N(\Omega))^{1/p_n^+} \geq \lim_{n\to\infty} \left(\int_\Omega |u(x)|^{p_n(x)} dx\right)^{1/p_n^+} \geq \lim_{n\to\infty} \left(\int_\Omega |u(x)|^{p_n^+} dx\right)^{1/p_n^+} = ||u||_\infty = 1$$

that is

$$\lim_{n\to\infty} \left(\int_\Omega |u(x)|^{p_n(x)} dx\right)^{1/p_n^+} = 1. \tag{3.5}$$

Due to (3.2), the sequence $\beta_n = \left(\frac{p_n^+}{p_n^-}\right)_n$ satisfies $1 \le \beta_n \le \beta$. Then (3.5) implies

$$\lim_{n\to\infty}\left(\int_\Omega |u(x)|^{p_n(x)}dx\right)^{1/p_n^-} = \lim_{n\to\infty}\left(\int_\Omega |u(x)|^{p_n(x)}dx\right)^{\beta_n/p_n^+} = 1. \tag{3.6}$$

Moreover, by (2.4), we have that for every $n \in \mathbb{N}$

$$\min\left\{\left(\int_\Omega |u(x)|^{p_n(x)}dx\right)^{1/p_n^-}, \left(\int_\Omega |u(x)|^{p_n(x)}dx\right)^{1/p_n^+}\right\} \le ||u||_{p_n(\cdot)}$$

$$\le \max\left\{\left(\int_\Omega |u(x)|^{p_n(x)}dx\right)^{1/p_n^-}, \left(\int_\Omega |u(x)|^{p_n(x)}dx\right)^{1/p_n^+}\right\}.$$

Then, taking into account (3.5) and (3.6), when we pass to the limit when $n \to \infty$, we get

$$\lim_{n\to\infty} ||u||_{p_n(\cdot)} = 1.$$

$\boxed{(ii) \Longrightarrow (i)}$ Assume now that $\lim_{n\to\infty} ||u||_{p_n(\cdot)} \in \mathbb{R}$. Let $q \ge 1$. Thanks to (3.1), there exists $n_0 = n_0(q) \in \mathbb{N}$ big enough such that $p_n^- > q$ for every $n \ge n_0$.
By Lemma 3.2, we get that

$$||u||_q \le \max\left\{\left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}-\frac{1}{p_n^-}}, \left(\mathcal{L}^N(\Omega)\right)^{\beta\left(\frac{1}{q}-\frac{1}{p_n^+}\right)}\right\}\left[1+\frac{q}{p_n^+(\cdot)}(\beta-1)\right]^{1/q}||u||_{p_n(\cdot)}$$

for every $n \ge n_0$.
Since for every $q \ge 1$ we have that

$$\lim_{n\to\infty}\left[1+\frac{q}{p_n^+(\cdot)}(\beta-1)\right]^{1/q} = 1 \tag{3.7}$$

by passing to the limit when $n \to \infty$ it follows that

$$\begin{aligned} ||u||_q &\le \lim_{n\to\infty}\max\left\{\left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}-\frac{1}{p_n^-}}, \left(\mathcal{L}^N(\Omega)\right)^{\beta\left(\frac{1}{q}-\frac{1}{p_n^+}\right)}\right\}\left[1+\frac{q}{p_n^+(\cdot)}(\beta-1)\right]^{1/q}||u||_{p_n(\cdot)} \\ &= \max\left\{\left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}}, \left(\mathcal{L}^N(\Omega)\right)^{\frac{\beta}{q}}\right\}\lim_{n\to\infty}||u||_{p_n(\cdot)} \in \mathbb{R} \quad \forall q \ge 1. \end{aligned}$$

This implies

$$\lim_{q\to\infty}||u||_q \le \lim_{q\to\infty}\left[\max\left\{\left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}}, \left(\mathcal{L}^N(\Omega)\right)^{\frac{\beta}{q}}\right\}\lim_{n\to\infty}||u||_{p_n(\cdot)}\right] \le \lim_{n\to\infty}||u||_{p_n(\cdot)} \in \mathbb{R}.$$

Then $u \in L^\infty(\Omega)$ and by the first part of this proof, it holds $||u||_\infty = \lim_{n\to\infty}||u||_{p_n(\cdot)}$.
□

We conclude this section with the following lemma, already shown in [1] when $f = f(x,\xi)$ is a Carathéodory integrand (see Lemma 4.5 therein). For the reader's convenience we report here the proof.

Lemma 3.4. *Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^+$ be a normal integrand. Then*

$$\lim_{q\to\infty} \left(\int_\Omega \int_{\mathbb{R}^k} f(x, v(x), \xi)^q d\mu_x(\xi) dx \right)^{1/q} = \operatorname*{ess\,sup}_{x\in\Omega} \left(\mu_x\text{-} \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^k} f(x, v(x), \xi) \right),$$

for every Young measure $\mu = (\mu_x)$ and for every measurable function $v : \Omega \to \mathbb{R}^d$.

Proof. Taking into account Theorem 2.8, part (1), the following inequality

$$\begin{aligned}
& \limsup_{q\to\infty} \left(\int_\Omega \int_{\mathbb{R}^k} f(x, v(x), \xi)^q d\mu_x(\xi) dx \right)^{1/q} \\
\leq\ & \limsup_{q\to\infty} \left(\int_\Omega \mu_x(\mathbb{R}^k) \mu_x\text{-} \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^k} (f(x, v(x), \xi))^q dx \right)^{1/q} \\
\leq\ & \limsup_{q\to\infty} \left(\int_\Omega \left(\mu_x\text{-} \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^k} f(x, v(x), \xi) \right)^q dx \right)^{1/q} \\
\leq\ & \operatorname*{ess\,sup}_{x\in\Omega} \left(\mu_x\text{-} \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^k} f(x, v(x), \xi) \right)
\end{aligned}$$

is straighforward, by the convergence of the L^q norms to the L^∞ norm. Let us prove that

$$\liminf_{q\to\infty} \left(\int_\Omega \int_{\mathbb{R}^k} f(x, v(x), \xi)^q d\mu_x(\xi) dx \right)^{1/q} \geq \operatorname*{ess\,sup}_{x\in\Omega} \left(\mu_x\text{-} \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^k} f(x, v(x), \xi) \right).$$

Without loss of generality we assume that

$$\liminf_{q\to\infty} \left(\int_\Omega \int_{\mathbb{R}^k} f(x, v(x), \xi)^q d\mu_x(\xi) dx \right)^{1/q} < +\infty. \tag{3.8}$$

For every fixed exponent r such that $q > r$, by applying Hölder's inequality we get that

$$\left(\int_\Omega \int_{\mathbb{R}^k} f(x, v(x), \xi)^q d\mu_x(\xi) dx \right)^{1/q} \geq \left(\int_\Omega \left(\int_{\mathbb{R}^k} f(x, v(x), \xi)^r d\mu_x(\xi) \right)^{q/r} dx \right)^{1/q}. \tag{3.9}$$

Passing to the limit as $q \to \infty$, by the convergence of the L^q-norm to the L^∞-norm, we have that

$$\lim_{q\to\infty} \left(\int_\Omega \left(\int_{\mathbb{R}^k} f(x, v(x), \xi)^r d\mu_x(\xi) \right)^{q/r} dx \right)^{1/q} = \operatorname*{ess\,sup}_{x\in\Omega} \left(\int_{\mathbb{R}^k} f(x, v(x), \xi)^r d\mu_x(\xi) \right)^{1/r}. \tag{3.10}$$

We now denote

$$g_r(x) := \left(\int_{\mathbb{R}^k} f(x, v(x), \xi)^r d\mu_x(\xi) \right)^{1/r}.$$

Then (g_r) is an increasing positive family pointwise converging to the function

$$g(x) := \mu_x\text{-} \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^k} f(x, v(x), \xi)$$

as $r \to \infty$. Moreover, by (3.8)-(3.10), we have that $\sup_r ||g_r||_\infty < +\infty$. In particular, by Lebesgue's dominated convergence theorem, we have that $g_r \rightharpoonup g$ weakly* in L^∞. By (3.9), (3.10) and the weak* lower semicontinuity of the L^∞-norm, we have that

$$\liminf_{q\to\infty} \left(\int_\Omega \int_{\mathbb{R}^k} f(x, v(x), \xi)^q d\mu_x(\xi) dx \right)^{1/q} \geq \operatorname*{ess\,sup}_{x\in\Omega} \left(\mu_x\text{- }\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^k} f(x, v(x), \xi) \right),$$

which concludes the proof. □

4 The L^p approximation

In this section we study the L^p-approximation, via Γ-convergence, of supremal functionals. In the following we consider a normal integrand $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ satisfying the following assumptions:

(H1) for a.e. $x \in \Omega$, $f(x, u, \cdot)$ is level convex for every $u \in \mathbb{R}^d$;

(H2) there exist $\alpha, \gamma > 0$ such that

$$f(x, u, \xi) \geq \alpha|\xi|^\gamma \qquad \text{for a.e } x \in \Omega, \text{ for every } (u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{Nd}. \tag{4.1}$$

4.1 Statement of the main results

We start by stating all theorems to easily compare the results obtained according to the different set of hypotheses and topologies considered.
The following result requires a regularity assumption of Ω in the proof of the Γ-liminf inequality since we use the Sobolev imbedding, but we drop the hypothesis that Ω is connected (used in the proof given in [9] when the authors use the Poincarè-Wirtinger inequality).

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ be a normal integrand satisfying assumptions* **(H1)** *and* **(H2)***. Let $F_n : L^1(\Omega, \mathbb{R}^d) \to [0, +\infty]$ be the functional defined by*

$$F_n(u) := \begin{cases} ||f(\cdot, u(\cdot), Du(\cdot))||_{p_n(\cdot)} & \text{if } u \in W^{1,p_n(\cdot)}(\Omega, \mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.2}$$

where $p_n : \Omega \to [1, +\infty)$ is a sequence of bounded variable exponents satisfying (3.1) *and* (3.2)*. Let $F : L^1(\Omega, \mathbb{R}^d) \to [0, +\infty]$ be the functional defined by*

$$F(u) := \begin{cases} \operatorname*{ess\,sup}_\Omega f(x, u(x), Du(x)) & \text{if } u \in W^{1,\infty}(\Omega, \mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.3}$$

Then,

(i) for every $u \in L^1(\Omega, \mathbb{R}^d)$ and $(u_n) \subset L^1(\Omega, \mathbb{R}^d)$ such that $u_n \rightharpoonup u$ in $L^1(\Omega, \mathbb{R}^d)$, we have

$$F(u) \leq \liminf_{n\to\infty} F_n(u_n);$$

(ii) for every $u \in L^1(\Omega, \mathbb{R}^d)$ there exists $(u_n) \subset L^1(\Omega, \mathbb{R}^d)$ such that $u_n \to u$ in $L^1(\Omega, \mathbb{R}^d)$ and

$$\limsup_{n\to\infty} F_n(u_n) \leq F(u).$$

In particular, (F_n) Γ- *converges to* F*, as* $n \to +\infty$*, with respect to the* L^1-**strong convergence**.

The following result instead does not require any regularity assumption of Ω.

Theorem 4.2. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a bounded open set. Let* $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ *be a normal integrand satisfying assumptions* **(H1)** *and* **(H2)**. *Let* $X \in \{L^\infty(\Omega, \mathbb{R}^d), C(\Omega, \mathbb{R}^d)\}$ *be endowed with the norm* $||\cdot||_\infty$. *Let* $F_n : X \to [0, +\infty]$ *be the functional defined by*

$$F_n(u) := \begin{cases} ||f(\cdot, u(\cdot), Du(\cdot))||_{p_n(\cdot)} & \text{if } u \in W^{1,p_n(\cdot)}(\Omega, \mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.4}$$

where $p_n : \Omega \to [1, +\infty)$ *is a sequence of bounded variable exponents satisfying* (3.1) *and* (3.2). *Then, as* $n \to +\infty$*, the sequence* (F_n) Γ- *converges, with respect to the* L^∞-**strong convergence**, *to the functional* $F : X \to [0, +\infty]$ *given by*

$$F(u) := \begin{cases} \operatorname*{ess\,sup}_{\Omega} f(x, u(x), Du(x)) & \text{if } u \in W^{1,\infty}(\Omega, \mathbb{R}^d), \\ +\infty & \text{otherwise in } X. \end{cases} \tag{4.5}$$

As a corollary, by applying the previous result when (p_n) is an arbitrary real sequence diverging to $+\infty$, we get the following improvement of Theorem 3.1 in [15].

Corollary 4.3. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a bounded open set. Let* $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ *be a normal integrand satisfying assumptions* **(H1)** *and* **(H2)**. *Let* $X \in \{L^\infty(\Omega, \mathbb{R}^d), C(\Omega, \mathbb{R}^d)\}$ *be endowed with the norm* $||\cdot||_\infty$. *For every* $p \geq 1$ *let* $F_p : X \to [0, +\infty]$ *be the functional defined by*

$$F_p(u) := \begin{cases} ||f(\cdot, u(\cdot), Du(\cdot))||_p & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.6}$$

and let F *be the functional defined by* (4.5). *Then,* (F_p) Γ*-converges to* F*, as* $p \to +\infty$*, with respect to the* L^∞-**strong convergence**.

Finally we show the following results:

Theorem 4.4. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a bounded open set with Lipschitz boundary. Let* $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ *be a normal integrand satisfying assumptions* **(H1)** *and* **(H2)**. *Let* $\mathcal{F}_n : L^1(\Omega, \mathbb{R}^d) \to [0, +\infty]$ *be the functional defined by*

$$\mathcal{F}_n(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{p_n(x)} f^{p_n(x)}(x, u(x), Du(x))dx & \text{if } u \in W^{1,p_n(\cdot)}(\Omega, \mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.7}$$

where $p_n : \Omega \to [1, +\infty)$ *is a sequence of bounded variable exponents satisfying* (3.1) *and* (3.2). *Let* $\mathcal{F} : L^1(\Omega, \mathbb{R}^d) \to [0, +\infty]$ *be the functional defined by*

$$\mathcal{F}(u) := \begin{cases} 0 & \text{if } u \in W^{1,\infty}(\Omega, \mathbb{R}^d) \text{ and } ||f(x, u(x), Du(x))||_\infty \leq 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.8}$$

Then,

(i) *for every* $u \in L^1(\Omega, \mathbb{R}^d)$ *and* $(u_n) \subset L^1(\Omega, \mathbb{R}^d)$ *such that* $u_n \rightharpoonup u$ *in* $L^1(\Omega, \mathbb{R}^d)$*, we have*

$$\mathcal{F}(u) \leq \liminf_{n \to \infty} \mathcal{F}_n(u_n);$$

(ii) for every $u \in L^1(\Omega, \mathbb{R}^d)$ there exists $(u_n) \subset L^1(\Omega, \mathbb{R}^d)$ such that $u_n \to u$ in $L^1(\Omega, \mathbb{R}^d)$ and

$$\limsup_{n\to\infty} \mathcal{F}_n(u_n) \le \mathcal{F}(u).$$

*In particular, $(\mathcal{F}_n)$ Γ- converges to $\mathcal{F}$, as $n \to +\infty$, with respect to the L^1-***strong convergence***.*

Theorem 4.5. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ be a normal integrand satisfying assumptions* **(H1)** *and* **(H2)***. Let $X \in \{L^\infty(\Omega, \mathbb{R}^d), C(\Omega, \mathbb{R}^d)\}$ be endowed with the norm $||\cdot||_\infty$. Let $\mathcal{F}_n : X \to [0, +\infty]$ be the functional defined by* (4.7) *with $p_n : \Omega \to [1, +\infty)$ a sequence of bounded variable exponents satisfying* (3.1) *and* (3.2)*. Let $\mathcal{F} : X \to [0, +\infty]$ be the functional defined by* (4.8)*. Then, as $n \to +\infty$, $(\mathcal{F}_n)$ Γ- converges $\mathcal{F}$, with respect to the L^∞-***strong convergence***.*

4.2 Proofs of Theorems

Proof of Theorem 4.1. First of all we consider the case when $\gamma \ge 1$. We observe that

$$\limsup_{n\to\infty}\ F_n(u) \le F(u) \tag{4.9}$$

for any $u \in L^1(\Omega, \mathbb{R}^d)$. Indeed, if $F(u) = +\infty$, there is nothing to prove, and if $F(u) < +\infty$, then, $u \in W^{1,\infty}(\Omega, \mathbb{R}^d)$ and $f(\cdot, u(\cdot), Du(\cdot)) \in L^\infty(\Omega)$. By Proposition 3.3 we have that

$$\lim_{n\to\infty} F_n(u) = ||f(\cdot, u(\cdot), Du(\cdot))||_\infty = F(u)$$

so that (4.9) follows. As a consequence, for any $u \in L^1(\Omega, \mathbb{R}^d)$ it holds

$$\Gamma(L^1)\text{-}\limsup_{n\to\infty} F_n(u) \le \limsup_{n\to\infty} F_n(u) \le F(u)\,.$$

We now deal with the liminf inequality. Let $(u_n) \in L^1(\Omega, \mathbb{R}^d)$ be a sequence weakly converging to $u \in L^1(\Omega, \mathbb{R}^d)$. Without loss of generality, we can assume that

$$\liminf_{n\to\infty} F_n(u_n) = \lim_{n\to\infty} F_n(u_n) = M < +\infty, \tag{4.10}$$

hence, by definition of the functionals F_n, we have that there exists $n_0 \in \mathbb{N}$ such that $u_n \in W^{1,p_n(\cdot)}(\Omega, \mathbb{R}^d)$ for every $n \ge n_0$. Fix $q > 1$ and let $n_1 \ge n_0$ be such that, in view of (3.1)

$$p_n^- \ge q \text{ and } F_n(u_n) \le M + 1 \qquad \forall\, n \ge n_1. \tag{4.11}$$

Then, by applying Lemma 3.2 to u_n and to Du_n with $p(\cdot) = p_n(\cdot)$, we get that $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$ for every $n \ge n_1$; on the other hand, still be Lemma 3.2, we also get the estimate

$$||u_n||_q \le c_{q,n} ||u_n||_{p_n(\cdot)} \qquad \forall n \ge n_1, \tag{4.12}$$

where

$$c_{q,n} := \max\left\{ \left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}-\frac{1}{p_n^-}}, \left(\mathcal{L}^N(\Omega)\right)^{\beta\left(\frac{1}{q}-\frac{1}{p_n^+}\right)} \right\} \left[1 + \frac{q}{p_n^+(\cdot)}(\beta - 1)\right]^{1/q}.$$

Moreover the function $v_n(\cdot) := f(x, u_n(\cdot), Du_n(\cdot)) \in L^{p_n(\cdot)}(\Omega)$ $\forall\, n \ge n_1$ and, by applying again Lemma 3.2, this time to v_n, we obtain that for every $n \ge n_1$

$$\begin{aligned} ||f(x, u_n(\cdot), Du_n(\cdot))||_q &\le c_{q,n} ||f(x, u_n(\cdot), Du_n(\cdot))||_{p_n(\cdot)} \\ &\le (M + 1) c_{q,n}, \end{aligned} \tag{4.13}$$

where we used (4.11). Note that for every fixed $q>1$ the sequence $(c_{q,n})_n$ is bounded since, by (3.1)

$$\lim_{n\to\infty} c_{q,n} = \max\left\{\left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}}, \left(\mathcal{L}^N(\Omega)\right)^{\frac{\beta}{q}}\right\} := c_q < +\infty.$$

Taking into account the growth condition (4.1), (4.13) implies that for every $n \geq n_1$

$$\begin{aligned} ||Du_n||_q^\gamma &\leq \left(\mathcal{L}^N(\Omega)\right)^{\frac{\gamma}{q}-\frac{1}{q}} ||Du_n||_{\gamma q}^\gamma \\ &\leq \left(\mathcal{L}^N(\Omega)\right)^{\frac{\gamma}{q}-\frac{1}{q}} \frac{1}{\alpha} ||f(x,u_n(\cdot),Du_n(\cdot))||_q \leq \left(\mathcal{L}^N(\Omega)\right)^{\frac{\gamma}{q}-\frac{1}{q}} \frac{M+1}{\alpha} c_{q,n}. \end{aligned}$$

that is

$$||Du_n||_q \leq \left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}-\frac{1}{\gamma q}} \left(\frac{M+1}{\alpha} c_{q,n}\right)^{\frac{1}{\gamma}}. \tag{4.14}$$

In particular

$$\sup_{n\geq n_1} ||Du_n||_q \leq \left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}-\frac{1}{\gamma q}} \left(\frac{M+1}{\alpha} \sup_{n\geq n_1} c_{q,n}\right)^{\frac{1}{\gamma}} < +\infty. \tag{4.15}$$

Then, up to a subsequence (depending on q), $(Du_n)_n$ weakly converges to a function w in $L^q(\Omega,\mathbb{R}^{Nd})$. Since $(u_n)_n$ weakly converges to u in $L^1(\Omega,\mathbb{R}^d)$, it is easy to show that w is the distributional gradient of u. In particular $u \in W^{1,1}(\Omega,\mathbb{R}^d)$ and, since every subsequence of $(Du_n)_n$ admits a subsequence converging to Du, we get that the whole sequence $Du_n \rightharpoonup Du$ weakly in $L^q(\Omega,\mathbb{R}^{Nd})$. Now we show that $u \in W^{1,q}(\Omega,\mathbb{R}^d)$. Note that, being $u \in W^{1,1}(\Omega,\mathbb{R}^d)$, thanks to the Sobolev immersion, we get that $u \in L^{1^*}(\Omega,\mathbb{R}^d) = L^{\frac{N}{N-1}}(\Omega,\mathbb{R}^d)$. Since $Du \in L^{\frac{N}{N-1}}(\Omega,\mathbb{R}^{Nd})$, we deduce that $u \in W^{1,\frac{N}{N-1}}(\Omega,\mathbb{R}^d)$. Then $u \in L^{(\frac{N}{N-1})^*}(\Omega,\mathbb{R}^d) = L^{\frac{N}{N-2}}(\Omega,\mathbb{R}^d)$. By going on, after $k=N-1$ steps we get that

$$u \in L^{(\frac{N}{N-(k-1)})^*}(\Omega,\mathbb{R}^d) = L^{\frac{N}{N-k}}(\Omega,\mathbb{R}^d) = L^N(\Omega,\mathbb{R}^d)$$

that is $u \in W^{1,N}(\Omega,\mathbb{R}^d)$. By Sobolev immersion, we can conclude that $u \in L^q(\Omega,\mathbb{R}^d)$ for every $q \geq N$ and, since $Du \in L^q(\Omega,\mathbb{R}^{Nd})$ for every $q \geq 1$, we obtain that $u \in W^{1,q}(\Omega,\mathbb{R}^d)$ for every $q \geq 1$ and $u_n \rightharpoonup u$ weakly in $W^{1,q}(\Omega,\mathbb{R}^d)$. In particular $u \in L^\infty(\Omega,\mathbb{R}^d)$.
Moreover, taking into account (4.14), we get

$$||Du||_q \leq \liminf_{n\to\infty} ||Du_n||_q \leq \left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}-\frac{1}{\gamma q}} \left(\frac{M+1}{\alpha} c_q\right)^{\frac{1}{\gamma}} \quad \forall q>1$$

that implies, taking into account that $c_q \to 1$ when $q \to \infty$,

$$\lim_{q\to\infty} ||Du||_q \leq \left(\frac{M+1}{\alpha}\right)^{\frac{1}{\gamma}} < \infty \tag{4.16}$$

i.e. $Du \in L^\infty(\Omega,\mathbb{R}^{Nd})$ and $u \in W^{1,\infty}(\Omega,\mathbb{R}^d)$.
By Remarks 2.12 and 2.9, we have that (Du_n) generates a Young measure $(\mu_x)_{x\in\Omega}$ such that $\mu_x(\mathbb{R}^{Nd}) = 1$ and

$$Du(x) = \int_{\mathbb{R}^{Nd}} \xi \, d\mu_x(\xi) \tag{4.17}$$

for a.e. $x \in \Omega$. Then, for any fixed $q > N$, by applying (4.13) and (2.13), we have that

$$\begin{aligned}\liminf_{n\to\infty} F_n(u_n) &\geq \liminf_{n\to+\infty} \frac{1}{c_{q,n}} ||f(\cdot, u_n(\cdot), Du_n(\cdot)||_q \\ &= \frac{1}{c_q} \liminf_{n\to\infty} \left(\int_\Omega f^q(x, u_n(x), Du_n(x))dx \right)^{1/q} \\ &\geq \left(\int_\Omega \int_{\mathbb{R}^{Nd}} f^q(x, u(x), \xi) d\mu_x(\xi) dx \right)^{1/q}.\end{aligned}$$

By applying Lemma 3.4 we obtain

$$\begin{aligned}\liminf_{n\to\infty} F_n(u_n) &\geq \liminf_{q\to\infty} \left(\int_\Omega \int_{\mathbb{R}^{Nd}} f^q(x, u(x), \xi) d\mu_x(\xi) dx \right)^{1/q} \\ &= \operatorname*{ess\,sup}_{x\in\Omega} \left(\mu_x\text{-} \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^{Nd}} f(x, u(x), \xi) \right).\end{aligned} \tag{4.18}$$

Since $f(x, u(x), \cdot)$ is level convex for a.e. $x \in \Omega$, taking into account (4.17), by Jensen's inequality (2.11) we have that

$$f(x, u(x), Du(x)) = f\left(x, u(x), \int_{\mathbb{R}^{Nd}} \xi \, d\mu_x(\xi) \right) \leq \mu_x\text{-} \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^{Nd}} f(x, u(x), \xi)$$

for a.e. $x \in \Omega$. In particular

$$\operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), Du(x)) \leq \operatorname*{ess\,sup}_{x\in\Omega} \left(\mu_x\text{-} \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^{Nd}} f(x, u(x), \xi) \right). \tag{4.19}$$

Then, by the very definition of F, we get

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), Du(x))$$

and (4.19) and (4.18) imply the Γ-liminf inequality.
Thus, the proof in the case $\gamma \geq 1$ is concluded. Assume now that $0 < \gamma < 1$. First of all we observe that, since the function $t \to t^{\frac{1}{\gamma}}$ is monotone on $[0, +\infty)$, then the function $g(x, u, \xi) := f^{\frac{1}{\gamma}}(x, u, \xi)$ is level convex too with respect to the gradient variable and satisfies the grouth condition

$$g(x, u, \xi) \geq \alpha' |\xi|$$

for a.e $x \in \Omega$, for every $(u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{Nd}$, with $\alpha' = \alpha^{\frac{1}{\gamma}}$.
Then, we get that the sequence of the functionals $G_n : L^1(\Omega, \mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$G_n(u) := \begin{cases} ||g(\cdot, u(\cdot), Du(\cdot))||_{\gamma p_n(\cdot)} & \text{if } u \in W^{1,\gamma p_n}(\Omega, \mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.20}$$

Γ-converges to $G : L^1(\Omega, \mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$G(u) := \begin{cases} \operatorname*{ess\,sup}_{\Omega} g(x, u(x), Du(x)) & \text{if } u \in W^{1,\infty}(\Omega, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.21}$$

with respect to the L^1- strong convergence. Since $p_n^- \to +\infty$, for n big enough we have that $\frac{1}{\gamma} < p_n^-$. Then, by Lemma 3.1 applied with $s = \frac{1}{\gamma}$, we get that

$$||g(\cdot, u(\cdot), Du(\cdot))||^{\gamma}_{\gamma p_n(\cdot)} = ||f^{\frac{1}{\gamma}}(\cdot, u(\cdot), Du(\cdot))||^{\gamma}_{\gamma p_n(\cdot)} = ||f(\cdot, u(\cdot), Du(\cdot))||_{p_n(\cdot)}. \tag{4.22}$$

Moreover, since $\gamma < 1$, we have that

$$W^{1,n}(\Omega, \mathbb{R}^d) \subseteq W^{1,\gamma n}(\Omega, \mathbb{R}^d).$$

Thus, taking into account (4.22), we get

$$G_n^{\gamma} \le F_n \le F.$$

By passing to the Γ-limit when $n \to \infty$ with respect to the L^1-convergence and noticing that G_n^{γ} Γ-converges to $G^{\gamma} = F$, we get the thesis. $\square$

Proof of Theorem 4.2. As in the proof of Theorem 4.1 it is sufficient to prove the result in the case $\gamma \ge 1$. The proof of the Γ-limsup inequality follows the same arguments as in Theorem 4.1. In order to get the Γ-liminf inequality, it is sufficient to note that if $(u_n) \subseteq X$ is a sequence L^{∞}-converging to u in X, then (u_n) weakly L^q-converges to u for every $q \ge 1$. By applying inequality (4.15) we get that the sequence $(Du_n)_n$ weakly converges to Du in $L^q(\Omega, \mathbb{R}^d)$ for every $q > 1$. In particular $(u_n)_n$ converges weakly to u in $W^{1,q}(\Omega, \mathbb{R}^{Nd})$ for every $q > N$. Then (Du_n) generates a Young measure $(\mu_x)_{x\in\Omega}$ such that $\mu_x(\mathbb{R}^{Nd}) = 1$ and $Du(x) = \int_{\mathbb{R}^{dN}} \xi \, d\mu_x(\xi)$ for a.e. $x \in \Omega$. Thus the Γ-liminf inequality follows by applying Jensen's inequality (2.11) and Lemma 3.4 in order to get (4.19). $\square$

Proof of Corollary 4.3. It is sufficient to apply Theorem 4.2 to get that, for every sequence (p_n) diverging to ∞ as $n \to \infty$, the sequence (F_n), defined by (4.4), Γ-converges to F with respect to the L^{∞} strong convergence. $\square$

Proof of Theorem 4.4. We observe that

$$\limsup_{n\to\infty} \ \mathcal{F}_n(u) \le \mathcal{F}(u) \tag{4.23}$$

for any $u \in L^1(\Omega, \mathbb{R}^d)$. Indeed, if $\mathcal{F}(u) = +\infty$, there is nothing to prove, and if $\mathcal{F}(u) < +\infty$, then $\mathcal{F}(u) = 0$ that implies $u \in W^{1,\infty}(\Omega, \mathbb{R}^d)$ and $||f(\cdot, u(\cdot), Du(\cdot))||_{\infty} \le 1$. In particular

$$\limsup_{n\to\infty} \mathcal{F}_n(u) = \limsup_{n\to\infty} \int_{\Omega} \frac{1}{p_n(x)} f^{p_n(x)}(\cdot, u(\cdot), Du(\cdot))dx \le \mathcal{L}^N(\Omega) \limsup_{n\to\infty} \frac{1}{p_n^-} = 0.$$

Then it is sufficient to take $u_n = u$ to get the Γ-limsup inequality. We now deal with the Γ-liminf inequality. Let $(u_n) \in L^1(\Omega, \mathbb{R}^d)$ be a sequence weakly converging to $u \in L^1(\Omega, \mathbb{R}^d)$. Without loss of generality, we can assume that

$$\liminf_{n\to\infty} \mathcal{F}_n(u_n) = \lim_{n\to\infty} \mathcal{F}_n(u_n) = M < +\infty, \tag{4.24}$$

hence, by definition of the functionals $\mathcal{F}_n$, we have that there exists $n_0 \in \mathbb{N}$ such that $\mathcal{F}_n(u_n) \le 2M$ for every $n \ge n_0$. In particular $u_n \in W^{1,p_n(\cdot)}(\Omega, \mathbb{R}^d)$ for every $n \ge n_0$. For each $n \in \mathbb{N}$, define

$$\Omega_n^+ := \{x \in \Omega : f(x, u_n(x), Du_n(x)) > 1\} \qquad \text{and} \qquad \Omega_n^- := \{x \in \Omega : f(x, u_n(x), Du_n(x)) \le 1\}.$$

Then, for every $n \geq n_0$ it holds

$$\begin{aligned}
&\frac{1}{p_n^+}\int_\Omega f^{p_n(x)}(x,u_n(x),Du_n(x))dx\\
\leq\ & \int_\Omega \frac{1}{p_n(x)} f^{p_n(x)}(x,u_n(x),Du_n(x))dx\\
=\ & \int_{\Omega_n^+} \frac{1}{p_n(x)} f^{p_n(x)}(x,u_n(x),Du_n(x))dx + \int_{\Omega_n^-} \frac{1}{p_n(x)} f^{p_n(x)}(x,u_n(x),Du_n(x))dx\\
\leq\ & \int_{\Omega_n^+} \frac{1}{p_n(x)} f^{p_n(x)}(x,u_n(x),Du_n(x))dx + \mathcal{L}^N(\Omega)\frac{1}{p_n^-}\\
\leq\ & \int_{\Omega} \frac{1}{p_n(x)} f^{p_n(x)}(x,u_n(x),Du_n(x))dx + \mathcal{L}^N(\Omega)\frac{1}{p_n^-}\\
\leq\ & 2M + \mathcal{L}^N(\Omega)\frac{1}{p_n^-}.
\end{aligned}$$

In particular

$$\int_\Omega f^{p_n(x)}(x,u_n(x),Du_n(x))dx \leq p_n^+\left(2M + \mathcal{L}^N(\Omega)\frac{1}{p_n^-}\right)$$

that implies

$$\left[\rho_{p_n(\cdot)}(f(x,u_n(\cdot),Du_n(\cdot)))\right]^{\frac{1}{p_n^+}} \leq \left[p_n^+\left(2M + \mathcal{L}^N(\Omega)\frac{1}{p_n^-}\right)\right]^{\frac{1}{p_n^+}} = \left[2Mp_n^+ + \mathcal{L}^N(\Omega)\frac{p_n^+}{p_n^-}\right]^{\frac{1}{p_n^+}} = M(n)$$

and also, replacing p_n^+ with p_n^- in the exponent of the modular in the left hand side

$$\left[\rho_{p_n(\cdot)}(f(x,u_n(\cdot),Du_n(\cdot)))\right]^{\frac{1}{p_n^-}} \leq \left[p_n^+\left(2M + \mathcal{L}^N(\Omega)\frac{1}{p_n^-}\right)\right]^{\frac{1}{p_n^-}} = (M(n))^{\frac{p_n^+}{p_n^-}}.$$

Taking into account (2.4), it follows

$$\|f(\cdot,u_n(\cdot),Du_n)\|_{p_n(\cdot)} \leq \max\left\{M(n),(M(n))^{\frac{p_n^+}{p_n^-}}\right\} \tag{4.25}$$

Since $(\frac{p_n^+}{p_n^-})_n$ is a bounded sequence, $M(n)\to 1$ and $\frac{1}{p_n^+}\to 0$ when $n\to\infty$, the previous inequality implies that the sequence $(\|f(\cdot,u_n(\cdot),Du_n)\|_{p_n(\cdot)})_n$ is bounded and

$$\liminf_{n\to\infty}\|f(\cdot,u_n(\cdot),Du_n)\|_{p_n(\cdot)} \leq 1.$$

Moreover, by applying the Γ-liminf inequality in Theorem 4.1, we obtain that $u\in W^{1,\infty}(\Omega)$ and

$$F(u) \leq \liminf_{n\to\infty} F_n(u_n) = \liminf_{n\to\infty}\|f(\cdot,u_n(\cdot),Du_n)\|_{p_n(\cdot)} \leq 1$$

where F is defined by (4.5). This implies $\mathcal{F}(u)=0$. □

Proof of Theorem 4.5. The proofs follows the lines of the previous result by applying Theorem 4.2 instead of Theorem 4.1. □

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