Nonlinear dynamics in a Cournot duopoly with relative profit delegation

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Abstract This paper analyses the dynamics of a nonlinear Cournot duopoly with managerial delegation and homogeneous players. We assume that the owners of both firms hire a manager and delegate output decisions to him or her. Each manager receives a fixed salary plus a bonus based on relative (profit) performance. Managers of both firms may collude or compete. In cases of both collusion and a low degree of competition, we find that synchronised dynamics take place. However, when the degree of competition is high, the dynamics may undergo symmetry-breaking bifurcations, which can cause significant global phenomena. Specifically, there is on-off intermittency and blow-out bifurcations for several parameter values. In addition, several attractors may coexist. The global behaviour of the noninvertible map is investigated through studying a transverse Lyapunov exponent and the folding action of the critical curves of the map. These phenomena are impossible under profit maximisation.

Keywords Cournot; Managerial delegation; Nonlinear dynamics; Oligopoly; Relative profits

JEL Classification C62; D43; L13

1. Introduction

Analysing a firm's objectives other than pure profit maximisation dates back at least to Baumol [1]. When ownership and management are separate (Fama and Jensen [2]), managers

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are likely to be driven by other motives than just maximising profits, thus owners may try to motivate them through incentives in order to achieve a competitive advantage in the market. This is the case of large companies, where governance is different from competitive firms.

On the one hand, the strategic use of managerial incentive contracts was studied in the pioneering papers of Vickers [3], Fershtman [4], Fershtman and Judd [5] and Sklivas [6]. According to these authors, owners compensate their managers with a bonus based on a weighted sum of profits and sales, in order to make them perform more aggressively on the market. In addition, more recent studies argue that managerial delegation can also lead to incentive contracts based on market share delegation and relative (profit) performance evaluation (Salas Fumas [7], Miller and Pazgal [8, 9], Jansen et al. [10]). With regard to relative profits, the general idea is that the performance of each firm is compared with competitors (see Gibbons and Murphy [11], Aggarwal and Samwick [12], for empirical evidence), and the incentive contracts are publicly available.

On the other hand, there also exists a burgeoning literature aiming at studying nonlinear dynamics in oligopoly models with quantity and price competition (see Bischi et al. [13]).

The novelty of this paper is the introduction of managerial delegation schemes (based on relative profits) in a nonlinear duopoly with quantity competition and *homogeneous* players, which are assumed to have limited information regarding their objective function (Dixit [14], Bischi and Naimzada [15]). Under the hypothesis of profit-maximising firms, it has been shown that when producers use the output currently regarded as a proxy of the output for the next period, the pure strategy Nash equilibrium in a Cournot duopoly can undergo either a flip bifurcation or Neimark-Sacker bifurcation, and more complicated dynamics can also occur (e.g. Puu [16], Kopel [17], Bischi et al. [18, 19], Tramontana [20]).

Despite the assumption of *homogeneous* players (symmetric map), in this study we found that a high degree of competition between managers causes on-off intermittency, blow-out phenomena and multistability. These two latter results have been found by Bischi et al. [18] to hold in a Cournot duopoly under profit-maximisation only when players are *heterogeneous* (asymmetric map).

The paper is organised as follows. Section 2 builds on the model. Section 3 studies both local and global dynamic phenomena. Section 4 presents the conclusion.

2. The model

We consider a Cournot duopoly for a single homogeneous product with normalised linear inverse demand given by $p = 1 - Q$,[1] where p is the market price of product Q, and $Q < 1$ is the sum of output $q_1 \geq 0$ and output $q_2 \geq 0$ produced by firms 1 and 2, respectively. The (average and marginal) cost of producing an additional unit of output is $0 < c < 1$ for every firm. Therefore, the production function of firm $i = \{1,2\}$ has constant marginal returns to labour, that is $q_i = L_i$, where L_i represents the labour force employed (Correa-López and Naylor [21]).

We assume that the owners of both firms hire a manager and delegate output decisions to him or her. Each manager receives a fixed salary plus a bonus offered in a contract which is freely available to the public. The bonus of the manager hired by firm i is given by the normalised linear combination $U_i = \Pi_i - \beta_i \Pi_j$, where $\Pi_i = (p - c) q_i$ are profits of the ith firm

[1] The inverse demand $p' = A - BQ'$ is normalised by using $p = p'/A$ and $Q = (B/A)Q'$.

and $-1<\beta_i<1$ denotes the manager's attitude ("type").[2] In other words, the utility of manager i depends on firm i's relative profitability. Negative (resp. positive) values of β_i imply that the owner of the ith firm is interested in cooperative (resp. competitive) behaviour with the manager hired by firm j.[3] Given the assumption of uniform type $\beta_i=\beta_j=\beta$, the bonus of manager i can be expressed as $U_i=(1-Q-c)(q_i-\beta q_j)$. Note that the manager's attitude β is exogenously given.[4] This implies that the owners hire managers of a competitive or cooperative type and the hiring decision is *exogenous*.

In the industrial economics literature (see Miller and Pazgal [8, 9], Jansen et al. [10], Kopel and Lambertini [25]), it is usual to assume that β_i is an endogenous variable chosen by the owners of each firm to maximise profits in the contract-stage of the game (i.e., the stage played by owners and managers to choose the size of weight β_i in the managerial contract), while being taken as given by managers in the market-stage of the game (i.e., the stage at which the manager makes decisions about production).

3. Dynamics

We now consider a dynamic version of the Cournot duopoly with relative profit delegation. Time is discrete and indexed by $t=0,1,2,\ldots$. We assume that both players have limited information regarding manager bonuses (no knowledge of the market), however they follow an adjustment process based on local estimates of the marginal bonus ($\partial U_i/\partial q_i$). We adopt the adjustment mechanism of quantities over time proposed by Bischi and Naimzada [15], that is:

$$q_i'=q_i+\alpha_i q_i\frac{\partial U_i}{\partial q_i}, \tag{1}$$

where q_i' is the unit-time advancement of variable q_i, $\frac{\partial U_i}{\partial q_i}=1-2q_i-q_j(1-\beta)-c$, $\alpha_i>0$ is a coefficient that captures the speed of adjustment of firm i's quantity with respect to a marginal change in the manager's bonus when q_i varies, and $\alpha_i q_i$ is the intensity of the reaction of every player. Therefore, the manager of firm i increases or decreases production in the future depending on whether $\partial U_i/\partial q_i$ is positive or negative.

By using Eq. (1) the two-dimensional system that characterises the dynamics of the economy is the following:

$$T=T(q_1,q_2):\begin{cases} q_1'=q_1+\alpha q_1[1-2q_1-q_2(1-\beta)-c] \\ q_2'=q_2+\alpha q_2[1-2q_2-q_1(1-\beta)-c]\end{cases}. \tag{2}$$

where $\alpha_1=\alpha_2=\alpha$. Eq. (2) highlights that when $\beta>0$ (resp. $\beta<0$), a rise in (the absolute value of) β increases (resp. reduces) the value of the marginal bonus of manager i, so that production at time $t+1$ increases (resp. reduces).

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[2] This is equivalent to assuming weight w attached to the company's own profits (Π_i) and weight $1-w$ attached to the difference between the company's own profits and the rival's profits ($\Pi_i-\Pi_j$), where $0<w<2$. Then, $U_i=w_i\Pi_i+(1-w_i)(\Pi_i-\Pi_j)=\Pi_i-\beta_i\Pi_j$, where $\beta_i:=1-w_i$.

[3] Note that the problems of partial cooperation in oligopolies have been studied, amongst others, by Cyert and DeGroot [22] in a static context, and Kopel and Szidarovszky [23] in a dynamic context.

[4] The case $-1<\beta<0$ could also be interpreted as if every firm held the shares of each other's company (Reitman [24]).

Remark 1. *From a mathematical point of view, it is important to stress that, unlike Bischi et al. [18], there is a parameter (β) that weights the mixed term $q_i q_j$ of map T without affecting the pure quadratic term q_i^2. This apparently negligible change in the definition of the map actually causes significant changes in the long-term dynamics with respect to Bischi et al. [18], as will be shown later in this paper.*

Remark 2. *Map (2) generates unbounded trajectories if the initial condition is taken with at least one negative coordinate or far enough[5] from the origin, i.e. in a suitable neighbourhood of ∞ (which is always an attractor of map 2). In fact, if we take large enough values of the initial conditions of q_1 and q_2, then the first iterate of map (2) gives negative values of q_1 and q_2 so that the subsequent iterates also gives negative and decreasing values.*

As a consequence of Remark 2, and knowing that $q_1 \geq 0$ and $q_2 \geq 0$, we rewrite Eq. (2) as follows:

$$T^+ = T^+(q_1, q_2): \begin{cases} q_1' = \max\{0, q_1 + \alpha q_1[1 - 2q_1 - q_2(1-\beta) - c]\} \\ q_2' = \max\{0, q_2 + \alpha q_2[1 - 2q_2 - q_1(1-\beta) - c]\} \end{cases}, \tag{3}$$

see, e.g., Cánovas et al. [26].

Remark 3. *Let an initial condition be given. If the trajectory generated by map (2) lies on the non-negative orthant for any t, then the same trajectory is generated by map (3). If the trajectory generated by map (2) has at least one non-positive coordinate, then a t^* exists such that the trajectory generated by map (3) lies on the non-negative half line axes for any $t \geq t^*$.*

In the light of Remark 3, we can focus on map (2) and we will then refer to map (3) when necessary (see Section 3.2).

Map (2) has four fixed points: $E_0 = (0,0)$, $E_1 = \left(0, \frac{1-c}{2}\right)$ and $E_2 = \left(\frac{1-c}{2}, 0\right)$ located on the invariant coordinate axes,[6] and

$$E^* = \left(\frac{1-c}{3-\beta}, \frac{1-c}{3-\beta}\right), \tag{4}$$

which is the unique Cournot-Nash equilibrium, where $q^*_1 = q^*_2 = q^*$. Then, equilibrium profits are $\Pi^* = \frac{(1-c)^2(1-\beta)}{(3-\beta)^2} > 0$; E_1 and E_2 represent the steady states for which only one firm serves the market (monopoly), while at E_0 both firms do not produce.

One important characteristic of map (2) is that the diagonal $\Delta = \{(q_1, q_2) : q_1 = q_2\}$ is an invariant manifold, that is $T(\Delta) \subseteq \Delta$. In fact, map T has a symmetric form, so that it does change if variables q_1 and q_2 are swapped, that is $T \circ S = S \circ T$, where $S : (q_1, q_2) \to (q_2, q_1)$. This implies that by starting with the same initial condition $q_1(0) = q_2(0)$, the dynamics lie on Δ for every t. In this case, the behaviour of the dynamic system is described by the restriction of map T on Δ, and the synchronised trajectories (i.e., $q_1(t) = q_2(t)$, for every t) are governed

5 However, in Section 3.2 we will see that other initial conditions can exist that generate unbounded trajectories.

6 Testing for the invariance of axis i is straightforward: if we start from $q_i = 0$, we get $q_i' = 0$. Thus, the dynamics lie on axis i for every t.

by $T_\Delta : \Delta \to \Delta$, where

$$T_\Delta : \quad q' = q + \alpha q[1 - q(3-\beta) - c]. \tag{5}$$

This map is conjugated to the logistic map

$$z' = \mu z(1-z). \tag{6}$$

by the linear transformation

$$q := \frac{1+\alpha(1-c)}{\alpha(3-\beta)} z. \tag{7}$$

with $\mu := 1 + \alpha(1-c)$. It follows that the dynamics on Δ can be obtained from the well known behaviour of the standard logistic map by a homeomorphism (see Devaney, 1989). If $0 < \alpha(1-c) < 2$, then $q^* = \dfrac{1-c}{3-\beta}$ is locally asymptotically stable; if we let $\alpha(1-c)$ increase, then period doubling cascade occurs. For $\alpha(1-c) > \upsilon^* - 1$ ($\upsilon^* \cong 3.57$), the fate of a generic trajectory starting in the interval $\left(0, \dfrac{1+\alpha(1-c)}{\alpha(3-\beta)}\right)$ is an attracting cycle or cyclic-invariant chaotic interval, or a Cantor set belonging to trapping intervals bounded by critical points. Finally, for $\alpha(1-c) > 3$, the generic trajectory of map (2) is divergent. With regard to map (3), this means that the corner solution $q = 0$ is reached and both producers exit the market.

Remark 4. *From Eq. (7) and the definition of μ it follows that β does not affect the dynamic properties of the map on the diagonal (for instance, existence and stability of cycles).*[7] *while affecting the coordinate of the fixed point, the attractors and their basins of attraction. From an economic point of view, this means that when $q_1(t) = q_2(t)$ every manager behaves as if he/she were not interested in the behaviour of the rival, i.e. the effects of β on the behaviour of managers are completely removed.*

Understanding whether an attractor for the one-dimensional map (5) also represents an attractor for the two dimensional map (2), entails studying the transverse attractiveness of attractors located on Δ. Therefore, we consider the following Jacobian matrix associated with map (2):

$$J(q_1, q_2) = \begin{pmatrix} 1 + \alpha - 4\alpha q_1 + \alpha q_1(\beta - 1) - \alpha c & \alpha q_1(\beta - 1) \\ \alpha q_2(\beta - 1) & 1 + \alpha - 4\alpha q_2 + \alpha q_1(\beta - 1) - \alpha c \end{pmatrix}, \tag{8}$$

evaluated at a generic point (q, q) on Δ:

$$J(q, q) = \begin{pmatrix} l(q) & m(q) \\ m(q) & l(q) \end{pmatrix}, \tag{9}$$

where $l(q) = 1 + \alpha - \alpha c + \alpha q(\beta - 5)$ and $m(q) = \alpha q(\beta - 1)$. The associated eigenvalues are:

$$\lambda_\parallel = l(q) + m(q) = 1 + \alpha - 2\alpha q(3-\beta) - \alpha c, \tag{10}$$

with eigenvector $(1,1)$ and

$$\lambda_\perp = l(q) - m(q) = 1 + \alpha - 4\alpha q - \alpha c, \tag{11}$$

with eigenvector $(1,-1)$. The eigenvalue $\lambda_\parallel$ is related to the invariant manifold Δ and coincides with the multiplier of the restriction of the map on Δ. The eigenvector associated with the eigenvalue $\lambda_\perp$ is always orthogonal to Δ regardless of q.

Starting from the fixed point E^*, the local dynamic properties are summarised in the

[7] However, we note that a change in β induces, for instance, a change in the coordinates of the fixed point q^*.

following proposition.

Proposition 1. *The Cournot-Nash equilibrium* E^* *is asymptotically stable if and only if* $\alpha < \frac{2}{1-c}$. *For* $\alpha = \frac{2}{1-c}$, *the Cournot-Nash equilibrium* E^* *is destabilised by a supercritical flip bifurcation occurring along the invariant diagonal.*

Proof. Due the symmetry of the map it follows that the eigenvalues defined by Eqs. (10) and (11) are real numbers. Then, by substituting the coordinates of E^* in the generic expression of the eigenvalues of a point on Δ, we have $\lambda_{\parallel}(E^*) = 1 - \alpha(1-c) < \lambda_{\perp}(E^*) = 1 - \frac{\alpha(1-c)(1+\beta)}{3-\beta} < 1$. It follows that the fixed point E^* is asymptotically stable if and only if $\lambda_{\parallel}(E^*) > -1 \Leftrightarrow \alpha < \frac{2}{1-c}$. **Q.E.D.**

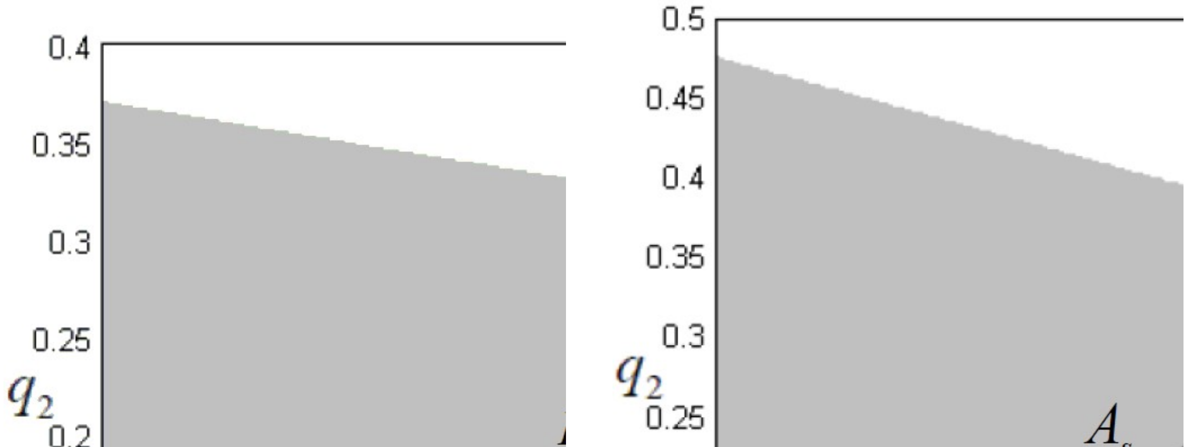


Figure 1. (a) Case $\alpha < 2/(1-c)$: the Cournot-Nash equilibrium E^* is the unique attractor of the system (parameter values: $\alpha = 4$, $\beta = 0.575$ and $c = 0.51$). (b) Case $\alpha > 2/(1-c)$: the Cournot-Nash equilibrium E^* is unstable and a chaotic set A_s exists along the diagonal. The properties of A_s will be analysed in Section 3.3 (parameter values: $\alpha = 4$, $\beta = 0.3$ and $c = 0.3$). Note that the grey region in both figures denotes the basin of attraction of the unique finite distance attractor; the white region represents the set of unfeasible trajectories.

Remark 5. *Parameter* β *does not matter even on local dynamics in the neighbourhood of* E^*. *From an economic point of view, this implies that if the initial condition is close to the Nash equilibrium, then not every manager takes the behaviour of the rival into account.*

Now, from Eqs. (10) and (11), we study the stability of cycles that can occur on Δ. For a k-cycle $\{(x_1,x_1),(x_2,x_2),...,(x_k,x_k)\}$ of map (2), corresponding to cycle $\{x_1,x_2,...,x_k\}$ of map (5), the two multipliers are:

$$\lambda_{\parallel}^{(k)} = \prod_{i=1}^{k}(l(x_i)+m(x_i)) = \prod_{i=1}^{k}(1+\alpha-2\alpha x_i(3-\beta)-\alpha c), \tag{12}$$

associated with the eigenvector $(1,1)$, and

$$\lambda_{\perp}^{(k)} = \prod_{i=1}^{k}(l(x_i)-m(x_i)) = \prod_{i=1}^{k}(1+\alpha-4\alpha x_i-\alpha c), \tag{13}$$

associated with the eigenvector $(1,-1)$.

The conditions for local stability and local bifurcations along Δ are discussed above. Hence, we can restrict our focus to transverse stability. The problem is trivial when a cycle of finite period exists on the diagonal Δ, as only the eigenvalue $\lambda_\perp^{(k)}$ needs to be evaluated. The problem becomes interesting when the dynamics restricted to the invariant sub-manifold are embedded in a chaotic set A_s. To study this issue, we now introduce the definition of the transverse Lyapunov exponent by which the "average" local behaviour of the trajectories in a neighbourhood of the invariant set $A_s \subset \Delta$ can be classified.

Definition 1. *Let* $\left\{x_i = f^i(x_0), i \geq 0\right\}$ *be a trajectory of Eq. (5) embedded in* $A_s \subset \Delta$. *Then,*

$$\Lambda_\perp = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln\left|\lambda_\perp(x_i)\right| = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln\left|1+\alpha-4\alpha x_i-\alpha c\right|, \tag{14}$$

is called a transverse Lyapunov exponent.

If x_0 belongs to a k-cycle, then $\Lambda_\perp = \ln\left|\lambda_\perp^{(k)}\right|$ and if $\Lambda_\perp < 0$, then the k-cycle is transversely stable. Otherwise, if A_s is a chaotic set and $\Lambda_\perp < 0$ for each trajectory starting inside A_s, then A_s is asymptotically stable.

In addition, the following classical definition of attractiveness can be established:

Definition 2. A *is an asymptotically stable attractor (or topological attractor) if it is Lyapunov stable, i.e. for every neighbourhood* U *of* A, *a neighbourhood* V *of* A *exists such that* $T^n(V) \subset U$ *for every* $n \geq 0$ *and the basin of attraction* $B(A)$ *contains a neighbourhood of* A.

Proving that an attractor is asymptotically stable entails evaluating the Lyapunov exponent over an infinity of trajectories. It is therefore easier to show whether or not a chaotic set along the diagonal *is* an attractor in the Lyapunov sense, because it is sufficient to find a trajectory (for instance, a cycle) with respect to which the condition $\Lambda_\perp < 0$ is violated (see Section 3.3).

3.1. Critical curves

An important feature of the map defined in Eq. (2) is that it is a noninvertible endomorphism. Indeed, for a given (q_1', q_2'), the rank-1 preimage (that is, the backward iterate defined as T^{-1}) may not exist or may be multivalued. By considering Eq. (2), if we want to compute (q_1, q_2) in terms of (q_1', q_2'), we need to solve a fourth degree algebraic system that may have four, two or no real solutions. Thus, we can divide the plane into regions Z_0, Z_2, Z_4, according to the number of preimages (where the subscript in Z indicates their number). Consequently, if we let (q_1', q_2') vary in the plane R^2, the number of rank-1 preimages changes as the point (q_1', q_2') crosses the boundary that separates these regions. These boundaries are generally characterised by the existence of two coincident preimages. Thus following the notation used by Mira et al. (1996), we introduce the definition of critical curves. The critical curve of rank-1, denoted by LC (from the French "Ligne Critique"), is defined as the locus of points with two (or more) coincident rank-1 preimages located on a set called LC_{-1}. It is quite intuitive to interpret (*i*) the set LC as the two-dimensional generalization of the notion of critical value, defined as the value corresponding to either the local minimum or maximum of a one-dimensional map, and (*ii*) LC_{-1} as the generalisation of the notion of critical point. Thus, arcs of LC separate the regions of the plane characterised by a different number of real preimages (see Mira et al. [27] for details). Since the map defined by Eq. (2) is continuously differentiable, LC_{-1} belongs to the locus of points where the Jacobian determinant of T vanishes (i.e. the points where T is not locally invertible), i.e.: $LC_{-1} \subseteq \{(q_1, q_2) \in R^2 : Det(T) = 0\}$, and LC is the rank-1 image of LC_{-1} under T, i.e. $LC = T(LC_{-1})$. From direct computations, we have that:

$$\begin{aligned} Det(T) = 0 \Leftrightarrow\ & 4\alpha^2 q_1^2(\beta-1) + 4\alpha^2 q_2^2(\beta-1) + 16\alpha^2 q_1 q_2 + \alpha q_1(5-\beta)(\alpha(c-1)-1) \\ & + \alpha q_2(5-\beta)(\alpha(c-1)-1) + (\alpha(c-1)-1)^2 = 0 \end{aligned}. \tag{15}$$

Eq. (15) is the equation of a hyperbola in the plane (q_1, q_2). Thus, LC_{-1} is made up two branches, denoted by $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$, respectively. This also implies that LC and subsequent iterates of critical curves can be interpreted as the union of two different branches indexed by a and b. Fig. 2 shows the critical curves $LC_{-1}^{(a)}$, $LC_{-1}^{(b)}$, $LC^{(a)}$ and $LC^{(b)}$. Each branch of critical curve LC separates the phase plane of T into regions whose points have the same number of distinct rank-1 preimages. Specifically, $LC^{(b)}$ separates region Z_0 from region Z_2, and $LC^{(a)}$ separates region Z_2 from region Z_4.

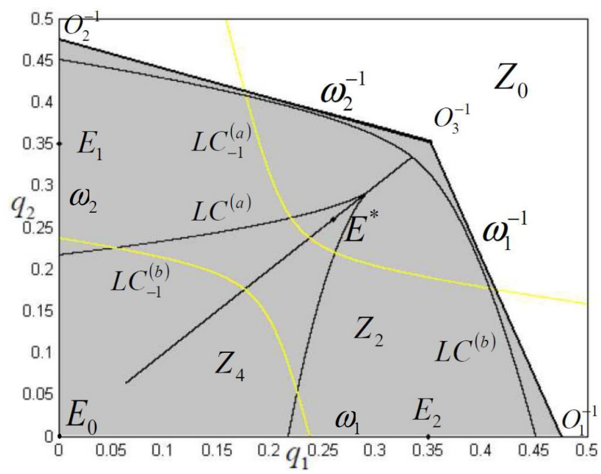


Figure 2. Critical curves are represented for the parameter set $\alpha = 4$, $\beta = 0.3$ and $c = 0.3$. The yellow lines $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$ are the two branches of the LC_{-1} set. The portion of the plane that lies between the axes and $LC^{(a)}$ defines region Z_4; the portion of the plane that lies between $LC^{(a)}$ and $LC^{(b)}$ defines region Z_2; the portion of the plane beyond $LC^{(b)}$ defines region Z_0. Moreover, E_0 and its preimages O_1^{-1}, O_2^{-1} and O_3^{-1} define the (grey) region of bounded trajectories. The region of unbounded trajectories is white-coloured. Notice that the sides $\omega_1^{-1} = [O_1^{-1}, O_3^{-1}]$ and $\omega_2^{-1} = [O_2^{-1}, O_3^{-1}]$ are the preimages of $\omega_1 = [E_0, O_1^{-1}]$ and $\omega_2 = [E_0, O_2^{-1}]$, respectively.

The forward images of rank-k of LC_{-1} give the critical sets of rank-k, denoted by $LC_{k-1} = T^k(LC_{-1}) = T^{k-1}(LC)$. Portions of critical curves of increasing rank can be used to obtain the boundary of absorbing and chaotic areas of non-invertible maps of the plane (see Mira et al. [27] for details). We recall here some basic definitions and refer to Agliari et al. [28] for further details.

Definition 3. *An absorbing area A is a bounded region of the plane whose boundary is given by portions of critical curves and their images of increasing order, such that a neighbourhood $U \supset A$ exists whose points enter A after a finite number of iterations and then never escape A*

since $T(A) \subseteq A$.

Definition 4. *A chaotic area is an absorbing area* A *such that* $T(A) = A$ *and chaotic dynamics occur inside* A.

The boundary of a chaotic area A is shown in Fig. 5.b and is obtained by following the procedure described in Mira et al. [27] and later by Bischi and Gardini [29]. First, we take subsequent images of the two segments of LC_{-1} included in A until a chaotic area is defined. Then, the procedure is repeated by taking only the images of $\gamma_i = A \cap LC_{-1}^i$, $i = \{a,b\}$ (this last step shows that the union of the first four iterates of γ_i covers the whole boundary of chaotic area A, which ensures that this selected area is invariant, i.e. $T(A) = A$).

3.2. Basins of attraction

This section describes the basins of attraction of map (3). Given Remark 3, we do this by analysing the properties of map (2).

By Following Bischi et al. [18], it is possible to show that the coordinate axes and their preimages of any rank form the boundary for initial conditions of any non-diverging trajectory. The dynamics of map (2) restricted to one of the axis is governed by the following one-dimensional map:

$$q_i' = q_i + \alpha q_i (1 - 2q_i - c), \tag{16}$$

which is conjugated to the logistic map (see Devaney [30]) by the linear transformation $\mu := 1 + \alpha(1-c)$, $q_i = \dfrac{1+\alpha(1-c)}{2\alpha z_i}$. Bounded trajectories along the invariant axes are obtained when $1 < \alpha(1-c) < 3$ provided that the initial value of the map lies on the segments

$$\omega_i = [O, O_i^{-1}], \tag{17}$$

where O_i^{-1} is the rank-1 preimage of the origin on the corresponding axis with non-null coordinate equals $\dfrac{1+\alpha(1-c)}{2\alpha}$. In contrast, negatively divergent trajectories along the invariant axis are obtained, on the basis of an initial condition. Furthermore, by computing the eigenvalues of the cycles that belong to ω_i the direction transverse to the coordinate axes is repelling. It thus follows that ω_i belongs to $\partial B(\infty)$ and to their preimages of any rank. Now, according to Bischi et al. [18], the next proposition gives an exact delimitation of $\partial B(\infty)$ for map (2).

Proposition 2. *Let* $0 < \alpha(1-c) < 3$ *and* ω_i, $i = \{1,2\}$ *be the segment lines defined in Eq. (17). Then,*

$$\partial B(\infty) = \left(\bigcup_{n=0}^{+\infty} T^{-n}(\omega_1) \right) \cup \left(\bigcup_{n=0}^{+\infty} T^{-n}(\omega_2) \right). \tag{18}$$

Proof. See Bischi et al. [18].

We are now in a position to characterise the basins of attraction of trajectories that lie on the non-negative half line axes of map (3).

Corollary 1. *Let* OAB *be the open triangle of vertices: the origin,* $A = (0,1)$ *and* $B = (1,0)$. *If the* *initial condition belongs to the set* $OAB \cap B(\infty)$, *then a* t^* *exists such that the trajectory "jumps" and lies on the half line axes for every* $t \ge t^*$. *If the initial condition lies on the set* $Int(OAB \setminus B(\infty))$, *then the trajectory lies on* $Int(OAB)$ *for every* t.

More specifically, with regard to the sets defined in Corollary 1, when the segment lines ω_i^{-1} of map (2) belong to Z_0, their structure is simple: the union of the basins of attraction at finite distance attractors is defined by the open quadrilateral of vertices $OO_1^{-1}O_3^{-1}O_2^{-1}$. If the economy starts on the portion of the diagonal that belongs to the set $Int(OAB \setminus OO_1^{-1}O_3^{-1}O_2^{-1})$, then both firms exit the market in the long run. In contrast, if the initial condition belongs to $Int(OAB \setminus OO_1^{-1}O_3^{-1}O_2^{-1})$ outside the diagonal, then the economy ends up at E_1 or E_2, or, alternatively, monopoly nonlinear dynamics can be generated (Cánovas et al. [26]).

The scenario changes dramatically when a contact between the segment line ω_i^{-1} and the critical curve $LC^{(b)}$ occurs. In this case, inside the quadrilateral there are also an infinite number of holes made up of portions of the attractors' basins of attraction on the coordinate axes . This phenomenon can be observed using the following parameter values: $\alpha = 5.9$, $\beta = c = 1/2$, and $q_1(0) = 0.1$ and $q_2(0) = 0.2$ as initial conditions.[8]

3.3. Intermittency, synchronisation and blow-out

In this section we show that interesting and complicated dynamic phenomena can occur when β varies. One of the most important findings is that although the model is characterised by a symmetric map, we observe synchronisation failure when β is high. We will focus on the case in which a chaotic set A_s exists on the diagonal (that is, $\alpha(1-c) > 2.57$). Then, we let β vary and study the dynamic properties of the map. First, in the case of collusion with a large enough value of $|\beta|$, numerical evidence exists such that A_s is Lyapunov stable: given any initial condition $q_i(0)$ within the quadrilateral of vertices $OO_1^{-1}O_3^{-1}O_2^{-1}$ (Fig. 2), this means that after a few iterations both managers coordinate their behaviour on the diagonal. Then, moving from negative to positive values of β, we find that a threshold value $\beta = \beta_{\mathrm{M}}$ exists (whose size depends on the other parameters of the model), so that there are some cycles on the diagonal with a positive (transverse) Lyapunov exponent. This implies that this object cannot be classified as an attractor in a Lyapunov sense.[9] In order to study this issue it is useful to define a spectrum of transverse Lyapunov exponents

$$\Lambda_\perp^{\min} < .... < \Lambda_\perp^{nat} < ... < \Lambda_\perp^{\max}, \tag{19}$$

each of which is associated with a specific trajectory, where $\Lambda_\perp^{nat}$ is the Lyapunov exponent evaluated on a generic trajectory taken in the chaotic attractor. Roughly speaking, $\Lambda_\perp^{nat}$ can be interpreted as a sort of "weighted balance" amongst the Lyapunov exponents associated with different cycles. Now, if a set is a Lyapunov attractor, it follows that $\Lambda_\perp^{\max} < 0$ (see Definition 2). If $\Lambda_\perp^{\max} > 0$, while $\Lambda_\perp^{nat}$ is still negative (see Maistrenko et al. [31], Bischi and Gardini [32]), A_s is no longer Lyapunov stable, however it continues to attract a positive (Lebesgue) measure set of points of the two dimensional phase space. In this case A_s is said to be *weak*

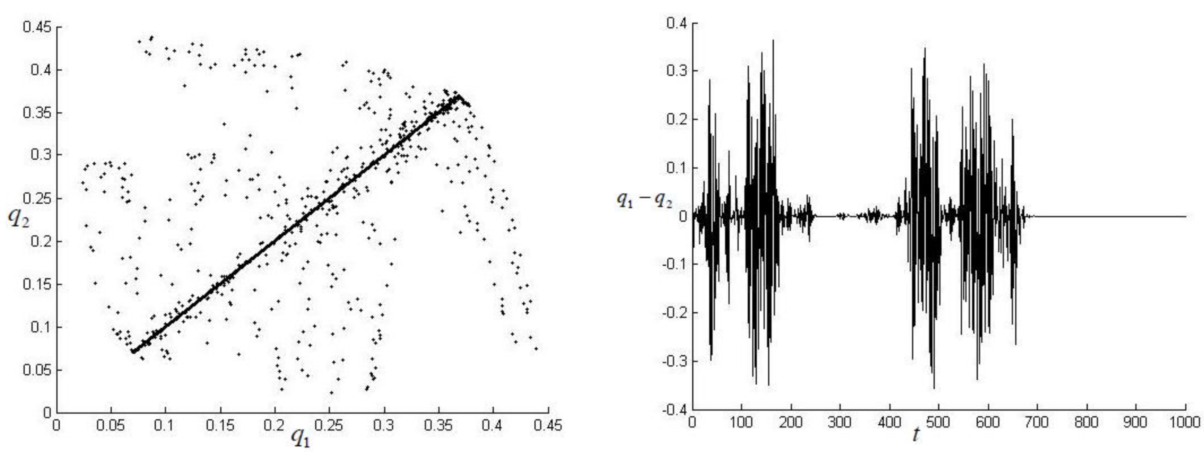
[8] Similar phenomena are described in Bischi et al. [18] where, however, non-negative constraints are not introduced. We thank an anonymous reviewer for pointing this out.

[9] This type of loss of stability may not be clear from Fig. 1.b, because in the long run all feasible trajectories synchronise on the diagonal.

stable or stable in a Milnor sense (Fig. 1.b).

Figure 3. Natural transverse Lyapunov exponent $\Lambda_{\perp}^{nat}$ (as a function of β) belonging to the interval $[-0.1, 0.7]$ (parameter set: $\alpha = 0.4$ and $c = 0.3$). The figure is obtained by dividing the interval $[-0.1, 0.7]$ into 1000 equidistant points. By taking an initial condition on the diagonal, each point has been iterated 10000 times through map T (to eliminate the transient) and then averaging over subsequent 50000 iterations.

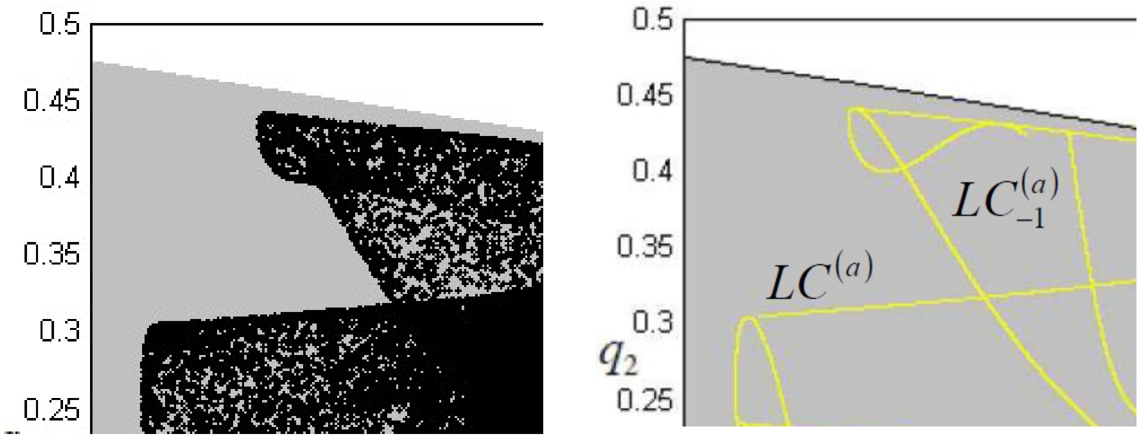
In the numerical example in Figs. 4.a and 4.b no attractors other than A_s exist, so that first trajectories move away from the diagonal and then approach A_s, and the dynamics are characterised by some bursts away from Δ before synchronising on it (on-off intermittency). The synchronisation is time consuming because $\Lambda_{\perp}^{nat}$ is negative, but close to zero (Fig. 3). This is in accordance with Bischi et al. [18] in a Cournot model with profit-maximising firms ($\beta = 0$).



(a) (b)

Figure 4. Parameter values: $\alpha = 0.4$, $c = 0.3$ and $\beta = 0.556$. (a) Typical trajectory that starts in the basin of attraction at finite distance (Milnor) attractor. (b) Intermittency (by displaying the difference between q_1 and q_2 versus time): the convergence towards the unique chaotic (Milnor) attractor of the system embedded in the diagonal occurs only after a very long transient.

If we let β increase, $\Lambda_{\perp}^{nat}$ becomes positive because the transversely unstable periodic orbits embedded in A_s weigh more than the transversely attracting orbits (Fig. 3, Figs. 5.a and 5.b). In this case, A_s becomes a chaotic saddle and an explosion of the attractor (no longer confined on the diagonal) is observed. This is the so-called blow-out bifurcation. Note that in the economic literature, this is usually observed both in models described by polynomial asymmetric maps (Bischi et al. [18]) and in maps with denominators (Bischi and Gardini [32], Kopel et al. [33], Bischi and Kopel [34]).



(a) (b)

Figure 5. Parameter values: $\alpha = 0.4$, $c = 0.3$ and $\beta = 0.6$. (a) Effect of a rise in β on long-run dynamics: the one dimensional Milnor attractor, as shown in Fig. 1.b for $\beta = 0.3$ and in Fig. 4.a for $\beta = 0.556$, is by a two-dimensional attractor. (b) Boundary of the chaotic area (simulated in Fig. 5.a) obtained by arcs of the critical curves LC, LC_1, LC_2 and LC_3.

In this case we can talk about coordination failure or symmetry-breaking of the dynamics. Although the long-run dynamics are no longer restricted on the diagonal, the borders of the phase plane in which the asymptotic dynamics are bounded can be delimited by using the critical curves and their iterates. The same technique can also be applied to give an upper boundary for the trajectories of the map when intermittency occurs.

3.4 Multistability and related basins of attraction

A dramatic change in the long run dynamics occurs if we take different parameter sets and let β vary. By assuming $\alpha = 3.8$ and $c = 0.3$, some interesting events take place when β belongs to the interval $[0.165, 0.5379]$. From calculations of the transverse Lyapunov exponents, we find that the set A_s is a Milnor attractor. In fact, through local bifurcations new stable equilibria arise outside the diagonal. Moreover the basin of attraction of A_s is (apparently) *riddled* with the basins of attraction the other attractor, that is several repelled trajectories (from the diagonal) belong to the basin of attractor outside of Δ.

Fig. 6.a ($\beta = 0.45$) shows that there is a four-period cycle along with the diagonal. If β increases, the cycle undergoes a Neimark-Sacker bifurcation and an attractor made up of four smooth curves coexists with the diagonal (see Fig. 6.b, where $\beta = 0.46$). Further increases in β lead to: (*i*) the loss of smoothness of the attractor formed by four curves (see Fig. 6.c, plotted for $\beta = 0.485$), and (*ii*) the birth of a four-piece chaotic attractor (see Fig. 6.d, plotted for $\beta = 0.53$).

The subsequent final bifurcation due to the contact between the four-piece chaotic attractor and its basin when $\beta \cong 0.5379$, causes the death of the attractor, and only the two-piece Milnor attractor exists as the unique attractor of the system (note that this phenomenon is not reported in any figures).

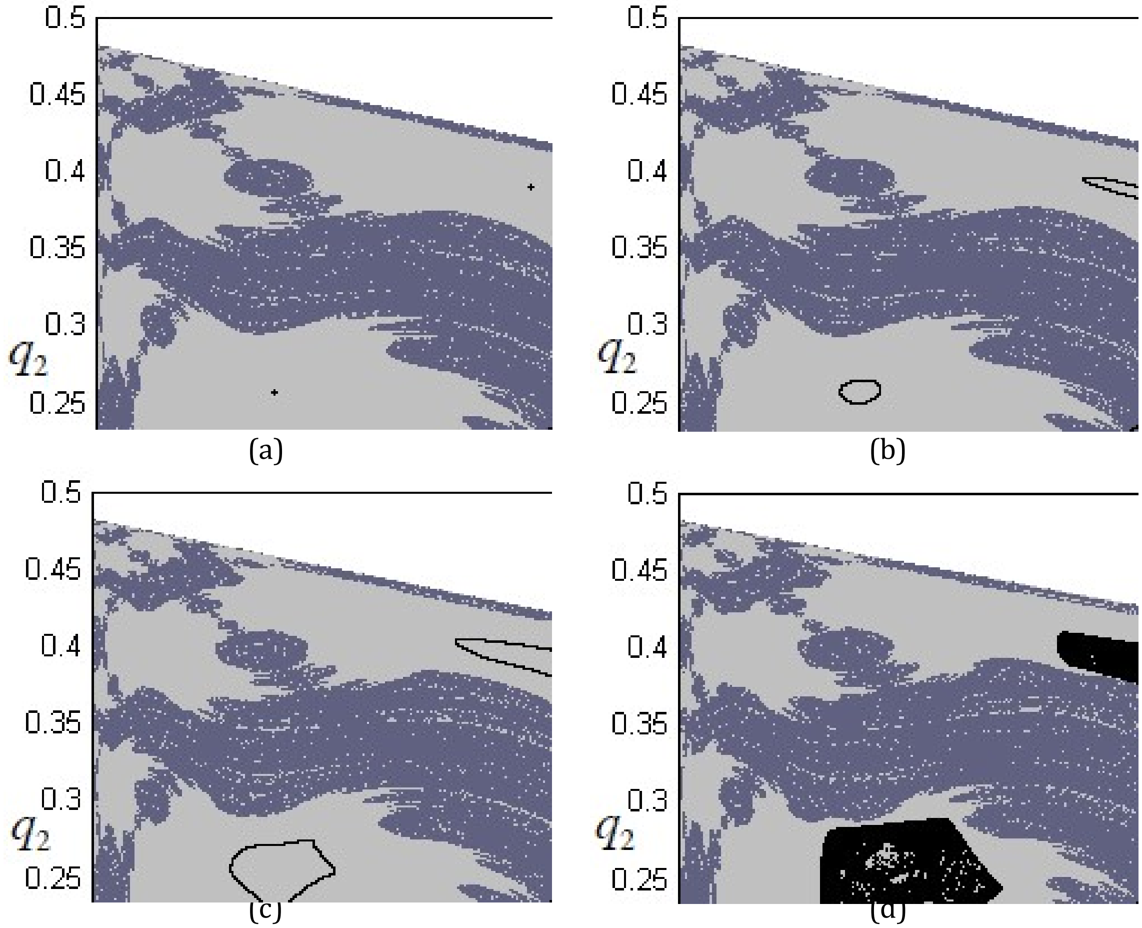


Figure 6. Parameter values: $\alpha = 3.8$ and $c = 0.3$. (a) For $\beta = 0.45$, an attracting four-period cycle exists together with the two-piece Milnor attractor on the diagonal. (b) For $\beta = 0.46$, the four-period cycle has undergone a Neimark-Sacker bifurcation and an attractor formed by four smooth curves exist together with the two-piece Milnor attractor. (c) For $\beta = 0.485$, the attractor formed by the four smooth curves has become non-smooth. (d) For $\beta = 0.53$, there exists a four-piece chaotic attractor outside of the diagonal.

Other interesting phenomena can be observed by slightly changing the parameter values. In particular, multistability occurs even if an attractor (à la Milnor) does not exist along the diagonal. To show this phenomenon clearly, we choose $\alpha = 3.9$, $c = 0.32$ and let β be 0.987 and 0.97.

When $\beta = 0.987$, two chaotic two-piece attractors exist whose basins of attraction are separated by some invariant manifolds associated with the unstable cycles. This is clear from Fig. 7.a, where the dark-grey (light-grey) region denotes the basin of attraction of the black-coloured (red-coloured) chaotic attractor. A more complicated structure of the basins of attraction is obtained by slightly reducing the parameter β, i.e. $\beta = 0.97$ (Fig. 7.b). In fact, Fig. 7.b reveals new portions of the basin of attraction of the black-coloured chaotic attractor which are embedded in the basin of attraction of the red-coloured chaotic attractor. We now use critical curves to explain this phenomenon. Fig. 7.c is an enlargement of Fig. 7.b – that refers to the region around P_1), and highlights that a global bifurcation of the basins of attraction has just occurred. A reduction in β increases the distance between the two critical curves $LC^{(a)}$ and $LC^{(b)}$, so that the branches of the rank-1 critical curve $LC^{(b)}$, which separates Z_0 from Z_2, move towards the regions denoted by P_1 and P_2 in Fig. 7.a. This implies that there is a threshold value of β (β_T)such that a tangency between $LC^{(b)}$ and the boundary of P_1 and P_2 occurs simultaneously (due to the symmetry of the map). Then, as shown in Figs. 7.b and 7.c, a portion of P_1 and P_2 (labelled as H in Fig. 7.c) enters the Z_2 area. Since P_1 and P_2 belong to the basin of attraction of the black-coloured attractor, their preimages thus also belong to this a basin. This creates an infinite number of dark-grey *lakes* (see Mira et al., [27]) embedded in the light-grey region inside the quadrilateral of vertices $OO_1^{-1}O_3^{-1}O_2^{-1}$. After the bifurcation (tangency), two main lakes lie inside Z_4. Hence, both lakes have further preimages which form smaller lakes within the light-grey region. These lakes lie inside the quadrilateral in the region complement to Z_0. This phenomenon is iterated indefinitely (see Mira et al. [27] for an analysis and detailed discussion of this type of basin bifurcation).

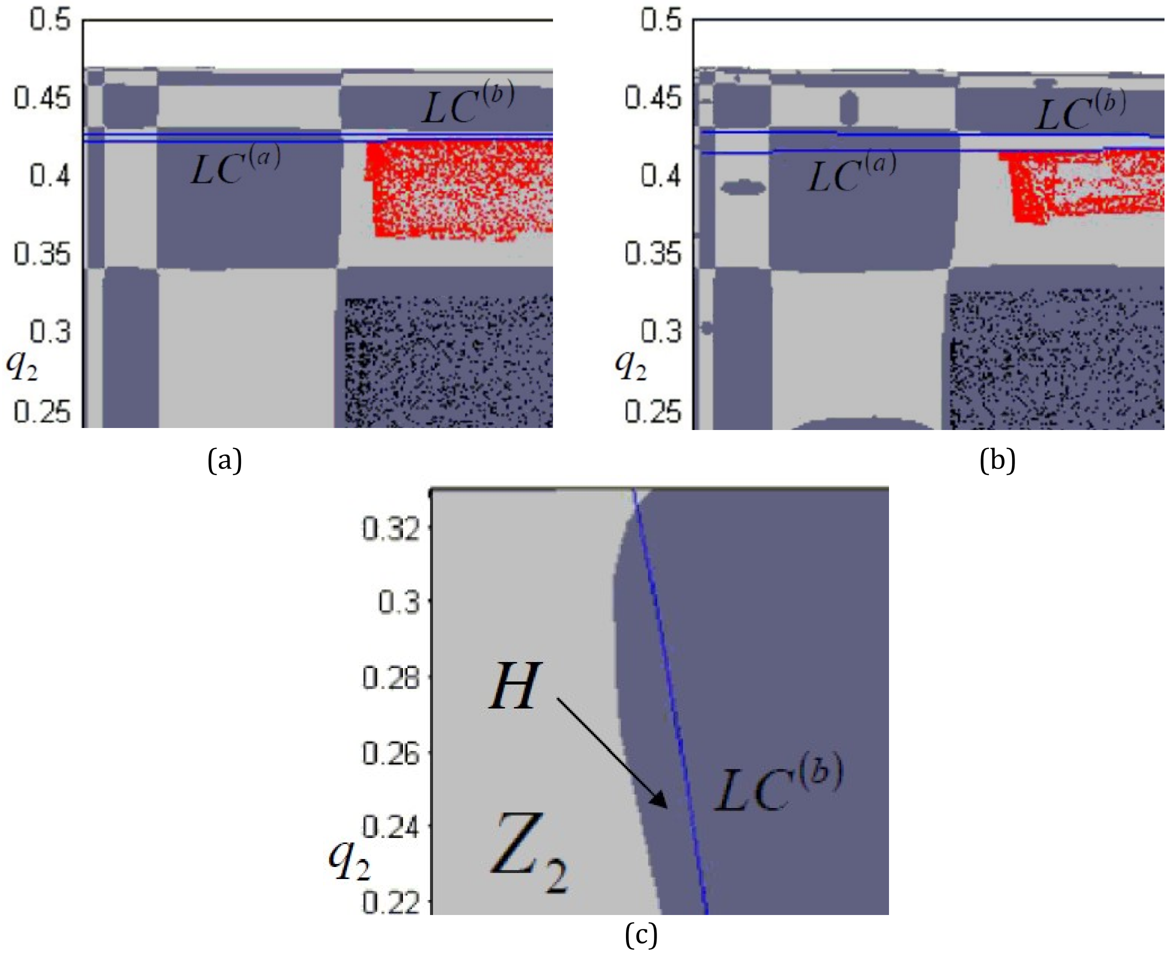


Figure 7. Parameter values: $\alpha = 3.9$ and $c = 0.32$ (a) For $\beta = 0.987$, we observe the coexistence of two chaotic attractors (black-coloured and red-coloured), whose basins of attraction are represented by the dark-grey region and the light-grey region, respectively. (b) For $\beta = 0.97$, a global bifurcation is just occurred and infinitely many dark-grey lakes embedded in the light-grey region are born. In both Figs. 7.a and 7.b also the critical curves $LC^{(a)}$ and $LC^{(b)}$, which separate the zones with number of preimages, are depicted. (c) An enlargement view of Fig. 7.b., showing a portion of the phase plane after the tangency between $LC^{(b)}$ and the boundary of P_1 and P_2 has occurred. The birth of the region between $LC^{(b)}$ and the boundary of P_1 (denoted by H) is also shown.

4. Conclusions

This origins of this paper are twofold: (1) the increasing interest in a refined analysis of the burgeoning literature on dynamic oligopolies (Bischi et al. [13]), and (2) the importance, emphasised by both the theoretical and empirical literature (e.g., Vickers [3], Gibbons and Murphy [11]), of compensation practices for managers based on sales, market shares and related profit schemes. The novelty of this paper lies in the introduction of managerial incentive contracts (relative profits) in a dynamic Cournot duopoly with homogeneous players.

We have shown that the existence of relative profit delegation in a Cournot duopoly can create interesting local and global dynamic events. When the basic parameters of the model are such that a chaotic set exists on the diagonal, the collusion between managers may favour a synchronisation of the dynamics in a few iterations. In contrast, in the case of competition, we observed on-off intermittency, blow out phenomena and multistability. Different from Bischi et al. [18, 19], this phenomena are observed even if players are homogeneous (symmetric map).

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