ABELIAN VARIETIES IN BRILL–NOETHER LOCI

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Abstract. In this paper, improving on results in [1, 6], we give the full classification of curves C of genus g such that a Brill–Noether locus $W^s_d(C)$, strictly contained in the jacobian $J(C)$ of C, contains a variety Z stable under translations by the elements of a positive dimensional abelian subvariety $A \subsetneq J(C)$ and such that $\dim(Z) = d - \dim(A) - 2s$, i.e., the maximum possible for such a Z.

Contents

1. Introduction

In [1] the authors posed the problem of studying, and possibly classifying, situations like this:

(*) C is a smooth, projective, complex curve of genus g, Z is an irreducible r–dimensional subvariety of a Brill–Noether locus $W^s_d(C) \subsetneq J^d(C)$, and Z is stable under translations by the elements of an abelian subvariety $A \subsetneq J(C)$ of dimension $a > 0$ (if so, we will say that Z is *A–stable*).

Actually in [1] the variety Z *is* the translate of a positive dimensional proper abelian subvariety of $J(C)$, while the above slightly more general formulation was given in [6].

The motivation for studying (*) resides, among other things, in a theorem of Faltings (see [7]) to the effect that if X is an abelian variety defined over a number field $\mathbb{K}$, and $Z \subsetneq X$ is a subvariety not containing any translate of a positive dimensional abelian subvariety of X, then the number of rational point of Z over $\mathbb{K}$ is finite. The idea in [1] was to apply Faltings' theorem to the d–fold symmetric product $C(d)$ of a curve C defined over a number field $\mathbb{K}$. If C has no positive dimensional linear series of degree d, then $C(d)$ is isomorphic to its Abel–Jacobi image $W_d(C)$ in $J^d(C)$, thus $C(d)$ has finitely many rational points over $\mathbb{K}$ if $W_d(C)$ does not contain any translate of a positive dimensional abelian subvariety of $J(C)$. The suggestion in [1] is that, if, by contrast, $W_d(C)$ contains the translate of a positive dimensional abelian subvariety of $J(C)$, then C should be *quite special*, e.g., it should admit a map to a curve of lower positive genus (curves of this kind clearly are in situation (*)). This idea was tested in [1], where a number of partial results were proven for low values of d.

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The problem was taken up in [6] where, among other things, it is proven that if (*) holds, then $r + a + 2s \leqslant d$, and, if in addition $d + r \leqslant g - 1$, then $r + a + 2s = d$ if and only if:

(a) there is a degree 2 morphism $\varphi\colon C \to C'$, with C' a smooth curve of genus a, such that $A = \varphi^*(J(C'))$ and $Z = W_{d-2a-2s}(C) + \varphi^*(J^{a+s}(C'))$.

In [6] there is also the following example with $(d, s) = (g - 1, 0)$:

(b) there is an (étale) degree 2 morphism $\varphi\colon C \to C'$, with C' a smooth curve of genus $g' = r + 1$, A is the Prym variety of φ and $Z \subset W_{g-1}(C)$ is the connected component of $\varphi_*^{-1}(K_{C'})$ consisting of divisor classes D with $h^0(\mathcal{O}_C(D))$ odd, where $\varphi_*\colon J^{g-1}(C) \to J^{g-1}(C')$ is the *norm map*. One has $Z \cong A$, hence $r = a$.

ho aggiunto qualche ulteriore dettaglio

One more family of examples is the following (see Corollary 3.9 below):

(c) C is hyperelliptic, there is a degree 2 morphism $\varphi\colon C \to C'$ with C' a smooth curve of genus a such that $g > 2a + 1$, $A = \varphi^*(J(C'))$, $0 < s < g - 1$ and $Z = \varphi^*(J^a(C')) + W_{d-2s-2a}(C) + W^s_{2s}(C)$ (notice that $W^s_{2s}(C)$ is a point).

piccole modifiche

The aforementioned result in [6] goes exactly in the direction indicated in [1]. The unfortunate feature of it is the hypothesis $d + r \leqslant g - 1$ which turns out to be *quite strong*. To understand how strong it is, consider the case $(d, s) = (g - 1, 0)$, which is indeed the crucial one (see [6, Proposition 3.3] and §3.2 below) and in which Debarre–Fahaloui's theorem is void.

The aim of the present paper is to give the full classification of the cases in which (*) happens and $d = r + a + 2s$. What we prove (see Theorem 3.1 and Corollary 3.9) is that if (*) holds then, with no further assumption, either (a) or (b) or (c) occurs.

The idea of the proof is not so different, in principle, from the one proposed in [6] in the restricted situation considered there. Indeed, one uses the A–stability of Z and its maximal dimension to produce linear series on C which are not birational, in fact composed with a degree 2 irrational involution. The main tool in [6], inspired by [1], is a Castelnuovo's type of analysis for the growth of the dimension of certain linear series.

Our approach also consists in producing a non birational linear series on C, but it is in a sense more direct. We consider (*) with $(d, s) = (g - 1, 0)$ and $a + r = g - 1$, i.e., the basic case (the others follow from this), in which Z is contained in $W_{g-1}(C)$, which is a translate of the theta divisor $\Theta \subset J(C)$. This immediately produces, restricting to Z the Gauss map of Θ, a base point free sublinear series L of dimension r of the canonical series of C. It turns out that Z is birational to an irreducible component of the variety $C(g - 1, L) \subset C(g - 1)$ consisting of all effective divisors of C of degree $g-1$ contained in some divisor of L. The A–stability of Z implies that $C(g - 1, L)$ has some other component besides the one birational to Z, and this forces L to be non–birational. Once one knows this, a (rather subtle) analysis of the map determined by L and of its image leads to the conclusion.

As for the contents of the paper, §2 is devoted to a few general facts about the varieties $C(k, L) \subset C(k)$ of effective divisors of degree k contained in a divisor of a linear series L on a curve C. These varieties, as we said, play a crucial role in our analysis. Section 3 is devoted to the proof of our main result.

One final word about our own interest in this problem, which is quite different from the motivation of [1, 6]. It is in fact related to the study of irregular surfaces S of general type, where situation (*) presents itself in a rather natural way. For example, let $C \subset S$ be a smooth, irreducible curve, and assume that C corresponds to the general point of an irreducible component $\mathcal{C}$ of the Hilbert scheme of curves on S which dominates $\mathrm{Pic}^0(S)$. Recall that there is also only one irreducible component $\mathcal{K}$ of the Hilbert scheme of curves algebraically equivalent to canonical curves on S which dominates $\mathrm{Pic}^0(S)$ (this is called the *main paracanonical system*). The curves of $\mathcal{C}$ cut out on C divisors which are residual, with respect to $|K_C|$, of divisors cut out by curves in $\mathcal{K}$. Consider now the one of the two systems $\mathcal{C}$ and $\mathcal{K}$ whose curves cut on C divisors of minimal degree $d = \min\{C^2, K_S \cdot C\}$, and denote by s the dimension of the general fibre of this system over $\mathrm{Pic}^0(S)$. Thus the image of the natural map $\mathrm{Pic}^0(S) \to J^d(C)$ defined by $\eta \mapsto \mathcal{O}_C(C+\eta)$ (or by $\eta \mapsto \mathcal{O}_C(K_S+\eta)$) is a q–dimensional abelian variety contained in $W^s_d(C)$, which is what happens in (*). Thus understanding (*) would provide us with the understanding of (most) curves on irregular surfaces.

The results in this paper, even if restricted to the very special case of (*) in which Z has maximal dimension, turn out to be useful in surface theory. For example, if S is a minimal irregular surface of general type, then $K_S^2 \geqslant 2p_g$ (see [5]). Using the results in this paper we are able to classify surfaces for which $K_S^2 = 2p_g$. This classification will appear in a forthcoming paper.

2. Linear series on curves and related subvarieties of symmetric products

2.1. **Generalities.** Let C be a smooth, projective, irreducible curve of genus g. For an integer $k \geqslant 0$, we denote by $C(k)$ the k-th symmetric product of C (by convention, $C(0)$ is a point).

Let L be a base point free g^r_d on C. We denote the corresponding morphism by $\phi_L\colon C \to \bar{C} \subset \mathbb{P}^r$ and by C' the normalization of $\bar{C}$. We let $f\colon C \to C'$ be the induced morphism and L' the linear series on C' such that $L = f^*(L')$. We set $\deg(f) = \nu \geq 1$, so that $d = \delta\nu$, with $\delta = \deg(\bar{C})$. We say that L is *birational* if $\phi_L\colon C \to \bar{C}$ is birational.

Let $k \leqslant d$ be a positive integer. We consider the incidence correspondence

$$\mathcal{C}(k, L) = \{(D, H) \in C(k) \times L \mid D \leqslant H\}$$

with projections p_i (with $1 \leqslant i \leqslant 2$) to the first and second factor. Set $C(k, L) = p_1(\mathcal{C}(k, L))$, which has a natural scheme structure (see [4] or [2, p. 341]). Note the isomorphism

$$\mathfrak{s}\colon (D, H) \in \mathcal{C}(k, L) \to (H - D, H) \in \mathcal{C}(d - k, L)$$

Lemma 2.1. (i) *If $k \leqslant r$, then $C(k, L) = C(k)$.*
(ii) *If $k \geqslant r$, then $C(k, L)$ has pure dimension r. If V is an irreducible component of $C(k, L)$, there is a unique component $\mathcal{V}$ of $\mathcal{C}(k, L)$ dominating V via p_1 and p_1 induces a birational morphism of $\mathcal{V}$ onto V. Finally $\mathcal{V}$ dominates L via p_2.*
(iii) *If $\min\{k, d-k\} \geqslant r$ the isomorphism $\mathfrak{s}$ induces a map $\mathfrak{r}\colon C(k, L) \dashrightarrow C(d-k, L)$, which is componentwise birational.*
(iv) *If L is birational, then $C(k, L)$ is irreducible.*

Proof. Part (i) is clear. The dimensionality assertion in (ii) follows from [4, § 1] or [2, Lemma (3.2), p. 342]. The rest of (ii) follows from these facts: the fibres of p_1 are isomorphic to linear subseries of L and p_2 is finite, so no component of $\mathcal{C}(k, L)$ has dimension larger than r. Part (iii) follows from (ii). Part (iv) follows from the Uniform Position Theorem (see [2, p. 112]). □

Next we look at the case L non–birational and $k \geqslant r$. Consider the induced finite morphism $f_*\colon C(k, L) \to C'(k)$. For each partition $\underline{m} = (m_1, \ldots m_s)$ of k with $\delta \geqslant s \geqslant r$ and $1 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_s \leqslant \nu$, we denote by $C(\underline{m}, L)$ the closure in $C(k, L)$ of the inverse image via f_* of the set of divisors of the form $m_1y_1 + \cdots + m_sy_s$, with $y_1 + \cdots + y_s$ a reduced divisor in $C'(s, L')$. To denote a partition $\underline{m}$ as above we may use the *exponential notation* $\underline{m} = (1^{\mu_1}, \ldots, \nu^{\mu_\nu})$, meaning that i is repeated μ_i times, with $1 \leqslant i \leqslant \nu$, and we may omit i^{μ_i} if $\mu_i = 0$. Note that $\sum_{i=1}^{\nu} \mu_i = s \leqslant \delta$. Set $\mu_0 := \delta - s$. Then $\underline{m}^c = (1^{\mu_{\nu-1}}, \ldots, (\nu-1)^{\mu_1}, \nu^{\mu_0})$ is a partition of $d - k$ which we call the *complementary partition* of $\underline{m}$.

Lemma 2.2. *In the above set up each irreducible component of $C(\underline{m}, L)$ has dimension r, hence it is an irreducible component of $C(k, L)$ and all irreducible components of $C(k, L)$ are of this type.*

Proof. Let V be an irreducible component of $C(k, L)$: by Lemma 2.1 there is a unique irreducible component $\mathcal{V}$ of $\mathcal{C}(k, L)$ dominating it and $p_1|_{\mathcal{V}} : \mathcal{V} \to V$ is birational. Let $D \in V$ be a general point and let (D, H) be the unique point of $\mathcal{V}$ mapping to D via p_1, so that H is a general divisor in L (cf. Lemma 2.1, (ii)). Hence H consists of δ distinct fibres $F_1, \ldots, F_\delta$ of f, each being a reduced divisor of degree ν on C. Then D consists of m_i points in F_i, for $1 \leqslant i \leqslant s \leqslant \delta$, where we may assume $1 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_s \leqslant \nu$. Moreover, since $\dim(V) = \dim(\mathcal{V}) = r$ and p_2 is finite, one has $s \geqslant r$. Hence $D \in C(\underline{m}, L)$, with $\underline{m} = (m_1, \ldots m_s)$, i.e. $V \subseteq C(\underline{m}, L) \subseteq C(k, L)$ hence V is a component of $C(\underline{m}, L)$.

The above considerations and Lemma 2.1, (ii), applied to $C'(s, L')$, imply that the image of $C(\underline{m}, L)$ in $C'(k)$ has dimension r, so each component of $C(\underline{m}, L)$ has dimension r. □

ho tolto il lemma, commentandolo: cosi' com'e' e' sbagliato, in quanto non e' detto affatto che la $L(-D)$ abbia punti di ramificazione semplice se ha dimensione 1. Ad esempio se una curva piana ha una cuspide non ordinaria...La cosa e' invece vera se L e' molto ampia

Si', ho fatto un errore stupido. In effetti avevo in mente il caso molto ampio

2.2. **Abel–Jacobi images.** We assume from now on that C has genus $g > 0$. For an integer k, we denote by $J^k(C) \subset \mathrm{Pic}(C)$ the set of linear equivalence classes of divisors of degree k on C. So $J(C) := J^0(C)$ is the Jacobian of C, which is a principally polarised abelian variety whose theta divisor class we denote by Θ_C, or simply by Θ.

The abelian variety $J(C)$ acts via translation on $J^k(C)$ for all k. If $X \subseteq J^k(C)$ and $Y \subseteq J(C)$, we say that X is Y*–stable*, if for all $x \in X$ and for all $y \in Y$, one has $x + y \in X$.

For all integers k, fixing the class of a divisor of degree k determines an isomorphism $J^k(C) \cong J(C)$. Given a subvariety V of $J^k(C)$, one says that it *generates* $J^k(C)$ if the image of V via one of the above isomorphisms generates $J(C)$ as an abelian variety. This definition does not depend on the choice of the isomorphism $J^k(C) \cong J(C)$.

For every $k \geqslant 1$, we denote by $j_k : C(k) \to J^k(C) \cong J(C)$ (or simply by j) the Abel–Jacobi map. We denote by $W^s_k(C)$ the subscheme of $J^k(C)$ corresponding to classes of divisors D such that $h^0(\mathcal{O}_C(D)) \geqslant s+1$ (these are the so–called *Brill–Noether loci*). One sets $W_k(C) := W^0_k(C) = \mathrm{Im}(j_k)$ and $W_{g-1}(C)$ maps to a theta divisor of $J(C)$, so we may abuse notation and write $W_{g-1}(C) = \Theta_C$.

We denote by $\Gamma_C(k, L)$ [resp. $\Gamma_C(\underline{m}, L)$] the image in $W_k(C)$ of $C(k, L)$ [resp. $C(\underline{m}, L)$] (we may drop the subscript C if there is no danger of confusion). The expected dimension of $\Gamma(k, L)$ is $\min\{r, g\}$ (by dimension of a scheme we mean the maximum of the dimensions of its components).

We set $\rho_C(k, L) := \dim(\Gamma_C(k, L))$ (simply denoted by $\rho(k, L)$ or by ρ if no confusion arises). By Lemma 2.1, (i), one has:

$$\text{if } k \leqslant r \text{ then } \Gamma(k, L) = W_k(C), \text{ hence } \rho = \min\{k, g\} \tag{2.1}$$

So we will consider next the case $k > r$, in which $\rho \leqslant \dim(C(k, L)) = r$, by Lemma 2.1, (ii). Then the class $c(k, L)$ of $C(k, L)$ in the Chow ring of $C(k)$ is computed in [2, Lemma VIII.3.2]. If x is the class of $C(k-1) \subset C(k)$ and $\theta := j^*(\Theta)$, one has

$$c(k, L) = \sum_{s=0}^{k-r} \binom{d-g-r}{s} \frac{x^s \theta^{k-r-s}}{(k-r-s)!} . \tag{2.2}$$

ho spostato di un capoverso la formula (2.2), che vale SOLO per $k \geqslant r$

Lemma 2.3. *Assume $k > r$ and $d - g - r \geqslant 0$. Then:*
(i) if $k - g \leqslant \min\{k - r, d - g - r\}$ one has $\rho = r \leqslant g$;
(ii) if $k - g \geqslant \min\{k - r, d - g - r\} = k - r$, one has $\rho = g \leqslant r$;
(iii) if $k - g \geqslant \min\{k - r, d - g - r\} = d - g - r$, one has $\rho = d - k \leqslant \min\{r, g\}$.

Proof. Note that x^s is the class of $C(k - s) \subset C(k)$, for $1 \leqslant s \leqslant k$. Applying the projection formula (cf. [8], Example 8.1.7) to (2.2), we find the class $\gamma(k, L)$ of $\Gamma(k, L)$

$$\gamma(k, L) = \sum_{s=0}^{k-r} \binom{d-g-r}{s} \frac{w_{k-s}\Theta^{k-r-s}}{(k-r-s)!} \tag{2.3}$$

where w_i is the class of $W_i(C)$ for any $i \geqslant 0$. By Poincaré's formula (cf. [3, §11.2]) one has

$$w_{k-s}\Theta^{k-r-s} = \begin{cases} \Theta^{k-r-s}, & \text{if } k - g \geqslant s \geqslant 0, \\ \frac{\Theta^{g-r}}{(g-k+s)!}, & \text{if } \max\{0, k - g\} \leqslant s \leqslant \min\{k - r, d - g - r\}, \end{cases}$$

whence the assertion follows. □

Lemma 2.4. *(i) If $\rho = g$ then $g \leqslant \min\{k, r\}$;*
(ii) if $r \geqslant k \geqslant g$, then $\rho = g$;
(iii) if $k > r \geqslant g$ and $d \geqslant k + g$ then $\rho = g$;
(iv) $\rho = 0$ if and only if $k = d$.

Proof. Parts (i) and (ii) follow from Lemma 2.1.

(iii) In (2.3) one has the summand corresponding to the index $s = k - r > 0$, which is Θ^0 with the positive coefficient $\binom{d-g-r}{k-r}$, and no other summand in (2.3) cancels it.

non capisco come vuoi cambiare, comunque procedi pure

non importa, lasciamo così

(iv) If $k = d$ then $C(k, L) = L$ and clearly $\rho = 0$. Conversely, if $\rho = 0$ then in (2.3) the term Θ^g has to appear with non–zero coefficient and no other term Θ^i with $0 \leqslant i < g$ appears with non–zero coefficient. By looking at the proof of Lemma 2.3, we see that the summand Θ^g appears in (2.3) only if $0 \leqslant s = k - r - g$. Then $d \geqslant k \geqslant r + g$. So we may apply Lemma 2.3, and conclude that $\rho = 0$ occurs only in case (iii), if $k = d$. □

Rita ha ragione, anzi di piu', il lemma successivo e' vero anche se $k < r$. Ho cambiato leggermente la dim per adattarla al caso generale. Ho cambiato conseguentemente i corollari successivi.

Lemma 2.5. *Let $A \subseteq J(C)$ be an abelian subvariety of dimension a and let $p\colon J^k(C) \to J' := J(C)/A$ be the map obtained by composing an isomorphism $J^k(C) \cong J(C)$ with the quotient map $J(C) \to J'$. Then*

$$\dim(p(\Gamma(k, L))) = \min\{g - a, \rho\}.$$

Proof. If $\rho = g$ the statement is obvious, hence we assume $\rho < g$.

Consider first the case $k > r$. Assume by contradiction that $\dim(p(\Gamma(k, L)) < \min\{\rho, g - a\}$. Let ξ be the class of the pull back to $J(C)$ of an ample line bundle of J'. We have $\bar{\gamma}(k, L)\xi^\rho = 0$, where $\bar{\gamma}(k, L)$ is the ρ–dimensional part of $\gamma(k, L)$. By (2.3) one has $\bar{\gamma}(k, L) = \alpha\Theta^{g-\rho}$, where $\alpha \in \mathbb{Q}$ is positive because $\Gamma(k, L)$ is an effective non–zero cycle of dimension ρ. Hence $\bar{\gamma}(k, L)\xi^\rho = \alpha\Theta^{g-\rho}\xi^\rho > 0$, because Θ is ample. Thus we have a contradiction.

If $k \leqslant r$, then $\Gamma(k, L) = W_k(C)$, $\rho = k$ and $\gamma(k, L)$ is again a rational multiple of Θ^{g-k} (by Poincaré's formula), so the proof proceeds as above. □

Corollary 2.6. *If $A \subseteq J(C)$ is an abelian subvariety of dimension $a > 0$ and $\Gamma(k, L)$ is A–stable, then the restriction of p to $\Gamma(k, L)$ is surjective onto $J' = J(C)/A$, hence $\Gamma(k, L) = J^k(C) = W_k(C)$, i.e., $\rho = g$.*

Corollary 2.7. *If $d - 1 \geqslant k \geqslant 1$, then $\Gamma(k, L)$ generates $J^k(C)$.*

Proof. If $k \leqslant r$ then $\Gamma(k, L) = W_k(C)$ and the assertion is clear. Assume $k > r$. By Lemma 2.5, $\Gamma(k, L)$ generates $J^k(C)$ as soon as $\rho > 0$, which is the case by Lemma 2.4, (iv). □

2.3. **A useful lemma.** Let L be a base point free g^1_d, let $\phi_L \colon C \to \mathbb{P}^1$ be the corresponding map and denote by G_L the Galois group of ϕ_L.

Lemma 2.8. *If L is a base point free g^1_4, then one of the following occurs:*
(a) $C(2, L)$ is irreducible.
(b) $C(2, L)$ has two components. This occurs if and only if $G_L \cong \mathbb{Z}_2,\ \mathbb{Z}_4$.
(c) $C(2, L)$ has 3 components. This occurs if and only if $G_L \cong \mathbb{Z}_2^2$.

Proof. We argue as in the proof of [3, Lemma 12.7.1]. Let $\Delta \subset \mathbb{P}^1$ be the set of critical values of $\phi_L \colon C \to \mathbb{P}^1$, let $\rho \colon \pi_1(\mathbb{P}^1 \setminus \Delta) \to \mathfrak{S}_4$ be the monodromy representation and let $\Sigma := \mathrm{Im}(\rho)$. The group Σ acts transitively on $\mathbb{I}_4 := \{1, 2, 3, 4\}$ (identified with the general divisor $x_1 + \ldots + x_4 \in L$) and its order s is divisible by 4. The irreducible components of $C(2, L)$ are in 1–to–1 correspondence with the Σ–orbits of the order two subsets of $\mathbb{I}_4$. If Σ contains an element of order 3, then s is divisible by 12. Hence $\mathfrak{A}_4 \subseteq \Sigma$, thus the action is transitive, $C(2, L)$ is irreducible and (a) holds.

Assume Σ is a 2–group. If $s = 8$ then Σ is a 2–Sylow subgroup of $\mathfrak{S}_4$, hence Σ is the dihedral group D_4. Then C is obtained from a Σ–cover $C_0 \to \mathbb{P}^1$ by moding out by a reflection $\sigma \in \Sigma$. In the Σ-action on $\mathbb{I}_4$ we may assume that an element of order 4 acts by sending i to $i + 1$ modulo 4, for $1 \leqslant i \leqslant 4$. So the order 2 element in the center of Σ induces the involution ι of C that maps i to $i + 2$ modulo 4, for $1 \leqslant i \leqslant 4$, and ι generates $G_L \cong \mathbb{Z}_2$. There are two orbits for the Σ–action on the set of order two subsets of $\mathbb{I}_4$: one of order 2 given by $\{\{1, 3\}, \{2, 4\}\}$, the other of order 4 given by $\{\{i, i + 1\}$, for $1 \leqslant i \leqslant 4\}$ (here i is taken modulo 4). These orbits respectively correspond to two components E_1, E_2 of $C(2, L)$ and we are in case (b).

Assume $s = 4$. Then ϕ_L is Galois with $G_L = \Sigma$. If $G_L \cong \mathbb{Z}_4$, then the Σ–orbits on the set of order two subsets of $\mathbb{I}_4$ are as in the previous case. If $G_L \cong \mathbb{Z}_2^2$, then one has $G_L = \{\mathrm{Id}, (12)(34), (13)(24), (14)(32)\}$. There are then three orbits, corresponding to three components E_1, E_2, E_3. □

One can be more precise about the components E_i of $C(2, L)$ in Lemma 2.8, whose geometric genera we denote by g_i (with $1 \leqslant i \leqslant 2 + \epsilon$, and $\epsilon = 0$ in case (b), $\epsilon = 1$ in case (c)).

Lemma 2.9. *Same setting and notation as in Lemma 2.8 and its proof. Then:*
(i) each component of $C(2, L)$ maps birationally to its image in $J^2(C)$ unless C is hyperelliptic and L is composed with the hyperelliptic involution $\mathcal{L}$: in this case one of the components of $\Gamma(2, L)$ is $\mathcal{L} = C(2, \mathcal{L})$, which is contracted to a point in $J^2(C)$;
(ii) in case (b) one has $E_1 \cong C/\iota$ (where ι is the non–trivial involution in G_L), which is hyperelliptic, and the abelian subvariety of $J^2(C) \cong J(C)$ generated by $j(E_1)$ is the ι–invariant part of $J(C)$. Moreover $2g_2 \geqslant g$;
(iii) In case (c) one has $E_i := C/\iota_i$ where ι_i are the three nonzero elements of G_L, for $1 \leqslant i \leqslant 3$.

Proof. We prove the only non–trivial assertion, i.e., $2g_2 \geqslant g$ in part (ii).

First assume $G_L = \mathbb{Z}_4 = \langle \rho \rangle$. Consider in $C(2)$ the curves $E_1 = \{P + \rho^2(P) \mid P \in C\}$ and $E_2 = \{P + \rho(P) \mid P \in C\}$. One has $C(2, L) = E_1 \cup E_2$ and $E_1 \cong C/\rho^2$. The curve E_2 is the image in $C(2)$ of the graph of ρ, so $g_2 = g$.

Suppose now $G_L = \mathbb{Z}_2$ and $\Sigma = D_4$. Recall that C is obtained from a D_4–Galois cover $f\colon C_0 \to \mathbb{P}^1$ by moding out by a reflection $\sigma \in D_4$ (see the proof of Lemma 2.8). Denote by g_0 the genus of C_0. Let $\rho \in D_4$ be an element of order 4, so that $D_4 = \langle \sigma, \rho \rangle$. Let n be the number of points of C_0 fixed by σ, n' the number of points fixed by $\sigma\rho$, m the number of points fixed by ρ and $m+\varepsilon$ the number of points fixed by ρ^2. The Hurwitz formula, applied to $C_0 \to \mathbb{P}^1$ and to $C_0 \to C = C_0/\sigma$, gives

$$g_0 = \frac{3}{2}m + n + n' + \frac{\varepsilon}{2} - 7, \quad \text{and} \quad g = \frac{n'}{2} + \frac{n}{4} + \frac{\varepsilon}{4} - \frac{3}{4}m - 3. \tag{2.4}$$

Claim 2.10. (i) *m, n, n' and ε are even.*
(ii) $n + m \equiv n' + m \equiv n + n' \equiv 0 \mod 4$*, and at most one among m, n, n' can be 0.*

Proof of the Claim. (i) The numbers n, n' and $m+\varepsilon$ are even because $\sigma, \sigma\rho$ and $\sigma\rho^2$ are involutions. If $P \in C_0$ is fixed by ρ, then $\sigma(P)$ is also fixed by ρ. Since the stabilizer of any point is cyclic, then $\sigma(P) \neq P$. This implies that m is even.

(ii) Consider the $\mathbb{Z}_2^2$–cover $D := C_0/\rho^2 \to \mathbb{P}^1$. The cardinalities of the images in $\mathbb{P}^1$ of the fixed loci of the three involutions are $n/2$, $n'/2$ and $m/2$. Indeed, denote by γ_1 [resp. by γ_2] the image of σ [resp. of ρ] in $D_4/\rho^2 \cong \mathbb{Z}_2^2$. Let $Q \in \mathbb{P}^1$ be a branch point whose preimage in D is fixed by γ_1. Then the preimage of Q in C_0 consists of 4 points, two of which fixed by σ and two by $\sigma\rho^2$, so the number of such points Q is $(2n)/4 = n/2$. Similarly, the image in $\mathbb{P}^1$ of the set of points of D fixed by γ_2 has cardinality $m/2$ and the image of the set of points fixed by $\gamma_1\gamma_2$ has cardinality $n'/2$. Hurwitz formula for $D_1 := D/\gamma_1 \to \mathbb{P}^1$ gives

$$2g(D_1) - 2 = \frac{m}{2} + \frac{n'}{2} - 4,$$

hence $m + n' > 0$ is divisible by 4. Similarly $n + n'$ and $m + n$ are positive and divisible by 4. □

We compute now the ramification of $f\colon C_0 \to \mathbb{P}^1$ and of L. Write the D_4–orbit of $P \in C_0$ general as

$$\begin{array}{llll} P & \rho(P) & \rho^2(P) & \rho^3(P) \\ \sigma(P) & \sigma\rho(P) & \sigma\rho^2(P) & \sigma\rho^3(P). \end{array} \tag{2.5}$$

Denote by $Q_1, \ldots, Q_4$ the images in C of the points in the first row (or, what is the same, in the second row) of (2.5). The singular fibers of f occur when P has non trivial stabilizer, i.e., when:
- $P \in C_0$ is fixed by ρ. The fiber of f is $4(P + \sigma(P))$ and the corresponding divisor of L is $4Q_1$. There are $m/2$ divisors of L of this type;
- P is fixed by ρ^2 but not by ρ. The fiber of f is $2(P + \sigma(P) + \rho(P) + \sigma\rho(P))$. Then $Q_1 = Q_3$, $Q_2 = Q_4$, and the corresponding divisor of L is $2(Q_1 + Q_2)$. There are $\varepsilon/4$ such divisors;
- P is fixed by σ. The fibre of f is $2(P + \rho(P) + \rho^2(P) + \rho^3(P))$, and $Q_2 = Q_4$, while Q_1, Q_2 and Q_3 are distinct, so the corresponding divisor of L is $Q_1 + 2Q_2 + Q_3$ and there are $n/2$ such divisors;
- P is fixed by $\sigma\rho^2$. This is the same as the previous case;
- P is fixed by $\sigma\rho$. The fibre of f is again $2(P + \rho(P) + \rho^2(P) + \rho^3(P))$, and $Q_1 = Q_2$, $Q_3 = Q_4$, so the corresponding divisor of L is $2Q_1 + 2Q_3$ and there are $n'/2$ such fibers;
- P is fixed by $\sigma\rho^3$. This is the same as the previous case.

Denote by ι the involution of C induced by ρ^2. Then $C(2, L)$ is the union of $E_1 = \{P + \iota(P) \mid P \in C\} = C/\iota$ and of the irreducible curve E_2. Keeping the above notation, if $Q_1 + \cdots + Q_4$ is the general divisor of L, then E_1 is described by the divisors $Q_1 + Q_3, Q_2 + Q_4$ and E_3 by $Q_1 + Q_2, Q_1 + Q_4, Q_2 + Q_3, Q_3 + Q_4$.

To compute g_2, define a map $\phi\colon C_0 \to E_2$ by sending $P \in C_0$ to the image via $C_0 \to C$ of the divisor $P + \rho(P)$, i.e., $Q_1 + Q_2 \in E_2$.

Claim 2.11. (i) $\deg(\phi) = 2$ *and* E_2 *is birational to* $C_0/\sigma\rho$*;*
(ii) $g_2 = \frac{3}{4}m + \frac{n}{2} + \frac{n'}{4} + \frac{\varepsilon}{4} - 3$.

Proof of the Claim. (i) Let $Q + Q' \in E_2$ be a general point and let $P, \sigma(P)$ [resp. $P', \sigma(P')$] be the preimages of Q [resp. of Q'] on C_0. Since $Q + Q'$ is of the form $Q_1 + Q_2, Q_1 + Q_4, Q_2 + Q_3$, or $Q_3 + Q_4$, we may assume that $P' = \rho(P)$, so that $\psi^{-1}(Q + Q') = \{P, \sigma\rho(P)\}$.

Part (ii) follows by applying Hurwitz formula. □

Finally, suppose by contradiction that $2g_2 \leqslant g - 1$. Then, by (ii) of Claim 2.11 and by (2.4), we would have $3(m + n) + \varepsilon \leq 8$, hence $m + n \leq 2$, contradicting (ii) of Claim 2.10. □

3. Abelian subvarieties of Brill–Noether loci

As in [1, 6], we consider $Z \subseteq W^s_d(C) \subsetneq J^d(C)$ an irreducible A–stable variety of dimension r, with $A \subsetneq J(C)$ an abelian subvariety of dimension $a > 0$. Note that $r \geqslant a$, with equality if and only if $Z \cong A$. Moreover, since $W^s_d(C) \subsetneq J^d(C)$, the general linear series $L \in Z$ is special, thus $s > d - g$ and $d \geqslant 2s$ by Clifford's theorem. From [6, Proposition 3.3], we have

$$r + a + 2s \leqslant d. \tag{3.1}$$

In this section, we classify the cases in which equality holds in (3.1), thus improving the partial results in [6] on this subject. Note that if equality holds in (3.1), then $Z \not\subseteq W^{s+1}_d(C)$ and A is a maximal abelian subvariety of $J(C)$ such that Z is A–stable.

3.1. The Theta divisor case.

As in [6], we first consider the case $(d, s) = (g - 1, 0)$, i.e., $Z \subset W_{g-1}(C) = \Theta$ is an irreducible A–stable variety of dimension $r = g - 1 - a$. Then $Z \not\subseteq W^1_{g-1}(C) = \mathrm{Sing}(\Theta)$, i.e., Z has a non–empty intersection with $\Theta_{\mathrm{sm}} := \Theta - \mathrm{Sing}(\Theta)$.

Theorem 3.1. *Let C be a curve of genus g. Let $A \subsetneq J(C)$ be an abelian variety of dimension $a > 0$ and $Z \subset \Theta$ an irreducible, A–stable variety of dimension $r = g - 1 - a$. Then there is a degree 2 morphism $\varphi \colon C \to C'$, with C' smooth of genus g', such that one of the following occurs:*
(a) $g' = a$, $A = \varphi^*(J(C'))$ and $Z = W_{g-1-2a}(C) + \varphi^*(J^a(C'))$;
(b) $g' = r+1$, φ is étale, A is the Prym variety of φ and $Z \subset W_{g-1}(C)$ is the connected component of $\varphi_*^{-1}(K_{C'})$ consisting of divisor classes D with $h^0(\mathcal{O}_C(D))$ odd, where $\varphi_* \colon J^{g-1}(C) \to J^{g-1}(C')$ is the norm map.

come nell'introduzione

In particular, $Z \cong A$ is an abelian variety if and only if either we are in case (a) and $g = 2a+1$, or in case (b).

il remark è nuovo

Remark 3.2. Cases (a) and (b) of Theorem 3.1 are not mutually exclusive. Indeed, if the curve C in case (b) is hyperelliptic, then the Abel-Prym map $C \to A$ induces a 2-to-1 map $\psi \colon C \to D$, where D is a smooth curve embedded into $A \cong J(D)$ by the Abel-Jacobi map. One can check that $A = \psi^*(J^r(D))$, namely this is also an instance of case (a) of Theorem 3.1.

The proof of Theorem 3.1 requires various preliminary lemmas. First, recall that the tangent space to $J(C)$ at 0 can be identified with $H^1(\mathcal{O}_C) \cong H^0(K_C)^*$. Denote by $T \subseteq H^0(K_C)^*$ the tangent space to A at 0 and by L the linear series $\mathbb{P}(T^\perp) \subseteq |K_C|$. One has $\dim(L) = g - 1 - a = r$.

Since $Z \cap \Theta_{\mathrm{sm}} \neq \emptyset$, the Gauss map of Θ restricts to a rational map $\gamma \colon Z \dashrightarrow \mathbb{P} := |K_C|$.

Lemma 3.3. *One has $\gamma(Z) = L$.*

Proof. Since Z is A–stable, one has $\gamma(Z) \subsetneq L$. A point of $\Theta_{\rm sm}$ can be identified with a divisor D of degree $g-1$ such that $h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_C(K_C - D)) = 1$. The Gauss map sends $D \in \Theta_{\rm sm}$ to the unique divisor of $|K_C|$ containing D. Then $\gamma^{-1}(\gamma(D))$ is finite. Hence $\dim(\gamma(Z)) = \dim(Z) = r = \dim(L)$. The assertion follows. □

Note the birational involution $\sigma\colon \Theta \dashrightarrow \Theta$, defined on $\Theta_{\rm sm}$, sending a divisor $D \in \Theta_{\rm sm}$ to the unique effective divisor $D' \in |K_C - D|$. Then σ restricts on Z to a birational map (still denoted by σ) onto its image Z', which is also A-stable.

Lemma 3.4. *The linear series L is base point free.*

Proof. Suppose $P \in C$ is a base point of L. If every $D \in Z$ contains P, then the map $D \to D - P$ defines an injection $Z \hookrightarrow W_{g-2}(C)$, contradicting (3.1). If the general $D \in Z$ does not contain P, then $\sigma(D)$ contains P for $D \in Z$ general, and we can apply the previous argument to Z'. □

ho cambiato l'enunciato (e un po' la dimostrazione) del lemma seguente

Lemma 3.5. *One has:*
(i) *Z is a component of $\Gamma(g-1, L)$.*
(ii) *$Z \subsetneq \Gamma(g-1, L)$.*
(iii) *The linear series L is not birational.*

Proof. By Lemma 3.3, we have $Z \subseteq \Gamma(g-1, L)$ and the dimension of $\Gamma(g-1, L)$ is equal to r (by Lemma 2.3). So (i) holds. If $Z = \Gamma(g-1, L)$, then $\Gamma(g-1, L)$ is A-stable. By Corollary 2.6, we have $r = g$, a contradiction. This proves (ii). Then (iii) holds by Lemma 2.1, (iv). □

Recall the notation $\phi_L\colon C \to \bar{C} \subset \mathbb{P}^r$, C' for the normalization of $\bar{C}$, $f\colon C \to C'$ the induced morphism and $\nu = \deg(f) > 1$. For any integer h one has the morphism $f^*\colon J^h(C') \to J^{h\nu}(C)$. Let g' be the genus of C' and L' the (birational) linear series on C' of dimension r such that $L = f^*(L')$, whose degree is $\delta = \frac{2g-2}{\nu}$.

ho aggiunto "birational" fra parentesi due righi su

Lemma 3.6. *Assume $\nu \leqslant 3$ and let $\underline{m} = (1^{\mu_1}, \ldots, \nu^{\mu_\nu})$ be the partition of $g-1$ such that $Z \subseteq \Gamma(\underline{m}, L)$. Set $\mu_1 = \mu$, $\mu_2 = \mu'$. Then $\nu = 2$ and there are the following possibilities:*
(i) *$\mu = g-1$ (hence $\mu' = 0$);*
(ii) *$\mu = g-1-2a$, $\mu' = a$, $A = f^*(J(C'))$ (hence $g' = a$) and $Z = W_\mu(C) + f^*(J^a(C'))$.*

ho apportato ulteriori modificazioni a causa della eliminazione del lemma 2.3.

Proof. We first consider the case $\nu = 2$.

If $\mu = 0$, then $g-1 = 2\mu'$ and $Z = f^*(\Gamma_{C'}(\mu', L'))$, since $\Gamma_{C'}(\mu', L')$ is irreducible by Lemma 2.1. The general point of Z corresponds to a linearly isolated divisor, hence $\mu' \leqslant g'$. Since $f^*\colon J^{\mu'}(C') \to J^{g-1}(C)$ is finite, one has $r = \rho_{C'}(\mu', L') \leqslant \mu' \leqslant g'$. Since Z is A–stable, we have an isogeny $A \to \bar{A} \subseteq J(C')$ (so that $a \leqslant g'$) and $\Gamma_{C'}(\mu', L')$ is $\bar{A}$–stable. Hence by Corollary 2.6 one has $\Gamma_{C'}(\mu', L') = J^{\mu'}(C') = W_{\mu'}(C')$ and $g' \leqslant \min\{\mu', r\}$ (see Lemma 2.4, (i)). Then Z is $f^*(J(C'))$–stable, and, since a is the maximal dimension of an abelian subvariety of $J(C)$ for which Z is stable and $a = \dim(\bar{A}) \leqslant g'$, it follows $a = g'$. In conclusion $\mu' = a = r = g'$, and we are in case (ii).

Assume now $\mu > 0$. The general point of Z (which is a component $\Gamma(\underline{m}, L)$) is smooth for Θ, hence it corresponds to a linearly isolated, effective divisor D of degree $g-1$, which is reduced (see Lemma 2.1, (ii)) and can be written in a unique way as $D = M + f^*(N)$, where M and N

are effective divisors, with $\deg(M) = \mu$, $\deg(N) = \mu'$ and $M' := f_*(M)$ reduced. So there is a rational map $h\colon Z \dashrightarrow J^\mu(C)$ defined by $D = M + f^*(N) \mapsto M$.

Assume $\mu \le r$. By Lemma 2.1 the image of h is $W_\mu(C)$. Identify A with its general translate inside Z. Then we have a morphism $h|_A\colon A \to J(C)$ whose image we denote by $\bar{A}$. Then $W_\mu(C)$ is $\bar{A}$–stable. Since $W_\mu(C)$ is birational to $C(\mu)$, which is of general type because $\mu < g$, then $\bar{A} = \{0\}$. It follows that each component of the general fibre of h is A–stable, in particular $r - \mu \geqslant a \geqslant 1$.

Take $M \in W_\mu(C)$ general and set $L'' := L'(-M')$. Since L' is birational and $M' \in W_\mu(C')$ is general, with $\mu < r = \dim(L')$, then L'' has dimension $r - \mu \geqslant 1$ and it is base point free. It is also birational as soon as $r - \mu \geqslant 2$.

Assume first $r - \mu \geqslant 2$. Then $C'(\mu', L'')$ is irreducible by Lemma 2.1, (iv), and there is a birational morphism $C'(\mu', L'') \to h^{-1}(M) \subset J^{g-1}(C)$ factoring through the map $C'(\mu', L'') \to J^{\mu'}(C') \hookrightarrow J^{g-1}(C)$, where the last inclusion is translation by M. Namely, up to a translation, $h^{-1}(M) = \Gamma(\mu', L'')$. In particular, $\dim(\Gamma(\mu', L'')) = \dim(C(\mu', L'')) = r - \mu$, hence $\mu' \ge r - \mu$.

Remember that $h^{-1}(M) = \Gamma(\mu', L'')$ is A–stable for $M \in W_\mu(C)$ general. By Corollary 2.6 one has $\Gamma_{C'}(\mu', L'') = W_{\mu'}(C') = J^{\mu'}(C')$, so $g' \leqslant \min\{\mu', r - \mu\} = r - \mu$. On the other hand, since $h^0(\mathcal{O}_C(D)) = 1$, one has also $h^0(\mathcal{O}_C(f^*(N))) = 1$, hence $\mu' \leqslant g'$ and we conclude that $\mu' = g' = r - \mu$. The same argument as above yields $a = g'$ and we are again in case (ii).

If $r - \mu = 1$, then $a = 1$, $r = g - 2$, $\mu = g - 3$ and $\mu' = 1$. On the other hand L is cut out on the canonical image of C by the hyperplanes through the point p_A which is the projectivized tangent space to A at the origin. Then $\phi_L : C \to \bar{C}$ is the projection from p_A, $\bar{C}$ it is a normal elliptic curve, and we are again in case (ii).

Assume now $\mu > r$ and keep the above notation. In this case the map $h\colon Z \dashrightarrow \bar{Z} := h(Z)$ is generically finite, $\bar{A}$ is isogenous to A and $\bar{Z}$ is $\bar{A}$–stable. By [6, Lemma 3.1], we have $g - 1 \geqslant \mu \geqslant \dim(\bar{Z}) + \dim(\bar{A}) = r + a = g - 1$, thus $\mu = g - 1$, so we are in case (i).

Finally consider the case $\nu = 3$. Write the general $D \in \Gamma(\underline{m}, L)$ as $D = M_1 + M_2 + f^*(N)$, where M_1 is reduced of degree μ, $f_*(M_2) = 2M_2'$ with M_2' reduced of degree μ' and, as above, $\mu_3 = \deg(N) \leqslant g'$. Set $\tau = \mu + \mu'$ and consider the rational map $h\colon Z \dashrightarrow W_\tau(C) \subseteq J^\tau(C)$ defined by $D \mapsto M_1 + f^*(M_2') - M_2$.

If $\tau \leqslant r$, arguing as above (and keeping a similar notation) one sees that $h(Z) = W_\tau(C)$, the general fibre of h is A–stable, hence $r - \tau \geqslant a$. However $a = 1$ and ϕ_L non–birational, forces, as we have seen, $\nu = 2$, which is not the case here. Hence we have $a \geqslant 2$. We consider now $L'' = L'(-f_*(M_1) - M_2')$, which has dimension $r - \tau \geqslant 2$ and is base point free and birational, so $C'(\mu_3, L'')$ is irreducible. The general fiber $h^{-1}(h(D))$ of h is isomorphic to $\Gamma_{C'}(\mu_3, L'')$ and is A–stable. In particular $\dim(\Gamma_{C'}(\mu_3, L'')) = r - \tau$, hence $\mu_3 \ge r - \tau$. By Corollary 2.6, we have $\Gamma_{C'}(\mu_3, L'') \cong W_{\mu_3}(C') \cong J^{\mu_3}(C')$, so $g' \leqslant \min\{\mu_3, r - \tau\} = r - \tau$, hence $g' \le r - \tau \leqslant \mu_3 \leqslant g'$, thus $\mu_3 = r - \tau = g'$. In addition, as above, we have $a = g'$, so that $A = f^*(J(C'))$. Then $g - 1 - a = r = \tau + \mu_3 = \mu + \mu' + \mu_3$. On the other hand $g - 1 = \mu + 2\mu' + 3\mu_3$. This yields $\mu' + 2\mu_3 = a = \mu_3$, hence $\mu' = \mu_3 = 0$, which is not possible.

If $\tau > r$ then $h\colon Z \dashrightarrow \bar{Z} := h(Z)$ is generically finite, $\bar{A}$ is isogenous to A and $\bar{Z}$ is $\bar{A}$–stable. By [6, Lemma 3.1], we have $g - 1 \geqslant \tau \geqslant \dim(\bar{Z}) + \dim(\bar{A}) = r + a = g - 1$, thus $\tau = g - 1$, contradicting $\tau \leqslant \deg(L') = \frac{2}{3}(g - 1)$. □

Lemma 3.7. *If $\nu \geqslant 4$ then $\nu = 4$ and either*
(i) there is a degree 2 map $\psi\colon C \to E_1$ with E_1 a genus r hyperelliptic curve such that $Z = A = \psi^*(J^r(E_1))$; or
(ii) there is a faithful $\mathbb{Z}_2^2$–action on C with rational quotient; denoting by $f_i\colon C \to E_i$ (for $1 \leqslant i \leqslant 3$) the quotient map for the three non–trivial involutions of $\mathbb{Z}_2^2$, with E_i of genus g_i, and

$g_1 \geqslant g_2 \geqslant g_3$, then $g_1 = r+1$, $g_2+g_3 = r$ and $Z = A = f_2^*(J^{g_2}(E_2)) \times f_3^*(J^{g_3}(E_3))$ is the Prym variety associated to f_1.

Proof. Since L is base point free by Lemma 3.4 and it is not birational by Lemma 3.5, we have $\delta = \frac{2g-2}{\nu} \geqslant r \geqslant \frac{g-1}{2}$, hence $\nu \leqslant 4$.

Assume $\nu = 4$. Then $\bar{C}$ is a curve of degree $\frac{g-1}{2}$ spanning a projective space of dimension $r \geqslant \frac{g-1}{2}$. Hence $r = a = \frac{g-1}{2}$, $Z = A$, and $\phi_L = f$ is the composition of a g^1_4 (that we denote by $\mathcal{L}$) with the degree r Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^r$.

Let $\underline{m} = (1^{\mu_1}, \dots, 4^{\mu_4})$ be the partition of $2r$ such that $Z \subseteq \Gamma(\underline{m}, L)$.

Claim 3.8. *One has $\underline{m} = (2^r)$.*

Proof of the Claim. Assume by contradiction this is not the case, so that one among μ_1, μ_3, μ_4 is non–zero. We have $r = \dim(\Gamma(\underline{m}, L)) = \mu_1 + \cdots + \mu_4$, because $\bar{C}$ is a rational normal curve of degree r in $\mathbb{P}^r$, and $2r = g-1 = \mu_1 + 2\mu_2 + 3\mu_3 + 4\mu_4$, hence $\mu_1 = \mu_3 + 2\mu_4$ and $\mu_1 > 0$. So we may write the general divisor $D \in Z$ as $D = M + N$, where $M' := f_*(M)$ is reduced of degree $\mu := \mu_1$. Then we proceed as in the proof of Lemma 3.6.

Consider the rational map $h\colon Z = A \dashrightarrow J^\mu(C)$ defined by $D = M + N \mapsto M$. It extends to a morphism and $\bar{A} := h(A)$ is an abelian variety contained in $W_\mu(C)$. Since $\mu \leqslant r = \frac{g-1}{2}$, one has $\bar{A} = W_\mu(C)$, which is impossible. □

Let $E := C(2, L)$. Assume first C is not hyperelliptic. Then there is a birational (dominant) morphism $E(r) \to \Gamma((2^r), L)$ (see Lemma 2.9, (i)). If E is irreducible, then by Corollary 2.7 its image generates $J^2(C)$, hence also $\Gamma((2^r), L) = Z$ generates $J^{2r}(C)$ and we obtain a contradiction.

So E is reducible and we apply Lemma 2.8 and 2.9. Suppose we are in case (b) of Lemma 2.8 and (ii) of Lemma 2.9. Then $A = Z = \Gamma((2^r), L)$ is birational to $E_1(r_1) \times E_2(r_2)$, for non–negative integers r_1, r_2. However, if $r_i > 0$, then $r_i = g_i$, for $i \in \{1, 2\}$. Since $2g_2 \geqslant g$, one has $r_2 = 0$ because $2r = g - 1 < g$. Hence A is birational to $E_1(g_1)$ and we are in case (i).

Consider now case (c) of Lemma 2.8 and (iii) of Lemma 2.9. We claim that $J(C)$ is isogenous to $J(E_1) \times J(E_2) \times J(E_3)$. Indeed, consider the representation of $\mathbb{Z}_2^2$ on $H^0(K_C)$. Since $C/\mathbb{Z}_2^2$ is rational, we have $H^0(K_C) = V_{\chi_1} \oplus V_{\chi_2} \oplus V_{\chi_3}$, where for $i = 1, 2, 3$ the non trivial character χ_i of $\mathbb{Z}_2^2$ is orthogonal to the involution ι_i such that $C/\iota_i \cong E_i$, and $\mathbb{Z}_2^2$ acts on V_{χ_i} as multiplication by χ_i. Thus V_{χ_i} is the tangent space to $f_i^*(J(E_i))$.

Recall that there are three non–negative integers r_1, r_2, r_3 such that A is birational to $E_1(r_1) \times E_2(r_2) \times E_3(r_3)$. If $r_i > 0$, then $r_i = g_i$ (for $1 \leqslant i \leqslant 3$). Since $g_1 + g_2 + g_3 = g = 2r + 1$, at least one of the integers r_i is zero. If two of them are zero, we are again in case (i). So assume $r_1 = 0$ and r_2, r_3 non–zero. Then $r = g_2 + g_3$ and $g_1 = r + 1$. Moreover the involution ι_1 acts on A as multiplication by -1, hence A is the Prym variety of $f_1\colon C \to E_1$, and we are in case (ii).

The case C hyperelliptic can be treated similarly. In case (b) of Lemma 2.8 and (ii) of Lemma 2.9, E_1 is rational (hence it is contracted to a point by j), and $A = Z = \Gamma((2^r), L)$ is birational to $E_2(r)$. Then $r = g_2$ and we reach a contradiction since $2g_2 \geqslant g$. In case (c) of Lemma 2.8 and (iii) of Lemma 2.9 one has $g_3 = 0$ and the above argument applies with no change. □

Proof of Theorem 3.1. By Lemma 3.7, we may assume $\nu = 2$. Let $\underline{m}$ be the partition of $2r$ such that $Z \subseteq \Gamma(\underline{m}, L)$. By Lemma 3.6, it is enough to consider the case $\nu = 2$ and $\underline{m} = (1^{g-1})$, i.e., case (i) of that lemma. Then Z is contained in the kernel P of $f_*\colon J^{g-1}(C) \to J^{g-1}(C')$ (namely P is the generalized Prym variety associated with f). The space $H^0(K_C)$ decomposes under the involution ι associated with f as $H^0(K_{C'}) \oplus V$, where V is the space of antiinvariant 1-forms. Hence V^* is the tangent space to P, the tangent space T to A is also contained in V^* and the linear series L is equal to $\mathbb{P}(T^\perp) \supseteq \mathbb{P}(H^0(K_{C'}))$. On the other hand, by construction ι acts trivially on L, hence $T = V^*$ and thus $A = P$. This implies that $Z = A = P$, $g = 2r + 1$ and f is unramified with $g' = r + 1$. □

3.2. **General Brill–Noether loci.** The proof of the general formula (3.1) in [6] uses an argument which is useful to briefly recall. Let $Z \subseteq W^s_d(C) \subsetneq J^d(C)$ be A–stable and $L \in Z$ general, so L is a special g^s_d, which we may assume to be complete, so $d \geqslant 2s$. Let F_L be the fixed divisor of L and, if $s > 0$, let L' be the base point free residual linear series. Set $d' = \deg(L')$. Since L' is also special, we have $d' \geqslant 2s$. Then, by Lemma 2.8, (iii), one has the birational map $\mathfrak{r}\colon C(d'-s, L') \dashrightarrow C(s, L') \cong C(s)$.

Consider the morphism $j\colon C(d'-s, L') \times C(g-1-d+s) \to J^{g-1}(C)$ such that $j(D, D')$ is the class of $D + F_L + D'$ (if $s = 0$, we define $j\colon C(g-1-d) \to J^{g-1}(C)$ by $D' \mapsto F_L + D'$). If (D, D') is general in $C(d'-s, L') \times C(g-1-d+s)$, the divisor $D + F_L + D'$ is linearly isolated, hence j is generically finite onto its image Z_L, which therefore has dimension $g-1-d+2s$. Consider the closure Z' of the union of the Z_L's, with $L \in Z$ general, which is A–stable. One has $Z' \subseteq \Theta$ and the above discussion yields $r' := \dim(Z') = r + g - 1 - d + 2s$. Therefore $r' + a \leqslant g - 1$ if and only if (3.1) holds, with equality if and only if equality holds in (3.1).

Corollary 3.9. *Let C be a curve of genus g. Let $A \subsetneq J(C)$ be an abelian variety of dimension $a > 0$ and $Z \subseteq W^s_d(C) \subsetneq J^d(C)$ an irreducible, A–stable variety of dimension $r = d - 2s - a$. Assume $(d, s) \neq (g-1, 0)$. Then there is a degree 2 morphism $\varphi\colon C \to C'$, with C' a smooth curve of genus a with $g > 2a + 1$, such that $A = \varphi^*(J(C'))$ and either*
(i) $Z = W_{d-2a-2s}(C) + \varphi^*(J^{a+s}(C'))$, or
(ii) C is hyperelliptic and $Z = \varphi^*(J^a(C')) + W_{d-2s-2a}(C) + W^s_{2s}(C)$.

Proof. We keep the same notation as above.

We apply Theorem 3.1 to Z'. Since $d = r + a + 2s \geqslant 2(a + s)$ and $(d, s) \neq (g-1, 0)$, then $2a < g - 1$, hence case (b) of Theorem 3.1 does not occur for Z'. So we have a degree 2 morphism $\varphi\colon C \to C'$, with C' a smooth curve of genus a with $g > 2a + 1$, such that $A = \varphi^*(J(C'))$ and $Z' = W_{g-1-2a}(C) + \varphi^*(J^a(C'))$. The case $s = 0$ follows right away and we are in case (i). So we assume $s > 0$ from now on.

If L' is composed with the involution ι determined by φ, we are again in case (i). So assume L' is not composed with ι. The general $D \in C(d'-s, L')$ contains no fibre of φ and the same happens for the general $D' \in C(g-1-d+s)$. Since $D + F_L + D'$ corresponds to the general point of Z, and contains exactly a general fibres of φ, then F_L has to contain these a fibres, whose union we denote by F. Moreover, the description of Z implies that $D + D' + (F_L - F)$ is a general divisor of degree $g - 1 - 2a$, in particular D is a general divisor of degree $d' - s$. But D is also general in $C(d'-s, L')$, and this implies $d' - s \leqslant s$. On the other hand $d' \geqslant 2s$, hence $d' = 2s$. Then, either L' is the canonical series of C or C is hyperelliptic and L' is the s–multiple of the g^1_2. However the former case does not occur since by construction $d' = \deg(L') \leq g - 1$, hence we are in case (ii). □

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