Hylomorphic solitons and charged Q-balls: existence and stability

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Abstract

In this paper we give an abstract definition of solitary wave and soliton and we develope an abstract existence theory. This theory provides a powerful tool to study the existence of solitons for the Klein-Gordon equations as well as for gauge theories. Applying this theory, we prove the existence of a continuous family of stable charged Q-balls.

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1 Introduction

Loosely speaking a solitary wave is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A *soliton* is a solitary wave which exhibits some strong form of stability so that it has a particle-like behavior (see e.g. [35], [38], [37] and the references therein contained). We are interested in a class of solitons which, following [6], [4], [5], [12], we call hylomorphic. Their existence is due to an interplay between energy and charge. These solitons include the Q-balls, which are spherically symmetric solutions of the nonlinear Klein-Gordon equation and which have been studied since the pioneering papers [36] and [20]. Q-balls arise in a theory of bosonic particles (see [29], [30]), when there is an attraction between the particles. Roughly speaking, a Q-ball is a finite-sized "bubble" containing a large number of particles. The Q-ball is stable against fission into smaller Q-balls since, due to the attractive interaction, the Q-ball is the lowest-energy configuration of that number of particles. Q-balls also play an important role in the study of the origin of the matter that fills the universe (see [25]).

In this paper we give an abstract definition of solitary wave and soliton and we develope an abstract existence theory. This theory provides a powerful tool to study the existence of solitons for the Klein-Gordon equations as well as for gauge theories (see [7]). Most of the existence results in the present literature can be deduced in the framework of this theory using Th.18 or 19, as it is shown in a forthcoming book [17]. We get a new result applying Th.19 to the study of *charged Q-balls*. Let us describe this result.

If the Klein-Gordon equations are coupled with the Maxwell equations (NKGM), then the relative solitary waves are called *charged*, or *gauged Q-balls* (see e.g.[20]). The existence of charged Q-balls is stated in [10], [11], [13], [33]. However, in these papers there are not stability results and hence the existence of solitons for NKGM (namely *stable* charged Q-balls) was an open question.

The problem with the stability of charged Q-balls is that the electric charge tends to brake them since charges of the same sign repel each other. In this respect Coleman, in his celebrated paper [20], says "*I have been unable to construct Q-balls when the continuous symmetry is gauged. I think what is happening physically is that the long-range force caused by the gauge field forces the charge inside the Q-ball to migrate to the surface, and this destabilizes the system, but I am not sure of this*".

A partial answer to this question is in [15] where the existence of stable charged Q-balls is established provided that the interaction between matter and gauge field is sufficiently small. Theorem 19 allows to extend this result and to prove the existence of a continuous family of stable charged Q-balls. More precisely, we prove that there is a family of Q-balls $\{\mathbf{u}_\delta\}_{\delta\in(0,\delta_\infty)}$ whose energy and charge are decreasing with δ.

2 Solitary waves and solitons

In this section we construct a functional abstract framework which allows to define solitary waves, solitons and hylomorphic solitons.

2.1 Definitions of solitons

Solitary waves and solitons are particular *states* of a dynamical system described by one or more partial differential equations. Thus, we assume that the states of this system are described by one or more *fields* which mathematically are represented by functions

$$\mathbf{u} : \mathbb{R}^N \to V$$

where V is a vector space with norm $|\cdot|_V$ and which is called the internal parameters space. We assume the system to be deterministic; this means that it can be described as a dynamical system (X, γ) where X is the set of the states and $\gamma : \mathbb{R} \times X \to X$ is the time evolution map. If $\mathbf{u}_0(x) \in X$, the evolution of the system will be described by the function

$$\mathbf{u}(t, x) := \gamma_t \mathbf{u}_0(x). \tag{1}$$

We assume that the states of X have "finite energy" so that they decay at ∞ sufficiently fast and that

$$X \subset L^1_{loc}\left(\mathbb{R}^N, V\right). \tag{2}$$

Thus we are lead to give the following definition:

Definition 1 *A dynamical system (X, γ) is called of FT type (field-theory-type) if X is a Hilbert space of fuctions of type (2).*

For every $\tau \in \mathbb{R}^N$, and $\mathbf{u} \in X$, we set

$$(T_\tau \mathbf{u})(x) = \mathbf{u}(x + \tau). \tag{3}$$

Clearly, the group

$$\mathcal{T} = \left\{T_\tau |\ \tau \in \mathbb{R}^N\right\} \tag{4}$$

is a unitary representation of the group of translations.

Definition 2 *A set $\Gamma \subset X$ is called compact up to space tanslations or $\mathcal{T}$-compact if for any sequence $\mathbf{u}_n(x) \in \Gamma$ there is a subsequence $\mathbf{u}_{n_k}$ and a sequence $\tau_k \in \mathbb{R}^N$ such that $\mathbf{u}_{n_k}(x - \tau_k)$ is convergent.*

Now, we want to give a very abstract definition of solitary wave. As we told in the introduction, a solitary wave is a field whose energy travels as a localized packet and which preserves this localization in time. For example, consider a solution of a field equation having the following form:

$$\mathbf{u}\,(t,x) = u_0(x - vt - x_0)e^{i(v\cdot x - \omega t)};\ u_0 \in L^2(\mathbb{R}^N); \tag{5}$$

for every $x_0, v \in \mathbb{R}^N, \omega \in \mathbb{R}$, $\mathbf{u}\,(t,x)$ is a solitary wave. The evolution of a solitary wave is a translation plus a mild change of the internal parameters (in this case the phase).

This situation can be formalized by the following definition:

Definition 3 *If $\mathbf{u}_0 \in X$, we define the closure of the orbit of $\mathbf{u}_0$ as follows:*

$$\mathcal{O}\,(\mathbf{u}_0) := \overline{\{\gamma_t \mathbf{u}_0(x) \mid t \in \mathbb{R}\}}.$$

A state $\mathbf{u}_0 \in X$ is called solitary wave if

- *(i)* $0 \notin \mathcal{O}\,(\mathbf{u}_0)$*;*
- *(ii)* $\mathcal{O}\,(\mathbf{u}_0)$ *is $\mathcal{T}$-compact.*

Clearly, (5) describes a solitary wave according to the definition above. The standing waves, namely objects of the form

$$\gamma_t \mathbf{u} = \mathbf{u}(t,x) = u(x)e^{-i\omega t},\quad u \in L^2(\mathbb{R}^N),\ u \neq 0, \tag{6}$$

probably are the "simplest" solitary waves. In this case the orbit $\mathcal{O}\,(\mathbf{u}_0)$ itself is compact.

Take $X = L^2(\mathbb{R}^N)$ and $u \in X$; if $\gamma_t u = u\left(\frac{x}{e^t}\right)$, u is not a solitary wave, since (i) of the above definition is violated; if $\gamma_t u = \frac{1}{e^t} u\left(\frac{x}{e^t}\right)$, u is not a solitary wave since (ii) of Def. 3 does not hold. Also, according to our definition, a "couple" of solitary waves is not a solitary wave: for example

$$\gamma_t \mathbf{u} = \left[u_0(x - vt) + u_0(x + vt)\right] e^{i(v\cdot x - \omega t)},$$

is not a solitary wave since (ii) is violated.

The *solitons* are solitary waves characterized by some form of stability. To define them at this level of abstractness, we need to recall some well known notions in the theory of dynamical systems.

Definition 4 *A set $\Gamma \subset X$ is called invariant if $\forall \mathbf{u} \in \Gamma, \forall t \in \mathbb{R},\ \gamma_t \mathbf{u} \in \Gamma$.*

Definition 5 *Let (X, d) be a metric space and let (X, γ) be a dynamical system. An invariant set $\Gamma \subset X$ is called stable, if $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall \mathbf{u} \in X$,*

$$d(\mathbf{u}, \Gamma) \leq \delta,$$

implies that

$$\forall t \geq 0, \; d(\gamma_t \mathbf{u}, \Gamma) \leq \varepsilon.$$

Now we are ready to give the definition of soliton:

Definition 6 *A state $\mathbf{u} \in X$ is called soliton if $\mathbf{u} \in \Gamma \subset X$ where*

- *(i) Γ is an invariant stable set,*
- *(ii) Γ is $\mathcal{T}$-compact*
- *(iii) $0 \notin \Gamma$.*

The above definition needs some explanation. First of all notice that every $\mathbf{u} \in \Gamma$ is a soliton and that every soliton is a solitary wave. Now for simplicity, we assume that Γ is a manifold (actually, in many concrete models, this is the generic case). Then (ii) implies that Γ is finite dimensional. Since Γ is invariant, $\mathbf{u}_0 \in \Gamma \Rightarrow \gamma_t \mathbf{u}_0 \in \Gamma$ for every time. Thus, since Γ is finite dimensional, the evolution of $\mathbf{u}_0$ is described by a finite number of parameters. The dynamical system (Γ, γ) behaves as a point in a finite dimensional phase space. By the stability of Γ, a small perturbation of $\mathbf{u}_0$ remains close to Γ. However, in this case, its evolution depends on an infinite number of parameters. Thus, this system appears as a finite dimensional system with a small perturbation.

Example. We will illustrate the definition 6 with an example. Consider the solitary wave (5) and the set

$$\Gamma_v = \left\{ u(x - x_0) e^{i(v \cdot x - \theta)} \in H^1\left(\mathbb{R}^N, \mathbb{C}\right) : x_0 \in \mathbb{R}^N; \; \theta \in \mathbb{R} \right\}.$$

(5) is a soliton provided that Γ_v is stable; in fact the following conditions are satisfied:

- The dynamics on Γ_v is given by the following equation:

  $$\gamma_t \left[u(x - x_0) e^{i(v \cdot x - \theta)} \right] = u(x - vt - x_0) e^{i(v \cdot x - \theta - \omega t)}.$$

  This dynamics implies that Γ_v is invariant and that (iii) holds.

- we have assumed that Γ_v is stable; in this case any perturbation of our soliton has the following structure:

  $$u(t, x) = u(x - vt - x_0(t)) e^{i(v \cdot x - \theta(t))} + w(t, x)$$

  where $x_0(t)$, $\theta(t)$ are suitable functions and $w(t, x)$ is a perturbation small in $H^1\left(\mathbb{R}^N, \mathbb{C}\right)$.

- Γ_v is $\mathcal{T}$-compact; actually it is isomorphic to $\mathbb{R}^N \times S^1$.

2.2 Definition of hylomorphic solitons

We now assume that the dynamical system (X, γ) has two constants of motion: the energy E and the charge C. At the level of abstractness of this section (and the next one), the name energy and charge are conventional, but in our applications, E and C will be the energy and the charge as defined in section 5.2.

Definition 7 *A solitary wave* $\mathbf{u}_0 \in X$ *is called* ***standing hylomorphic soliton*** *if it is a soliton according to Def. 6 and if* Γ *has the following structure*

$$\Gamma = \Gamma(e_0, c_0) = \{\mathbf{u} \in X \mid E(\mathbf{u}) = e_0,\ |C(\mathbf{u})| = c_0\} \tag{7}$$

where

$$e_0 = \min\{E(\mathbf{u}) \mid |C(\mathbf{u})| = c_0\}. \tag{8}$$

Notice that, by (8), we have that a hylomorphic soliton $\mathbf{u}_0$ satisfies the following nonlinear eigenvalue problem:

$$E'(\mathbf{u}_0) = \lambda C'(\mathbf{u}_0).$$

In general, a minimizer $\mathbf{u}_0$ of E on $\mathfrak{M}_{c_0} = \{\mathbf{u} \in X \quad |C(\mathbf{u})| = c_0\}$ is not a soliton; in fact, according to Def. 6, it is necessary to prove the following facts:

- (i) The set $\Gamma(e_0, c_0)$ is stable.
- (ii) The set $\Gamma(e_0, c_0)$ is $\mathcal{T}$-compact (i.e. compact up to translations).
- (iii) $0 \notin \Gamma(e_0, c_0)$, since otherwise, some $\mathbf{u} \in \Gamma(e_0, c_0)$ is not even a solitary wave (see Def. 3,(i)).

In concrete cases, the point (i) is the most delicate point to prove. If (i) does not hold, according to our definitions, $\mathbf{u}_0$ is a solitary wave but not a soliton.

Now let us see the general definition of hylomorphic soliton.

Definition 8 *Let* (X, γ) *be a dynamical system of type FT and invariant for the action of a Lie group* G*, namely, for any* $\mathbf{u} \in X,\ \forall g \in G$,

$$g\gamma_t\mathbf{u} = \gamma_t g\mathbf{u}.$$

$\mathbf{u}$ *is called* ***hylomorphic soliton*** *if* $\mathbf{u} = g\mathbf{u}_0$ *where* $\mathbf{u}_0$ *is a standing hylomorphic soliton and* g *is a suitable element of* G.

In the application G will be a representation of the Galileo or of the Lorentz group. Now let us illustrate with an example Def. 7 and Def. 8.

Example. Let us consider the example (6). The standing wave $u(x)e^{-i\omega t}$ is a hylomorphic soliton if

$$\Gamma_{\mathbf{0}} = \left\{u(x - x_0)e^{-i\theta} \in H^1\left(\mathbb{R}^N, \mathbb{C}\right) : x_0 \in \mathbb{R}^N;\ \theta \in \mathbb{R}\right\}$$

satisfies the request in Definition 6 and if $\Gamma_{\mathbf{0}} = \Gamma(e_0, c_0)$ (see (7)) for a suitable c_0.

3 Existence results of hylomorphic solitons

In the previous section, we have seen that the existence of hylomorphic soliton is related to the existence of minimizers of the energy. In this section we will investigate the following minimization problem

$$\min_{\mathbf{u}\in\mathfrak{M}_c} E(\mathbf{u}) \ \text{ where } \ \mathfrak{M}_c := \{\mathbf{u} \in X \mid |C(\mathbf{u})| = c\} \tag{9}$$

and under which conditions the set of the minimizers

$$\Gamma(e,c) = \{\mathbf{u} \in X \mid E(\mathbf{u}) = e, \ |C(\mathbf{u})| = c\}\,; \ e = \min_{\mathbf{u}\in\mathfrak{M}_c} E(\mathbf{u})$$

is stable.

3.1 The abstract framework

The following definitions could be given in a more abstract framework. Nevertheless, for the sake of definitess, in the following we shall assume that

$$(X,\gamma) \text{ is a dynamical system of FT-type (see Def 1)}$$

and that

$$G \text{ is a subgroup of } \mathcal{T} \text{ (see(4))}.$$

Definition 9 *A subset* $\Gamma \subset X$ *is called* G*-invariant if*

$$\forall \mathbf{u} \in \Gamma, \ \forall g \in G, \ g\mathbf{u} \in \Gamma.$$

Definition 10 *A sequence* $\mathbf{u}_n$ *in* X *is called* G-compact *if there is a subsequence* $\mathbf{u}_{n_k}$ *and a sequence* $g_k \in G$ *such that* $g_k\mathbf{u}_{n_k}$ *is convergent. A subset* $\Gamma \subset X$ *is called* G*-compact if every sequence in* Γ *is* G-compact.

Observe that the above definition reduces to Definition 2 if $G = \mathcal{T}$. If $G = \{Id\}$ or, more in general, it is a compact group, G-compactness implies compactness. If G is not compact such as the tranlation group $\mathcal{T}$, G-compactness is a weaker notion than compactness.

Definition 11 *A* G*-invariant functional* J *on* X *is called* G*-compact if any minimizing sequence* $\mathbf{u}_n$ *is* G*-compact.*

Clearly a G-compact functional has a G-compact set of minimizers.

Definition 12 *We say that a functional* F *on* X *has the splitting property if given a sequence* $\mathbf{u}_n = \mathbf{u} + \mathbf{w}_n \in X$ *such that* $\mathbf{w}_n$ *converges weakly to* 0*, we have that*

$$F(\mathbf{u}_n) = F(\mathbf{u}) + F(\mathbf{w}_n) + o(1).$$

Remark 13 *Every quadratic form, which is continuous and symmetric, satisfies the splitting property; in fact, in this case, we have that* $F(\mathbf{u}) := \langle L\mathbf{u}, \mathbf{u} \rangle$ *for some continuous selfajoint operator* L*; then, given a sequence* $\mathbf{u}_n = \mathbf{u} + \mathbf{w}_n$ *with* $\mathbf{w}_n \rightharpoonup 0$ *weakly, we have that*

$$\begin{aligned} F(\mathbf{u}_n) &= \langle L\mathbf{u}, \mathbf{u} \rangle + \langle L\mathbf{w}_n, \mathbf{w}_n \rangle + 2 \langle L\mathbf{u}, \mathbf{w}_n \rangle \\ &= F(\mathbf{u}) + F(\mathbf{w}_n) + o(1). \end{aligned}$$

Definition 14 *A sequence* $\mathbf{u}_n \in X$ *is called G-**vanishing sequence** if it is bounded and if for any subsequence* $\mathbf{u}_{n_k}$ *and for any sequence* $g_k \subset G$ *the sequence* $g_k \mathbf{u}_{n_k}$ *converges weakly to* 0*.*

So, if $\mathbf{u}_n \to 0$ strongly, $\mathbf{u}_n$ is a G-vanishing sequence. However, if $\mathbf{u}_n \rightharpoonup 0$ weakly, it might happen that it is not a G-vanishing sequence; namely it might exist a subsequence $\mathbf{u}_{n_k}$ and a sequence $g_k \subset G$ such that $g_k \mathbf{u}_{n_k}$ is weakly convergent to some $\bar{\mathbf{u}} \neq 0$. Let see an example; if $\mathbf{u}_0 \in X$ is a solitary wave and $t_n \to +\infty$, then the sequence $\gamma_{t_n} \mathbf{u}_0$ is not a $\mathcal{T}$-vanishing sequence.

In the following E and C will denote two constants of the motion for the dynamical system (in the applications they will be the energy and the charge). We will assume that

$$E \text{ and } C \text{ are } C^1 \text{ and bounded functionals on } X.$$

We set

$$\Lambda(\mathbf{u}) := \frac{E(\mathbf{u})}{|C(\mathbf{u})|}, \tag{10}$$

Since E and C are constants of motion, also Λ is a constant of motion; it will be called **hylenic ratio** (see the definition of charge, sec. 5.2) and, as we will see it will play a central role in this theory.

The notions of G-vanishing sequence and of hylenic ratio allow to introduce the following (important) definition:

Definition 15 *We say that the hylomorphy condition holds if*

$$\inf_{\mathbf{u} \in X} \frac{E(\mathbf{u})}{|C(\mathbf{u})|} < \Lambda_0. \tag{11}$$

where

$$\Lambda_0 := \inf \left\{ \liminf \ \Lambda(\mathbf{u}_n) \mid \mathbf{u}_n \textit{ is a G-vanishing sequence} \right\}. \tag{12}$$

Moreover, we say that $\mathbf{u}_0 \in X$ *satisfies the hylomorphy condition if,*

$$\frac{E(\mathbf{u}_0)}{|C(\mathbf{u}_0)|} < \Lambda_0. \tag{13}$$

By this definition, using the above notation, we have the following:

$$\lim \Lambda(\mathbf{u}_n) < \Lambda_0 \Rightarrow \exists \mathbf{u}_{n_k}, g_k \in G : g_k \mathbf{u}_{n_k} \rightharpoonup \bar{\mathbf{u}} \neq 0.$$

In order to apply the existence theorems of sect. 3.2, it is necessary to estimate Λ_0; the following propositons may help to do this.

Proposition 16 *Assume that there exists a seminorm* $\|\|_\sharp$ *on* X *such that*

$$\{\mathbf{u}_n \textit{ is a } G-\textit{vanishing sequence}\} \Rightarrow \|\mathbf{u}_n\|_\sharp \to 0. \tag{14}$$

Then

$$\liminf_{\|\mathbf{u}\|_\sharp \to 0} \Lambda(\mathbf{u}) \leq \Lambda_0 \leq \liminf_{\|\mathbf{u}\| \to 0} \Lambda(\mathbf{u}). \tag{15}$$

Proof. It follows directly from the definition (12) of Λ_0 and (14).

□

Proposition 17 *If* E *and* C *are twice differentiable in* 0 *and*

$$E(0) = C(0) = 0;\ E'(0) = C'(0) = 0,$$

then we have that

$$\Lambda_0 \leq \inf_{\mathbf{u} \neq 0} \frac{E''(0)[\mathbf{u}, \mathbf{u}]}{|C''(0)[\mathbf{u}, \mathbf{u}]|}.$$

Proof. By the above proposition,

$$\begin{aligned} \Lambda_0 \quad &\leq \quad \liminf_{\|\mathbf{u}\| \to 0} \Lambda(\mathbf{u}) = \liminf_{\|\mathbf{u}\| \to 0} \frac{E(0) + E'(0)[\mathbf{u}] + E''(0)[\mathbf{u}, \mathbf{u}] + o(\|\mathbf{u}\|^2)}{\left|C(0) + C'(0)[\mathbf{u}] + C''(0)[\mathbf{u}, \mathbf{u}] + o(\|\mathbf{u}\|^2)\right|} \\ &= \quad \inf_{\mathbf{u} \neq 0} \frac{E''(0)[\mathbf{u}, \mathbf{u}]}{|C''(0)[\mathbf{u}, \mathbf{u}]|}. \end{aligned}$$

□

Now, finally, we can give some abstract theorems relative to the existence of hylomorphic solitons.

3.2 Statement of the abstract existence theorems

We formulate the assumptions on E and C:

- *(EC-0)* ***(Values at 0)***

$$E(0) = C(0) = 0;\ E'(0) = C'(0) = 0.$$

- *(EC-1)****(Invariance)*** $E(\mathbf{u})$ *and* $C(\mathbf{u})$ *are* G*-invariant.*
- *(EC-2)****(Splitting property)*** E *and* C *satisfy the splitting property.*
- *(EC-3)****(Coercivity)*** *We assume that*
  - (i) $\forall \mathbf{u} \neq 0,\ E(\mathbf{u}) > 0.$
  - (ii) *if* $\|\mathbf{u}_n\| \to \infty$, *then* $E(\mathbf{u}_n) \to \infty$;

– (iii) *if* $E(\mathbf{u}_n) \to 0$, *then* $\|\mathbf{u}_n\| \to 0$.

Now we can state the main results:

Theorem 18 *Assume that E and C satisfy (EC-0),...,(EC-2) and (EC-3). Moreover assume that the hylomorphy condition of Def. 15 is satisfied. Then there exists a family of hylomorphic solitons $\mathbf{u}_\delta$, $\delta \in (0, \delta_\infty)$, $\delta_\infty > 0$.*

Theorem 19 *Let the assumptions of theorem 18 hold. Moreover assume that*

$$\|E'(\mathbf{u})\| + \|C'(\mathbf{u})\| = 0 \Leftrightarrow \mathbf{u} = 0. \tag{16}$$

Then for every $\delta \in (0, \delta_\infty)$, $\delta_\infty > 0$, there exists a hylomorphic soliton $\mathbf{u}_\delta$. Moreover, if $\delta_1 < \delta_2$, the corresponding solitons $\mathbf{u}_{\delta_1}, \mathbf{u}_{\delta_2}$ are distinct, namely we have that

- *(a)* $\Lambda(\mathbf{u}_{\delta_1}) < \Lambda(\mathbf{u}_{\delta_2})$
- *(b)* $|C(\mathbf{u}_{\delta_1})| > |C(\mathbf{u}_{\delta_2})|$.
- *(c)* $E(\mathbf{u}_{\delta_1}) > E(\mathbf{u}_{\delta_2})$

The proofs of the above results are in the remaining part of this section. In subsection 3.3 we prove the existence of minimizers, namely that $\Gamma(e, c) \neq \varnothing$ (see (7)) and in subsection 3.4, we prove the stability of $\Gamma(e, c)$, namely that the minimizers are hylomorphic solitons.

3.3 A minimization result

We start with a technical lemma.

Lemma 20 *Let $\mathbf{u}_n = \mathbf{u} + \mathbf{w}_n \in X$ be a sequence such that $\mathbf{w}_n$ converges weakly to 0. Then, up to a subsequence, we have*

$$\lim \Lambda(\mathbf{u} + \mathbf{w_n}) \geq \min(\Lambda(\mathbf{u}), \lim \Lambda(\mathbf{w_n}))$$

and the equality holds if and only if $\Lambda(\mathbf{u}) = \lim \Lambda(\mathbf{w_n})$.

Proof. Given four real numbers A, B, a, b, (with $B, b > 0$), we have that

$$\frac{A + a}{B + b} \geq \min\left(\frac{A}{B}, \frac{a}{b}\right). \tag{17}$$

In fact, suppose that $\frac{A}{B} \geq \frac{a}{b}$; then

$$\frac{A + a}{B + b} = \frac{\frac{A}{B}B + \frac{a}{b}b}{B + b} \geq \frac{\frac{a}{b}B + \frac{a}{b}b}{B + b} = \frac{a}{b} \geq \min\left(\frac{A}{B}, \frac{a}{b}\right).$$

Notice that the equality holds if and only if $\frac{A}{B} = \frac{a}{b}$. Using the splitting property and the above inequality, up to a subsequence, we have that

$$\begin{aligned} \lim \Lambda\left(\mathbf{u}+\mathbf{w_n}\right) &= \frac{\lim E\left(\mathbf{u}+\mathbf{w_n}\right)}{\lim\left|C\left(\mathbf{u}+\mathbf{w_n}\right)\right|} = \frac{E\left(\mathbf{u}\right)+\lim E\left(\mathbf{w_n}\right)}{\left|C\left(\mathbf{u}\right)+\lim C\left(\mathbf{w_n}\right)\right|} \geq \\ & \frac{E\left(\mathbf{u}\right)+\lim E\left(\mathbf{w_n}\right)}{\left|C\left(\mathbf{u}\right)\right|+\left|\lim C\left(\mathbf{w_n}\right)\right|} \\ &\geq \min\left(\frac{E\left(\mathbf{u}\right)}{\left|C\left(\mathbf{u}\right)\right|}, \frac{\lim E\left(\mathbf{w_n}\right)}{\lim\left|C\left(\mathbf{w_n}\right)\right|}\right) = \min\left(\Lambda\left(\mathbf{u}\right), \lim \Lambda\left(\mathbf{w_n}\right)\right). \end{aligned}$$

□

For any $\delta > 0$, set

$$J_\delta(\mathbf{u}) = \Lambda\left(\mathbf{u}\right) + \delta E(\mathbf{u})$$

By the hylomorphy condition (11) we have

$$\delta_\infty = \sup\left\{\delta > 0 \mid \exists \mathbf{v} : \Lambda\left(\mathbf{v}\right) + \delta E(\mathbf{v}) < \Lambda_0 \right\} \in \mathbb{R}^+ \cup \{+\infty\}. \tag{18}$$

Clearly, if $\delta \in [0, \delta_\infty)$, $\exists \mathbf{v} : \Lambda\left(\mathbf{v}\right) + \delta E(\mathbf{v}) < \Lambda_0$.

Theorem 21 *Assume that E and C satisfy (EC-0),...,(EC-3) and the hylomorphy condition (11). Then, for every $\delta \in (0, \delta_\infty)$, J_δ is G-compact and it has a minimizer $\mathbf{u}_\delta \neq 0$. Moreover, $\mathbf{u}_\delta \in \Gamma\left(e_\delta, c_\delta\right)$ (see (7)) where $e_\delta = E(\mathbf{u}_\delta)$, $c_\delta = |C(\mathbf{u}_\delta)| > 0$.*

Proof. Let $\mathbf{u}_n$ be a minimizing sequence of J_δ ($\delta \in (0, \delta_\infty)$). This sequence $\mathbf{u}_n$ is bounded in X. In fact, arguing by contradiction, assume that, up to a subsequence, $\|\mathbf{u}_n\| \to \infty$. Then by (EC-3) (ii), $E(\mathbf{u}_n) \to \infty$ and hence $J_\delta(\mathbf{u}_n) \to \infty$ which contradicts the fact that $\mathbf{u}_n$ is a minimizing sequence of J_δ.

We now set

$$j_\delta := \inf_{\mathbf{u} \in X} J_\delta\left(\mathbf{u}\right).$$

Since $\delta \in (0, \delta_\infty)$, where δ_∞ is defined in (18), we have that

$$j_\delta < \Lambda_0. \tag{19}$$

Moreover, since $E \geq 0$,we have

$$0 \leq \Lambda\left(\mathbf{u}_n\right) \leq J_\delta(\mathbf{u}_n)$$

and

$$J_\delta(\mathbf{u}_n) \to j_\delta < \Lambda_0.$$

Then, up to a subsequence, $\Lambda\left(\mathbf{u}_n\right) \to \lambda < \Lambda_0$. So, by definition (12) of Λ_0 , $\mathbf{u}_n$ is not a $G-$vanishing sequence. Hence, by Def. 14, we can extract a subsequence $\mathbf{u}_{n_k}$ and we can take a sequence $g_k \subset G$ such that $\mathbf{u}'_k := g_k \mathbf{u}_{n_k}$ is weakly convergent to some

$$\mathbf{u}_\delta \neq 0. \tag{20}$$

We can write

$$\mathbf{u}_n' = \mathbf{u}_\delta + \mathbf{w}_n$$

with $\mathbf{w}_n \rightharpoonup 0$ weakly. We want to prove that $\mathbf{w}_n \to 0$ strongly.

By lemma 20 and by the splitting property of E, we have, up to a subsequence, that

$$\begin{aligned} j_\delta &= \lim J_\delta(\mathbf{u}_\delta + \mathbf{w}_n) = \lim\left[\Lambda(\mathbf{u}_\delta + \mathbf{w}_n) + \delta E(\mathbf{u}_\delta + \mathbf{w}_n)\right] \\ &\geq \left[\min\left\{\Lambda(\mathbf{u}_\delta), \lim \Lambda(\mathbf{w}_n)\right\}\right] + \delta E(\mathbf{u}_\delta) + \delta \lim E(\mathbf{w}_n). \end{aligned}$$

Now there are two possibilities (up to subsequences): 1- $\min\left\{\Lambda(\mathbf{u}_\delta), \lim \Lambda(\mathbf{w}_n)\right\} = \lim \Lambda(\mathbf{w}_n)$; 2- $\min\left\{\Lambda(\mathbf{u}_\delta), \lim \Lambda(\mathbf{w}_n)\right\} = \Lambda(\mathbf{u}_\delta)$. We will show that the possibility 1 cannot occur. In fact, if it holds, we have that

$$\begin{aligned} j_\delta &\geq \lim \Lambda(\mathbf{w}_n) + \delta E(\mathbf{u}_\delta) + \delta \lim E(\mathbf{w}_n) \\ &= \lim J_\delta(\mathbf{w}_n) + \delta E(\mathbf{u}_\delta) \\ &\geq j_\delta + \delta E(\mathbf{u}_\delta) \end{aligned}$$

and hence, we get that $E(\mathbf{u}_\delta) \leq 0$, contradicting (20). Then possibility 2 holds and we have that

$$\begin{aligned} j_\delta &\geq \Lambda(\mathbf{u}_\delta) + \delta E(\mathbf{u}_\delta) + \delta \lim E(\mathbf{w}_n) \\ &= J_\delta(\mathbf{u}_\delta) + \delta \lim E(\mathbf{w}_n) \\ &\geq j_\delta + \delta \lim E(\mathbf{w}_n). \end{aligned}$$

Then, $\lim E(\mathbf{w}_n) = 0$ and, by (EC-3)(iii), $\mathbf{w}_n \to 0$ strongly. Then $J_\delta(\mathbf{u}_n') \to J_\delta(\mathbf{u}_\delta)$. So J_δ is G.-compact and $\mathbf{u}_\delta$ is a minimizer.

To prove the second part of the theorem, we set:.

$$\begin{aligned} e_\delta &= E(\mathbf{u}_\delta) \\ c_\delta &= |C(\mathbf{u}_\delta)| \\ \mathfrak{M}_\delta &:= \left\{\mathbf{u} \in X \mid |C(\mathbf{u})| = c_\delta\right\}. \end{aligned}$$

Since

$$J_\delta|_{\mathfrak{M}_\delta} = \frac{E}{c_\delta} + \delta E = \left(\frac{1}{c_\delta} + \delta\right) E$$

it follows that $\mathbf{u}_\delta$ minimizes also $E|_{\mathfrak{M}_\delta}$.

□

In the following $\mathbf{u}_\delta$ will denote a minimizer of J_δ.

Lemma 22 *Let the assumptions of Theorem 21 be satisfied. If $\delta_1, \delta_2 \in (0, \delta_\infty)$ $\delta_1 < \delta_2$ (δ_∞ as in (18)), then the minimizers $\mathbf{u}_{\delta_1}, \mathbf{u}_{\delta_2}$ of J_{δ_1}, J_{δ_2} respectively satisfy the following inequalities:*

- *(a)* $J_{\delta_1}(\mathbf{u}_{\delta_1}) < J_{\delta_2}(\mathbf{u}_{\delta_2})$

- *(b)* $E(\mathbf{u}_{\delta_1}) \geq E(\mathbf{u}_{\delta_2})$,
- *(c)* $\Lambda(\mathbf{u}_{\delta_1}) \leq \Lambda(\mathbf{u}_{\delta_2})$,
- *(d)* $|C(\mathbf{u}_{\delta_1})| \geq |C(\mathbf{u}_{\delta_2})|$.

Proof. (a)

$$\begin{aligned} J_{\delta_1}(\mathbf{u}_{\delta_1}) &= \Lambda(\mathbf{u}_{\delta_1}) + \delta_1 E(\mathbf{u}_{\delta_1}) \\ &\leq \Lambda(\mathbf{u}_{\delta_2}) + \delta_1 E(\mathbf{u}_{\delta_2}) \quad \text{(since } \mathbf{u}_{\delta_1} \text{ minimizes } J_{\delta_1}) \\ &< \Lambda(\mathbf{u}_{\delta_2}) + \delta_2 E(\mathbf{u}_{\delta_2}) \quad \text{(since } E \text{ is positive)} \\ &= J_{\delta_2}(\mathbf{u}_{\delta_2}). \end{aligned}$$

(b) We set

$$\begin{aligned} \Lambda(\mathbf{u}_{\delta_1}) &= \Lambda(\mathbf{u}_{\delta_2}) + a \\ E(\mathbf{u}_{\delta_1}) &= E(\mathbf{u}_{\delta_2}) + b. \end{aligned}$$

We need to prove that $b \geq 0$ and $a \leq 0$. We have

$$\begin{aligned} J_{\delta_2}(\mathbf{u}_{\delta_2}) &\leq J_{\delta_2}(\mathbf{u}_{\delta_1}) \Rightarrow \\ \Lambda(\mathbf{u}_{\delta_2}) + \delta_2 E(\mathbf{u}_{\delta_2}) &\leq \Lambda(\mathbf{u}_{\delta_1}) + \delta_2 E(\mathbf{u}_{\delta_1}) \Rightarrow \\ \Lambda(\mathbf{u}_{\delta_2}) + \delta_2 E(\mathbf{u}_{\delta_2}) &\leq (\Lambda(\mathbf{u}_{\delta_2}) + a) + \delta_2 (E(\mathbf{u}_{\delta_2}) + b) \Rightarrow \\ 0 &\leq a + \delta_2 b. \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned} J_{\delta_1}(\mathbf{u}_{\delta_2}) &\geq J_{\delta_1}(\mathbf{u}_{\delta_1}) \Rightarrow \\ \Lambda(\mathbf{u}_{\delta_2}) + \delta_1 E(\mathbf{u}_{\delta_2}) &\geq \Lambda(\mathbf{u}_{\delta_1}) + \delta_1 E(\mathbf{u}_{\delta_1}) \Rightarrow \\ \Lambda(\mathbf{u}_{\delta_2}) + \delta_1 E(\mathbf{u}_{\delta_2}) &\geq (\Lambda(\mathbf{u}_{\delta_2}) + a) + \delta_1 (E(\mathbf{u}_{\delta_2}) + b) \Rightarrow \\ 0 &\geq a + \delta_1 b. \end{aligned} \tag{22}$$

From (21) and (22) we get

$$(\delta_2 - \delta_1)\, b \geq 0$$

and hence $b \geq 0$.

Moreover by (21) and (22) we also get

$$\left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right) a \geq 0$$

and hence $a \leq 0$. Since $|C(\mathbf{u})| = \frac{E(\mathbf{u})}{\Lambda(\mathbf{u})}$, also inequality (d) follows.

□

Lemma 23 *Let the assumptions of Theorem 21 be satisfied and assume that also (16) is satisfied. If $\delta_1, \delta_2 \in (0, \delta_\infty)$ (δ_∞ as in (18)), $\delta_1 < \delta_2$, then the minimizers $\mathbf{u}_{\delta_1}, \mathbf{u}_{\delta_2}$ of J_{δ_1}, J_{δ_2} respectively satisfy the following inequalities:*

- *(a)* $E(\mathbf{u}_{\delta_1}) > E(\mathbf{u}_{\delta_2})$,
- *(b)* $\Lambda(\mathbf{u}_{\delta_1}) < \Lambda(\mathbf{u}_{\delta_2})$
- *(c)* $|C(\mathbf{u}_{\delta_1})| > |C(\mathbf{u}_{\delta_2})|$.

Proof: Let $\delta_1, \delta_2 \in (0, \delta_\infty)$ and assume that $\delta_1 < \delta_2$.

(a) It is sufficient to prove that $E(\mathbf{u}_{\delta_1}) \neq E(\mathbf{u}_{\delta_2})$. We argue indirectly and assume that

$$E(\mathbf{u}_{\delta_1}) = E(\mathbf{u}_{\delta_2}). \tag{23}$$

By the previous lemma, we have that

$$\Lambda(\mathbf{u}_{\delta_1}) \leq \Lambda(\mathbf{u}_{\delta_2}). \tag{24}$$

Also, we have that

$$\begin{aligned} \Lambda(\mathbf{u}_{\delta_2}) + \delta_2 E(\mathbf{u}_{\delta_2}) &\leq \Lambda(\mathbf{u}_{\delta_1}) + \delta_2 E(\mathbf{u}_{\delta_1}) \quad \text{(since } \mathbf{u}_{\delta_2} \text{ minimizes } J_{\delta_2}) \\ &= \Lambda(\mathbf{u}_{\delta_1}) + \delta_2 E(\mathbf{u}_{\delta_2}) \quad \text{(by (23))} \end{aligned}$$

and so

$$\Lambda(\mathbf{u}_{\delta_2}) \leq \Lambda(\mathbf{u}_{\delta_1})$$

and by (24) we get

$$\Lambda(\mathbf{u}_{\delta_1}) = \Lambda(\mathbf{u}_{\delta_2}). \tag{25}$$

Then, it follows that $\mathbf{u}_{\delta_1}$ is a minimizer of J_{δ_2};in fact, by (25) and (23))

$$\begin{aligned} J_{\delta_2}(\mathbf{u}_{\delta_1}) &= \Lambda(\mathbf{u}_{\delta_1}) + \delta_2 E(\mathbf{u}_{\delta_1}) \\ &= \Lambda(\mathbf{u}_{\delta_2}) + \delta_2 E(\mathbf{u}_{\delta_2}) = J_{\delta_2}(\mathbf{u}_{\delta_2}). \end{aligned}$$

Then, we have that $J'_{\delta_2}(\mathbf{u}_{\delta_1}) = 0$ as well as $J'_{\delta_1}(\mathbf{u}_{\delta_1}) = 0$ which esplicitely give

$$\begin{aligned} \Lambda'(\mathbf{u}_{\delta_1}) + \delta_2 E'(\mathbf{u}_{\delta_1}) &= 0 \\ \Lambda'(\mathbf{u}_{\delta_1}) + \delta_1 E'(\mathbf{u}_{\delta_1}) &= 0. \end{aligned}$$

The above equations imply that $E'(\mathbf{u}_{\delta_1}) = 0$ and $\Lambda'(\mathbf{u}_{\delta_1}) = 0$, and since $\Lambda(\mathbf{u}) = \frac{E(\mathbf{u})}{|C(\mathbf{u})|}$, we get that $C'(\mathbf{u}_{\delta_1}) = 0$. Then by (16) $\mathbf{u}_{\delta_1} = 0$, and this fact contradicts Th. 21.

(b) We argue indirectly and assume that

$$\Lambda(\mathbf{u}_{\delta_1}) = \Lambda(\mathbf{u}_{\delta_2}). \tag{26}$$

By (a), we have that

$$E(\mathbf{u}_{\delta_1}) > E(\mathbf{u}_{\delta_2}). \tag{27}$$

Also, we have that

$$\begin{aligned} \Lambda(\mathbf{u}_{\delta_1}) + \delta_1 E(\mathbf{u}_{\delta_1}) &\leq \Lambda(\mathbf{u}_{\delta_2}) + \delta_1 E(\mathbf{u}_{\delta_2}) \quad \text{(since } \mathbf{u}_{\delta_1} \text{ minimizes } J_{\delta_1}) \\ &= \Lambda(\mathbf{u}_{\delta_1}) + \delta_1 E(\mathbf{u}_{\delta_2}) \quad \text{(by (26))} \end{aligned}$$

and so

$$E\left(\mathbf{u}_{\delta_1}\right) \leq E\left(\mathbf{u}_{\delta_2}\right)$$

and by (27) we get a contradiction.

(c) Since

$$\left|C\left(\mathbf{u}_{\delta}\right)\right| = \frac{E\left(\mathbf{u}_{\delta}\right)}{\Lambda\left(\mathbf{u}_{\delta}\right)},$$

the conclusion follows from (a) and (b).

□

3.4 The stability result

In order to prove Theorem 18 it is sufficient to show that the minimizers in Th. 21 provide solitons, so we have to prove that the set $\Gamma\left(e,c\right)$ is stable. To do this, we need the (well known) Liapunov theorem in following form:

Theorem 24 *Let Γ be an invariant set and assume that there exists a differentiable function V (called Liapunov function) defined on a neighborhood of Γ such that*

- *(a)* $V(\mathbf{u}) \geq 0$ *and* $V(\mathbf{u}) = 0 \Leftrightarrow u \in \Gamma$
- *(b)* $\partial_t V(\gamma_t\left(\mathbf{u}\right)) \leq 0$
- *(c)* $V(\mathbf{u}_n) \to 0 \Leftrightarrow d(\mathbf{u}_n, \Gamma) \to 0.$

Then Γ is stable.

Proof. For completeness, we give a proof of this well known result. Arguing by contradiction, assume that Γ, satisfying the assumptions of Th. 24, is not stable. Then there exists $\varepsilon > 0$ and sequences $\mathbf{u}_n \in X$ and $t_n > 0$ such that

$$d(\mathbf{u}_n, \Gamma) \to 0 \text{ and } d(\gamma_{t_n}\left(\mathbf{u}_n\right), \Gamma) > \varepsilon. \tag{28}$$

Then we have

$$d(\mathbf{u}_n, \Gamma) \to 0 \Longrightarrow V(\mathbf{u}_n) \to 0 \Longrightarrow V(\gamma_{t_n}\left(\mathbf{u}_n\right)) \to 0 \Longrightarrow d(\gamma_{t_n}\left(\mathbf{u}_n\right), \Gamma) \to 0$$

where the first and the third implications are consequence of property (c). The second implication follows from property (b). Clearly, this fact contradicts (28).

□

Lemma 25 *Let V be G-compact, continuous functional, $V \geq 0$ and let $\Gamma = V^{-1}(0)$ be the set of minimizers of V. If $\Gamma \neq \varnothing$, then Γ is G-compact and V satisfies the point (c) of the previous theorem.*

Proof: The fact that Γ is G-compact, is a trivial consequence of the fact that Γ is the set of minimizers of a G-compact functional V. Now we prove (c). First we show the implication $\Rightarrow$. Let $\mathbf{u}_n$ be a sequence such that $V(\mathbf{u}_n) \to 0$. By contradiction, we assume that $d(\mathbf{u}_n, \Gamma) \nrightarrow 0$, namely that there is a subsequence $\mathbf{u}_n^{'}$ such that

$$d(\mathbf{u}_n', \Gamma) \geq a > 0. \tag{29}$$

Since $V(\mathbf{u}_n) \to 0$ also $V(\mathbf{u}_n') \to 0$, and, since V is G compact, there exists a sequence g_n in G such that, for a subsequence $\mathbf{u}_n''$, we have $g_n \mathbf{u}_n'' \to \mathbf{u}_0$. Then

$$d(\mathbf{u}_n'', \Gamma) = d(g_n \mathbf{u}_n'', \Gamma) \leq d(g_n \mathbf{u}_n'', \mathbf{u}_0) \to 0$$

and this contradicts (29).

Now we prove the other implication $\Leftarrow$. Let $\mathbf{u}_n$ be a sequence such that $d(\mathbf{u}_n, \Gamma) \to 0$, then there exists $\mathbf{v}_n \in \Gamma$ s.t.

$$d(\mathbf{u}_n, \Gamma) \geq d(\mathbf{u}_n, \mathbf{v}_n) - \frac{1}{n}. \tag{30}$$

Since V is G-compact, also Γ is G-compact; so, for a suitable sequence g_n, we have $g_n \mathbf{v}_n \to \bar{\mathbf{w}} \in \Gamma$. We get the conclusion if we show that $V(\mathbf{u}_n) \to 0$. We have by (30), that $d(\mathbf{u}_n, \mathbf{v}_n) \to 0$ and hence $d(g_n \mathbf{u}_n, g_n \mathbf{v}_n) \to 0$ and so, since $g_n \mathbf{v}_n \to \bar{\mathbf{w}}$, we have $g_n \mathbf{u}_n \to \bar{\mathbf{w}} \in \Gamma$. Therefore, by the continuity of V and since $\bar{\mathbf{w}} \in \Gamma$, we have $V(g_n \mathbf{u}_n) \to V(\bar{\mathbf{w}}) = 0$ and we can conclude that $V(\mathbf{u}_n) \to 0$.
□

Proof of Th. 18. By Theorem 21 for every $\delta \in (0, \delta_\infty)$ J_δ is $G-$compact and it has a minimizer $\mathbf{u}_\delta \neq 0$ with

$$E(\mathbf{u}_\delta) = e_\delta$$

where

$$e_\delta = \min \{E(\mathbf{u}) : |C(\mathbf{u})| = c_\delta\}, \ c_\delta = |C(\mathbf{u}_\delta)|.$$

So, in order to show that $\mathbf{u}_\delta$ is an hylomorphic soliton, we need to show that

$$\Gamma(e_\delta, c_\delta) = \{\mathbf{u} \in X \mid |C(\mathbf{u})| = c_\delta, E(\mathbf{u}) = e_\delta\}$$

is G- compact and stable (see Definitions 6 and 7).

We set

$$V(\mathbf{u}) = (E(\mathbf{u}) - e_\delta)^2 + (|C(\mathbf{u})| - c_\delta)^2.$$

Clearly

$$\Gamma(e_\delta, c_\delta) = V^{-1}(0)$$

V is G compact. In fact:

Let $\mathbf{w}_n$ be a minimizing sequence for V, then $V(\mathbf{w}_n) \to 0$ and consequently $E(\mathbf{w}_n) \to e_\delta$ and $C(\mathbf{w}_n) \to c_\delta$. Now, since

$$\min J_\delta = \frac{e_\delta}{c_\delta} + \delta e_\delta,$$

we have that $\mathbf{w}_n$ is a minimizing sequence also for J_δ. Then, since by Theorem 21 J_δ is G-compact, we get

$$\mathbf{w}_n \text{ is } G\text{-compact.} \tag{31}$$

So we conclude that V is G-compact.

Then, by Lemma 25, we deduce that $V^{-1}(0) = \Gamma(e_\delta, c_\delta)$ is G compact and that V satisfies the point (c) in Theorem 24. Moreover V satisfies also the points (a) and (b) in Theorem 24. So we conclude that $\Gamma(e_\delta, c_\delta)$ is stable.

□

Proof of Th. 19 By Theorem 18 for any $\delta \in (0, \delta_\infty)$ there exists a hylomorphic soliton $\mathbf{u}_\delta$. By using Lemma 23, we get different solitons for different values of δ. Namely for $\delta_1 < \delta_2$ we have $\Lambda(\mathbf{u}_{\delta_1}) < \Lambda(\mathbf{u}_{\delta_2})$, $|C(\mathbf{u}_{\delta_1})| > |C(\mathbf{u}_{\delta_2})|$ and $E(\mathbf{u}_{\delta_1}) > E(\mathbf{u}_{\delta_2})$.

□

4 The structure of hylomorphic solitons

4.1 The meaning of the hylenic ratio

Let (X, γ) be a dynamical system of type FT. If $\mathbf{u} \in X$ is a finite energy field, usually it disperses as time goes on, namely

$$\lim_{t\to\infty} \|\gamma_t \mathbf{u}\|_\bigstar = 0.$$

where

$$\|\mathbf{u}\|_\bigstar = \sup_{x\in\mathbb{R}^N} \int_{B_1(x)} |\mathbf{u}|_V \, dx,$$

V is the internal parameter space (cf. pag. 3) and $B_1(x) = \left\{y \in \mathbb{R}^N : |x-y| < 1\right\}$. However, if the hylomorphy condition (11) is satisfied, this dispersion in general does not occur. In fact we have the following result:

Proposition 26 *Assume that X is compactly embedded into $L^1_{loc}\left(\mathbb{R}^N, V\right)$. Let $\mathbf{u}_0 \in X$ such that $\Lambda(\mathbf{u}_0) < \Lambda_0$, then*

$$\min_{t\to\infty}\lim \|\mathbf{u}(t)\|_\bigstar > 0$$

where $\mathbf{u}(t) = \gamma_t \mathbf{u}$ and $\gamma_0 \mathbf{u} = \mathbf{u}_0$.

Proof: Let $t_n \to \infty$ be a sequence of times such that

$$\lim_{n\to\infty} \|\mathbf{u}(t_n)\|_\bigstar = \min_{t\to\infty}\lim \|\mathbf{u}(t)\|_\bigstar . \tag{32}$$

Since Λ is a constant of motion

$$\Lambda(\mathbf{u}(t_n)) = \Lambda(\mathbf{u}_0) < \Lambda_0$$

then, by the definition of Λ_0, may be taking a subsequence, there is a sequence of translations T_{x_n} such that

$$T_{x_n}\mathbf{u}(t_n) = \mathbf{u}(t_n, x - x_n) = \bar{\mathbf{u}} + \mathbf{w}_n \tag{33}$$

where $\bar{\mathbf{u}} \neq 0$ and $\mathbf{w}_n \rightharpoonup 0$ in X. Without loss of generality, we may assume that $\bar{\mathbf{u}} \neq 0$ in $B_1(0)$. Since X is compactly embedded into $L^1_{loc}\left(\mathbb{R}^N, V\right)$, we have that

$$\int_{B_1(0)} |\mathbf{w}_n|_V \; dx \to 0. \tag{34}$$

By (33), we have that

$$|T_{x_n}\mathbf{u}(t_n)|_V \geq |\bar{\mathbf{u}}|_V - |\mathbf{w}_n|_V \,. \tag{35}$$

Then, using (35), (34), we have that

$$\begin{aligned} & \min_{n\to\infty}\lim \int_{B_1(0)} |T_{x_n}\mathbf{u}(t_n)|_V \; dx \\ \geq & \lim_{n\to\infty}\left(\int_{B_1(0)} |\bar{\mathbf{u}}|_V \; dx - \int_{B_1(0)} |\mathbf{w}_n|_V \; dx\right) \\ = & \int_{B_1(0)} |\bar{\mathbf{u}}|_V \; dx > 0 \end{aligned}$$

Then

$$\min_{n\to\infty}\lim \int_{B_1(0)} |T_{x_n}\mathbf{u}(t_n)|_V \; dx > 0. \tag{36}$$

Finally, by (32) and (36), we get

$$\begin{aligned} \min_{t\to\infty}\lim \; \|\mathbf{u}(t)\|_\bigstar &= \lim_{n\to\infty} \|\mathbf{u}(t_n)\|_\bigstar \geq \min_{n\to\infty}\lim \int_{B_1(x_n)} |\mathbf{u}(t_n)|_V \; dx \\ &= \min_{n\to\infty}\lim \int_{B_1(0)} |T_{x_n}\mathbf{u}(t_n)|_V \; dx > 0. \end{aligned}$$

□

Thus the hylomorphy condition prevents the dispersion. As we have seen in the preceding section, (11) is also a fundamental assumption in proving the existence of hylomorphic solitons.

Now, we assume E and C to be local quantities, namely, given $\mathbf{u} \in X$, there exist the density functions $\rho_{E,\mathbf{u}}(x)$ and $\rho_{C,\mathbf{u}}(x) \in L^1(\mathbb{R}^N)$ such that

$$E(\mathbf{u}) = \int \rho_{E,\mathbf{u}}(x) \; dx$$

$$C(\mathbf{u}) = \int \rho_{C,\mathbf{u}}(x) \; dx$$

Energy and hylenic densities $\rho_{E,\mathbf{u}}$, $\rho_{C,\mathbf{u}}$ allow to define the density of *binding energy* as follows:

$$\beta(t,x) = \beta_{\mathbf{u}}(t,x) = \left[\rho_{E,\mathbf{u}}(t,x) - \Lambda_0 \cdot \left|\rho_{C,\mathbf{u}}(t,x)\right|\right]^{-} \tag{37}$$

where $[f]^{-}$ denotes the negative part of f.

If $\mathbf{u}$ satisfies the hylomorphy condition, we have that $E(\mathbf{u}) < \Lambda_0 |C(\mathbf{u})|$ and hence he have that $\beta_{\mathbf{u}}(t,x) \neq 0$ for some $x \in \mathbb{R}^N$.

The support of the binding energy density is called *bound matter region;* more precisely we have the following definition

Definition 27 *Given any configuration* $\mathbf{u}$*, we define the* ***bound matter region*** *as follows*

$$\Sigma(\mathbf{u}) = \overline{\{x : \beta_{\mathbf{u}}(t,x) \neq 0\}}.$$

If $\mathbf{u}_0$ *is a soliton, the set* $\Sigma(\mathbf{u}_0)$ *is called* ***support of the soliton*** *at time* t.

In the situation considered in this article, we will see that the solitons satisfy the hylomorphy condition. Thus we may think that a soliton $\mathbf{u}_0$ consists of bound matter localized in a precise region of the space, namely $\Sigma(\mathbf{u}_0)$. This fact gives the name to this type of soliton from the Greek words "*hyle*"="*matter*" and "*morphe*"="*form*".

4.2 The *swarm interpretation*

Clearly the physical interpretation of hylomorphic solitons depends on the model which we are considering. However we can always assume a *conventional interpretations* which we will call *swarm interpretation* since the soliton is regarded as a *swarm* of particles bound together. This iterpretation is consitent with the model of the Q-ball.

We assume that $\mathbf{u}$ is a field which describes a fluid consisting of particles; the particles density is given by the function $\rho_C(t,x) = \rho_{C,\mathbf{u}}(t,x)$ which, of course satisfies a continuity equation

$$\partial_t \rho_C + \nabla \cdot \mathbf{J}_C = 0 \tag{38}$$

where $\mathbf{J}_C$ is the flow of particles. Hence C is the total number of particles. Here the particles are not intended to be as in "particle theory" but rather as in fluid dynamics, so that C does not need to be an integer number. Alternatively, if you like, you may think that C is not the number of particles but it is proportional to it. Also, in some equations as for example in NKG, C can be negative; in this case, the existence of *antiparticles* is assumed.

Thus, the hylomorphy ratio

$$\Lambda(\mathbf{u}) = \frac{E(\mathbf{u})}{|C(\mathbf{u})|}$$

represents the average energy of each particle (or antiparticle). The number Λ_0 defined in (12) is interpreted as the rest energy of each particle when they do not interact with each other. If $\Lambda(\mathbf{u}) > \Lambda_0$, then the average energy of each particle is bigger than the rest energy; if $\Lambda(\mathbf{u}) < \Lambda_0$, the opposite occurs and this fact means that particles act with each other with an attractive force.

If the particles were at rest and they were not acting on each other, their energy density would be

$$\Lambda_0 \cdot |\rho_C(t,x)| .$$

If $\rho_E(t,x)$ denotes the energy density and if

$$\rho_E(t,x) < \Lambda_0 \cdot |\rho_C(t,x)| ;$$

then, in the point x at time t, the particles attract each other with a force which is stronger than the repulsive forces; this explains the name *density of binding energy* given to $\beta(t,x)$ in (37).

Thus a soliton relative to the state $\mathbf{u}$ can be considered as a "rigid" object occupying the region of space $\Sigma(\mathbf{u})$ (cf. Def. 27); it consists of particles which stick to each other; the energy to destroy the soliton is given by

$$\int \beta_{\mathbf{u}}(t,x)dx = \int_{\Sigma(\mathbf{u})} (\Lambda_0 |\rho_C(t,x)| - \rho_E(t,x))\, dx.$$

5 The Nonlinear Klein-Gordon-Maxwell equations

Existence results of solitary waves for the Nonlinear Klein-Gordon-Maxwell (NKGM) are stated in many papers (besides the papers quoted in the introduction see also [8], [19], [23], [21], [22], [1], [2], [13], [15], [31]). As stated in the introduction, in this section we prove the existence of a *continuous family* of *stable* solitary waves of NKGM.

5.1 Basic features

The nonlinear Klein-Gordon equation for a complex valued field ψ, defined on the space-time $\mathbb{R}^4$, can be written as follows:

$$\Box\psi + W'(\psi) = 0 \tag{39}$$

where

$$\Box\psi = \frac{\partial^2\psi}{\partial t^2} - \Delta\psi, \quad \Delta\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

and, with some abuse of notation,

$$W'(\psi) = W'(|\psi|)\frac{\psi}{|\psi|}$$

for some smooth function $W : [0, \infty) \to \mathbb{R}$. Hereafter $x = (x_1, x_2, x_3)$ and t will denote the space and time variables. The field $\psi : \mathbb{R}^4 \to \mathbb{C}$ will be called *matter field.* If $W'(s)$ is linear, $W'(s) = m^2 s$, $m \neq 0$, equation (39) reduces to the Klein-Gordon equation.

Consider the Abelian gauge theory in $\mathbb{R}^4$ equipped with the Minkowski metric and described by the Lagrangian density (see e.g. [9], [38], [37])

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 - W(|\psi|) \tag{40}$$

where

$$\mathcal{L}_0 = \frac{1}{2}\left[|D_t \psi|^2 - |\mathbf{D}_x \psi|^2 \right]$$

$$\mathcal{L}_1 = \frac{1}{2}\left| \partial_t \mathbf{A} + \nabla \varphi \right|^2 - \frac{1}{2}\left| \nabla \times \mathbf{A} \right|^2 .$$

Here q denotes a positive parameter, $\nabla\times$ and ∇ denote respectively the curl and the gradient operators with respect to the x variable,

$$D_t = \frac{\partial}{\partial t} + iq\varphi,\ D_j = \frac{\partial}{\partial x^j} - iqA_j,\ \mathbf{D}_x \psi = (D_1\psi, D_2\psi, D_3\psi) \tag{41}$$

are the covariant derivatives and finally $\varphi \in \mathbb{R}$ and $\mathbf{A}= (A_1, A_2, A_3) \in \mathbb{R}^3$ are the gauge potentials.

Now consider the total action

$$\mathcal{S} = \int \left(\mathcal{L}_0 + \mathcal{L}_1 - W(|\psi|) \right) dxdt. \tag{42}$$

Making the variation of $\mathcal{S}$ with respect to ψ, φ and $\mathbf{A}$ we get the following system of equations

$$D_t^2 \psi - \mathbf{D}_x^2 \psi + W'(\psi) = 0 \tag{43}$$

$$\nabla \cdot \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right) = q \left(\operatorname{Im} \frac{\partial_t \psi}{\psi} + q\varphi \right) |\psi|^2 \tag{44}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right) = q \left(\operatorname{Im} \frac{\nabla \psi}{\psi} - q\mathbf{A} \right) |\psi|^2 . \tag{45}$$

The abelian gauge theory, namely equations (43, 44, 45), provides a very elegant way to couple the Maxwell equation with matter if we interpret ψ as a matter field.

In order to give a more meaningful form to these equations, we will write ψ in polar form

$$\psi(x,t) = u(x,t)\, e^{iS(x,t)},\ \ u \geq 0,\ \ S \in \mathbb{R}/2\pi\mathbb{Z}$$

So (42) takes the following form

$$\mathcal{S}(u,S,\varphi,\mathbf{A}) = \int\int\left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\left|\nabla u\right|^2 - W(u)\right]dxdt +$$
$$+\frac{1}{2}\int\int\left[\left(\frac{\partial S}{\partial t}+q\varphi\right)^2 - \left|\nabla S - q\mathbf{A}\right|^2\right]u^2 dxdt$$
$$+\frac{1}{2}\int\int\left(\left|\frac{\partial \mathbf{A}}{\partial t}+\nabla\varphi\right|^2 - \left|\nabla\times\mathbf{A}\right|^2\right)dxdt$$

and the equations (43, 44, 45) take the form:

$$\Box u + W'(u) + \left[\left|\nabla S - q\mathbf{A}\right|^2 - \left(\frac{\partial S}{\partial t}+q\varphi\right)^2\right]u = 0 \tag{46}$$

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial S}{\partial t}+q\varphi\right)u^2\right] - \nabla\cdot\left[\left(\nabla S - q\mathbf{A}\right)u^2\right] = 0 \tag{47}$$

$$\nabla\cdot\left(\frac{\partial \mathbf{A}}{\partial t}+\nabla\varphi\right) = q\left(\frac{\partial S}{\partial t}+q\varphi\right)u^2 \tag{48}$$

$$\nabla\times\left(\nabla\times\mathbf{A}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{A}}{\partial t}+\nabla\varphi\right) = q\left(\nabla S - q\mathbf{A}\right)u^2. \tag{49}$$

In order to show the relation of the above equations with the Maxwell equations, we make the following change of variables:

$$\mathbf{E} = -\left(\frac{\partial \mathbf{A}}{\partial t}+\nabla\varphi\right) \tag{50}$$

$$\mathbf{H} = \nabla\times\mathbf{A} \tag{51}$$

$$\rho = -\left(\frac{\partial S}{\partial t}+q\varphi\right)qu^2 \tag{52}$$

$$\mathbf{j} = \left(\nabla S - q\mathbf{A}\right)qu^2. \tag{53}$$

So (48) and (49) are the second couple of the Maxwell equations with respect to a matter distribution whose charge and current density are respectively ρ and $\mathbf{j}$:

$$\nabla\cdot\mathbf{E} = \rho \tag{GAUSS}$$

$$\nabla\times\mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j} \tag{AMPERE}$$

(50) and (51) give rise to the first couple of the Maxwell equations

$$\nabla\times\mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0 \tag{FARADAY}$$

$$\nabla\cdot\mathbf{H} = 0. \tag{NOMONOPOLE}$$

Using the variables $\mathbf{j}$ and ρ, equation (46) can be written as follows

$$\Box u + W'(u) + \frac{\mathbf{j}^2 - \rho^2}{q^2 u} = 0 \qquad \text{(MATTER)}$$

and finally Equation (47) is the charge continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j} = 0. \qquad (54)$$

Notice that equation (54) is a consequence of (GAUSS) and (AMPERE). In conclusion, an Abelian gauge theory, via equations (GAUSS,..,MATTER), provides a model of interaction of the matter field ψ with the electromagnetic field $(\mathbf{E}, \mathbf{H})$. In fact that equations (GAUSS,..,MATTER) are equivalent to (46,..,49).

5.2 Energy and charge

Let examine the invariants of NKGM which are relevant for us, namely the energy and the charge. In this subsection we compute these invariants using the gauge invariant variables $u, \rho, \mathbf{j}, \mathbf{E}$, $\mathbf{H}$.

Energy. Energy, by definition, is the quantity which is preserved by the time invariance of the Lagrangian. Using the gauge invariant variables, the energy E calculated along the solutions of equation (GAUSS) takes the following form

$$E = E_m + E_f \qquad (55)$$

where

$$E_m = \frac{1}{2}\int \left[\left(\frac{\partial u}{\partial t}\right)^2 + |\nabla u|^2 + W(u) + \frac{\rho^2 + \mathbf{j}^2}{2q^2u^2}\right] dx$$

and

$$E_f = \frac{1}{2}\int \left(\mathbf{E}^2 + \mathbf{H}^2\right) dx.$$

Proof. By the Noether's theorem (see e.g. [28] or [6]), we have that, given the Lagrangian

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}|\nabla u|^2 - W(u) + \\ &+ \frac{1}{2}\left(\frac{\partial S}{\partial t} + q\varphi\right)^2 - \frac{1}{2}|\nabla S - q\mathbf{A}|^2 u^2 \\ &+ \frac{1}{2}\left|\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right|^2 - \frac{1}{2}|\nabla \times \mathbf{A}|^2)\end{aligned}$$

the density of energy takes the following form:

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)} \cdot \frac{\partial u}{\partial t} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial S}{\partial t}\right)} \cdot \frac{\partial S}{\partial t} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)} \cdot \frac{\partial \varphi}{\partial t} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \mathbf{A}}{\partial t}\right)} \cdot \frac{\partial \mathbf{A}}{\partial t} - \mathcal{L}$$

Now we will compute each term. We have:

$$\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial u}{\partial t}\right)} \cdot \frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial t}\right)^2 \tag{56}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial S}{\partial t}\right)} \cdot \frac{\partial S}{\partial t} &= \left(\frac{\partial S}{\partial t} + q\varphi\right) \frac{\partial S}{\partial t} u^2 \\
&= \left(\frac{\partial S}{\partial t} + q\varphi\right) \frac{\partial S}{\partial t} u^2 + \left(\frac{\partial S}{\partial t} + q\varphi\right) q\varphi u^2 - \left(\frac{\partial S}{\partial t} + \varphi\right) q\varphi u^2 \\
&= \left(\frac{\partial S}{\partial t} + q\varphi\right)^2 u^2 - \left(\frac{\partial S}{\partial t} + q\varphi\right) q\varphi u^2 \\
&= \frac{\rho^2}{q^2 u^2} + \rho\varphi.
\end{aligned}$$

Multiplying by φ equation (GAUSS) and integrating, we get

$$-\int \mathbf{E} \cdot \nabla\varphi = \int \rho\varphi$$

Thus, replacing this expression in the above formula, we get

$$\int \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial S}{\partial t}\right)} \cdot \frac{\partial S}{\partial t} = \int \frac{\rho^2}{q^2 u^2} - \mathbf{E} \cdot \nabla\varphi \tag{57}$$

Also we have

$$\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \varphi}{\partial t}\right)} \cdot \frac{\partial \varphi}{\partial t} = 0 \tag{58}$$

and

$$\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \mathbf{A}}{\partial t}\right)} \cdot \frac{\partial \mathbf{A}}{\partial t} = \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right) \cdot \frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} \cdot \frac{\partial \mathbf{A}}{\partial t} \tag{59}$$

Moreover, using the notation (50, 51, 52, 53), we have that

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}|\nabla u|^2 - W(u) + \frac{\rho^2 - \mathbf{j}^2}{2q^2 u^2} + \frac{\mathbf{E}^2 - \mathbf{H}^2}{2}$$

Then, by (56,...,59) and the above expression for $\mathcal{L}$ we get

$$\begin{aligned} E(u,S,\varphi,\mathbf{A}) &= \int \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)}\cdot\frac{\partial u}{\partial t}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial S}{\partial t}\right)}\cdot\frac{\partial S}{\partial t}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \mathbf{A}}{\partial t}\right)}\cdot\frac{\partial \mathbf{A}}{\partial t}-\mathcal{L} \\ &= \int \left(\frac{\partial u}{\partial t}\right)^2+\frac{\rho^2}{q^2u^2}-\mathbf{E}\cdot\nabla\varphi-\mathbf{E}\cdot\frac{\partial \mathbf{A}}{\partial t}-\mathcal{L} \\ &= \int \left(\frac{\partial u}{\partial t}\right)^2+\frac{\rho^2}{q^2u^2}+\mathbf{E}^2 \qquad (60) \\ &\quad -\int\left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2-\frac{1}{2}\left|\nabla u\right|^2-W(u)+\frac{\rho^2-\mathbf{j}^2}{2q^2u^2}+\frac{\mathbf{E}^2-\mathbf{H}^2}{2}\right] \\ &= \int\left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2+\frac{1}{2}\left|\nabla u\right|^2+W(u)+\frac{\rho^2+\mathbf{j}^2}{2q^2u^2}+\frac{\mathbf{E}^2+\mathbf{H}^2}{2}\right] \end{aligned}$$

□

Charge. Using (54), we see that the electric charge has the following expression

$$Q=\int \rho dx=-q\int(\partial_t S+q\varphi)u^2dx \tag{61}$$

In order to be consistent with the previous literature ([6], [4], [5], [12], [16], [15], [13], [17]), we will call *charge* the following quantity:

$$C(\mathbf{u})=\frac{Q}{q}=-\int(\partial_t S+q\varphi)u^2dx.$$

$C(\mathbf{u})$ is a dimensionless quantity which, in some interpretation of NKGM, represents the number of particles (see [20], [6], [17]). In some of the quoted papers, $C(\mathbf{u})$ is called hylenic charge and hence the ratio (10) is called hylenic ratio.

5.3 Existence of charged Q-balls

We shall make the following assumptions on W:

- **(W-0) (Positivity)**

$$W(s)\geq 0; \tag{62}$$

- **(W-i) (Nondegeneracy)** W is a C^2 function s.t. $W(0)=W'(0)=0$ and

$$W''(0)=m^2>0; \tag{63}$$

- **(W-ii) (Hylomorphy**) if we set

$$W(s) = \frac{1}{2} m^2 s^2 + N(s) \tag{64}$$

then

$$\exists s_0 \in \mathbb{R}^+ \text{ such that } N(s_0) < 0 \tag{65}$$

- **(W-iii) (Growth condition**)

$$|N'(s)| \leq c_1 s^{r-1} + c_2 s^{q-1} \text{ for } q,\ r \ \in \ (2, 6) \tag{66}$$

(W-0) implies that the energy E in (55) is positive; if this condition does not hold, it is possible to have solitary waves, but not hylomorphic solitons.

(W-i) In order to have solitary waves it is necessary to have $W''(0) \geq 0$. There are some results also when $W''(0) = 0$ (null-mass case, see e.g. [18] and [3]), however the most interesting situations occur when $W''(0) > 0$.

(W-ii) is the crucial assumption which characterizes the nonlinearity which might produce hylomorphic solitons.

The hylomorphy condition (W-ii) can also be written as follows:

$$\alpha_0 := \inf_{s \in \mathbb{R}^+} \frac{W(s)}{\frac{1}{2} |s|^2} < m^2 \tag{67}$$

(W-iii) if W and hence N is of class C^3, then (66) reduces to $|N'(s)| \leq cs^{q-1}$ for $q < 6$, and this is the usual subcritical growth condition.

Now we introduce the phase space X.

First observe that the term $\left(\rho^2 + \mathbf{j}^2\right)/u^2$ in (55) is singular, so we introduce new gauge invariant variables which eliminate this singularity:

$$\theta = \frac{\rho}{qu};\ \Theta = \frac{\mathbf{j}}{qu}.$$

Using these new variables the energy takes the form:

$$\begin{aligned} E(\mathbf{u}) &= \frac{1}{2} \int \left(|\partial_t u|^2 + |\nabla u|^2 + \theta^2 + \Theta^2 + \mathbf{E}^2 + \mathbf{H}^2 \right) dx + \int W(u) dx \\ &= \frac{1}{2} \int \left[|\partial_t u|^2 + |\nabla u|^2 + m^2 u^2 + \theta^2 + \Theta^2 + \mathbf{E}^2 + \mathbf{H}^2 \right] + \int N(u). \end{aligned}$$

The generic point in the phase space X is given by

$$\mathbf{u} = (u, \hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H})$$

where $\hat{u} = \partial_t u$ is considered as independent variable; the phase space is given by

$$X = \{\mathbf{u} \in \mathcal{H} : \nabla \cdot \mathbf{E} = q\theta u,\ \nabla \cdot \mathbf{H} = 0\} \tag{68}$$

where $\mathcal{H}$ is the Hilbert space of the functions

$$\mathbf{u} = (u, \hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H}) \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)^{11}$$

equipped with the norm defined by the quadratic part of the energy, namely:

$$\|\mathbf{u}\|^2 = \int \left[\hat{u}^2 + |\nabla u|^2 + m^2 u^2 + \theta^2 + \Theta^2 + \mathbf{E}^2 + \mathbf{H}^2\right] dx \tag{69}$$

In these new variables the energy and the hylenic charge become two continuous functionals on X having the form

$$E(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|^2 + \int N(u)dx \tag{70}$$

$$C(\mathbf{u}) = \int \theta u \; dx. \tag{71}$$

Our equations (MATTER, GAUSS, AMPERE, FARADAY, NOMONOPOLE) become

$$\begin{aligned} \Box u + W'(u) + \frac{\Theta^2 - \theta^2}{u} &= 0 \\ \nabla \cdot \mathbf{E} &= q\theta u \\ \nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} &= q\Theta u \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{H} &= 0. \end{aligned} \tag{72}$$

Remark 28 *In the following we shall assume that the Cauchy problem for (NKGM) is well posed in X. Actually, in the literature there are few results relative to this problem (we know only [26], [32], [34]) and we do not know which are the assumptions that W should satisfy. Also we refer to [14] for a discussion and some partial results on this issue.*

We have the following existence results.

Theorem 29 *Assume that W satisfies assumptions (W-0),(W-i),(W-ii),(W-iii). Then there exists $\bar{q} > 0$ such that for every $q \in [0, \bar{q}]$ the dynamical system described by (72) has a family $\mathbf{u}_\delta$ $(\delta \in (0, \delta_\infty), \delta_\infty > 0)$ of standing hylomorphic solitons (Definition 7). Moreover if $\delta_1 < \delta_2$ we have that*

- *(a)* $\Lambda(\mathbf{u}_{\delta_1}) < \Lambda(\mathbf{u}_{\delta_2})$
- *(b)* $|C(\mathbf{u}_{\delta_1})| > |C(\mathbf{u}_{\delta 2})|$
- *(c)* $.E(\mathbf{u}_{\delta_1}) > E(\mathbf{u}_{\delta 2})$

Theorem 30 *The solitons* $\mathbf{u}_\delta = (u_\delta, \hat{u}_\delta, \theta_\delta, \Theta_\delta, \mathbf{E}_\delta, \mathbf{H}_\delta)$ *in Theorem 29 are stationary solutions of (72), this means that* $\hat{u}_\delta = \Theta_\delta = \mathbf{H}_\delta = 0$, $\mathbf{E}_\delta = -\nabla\varphi_\delta$ *and* $u_\delta, \theta_\delta, \varphi_\delta$ *solve the equations*

$$-\Delta u_\delta + W'(u_\delta) - \frac{\theta_\delta^2}{u_\delta} = 0 \tag{73}$$

$$-\Delta\varphi_\delta = -q\theta_\delta u_\delta. \tag{74}$$

We shall prove Theorem 29 by using the abstract Theorem 19. First of all observe that the energy and the hylenic charge E and C, defined in (70) and (71) are invariant under translations i. e. under the action of the group $\mathcal{T}$ defined in (4).

We shall see that assumptions (62),...,(66) on W permit to show that assumptions (EC-0), (EC-1), (EC-2), (EC-3), (16) and (11) of the abstract theorem 19 are satisfied.

The next two lemmas, whose proofs follow standard arguments, state that E satisfies the coercivity assumption (EC-3) and that both E and C satisfy the splitting property (EC-2).

Lemma 31 *Let the assumptions of Theorem 29 be satisfied, then E defined by (70) satisfies (EC-3), namely for any sequence* $\mathbf{u}_n = (u_n, \hat{u}_n, \theta_n, \Theta_n, \mathbf{E}_n, \mathbf{H}_n)$ *in* $\mathcal{H}$ *such that* $E(\mathbf{u}_n) \to 0$ *(respectively* $E(\mathbf{u}_n)$ *bounded), we have* $\|\mathbf{u}_n\| \to 0$ *(respectively* $\|\mathbf{u}_n\|$ *bounded), where* $\|\cdot\|$ *is defined in (69).*

Proof. See proof of Lemma 23 in [15].
□

Lemma 32 *Let the assumptions of Theorem 29 be satisfied, then E and C satisfy the splitting property (EC-2).*

Proof. See proof of Lemma 22 in [15].
□

It remains to prove that the hylomorphy condition (11) holds.

5.4 Analysis of the hylenic ratio

First of all we set:

$$\|\mathbf{u}\|_\sharp = \|(u, \hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H})\|_\sharp = \max\left(\|u\|_{L^r}, \|u\|_{L^q}\right) \tag{75}$$

where r, q are introduced in (66). With some abuse of notation we shall write $\max\left(\|u\|_{L^r}, \|u\|_{L^q}\right) = \|u\|_\sharp$.

Lemma 33 *The seminorm* $\|\mathbf{u}\|_\sharp$ *defined in (75) satisfies the property (14), namely*

$$\{\mathbf{u}_n \text{ is a } \mathcal{T}\text{−vanishing sequence}\} \Rightarrow \|u_n\|_\sharp \to 0.$$

where $\mathcal{T}$ *is defined in (4).*

Proof. Let u_n be a bounded sequence in $H^1\left(\mathbb{R}^3\right)$

$$\|u_n\|^2_{H^1(\mathbb{R}^3)} \leq M \tag{76}$$

such that, up to a subsequence,

$$\|u_n\|_\sharp \geq a > 0. \tag{77}$$

. We need to show that u_n is not $\mathcal{T}-$ vanishing.

May be taking a subsequence, we have that at least one of the following holds:

- (i) $\|u_n\|_\sharp = \|u_n\|_{L^r}$
- (ii) $\|u_n\|_\sharp = \|u_n\|_{L^q}$

Now suppose that (i) holds (If (ii) holds, we will argue in the same way replacing r with q).

We set for $j \in \mathbb{Z}^3$

$$Q_j = j + Q = \{j + q : q \in Q\}$$

where Q is now the cube defined as follows

$$Q = \left\{(x_1, .., x_n) \in \mathbb{R}^3 : 0 \leq x_i < 1\right\}.$$

Clearly

$$\mathbb{R}^3 = \bigcup_j Q_j.$$

Now let c be the constant for the Sobolev embedding $H^1(Q_j) \subset L^t(Q_j)$. We have

$$\begin{aligned}
0 < a^r \leq \int |u_n|^r &= \sum_j \int_{Q_j} |u_n|^r = \sum_j \|u_n\|^{r-2}_{L^t(Q_j)} \|u_n\|^2_{L^t(Q_j)} \\
&\leq \left(\sup_j \|u_n\|^{r-2}_{L^t(Q_j)}\right) \cdot \sum_j \|u_n\|^2_{L^r(Q_j)} \\
&\leq c\left(\sup_j \|u_n\|^{r-2}_{L^t(Q_j)}\right) \cdot \sum_j \|u_n\|^2_{H^1(Q_j)} \\
&= c\left(\sup_j \|u_n\|^{r-2}_{L^r(Q_j)}\right) \|u_n\|^2_{H^1} \leq cM\left(\sup_j \|u_n\|^{r-2}_{L^t(Q_j)}\right).
\end{aligned}$$

where M and a are the constants respectively in (76) and (77). Then

$$\left(\sup_j \|u_n\|_{L^t(Q_j)}\right) \geq \left(\frac{a^t}{cM}\right)^{1/(t-2)}$$

Then, for any n, there exists $j_n \in \mathbb{Z}^3$ such that

$$\|u_n\|_{L^r(Q_{j_n})} \geq \alpha > 0. \tag{78}$$

Then, since $(T_{j_n} u_n)(x) = u_n(x + j_n)$ (see (3)), we have

$$\|T_{j_n} u_n\|_{L^r(Q_0)} = \|u_n\|_{L^r(Q_{j_n})} \geq \alpha > 0. \tag{79}$$

Since u_n is bounded, also $T_{j_n} u_n$ is bounded in $H^1(\mathbb{R}^3)$. Then we have, up to a subsequence, that $T_{j_n} u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ and hence strongly in $L^r(Q)$. By (79), $u_0 \neq 0$.

So we conclude that u_n is not $\mathcal{T}-$ vanishing.

$\square$

Now, as usual, we set

$$\Lambda_0 := \inf \{\liminf \ \Lambda(\mathbf{u}_n) \mid \mathbf{u}_n \text{ is a } \mathcal{T}\text{−vanishing sequence}\}$$

$$\begin{aligned} \Lambda_\sharp &= \liminf_{\|\mathbf{u}\|_\sharp \to 0} \Lambda(\mathbf{u}) = \\ & \lim_{\varepsilon \to 0} \ \inf \left\{ \Lambda(\varepsilon u, \hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H}) \mid u \in H^1, \ (\hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H}) \in \left(L^2\right)^{11}; \ \|u\|_\sharp = 1 \right\}. \end{aligned} \tag{80}$$

By the definition of Λ_0 and $\Lambda_\sharp$ and lemma 33, we have that

$$\Lambda_0 \geq \Lambda_\sharp \tag{81}$$

The following lemma holds:

Lemma 34 *Let W satisfy assumption (66), then the following inequality holds*

$$\Lambda_\sharp \geq m. \tag{82}$$

Proof. First of all observe that by (66) we have

$$\begin{aligned} \left| \int N(|u|) dx \right| &\leq k_1 \int |u|^r + k_2 \int |u|^q \\ &\leq k_1 \|u\|_\sharp^r + k_2 \|u\|_\sharp^q . \end{aligned}$$

So, if we take $\|u\|_\sharp = 1$ and $\varepsilon > 0$,we get

$$\left| \int N(|\varepsilon u|) dx \right| \leq k_1 \varepsilon^r + k_2 \varepsilon^q. \tag{83}$$

By the Sobolev embeddings, there is $k_3 > 0$ such that

$$\int \left(|\nabla u|^2 + m^2 u^2 \right) dx \geq k_3 \|u\|_\sharp^2 \tag{84}$$

Now, choose

$$2 < s < \min(r,q).$$

Since $r, q > s$, we have, by (84), (83) and taking $\varepsilon > 0$ small enough, that

$$\varepsilon^s \int \left(|\nabla u|^2 + m^2 u^2 \right) dx - \left| \int N(|\varepsilon u|) dx \right|$$
$$\geq \varepsilon^s k_3 \|\psi\|_\sharp^2 - k_1 \varepsilon^r - k_2 \varepsilon^q = k_3 \varepsilon^s - k_1 \varepsilon^r - k_2 \varepsilon^q \geq 0$$

So

$$\left| \int N(\varepsilon |u|) dx \right| \leq \varepsilon^s \int \left(|\nabla u|^2 + m^2 u^2 \right) dx \text{ where } s > 2. \tag{85}$$

Then, by using (85), for any $\mathbf{u} = (u, \hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H})$, with $u \in H^1$, $\|u\|_\sharp = 1$ and any $(\hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H}) \in \left(L^2\right)^{11}$, we have, for $\varepsilon > 0$ small

$$\begin{aligned}
\Lambda(\varepsilon u, \hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H}) &\geq \frac{\frac{1}{2} \int \left(\theta^2 + \varepsilon^2 |\nabla u|^2 + \varepsilon^2 m^2 |u|^2 \right) dx + \int N(|\varepsilon u|) dx}{\varepsilon \left| \int \theta u \right|} \\
&\geq \frac{\frac{1}{2} \int \theta^2 + \left(\frac{\varepsilon^2}{2} - \varepsilon^s \right) \int \left(|\nabla u|^2 + m^2 |u|^2 \right)}{\varepsilon \|\theta\|_{L^2} \|u\|_{L^2}} \\
&\geq \frac{\left(\int \theta^2 dx \right)^{1/2} \cdot \varepsilon m \sqrt{1 - 2\varepsilon^{s-2}} \left(\int u^2 dx \right)^{1/2}}{\varepsilon \|\theta\|_{L^2} \|u\|_{L^2}} = m\sqrt{1 - 2\varepsilon^{s-2}}.
\end{aligned}$$

Then, since $s > 2$, we have

$$\Lambda_\sharp = \lim_{\varepsilon \to 0} \inf \left\{ \Lambda(\varepsilon u, \hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H}) \mid u \in H^1,\ (\hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H}) \in \left(L^2\right)^{11};\ \|u\|_\sharp = 1 \right\} \geq m \tag{86}$$

□

Next we will show that the hylomorphy assumption (11) is satisfied.

Lemma 35 *Assume that W satisfies (W-0),...,(W-iii) and (66) then*

$$\inf_{\mathbf{u} \in X} \Lambda(\mathbf{u}) < \Lambda_0. \tag{87}$$

Proof. We shall prove that

$$\Lambda_* = \inf_{\mathbf{u} \in X} \Lambda(\mathbf{u}) < m \tag{88}$$

So (87) will follow from (81) and (82) and (88).

Let $R > 0$; set

$$u_R = \begin{cases} s_0 & if \ \ |x| < R \\ 0 & if \ \ |x| > R + 1 \\ \frac{|x|}{R} s_0 - (|x| - R) \frac{R+1}{R} s_0 & if \ \ R < |x| < R + 1 \end{cases}. \tag{89}$$

where $R > 1$.

By the hylomorphy assumption (63) there exist $\alpha \in (0, m)$ such that

$$W(s_0) \leq \frac{1}{2}\alpha^2 s_0^2 \tag{90}$$

Now let $\varphi_R \in \mathcal{D}^{1,2}$ denote the solution of the following equation

$$\Delta\varphi = -q\alpha u_R^2. \tag{91}$$

We have

$$\begin{aligned}
\Lambda_* = & \inf_{\mathbf{u}\in X} \frac{\frac{1}{2}\|\mathbf{u}\|^2 + \int N(u)dx}{|C(\mathbf{u})|} \\
= & \inf_{\mathbf{u}\in X} \frac{\frac{1}{2}\int \left[\hat{u}^2 + |\nabla u|^2 + \theta^2 + \Theta^2 + \mathbf{E}^2 + \mathbf{H}^2\right] dx + \int W(u)dx}{\left|\int \theta u \ dx\right|}.
\end{aligned}$$

Now remember that $\mathbf{u} = (u, \hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H})$ and take $\mathbf{u} = \mathbf{u}_R$ with

$$\mathbf{u}_R = (u_R, 0, \alpha u_R, 0, \nabla\varphi_R, \mathbf{0}).$$

By (91), $\mathbf{u}_R \in X$; then we have

$$\begin{aligned}
\Lambda_* = & \inf_{\mathbf{u}\in X} \frac{\frac{1}{2}\|\mathbf{u}\|^2 + \int N(u)dx}{|C(\mathbf{u})|} \leq \frac{\frac{1}{2}\|\mathbf{u}_R\|^2 + \int N(u_R)dx}{|C(\mathbf{u}_R)|} \\
= & \frac{\frac{1}{2}\int \left[|\nabla u_R|^2 + \alpha^2 u_R^2 + |\nabla\varphi_R|^2\right] dx + \int W(u_R)dx}{\alpha \int u_R^2 \ dx} \\
\leq & \frac{\frac{1}{2}\int_{|x|<R} \left[|\nabla u_R|^2 + \alpha^2 u_R^2\right] + \int_{|x|<R} W(u_R)}{\alpha \int_{|x|<R} u_R^2 \ dx} \\
+ & \frac{\frac{1}{2}\int_{R<|x|<R+1} \left[|\nabla u_R|^2 + \alpha^2 u_R^2\right] + \int_{R<|x|<R+1} W(u_R)}{\alpha \int_{|x|<R} u_R^2} + \frac{\frac{1}{2}\int |\nabla\varphi_R|^2}{\alpha \int_{|x|<R} u_R^2} \\
= & \frac{\frac{1}{2}\int_{|x|<R} \alpha^2 s_0^2 + \int_{|x|<R} W(s_0)}{\alpha \int_{|x|<R} s_0^2} + \frac{c_1 R^2}{\alpha \int_{|x|<R} s_0^2} + \frac{\frac{1}{2}\int |\nabla\varphi_R|^2}{\alpha \int_{|x|<R} s_0^2} \\
\leq & \alpha + \frac{c_2}{\alpha R} + \frac{\frac{1}{2}\int |\nabla\varphi_R|^2}{\frac{4}{3}\pi\alpha s_0^2 R^3}
\end{aligned} \tag{92}$$

where the last inequality is a consequence of (90).

In order to estimate the term containing φ_R in (92), we remember that φ_R is the solution of (91). Observe that u_R^2 has radial symmetry and that the electric field outside any spherically symmetric charge distribution is the same as if all of the charge were concentrated into a point. So $|\nabla\varphi_R(r)|$ corresponds to the strength of an electrostatic field at distance r, created by an electric charge given by

$$|C_{el}| = \int_{|x|\leq r} q\alpha u_R^2 dx = 4\pi \int_0^r q\alpha u_R^2 v^2 dv$$

and located at the origin. So we have

$$|\nabla\varphi_R(r)| = \frac{|C_{el}|}{r^2}\begin{cases} = \frac{4}{3}\pi q\alpha s_0^2 r & if\ r < R \\ \leq \frac{4}{3}\pi q\alpha s_0^2 \frac{(R+1)^3}{r^2} & if\ r \geq R \end{cases}$$

Then

$$\begin{aligned}\int |\nabla\varphi_R|^2 dx &\leq c_3 q^2\alpha^2 s_0^4 \left(\int_{r<R} r^4 dr + \int_{r>R} \frac{(R+1)^6}{r^2} dr\right) \\ &\leq c_4 q^2\alpha^2 s_0^4 \left(R^5 + \frac{(R+1)^6}{R}\right) \leq c_5 q^2\alpha^2 s_0^4 R^5.\end{aligned}$$

Then

$$\frac{\frac{1}{2}\int |\nabla\varphi_R|^2}{\frac{4}{3}\pi\alpha s_0^2 R^3} \leq c_6 q^2\alpha s_0^2 R^2. \tag{93}$$

By (92), we get

$$\Lambda_* \leq \alpha + \frac{c_1}{\alpha R} + c_6 q^2\alpha s_0^2 R^2. \tag{94}$$

Now set

$$m - \alpha = 2\varepsilon$$

and take

$$R = \frac{c_1}{\alpha\varepsilon},\ 0 < q < \sqrt{\frac{\varepsilon^3\alpha}{s_0^2 c_1^2 c_6}}.$$

With these choices of R and q, a direct calculation shows that

$$\alpha + \frac{c_1}{\alpha R} + c_6 q^2\alpha s_0^2 R^2 < m. \tag{95}$$

Then, by (94) and (95), we get that there exists a positive constant c such that, for $0 < q < \frac{c}{s}\sqrt{(m-\alpha)^3\alpha}$, we have

$$\inf_{\mathbf{u}\in X} \Lambda(\mathbf{u}) < m. \tag{96}$$

□

.

Proof of Th.29 Assumptions (EC-0), (EC-1) of Theorem19 are clearly satisfied. By Lemma 32 E and C satisfy the splitting property (EC-2) . . By Lemma 31 and by Lemma 35 also the coercitivity assumption (EC-3) and hylomorphy condition (11) are satisfied.

Finally it remains to show that also (16) is satisfied. To this end let

$$\mathbf{u} = (u, \hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H}) \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)^{11}$$

be a solution of $E'(\mathbf{u}) = 0$, then it is easy to see that $(\hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H}) = 0$ and $u \in H^1\left(\mathbb{R}^3\right)$ solves the equation

$$-\Delta u + W'(u) = 0.$$

So, since $W \geq 0$, we have by the Derrick-Pohozaev identity [24] that also $u = 0$.We conclude that

$$\mathbf{u} = (u, \hat{u}, \theta, \Theta, \mathbf{E}, \mathbf{H}) = 0.$$

So all the assumptions of the Theorem 19 are satisfied and the conclusion follows.

□.

Proof of Th. 30. Let

$$\mathbf{u}_\delta = (u_\delta, 0, \theta_\delta, 0, \mathbf{E}_\delta, \mathbf{0}) \in X = \{\mathbf{u} \in \mathcal{H} : \nabla \cdot \mathbf{E} = q\theta u,\ \nabla \cdot \mathbf{H} = 0\}$$

be as in Theorem 30.

So $\mathbf{u}_\delta$ minimizes the energy E (see (70)) on the manifold

$$X_\delta = \{\mathbf{u} \in X : C(\mathbf{u}) = C(\mathbf{u}_\delta) = \sigma_\delta\}.$$

If we write $\mathbf{E} = -\nabla\varphi$, the constraint $\nabla \cdot \mathbf{E} = q\theta u$ becomes

$$-\Delta\varphi = q\theta u.$$

So $\mathbf{u}_\delta$ is a critical point of E on the manifold (in $\mathcal{H}$) made up by those $\mathbf{u} = (u, 0, \theta, 0, \nabla\varphi, \mathbf{0})$ satisfying the constraints

$$\Delta\varphi = q\theta u \tag{97}$$

$$C(\mathbf{u}) = \int \theta u \, dx = \sigma_\delta. \tag{98}$$

Therefore, for suitable Lagrange multipliers $\lambda \in \mathbb{R}$, $\xi \in \mathcal{D}^{1,2}$ ($\mathcal{D}^{1,2}$ is the closure of C_0^∞ with respect to the norm $\|\nabla\varphi\|_{L^2}$), we have that $\mathbf{u}_\delta$ is a critical point of

$$E_{\lambda,\xi}(\mathbf{u}) = E(\mathbf{u}) + \lambda\left(\int \theta u \, dx - \sigma_\delta\right) + \langle \xi, -\Delta\varphi + q\theta u\rangle \tag{99}$$

where $\langle \cdot\ ,\ \cdot \rangle$ denotes the duality map in $\mathcal{D}^{1,2}$. It is easy to show that $E'_{\lambda,\xi}(\mathbf{u}_\delta) = 0$ gives the equations

$$-\Delta u_\delta + W'(u_\delta) + \lambda\theta_\delta + q\xi\theta_\delta = 0 \tag{100}$$

$$-\Delta\varphi_\delta = \Delta\xi \tag{101}$$

$$\theta_\delta + \lambda u_\delta + q\xi u_\delta = 0. \tag{102}$$

From (101) we get $\xi = -\varphi_\delta$, so (100) and (102) become

$$\begin{aligned} -\Delta u_\delta + W'(u_\delta) + \theta_\delta(\lambda - q\varphi_\delta) &= 0 \\ (\lambda - q\varphi_\delta)\, u_\delta &= -\theta_\delta. \end{aligned}$$

From the above equations we clearly get (73). (74) is given by the constraint (97).

□

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