Image Space Analysis and Separation for G-Semidifferentiable Vector Problems

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Abstract. This paper aims at studying, in the image space, an approximation of a vector optimization problem obtained by substituting the involved functions with their G-derivatives. It is shown that, under the hypothesis of G-differentiability, the existence of a lower semistationary point is equivalent to the linear separation between the image of the approximated problem and a suitable convex subset of the image space. Applications to optimality conditions are provided.

Keywords: Vector optimization; image space; separation

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1 Introduction and preliminaries

The concept of G-semidifferentiability was introduced by Giannessi in [4] to develop in an axiomatic way the theory of generalized directional derivatives; further important results concerning this approach have been obtained in [12, 14, 18].

In [6] it has been stressed the importance of separation arguments in the image space for the analysis of many topics in Vector Optimization, such as optimality conditions, scalarization and duality. In order to obtain necessary optimality conditions, in this paper we develop the study of the separation between the image of an approximation of the vector problem given by the G-derivatives of the involved functions and a suitable convex cone in the image space. This approach has been exploited in [4, 13] to analyze optimality conditions for a scalar optimization problem and in [6] for a vector one; subsequently, a generalization to Vector Variational Inequalities has been proposed in [9].

Let us mention some notations and definitions that will be used in what follows. If $M \subseteq \mathbb{R}^n$, clM denotes the closure of M, $intM$ the interior and riM the relative interior of M, $convM$ the convex hull of M and $coneM := \{y \in \mathbb{R}^n : y = \lambda x, \lambda \geq 0, x \in M\}$ the cone generated by M; if $A, B \subseteq \mathbb{R}^n$, $A \pm B := \{x \in \mathbb{R}^n \ : x = a \pm b, a \in A, b \in B\}$. The set $P \subseteq \mathbb{R}^n$ is a cone iff $\lambda x \in P$, $\forall x \in P$ and $\forall \lambda > 0$; the conic extension of M with respect to the cone P, denoted with $\mathcal{E}(M, P)$, is defined by $M - P$. The polar cone of P is $P^* := \{y \in \mathbb{R}^n : \langle y, x\rangle \geq 0, \ \forall x \in P\}$. Let $x \in clM$, $T(x; M) := \{v \in \mathbb{R}^n : \exists t_k > 0, \exists x_k \in M, x_k \to x, t_k(x_k - x) \to v\}$ is the Bouligand tangent cone to M at x.

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With O_n we denote the n-uple, whose entries are zero; when there is no fear of confusion, the suffix is omitted; for $n = 1$, we set $O_1 = 0$. Let D be a convex cone in $\mathbb{R}^m$, $a, b \in \mathbb{R}^m$; $a \geq_D b$ iff $a - b \in D$, $a \not\geq_D b$ iff $a - b \notin D$; $a \geq O$ iff $a_i \geq 0$, $i = 1, \ldots, m$. $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m : x \geq O\}$. The function $g : X \longrightarrow \mathbb{R}^m$ is called *D–convexlike* on $X \subseteq \mathbb{R}^n$ iff $\forall x_1, x_2 \in X,\ \forall \alpha \in [0,1]$,

$$\exists \hat{x} \in X \quad s.t. \quad (1-\alpha)g(x_1) + \alpha g(x_2) - g(\hat{x}) \in D.$$

It is known [17] that g is a D–convexlike function on X, iff the set $g(X) + D$ is convex.

Let $\phi : \mathbb{R}^n \to \mathbb{R}$; the set $epi\ \phi := \{(x,u) \in \mathbb{R}^n \times \mathbb{R}\ : u \geq \phi(x)\}$ is the epigraph of ϕ.

We will denote with $\phi^-(\bar{x}; z)$ the lower Dini directional derivative of ϕ at $\bar{x} \in \mathbb{R}^n$ in the direction $z \in \mathbb{R}^n$:

$$\phi^-(\bar{x}; z) = \liminf_{t \to 0^+} \frac{\phi(\bar{x} + tz) - \phi(\bar{x})}{t}.$$

The lower Dini-Hadamard directional derivative of ϕ at $\bar{x} \in \mathbb{R}^n$ in the direction z is

$$\phi^-_{DH}(\bar{x}; z) = \liminf_{\substack{t \to 0^+ \\ y \to z}} \frac{\phi(\bar{x} + ty) - \phi(\bar{x})}{t}.$$

The upper Dini and upper Dini-Hadamard derivatives are obtained replacing "liminf" with "limsup" and are denoted by ϕ^+ and ϕ^+_{DH}, respectively. When the limit exists we will say that ϕ admits the Dini or the Dini-Hadamard derivative and this latter will be denoted by ϕ_{DH}; the Dini-Hadamard derivative is also known as Neustadt derivative [10].

Consider the following Vector Optimization Problem (for short, VOP):

$$\min{}_C\ f(x), \quad x \in R := \{x \in X : g_i(x) = 0, i \in I^0;\ g_i(x) \geq 0, i \in I^+\} \tag{1.1}$$

where $X \subseteq \mathbb{R}^n$, $C := \mathbb{R}^\ell_+$, ℓ, m and p are positive integers with $p \leq m$, $J := \{1, ..., \ell\}$, $I^0 := \{1, \ldots, p\}$, $I^+ := \{p+1, \ldots, m\}$, $f_j : X \to \mathbb{R}$, $j = 1, ..., \ell$, and $f(x) = (f_1(x), \ldots, f_\ell(x))$, $g_i : X \to \mathbb{R}$, $i = 1, \ldots, m$ and $g(x) = (g_1(x), \ldots, g_m(x))$. Let $C_0 := C \setminus \{O\}$. $\bar{x} \in R$ is a (global) vector minimum point (for short, v.m.p.) of VOP, iff

$$f(\bar{x}) \not\geq_{C_0} f(x)\ , \quad \forall x \in R, \tag{1.2}$$

$\bar{x} \in R$ is a (global) weak v.m.p. of VOP iff

$$f(\bar{x}) \not\geq_{intC} f(x)\ \ , \ \ \forall x \in R. \tag{1.3}$$

We recall the main definitions concerning the image space. Let $D := O_p \times \mathbb{R}^{m-p}_+$. Observe that (1.2) is satisfied iff the system (in the unknown x):

$$f(\bar{x}) - f(x) \geq_{C_0} O\ \ ,\ \ g(x) \geq_D O\ \ ,\ \ x \in X, \tag{1.4}$$

is impossible.

Suppose that $\bar{x} \in R$ and define $F(x) := (f(\bar{x}) - f(x), g(x))$. The set

$$\mathcal{K}_{\bar{x}} := \{(u, v) \in \mathbb{R}^{\ell} \times \mathbb{R}^m : u = f(\bar{x}) - f(x), v = g(x), x \in X\} = F(X)$$

is called the *image* of VOP and the space $\mathbb{R}^{\ell} \times \mathbb{R}^m$ the *image space*. Consider the following subsets of the image space:

$$\mathcal{H}_{C_0} := \{(u, v) \in \mathbb{R}^{\ell} \times \mathbb{R}^m : u \geq_{C_0} O, v \geq_D O \}, \quad \mathcal{E}(\mathcal{K}_{\bar{x}}) := \mathcal{E}(\mathcal{K}_{\bar{x}}, cl\mathcal{H}_{C_0}).$$

Obviously, $\bar{x} \in R$ is a v.m.p. of VOP iff

$$\mathcal{K}_{\bar{x}} \cap \mathcal{H}_{C_0} = \emptyset, \tag{1.5}$$

which is proved [6] to be equivalent to

$$\mathcal{E}(\mathcal{K}_{\bar{x}}) \cap \mathcal{H}_{C_0} = \emptyset. \tag{1.6}$$

Similarly, defining:

$$\mathcal{H}_{intC} := \{(u, v) \in \mathbb{R}^{\ell} \times \mathbb{R}^m : u \geq_{intC} O, v \geq_D O \},$$

$\bar{x} \in R$ is a weak v.m.p. of VOP iff

$$\mathcal{K}_{\bar{x}} \cap \mathcal{H}_{intC} = \emptyset, \tag{1.7}$$

or, equivalently [6], iff

$$\mathcal{E}(\mathcal{K}_{\bar{x}}) \cap \mathcal{H}_{intC} = \emptyset. \tag{1.8}$$

In order to consider necessary optimality conditions for nondifferentiable VOP, we will use an approximation scheme based on the concept of G–semidifferentiability introduced in [4]. Denote by G a given subset of the set, say $\mathcal{G}$, of positively homogeneous functions of degree one on $X - \bar{x}$; by $\mathcal{C} \subseteq \mathcal{G}$ the set of convex positively homogeneous functions, by $\mathcal{L}$ the set of linear functions, provided that X is a convex cone.

Definition 1.1 [4] *A function $\phi : X \to \mathbb{R}$ is said lower G–semidifferentiable at $\bar{x} \in X$ iff there exist functions $\underline{\mathcal{D}}_G\phi : X \times (X - \bar{x}) \to \mathbb{R}$ and $\varepsilon_\phi : X \times (X - \bar{x}) \to \mathbb{R}$ such that:*

(i) $\underline{\mathcal{D}}_G\phi(\bar{x}; \cdot) \in G$;

(ii) $\phi(x) - \phi(\bar{x}) = \underline{\mathcal{D}}_G\phi(\bar{x}; x - \bar{x}) + \varepsilon_\phi(\bar{x}; x - \bar{x})\ , \quad \forall x \in X$;

(3i) $\liminf_{x \to \bar{x}} \dfrac{\varepsilon_\phi(\bar{x}; x - \bar{x})}{\|x - \bar{x}\|} \geq 0$;

(4i) *for every pair (h, ε) of functions which satisfy* (i)–(3i) *(in place of $\underline{\mathcal{D}}_G\phi$ and ε_ϕ, respectively), we have $epih \supseteq epi\underline{\mathcal{D}}_G\phi$.*

$\underline{\mathcal{D}}_G\phi(\bar{x}; \frac{x - \bar{x}}{\|x - \bar{x}\|})$ *is called the lower G–semiderivative of ϕ at $\bar{x}$.*

A function $\phi : X \to \mathbb{R}$ is said upper G–semidifferentiable at $\bar{x} \in X$ iff there exist functions $\overline{\mathcal{D}}_G\phi : X \times (X - \bar{x}) \to \mathbb{R}$ and $\varepsilon_\phi : X \times (X - \bar{x}) \to \mathbb{R}$ such that:

(i') $\overline{\mathcal{D}}_G\phi(\bar{x}; \cdot) \in G$;

(ii') $\phi(x) - \phi(\bar{x}) = \overline{\mathcal{D}}_G\phi(\bar{x}; x-\bar{x}) + \varepsilon_\phi(\bar{x}; x-\bar{x})$, $\forall x \in X$;

(3i') $\limsup_{x\to\bar{x}} \dfrac{\varepsilon_\phi(\bar{x}; x-\bar{x})}{\|x-\bar{x}\|} \le 0$;

(4i') *for every pair* (h, ε) *of functions which satisfy* (i')–(3i') *(in place of* $\overline{\mathcal{D}}_G\phi$ *and* ε_ϕ*, respectively), we have epi*$h \subseteq$ *epi*$\overline{\mathcal{D}}_G\phi$.

$\overline{\mathcal{D}}_G\phi(\bar{x}; \frac{x-\bar{x}}{\|x-\bar{x}\|})$ *is called the upper* G*–semiderivative of* ϕ *at* $\bar{x}$.

When lower and upper G*–semiderivatives coincide at* $\bar{x}$*, then* ϕ *is said to be* G*–differentiable at* $\bar{x}$ *and its* G*–derivative is denoted by* $\mathcal{D}_G\phi(\bar{x}; \frac{x-\bar{x}}{\|x-\bar{x}\|})$*. In such a case, we have* $\lim_{x\to\bar{x}} \dfrac{\varepsilon_\phi(\bar{x}; x-\bar{x})}{\|x-\bar{x}\|} = 0$.

Remark 1.1 From Definition 1.1, it is immediate that

$$\underline{\mathcal{D}}_G\phi = -\overline{\mathcal{D}}_{(-G)}(-\phi), \tag{1.9}$$

so that ϕ is lower G–semidifferentiable at $\bar{x}$ iff $-\phi$ is upper $(-G)$–semidifferentiable at $\bar{x}$. Moreover, we observe that the set of $\mathcal{L}$–differentiable functions coincides with the set of Fréchet differentiable functions; for $G = \mathcal{G}$, the class of G-differentiable functions at $\bar{x}$ coincides with the class of B-differentiable functions at $\bar{x}$, in the sense of Robinson [16].

Definition 1.2 *Let* $G \subseteq \mathcal{C}$*. The generalized subdifferential of a lower (upper)* G*–semidifferentiable function* ϕ *at* $\bar{x}$*, denoted by* $\partial_G\phi(\bar{x})$*, is defined as the subdifferential at* $\bar{x}$ *of the convex function* $\underline{\mathcal{D}}_G\phi(\bar{x}; x-\bar{x})$ *(*$\overline{\mathcal{D}}_G\phi(\bar{x}; x-\bar{x})$*); that is*

$$\partial_G\phi(\bar{x}) = \partial\underline{\mathcal{D}}_G\phi(\bar{x}, 0) \quad (\textit{or } \partial_G\phi(\bar{x}) = \partial\overline{\mathcal{D}}_G\phi(\bar{x}, 0)).$$

If $G \subseteq (-\mathcal{C})$ then the generalized superdifferential of a lower (or upper) G–semidifferentiable function ϕ is defined as the superdifferential of its concave approximation $\underline{\mathcal{D}}_G\phi$ (or $\overline{\mathcal{D}}_G\phi$).

Let us recall the main properties of G-semidifferentiable functions, that will be used in what follows.

Proposition 1.1 [11] *Suppose that* G *satisfies the following conditions:*

$$\psi_1, \psi_2 \in G \quad \textit{implies} \quad \psi_1 + \psi_2 \in G; \qquad \psi \in G \quad \textit{implies} \quad \alpha\psi \in G, \quad \forall \alpha > 0.$$

(i) *If* ϕ_1, ϕ_2 *and* $\phi_1 + \phi_2$ *are lower (upper)* G*-semidifferentiable at* $\bar{x}$ *then*

$$\underline{\mathcal{D}}_G\phi_1 + \underline{\mathcal{D}}_G\phi_2 \ge \underline{\mathcal{D}}_G(\phi_1 + \phi_2) \quad (\overline{\mathcal{D}}_G\phi_1 + \overline{\mathcal{D}}_G\phi_2 \le \overline{\mathcal{D}}_G(\phi_1 + \phi_2)).$$

(ii) *If* ϕ *is lower (upper)* G*-semidifferentiable at* $\bar{x}$ *then* $\forall \alpha > 0$*,* $\alpha\phi$ *is lower (upper)* G*-semidifferentiable at* $\bar{x}$ *with* $\alpha\underline{\mathcal{D}}_G\phi$ $(\alpha\overline{\mathcal{D}}_G\phi)$ *as lower (upper)* G*-semiderivative.*

Throughout the paper we will assume that the properties of Proposition 1.1 are fulfilled.

We propose to study the linear separation of an approximation of the image of VOP obtained by substituting the functions $-f$ and g with their G–approximations. If $-f$ and g are lower G-semidifferentiable at $\bar{x}$, then we define:

$$\underline{\mathcal{K}}_G := \underline{\mathcal{D}}_G F(X) + F(\bar{x}). \tag{1.10a}$$

We adopt a similar definition in the case where $-f$ and g are upper G-semidifferentiable; in such a case, we set:

$$\overline{\mathcal{K}}_G := \overline{\mathcal{D}}_G F(X) + F(\bar{x}). \tag{1.10b}$$

When $-f$ and g are G-differentiable at $\bar{x}$, (1.10a) and (1.10b) are equivalent, i.e., $\underline{\mathcal{K}}_G = \overline{\mathcal{K}}_G$ and they will be both denoted by $\mathcal{K}_G$. We observe that $\underline{\mathcal{K}}_G$ and $\overline{\mathcal{K}}_G$ are cones with vertex at $F(\bar{x})$; they are also called *homogeneizations* of the image $\mathcal{K}_{\bar{x}}$. In [4], where the scalar case with inequality constraints is considered, G is the set $\mathcal{C}$ of the convex positively homogeneous functions; in such a case, X is assumed to be a convex cone and f and $-g$ upper $\mathcal{C}$-semidifferentiable. Such a choice allows one to obtain a convex approximation of the problem, whose image is linearly separable from $\mathcal{H}_{C_0}$. Linear separation in the image space is a source of optimality conditions; in particular, it has been shown that, under G-differentiability assumptions, the homogeneization of the image of a scalar problem with equality and inequality constraints, is linearly separable from $\mathcal{H}_{C_0}$ iff the point $\bar{x}$ is a semistationary point for the problem [13].

In this paper, we will extend such results to the vector case by means of the analysis of the homogeneization associated with VOP. A first step in this direction can be found in [6] where, by means of a generalization of the classic linearization lemma of Abadie [1], necessary optimality conditions are established.

In section 2, we analyse the properties of the homogeneized image and its relationships with the image $\mathcal{K}_{\bar{x}}$. In section 3, we study the connections between the existence of a linear separation for $\underline{\mathcal{K}}_G$ (or $\overline{\mathcal{K}}_G$) and $\mathcal{H}_{C_0}$ and the semistationarity of the point $\bar{x}$ (see Definition 3), comparing our results with those obtained in [4, 6, 13]. Section 4 is devoted to the applications to Lagrangian type necessary optimality conditions for VOP.

2 Some properties of the homogeneized image

In this section, we will consider the relationships between the set $\mathcal{K}_{\bar{x}}$ and its homogeneization $\underline{\mathcal{K}}_G$. We will suppose that X is convex.

Proposition 2.1 *Let $-f_j$, $j \in J$, and g_i, $i \in I^+$ be lower G-semidifferentiable at $\bar{x}$ and assume that the lower limits in (3i) of Definition 1 are finite; let g_i, $i \in I^0$ be G-differentiable at $\bar{x}$. Then*

$$\underline{\mathcal{K}}_G \subseteq (\bar{u}, \bar{v}) + \mathcal{E}[T((\bar{u}, \bar{v}), \mathcal{K}_{\bar{x}})], \tag{2.1}$$

where $(\bar{u}, \bar{v}) := F(\bar{x})$.

Proof Let $(\tilde{u},\tilde{v}) \in \underline{\mathcal{K}}_G = \{(u,v) \in \mathbb{R}^\ell \times \mathbb{R}^m : u_j = \underline{\mathcal{D}}_G(-f_j(\bar{x}; x-\bar{x})), j \in J, v_i = \mathcal{D}_G(g_i(\bar{x}; x-\bar{x})), i \in I^0, v_i = g_i(\bar{x}) + \underline{\mathcal{D}}_G(g_i(\bar{x}; x-\bar{x})), i \in I^+, x \in X\}$; hence there exists $\tilde{x} \in X$ such that $(\tilde{u},\tilde{v}) = \underline{\mathcal{D}}_G F(\tilde{x}) + F(\bar{x})$. We observe that (2.1) is equivalent to prove that there exist $(u',v') \in T((\bar{u},\bar{v}),\mathcal{K}_{\bar{x}})$ and $(c,d) \in C \times D$ such that

$$(\tilde{u},\tilde{v}) = (\bar{u},\bar{v}) + (u',v') - (c,d). \tag{2.2}$$

Since $X - \bar{x}$ is convex, then

$$(\tilde{u}^r,\tilde{v}^r) \in \underline{\mathcal{K}}_G, \quad \forall r \in \mathbb{N} \setminus \{0\},$$

where $\tilde{u}^r_j = \underline{\mathcal{D}}_G(-f_j(\bar{x}; \frac{\tilde{x}-\bar{x}}{r})), j \in J$, $\tilde{v}^r_i = \mathcal{D}_G g_i(\bar{x}; \frac{\tilde{x}-\bar{x}}{r}), i \in I^0$, $\tilde{v}^r_i$
$= \underline{\mathcal{D}}_G g_i(\bar{x}; \frac{\tilde{x}-\bar{x}}{r}), i \in I^+$.
Consider the sequence $\{(u^r,v^r)\}$ defined by:

$$\begin{cases} u^r_j := \underline{\mathcal{D}}_G(-f_j(\bar{x}; \frac{\tilde{x}-\bar{x}}{r})) + \varepsilon_{(-f_j)}(\bar{x}; \frac{\tilde{x}-\bar{x}}{r}), & j \in J \\ v^r_i := \mathcal{D}_G g_i(\bar{x}; \frac{\tilde{x}-\bar{x}}{r}) + \varepsilon_{g_i}(\bar{x}; \frac{\tilde{x}-\bar{x}}{r}), & i \in I^0 \\ v^r_i := g_i(\bar{x}) + \underline{\mathcal{D}}_G g_i(\bar{x}; \frac{\tilde{x}-\bar{x}}{r}) + \varepsilon_{g_i}(\bar{x}; \frac{\tilde{x}-\bar{x}}{r}), & i \in I^+. \end{cases}$$

By Definition 1.1, we have that $(u^r,v^r) \in \mathcal{K}_{\bar{x}}, \forall r \in \mathbb{N} \setminus \{0\}$ and that

$$\liminf_{r \to +\infty} r\varepsilon_{(-f_j)}\left(\bar{x}; \frac{\tilde{x}-\bar{x}}{r}\right) \geq 0, \quad j \in J; \tag{2.3a}$$

$$\lim_{r \to +\infty} r\varepsilon_{g_i}\left(\bar{x}; \frac{\tilde{x}-\bar{x}}{r}\right) = 0, \quad i \in I^0; \tag{2.3b}$$

$$\liminf_{r \to +\infty} r\varepsilon_{g_i}\left(\bar{x}; \frac{\tilde{x}-\bar{x}}{r}\right) \geq 0, \quad i \in I^+. \tag{2.3c}$$

By assumptions, the lower limits in (2.3a) and (2.3c) are finite and hence conditions (2.3) imply the existence of a subsequence $\{r_h\} \subseteq \mathbb{N} \setminus \{0\}$ such that

$$\lim_{h \to +\infty} r_h \varepsilon_{(-f_j)}(\bar{x}; \frac{\tilde{x}-\bar{x}}{r_h}) \geq 0, \quad j \in J; \tag{2.4a}$$

$$\lim_{h \to +\infty} r_h \varepsilon_{g_i}(\bar{x}; \frac{\tilde{x}-\bar{x}}{r_h}) = 0, \quad i \in I^0; \tag{2.4b}$$

$$\lim_{h \to +\infty} r_h \varepsilon_{g_i}(\bar{x}; \frac{\tilde{x}-\bar{x}}{r_h}) \geq 0, \quad i \in I^+ \tag{2.4c}$$

exist and are finite.
Recalling that $(\bar{u},\bar{v}) = (0, g(\bar{x}))$ and that the $G-$semiderivatives are positively homogeneous, we have

$$\lim_{h \to +\infty} r_h[(u^{r_h},v^{r_h}) - (\bar{u},\bar{v})] \geq_{C \times D} (\underline{\mathcal{D}}_G(-f_j(\bar{x}; \tilde{x}-\bar{x})), j \in J, \underline{\mathcal{D}}_G g_i(\bar{x}; \tilde{x}-\bar{x}), i \in I^0 \cup I^+), \tag{2.5}$$

so that, setting

$$(u',v') := \lim_{h \to +\infty} r_h[(u^{r_h},v^{r_h}) - (\bar{u},\bar{v})] \in T((\bar{u},\bar{v}),\mathcal{K}_{\bar{x}}),$$

from (2.5) we obtain that

$$(\bar{u},\bar{v})+(u',v') \geq_{C\times D} (\bar{u},\bar{v})+(\underline{\mathcal{D}}_G(-f_j(\bar{x};\tilde{x}-\bar{x}))), j\in J, \underline{\mathcal{D}}_G g_i(\bar{x};\tilde{x}-\bar{x}), i\in I^0\cup I^+) = (\tilde{u},\tilde{v}).$$

Hence (2.2) holds.

Remark 2.1 In the statement of Proposition 2.1, the finiteness of the lower limits in (3i) of Definition 1.1 can be replaced by the assumption that the lower Dini derivatives $(-f_j)^-(\bar{x};x-\bar{x})$, $j\in J$ and $(g_i)^-(\bar{x};x-\bar{x})$, $i\in I^+$ exist and are finite $\forall x\in X$, which guarantees that the limits (2.3a) and (2.3c) are finite. Indeed, since $-f_j$, $j\in J$ are lower G-semidifferentiable at $\bar{x}$, considering in (ii) of Definition 1.1 $x=x_r:=\bar{x}+\frac{1}{r}(\tilde{x}-\bar{x})$, we obtain

$$-f_j(\bar{x}+\frac{1}{r}(\tilde{x}-\bar{x}))+f_j(\bar{x}) = \underline{\mathcal{D}}_G(-f_j(\bar{x};\frac{\tilde{x}-\bar{x}}{r}))+\varepsilon_{(-f_j)}(\bar{x};\frac{\tilde{x}-\bar{x}}{r}),\ j\in J.$$

Multiplying both sides by r yields

$$\frac{-f_j(\bar{x}+\frac{1}{r}(\tilde{x}-\bar{x}))+f_j(\bar{x})}{1/r} = \underline{\mathcal{D}}_G(-f_j(\bar{x};\tilde{x}-\bar{x}))+r\varepsilon_{(-f_j)}(\bar{x};\frac{\tilde{x}-\bar{x}}{r}),\ j\in J. \tag{2.6}$$

Taking the liminf in (2.6) as $r\to+\infty$ and recalling that $(-f_j)^-(\bar{x};\tilde{x}-\bar{x})$ is finite, we prove that the limits in (2.3a) exist finite. The proof is analogous for (2.3c).

Lemma 2.1 *Let $\bar{y}\in\mathcal{K}_{\bar{x}}\cap cl\mathcal{H}_{C_0}$; then*

$$\mathcal{E}[T(\bar{y},\mathcal{K}_{\bar{x}})]\subseteq T(0,\mathcal{E}(\mathcal{K}_{\bar{x}})).$$

Proof $y\in\mathcal{E}[T(\bar{y},\mathcal{K}_{\bar{x}})]$ iff $\exists\{\alpha_n>0\},\{y_n\}\subseteq\mathcal{K}_{\bar{x}}, h\in cl\mathcal{H}_{C_0}$, $y_n\to\bar{y}$ such that

$$y=\lim_{n\to+\infty}\alpha_n(y_n-\bar{y})-h.$$

If $\{\alpha_n\}$ is bounded, then $y=-h\in -cl\mathcal{H}_{C_0}\subseteq T(0,\mathcal{E}(\mathcal{K}_{\bar{x}}))$. Otherwise, (taking a subsequence, if necessary) we can suppose that $\lim_{n\to+\infty}\alpha_n=+\infty$. Since $y=\lim_{n\to+\infty}\alpha_n(y_n-\bar{y})-h=\lim_{n\to+\infty}\alpha_n(y_n-\bar{y}-\frac{h}{\alpha_n})$, then the thesis follows, taking into account that $\bar{y}+\frac{h}{\alpha_n}\in cl\mathcal{H}_{C_0}$ and, therefore, $y_n-(\bar{y}+\frac{h}{\alpha_n})\in\mathcal{E}(\mathcal{K}_{\bar{x}})$ $\forall n$, and $y_n-(\bar{y}+\frac{h}{\alpha_n})\to 0, n\to+\infty$.

Proposition 2.2 *Let $(\bar{u},\bar{v})=F(\bar{x})\in\mathcal{K}_{\bar{x}}\cap cl\mathcal{H}_{C_0}$. Under the same assumptions of Proposition 2.1, the following inclusion holds:*

$$\underline{\mathcal{K}}_G\subseteq(\bar{u},\bar{v})+T(0,\mathcal{E}(\mathcal{K}_{\bar{x}})).$$

Proof By Proposition 2.1 and Lemma 2.1 with $\bar{y}=(\bar{u},\bar{v})$ it follows

$$\underline{\mathcal{K}}_G\subseteq(\bar{u},\bar{v})+\mathcal{E}[T((\bar{u},\bar{v}),\mathcal{K}_{\bar{x}})]\subseteq(\bar{u},\bar{v})+T(0,\mathcal{E}(\mathcal{K}_{\bar{x}})).$$

Remark 2.2 If $\mathcal{E}(\mathcal{K}_{\bar{x}})$ is convex, then $T(0,\mathcal{E}(\mathcal{K}_{\bar{x}}))$ is convex and, since $(\bar{u},\bar{v})\in\mathcal{E}(\mathcal{K}_{\bar{x}})$, we have

$$(\bar{u},\bar{v})+T(0,\mathcal{E}(\mathcal{K}_{\bar{x}}))\subseteq\mathcal{E}(\mathcal{K}_{\bar{x}})+T(0,\mathcal{E}(\mathcal{K}_{\bar{x}}))\subseteq T(0,\mathcal{E}(\mathcal{K}_{\bar{x}}))+T(0,\mathcal{E}(\mathcal{K}_{\bar{x}}))=T(0,\mathcal{E}(\mathcal{K}_{\bar{x}})).$$

3 Separation and semistationarity conditions

The definition of semistationarity has been considered in literature both for a function [4] and for a constrained optimization problem [13].

Now we extend to a VOP the definition introduced in [13] of lower semistationary point for a scalar problem by considering the generalized Lagrangian function $L(x, \vartheta, \lambda) := \langle \vartheta, f(x) \rangle - \langle \lambda, g(x) \rangle$.

Definition 3.1 *$\bar{x} \in X$ will be called a lower semistationary point for* VOP iff $\exists\ (\bar{\vartheta}, \bar{\lambda}) \in \mathbb{R}^\ell_+ \times (\mathbb{R}^p \times \mathbb{R}^{m-p}_+)$, $(\bar{\vartheta}, \bar{\lambda}) \neq O$ such that:

$$\liminf_{x \to \bar{x}} \frac{L(x, \bar{\vartheta}, \bar{\lambda}) - L(\bar{x}, \bar{\vartheta}, \bar{\lambda})}{\|x - \bar{x}\|} \geq 0; \tag{3.1a}$$

$$\langle \bar{\lambda}, g(\bar{x}) \rangle = 0; \tag{3.1b}$$

$$g_i(\bar{x}) = 0, \quad i \in I^0; \quad g_i(\bar{x}) \geq 0, \quad i \in I^+. \tag{3.1c}$$

Let $\bar{h} = (\bar{u}, \bar{v}) = F(\bar{x})$, $\bar{x} \in R$; we say that $\underline{\mathcal{K}}_G$ ($\overline{\mathcal{K}}_G$) is *linearly separable* from $cl\mathcal{H}_{C_0}$ iff $\exists\ \bar{\omega} = (\bar{\vartheta}, \bar{\lambda}) \in \mathbb{R}^\ell \times \mathbb{R}^m, \bar{\omega} \neq O$, such that:

$$\langle \bar{\omega}, k - \bar{h} \rangle \leq 0 \ \ \forall k \in \underline{\mathcal{K}}_G \ (\overline{\mathcal{K}}_G); \quad \langle \bar{\omega}, k - \bar{h} \rangle \geq 0 \ \ \forall k \in cl\mathcal{H}_{C_0}. \tag{3.2}$$

Since $\bar{h} \in cl\mathcal{H}_{C_0}$ and $cl\mathcal{H}_{C_0}$ is a convex cone then $\bar{h} + k \in cl\mathcal{H}_{C_0}$, $\forall k \in cl\mathcal{H}_{C_0}$. By the second inequality in (3.2) it follows that

$$\langle \bar{\omega}, k \rangle \geq 0, \quad \forall k \in cl\mathcal{H}_{C_0},$$

i.e., $\bar{\omega} \in (cl\mathcal{H}_{C_0})^* = \mathbb{R}^\ell_+ \times \mathbb{R}^p \times \mathbb{R}^{m-p}_+$.

The following theorem gives a sufficient condition for the semistationarity of $\bar{x}$ for VOP in terms of the linear separation between $\overline{\mathcal{K}}_G$ and $cl\mathcal{H}_{C_0}$.

Theorem 3.1 *Let $-f$ and g_i, $i \in I^+$ be upper G–semidifferentiable functions at $\bar{x} \in R$ and g_i, $i \in I^0$ be G–differentiable at $\bar{x}$. If $\overline{\mathcal{K}}_G$ and $cl\mathcal{H}_{C_0}$ are linearly separable, then $\bar{x}$ is a lower semistationary point for* VOP.

Proof Let us suppose that $\overline{\mathcal{K}}_G$ and $cl\mathcal{H}_{C_0}$ are linearly separable, hence $\exists\ \bar{\omega} = (\bar{\vartheta}, \bar{\lambda}) \in (cl\mathcal{H}_{C_0})^* \setminus \{O\}$ such that

$$\langle \bar{\omega}, (\overline{\mathcal{D}}_G(-f)(\bar{x}; x - \bar{x}), \overline{\mathcal{D}}_G g(\bar{x}; x - \bar{x})) \rangle \leq 0, \quad \forall x \in X,$$

or, equivalently, by property (1.9),

$$\langle \bar{\omega}, (\underline{\mathcal{D}}_{(-G)} f(\bar{x}; x - \bar{x}), \underline{\mathcal{D}}_{(-G)}(-g)(\bar{x}; x - \bar{x})) \rangle \geq 0, \quad \forall x \in X.$$

This inequality implies that

$$\liminf_{x \to \bar{x}} \frac{\langle \bar{\omega}, (\underline{\mathcal{D}}_{(-G)} f(\bar{x}; x - \bar{x}), \underline{\mathcal{D}}_{(-G)}(-g)(\bar{x}; x - \bar{x})) \rangle}{\|x - \bar{x}\|} \geq 0. \tag{3.3}$$

We observe that the assumptions on $-f$ and g are equivalent to affirm that f and $-g$ are lower $(-G)$–semidifferentiable functions; hence, recalling that $\bar{\vartheta} \geq O_\ell$, $\bar{\lambda}_i \geq 0, i \in I^+$, (3.3) implies that:

$$0 \leq \liminf_{x \to \bar{x}} \frac{\langle \bar{\omega}, (\underline{\mathcal{D}}_{(-G)} f(\bar{x}; x - \bar{x}), \underline{\mathcal{D}}_{(-G)}(-g)(\bar{x}; x - \bar{x})) \rangle}{\|x - \bar{x}\|}$$

$$+ \liminf_{x \to \bar{x}} \frac{\sum_{i \in J} \bar{\vartheta}_i \varepsilon_{f_i}(\bar{x}; x - \bar{x}) + \sum_{i \in I} \bar{\lambda}_i \varepsilon_{(-g_i)}(\bar{x}; x - \bar{x})}{\|x - \bar{x}\|} \leq$$

$$\liminf_{x \to \bar{x}} \frac{\sum_{i \in J} \bar{\vartheta}_i \Big[\underline{\mathcal{D}}_{(-G)} f_i(\bar{x}; x - \bar{x}) + \varepsilon_{f_i}(\bar{x}; x - \bar{x}) \Big] + \sum_{i \in I} \Big[\bar{\lambda}_i (\underline{\mathcal{D}}_{(-G)}(-g_i)(\bar{x}; x - \bar{x}) + \varepsilon_{(-g_i)}(\bar{x}; x - \bar{x}) \Big]}{\|x - \bar{x}\|} =$$

$$= \liminf_{x \to \bar{x}} \frac{\langle \bar{\vartheta}, f(x) - f(\bar{x}) \rangle + \langle \bar{\lambda}, -g(x) + g(\bar{x}) \rangle}{\|x - \bar{x}\|} = \liminf_{x \to \bar{x}} \frac{L(x, \bar{\vartheta}, \bar{\lambda}) - L(\bar{x}, \bar{\vartheta}, \bar{\lambda})}{\|x - \bar{x}\|}$$

that is (3.1a).
To complete the proof of the lower semistationarity of $\bar{x}$, since $\bar{x} \in R$ and $\bar{h} := F(\bar{x}) = (0, \dots, 0; 0, \dots, 0;\ g_i(\bar{x}), i \in I^+)$, it is enough to prove that $\langle \bar{\omega}, \bar{h} \rangle = 0$. From $\bar{h} \in cl\mathcal{H}_{C_0}$ and $\bar{\omega} \in (cl\mathcal{H}_{C_0})^*$ we have $\langle \bar{\omega}, \bar{h} \rangle \geq 0$; computing the second inequality in (3.2) at $k = O \in cl\mathcal{H}_{C_0}$, we obtain $\langle \bar{\omega}, -\bar{h} \rangle \geq 0$, so that $\langle \bar{\omega}, \bar{h} \rangle = 0$.

Under analogous assumptions, the linear separation between $\underline{\mathcal{K}}_G$ and $cl\mathcal{H}_{C_0}$ is a necessary condition for the lower semistationarity of $\bar{x}$. To prove this result we need the following lemma.

Lemma 3.1 *Let $\phi : X \to \mathbb{R}$ and $\psi : X \to \mathbb{R}$ be such that*

$$\liminf_{x \to \bar{x}} (\phi(x) + \psi(x)) \geq 0 \quad \textit{and} \quad \limsup_{x \to \bar{x}} \psi(x) \leq 0.$$

Then, $\liminf_{x \to \bar{x}} \phi(x) \geq 0$.

Proof Ab absurdo, assume that $\liminf_{x \to \bar{x}} \phi(x) < 0$. Then, there exists a sequence $\{x_k\} \to \bar{x}$ such that $\lim_{k \to \infty} \phi(x_k) < 0$. Therefore, taking into account the assumption $\limsup_{x \to \bar{x}} \psi(x) \leq 0$, we get $\limsup_{k \to \infty} \psi(x_k) \leq 0$ and there exists a subsequence $\{x_{k_h}\}$ such that $\lim_{h \to \infty} \psi(x_{k_h}) \leq 0$. Finally, we have

$$\lim_{h \to \infty} (\phi(x_{k_h}) + \psi(x_{k_h})) = \lim_{h \to \infty} \phi(x_{k_h}) + \lim_{h \to \infty} \psi(x_{k_h}) < 0,$$

which contradicts the assumptions.

Theorem 3.2 *Let $-f$ and g_i, $i \in I^+$ be lower G–semidifferentiable functions at $\bar{x} \in R$ and g_i, $i \in I^0$ be G–differentiable at $\bar{x}$. If $\bar{x}$ is a lower semistationary point for* VOP, *then $\underline{\mathcal{K}}_G$ and $cl\mathcal{H}_{C_0}$ are linearly separable.*

Proof Let us suppose that $\bar{x}$ is a lower semistationary point for VOP. Then (3.1a) holds; i.e., we have

$$0 \leq \liminf_{x \to \bar{x}} \frac{L(x, \bar{\vartheta}, \bar{\lambda}) - L(\bar{x}, \bar{\vartheta}, \bar{\lambda})}{\|x - \bar{x}\|} = \liminf_{x \to \bar{x}} \frac{\langle \bar{\vartheta}, f(x) - f(\bar{x}) \rangle - \langle \bar{\lambda}, g(x) - g(\bar{x}) \rangle}{\|x - \bar{x}\|}$$

or, equivalently, from Definition 1.1

$$0 \leq \liminf_{x \to \bar{x}} \left[\frac{\langle \bar{\vartheta}, -\underline{\mathcal{D}}_G(-f)(\bar{x}; x - \bar{x}) \rangle - \langle \bar{\lambda}, \underline{\mathcal{D}}_G g(\bar{x}; x - \bar{x}) \rangle}{\|x - \bar{x}\|} + \right.$$

$$\left. \frac{\sum_{i \in J} \bar{\vartheta}_i (-\varepsilon_{(-f_i)})(\bar{x}; x - \bar{x}) - \sum_{i \in I} \bar{\lambda}_i \varepsilon_{g_i}(\bar{x}; x - \bar{x})}{\|x - \bar{x}\|} \right].$$

Now we prove that the lower G–semidifferentiability implies that the upper limit of the 2-nd term in square brackets is ≤ 0. From Definition 1.1, it follows that

$$\liminf_{x \to \bar{x}} \frac{\varepsilon_{(-f_i)}(\bar{x}; x - \bar{x})}{\|x - \bar{x}\|} \geq 0, \quad i \in J$$

or equivalently,

$$\limsup_{x \to \bar{x}} \frac{(-\varepsilon_{(-f_i)})(\bar{x}; x - \bar{x})}{\|x - \bar{x}\|} \leq 0, \quad i \in J.$$

Similarly, we obtain

$$\limsup_{x \to \bar{x}} \frac{(-\varepsilon_{g_i})(\bar{x}; x - \bar{x})}{\|x - \bar{x}\|} \leq 0, \quad i \in I^+.$$

By the properties of the upper limit our claim is proved.
Exploiting the above inequalities and applying Lemma 3.1 we deduce

$$0 \leq \liminf_{x \to \bar{x}} \frac{\langle \bar{\vartheta}, -\underline{\mathcal{D}}_G(-f)(\bar{x}; x - \bar{x}) \rangle - \langle \bar{\lambda}, \underline{\mathcal{D}}_G g(\bar{x}; x - \bar{x}) \rangle}{\|x - \bar{x}\|}$$

$$= \lim_{\rho \downarrow 0} \inf_{x \in N_\rho \setminus \{\bar{x}\}} \frac{\langle \bar{\vartheta}, -\underline{\mathcal{D}}_G(-f)(\bar{x}; x - \bar{x}) \rangle - \langle \bar{\lambda}, \underline{\mathcal{D}}_G g(\bar{x}; x - \bar{x}) \rangle}{\|x - \bar{x}\|},$$

where N_ρ is a neighbourhood of $\bar{x}$, with radius ρ. Hence, if $x^* \in X$ and if we set $x(\alpha) = \alpha x^* + (1 - \alpha)\bar{x}$, with $0 < \alpha \leq 1$, it turns out that

$$0 \leq \lim_{\alpha \downarrow 0} \inf_{x \in (\bar{x}, x(\alpha)]} \frac{\langle \bar{\vartheta}, -\underline{\mathcal{D}}_G(-f)(\bar{x}; x - \bar{x}) \rangle - \langle \bar{\lambda}, \underline{\mathcal{D}}_G g(\bar{x}; x - \bar{x}) \rangle}{\|x - \bar{x}\|}.$$

Since every $x \in (\bar{x}, x(\alpha)]$ is expressed as $(1-t)\bar{x} + tx(\alpha)$, with $0 < t \le 1$, we have $x - \bar{x} = t(x(\alpha) - \bar{x})$; from the positive homogeneity of $\underline{\mathcal{D}}_G(-f)$ and $\underline{\mathcal{D}}_G g$, it follows that:

$$\lim_{\alpha \downarrow 0} \inf_{x \in (\bar{x}, x(\alpha)]} \frac{\langle \bar{\vartheta}, -\underline{\mathcal{D}}_G(-f)(\bar{x}; x - \bar{x})\rangle - \langle \bar{\lambda}, \underline{\mathcal{D}}_G g(\bar{x}; x - \bar{x})\rangle}{\|x - \bar{x}\|} =$$

$$= \lim_{\alpha \downarrow 0} \frac{\langle \bar{\vartheta}, -\underline{\mathcal{D}}_G(-f)(\bar{x}; x(\alpha) - \bar{x})\rangle - \langle \bar{\lambda}, \underline{\mathcal{D}}_G g(\bar{x}; x(\alpha) - \bar{x})\rangle}{\|x(\alpha) - \bar{x}\|} =$$

$$= \lim_{\alpha \downarrow 0} \frac{\langle \bar{\vartheta}, -\underline{\mathcal{D}}_G(-f)(\bar{x}; \alpha(x^* - \bar{x}))\rangle - \langle \bar{\lambda}, \underline{\mathcal{D}}_G g(\bar{x}; \alpha(x^* - \bar{x}))\rangle}{\|\alpha(x^* - \bar{x})\|}.$$

Again from the positive homogeneity of $\underline{\mathcal{D}}_G(-f)$ and $\underline{\mathcal{D}}_G g$, we have

$$0 \le \langle \bar{\vartheta}, -\underline{\mathcal{D}}_G(-f)(\bar{x}; x^* - \bar{x})\rangle - \langle \bar{\lambda}, \underline{\mathcal{D}}_G g(\bar{x}; x^* - \bar{x})\rangle \quad \forall x^* \in X,$$

or, equivalently,

$$0 \ge \langle \bar{\vartheta}, \underline{\mathcal{D}}_G(-f)(\bar{x}; x^* - \bar{x})\rangle + \langle \bar{\lambda}, \underline{\mathcal{D}}_G g(\bar{x}; x^* - \bar{x})\rangle \quad \forall x^* \in X. \tag{3.4}$$

From the definition of $\underline{\mathcal{K}}_G$ and from (3.4), recalling that $\bar{h} = F(\bar{x})$, we have $\langle \bar{\omega}, k - \bar{h}\rangle \le 0,\ \forall k \in \underline{\mathcal{K}}_G$. From the definition of $\mathcal{H}$, it results:

$$\langle \bar{\omega}, k\rangle \ge 0 \quad \forall k \in cl\mathcal{H}_{C_0}; \tag{3.5}$$

moreover, from (3.1b), we have $\langle \bar{\vartheta}, f(\bar{x}) - f(\bar{x})\rangle + \langle \bar{\lambda}, g(\bar{x})\rangle = 0$, or, equivalently, $\langle \bar{\omega}, \bar{h}\rangle = 0$, that, together with (3.5), implies $\langle \bar{\omega}, k - \bar{h}\rangle \ge 0,\ \forall k \in cl\mathcal{H}_{C_0}$. Hence the linear separation between $\underline{\mathcal{K}}_G$ and $cl\mathcal{H}_{C_0}$ follows.

A straightforward consequence of Theorems 3.1 and 3.2 is the following theorem, that characterizes the semistationarity of $\bar{x}$ in terms of linear separation between $\mathcal{K}_G$ and $cl\mathcal{H}_{C_0}$.

Theorem 3.3 *If $-f$ and g are G–differentiable at $\bar{x} \in R$, then $\bar{x}$ is a lower semistationary point for VOP iff $\mathcal{K}_G$ and $cl\mathcal{H}_{C_0}$ are linearly separable.*

Under G-differentiability assumptions, Proposition 2.2 implies that the approximation $\mathcal{K}_G$ is included in the set $(\bar{u}, \bar{v}) + T(0; \mathcal{E}(\mathcal{K}_{\bar{x}}))$; therefore, it can be employed as a source of sufficient conditions for the semistationarity of $\bar{x}$.

Theorem 3.4 *Let $-f_j$, $j \in J$, and g_i, $i \in I$ be G–differentiable at $\bar{x}$. If*

$$conv[(\bar{u}, \bar{v}) + T(0; \mathcal{E}(\mathcal{K}_{\bar{x}}))] \cap ri\mathcal{H}_{C_0} = \emptyset, \tag{3.6}$$

then $\bar{x}$ is a lower semistationary point for VOP.

Proof Under G-differentiability assumptions, by Proposition 2.2 and (3.6), we obtain:

$$conv\mathcal{K}_G \cap ri\mathcal{H}_{C_0} = \emptyset. \tag{3.7}$$

Recalling that the convexity of $\mathcal{H}_{C_0}$ implies $ri\mathcal{H}_{C_0} = ri(cl\mathcal{H}_{C_0})$ (see Theorem 6.3 of [15]), (3.7) yields that $\mathcal{K}_G$ and $cl\mathcal{H}_{C_0}$ are linearly separable. The conclusion follows from Theorem 3.3.

Unlike the analogous condition (6) in [2], the assumption (3.6) does not imply the optimality of the point $\bar{x}$ as proved by the following example.

Example 3.1 *In* (1.1) *set* $\ell = p = 1$ *and* $m = 2$, $X = \mathbb{R}$, $f(x) = -x$, $g_1(x) = x(1-x)$, $g_2(x) = -x^2(1-x)^2$; $\bar{x} = 0$ *is a feasible point, but it is not optimal, in fact* $f(1) < f(0)$; $(\bar{u}, \bar{v_1}, \bar{v_2}) = (0,0,0)$. $\mathcal{K}_{\bar{x}} = \{(u, v_1, v_2) \in \mathbb{R}^3 : u = x, v_1 = x(1-x), v_2 = -x^2(1-x)^2, x \in \mathbb{R}\}$. *Hence,* $\mathcal{K}_{\bar{x}} \subseteq \{(u, v_1, v_2) \in \mathbb{R}^3 : v_2 \leq 0\}$ *and the same inclusion holds for* $(\bar{u}, \bar{v}) + T(0; \mathcal{E}(\mathcal{K}_{\bar{x}}))$ *and for its convex hull. This proves that the intersection with* $ri\mathcal{H}_{C_0}$ *is empty and* (3.6) *is fulfilled.*

Under suitable convexity assumptions, (3.6) can be simplified as shown in the next results.

Proposition 3.1 *Let* $-f_j$, $j \in J$, *and* g_i, $i \in I$ *be G-differentiable at* $\bar{x}$, $-F$ *be* $cl\mathcal{H}_{C_0}$*-convexlike. If*

$$T(0; \mathcal{E}(\mathcal{K}_{\bar{x}})) \cap ri\mathcal{H}_{C_0} = \emptyset, \tag{3.8}$$

then $\bar{x}$ *is a lower semistationary point for VOP.*

Proof Since $-F$ is $cl\mathcal{H}_{C_0}$-convexlike, then $\mathcal{E}(\mathcal{K}_{\bar{x}})$ is convex [17]. By Remark 2.2 we have $[(\bar{u}, \bar{v}) + T(0; \mathcal{E}(\mathcal{K}_{\bar{x}}))] \subseteq T(0; \mathcal{E}(\mathcal{K}_{\bar{x}}))$ and, hence, (3.8) implies $[(\bar{u}, \bar{v}) + T(0; \mathcal{E}(\mathcal{K}_{\bar{x}}))] \cap ri\mathcal{H}_{C_0} = \emptyset$. Observing that also $T(0; \mathcal{E}(\mathcal{K}_{\bar{x}}))$ is convex, we have that (3.6) holds and applying Theorem 3.4 we complete the proof.

Proposition 3.2 *Let* $-f_j$, $j \in J$, *and* g_i, $i \in I$ *be G–differentiable at* $\bar{x}$ *with* $G \subseteq (-\mathcal{C})$. *If*

$$[(\bar{u}, \bar{v}) + T(0; \mathcal{E}(\mathcal{K}_{\bar{x}}))] \cap ri\mathcal{H}_{C_0} = \emptyset, \tag{3.9}$$

then $\bar{x}$ *is a lower semistationary point for VOP.*

Proof By Proposition 2.2 and (3.9), we obtain that $\mathcal{K}_G$ and $ri\mathcal{H}_{C_0}$ have an empty intersection. Now we prove that $\mathcal{K}_G \cap ri\mathcal{H}_{C_0} = \emptyset \Leftrightarrow \mathcal{E}(\mathcal{K}_G) \cap ri\mathcal{H}_{C_0} = \emptyset$. To this aim, observe that, by Theorem 6.1 of [15] and since $\mathcal{H}_{C_0}$ is a convex cone, the following equalities hold:

$$\mathcal{K}_G - ri\mathcal{H}_{C_0} = \mathcal{K}_G - (cl\mathcal{H}_{C_0} + ri\mathcal{H}_{C_0}) = \mathcal{E}(\mathcal{K}_G) - ri\mathcal{H}_{C_0}.$$

Therefore, $0 \notin \mathcal{K}_G - ri\mathcal{H}_{C_0} \Leftrightarrow 0 \notin \mathcal{E}(\mathcal{K}_G) - ri\mathcal{H}_{C_0}$. The assumption $G \subseteq (-\mathcal{C})$ implies that $\mathcal{E}(\mathcal{K}_G)$ is convex [4]; hence, $\mathcal{K}_G$ and $cl\mathcal{H}_{C_0}$ are linearly separable and applying Theorem 3.3 we complete the proof.

Let us observe that, if all the inequality constraints are binding at $\bar{x}$, i.e., $g_i(\bar{x}) = 0, \forall i \in I^+$, then (3.9) collapses to (3.8).

4 Applications to optimality conditions

In this section we consider problem (1.1) with inequality constraints only, and we prove that the semistationarity of a solution is a necessary optimality condition that can be expressed by means of the generalized subdifferential of the Lagrangian function.

Proposition 4.1 *Let $I^0 = \emptyset$ and assume that $-f_j$, $j \in J$ and g_i, $i \in I^+$ are G–differentiable at $\bar{x}$, with $G \subseteq (-\mathcal{C})$. If $\bar{x}$ is a weak v.m.p. of VOP, then*
(i) *$\bar{x}$ is a lower semistationary point for VOP;*
(ii) *there exists $(\bar{\vartheta}, \bar{\lambda}) \in (\mathbb{R}^\ell_+ \times \mathbb{R}^m_+) \setminus \{O\}$ such that $O \in \partial_{(-G)} L(\bar{x}, \bar{\vartheta}, \bar{\lambda})$.*

Proof (i) The assumptions of the proposition imply those of Theorem 5 in [6] (see also Theorem 6.19 in [8]) , i.e. f_j, $j \in J$ are upper Φ–semidifferentiable at $\bar{x}$, with $\Phi \subseteq \mathcal{C}$, and g_i, $i \in I^+$ are lower Γ–semidifferentiable at $\bar{x}$, with $\Gamma \subseteq (-\mathcal{C})$. By such a theorem, it follows that there exists a vector $(\bar{\vartheta}, \bar{\lambda}) \in (\mathbb{R}^\ell_+ \times \mathbb{R}^m_+) \setminus \{O\}$, such that

$$\langle \bar{\vartheta}, \overline{\mathcal{D}}_\Phi f(\bar{x}; x - \bar{x}) \rangle - \langle \bar{\lambda}, \overline{\mathcal{D}}_\Gamma g(\bar{x}; x - \bar{x}) \rangle \geq 0, \quad \forall x \in X,$$

or, equivalently by (1.9), that

$$\langle \bar{\vartheta}, \underline{\mathcal{D}}_{(-\Phi)}(-f)(\bar{x}; x - \bar{x}) \rangle + \langle \bar{\lambda}, \overline{\mathcal{D}}_\Gamma g(\bar{x}; x - \bar{x}) \rangle \leq 0, \quad \forall x \in X, \tag{4.1}$$

If we set $-\Phi = \Gamma = G$ since by assumptions $\underline{\mathcal{D}}_{(-\Phi)} = \mathcal{D}_G$ and $\overline{\mathcal{D}}_\Gamma = \mathcal{D}_G$, then (4.1) is equivalent to the linear separation between $\mathcal{K}_G$ and $cl\mathcal{H}_{C_0}$. By Theorem 3.3, $\bar{x}$ is a lower semistationary point for VOP.
(ii) As shown in the proof of (i), $\mathcal{K}_G$ and $cl\mathcal{H}_{C_0}$ are linearly separable, i.e., there exist $(\bar{\vartheta}, \bar{\lambda}) \in (\mathbb{R}^\ell_+ \times \mathbb{R}^m_+) \setminus \{O\}$ such that

$$\langle \bar{\vartheta}, \mathcal{D}_G(-f)(\bar{x}; x - \bar{x}) \rangle + \langle \bar{\lambda}, \mathcal{D}_G g(\bar{x}; x - \bar{x})) \rangle \leq 0, \quad \forall x \in X. \tag{4.2}$$

Then, by Proposition 1.1, it follows that

$$\langle \bar{\vartheta}, \mathcal{D}_G(-f)(\bar{x}; x - \bar{x}) \rangle + \langle \bar{\lambda}, \mathcal{D}_G g(\bar{x}; x - \bar{x})) \rangle = \mathcal{D}_G(\langle \bar{\vartheta}, -f \rangle + \langle \bar{\lambda}, g \rangle)(\bar{x}; x - \bar{x}) \leq 0, \quad \forall x \in X,$$

which, by (1.9), is equivalent to

$$\mathcal{D}_{(-G)}(\langle \bar{\vartheta}, f \rangle - \langle \bar{\lambda}, g \rangle)(\bar{x}; x - \bar{x}) \geq 0, \quad \forall x \in X,$$

i.e., $O \in \partial_{(-G)} L(\bar{x}, \bar{\vartheta}, \bar{\lambda})$.

The following corollary improves Proposition 3.2 by assuming a Slater type regularity condition.

Corollary 4.1 *Let $I^0 = \emptyset$ and assume that $-f_j$, $j \in J$, g_i, $i \in I^+$ are G–differentiable at $\bar{x}$, with $G \subseteq (-\mathcal{C})$, and that there exists $\tilde{x} \in X$ such that*

$$\mathcal{D}_G g(\bar{x}; \tilde{x} - \bar{x}) > O. \tag{4.3}$$

If $\bar{x}$ is a weak v.m.p. of VOP, then the thesis of Proposition 4.1 holds with $\bar{\vartheta} \neq O$.

Proof It is enough to show that $\bar{\vartheta} \neq O$ in (4.2). Ab absurdo, assume that (4.2) holds with $\bar{\vartheta} = O$; then $\bar{\lambda} \neq O$ and, taking into account (4.3), the following inequalities hold:

$$0 < \langle \bar{\lambda}, \mathcal{D}_G g(\bar{x}; \tilde{x} - \bar{x}) \rangle \leq 0,$$

and a contradiction is achieved.

Finally, we consider the relationships with known results in the literature obtained by means of Dini-Hadamard derivatives [7]. To this aim, we first provide a characterization of the concept of semistationary point in terms of the lower Dini-Hadamard derivative.

Proposition 4.2 *Given a function $\phi : \mathbb{R}^n \to \mathbb{R}$, $\bar{x} \in \mathbb{R}^n$ is a semistationary point for ϕ, i.e.,*

$$\liminf_{x \to \bar{x}} \frac{\phi(x) - \phi(\bar{x})}{\|x - \bar{x}\|} \geq 0, \tag{4.4}$$

iff

$$\phi^-_{DH}(\bar{x}; z) \geq 0, \qquad \forall z \in S := \{z \in \mathbb{R}^n : \|z\| = 1\}.$$

Proof (4.4) is equivalent to the following condition: $\liminf\limits_{k \to +\infty} \frac{\phi(x_k) - \phi(\bar{x})}{\|x_k - \bar{x}\|} \geq 0$, $\forall \{x_k\} \to \bar{x}$. Observe that every element of the sequence $\{x_k\} \to \bar{x}$ can be equivalently expressed as $x_k = \bar{x} + t_k z_k$ with $z_k \in S$ and $t_k > 0$. Since $\{x_k\} \to \bar{x}$ then $\{t_k\} \to 0^+$ and, if we observe that $\|x_k - \bar{x}\| = t_k$, we have that (4.4) is equivalent to the statement

$$\liminf_{k \to +\infty} \frac{\phi(\bar{x} + t_k z_k) - \phi(\bar{x})}{t_k} \geq 0, \quad \forall \{z_k\} \subset S, \quad \forall \{t_k\} \to 0^+. \tag{4.5}$$

Moreover, the condition $\phi^-_{DH}(\bar{x}; z) \geq 0$, $\forall z \in S$ is equivalent to the following:

$$\forall z \in S, \quad \liminf_{k \to +\infty} \frac{\phi(\bar{x} + t_k z_k) - \phi(\bar{x})}{t_k} \geq 0, \quad \forall \{z_k\} \to z, \quad \forall \{t_k\} \to 0^+. \tag{4.6}$$

Hence the statement of the proposition is equivalent to prove that (4.5) $\Leftrightarrow$ (4.6).
Only if ((4.5) $\Rightarrow$ (4.6)). Ab absurdo, suppose that (4.6) does not hold; hence, assume that there exist $\bar{z} \in S$ and sequences $\{z_k\} \to \bar{z}$, $\{t_k\} \to 0^+$ such that $\liminf\limits_{k \to +\infty} \frac{\phi(\bar{x} + t_k z_k) - \phi(\bar{x})}{t_k} < 0$. Since $\{z_k\} \to \bar{z} \in S$, then $\{\|z_k\|\} \to 1$ and the normalized sequence $\left\{\frac{z_k}{\|z_k\|}\right\}$ converges to $\bar{z} \in S$. Let $h_k := t_k \|z_k\|$, we have that $\{h_k\} \to 0^+$. Therefore it follows that

$$0 > \liminf_{k \to +\infty} \frac{\phi(\bar{x} + t_k z_k) - \phi(\bar{x})}{t_k \|z_k\|} = \liminf_{k \to +\infty} \frac{\phi\left(\bar{x} + \frac{h_k z_k}{\|z_k\|}\right) - \phi(\bar{x})}{h_k}.$$

Since $\left\{\frac{z_k}{\|z_k\|}\right\} \subset S$ and $\{h_k\} \to 0^+$, we contradict (4.5).
If ((4.5) $\Leftarrow$ (4.6)). Ab absurdo, if (4.5) does not hold, there exist sequences $\{z_k\} \subset S$, $\{t_k\} \to 0^+$ such that $\liminf\limits_{k \to +\infty} \frac{\phi(\bar{x} + t_k z_k) - \phi(\bar{x})}{t_k} < 0$. Since S is compact,

then $\{z_k\}$ admits a subsequence $\{s_h\} := \{z_{k_h}\}$ converging to a point $\bar{z} \in S$ and such that, letting $\{t_h\} := \{t_{k_h}\}$, $\liminf_{k \to +\infty} \frac{\phi(\bar{x}+t_h s_h)-\phi(\bar{x})}{t_h} < 0$. This is against (4.6) and we achieve a contradiction.

From Proposition 4.2 and since the Dini-Hadamard derivative is a positively homogeneous function, it follows that (3.1a) can be replaced by to $L^-_{DH}(\bar{x}, \bar{\vartheta}, \bar{\lambda}; z) \geq 0, \ \forall z \neq 0$.

Now, let us recall Theorem 2.1 of [18], where it is affirmed that, if G is a subset of the continuous positively homogeneous functions and ϕ is upper G-semidifferentiable at $\bar{x}$, then

$$\phi^+_{DH}(\bar{x}; z) \leq \overline{\mathcal{D}}_G \phi(\bar{x}; z), \quad \forall z \in X - \bar{x}.$$

Under the same assumption on G, if ϕ is lower G-semidifferentiable at $\bar{x}$, then

$$\phi^-_{DH}(\bar{x}; z) \geq \underline{\mathcal{D}}_G \phi(\bar{x}; z), \quad \forall z \in X - \bar{x}.$$

Hence, by definition of Dini-Hadamard derivatives, it turns out that

$$\underline{\mathcal{D}}_G \phi(\bar{x}; z) \leq \phi^-_{DH}(\bar{x}; z) \leq \phi^+_{DH}(\bar{x}; z) \leq \overline{\mathcal{D}}_G \phi(\bar{x}; z), \quad \forall z \in X - \bar{x}. \tag{4.7}$$

If, furthermore, ϕ is G-differentiable at $\bar{x}$, then $\underline{\mathcal{D}}_G \phi(\bar{x}; z) = \overline{\mathcal{D}}_G \phi(\bar{x}; z)$; from (4.7) it follows that

$$\mathcal{D}_G \phi(\bar{x}; z) = \lim_{\substack{t \to 0^+ \\ y \to z}} \frac{\phi(\bar{x} + ty) - \phi(\bar{x})}{t} =: \phi_{DH}(\bar{x}, z), \quad \forall z \in X - \bar{x}. \tag{4.8}$$

By Theorem 3.3 the semistationarity of $\bar{x}$ is characterized by the linear separation between $\mathcal{K}_G$ and $\mathcal{H}_{C_0}$; therefore, there exist $\bar{\theta} \in \mathbb{R}^\ell_+$; $\bar{\lambda}_i \in \mathbb{R}$, $i \in I^0$; $\bar{\lambda}_i \geq 0$, $i \in I^+$, with $(\bar{\theta}, \bar{\lambda}) \neq O$ such that

$$\langle \bar{\theta}, \mathcal{D}_G(-f(\bar{x}; z)) \rangle + \langle \bar{\lambda}, \mathcal{D}_G g(\bar{x}; z) \rangle \leq 0, \quad \forall z \in X - \bar{x},$$

or equivalently, by the positively homogeneity of $\mathcal{D}_G f$ and $\mathcal{D}_G g$,

$$\langle \bar{\theta}, \mathcal{D}_G(-f(\bar{x}; z)) \rangle + \langle \bar{\lambda}, \mathcal{D}_G g(\bar{x}; z) \rangle \leq 0, \quad \forall z \in cl(cone(X - \bar{x})).$$

Since X is convex, then $cl(cone(X - \bar{x})) = T(\bar{x}; X)$. Moreover, if in Theorem 3.3 the G is a subset of the continuous positively homogeneous functions, then by (4.8), the previous condition collapses to

$$\langle \bar{\theta}, (-f_{DH}(\bar{x}; z)) \rangle + \langle \bar{\lambda}, g_{DH}(\bar{x}; z) \rangle \leq 0, \quad \forall z \in T(\bar{x}; X).$$

Hence, we have obtained the Fritz John conditions as defined in Remark 3.2 of [7].

5 Concluding remarks

We have considered an approximation of a vector optimization problem obtained by replacing the objective and the constraint functions by means of their G-derivatives. After analysing in details such an approximation scheme, we have deepened the connections with the semistationarity of the generalized Lagrangian function associated with the vector optimization problem; furthermore, we have studied the relationships with classic necessary optimality conditions obtained by means of Dini-Hadamard derivatives.

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