Abstract
The Hilbert function, its generating function and the Hilbert polynomial of a graded ring \(K[x_1, \ldots, x_n]\) have been extensively studied since the famous paper of Hilbert: "Ueber die Theorie der algebraischen Formen" (Hilbert, 1890). In particular the coefficients and the degree of the Hilbert polynomial play an important role in Algebraic Geometry.

If the grading is non-standard, then its Hilbert function is not definitely equal to a polynomial but to a quasi-polynomial.

It turns out that a Hilbert quasi-polynomial \(P_R/I\) associated to \((R/I, W)\) is given by \(P_R/I = P \sum_{n=0}^{\infty} t^n\) where \(P_R/I \in (R/I)_W\) and the coefficients of \(P\) associated to \((R/I, W)\) are the elements \(P_{R/I}^n\).

From now on, \((R/I, W)\) stands for the graded ring ring \(R/I\), where \(I\) is a homogeneous ideal of \(R\) and \(I\) is graded by \(W = \langle d_1, \ldots, d_k \rangle \in N^e\), where \(e = \text{deg}(I) = \text{max}(\{\text{deg}(x_i) \mid x_i \in I\})\).

Let \(P_R/I\) and the Poincaré series of \((R/I, W)\) be a polynomial standard graded ring and \(I\) a homogeneous ideal of \(R\). Then there exists a polynomial \(P_{R/I}(x) \in \mathbb{Q}[x]\) such that \(H_{R/I}(n) = P_{R/I}(n) \forall n \geq 0\).

This polynomial is called Hilbert polynomial of \((R/I, W)\).

**Hilbert Quasi-Polynomials**

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Let \(R = K[x_1, \ldots, x_n]\) be a polynomial standard graded ring and \(I\) a homogeneous ideal of \(R\). Then there exists a polynomial \(P_{R/I}(x) \in \mathbb{Q}[x]\) such that \(H_{R/I}(n) = P_{R/I}(n) \forall n \geq 0\).

**Definition 2** Let \(L\) be a polynomial standard graded ring and \(I\) a homogeneous ideal of \(L\). Then there exists a polynomial \(P_{L/I}(x) \in \mathbb{Q}[x]\) such that \(H_{L/I}(n) = P_{L/I}(n) \forall n \geq 0\).

This polynomial is called Hilbert polynomial of \((L/I, W)\).

**Numerical Examples**

Example 8. Let \(R = \mathbb{Q}[x_1, \ldots, x_k]\) be graded by \(W = \{1, 2, 3, 4, 6\}\). Then its Hilbert quasi-polynomial \(P_{R/I}^n\) is given by:

\[
P_{R/I}(x) = \begin{cases} 
\frac{1}{2}x^2 + \frac{1}{2}x + 1, & \text{if } x = 1 \\
\frac{1}{2}x^2 + \frac{1}{2}x + 1, & \text{if } x = 2 \\
\frac{1}{2}x^2 + \frac{1}{2}x + 1, & \text{if } x = 3 \\
\frac{1}{2}x^2 + \frac{1}{2}x + 1, & \text{if } x = 4 \\
\frac{1}{2}x^2 + \frac{1}{2}x + 1, & \text{if } x = 6 \\
\end{cases}
\]

**Conclusions and further work**

We have provided a partial characterization of Hilbert quasi-polynomials for \(K[x_1, \ldots, x_n]\). We want to complete this characterization. Specifically, we want to find the closed formulas for as many as possible coefficients of the Hilbert quasi-polynomial, periodic part included. Moreover, we want to extend our work to \(R/I\) graded quotient rings \(K[x_1, \ldots, x_n]/I\). This will allow us to write more efficient procedures for the computation of Hilbert quasi-polynomials.

**References**

