A matrix approach to Sheffer polynomials

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Abstract

This paper deals with a unified matrix representation for the Sheffer polynomials. The core of the proposed approach is the so-called creation matrix, a special subdiagonal matrix having as nonzero entries positive integer numbers, whose exponential coincides with the well-known Pascal matrix. In fact, Sheffer polynomials may be expressed in terms of two matrices both connected to it. As we will show, one of them is strictly related to Appell polynomials, while the other is linked to a binomial type sequence. Consequently, different types of Sheffer polynomials corresponds to different choices of these two matrices.

Keywords: Sheffer polynomials, binomial type polynomials, Appell polynomials, creation matrix, generalized Pascal matrix

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1. Introduction

In 1939 I. M. Sheffer in [17] studied in detail the so-called *polynomials sets of type zero* $s_n(x)$, $n \in \mathbb{N}_0$. Given the formal power series $f$ and $g$ such that $f(0) \neq 0$, $g(0) = 0$, $g'(0) \neq 0$, they can be characterized by a generating function of the form $G(x,t) = f(t) \exp(xg(t))$, $x,t \in \mathbb{R}$. Since then, these polynomial sets, nowadays called simply Sheffer polynomials, have been extensively studied not only due to the fact that they arise in various branches of Mathematics but also because of their importance in applied sciences, like Chemistry and Engineering. In the last decades, a renewed interest has been paid to those sequences and to their different representations (see, for example, [10] and references therein). A close connection with Riordan arrays have been established in [11] by proving the isomorphism between the Sheffer group and the Riordan group. Using those results a determinantal approach has been proposed in [19] by extending existent results on determinantal representations of Appell polynomials to Sheffer polynomials (see [8, 21]). A generalization of this determinantal approach to mixed special polynomials of two variables related to Gould-Hopper polynomials has been proposed in [13], where properties and operational relations between Sheffer and those so-called Gould-Hopper-Sheffer polynomials are derived. Moreover, each Sheffer polynomial sequence is related to a binomial type polynomial sequence (cf. [16]) that is proved to be tied to the Bell polynomials, cf. [14, 20]. An algebraic approach to the Sheffer polynomial sequences is developed in [9], where the authors also consider matrix representations. In the present paper, we propose an alternative matrix representation to Sheffer polynomials which combines the matrix approach for the representation of Appell polynomials
in a real variable, proposed in [2] and recently extended in [3] to the hyper-
complex case, with the matrix representation of a binomial type polynomial
sequence by the so-called Bell matrix. Thus, the matrix which represents
the Sheffer polynomial coefficients can be factorized into two matrices, one
associated to Appell polynomials and the other linked to the binomial type
polynomial sequence, both closely related to the so-called creation matrix
introduced in [4]. The result shows clearly the role of Appell and binomial
type polynomial sequences into the characterization of Sheffer polynomials.
In addition, the special structure of the creation matrix (a subdiagonal ma-
trix which contains as nonzero entries only positive integer numbers) sheds
light on the most fundamental arithmetical origins of the class of Sheffer
polynomials.

The paper is organized as follows. In Section 2, we present the character-
ization of Sheffer polynomial sequences by their generating functions and the
connection to the Bell polynomials and binomial type polynomial sequences.
In Section 3, we propose the matrix approach for the Sheffer polynomials by
introducing the generalized Pascal matrix and the Bell matrix and by show-
ing that both are related with the creation matrix. In Section 4, we prove
some known properties of Sheffer polynomials by using the proposed matrix
representation. In Section 5, we consider some classical Sheffer polynomials
as examples and, finally, in Section 6 we present some conclusions.
2. Preliminaries

Let us consider a real numerical sequence \( \{b_n\}_{n \geq 0} \), with \( b_0 = 0 \) and \( b_1 \neq 0 \) and associate to such numerical sequence the formal power series

\[
g(t) = \sum_{n=1}^{+\infty} b_n \frac{t^n}{n!}, \quad b_1 \neq 0.
\]  

(1)

Then, the function

\[
\Phi(x, g(t)) := \exp(xg(t)) = \sum_{n=0}^{+\infty} p_n(x) \frac{t^n}{n!}
\]

(2)

generates the polynomials \( \{p_n(x)\}_{n \geq 0} \) defined as follows [7, Section 3.3, p.133]

\[
p_0(x) = 1, \quad p_n(x) = \sum_{k=1}^{n} B_{n,k}(b_1, b_2, \ldots, b_{n-k+1}) x^k \quad n \geq 1,
\]

(3)

where

\[
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})
\]

are the (exponential) partial Bell polynomials in the variables \( x_1, x_2, \ldots, x_{n-k+1} \).

Explicitly,

\[
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{j_1 + \cdots + j_{n-k+1} = k} \frac{n!}{j_1! \cdots j_{n-k+1}!} \left( x_1 \right)^{j_1} \cdots \left( \frac{x_{n-k+1}}{(n-k+1)!} \right)^{j_{n-k+1}},
\]

where the sum is taken over all integers \( j_1, j_2, \ldots, j_{n-k+1} \) such that

\[
j_1 + j_2 + \ldots + j_{n-k+1} = k \quad \text{and} \quad j_1 + 2j_2 + 3j_3 + \ldots + (n-k+1)j_{n-k+1} = n.
\]

**Remark 1.** By virtue of the fact that \( B_{n,n}(x_1) = x_1^n \) and that \( b_1 \neq 0 \), it is immediate to check that \( p_n(x) \) is of exact degree \( n \).

**Remark 2.** The monomial sequence \( \{x^n\}_{n \geq 0} \) is a particular case of the sequence defined by (3), corresponding to the choice of \( g(t) = t \).
It is worth to point out the following interesting properties of the Bell polynomials that will be used in the paper cf. [7, p.135-136]:

\[ B_{n,k}(abx_1, ab^2 x_2, ab^3 x_3, \ldots) = a^k b^n B_{n,k}(x_1, x_2, x_3, \ldots), \quad a, b \in \mathbb{R}, \quad (4) \]

and

\[ B_{n,k}(0, \ldots, 0, x_j, 0, \ldots) = \frac{(jk)!}{k!(j!)^k}x^k \delta_{n,jk}, \quad (5) \]

where \( \delta_{n,jk} \) denotes the Kronecker symbol.

**Remark 3.** Some special combinatorial sequences can be obtained from the Bell polynomials by appropriately choosing the variables \( x_1, x_2, \ldots \). In particular, cf. [7, Theorem B, p.135],

- the Stirling numbers of the first kind \( s(n, k) = (-1)^{n-k}B_{n,k}(0!, 1!, 2!, \ldots) \),
- the Stirling numbers of the second kind \( S(n, k) = B_{n,k}(1, 1, 1, \ldots) \),
- the Lah numbers (or Stirling numbers of the third kind) \( \binom{n-1}{k-1}n!/k! = B_{n,k}(1!, 2!, 3! \ldots) \).

In addition cf. [1, Corollary 4],

\[ B_{n,k}((1a)^0, (2a)^1, (3a)^2, \ldots) = \binom{n-1}{k-1}(an)^{n-k}, \quad a \in \mathbb{R}. \quad (6) \]

Given the formal power series of the form

\[ f(t) = \sum_{n=0}^{+\infty} c_n \frac{t^n}{n!}, \quad c_0 \neq 0, \quad (7) \]

it is well-known that

\[ f(t) \Phi(x, g(t)) = \sum_{n=0}^{+\infty} s_n(x) \frac{t^n}{n!} \]
where
\[ s_n(x) = \sum_{k=0}^{n} \binom{n}{k} c_k p_{n-k}(x), \quad n = 0, 1, 2, \ldots \]  
(8)

**Definition 1.** The polynomials \( s_n(x) \) of degree \( n \) given by (8) are called Sheffer type polynomials for the pair \((f(t), g(t))\) or, shortly, Sheffer polynomials.

The Sheffer polynomials for the pair \((1, g(t))\) correspond to \( p_n(x) \) given by (3) and they are usually called of binomial type. These polynomials can be alternatively defined by the identity:
\[ p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y), \quad \forall n \in \mathbb{N}_0, \ x, y \in \mathbb{R}. \]  
(9)

The choice of \( g(t) = t \) leads to Sheffer polynomials for the pair \((f(t), t)\), well-known as Appell polynomials. These polynomials can be alternatively defined by (8) setting \( p_n(x) = x^n \), cf. [5].

### 3. Sheffer polynomials: a matrix approach

The main tool for the matrix approach we are proposing in this paper relies in the so-called creation matrix defined by
\[ (H)_{ij} = \begin{cases} 
  i, & i = j + 1 \\
  0, & \text{otherwise}, \quad i, j = 0, 1, \ldots, m.
\end{cases} \]  
(10)

As already observed in [4], it could also be called derivation matrix because
\[ He_j = (j + 1)e_{j+1}, \quad j = 0, 1, \ldots, m, \]

where \( e_j \) denote the standard unit basis vectors in \( \mathbb{R}^{m+1} \). Our notational convention is that, whenever \( j > m \), \( e_j = 0 \) (the null vector); this is consistent with the fact that the creation matrix is nilpotent of degree \( m + 1 \), i.e.
\[ H^j = O, \quad \text{for all } j \geq m + 1. \]
Recently, this matrix has been used in [2] to obtain a unified matrix representation of Appell polynomials. In this context, we want to extend this approach to the more general class of Sheffer polynomials. At this aim, let us consider the generalized Pascal matrix of the sequence \( \{ q_n(x) \}_{n \geq 0} \) defined by

\[
(P[q_n(x)])_{ij} = \begin{cases} 
\binom{i}{j} q_{i-j}(x), & i \geq j \\
0, & \text{otherwise},
\end{cases} \quad i, j = 0, 1, \ldots, m.
\]

It is worth to mention that such matrix was introduced in [22] for binomial type polynomial sequences and generalizes also the Pascal matrix defined in [6] for the sequence of monomials \( \{ x^n \}_{n \geq 0} \).

The next result shows that the generalized Pascal matrix of a sequence of binomial type is a suitable function of the creation matrix, fact not evident at a first glance.

**Theorem 1.** Let \( H \) be the creation matrix defined by (10). If \( P[p_n(x)] \) is the generalized Pascal matrix of the sequence of binomial type \( \{ p_n(x) \}_{n \geq 0} \) given in (3) then

\[
\frac{d^l}{dx^l} \Phi(x, g(H)) = P[p_n^{(l)}(x)], \quad l = 0, 1, \ldots, m.
\]

**Proof.** According to (2), for each \( l = 0, 1, \ldots, m \), the entries of \( \frac{d^l}{dx^l} \Phi(x, g(H)) \) are obtained by

\[
\left( \frac{d^l}{dx^l} \Phi(x, g(H)) \right)_{ij} = \left( \frac{d^l}{dx^l} \exp(xg(H)) \right)_{ij} = e_i^T \left( \sum_{n=0}^m \frac{p_n^{(l)}(x)}{n!} H^n \right) e_j \\
= \sum_{n=0}^m \frac{p_n^{(l)}(x)}{n!} e_i^T H^n e_j = \sum_{n=0}^m \frac{p_n^{(l)}(x)}{n!} (j + 1)^{(n)} e_i^T e_{j+n} \\
= \sum_{n=0}^m p_n^{(l)}(x) (j + 1)^{(n)} \delta_{i,j+n},
\]

(12)
where \( \delta_{i,j+n} \) is the Kronecker symbol and \((j+1)^{(n)} = (j+1)(j+2) \cdots (j+n)\) is the ascending factorial with \((j+1)^{(0)} := 1\). Thus, \((\Phi(x,g(H)))_{ij} = 0\) if \(i < j\), and when \(i = j + n\), i.e. \(i \geq j\), then follows

\[
\left( \frac{d^l}{dx^l} \Phi(x,g(H)) \right)_{ij} = p_{i-j}^{(l)}(x) \frac{i!}{(i-j)! j!} = \binom{i}{j} p_{i-j}^{(l)}(x)
\]

which completes the proof.

**Remark 4.** For the choice \(g(t) = t\), according to Remark 2, and by taking \(l = 0\) in Theorem 1, we get the known result, cf. [6]

\[\exp(xH) = P[x^n], \quad x \in \mathbb{R}.\]  

(13)

**Corollary 1.** In the conditions of Theorem 1,

\[\left( g(H) \right)^l = P[p_n^{(l)}(0)], \quad l = 0, 1, \ldots, m.\]

**Proof.** By observing that

\[\left( g(H) \right)^l = \frac{d^l}{dx^l} \Phi(0, g(H)), \quad l = 0, 1, \ldots, m,
\]

from (11) the statement follows immediately.

Consider now the matrix \(B\), hereafter called **Bell matrix**, whose entries are related to the sequence \(\{b_n\}_{n \geq 0}\) with \(b_0 = 0\) and \(b_1 \neq 0\), as follows

\[
(B)_{ij} = \begin{cases} 
B_{i,j} := B_{i,j}(b_1, b_2, \ldots, b_{i-j+1}), & \text{if } i \geq j \geq 1 \\
1, & \text{if } i = j = 0 \\
0, & \text{otherwise, } i, j = 0, 1, \ldots, m.
\end{cases}
\]

It is worth to note that this lower triangular matrix is nonsingular due to the fact that \(\text{diag}(B) = (1, b_1, b_1^2, \ldots, b_1^n)\) (see Remark 1).
The creation matrix $H$ plays also a role on the construction of the Bell matrix $B$, as we can see in the following result.

**Theorem 2.** Let $H$ be the creation matrix defined by (10) and $g$ the given formal power series (1). The Bell matrix $B$ can be obtained by adjoining the first column of each matrix $\left(\frac{g(H)}{k!}\right)^k$, $k = 0, 1, \ldots, m$, i.e.,

$$B = \left[ (g(H))_0^0 e_0 | (g(H))_0^1 e_0 | (g(H))_0^2 e_0 | \cdots | (g(H))_m^m e_0 \right] \Lambda^{-1}, \quad (14)$$

where $\Lambda = \text{diag}(0!, 1!, 2!, \ldots, m!)$.

**Proof.** For $k = 0$, the result is trivial. For $k = 1, 2, \ldots, m$, using Corollary 1, we have

$$\left(\frac{g(H)}{k!}\right)^k_{i0} = \left( P[p_n^{(k)}(0)]\right)_{i0} = \left( \begin{array}{c} i \\ 0 \end{array} \right) p_i^{(k)}(0), \quad i = 0, 1, \ldots, m.$$  

By recalling that $p_i(x)$ is of exact degree $i$ (see Remark 1), for $k > i$, $p_i^{(k)}(0) = 0$. In addition, by using (3), for $1 \leq k \leq i$, $p_i^{(k)}(0) = k! B_{i,k}$. Therefore,

$$\left(\frac{g(H)}{k!}\right)^k_{i0} = \begin{cases} k! B_{i,k}, & i \geq k \\ 0, & \text{otherwise} \end{cases}, \quad i = 0, 1, \ldots, m, \quad k = 1, \ldots, m,$$

which completes the proof.

**Example 1.** Let $m = 4$. In this case, the creation matrix is

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}$$
and the Bell matrix is

\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & B_{1,1} & 0 & 0 & 0 \\
0 & B_{2,1} & B_{2,2} & 0 & 0 \\
0 & B_{3,1} & B_{3,2} & B_{3,3} & 0 \\
0 & B_{4,1} & B_{4,2} & B_{4,3} & B_{4,4}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & b_1 & 0 & 0 & 0 \\
0 & b_2 & b_1^2 & 0 & 0 \\
0 & b_3 & 3b_1b_2 & b_1^3 & 0 \\
0 & b_4 & 4b_1b_3 + 3b_2^2 & 6b_1b_2 & b_1^4
\end{bmatrix}.
\]

Now, given the formal power series (1), we have

\[
g(H) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
b_1 & 0 & 0 & 0 & 0 \\
b_2 & 2b_1 & 0 & 0 & 0 \\
b_3 & 3b_2 & 3b_1 & 0 & 0 \\
b_4 & 4b_3 & 6b_2 & 4b_1 & 0
\end{bmatrix},
\]

\[
(g(H))^2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
2b_1^2 & 0 & 0 & 0 & 0 \\
6b_1b_2 & 6b_1^2 & 0 & 0 & 0 \\
8b_1b_3 + 6b_2^2 & 24b_1b_2 & 12b_1^2 & 0 & 0
\end{bmatrix}.
\]
By taking the first column of the identity matrix of order 5 and the first column of each of the above matrices and multiplying by the inverse of $\Lambda$, we form the matrix $B$ as expected.

**Remark 5.** The entries of the second column of the matrix $B$ are the given constants $b_0, b_1, b_2, \ldots, b_m$. Indeed, from (14) and Corollary 1, it is clear that

$$
(B)_{i1} = (g(H))_{i0} = (P[p_n'(0)])_{i0} = b_i, \quad i = 0, 1, \ldots, m.
$$

**Corollary 2.** In the conditions of Theorem 2, if $g(t) = b_1 t$ ($b_1 \neq 0$) and defining

$$
D(\ell) = \text{diag}(1, \ell, \ell^2, \ldots, \ell^m), \quad \ell \in \mathbb{R},
$$

the corresponding Bell matrix is given by

$$
B = D(b_1).
$$
Proof. The assertion follows by taking into account that $(g(H))^k e_0 = b_k^H e_0 = k! b_k^H e_k$, for each $k = 0, 1, \ldots, m$.

Consider now the nonsingular matrix

$$f(H) = \sum_{n=0}^{m} c_n \frac{H^n}{n!}, \quad c_0 \neq 0,$$

obtained substituting $t$ by $H$ in (7) and $M$, the so-called transfer matrix whose entries are given by

$$(M)_{ij} = \begin{cases} \binom{i}{j} c_{i-j}, & i \geq j \\ 0, & \text{otherwise}, \quad c_0 \neq 0, \quad i, j = 0, 1, \ldots, m. \end{cases}$$

As proved in [2, Theorem 3.2]

$$M = f(H). \quad (16)$$

Therefore, by introducing the vector

$$s(x) = (s_0(x) \ s_1(x) \ \cdots \ s_m(x))^T,$$

hereafter called Sheffer vector, we are able to write relation (8) in matrix form as follows

$$s(x) = P[p_n(x)] M e_0. \quad (17)$$

Due to Theorem 1 and (16), the matrices $P[p_n(x)]$ and $M$ are both functions of $H$ and then commute. Consequently, the previous relation can be rewritten as

$$s(x) = M P[p_n(x)] e_0 = M p(x), \quad (18)$$

where $p(x) = (p_0(x) \ p_1(x) \ \cdots \ p_m(x))^T$. We observe that this is the matrix form stated in Theorem 4.6 of [9], which constitutes the matrix form counterpart of the result given in Theorem 3.2 of the same paper (for the sake
of clarity, we stress that the matrix here denoted by $M$ coincides with its
inverse in [9]).

From (3) it is easy to check that $p(0) = e_0$ and therefore the entries of
$s(0)$, which are the constant terms of the Sheffer polynomials, coincide with
the first column of the matrix $M$, i.e.

$$s(0) = Me_0.$$  

By introducing the vector of monomial powers

$$\xi(x) = (1 \ x \ \cdots \ x^m)^T,$$
the algebraic relation (3) can be written in matrix form as

$$p(x) = B\xi(x).$$  \hspace{1cm} (19)

Therefore (18) becomes

$$s(x) = MB\xi(x).$$  \hspace{1cm} (20)

From this relation it is visible in a very clear way that a Sheffer sequence is
determined by assigning an Appell sequence and a polynomial sequence of
bynomial type. For this reason, from now on we shall refer to $s(x)$ as the

**Sheffer vector for the pair** $(M,B)$.

**Remark 6.** It is easy to check that the Sheffer vector for the pair $(I,B)$
just contains polynomials of binomial type and the entries of the matrix $B$
determine the different polynomial sequences (see (19)). In addition, when
$g(t) = t$ (which is the case of the Appell polynomials), $B$ coincides with the
identity matrix (see Corollary 2). In this case, the matrix form for the Appell
polynomials given in [2, eq.(3.9)] is recovered from (20) and the Sheffer vector
for the pair \((M, I)\) just contains Appell polynomials, determined by the entries of \(M\).

Recalling that both matrices \(M\) and \(B\) are nonsingular, from (20) the vector \(\xi(x)\) can be expressed in terms of Sheffer polynomials as

\[
\xi(x) = B^{-1}M^{-1}s(x)
\]

cf.[9, Theorem 4.7] and therefore, the Appell vector \(M\xi(x)\) can be related to the Sheffer vector \(s(x)\) for the pair \((M, B)\) by

\[
M\xi(x) = MB^{-1}M^{-1}s(x).
\]

4. Some properties of Sheffer polynomials

By considering the sequence of binomial type \(\{p_n(x)\}_{n \geq 0}\) and the corresponding generalized Pascal matrix \(P[p_n(x)]\), we can rewrite in matrix form the relation (9) as follows

\[
\mathbf{p}(x + y) = P[p_n(y)]\mathbf{p}(x), \quad \forall x, y \in \mathbb{R}. \tag{21}
\]

This enable us to obtain the matrix representation for the well-known Sheffer identity

\[
s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) s_{n-k}(y), \quad \forall n \in \mathbb{N}_0, \ x, y \in \mathbb{R},
\]

as follows:

**Theorem 3.** The vector \(\mathbf{s}(x)\) is the Sheffer vector for the pair \((M, B)\) if and only if the following identity holds

\[
\mathbf{s}(x + y) = P[p_n(y)]\mathbf{s}(x). \tag{22}
\]
Proof. Let \( s(x) \) be the Sheffer vector for the pair \((M, B)\). By using (18), (21), and the fact that \( P([p_n(y)]) \) and \( M \) commute (see Theorem 1 and (16)), we obtain

\[
 s(x + y) = M p(x + y) = MP([p_n(y)]) p(x) = P([p_n(y)]) M p(x) = P([p_n(y)]) s(x).
\]

Reciprocally, the Sheffer vector is obtained setting \( x = 0 \) in (22) and recalling that \( s(0) = Me_0 \).

From (22), choosing \( y = (m - 1)x \), it is immediate to derive the multiplication theorem

\[
 s(mx) = P([p_n((m - 1)x)]) s(x)
\]

which is the matrix form of the algebraic relation given in [9, Theorem 3.5].

Some other properties of Sheffer polynomials can be obtained by making explicitly use of the inverse of the transfer matrix \( M \). Setting

\[
 M^{-1} = \sum_{k=0}^{m} \gamma_k \frac{H^k}{k!},
\]

by straightforward calculations it can be checked that

\[
 \gamma_0 = \frac{1}{c_0}, \quad \gamma_k = -\frac{1}{c_0} \sum_{s=0}^{k-1} \binom{k}{s} c_{k-s} \gamma_s, \quad k = 1, 2, \ldots, m.
\]

In addition,

\[
 (M^{-1})_{ij} = \begin{cases} 
 (i^j) \gamma_{i-j}, & i \geq j \\
 0, & \text{otherwise,} \quad \gamma_0 \neq 0, \quad i, j = 0, 1, \ldots, m.
\end{cases}
\]

In fact,

\[
 (M^{-1})_{ij} = \sum_{k=0}^{m} \frac{\gamma_k}{k!} e_i^T H^k e_j = \sum_{k=0}^{m} \frac{\gamma_k}{k!} (j + 1)^{(k)} \frac{j^k}{n!} \delta_{i,j+k}.
\]
Thus, $(M^{-1})_{ij} = 0$ if $i < j$, and when $i = j + k$, i.e. $i \geq j$, then follows

$$(M^{-1})_{ij} = \gamma_{i-j} \frac{i!}{(i-j)!j!} = \binom{i}{j} \gamma_{i-j}.$$  

Consequently, (18) implies that

$$M^{-1} s(x) = p(x).$$

or, equivalently, (see [9, Corollary 4.3]):

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_{n-k} s_k(x), \quad n = 0, 1, \ldots, m,$$

from which we deduce a general recurrence relation for Sheffer polynomials (see [9, Theorem 4.2]):

$$s_n(x) = \frac{1}{\gamma_0} \left( p_n(x) - \sum_{k=0}^{n-1} \binom{n}{k} \gamma_{n-k} s_k(x) \right), \quad n = 0, 1, \ldots.$$

5. Examples

By appropriately choosing the pair $(f(t), g(t))$ many of the classical Sheffer polynomials can be derived. In particular, we consider

**i** the generalized Hermite polynomials $\{H_n^{(\mu, \nu)}(x)\}_{n \geq 0}$, with $\mu$ a positive integer, when

$$(f(t), g(t)) = (\exp(-t^\mu), \nu t);$$

\^\footnote{The expression of the pair $(f(t), g(t))$ associated to the Abel polynomials may be found in [15, Example 16.6, p.341], while for all the other reported sequences of classical Sheffer polynomials one can refer to [12, Table 2].}
(ii) the generalized Laguerre polynomials \( \{n!L_n^{(\alpha)}(x)\}_{n \geq 0}, \alpha > -1, \) when
\[
(f(t), g(t)) = \left( (1-t)^{-\alpha-1}, \frac{t}{t-1} \right);
\]

(iii) the actuarial polynomials \( \{a_n^{(\beta)}(x)\}_{n \geq 0}, \beta \geq 0, \) when
\[
(f(t), g(t)) = (\exp(\beta t), 1 - \exp(t));
\]

(iv) the Poisson-Charlier polynomials \( \{c_n(x; a)\}_{n \geq 0}, a \neq 0, \) when
\[
(f(t), g(t)) = \left( \exp(-t), \ln \left( 1 + \frac{t}{a} \right) \right);
\]

(v) the Bernoulli polynomials of the second kind \( \{b_n(x)\}_{n \geq 0}, \) when
\[
(f(t), g(t)) = \left( \frac{t}{\ln(1+t)}, \ln(1+t) \right).
\]

(vi) the Abel polynomials \( \{p_n(x; a)\}_{n \geq 0}, a \neq 0, \) when
\[
(f(t), g(t)) = \left( 1, \sum_{n=1}^{+\infty} (-an)^{n-1} \frac{t^n}{n!} \right).
\]

We saw that the knowledge of the pair \((M, B)\) is sufficient for a concrete matrix representation formula for any Sheffer sequence (see (20)). We now specify for the cases referred above the corresponding pairs.

(i) For the generalized Hermite polynomials \( \{H_n^{(\mu,\nu)}(x)\}_{0 \leq n \leq m}, \) since \( g(t) = \nu t, \) the given sequence \( \{b_n\}_{n \in \mathbb{N}} \) is defined by \( b_1 = \nu \) and \( b_n = 0, n \geq 2. \)

Thus, the use of (5) yields to
\[
(B)_{ij} = \begin{cases} 
B_{i,j}(\nu, 0, \ldots, 0) = \nu^j \delta_{i,j}, & i \geq j \geq 1 \\
1, & i = j = 0 \\
0, & \text{otherwise, } i, j = 0, 1, \ldots, m,
\end{cases}
\]
i.e., \( B = D(\nu) \) is the diagonal matrix defined by (15). Therefore, we have

\[
(M, B) = (\exp(-H^\mu), D(\nu)).
\]

It is worth to note that the change of variable \( t = \tau/\nu \) transforms the sequence of the generalized Hermite polynomials of Sheffer type into the one of Appell type. In particular, for the polynomials associated to the choice \( \mu = \nu = 2 \) the pair becomes

\[
(f(\tau), g(\tau)) = \left( \exp \left( -\frac{\tau^2}{4} \right), \tau \right),
\]

or, in matrix form,

\[
(M, B) = \left( \exp \left( -\frac{H^2}{4} \right), I \right).
\]

This is the pair corresponding to the Appell sequence of polynomials known as monic Hermite polynomials \( \{\tilde{H}_n(x)\}_{0 \leq n \leq m} \) considered in [2, eq. (3.14)].

(ii) For the generalized Laguerre polynomials \( \{n!L_n^{(\alpha)}(x)\}_{0 \leq n \leq m} \), considering that

\[
g(t) = -\frac{t}{1-t} = -\sum_{n=1}^{+\infty} t^n,
\]

the given sequence \( \{b_n\}_{n \in \mathbb{N}} \) is defined by \( b_n = -n! \). Then, from (4) we obtain that the corresponding Bell matrix has the following entries, for each \( i, j = 0, 1, \ldots, m \),

\[
(B)_{ij} = \begin{cases} 
(-1)^j B_{i,j}(1!, 2!, \ldots, (i-j+1)!), & i \geq j \geq 1 \\
1, & i = j = 0 \\
0, & \text{otherwise.}
\end{cases}
\]
Denoting by $\mathcal{L}$ the matrix whose nonzero entries are the Lah numbers, i.e.,

$$
(\mathcal{L})_{ij} = \begin{cases} 
\frac{(-1)^{i-1}i!}{j!}, & i \geq j \geq 1 \\
1, & i = j = 0 \\
0, & \text{otherwise, } i, j = 0, 1, \ldots, m,
\end{cases}
$$

and by taking into account Remark 3 and (15), we can rewrite the Bell matrix as follows:

$$
B = \mathcal{L}D(-1).
$$

Consequently, for the generalized Laguerre polynomials the pair is

$$(M, B) = ((I - H)^{-\alpha - 1}, \mathcal{L}D(-1)).$$

(iii) For the actuarial polynomials, also known as Toscano polynomials [18], $M = P[\beta^n], \text{ by (13)}. \text{ Now, considering that}$

$$
g(t) = 1 - \exp(t) = -\sum_{n=1}^{+\infty} \frac{t^n}{n!},
$$

the sequence $\{b_n\}_{n \in \mathbb{N}}$ in this case is defined by $b_n = -1$. This fact, together with (4), implies that the corresponding Bell matrix is given by

$$
(B)_{ij} = \begin{cases} 
(-1)^j B_{i,j}(1, 1, \ldots, 1), & i \geq j \geq 1 \\
1, & i = j = 0 \\
0, & \text{otherwise, } i, j = 0, 1, \ldots, m.
\end{cases}
$$

Then, by introducing $\mathcal{S}$, the matrix whose nonzero entries are the Stir-
ling numbers of the second kind, i.e.,

\[
(S)_{ij} = \begin{cases} 
S(i, j), & i \geq j \geq 1 \\
1, & i = j = 0 \\
0, & \text{otherwise, } i, j = 0, 1, \ldots, m,
\end{cases}
\]

from Remark 3 and (15), we obtain that in this case

\[B = SD(-1).\]

Thus, the pair characterizing the actuarial polynomials is the following:

\[(M, B) = (P[\beta^n], SD(-1)). \quad (23)\]

(iv) For the Poisson-Charlier polynomials \(\{c_n(x; a)\}_{0 \leq n \leq m}\), considering that
g(t) = \ln \left(1 + \frac{t}{a}\right) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \left(\frac{t}{a}\right)^n,

the sequence \(\{b_n\}_{n \in \mathbb{N}}\) is defined by \(b_n = (-1)^{n+1}(n-1)!/a^n\). Using property (4), we obtain the following entries for the Bell matrix:

\[
(B)_{ij} = \begin{cases} 
(-1)^{i-j} \left(\frac{1}{a}\right)^i B_{i,j}(0!, 1!, \ldots, (i-j)!), & i \geq j \geq 1 \\
1, & i = j = 0 \\
0, & \text{otherwise,}
\end{cases}
\]

\(i, j = 0, 1, \ldots, m\). Recalling that \((-1)^{i-j}B_{i,j}(0!, 1!, \ldots, (i-j)!)) = s(i, j),\) the Stirling numbers of the first kind (see Remark 3), and taking into account the fact that the Stirling numbers of the first and second kinds can be considered inverses of one another cf. [7, p.144], the matrix \(B\) can be decomposed as

\[B = D(a)^{-1}S^{-1}.\]
Then, from (13), the pair for the Poisson-Charlier polynomials can be written as

$$ (M, B) = (P[(-1)^n], (SD(a))^{-1}). \tag{24} $$

It is worth to observe that, setting $\beta = 1$ in example (iii), the function $f$ of the actuarial polynomials is the inverse of the corresponding function associated to the Poisson-Charlier polynomials. Moreover, in the case $a = -1$, this happens also for the corresponding functions $g$. Both these facts become now immediately visible looking on the matrices $M$ and $B$ assigned to each set of polynomials (see (23) and (24)).

(v) For the Bernoulli polynomials of the second kind $\{b_n(x)\}_{0 \leq n \leq m}$, as

$$ f(t) = \left( \ln(1 + t) \right)^{-1} = \left( \sum_{n=0}^{+\infty} \frac{(-t)^n}{n+1} \right)^{-1} $$

and the function $g$ is a particular case of the one associated to the Poisson-Charlier polynomials corresponding to $a = 1$, we have

$$ (M, B) = \left( \left( \sum_{n=0}^{m} \frac{(-H)^n}{n+1} \right)^{-1}, S^{-1} \right). $$

Let us consider now the change of variable $t = \exp(\tau) - 1$. It transforms the associated pair $(f(t), g(t))$ into the pair

$$ (f(\tau), g(\tau)) = \left( \frac{\exp(\tau) - 1}{\tau}, \tau \right). $$

Therefore, we obtain Appell polynomials with

$$ M = \sum_{n=0}^{m} \frac{H^n}{(n+1)!}. $$
It is worth to note that such matrix $M$ is the inverse of the transfer matrix associated to the classical \textit{Bernoulli polynomials} (see [2, eq. (3.12)]).

\textbf{(vi)} For the Abel polynomials the transfer matrix $M$ is the identity (they are of binomial type). Concerning the matrix $B$, we observe that

$$g(t) = \sum_{n=1}^{+\infty} (-an)^{n-1} \frac{t^n}{n!}$$

gives the sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by $b_n = (-an)^{n-1}$. Therefore, for each $i, j = 0, 1, \ldots, m$,

$$(B)_{ij} = \begin{cases} B_{i,j} (1, -2a, \ldots, -(i - j + 1)a)^{i-j),} & i \geq j \geq 1 \\ 1, & i = j = 0 \\ 0, & \text{otherwise}, \end{cases}$$

and using (6), $(B)_{ij} = \binom{i-1}{j-1} (-ai)^{i-j}, i \geq j \geq 1, i, j = 0, 1, \ldots, m$.

\section*{6. Conclusion}

The new matrix representation here introduced for Sheffer polynomial sets highlights that such sequences generalize both the Appell and binomial type polynomials. The key point of this unified approach is the so-called creation matrix. In fact, the paper shows how the two matrices related to Appell and binomial type polynomials occurring in the definition of Sheffer polynomials, are functions of it. In this framework, using only elementary linear algebra and combinatorics, the vector whose entries are Sheffer polynomials can be produced easily, stressing the effectiveness of our approach in practical calculations. In addition, the proposed matrix representation allows
to discover some connections between different kinds of Sheffer polynomials which could be not evident at a first glance.

References


